# High-Efficiency Super-Gain Antenna Arrays

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***Abstract*—It is an intriguing fact that the array gain of densely packed antenna arrays can be much larger than the number of antennas which comprise the antenna array. However, their large array gain seems to be inaccessible in practice, for it tends to be all eaten up by a loss of efficiency that accompanies such super-gain effects. In this paper, the authors argue that the assertion given above is based on a less than optimum choice of antenna separation inside the array as well as on less than optimum antenna excitation currents. We demonstrate that if both those issues are addressed optimally super-gain actually can be obtained with high efficiency. Compact antenna arrays therefore deserve to be given more attention in both the antenna-, and the signal processing layers of abstraction, for the successful application of such arrays requires optimum design in both layers.**

## I. Introduction

It is well known that arrays of closely spaced antennas can provide array gain which grows super-linearly with the number $N$, of antennas [1], and approaches $N^2$ from below as the distance between neighboring antennas is reduced more and more [2], [3]. However, such »super-gain« arrays have a bad reputation of being excessively inefficient [4]. The reason for the bad efficiency lies in the fact that the optimum antenna excitation currents which are necessary for achieving high array gains can have comparatively huge magnitude, which causes excessive dissipation in lossy antenna elements. This means that almost all power which is supplied to the array is dissipated into heat, and only but very little (as little as a fraction of $10^{-14}$, is exemplified in [4]) can actually be radiated, rendering super-gain arrays essentially useless [5].

In this paper, we argue that such bad efficiency reputation of super-gain arrays comes about because of two effects: non-optimum antenna spacing, and non-optimum excitation current. In fact, when both the separation between antennas, and the currents that are used to excite them are chosen optimally, high array gains actually can be obtained from lossy antennas with high efficiencies (like 90%). Indeed, recent results by Yaghijan et al., show that highly efficient super-gain arrays can be realized in practice [6]. We suggest that super-gain arrays should be given more attention from both the antenna-, and the signal processing communities, for optimum solutions in both fields are necessary.

## II. Super-Gain with Lossless Antennas

Let us consider a uniform linear array (ULA) of $N$ isotropic radiators. It can be described as a linear $N$-port. With lossless radiators, the radiated power is equal to the electric input power: $P_{\mathrm{rad}} = \mathrm{Re}\{\boldsymbol{v}^{\mathrm{H}}\boldsymbol{i}\}$, where $\boldsymbol{v} \in \mathbb{C}^{N\times 1}\cdot \mathrm{V}$, and $\boldsymbol{i} \in \mathbb{C}^{N\times 1}\cdot \mathrm{A}$, are the complex voltage and current envelopes, respectively, while $\mathrm{Re}\{.\}$, and $(.)^{\mathrm{H}}$ are the real-part, and the Hermitian operations, respectively. Due to linearity, we have that $\boldsymbol{v} = \boldsymbol{Z}\boldsymbol{i}$, where $\boldsymbol{Z} \in \mathbb{C}^{N\times N}\cdot\Omega$ is the impedance matrix of the antenna array. Because of reciprocity, there is $\boldsymbol{Z} = \boldsymbol{Z}^{\mathrm{T}}$, and hence, the radiated power can be written:

$$P_{\mathrm{rad}} = R_{\mathrm{r}}\cdot \boldsymbol{i}^{\mathrm{H}}\boldsymbol{C}\boldsymbol{i}, \tag{1}$$

