High-Efficiency Super-Gain Antenna Arrays

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***Abstract*—It is an intriguing fact that the array gain of densely packed antenna arrays can be much larger than the number of antennas which comprise the antenna array. However, their large array gain seems to be inaccessible in practice, for it tends to be all eaten up by a loss of efficiency that accompanies such super-gain effects. In this paper, the authors argue that the assertion given above is based on a less than optimum choice of antenna separation inside the array as well as on less than optimum antenna excitation currents. We demonstrate that if both those issues are addressed optimally super-gain actually can be obtained with high efficiency. Compact antenna arrays therefore deserve to be given more attention in both the antenna-, and the signal processing layers of abstraction, for the successful application of such arrays requires optimum design in both layers.**

I. Introduction

It is well known that arrays of closely spaced antennas can provide array gain which grows super-linearly with the number N of antennas [1], and approaches N^2 from below as the distance between neighboring antennas is reduced more and more [2]. However, such »super-gain« arrays have a bad reputation of being excessively inefficient [3]. The main reason for the bad efficiency lies in the fact that the optimum antenna excitation currents which are necessary for achieving high array gains can have comparatively huge magnitude, which causes excessive dissipation in the lossy antenna elements. This means that almost all power which is supplied to the antenna array is dissipated into heat, and only but very little (as little as a fraction of 10^{-14} is exemplified in [3]) can actually be radiated, rendering super-gain arrays essentially useless. In [4] it is argued that the term »super-gain« is a misnomer, for the gain is all eaten up by the loss in efficiency.

In this paper, we argue that such bad efficiency reputation of super-gain arrays comes about because of two effects: non-optimum antenna spacing, and non-optimum excitation current. In fact, when both the separation between antennas and the currents that are used to excite them are chosen optimally, high array gains actually *can* be obtained from lossy antennas with high efficiencies (something around 90%, or so). We suggest that super-gain arrays should be given more attention from both the antenna-, and the signal processing communities, for optimum solutions in both fields are necessary to successfully apply super-gain antenna arrays.

In this paper, we first provide a brief introduction to super-gain effects for lossless antenna arrays. The loss of array efficiency is then discussed which occurs when the antennas are allowed to be lossy. It will become clear that efficiency can be greatly improved if both optimum antenna spacing and optimum excitation currents – which take the antenna losses into account – are applied. The performance of the optimum configuration with respect to array gain is compared to the standard approach and shown to deliver huge improvement. These promising results indicate that compact antenna arrays may deserve more attention in future research from both the antenna- and the signal processing communities.

II. Radiated Power

In order to understand super-gain effects it is helpful to recall how the radiated power depends on the antenna excitation currents. Let us first consider only a single antenna. From electromagnetic field theory, the radiated power can be obtained by integrating the Poynting vector over any closed surface ∂V, that completely surrounds the antenna [5]. With $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{H}}$, denoting the vectors of complex phasors of the electric and magnetic field, this becomes:

$$P_{\text{rad}} = \int_{\partial V} \operatorname{Re}\left\{ \vec{\boldsymbol{E}}^* \times \vec{\boldsymbol{H}} \right\} \mathrm{d}\vec{\boldsymbol{A}}. \qquad (1)$$

We can make the surface ∂V large enough such that it is located in the antenna's far-field. Assuming the antenna is made of thin wire, the electromagnetic far-field resembles a spherical transversal-electric wave [4]. In spherical coordinates (see left hand side of Figure 1):

$$\begin{bmatrix} E_r & E_\theta & E_\phi \end{bmatrix} = \frac{\mathrm{e}^{-\mathrm{j}kr}}{r} \begin{bmatrix} 0 & F_\theta(\theta,\phi) & F_\phi(\theta,\phi) \end{bmatrix}, \qquad (2)$$

wherein $k = 2\pi/\lambda$ is the wave number, λ is the wave length, and F_θ, and F_ϕ are functions specific to the antenna used. The empty-space Maxwell equation: $\vec{\boldsymbol{H}} = \frac{\mathrm{j}}{\omega\mu_0} \nabla \times \vec{\boldsymbol{E}}$ used on equation (2), with $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{H/m}$ and ω denoting the angular frequency, requires that

$$H_\theta = -E_\phi / Z_0, \quad \text{and} \quad H_\phi = E_\theta / Z_0. \qquad (3)$$

Herein, $Z_0 = \mu_0 \mathrm{c} \approx 377\,\Omega$ and c is the speed of light. Choosing the surface ∂V to be a sphere with radius r centered in

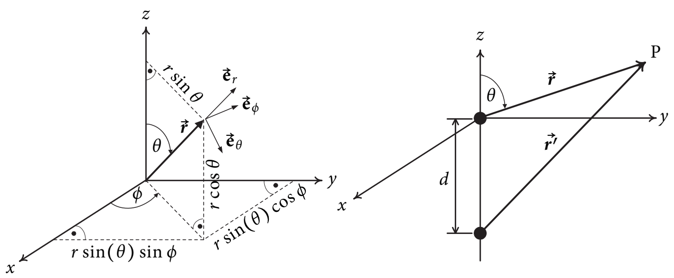

Figure 1. Left: Definition of the spherical coordinate system. Right: Array of two antennas and a point P in the far-field.

