# Belief Propagation based MIMO Detection Operating on Quantized Channel Output

Amine Mezghani and Josef A. Nossek

Institute for Circuit Theory and Signal Processing

Munich University of Technology, 80290 Munich, Germany

E-Mail: {Mezghani, Nossek}@nws.ei.tum.de

*Abstract*— In multiple-antenna communications, as bandwidth and modulation order increase, system components must work with demanding tolerances. In particular, high resolution and high sampling rate analog-to-digital converters (ADCs) are often prohibitively challenging to design. Therefore ADCs for such applications should be low-resolution. This paper provides new insights into the problem of optimal signal detection based on quantized received signals for multiple-input multiple-output (MIMO) channels. It capitalizes on previous works [1], [2], [3], [4] which extensively analyzed the unquantized linear vector channel using graphical inference methods. In particular, a "loopy" belief propagation-like (BP) MIMO detection algorithm, operating on quantized data with low complexity, is proposed. In addition, we study the impact of finite receiver resolution in fading channels in the large-system limit by means of a state evolution analysis of the BP algorithm, which refers to the limit where the number of transmit and receive antennas go to infinity with a fixed ratio. Simulations show that the theoretical findings might give accurate results even with moderate number of antennas.

## I. INTRODUCTION

Most of the contributions on signal detection for multiple-input multiple-output (MIMO) systems assume that the receiver has access to the channel data with infinite precision. In practice, however, a quantizer (A/D-converter) is applied to the received analog signal, so that the channel measurements can be processed in the digital domain. In ultra-wideband and/or high-speed applications, both the required resolution and speed of the ADCs tend to rise, making them expensive, power intensive and even infeasible [5]. This work deals with MIMO channels with quantized outputs, which we will refer to as quantized MIMO systems.

Recently, graphical inference methods have been applied to the usual unquantized MIMO detection problem. In [1], [2] "loopy" BP-like detection algorithms were derived as low complexity heuristics for computing the marginal distribution of each signal component. Hereby, we provide an extended version of the approximative BP based algorithm relying on the commonly used Gaussian approximation, while taking into account the quantizer operation. Then, we provide a state evolution formalism for analyzing the BP algorithm in the large-system limit, and also several theoretical results on the impact of finite receiver resolution that can be drawn from it. We note that a similar problem has been considered in [6], however based on the *replica method* from statistical physics [7]. In [8], linear systems with general separable output channels have been considered in the large-system limit. Although [8] could include our quantized MIMO case, only sparse systems have been considered and no simulation results have been

provided to validate the theoretical results. In this work, the "loopy" BP algorithm operating on quantized dense linear systems is studied theoretically and experimentally. Moreover, our derivation steps for the large-system limit are quite straightforward and well justified. The main advantage of the BP approach compared to [6] is that it is more intuitive and allows to find efficient algorithms and analyze their performance and convergence behavior. In order to ease calculations, we restrict ourselves to real-valued systems. However, the results can be extended to the complex case.

Our paper is organized as follows. Section II describes the system model. In Section III, an approximative BP-like detection algorithm operating on quantized data is derived; then, we provide a state evolution analysis of the BP algorithm and study the effects of quantization in the large-system limit in Section IV. Finally, in Section V, some simulation results are presented to numerically validate the theoretical findings.

## II. SYSTEM MODEL

We consider a point-to-point MIMO channel where the transmitter employs $K$ antennas and the receiver has $N$ antennas. Let the vector $\boldsymbol{x} \in \mathbb{R}^K$ comprises the $K$ transmitted i.i.d. symbols, each drawn from a certain distribution $q_0(x)$ with zero mean and variance $c_x$. The unquantized (analog) output vector $\boldsymbol{y} \in \mathbb{R}^N$ is related to the input as

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{\eta}, \tag{1}$$

where $\boldsymbol{H} \in \mathbb{R}^{N \times K}$ is the channel matrix assumed to be perfectly known at the receiver, and $\boldsymbol{\eta}$ refers to Gaussian noise vector with covariance $\boldsymbol{R}_{\eta\eta} = \mathrm{E}[\boldsymbol{\eta}\boldsymbol{\eta}^{\mathrm{T}}] = \sigma_0^2 \mathbf{I}$.

In a practical system, each receive signal component $y_l$, $1 \leq l \leq N$, is quantized by a $b$-bit resolution scalar quantizer (A/D-converter). Thus, the resulting quantized signals read as

$$r_l = \mathcal{Q}(y_l), \tag{2}$$

where $\mathcal{Q}(\cdot)$ denotes the quantization operation. For the case that we use a uniform symmetric mid-riser type quantizer [9], the quantized receive alphabet for each dimension is given by

$$r_l \in \{(-\frac{2^b}{2} - \frac{1}{2} + k)\Delta; \ k = 1, \cdots, 2^b\} = \mathcal{R}, \tag{3}$$

where $\Delta$ is the quantizer step-size and $b$ the number of quantizer bits, which are set the same for all the quantizers.