where $\boldsymbol{C} = \mathrm{Re}\{\boldsymbol{Z}\}/R_{\mathrm{r}}$, and $R_{\mathrm{r}} = (\mathrm{Re}\{\boldsymbol{Z}\})_{k,k}$, is the so-called *radiation resistance* [5]. Let the ULA be aligned with the $z$-axis of a Cartesian coordinate system. The strength of the electric far-field which is excited at a point in the direction of the elevation angle $\theta$, is given by (see e.g., [5] on pp 250 and 258): $E = \tilde{\alpha}\cdot\boldsymbol{a}^{\mathrm{H}}\boldsymbol{i}$, where $\boldsymbol{a} = \begin{bmatrix} 1 & \mathrm{e}^{\mathrm{j}\mu} & \cdots & \mathrm{e}^{\mathrm{j}(N-1)\mu}\end{bmatrix}^{\mathrm{T}}$, with $\mu = kd\cos\theta$, where $d$ is the distance between the radiators, $k = 2\pi/\lambda$, with $\lambda$ denoting the wavelength, and $\tilde{\alpha}$ is a constant. The receive power $P_{\mathrm{rx}}$, extractable by an antenna at a point in the far-field, is proportional to $|E|^2$, such that

$$P_{\mathrm{rx}} = \alpha\cdot \boldsymbol{i}^{\mathrm{H}}\boldsymbol{a}(\theta)\boldsymbol{a}^{\mathrm{H}}(\theta)\boldsymbol{i}, \tag{2}$$

where $\alpha$ is another constant. The optimum excitation current for beamforming into the direction $\theta$, and lossless (ideal) antennas, is given by:

$$\boldsymbol{i}_{\mathrm{opt}}^{\mathrm{ideal}} = \arg\max_{\boldsymbol{i}} \frac{P_{\mathrm{rx}}}{P_{\mathrm{rad}}} = \sqrt{\frac{P_{\mathrm{rad}}/R_{\mathrm{r}}}{\boldsymbol{a}^{\mathrm{H}}(\theta)\boldsymbol{C}^{-1}\boldsymbol{a}(\theta)}}\cdot\boldsymbol{C}^{-1}\boldsymbol{a}(\theta). \tag{3}$$

With $P_{\mathrm{rx}}^{\max} = P_{\mathrm{rx}}|_{\boldsymbol{i}=\boldsymbol{i}_{\mathrm{opt}}^{\mathrm{ideal}}}$, the array gain is defined as:

$$A^{\mathrm{ideal}} = \left.\frac{P_{\mathrm{rx}}^{\max}}{(P_{\mathrm{rx}}^{\max})\,|_{N=1}}\right|_{P_{\mathrm{rad}}=\mathrm{const}} = N\frac{\boldsymbol{a}^{\mathrm{H}}(\theta)\boldsymbol{C}^{-1}\boldsymbol{a}(\theta)}{\boldsymbol{a}^{\mathrm{H}}(\theta)\boldsymbol{a}(\theta)}, \tag{4}$$

that is, as the ratio of the maximum receive power using all $N$ antennas and the receive power obtainable by using only one antenna, while the radiated power is the same in both cases. For a ULA of isotropic radiators, $(\boldsymbol{C})_{n,m} = j_0\left(kd(n-m)\right)$, [2], [7], where $j_0(x) = \sin(x)/x$. The array gain depends on the direction ($\theta$) of beamforming, and on the antenna separation $d$ (mostly via $\boldsymbol{C}$). The largest array gain is obtained in the direction $\theta = 0$, the so-called »end-fire« direction, and approaches $N^2$, for small antenna separation [2], [3].

## III. Super-Gain with Lossy Antennas

When the antennas are *not* lossless, the approach from Section II can be disastrous. From (3), we have

$$\left\|\boldsymbol{i}_{\mathrm{opt}}^{\mathrm{ideal}}\right\|_2^2 = \frac{P_{\mathrm{rad}}}{R_{\mathrm{r}}}\cdot\frac{\boldsymbol{a}^{\mathrm{H}}(\theta)\boldsymbol{C}^{-2}\boldsymbol{a}(\theta)}{\boldsymbol{a}^{\mathrm{H}}(\theta)\boldsymbol{C}^{-1}\boldsymbol{a}(\theta)}, \tag{5}$$