the origin, we have $\mathrm{d}\vec{A} = \vec{\mathbf{e}}_r\, \mathrm{d}\theta \mathrm{d}\phi \sin\theta$, such that (1) can be written with the help of (2) and (3) explicitly as:

$$P_{\mathrm{rad}} = \frac{1}{Z_0} \int_0^{2\pi} \int_0^{\pi} |E|^2\, r^2 \sin(\theta)\, \mathrm{d}\theta \mathrm{d}\phi, \tag{4}$$

where $|E|^2 = |E_\theta|^2 + |E_\phi|^2 = \|\vec{E}\|_2^2$ is the intensity of the electric field. Let us write (2) in the following equivalent way:

$$\vec{E} = \tilde{\alpha}(\theta,\phi) \cdot \frac{\mathrm{e}^{-\mathrm{j}kr}}{r} \cdot \vec{\mathbf{e}}_0(\theta,\phi), \quad \vec{\mathbf{e}}_0 \bullet \vec{\mathbf{e}}_0 = 1, \;\; \vec{\mathbf{e}}_0 \bullet \vec{\mathbf{e}}_r = 0, \tag{5}$$

where $\tilde{\alpha}(\theta,\phi)$ describes the directional characteristics of the antenna, while the unity vector $\vec{\mathbf{e}}_0(\theta,\phi)$ describes the wave's polarization. Substituting (5) into (4) then yields:

$$P_{\mathrm{rad}} = \frac{1}{Z_0} \int_0^{2\pi} \int_0^{\pi} |\tilde{\alpha}(\theta,\phi)|^2 \sin(\theta)\, \mathrm{d}\theta \mathrm{d}\phi. \tag{6}$$

Notice that $|\tilde{\alpha}(\theta,\phi)|^2 = |F_\theta(\theta,\phi)|^2 + |F_\phi(\theta,\phi)|^2$, as follows by comparing the squared Euclidean norms of (5) and (2). Now, consider two identical antennas, one located in the origin, as before, and another one which is displaced by the distance d along the negative z-axis, as shown on the right hand side of Figure 1. The electric field at a point P, far away from the antennas, can be written as the linear superposition $\vec{E} = \vec{E}_1 + \vec{E}_2$, of the electric fields generated by each antenna:

$$\left.\begin{aligned} \vec{E}_1 &= \alpha_1(\theta,\phi) \cdot \frac{\mathrm{e}^{-\mathrm{j}kr}}{r} \cdot \vec{\mathbf{e}}_{0,1}(\theta,\phi), \\ \vec{E}_2 &= \alpha_2(\theta,\phi) \cdot \frac{\mathrm{e}^{-\mathrm{j}kr'}}{r'} \cdot \vec{\mathbf{e}}_{0,2}(\theta,\phi), \end{aligned}\right\} \tag{7}$$

where r and r' are the distances between the point P, and the first and the second antenna, respectively (see right hand side of Figure 1). Calling i_1 and i_2 the excitation current of the first and second antenna, respectively, we have

$$\left.\begin{aligned} \alpha_1(\theta,\phi) &= \alpha(\theta,\phi) \cdot i_1, \\ \alpha_2(\theta,\phi) &= \alpha(\theta,\phi) \cdot i_2. \end{aligned}\right\} \tag{8}$$

If the antenna is made of thin (radius less than about $\lambda/200$) wire that is not too long ($< \lambda/2$), then a vanishing excitation current also implies that the current vanishes everywhere on the thin and short wire (see e.g., [4], pages 17–19). When no current flows in the whole wire it does not alter the electromagnetic field – it essentially becomes »invisible« [6]. Assuming this so-called *canonical minimum scattering* [7] property holds true for both antennas, the function $\alpha(\theta,\phi)$ in (8) is *not* influenced by the neighboring antenna – it is the same function one would obtain if only one antenna was present in the first place. The distance r' can be expressed in terms of r, and elevation θ (see right hand side of Figure 1):

$$r' = r\sqrt{1 + \frac{d^2}{r^2} + \frac{2d}{r}\cos\theta} \approx r + d\cos\theta, \quad r \gg d. \tag{9}$$

Now let the two identical canonical minimum scattering antennas be oriented in the same way. This means that in the point P far removed from the array, both antennas excite a field with the same polarization, hence $\vec{\mathbf{e}}_{0,1} = \vec{\mathbf{e}}_{0,2} = \vec{\mathbf{e}}_0$. In a large enough distance $r \gg d$ in the far-field we therefore obtain from $\vec{E} = \vec{E}_1 + \vec{E}_2$, using (7), (8) and (9):

$$\vec{E} = \alpha(\theta,\phi) \frac{\mathrm{e}^{-\mathrm{j}kr}}{r} \left(i_1 + i_2 \mathrm{e}^{-\mathrm{j}kd\cos\theta}\right) \vec{\mathbf{e}}_0(\theta,\phi) \tag{10}$$

$$= \alpha(\theta,\phi)\, \boldsymbol{a}^{\mathrm{H}}(\theta) \boldsymbol{i} \cdot \frac{\mathrm{e}^{-\mathrm{j}kr}}{r} \vec{\mathbf{e}}_0(\theta,\phi), \tag{10a}$$

where we have collected the current phasors into the vector $\boldsymbol{i} = [\, i_1 \; i_2 \,]^{\mathrm{T}}$ and defined the *array steering vector:*

$$\boldsymbol{a}(\theta) = \left[\, 1 \;\; \mathrm{e}^{-\mathrm{j}kd\cos\theta} \,\right]^{\mathrm{H}}. \tag{11}$$