With these definitions, the conditional probability distribution of the quantized output given an input $\boldsymbol{x}$ reads as

$$p_{\boldsymbol{r}|\boldsymbol{x}}(\boldsymbol{r}|\boldsymbol{x}) = \prod_{l=1}^{N} \rho_0(r_l | \boldsymbol{h}_l^{\mathrm{T}} \boldsymbol{x}), \tag{4}$$

where $\boldsymbol{h}_l^{\mathrm{T}}$ is the $l$-th row of $\boldsymbol{H}$ and

$$
\begin{aligned}
\rho_0(r_l|\boldsymbol{h}_l^{\mathrm{T}}\boldsymbol{x}) &= \frac{1}{\sqrt{2\pi\sigma_0^2}}\int_{r_l^{\mathrm{low}}}^{r_l^{\mathrm{up}}} \mathrm{e}^{\frac{(y-\boldsymbol{h}_l^{\mathrm{T}}\boldsymbol{x})^2}{2\sigma_0^2}}\,dy \\
&= \Phi\left(\frac{r_l^{\mathrm{up}}-\boldsymbol{h}_l^{\mathrm{T}}\boldsymbol{x}}{\sigma_0}\right) - \Phi\left(\frac{r_l^{\mathrm{low}}-\boldsymbol{h}_l^{\mathrm{T}}\boldsymbol{x}}{\sigma_0}\right)
\end{aligned}
\tag{5}
$$

with $\Phi(x)$ represents the cumulative Gaussian distribution given by

$$
\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\exp\left(-\frac{t^2}{2}\right)\mathrm{d}t.
\tag{6}
$$

Hereby the lower and upper quantization boundaries are

$$
r_l^{\mathrm{low}} = \begin{cases} r_l - \frac{\Delta}{2} & \text{for } r_l \geq -\frac{\Delta}{2}(2^b-2) \\ -\infty & \text{otherwise,} \end{cases}
$$

and

$$
r_l^{\mathrm{up}} = \begin{cases} r_l + \frac{\Delta}{2} & \text{for } r_l \leq \frac{\Delta}{2}(2^b-2) \\ +\infty & \text{otherwise.} \end{cases}
$$

### III. APPROXIMATIVE BP DETECTION

Our goal is to derive a low complexity detector computing the conditional mean estimate

$$
\hat{\boldsymbol{x}} = \mathrm{E}[\boldsymbol{x}|\boldsymbol{r}],
\tag{7}
$$

based on the knowledge of $q_0(x)$, $\rho_0(\cdot|\cdot)$ and $\boldsymbol{H}$. This problem is related to the problem of finding the marginal probabilities $p_{x_k|\boldsymbol{r}}(x_k|\boldsymbol{r})$, for which belief propagation can provide low-complexity approximations. To this end, a factor graph representation is needed.

#### A. Factor Graph Representation

In analogy to [4], a factor graph representation of the quantized MIMO system is shown in Fig. 1. Each data stream $x_k$ is represented by a circle, referred to symbol node, and each received quantized signal $r_l$ corresponds to a square, called the signal node. Each edge connecting $k$ and $l$ represents the corresponding gain factor $h_{lk}$, if $h_{lk} \neq 0$. Ignoring the cycles in the graph, let us derive the so called "loopy" BP algorithm (or sum-product algorithm) from the factor graph representation.[1]
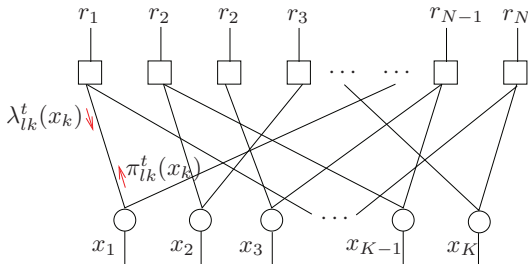


Fig. 1.   Factor-graph representation of the quantized MIMO channel.

[1]We note that the BP is optimal for cycle free graphs and performs nearly optimal in sparse graphs. In the case of dense, large enough, channel matrices, it may provide good approximate posteriors as we will see later.

#### B. BP based Detection

Each iteration $t$ of BP consists in, first sending messages from each signal node $l$ to each symbol node $k$ (horizontal step), and then vice versa (vertical step). The messages contain the extrinsic information of $x_k$ in form of density functions computed based on the previously received messages. We denote the symbol-to-signal messages by $\pi_{lk}^t(x_k)$ and the signal-to-symbol messages by $\lambda_{lk}^t(x_k)$:

Horizontal step:

$$
\lambda_{lk}^t(x_k) = \int \rho_0(r_l|\boldsymbol{h}_l^{\mathrm{T}}\boldsymbol{x}) \prod_{k'\neq k}[\pi_{lk'}^t(x_{k'})\mathrm{d}x_{k'}]
\tag{8}
$$