which grows unboundedly as $d \to 0$. Let each lossy antenna have a dissipation resistance $R_{\mathrm{d}}$. The power $P_{\mathrm{d}} = R_{\mathrm{d}}\cdot\|\boldsymbol{i}_{\mathrm{opt}}^{\mathrm{ideal}}\|_2^2$, which is dissipated in the array then also grows unboundedly as $d \to 0$, hence, causing excessively low efficiency. To illustrate, consider an array of $N = 4$ antennas with $R_{\mathrm{d}}/R_{\mathrm{r}} = 10^{-3}$, spaced a distance $d = \lambda/128$ apart, and excited for beamforming into the »end-fire« direction. We obtain $P_\mathrm{d} \approx 3 \times 10^7 P_\mathrm{rad}$. Hence, to radiate just one Microwatt, we would have to dissipate 30 Watts in the antennas. This translates into an array efficiency of approximately $3 \times 10^{-8}$. Clearly, such an array is useless, despite it has a high array gain of $A^\mathrm{ideal} \approx 16$. The problem here is that we should *not* have placed the antennas so closely as $\lambda/128$. Furthermore, the excitation current vector should have taken into account that the antennas are lossy:

$$\boldsymbol{i}_\mathrm{opt}^\mathrm{lossy} = \arg\max_{\boldsymbol{i}} \frac{P_\mathrm{rx}}{P_\mathrm{tot}} = \sqrt{\frac{P_\mathrm{tot}/R_\mathrm{r}}{\boldsymbol{a}^\mathrm{H}(\theta)\left(\boldsymbol{C} + \frac{R_\mathrm{d}}{R_\mathrm{r}}\mathbf{I}_N\right)^{-1}\boldsymbol{a}(\theta)}} \cdot \left(\boldsymbol{C} + \frac{R_\mathrm{d}}{R_\mathrm{r}}\mathbf{I}_N\right)^{-1}\boldsymbol{a}(\theta), \tag{6}$$

where $P_\mathrm{tot} = P_\mathrm{rad} + P_\mathrm{d}$, is the total power that is supplied into the array. With $P_\mathrm{rx}^\mathrm{max} = P_\mathrm{rx}|_{\boldsymbol{i}=\boldsymbol{i}_\mathrm{opt}^\mathrm{lossy}}$, the array gain for lossy antennas is defined as:

$$A^\mathrm{lossy} = \left.\frac{P_\mathrm{rx}^\mathrm{max}}{(P_\mathrm{rx}^\mathrm{max})\,|_{N=1,R_\mathrm{d}=0}}\right|_{P_\mathrm{tot}=\mathrm{const}} = N \cdot \frac{\boldsymbol{a}^\mathrm{H}(\theta)\left(\boldsymbol{C} + \frac{R_\mathrm{d}}{R_\mathrm{r}}\mathbf{I}_N\right)^{-1}\boldsymbol{a}(\theta)}{\boldsymbol{a}^\mathrm{H}(\theta)\boldsymbol{a}(\theta)}. \tag{7}$$

This quantifies how much more receive power we can obtain when all $N$ lossy antennas are used, compared to the case of having only one, but lossless, antenna, while the total supplied power is the same in both cases.

Figure 1 shows the array gain in »end-fire« direction, for a ULA of $N = 4$ isotropic radiators, for several values of $R_\mathrm{d}/R_\mathrm{r}$, as function of the antenna separation $d$. For lossless antennas ($R_\mathrm{d} = 0$), the largest array gain is achieved as $d \to 0$, and approaches $N^2$ from below. However, as $R_\mathrm{d} > 0$, we see that too small values of $d$ are disastrous for the array gain, because of the huge loss of efficiency. On the other hand, we see that there is an optimum antenna separation, which depends on $R_\mathrm{d}/R_\mathrm{r}$, for which the array gain is maximized. We can see from Figure 1, that this optimum distance is always less than half of the wavelength. Moreover, provided the antennas are spaced this optimum distance apart, the array gain is always larger than it would be for uncoupled isotrops (large, or half wavelength spacing). In other words, when done sensibly, the super-gain always outweights the antenna loss, and an array gain larger than that obtainable for uncoupled antennas can be obtained. This is true regardless of the amount of loss, that is, for every value of $R_\mathrm{d}/R_\mathrm{r}$. To analyze the array efficiency, note that from (6), we have