Herein the superscripts $^{\mathrm{T}}$ and $^{\mathrm{H}}$ denote the transpose and the complex conjugate transpose operation, respectively. Comparing (10a) with (5), we see that

$$\tilde{\alpha}(\theta,\phi) = \alpha(\theta,\phi) \cdot \boldsymbol{a}^{\mathrm{H}}(\theta) \boldsymbol{i}. \tag{12}$$

Substituting (12) into (6) we find:

$$P_{\mathrm{rad}} = \boldsymbol{i}^{\mathrm{H}} \left(Z_0^{-1} \int_0^{2\pi}\!\!\int_0^{\pi} |\alpha(\theta,\phi)|^2 \boldsymbol{a}(\theta) \boldsymbol{a}^{\mathrm{H}}(\theta) \sin(\theta)\, \mathrm{d}\theta \mathrm{d}\phi \right) \boldsymbol{i}, \tag{13}$$

One can easily generalize this result to the case when more than two, say N antennas are arranged into a uniform linear array (ULA). All we have to do is to define the array steering vector and the excitation vector accordingly:

$$\boldsymbol{a}(\theta) = \left[\, 1 \;\; \mathrm{e}^{-\mathrm{j}kd\cos\theta} \;\; \mathrm{e}^{-2\mathrm{j}kd\cos\theta} \;\; \cdots \;\; \mathrm{e}^{-(N-1)\mathrm{j}kd\cos\theta} \,\right]^{\mathrm{H}}, \tag{14}$$

$$\boldsymbol{i} = \left[\, i_1 \;\; i_2 \;\; i_3 \;\; \cdots \;\; i_N \,\right]^{\mathrm{T}}. \tag{15}$$

III. Array Impedance Matrix

We can also think of the N-element antenna array as a linear N-port circuit, where each port corresponds to the feedpoint of each antenna. Each port is characterized by two quantities: a complex voltage phasor $\boldsymbol{v} \in \mathbb{C}^{N\times 1} \cdot \mathrm{V}$ and a complex current phasor $\boldsymbol{i} \in \mathbb{C}^{N\times 1} \cdot \mathrm{A}$. The latter is identical to the excitation current vector (15). Because of linearity, the relationship between voltage and current at the ports must be a linear one:

$$\boldsymbol{v} = \boldsymbol{Z}\boldsymbol{i}, \tag{16}$$

where $\boldsymbol{Z} \in \mathbb{C}^{N\times N} \cdot \Omega$, is the so-called impedance matrix [8] of the antenna array. Using the two port variables, the total active power flowing into the N-port is given by:

$$P_{\mathrm{in}} = \mathrm{Re}\left\{\boldsymbol{v}^{\mathrm{H}} \boldsymbol{i}\right\}, \tag{17}$$

where $\mathrm{Re}\{\cdot\}$ is the realpart operation. Because antennas are reciprocal [4], we have that $\boldsymbol{Z} = \boldsymbol{Z}^{\mathrm{T}}$. Using this fact, we obtain by substituting (16) into (17):

$$P_{\mathrm{in}} = R_{\mathrm{r}} \cdot \boldsymbol{i}^{\mathrm{H}} \boldsymbol{C} \boldsymbol{i}, \tag{18}$$

where

$$\boldsymbol{C} = \mathrm{Re}\{\boldsymbol{Z}\}/R_{\mathrm{r}} = \begin{bmatrix} 1 & * & * & * \\ * & 1 & * & * \\ \vdots & \cdots & \ddots & \vdots \\ * & * & * & 1 \end{bmatrix} \in \mathbb{R}^{N\times N} \tag{19}$$

is the normalized realpart of the impedance matrix which contains unity on its main diagonal, and

$$R_\mathrm{r} = \mathrm{Re}\left\{(\boldsymbol{Z})_{k,k}\right\}, \quad \forall k \tag{20}$$

is the radiation resistance of each antenna [4]. Let us for the moment assume that the antennas are *lossless*. Because then no power is dissipated there must be

$$P_\mathrm{in} = P_\mathrm{rad}, \quad \text{for lossless antennas.} \tag{21}$$

That is, all power that is supplied to the antenna array is radiated. From (21) then follows with (18) and (13):

$$\boldsymbol{C} = \frac{\int_0^{2\pi}\int_0^{\pi} |\alpha(\theta,\phi)|^2\,\boldsymbol{a}(\theta)\boldsymbol{a}^\mathrm{H}(\theta)\sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\phi}{\int_0^{2\pi}\int_0^{\pi} |\alpha(\theta,\phi)|^2\sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\phi}. \tag{22}$$

The (normalized) realpart $\boldsymbol{C}$ of the array impedance matrix $\boldsymbol{Z}$ can, therefore, be computed purely from the *far-field* radiation pattern $|\alpha(\theta,\phi)|^2$ of each antenna, and the array steering vector $\boldsymbol{a}(\theta,\phi)$. That is, even though the off-diagonal elements of $\boldsymbol{Z}$ are a function of the mutual *near-field* coupling of the antennas in the array, the realpart of $\boldsymbol{Z}$ can be obtained purely from *far-field* considerations! On the other hand, the imaginary part of $\boldsymbol{Z}$ can only be obtained from the near-field. However, since the computation of the active input power (here equal to radiated active power) only requires the knowledge of the realpart of $\boldsymbol{Z}$, we do not have to worry about its imaginary part in this context.