Vertical step:

$$
\pi_{lk}^{t+1}(x_k) = \alpha_{lk}q_0(x_k)\prod_{l'\neq l}\lambda_{lk}^t(x_k),
\tag{9}
$$

where $\alpha_{lk}$ is a normalization factor so that $\pi_{lk}^{t+1}(x_k)$ is a valid density function. The algorithm is initialized by

$$
\pi_{lk}^0(x_k) = q_0(x_k), \ l=1,\ldots,N, \ k=1,\ldots,K.
\tag{10}
$$

#### C. Approximative BP based Detection Algorithm

In the case of dense matrices, the complexity of the BP algorithm grows enormously with $K$, since a $(K-1)$-dimensional integration (or summation) has to be performed in the horizontal step. Thus we discuss now an approximation scheme, in analogy to [1] (see also [10]), that may be justified in the large-system limit. For this we split the quantity $\boldsymbol{h}_l^{\mathrm{T}}\boldsymbol{x}$ as

$$
\boldsymbol{h}_l^{\mathrm{T}}\boldsymbol{x} = \sum_{k'\neq k} h_{lk'}x_{k'} + h_{lk}x_k = w_{lk} + h_{lk}x_k.
\tag{11}
$$

Given the distribution $p_{w_{lk}}(w_{lk})$, (8) can be rewritten as

$$
\lambda_{lk}^t(x_k) = \int_{-\infty}^{\infty} \rho_0(r_l|w_{lk}+h_{lk}x_k)p_{w_{lk}}(w_{lk})\mathrm{d}w_{lk}.
\tag{12}
$$

The key remark is that $w_{lk}$ is a weighted sum of the independent random variables $x_{k'}$ distributed according to $\pi_{lk'}^t(x_{k'})$. Due to the central-limit theorem, it can be regarded as a Gaussian random variable in the large-system regime, having the mean and the variance

$$
\mu_{lk}^t = \sum_{k'\neq k} h_{lk'}m_{lk'}^t,
\tag{13}
$$

$$
C_{lk}^t = \sum_{k'\neq k} h_{lk'}^2 V_{lk'}^t,
\tag{14}
$$

respectively, where $m_{lk'}^t$ and $V_{lk'}^t$ are the mean and the variance of $x_{k'}$ according to the distribution $\pi_{lk'}^t(x_{k'})$. The horizontal step can be now written as follows

$$
\begin{aligned}
\lambda_{lk}^t(x_k) &\approx \int_{-\infty}^{\infty} \rho_0(r_l|w_{lk}+h_{lk}x_k)\frac{\mathrm{e}^{-\frac{(w_{lk}-\mu_{lk}^t)^2}{2C_{lk}^t}}}{\sqrt{2\pi C_{lk}^t}}\mathrm{d}w_{lk} \\
&= \Phi\left(\frac{r_l^{\mathrm{up}}-\mu_{lk}^t-h_{lk}x_k}{\sqrt{C_{lk}^t+\sigma_0^2}}\right) - \Phi\left(\frac{r_l^{\mathrm{low}}-\mu_{lk}^t-h_{lk}x_k}{\sqrt{C_{lk}^t+\sigma_0^2}}\right) \\
&\doteq \rho_{lk}^t(r_l|\mu_{lk}^t+h_{lk}x_k).
\end{aligned}
\tag{15}
$$

Obviously the approximate BP is a kind of modified parallel interference cancellation (PIC) as in the unquantized case [1], where $\mu_{lk}^t$ represents an estimate of the interference

component in $r_l$ for the stream $k$, and $C_{lk}^t$ quantifies its MSE. Even if this approximate BP iteration holds in the limit of infinite number of antennas, it usually performs well for systems of moderate sizes.

## IV. STATE EVOLUTION ANALYSIS

State evolution analysis (also known as density evolution for the case of sparse matrices) is a powerful tool to study the behavior of belief propagation in the large-system limit [11]. The large-system limit means that we consider the limit when $K$ and $N$ go to infinity, while the ratio $\beta = K/N$ is kept fixed. Under the conjecture that the presented BP based detector would be asymptotically optimal, this analysis would deliver useful theoretical results about the MIMO system performance under quantization. For the analysis, we assume a random channel matrix $\boldsymbol{H}$, where the entries $\{h_{l,k}\}$ are i.i.d. with zero mean and variance $1/N$. The main idea is to approximate the messages by Gaussian densities, which holds exactly in the large-system limit. In fact, given that $h_{lk}x_k$ in (15) scales as $1/\sqrt{N}$, it becomes small as $N$ becomes large, and as such we take the second-order expansion of the messages (15) as

$$\rho_{lk}^t(x_k|\mu_{lk}^t + h_{lk}x_k) \approx \rho_{lk}^t(x_k|\mu_{lk}^t) + \dot{\rho}_{lk}^t(x_k|\mu_{lk}^t)h_{lk}x_k +$$
$$\frac{1}{2}\ddot{\rho}_{lk}^t(x_k|\mu_{lk}^t)h_{lk}^2x_k^2 + \mathcal{O}(N^{-3/2}),$$
(16)

where $\dot{\rho}_{lk}^t(x_k|\mu_{lk}^t)$ and $\ddot{\rho}_{lk}^t(x_k|\mu_{lk}^t)$ denote the first order and the second order derivatives of $\rho_{lk}^t(x_k|\mu_{lk}^t)$ with respect to $\mu_{lk}^t$. Keeping the terms up to the order of $N^{-1}$, and using now the following approximation

$$1 + ax + \frac{1}{2}bx^2 = e^{ax + \frac{1}{2}(b-a^2)x^2} + \mathcal{O}(x^3), \quad (17)$$

the horizontal and vertical steps can be represented as

$$\lambda_{lk}^t(x) \propto \exp\left[\theta_{lk}^t x - \frac{1}{2}(\theta_{lk}^{t,2} - \Xi_{lk}^t)x^2\right], \text{ and} \quad (18)$$

$$\pi_{lk}^{t+1}(x) \propto q_0(x)\exp\left[\sum_{l'\neq l}\theta_{l'k}^t x - \frac{1}{2}\sum_{l'\neq l}(\theta_{l'k}^{t,2} - \Xi_{l'k}^t)x^2\right], \quad (19)$$

respectively, where we introduced the definitions

$$\theta_{lk}^t \doteq \frac{\dot{\rho}_{lk}^t(r_l|\mu_{lk}^t)}{\rho_{lk}^t(r_l|\mu_{lk}^t)}h_{lk} \quad \text{and} \quad \Xi_{lk}^t \doteq \frac{\ddot{\rho}_{lk}^t(r_l|\mu_{lk}^t)}{\rho_{lk}^t(r_l|\mu_{lk}^t)}h_{lk}^2. \quad (20)$$

For the following, we assume a random matrix $\boldsymbol{H}$ with i.i.d. entries and we consider the large-system limit. The state evolution analysis aims to study the dynamics of the BP detector, i.e. to track the evolution of the densities parameters, namely $\mu_{lk}^t$, $C_{lk}^t$ from (13) and (14), respectively, and additionally

$$z_{lk}^t \doteq \sum_{l'\neq l}\theta_{l'k}^t, \quad F_{lk}^t \doteq \sum_{l'\neq l}\theta_{l'k}^{t,2}, \quad G_{lk}^t \doteq \sum_{l'\neq l}\Xi_{l'k}^t \quad (21)$$

over the iterations. Strictly speaking, these variables are random. Relying on a heuristic assumption that the incoming messages $(\theta_{lk}^t, \Xi_{lk}^t)$ to each symbol node remain independent from iteration to iteration, when conditioned on a given transmit symbol $x_k$ and channel $\boldsymbol{H}$, and by the central limit theorem, we conclude that $\mu_{lk}^t$, $C_{lk}^t$ (from (13) and (14)), $z_{lk}^t$,

$F_{lk}^t$ and $G_{lk}^t$ become asymptotically Gaussian. In particular $C_{lk}^t$, $F_{lk}^t$ and $G_{lk}^t$ are sums of terms of order $1/N$, thus they admit asymptotically zero variance. In other words, they become deterministic in the large-system limit at given iteration provided that $\boldsymbol{H}$ has i.i.d. entries, i.e. independent of the given realization of $\boldsymbol{H}$, the received vector $\boldsymbol{r}$ and the indexes $l$ and $k$; that is

$$\sum_{k'\neq k}h_{lk'}^2 V_{lk'}^t \to \sum_{k'\neq k}\mathrm{E}_{\boldsymbol{r}|x_k}[h_{lk'}^2 V_{lk'}^t|\boldsymbol{H}, x_k] \doteq C^t, \quad (22)$$

$$\sum_{l'\neq l}\theta_{l'k}^{t,2} \to \sum_{l'\neq l}\mathrm{E}_{\boldsymbol{r}|x_k}[\theta_{l'k}^{t,2}|\boldsymbol{H}, x_k] \doteq F^t, \quad (23)$$

$$\sum_{l'\neq l}\Xi_{l'k}^t \to \mathrm{E}_{\boldsymbol{r}|x_k}[\Xi_{l'k}^t|\boldsymbol{H}, x_k] \doteq G^t, \quad (24)$$

where $\to$ symbolizes the convergence to the asymptotic limit. Therefore, we can write the messages $\pi_{lk}^{t+1}(x)$ as

$$\pi_{lk}^{t+1}(x) \to \alpha q_0(x)\exp\left[z_{lk}^t x - \frac{1}{2}(F^t - G^t)x^2\right]. \quad (25)$$

We will return to the calculation of these parameters later on. Now let us consider the joint distribution $p_{w_{lk},\mu_{lk}^t}(w, \mu)$ of the interference term in antenna $l$ for symbol $k$, $w_{lk}$ from (11), and its estimate $\mu_{lk}^t$ in (13), given $\boldsymbol{H}$ and $x_k$. Again by the central limit argument, $w_{lk}$ and $\mu_{lk}^t$ are asymptotically jointly Gaussian

$$(w_{lk}, \mu_{lk}^t) \sim \mathcal{N}\left(\boldsymbol{0}, \boldsymbol{R}_{w,\mu}^t\right),$$