In the remaining of this paper we will assume that the individual antennas of the ULA are *isotropic*. While isotropic antennas do not really exist, it can be shown that their theoretical application is justified from a field-theoretic view point [9], because they lead to *qualitatively the same* realpart of the impedance matrix as obtained for a ULA of Hertzian dipoles. For isotropic radiators

$$\alpha(\theta,\phi) = \text{const}, \tag{23}$$

such that we obtain from (23), (22) and (14):

$$\boldsymbol{C} = \begin{bmatrix} 1 & j_0(kd) & j_0(2kd) & j_0(3kd) & \cdots \\ j_0(kd) & 1 & j_0(kd) & j_0(2kd) & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{24}$$

where the function j_0 is defined as:

$$j_0(x) = \frac{\sin x}{x}. \tag{25}$$

Recall that $k = 2\pi/\lambda$. Because (25) has equidistant roots at integer multiples of π, we see from (24), that

$$d/\lambda \in \frac{1}{2}\cdot\mathbb{N} \implies \boldsymbol{C} = \mathbf{I}_N, \tag{26}$$

where $\mathbf{I}_N$ is the $N \times N$ identity matrix. In a ULA of isotrops with inter-element spacing d equal to integer multiples of half the wavelength, the isotrops are uncoupled. For all other values of d the isotrops are, however, coupled. For $d < \lambda/2$ the coupling is strong and ultimately leads to a rank deficient all-ones matrix $\boldsymbol{C}$, when $d \to 0$. The matrix $\boldsymbol{C}$ has the property

$$\boldsymbol{C} = \boldsymbol{C}^\mathrm{T} = \boldsymbol{C}^\mathrm{H} > \mathbf{0}, \quad \forall d > 0. \tag{27}$$

The last property means that the array always radiates positive active power for any non-zero excitation current $\boldsymbol{i}$.

IV. Super-Gain with Lossless Antennas

The receive power at a point P in the far-field (see right hand side of Figure 1) is proportional to the *squared* magnitude of the electric field strength. Thus, from (5), (12) and (23), the receive power can be expressed in the following way:

$$P_\mathrm{Rx} = \gamma \cdot \left|\boldsymbol{a}^\mathrm{H}(\theta)\boldsymbol{i}\right|^2, \tag{28}$$

where $\gamma > 0$ is a (distance depending) constant. The optimum excitation current for beamforming into the direction θ with lossless (ideal) antennas is given by:

$$\boldsymbol{i}_\mathrm{opt}^\mathrm{ideal} = \arg\max_{\boldsymbol{i}} \frac{P_\mathrm{Rx}}{P_\mathrm{rad}} = \arg\max_{\boldsymbol{i}} \frac{\left|\boldsymbol{a}^\mathrm{H}(\theta)\boldsymbol{i}\right|^2}{\boldsymbol{i}^\mathrm{H}\boldsymbol{C}\boldsymbol{i}}. \tag{29}$$

That is, the optimum excitation current yields the largest possible receive power for a given radiated power. Alternatively, it minimizes the radiated power for a given receive power. The second equality in (29) is due to (18), (21) and (28). Because $\boldsymbol{C} = \boldsymbol{C}^\mathrm{H} > \mathbf{0}$, there also is $\boldsymbol{C}^{1/2} = \boldsymbol{C}^{\mathrm{H}/2}$, where $\boldsymbol{C}^{1/2}$ is a matrix square root of $\boldsymbol{C}$, that is, $\boldsymbol{C}^{1/2}\boldsymbol{C}^{1/2} = \boldsymbol{C}$. In the new variable $\boldsymbol{x} = \boldsymbol{C}^{1/2}\boldsymbol{i}$, (29) can be rewritten in the form:

$$\boldsymbol{x}_\mathrm{opt}^\mathrm{ideal} = \arg\max_{\boldsymbol{x}} \frac{\left|\boldsymbol{a}^\mathrm{H}(\theta)\boldsymbol{C}^{-1/2}\boldsymbol{x}\right|^2}{\|\boldsymbol{x}\|_2^2} = \text{const}\cdot\boldsymbol{C}^{-1/2}\boldsymbol{a}(\theta),$$

where again $\boldsymbol{C}^{-H/2} = \boldsymbol{C}^{-1/2}$ is used. With the inverse transformation $\boldsymbol{i} = \boldsymbol{C}^{-1/2}\boldsymbol{x}$, the optimum excitation current is given by $\boldsymbol{i}_\mathrm{opt}^\mathrm{ideal} = \text{const}\cdot\boldsymbol{C}^{-1}\boldsymbol{a}(\theta)$. Expressing the constant in terms of radiated power (by application of (18 and (21)), we find:

$$\boldsymbol{i}_\mathrm{opt}^\mathrm{ideal} = \sqrt{\frac{P_\mathrm{rad}/R_\mathrm{r}}{\boldsymbol{a}^\mathrm{H}(\theta)\boldsymbol{C}^{-1}\boldsymbol{a}(\theta)}}\cdot\boldsymbol{C}^{-1}\boldsymbol{a}(\theta). \tag{30}$$