with the covariance matrix $\boldsymbol{R}_{w,\mu}^t$, having the entries

$$\mathrm{E}[w_{lk}^2|\boldsymbol{H}, x_k] = \mathrm{E}\left[\left(\sum_{k'\neq k}h_{lk'}x_{k'}\right)^2\Big|\boldsymbol{H}, x_k\right]$$
$$= \sum_{k'\neq k}h_{lk'}^2\mathrm{E}[x_{k'}^2|\boldsymbol{H}] \quad (26)$$
$$\to \beta\int x_{k'}q_0(x_{k'})\mathrm{d}x_{k'} \doteq \beta c_x,$$

$$\mathrm{E}[\mu_{lk}^{t,2}|\boldsymbol{H}, x_k] = \mathrm{E}[(\sum_{k'\neq k}h_{lk'}m_{lk'}^t)^2|\boldsymbol{H}, x_k]$$
$$= \sum_{k'\neq k}h_{lk'}^2\mathrm{E}[m_{lk'}^{t,2}|\boldsymbol{H}] \quad (27)$$
$$\to \beta\mathrm{E}_{z_{lk'}^t}\left[\left(\int x\pi_{lk'}^t(x)\mathrm{d}x\right)^2\right] \doteq \beta c_m^t,$$

$$\mathrm{E}[w_{lk}\mu_{lk}^t|\boldsymbol{H}, x_k] = \sum_{k', k''\neq k}h_{lk'}h_{lk''}\mathrm{E}[x_{k''}m_{lk'}^t|\boldsymbol{H}]$$
$$= \sum_{k'\neq k}h_{lk'}^2\mathrm{E}[x_{k'}m_{lk'}^t|\boldsymbol{H}] \quad (28)$$
$$\to \beta\mathrm{E}_{x_{k'}, z_{lk'}^t}\left[x_{k'}\int x\pi_{lk'}^t(x)\mathrm{d}x\right] \doteq \beta c_{x,m}^t.$$

In the steps above, we used the fact that the expectations with respect to $x_{k'}$ and $z_{lk'}^t$ become asymptotically independent of the indexes $l$ and $k'$. Thus, we obtain the bivariate Gaussian distribution

$$p_{w_{lk},\mu_{lk}^t}(w, \mu) = \frac{e^{-\frac{(w - \frac{c_{x,m}^t}{c_m^t}\mu)^2}{2\beta(c_x - \frac{c_{x,m}^{t,2}}{c_m^t})}}}{\sqrt{2\pi\beta(c_x - \frac{c_{x,m}^{t,2}}{c_m^t})}}\frac{e^{-\frac{\mu^2}{2\beta c_m^t}}}{\sqrt{2\pi\beta c_m^t}}. \quad (29)$$

Next, the MSE parameter $C^t$ in (22) can be expressed as

$$C^t = \sum_{k' \neq k} \mathrm{E}\left[h_{lk'}^2 \int x^2 \pi_{lk'}^t(x)\mathrm{d}x - m_{lk'}^{t,2} \middle| \boldsymbol{H}, x_k\right]$$

$$= \sum_{k' \neq k} h_{lk'}^2 \mathrm{E}\left[\int x^2 \pi_{lk'}^t(x)\mathrm{d}x \middle| \boldsymbol{H}, x_k\right] - \sum_{k' \neq k} h_{lk'}^2 \mathrm{E}[m_{lk'}^{t,2}|\boldsymbol{H}]$$

$$\to \beta \mathrm{E}_{x_{k'}, z_{lk}^t}\left[\int x^2 \pi_{lk'}^t(x)\mathrm{d}x\right] - \beta c_m^t \doteq \beta(c_{\hat{x}}^t - c_m^t). \quad (30)$$

Note that the densities $\rho_{lk}^t(\cdot|\cdot)$ defined for the horizontal step in (15) become independent of $l$ and $k$, and read as

$$\rho^t(r|\mu) = \Phi\left(\frac{r^{\mathrm{up}} - \mu}{\sqrt{\beta(c_{\hat{x}}^t - c_m^t) + \sigma_0^2}}\right) - \Phi\left(\frac{r^{\mathrm{low}} - \mu}{\sqrt{\beta(c_{\hat{x}}^t - c_m^t) + \sigma_0^2}}\right). \quad (31)$$

Afterwards, we compute $F^t$ from (23). For that we need the joint distribution $p_{\boldsymbol{r}, \mu_{lk}^t | x_k}(\boldsymbol{r}, \mu_{lk}^t | x_k)$ for fixed channel $\boldsymbol{H}$

$$p_{\boldsymbol{r}, \mu_{lk}^t | x_k}(\boldsymbol{r}, \mu_{lk}^t | x_k) = \int p_{\boldsymbol{r}, \mu_{lk}^t, w_{lk} | x_k}(\boldsymbol{r}, \mu_{lk}^t, w_{lk} | x_k)\mathrm{d}w_{lk}$$

$$= p_{\boldsymbol{r}_l | x_k}(\boldsymbol{r}_l | x_k) \int p_{r_l, \mu_{lk}^t, w_{lk} | x_k}(r_l, \mu_{lk}^t, w_{lk} | x_k)\mathrm{d}w_{lk}$$