Substituting (30) for $\boldsymbol{i}$ in (28) yields for the maximum possible receive power

$$\max P_\mathrm{Rx} = \frac{\gamma P_\mathrm{rad}}{R_\mathrm{r}}\cdot\boldsymbol{a}^\mathrm{H}(\theta)\boldsymbol{C}^{-1}\boldsymbol{a}(\theta). \tag{31}$$

For reference, the receive power that would result if only one antenna was present in the array is given by:

$$P_\mathrm{Rx}\big|_{N=1} = \frac{\gamma P_\mathrm{rad}}{R_\mathrm{r}}\cdot|(\boldsymbol{a})_k|^2 = \frac{\gamma P_\mathrm{rad}}{R_\mathrm{r}}\boldsymbol{a}^\mathrm{H}\boldsymbol{a}/N, \quad \forall k. \tag{32}$$

This comes about because $\boldsymbol{C} = 1$ for $N = 1$, and all elements of the steering vector $\boldsymbol{a}$ have the same magnitude. We define the *array gain* A^ideal of the lossless array as the ratio of maximum possible receive power using optimum excitation of all N antennas, and the reference receive power which results from

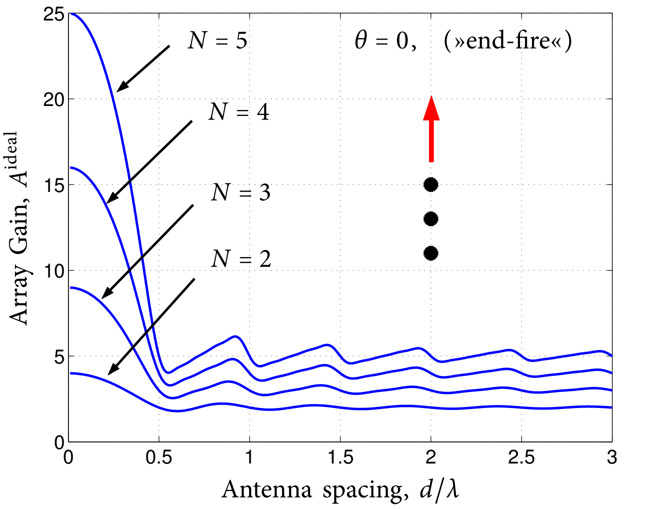

Figure 2. Array gain A^{ideal} in the »end-fire« direction, as function of the antenna separation for different number N of antennas.

having only one antenna excited but radiating the same power in both cases. Hence,

$$A^{\text{ideal}} = \left. \frac{\max P_{\text{Rx}}}{P_{\text{Rx}}|_{N=1}} \right|_{P_{\text{rad}}=\text{const}} \tag{33}$$

$$= N \frac{\boldsymbol{a}^{\text{H}}(\theta)\boldsymbol{C}^{-1}\boldsymbol{a}(\theta)}{\boldsymbol{a}^{\text{H}}(\theta)\boldsymbol{a}(\theta)}. \tag{33a}$$

The array gain depends on the direction (θ) of beamforming, and on the antenna separation d (mostly via $\boldsymbol{C}$). The largest array gain is obtained in the direction $\theta = 0$, the so-called »end-fire« direction, and approaches N^2, for small antenna separation [2], [10]. Figure 2 illustrates the dependencies of the array gain (33a) on the antenna separation d, and the number N of lossless antennas.

V. Super-Gain with Lossy Antennas

Let us now investigate the case when we allow the antennas to be lossy. A lossy antenna dissipates power when a non-zero excitation current flows. With lossy antennas the approach from Section IV to achieve super-gain can easily lead to a complete *disaster*. To see why this is so, have a look at the squared Euclidean norm of the excitation current (30):

$$\left\|\boldsymbol{i}_{\text{opt}}^{\text{ideal}}\right\|_2^2 = \frac{P_{\text{rad}}}{R_{\text{r}}} \cdot \frac{\boldsymbol{a}^{\text{H}}(\theta)\boldsymbol{C}^{-2}\boldsymbol{a}(\theta)}{\boldsymbol{a}^{\text{H}}(\theta)\boldsymbol{C}^{-1}\boldsymbol{a}(\theta)}. \tag{34}$$

Because the matrix $\boldsymbol{C}$ approaches the rank deficient all-ones matrix as $d \to 0$, it turns out that

$$\lim_{d\to 0} \left\|\boldsymbol{i}_{\text{opt}}^{\text{ideal}}\right\|_2^2 \to \infty.$$

Let each lossy antenna have a dissipation resistance R_{d}. Then the total power $P_{\text{diss}} = R_{\text{d}} \cdot \|\boldsymbol{i}_{\text{opt}}^{\text{ideal}}\|_2^2$, which is dissipated by the antenna array grows unboundedly, too, as $d \to 0$. This causes excessive heating and it is the reason why super-gain arrays may have terribly low array efficiency.

To illustrate this phenomenon, let us consider a uniform linear array of $N = 4$ lossy isotrops for which $R_{\text{d}}/R_{\text{r}} = 10^{-3}$. Imagine the isotrops are very densely spaced such that they are separated by the distance $d = \lambda/100$, and excited for beamforming in the »end-fire« direction using the excitation current from (30). We obtain $P_{\text{diss}} \approx 5 \times 10^7 P_{\text{rad}}$. In this way, to radiate just one microwatt of active power, one would have to dissipate 50 Watts in the antennas. This translates into a horribly low array efficiency of approximately 2×10^{-8}. Clearly, this array is next to useless, despite it has a high array gain of $A^{\text{ideal}} \approx 16$. In other words, even though the antenna array makes nearly the most out of its radiated power, it unfortunately radiates almost nothing of the supplied power but rather transforms practically all the delivered power irreversibly into heat. This is the reason why [4] calls »super-gain« a misnomer, for the product of array gain and array efficiency can be much less than unity.