$$= p_{\boldsymbol{r}_l | x_k}(\boldsymbol{r}_l | x_k) \int \rho_0(r_l | w_{lk} + h_{lk}x_k) p_{w_{lk}, \mu_{lk}^t}(w_{lk}, \mu_{lk}^t)\mathrm{d}w_{lk}$$

$$\to p_{\boldsymbol{r}_l | x_k}(\boldsymbol{r}_l | x_k) \bar{\rho}^t(r_l | \mu_{lk}^t + h_{lk}x_k) \frac{e^{-\frac{\mu_{lk}^{t,2}}{2\beta c_m^t}}}{\sqrt{2\pi\beta c_m^t}}, \quad (32)$$

where $\boldsymbol{r}_l$ is the vector containing the elements of $\boldsymbol{r}$ excluding $r_l$ and the density function $\bar{\rho}^t(\cdot|\cdot)$ is obtained using (29) by performing the integration as

$$\bar{\rho}^t(r|\mu) = \Phi\left(\frac{r^{\mathrm{up}} - \mu}{\sqrt{\beta(c_x - \frac{c_{x,m}^{t,2}}{c_m^t}) + \sigma_0^2}}\right) - \Phi\left(\frac{r^{\mathrm{low}} - \mu}{\sqrt{\beta(c_x - \frac{c_{x,m}^{t,2}}{c_m^t}) + \sigma_0^2}}\right). \quad (33)$$

From (23), (20) and (32) we identify after some straightforward steps the asymptotic non-vanishing term for the parameter $F^t$

$$F^t = \sum_{r \in \mathcal{R}} \int \bar{\rho}^t(r|\mu) \left[\frac{\dot{\rho}^t(r|\mu)}{\rho^t(r|\mu)}\right]^2 \frac{e^{-\frac{\mu^2}{2\beta c_m^t}}}{\sqrt{2\pi\beta c_m^t}}\mathrm{d}\mu, \quad (34)$$

where we dropped the indexes $l$ and $k$ due to asymptotic independence in the large-system limit. Similarly, we obtain the parameter $G^t$ in (25) as follows

$$G^t = \sum_{r \in \mathcal{R}} \int \bar{\rho}^t(r|\mu) \frac{\ddot{\rho}^t(r|\mu)}{\rho^t(r|\mu)} \frac{e^{-\frac{\mu^2}{2\beta c_m^t}}}{\sqrt{2\pi\beta c_m^t}}\mathrm{d}\mu. \quad (35)$$

We turn now to determine the distribution of $z_{lk}^t$ defined in (21), which, as mentioned before, follows a Gaussian distribution conditioned on $x_k$. Again by (20) and (32), we show that its mean is

$$\mathrm{E}_{\boldsymbol{r}|x_k, \boldsymbol{H}}[z_{lk}^t | x_k, \boldsymbol{H}] =$$

$$= \sum_{r \in \mathcal{R}} \int \sum_{l' \neq l} \bar{\rho}^t(r|\mu + h_{l'k}x_k) \frac{\dot{\rho}^t(r|\mu)}{\rho^t(r|\mu)} h_{l'k} \frac{e^{-\frac{\mu^2}{2\beta c_m^t}}}{\sqrt{2\pi\beta c_m^t}}\mathrm{d}\mu$$

$$\to \sum_{r \in \mathcal{R}} \int \dot{\rho}^t(r|\mu) \frac{\dot{\rho}^t(r|\mu)}{\rho^t(r|\mu)} \frac{e^{-\frac{\mu^2}{2\beta c_m^t}}}{\sqrt{2\pi\beta c_m^t}}\mathrm{d}\mu \cdot x_k \doteq E^t \cdot x_k, \quad (36)$$

and its variance is

$$\mathrm{E}_{\boldsymbol{r}|x_k, \boldsymbol{H}}[z_{lk}^{t,2} | x_k, \boldsymbol{H}] - (\mathrm{E}_{\boldsymbol{r}|x_k, \boldsymbol{H}}[z_{lk}^t | x_k, \boldsymbol{H}])^2$$

$$\to \sum_{l' \neq l} \mathrm{E}_{\boldsymbol{r}|x_k, \boldsymbol{H}}[\theta_{l'k}^{t,2} | x_k, \boldsymbol{H}] \stackrel{!}{=} F^t. \quad (37)$$

Note that $z_{lk}^t \approx \sum_{l'=1}^N \theta_{l'k}^t = z_k^t$ and thus $\pi_{lk}^t(x_k) \approx \pi_k^t(x_k)$ for all $l$ because each $\theta_{lk}$ has a vanishingly small effect on the sum. In summary, we get the conditional density

$$\rho_G^t(z_k^t | x_k, \boldsymbol{H}) = \frac{1}{\sqrt{2\pi F^t}} e^{-\frac{(z_k^t - E^t x_k)^2}{2F^t}}. \quad (38)$$

Since the BP iteration is initialized by $\pi_{lk}(x_k) = q_0(x_k)$, we have from (27), (28) and (30) the initial parameters $c_m^0 = c_{x,m}^0 = 0$ and $c_{\hat{x}}^0 = c_x$, it can be shown by mathematical induction that for all $t$

$$c_m^t = c_{x,m}^t, \quad c_{\hat{x}}^t = c_x, \quad \bar{\rho}^t(r|\mu) \equiv \rho^t(r|\mu),$$
$$E^t = F^t, \quad G^t = 0, \quad \pi_k^{t+1}(x_k) \propto q_0(x_k) \cdot \rho_G^t(z_k | x_k). \quad (39)$$

Finally, we conclude that the performance of the large-system regime is fully described by the sequential application of two updates for the parameters $F^t$ and $c_m^t$ (cf. (34) and (27))

$$F^t = \sum_{r \in \mathcal{R}} \int \frac{\dot{\rho}^t(r|\mu)^2}{\rho^t(r|\mu)} \frac{e^{-\frac{\mu^2}{2\beta c_m^t}}}{\sqrt{2\pi\beta c_m^t}}\mathrm{d}\mu,$$
$$c_m^{t+1} = \int \frac{[\int x \rho_G^t(z|x)q_0(x)\mathrm{d}x]^2}{\int \rho_G^t(z|x)q_0(x)\mathrm{d}x}\mathrm{d}z, \quad (40)$$

with the initial state $c_m^0 = 0$. Interestingly, the stationary conditions of the state evolution equations coincides with the fixed point equation found in [7] with the replica method.

## V. NUMERICAL RESULTS

Let us consider BPSK transmission, i.e. $x_k \in \{-1, +1\}$. We have $c_x = 1$ and from (40) we get

$$c_m^{t+1} = \int_{-\infty}^\infty \frac{1}{2\sqrt{2\pi F^t}}\left(e^{-\frac{(z - F^t)^2}{2F^t}} - e^{-\frac{(z + F^t)^2}{2F^t}}\right)\tanh(z)\mathrm{d}z$$

$$= \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi F^t}} e^{-\frac{(z - F^t)^2}{2F^t}}\tanh(z)\mathrm{d}z, \quad (41)$$

where the second line follows from the antisymmetric property of the $\tanh(z)$ function. The equations (40) of the state evolution as well as the typical trajectory of the BP iteration are shown in Fig. 2, for $\beta = 1.8$, $\sigma_0 = 0.1$ and $b = 4$. The quantizer step size $\Delta$ has been chosen to minimize the distortion under Gaussian input. We distinguish different fixed points. Clearly, the performance of BP based detection algorithm is characterized by the poor solution as shown in Fig. 2. Besides, from the distribution of $z_k$ given $x_k$, $\rho_G^t(z_k | x_k)$ in (38), which directly affects the decision of $x_k$ through $\pi_k(x_k)$, it immediately follows that the bit error probability (BER) after performing $t$ iterations is given by

$$P_b^t = \mathrm{Pr}(z^t \leq 0) = Q(\sqrt{F^t}), \quad (42)$$

where $Q(z) = \int_y^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}dt$ is the error function. The analytical BER performance at the fixed point for $\beta = 1$ ($N = K$) is shown in Fig. 3 as function of the SNR=

$10 \log_{10} \frac{1}{\sigma_0^2}$ for different number of bits. The experimental BERs, carried out for a $20 \times 20$ system with $b \in \{1, 4\}$ and using 10 BP iterations, are also shown for comparison. Obviously, there is a good match between the theoretical results and the Monte Carlo results for $b = 1$, while there is a gap for $b = 4$ due to the insufficiently high number of antennas. Nevertheless, the analytical curve for $b = 4$ still predicts correctly that the performance loss with four bits compared to the ideal case ($b = \infty$) becomes negligible. We note that (40) for the noiseless case ($\sigma_0 = 0$) always admits a possible perfect detection solution, i.e. $c_m^\infty = 1$ and $F^\infty = \infty$ [6]. However, since the fixed point solution is not unique and BP usually converges to the worst solution, the BER behavior over the SNR might exhibit an error-floor as shown in Fig. 3 for the case $b = 1$. The minimum number of bits needed to ensure perfect detection in the noiseless case, i.e. where the recursion (40) evolves to the unique fixed point solution at $c_m^\infty = 1$, is depicted in Fig. 4 as function of the load factor $\beta$. We observe that for a system load $\beta \leq 1$ even 2-bit ADCs might be sufficient for symmetrical systems.
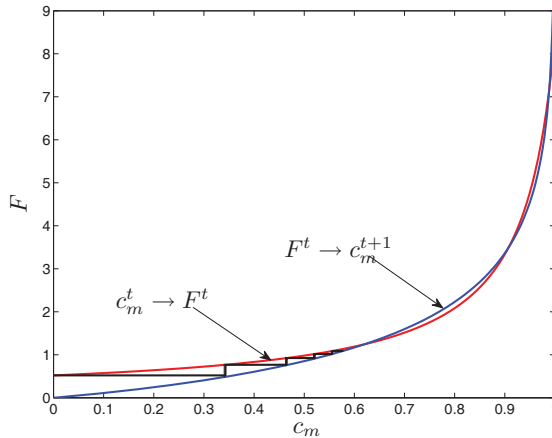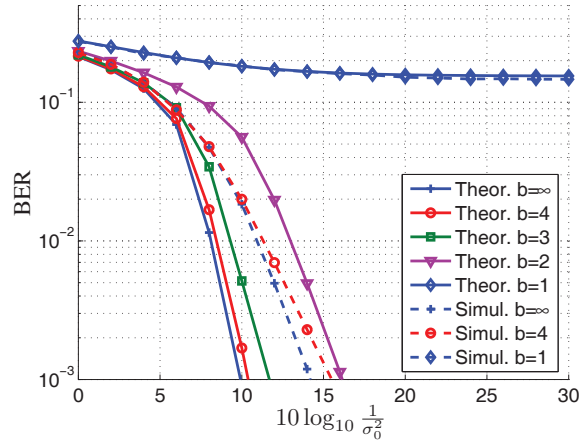