The main problem here is that one should *not* have placed the antennas as closely as $\lambda/100$ in the example case above. Moreover, the excitation current vector should have taken into account that the antennas are lossy. We suggest that a better way to choose the excitation current is the following:

$$\boldsymbol{i}_{\text{opt}}^{\text{lossy}} = \arg\max_{\boldsymbol{i}} \frac{P_{\text{Rx}}}{P_{\text{rad}} + P_{\text{diss}}}. \tag{35}$$

The excitation is chosen to achieve the largest possible receive power given a total supplied power, or alternatively, to minimize total supplied power for given receive power. Note that:

$$P_{\text{tot}} = P_{\text{rad}} + P_{\text{diss}} \tag{36}$$

$$= R_r \cdot \boldsymbol{i}^{\text{H}}\boldsymbol{C}\boldsymbol{i} + R_{\text{d}} \cdot \boldsymbol{i}^{\text{H}}\boldsymbol{i}$$

$$= R_r \cdot \boldsymbol{i}^{\text{H}} \left(\boldsymbol{C} + \frac{R_{\text{d}}}{R_{\text{r}}}\mathbf{I}_{\text{N}}\right)\boldsymbol{i}. \tag{37}$$

The optimum excitation current for the lossy array equals:

$$\boldsymbol{i}_{\text{opt}}^{\text{lossy}} = \sqrt{\frac{P_{\text{tot}}/R_{\text{r}}}{\boldsymbol{a}^{\text{H}}(\theta)\left(\boldsymbol{C} + \frac{R_{\text{d}}}{R_{\text{r}}}\mathbf{I}_N\right)^{-1}\boldsymbol{a}(\theta)}} \cdot \left(\boldsymbol{C} + \frac{R_{\text{d}}}{R_{\text{r}}}\mathbf{I}_N\right)^{-1}\boldsymbol{a}(\theta). \tag{38}$$

When we compare (38) with (30), we can observe that there is a diagonal loading of the matrix $\boldsymbol{C}$. Because $\boldsymbol{C} > \boldsymbol{0}$, this diagonal loading leads to a regularization which reduces the condition number of the diagonally loaded matrix. For $R_{\text{d}} > 0$, no rank deficiency occurs and the optimum excitation current remains finite even when $d \to 0$. Substituting (38) for $\boldsymbol{i}$ in (28) yields the maximum receive power that can be obtained for a given total supplied power P_{tot}:

$$\max P_{\text{Rx}} = \frac{\gamma P_{\text{tot}}}{R_{\text{r}}} \cdot \boldsymbol{a}^{\text{H}}(\theta)\left(\boldsymbol{C} + \frac{R_{\text{d}}}{R_{\text{r}}}\mathbf{I}_N\right)^{-1}\boldsymbol{a}(\theta). \tag{39}$$

We again choose, as a reference, the receive power that would be obtained when only one *but lossless* transmit antenna was excited:

$$P_{\text{Rx}}|_{N=1,R_{\text{d}}=0} = \frac{\gamma P_{\text{rad}}}{R_{\text{r}}} \cdot |(\boldsymbol{a})_k|^2 = \frac{\gamma P_{\text{rad}}}{R_{\text{r}}} \boldsymbol{a}^{\text{H}}\boldsymbol{a}/N, \quad \forall k. \tag{40}$$

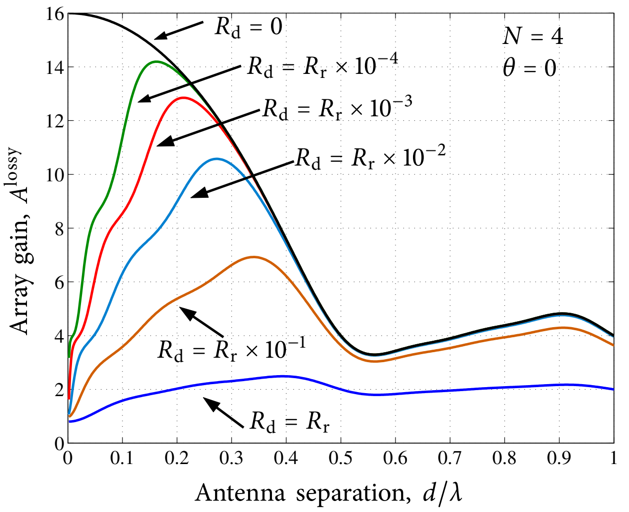

Figure 3. Array gain (as function of antenna separation) of a uniform linear array of $N = 4$ lossy isotrops, when beamforming in »end fire« direction is applied. The different curves correspond to different amounts of antenna loss, quantified by the ratio R_d/R_r, of dissipation and radiation resistance.

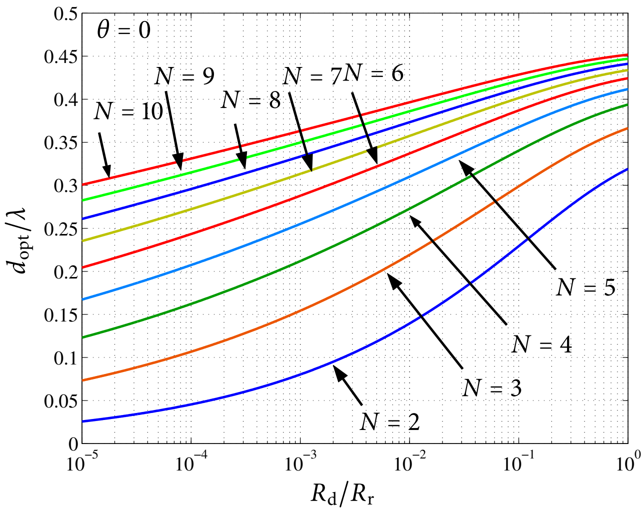

Figure 4. The optimum antenna separation for »end-fire« beamforming as function of the amount of loss (R_d/R_r).

In this way, we can generalize the definition of the array gain from (33) to include the case of lossy antennas:

$$A^{lossy} = \left. \frac{\max P_{Rx}}{P_{Rx}|_{N=1,R_d=0}} \right|_{P_{tot}=\text{const}} = N \cdot \frac{\boldsymbol{a}^H(\theta)\left(\boldsymbol{C} + \frac{R_d}{R_r}\mathbf{I}_N\right)^{-1}\boldsymbol{a}(\theta)}{\boldsymbol{a}^H(\theta)\boldsymbol{a}(\theta)}. \tag{41}$$

This quantifies how much more receive power we can obtain when all N lossy antennas are used, compared to the case of having only one but lossless antenna, while the total supplied power is the same in both cases. Note that

$$A^{lossy} \le A^{ideal}, \tag{42}$$

where equality holds only if $R_d = 0$. This comes about because $\boldsymbol{C}^{-1}$ and $(\boldsymbol{C} + (R_d/R_r)\mathbf{I}_N)^{-1}$ have the same eigenvectors, but the latter has smaller corresponding eigenvalues, because $\boldsymbol{C}$ is positive definite, and $R_d/R_r > 0$.

Figure 3 shows the array gain from (41) in »end-fire« direction for a ULA of $N = 4$ isotropic radiators for several values of R_d/R_r as function of the antenna separation d. For lossless antennas ($R_d = 0$) the largest array gain is achieved as $d \to 0$ and approaches N^2 from below. However, as $R_d > 0$, we see that too small values of d are unfavorable for the array gain because of the unavoidable loss of efficiency that occurs despite that the excitation is optimized with the losses in mind. On the other hand we see that there is an optimum antenna separation which depends on R_d/R_r for which the array gain is maximized. We can see from Figure 3 that this optimum distance is always less than half of the wavelength. Moreover, provided the antennas are spaced this optimum distance apart the array gain is always larger than it would be for uncoupled isotrops (large, or half wavelength spacing; recall (26)). In other words, when done sensibly, the super-gain always outweights the antenna loss, and an array gain larger than that obtainable for uncoupled antennas can be obtained. This is true regardless of the amount of loss, that is, for every value of R_d/R_r. Note from Figure 3, that for $R_d/R_r \le 10^{-2}$, one can achieve with $N = 4$ antennas a transmit array gain $A^{lossy} > 10$, provided one uses the optimum antenna separation and applies the optimum antenna excitation.

The optimum antenna separation depends on the direction of beamforming, number of antennas, and the ratio R_d/R_r. Figure 4 shows the results for the »end-fire« direction and a uniform linear array of isotropic radiators. E.g., with $N = 4$ antennas which have $R_d = 10^{-3} \times R_r$ we have $d_{opt} \approx 0.21\lambda$. However, $N = 8$ antennas with an $R_d = 10^{-2} \times R_r$ need a little bit more room to breath. They are most happy with 0.37λ space between neighbors. Note from Figure 4 that the more antennas we have, or the more lossy they are, the more close d_{opt} comes to $\lambda/2$.

VI. Array Efficiency

With (38), the power dissipated in the array when its antennas are optimally excited equals:

$$P_{diss} = R_d \cdot \left\|\boldsymbol{i}_{opt}^{lossy}\right\|_2^2 = P_{tot}\frac{R_d}{R_r} \cdot \frac{\boldsymbol{a}^H(\theta)\left(\boldsymbol{C} + \frac{R_d}{R_r}\mathbf{I}_N\right)^{-2}\boldsymbol{a}(\theta)}{\boldsymbol{a}^H(\theta)\left(\boldsymbol{C} + \frac{R_d}{R_r}\mathbf{I}_N\right)^{-1}\boldsymbol{a}(\theta)}. \tag{43}$$

The array efficiency is then defined as:

$$\eta = \frac{P_{rad}}{P_{rad} + P_{diss}} = \frac{P_{rad}}{P_{tot}} = \frac{P_{tot} - P_{diss}}{P_{tot}}, \tag{44}$$

the ratio of radiated power and total power supplied into the antenna array. Applying (43) in (44) then yields for the efficiency of the optimally excited lossy antenna array:

$$\eta = 1 - \frac{R_d}{R_r} \cdot \frac{\boldsymbol{a}^H(\theta)\left(\boldsymbol{C} + \frac{R_d}{R_r}\mathbf{I}_N\right)^{-2}\boldsymbol{a}(\theta)}{\boldsymbol{a}^H(\theta)\left(\boldsymbol{C} + \frac{R_d}{R_r}\mathbf{I}_N\right)^{-1}\boldsymbol{a}(\theta)}. \tag{45}$$