Fig. 3. BER for BPSK as function of $\sigma_0^2$ for $\beta = 1$ and different bit resolutions. The simulative results were obtained from the approximative BP detection applied on a $20 \times 20$ system.
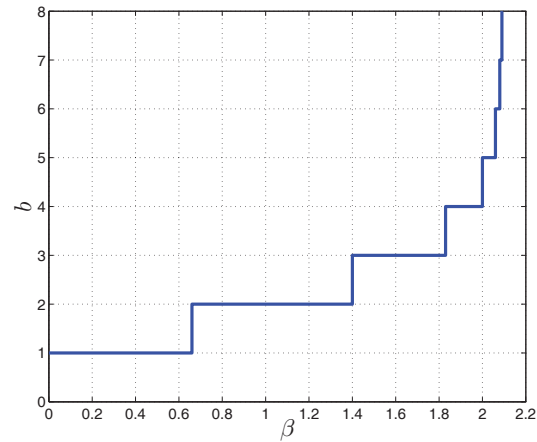


Fig. 2. State evolution chart, $\beta = 1.8$, $\sigma_0 = 0.1$ and $b = 4$.



Fig. 4. Minimum number of bits needed for BP perfect detection ($P_b \rightarrow 0$ as $\sigma_0 \rightarrow 0$) as function of $\beta$.

## VI. CONCLUSION

We studied a low complexity detection algorithm based on belief propagation for quantized MIMO systems. Additionally, a state evolution formalism has been presented to analyze the performance of the BP based detector when operating on quantized data in the large-system limit of fading channels. A set of simulation results was provided, which, in agreement with the analytical results, shows that the BP approach achieves good performance even with low resolution ADCs and moderate number of antennas.

## REFERENCES

[1] Y. Kabashima, "A CDMA multiuser detection algorithm on the basis of belief propagation," *J. Phys. A: Math. Gen.*, vol. 36, pp. 11111–11121, 2003.

[2] T. Tanaka and M. Okada, "Approximate belief propagation, density evolution, and statistical neurodynamics for CDMA multiuser detection," *IEEE Trans. Inform. Theory*, vol. 51, no. 2, pp. 700–706, Feb. 2005.

[3] A. Montanari and D. Tse, "Analysis of belief propagation for non-linear problems: The example of CDMA (or: How to prove Tanakas formula)," in *Proc. IEEE Inform. Theory Workshop*, Punta del Este, Uruguay, Mar. 2006, p. 122126.

[4] D. Guo and C.-C. Wang, "Multiuser Detection of Sparsely Spread CDMA," *IEEE J. Sel. Areas Commun.*, vol. 26, no. 3, pp. 421–431, April 2008.

[5] D. D. Wentzloff, R. Blázquez, F. S. Lee, B. P. Ginsburg, J. Powell, and A. P. Chandrakasan, "System design considerations for ultra-wideband communication," *IEEE Commun. Mag.*, vol. 43, no. 8, pp. 114–121, Aug. 2005.

[6] K. Nakamura and T. Tanaka, "Performance analysis of signal detection using quantized received signals of linear vector channel," in *Proc. Inter. Symp. Inform. Theory and its Applications (ISITA)*, Auckland, New Zealand, Dec. 2008.

[7] K. Nakamura and T. Tanaka, "Microscopic analysis for decoupling principle of linear vector channel," in *IEEE Intern. Symp. Inform. Theory (ISIT)*, Toronto, Canada, Jul. 2008, pp. 519–523.

[8] D. Guo and C.-C. Wang, "Random Sparse Linear Systems Observed Via Arbitrary Channels: A Decoupling Principle," in *IEEE Intern. Symp. Inform. Theory (ISIT)*, Nice, France, June 2007, pp. 946–950.

[9] J. G. Proakis, *Digital Communications*, McGraw Hill, New York, third edition, 1995.

[10] D. Guo and T. Tanaka, *Generic multiuser detection and statistical physics*, in Advances in Multiuser Detection, edited by M. Honig. Wiley-IEEE Press, 2009.

[11] T. J. Richardson and R. L. Urbanke, "The capacity of low-density parity check codes under message-passing decoding," *IEEE Trans. Inform. Theory, vol.*, vol. 47, no. 2, pp. 599–618, Feb. 2001.