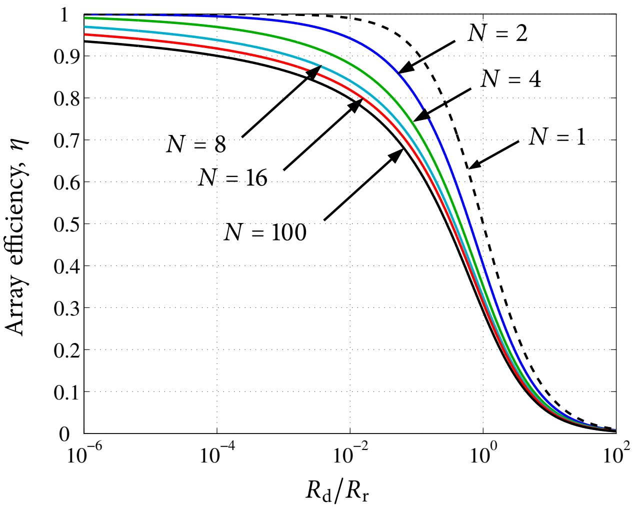

Figure 5. Array efficiency as function of R_d/R_r for different number of antennas. Beamforming is done in the »end-fire« direction.

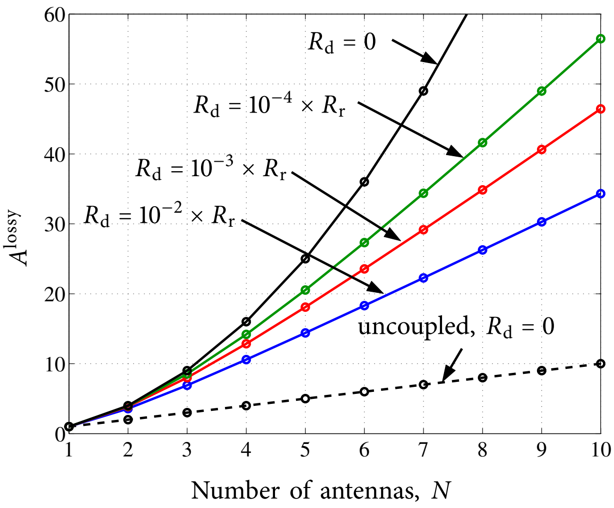

Figure 6. Array gain as function of antenna number for beamforming in »end-fire« direction.

For the purpose of illustration, consider again our array of $N = 4$ lossy antennas with $R_d/R_r = 10^{-3}$. This time, however, the antenna spacing is *not* chosen to be so small as $\lambda/100$, but instead equals the optimum distance $d = 0.212\lambda$ (see Figure 4). Using (43), we see that the dissipated power this time is given by $P_{\text{diss}} \approx 0.06 \times P_{\text{tot}}$, which translates into an array efficiency of about 94%. With this array, we can radiate a large power of, say 10 Watts, while dissipating only about 0.6 Watts power in the antennas. At the same time, we see from (41), that we have an array gain of $A^{\text{lossy}} \approx 12.85$ which is less than 1dB below the maximum array gain of 16, but more than 5dB larger than the number of antennas. This example suggests that, by optimum spacing of the antennas in conjunction with optimum antenna excitation (beamforming), it is possible to extract a fairly large amount of super-gain without losing much in array efficiency.

In Figure 5, the array efficiency of a ULA of lossy isotropic radiators is shown. The array is employed for beamforming in the »end-fire« direction (that is, $\theta = 0$). The distance between antennas is chosen such as to maximize the array gain (see Figure 4). One can see that the array efficiency depends only very little on the number of antennas. It mostly depends on the ratio R_d/R_r. From Figure 5 we can observe that the array efficiency is actually quite high, provided that the optimum antenna separation is chosen and the optimum excitation (38) is used. For $10^{-4} < R_d/R_r < 10^{-2}$, the array efficiency ranges between 80% and 99.5%.

Assuming optimum antenna spacing is chosen, we obtain the array gain of a uniform linear array of isotrops excited for beamforming in the »end-fire« direction shown as the solid curves in Figure 6 for different ratios R_d/R_r. When the radiators are lossless (that is, $R_d = 0$), the maximum array gain equals the square of the number of radiators. When $R_d > 0$, but still small compared to R_r, the array gain first starts growing almost quadratically with the number of radiators, but the growth flattens out into a linear growth for large number of antennas. E.g., if $R_d = 10^{-4} \times R_r$, then for $N \geq 5$, the growth is practically linear. Nevertheless, when we compare to the case of uncoupled antennas (dashed curve in Figure 6), we see that even for very lossy radiators, the absolute values of array gain are pretty large. E.g., if $R_{\text{diss}} = 10^{-2} \times R_{\text{rad}}$, and $M = 10$, we see that the super-gain array achieves an array gain which is 3.4 times larger than with uncoupled antennas.

VII. Conclusion

The common belief that super-gain is regularly »eaten up« by the simultaneous loss in antenna efficiency is found to be true only if the antenna separation is chosen to be too small. Optimum antenna separation in conjunction with optimum antenna excitation (beamforming), makes it possible to obtain a large amount of super-gain while still maintaining a reasonably large array efficiency. Consequently, compact antenna arrays deserve more attention in future research.

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