$$\left\|\boldsymbol{i}_\mathrm{opt}^\mathrm{lossy}\right\|_2^2 = \frac{P_\mathrm{tot}}{R_\mathrm{r}} \cdot \frac{\boldsymbol{a}^\mathrm{H}(\theta)\left(\boldsymbol{C} + \frac{R_\mathrm{d}}{R_\mathrm{r}}\mathbf{I}_N\right)^{-2}\boldsymbol{a}(\theta)}{\boldsymbol{a}^\mathrm{H}(\theta)\left(\boldsymbol{C} + \frac{R_\mathrm{d}}{R_\mathrm{r}}\mathbf{I}_N\right)^{-1}\boldsymbol{a}(\theta)}, \tag{8}$$

which is finite for any $d$. To illustrate, consider again our array of $N = 4$ lossy antennas with $R_\mathrm{d}/R_\mathrm{r} = 10^{-3}$. This time, however, the antenna spacing is *not* chosen as $\lambda/128$, but instead equals the optimum distance $d = 0.212\lambda$. Using (8), we see that the power $P_\mathrm{d} = R_\mathrm{d} \cdot ||\boldsymbol{i}_\mathrm{opt}^\mathrm{lossy}||_2^2$, which is dissipated in the array, is

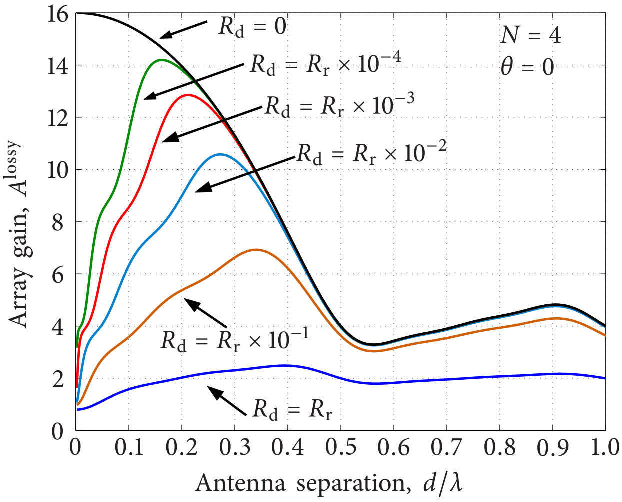


Figure 1. Array gain (as function of antenna separation) of a uniform linear array of $N = 4$ lossy isotrops, when beamforming in »end fire« direction is applied. The different curves correspond to different amounts of antenna loss, quantified by the ratio $R_\mathrm{d}/R_\mathrm{r}$.

now given by $P_\mathrm{d} \approx 0.06 \times P_\mathrm{tot}$, which translates into an array efficiency of more than 94%. With this array, we can radiate a large power of, say, 10 Watts, while dissipating only about 0.6 Watts power in the antennas. At the same time, we see from (7), that we have an array gain of $A^\mathrm{lossy} \approx 12.85$, which is less than 1dB below the maximum array gain of 16, but more than 5dB larger than the number of antennas. This example clearly demonstrates that by optimum spacing of the antennas, and by optimum beamforming (excitation current), one can extract a fairly large amount of super-gain without losing much of array efficiency. These claims are backed by recent experimental results [6].

## IV. Conclusion

The common belief that super-gain is regularly »eaten up« by the simultaneous loss in antenna efficiency is found to be true only if the antenna separation is chosen to be too small. Optimum antenna separation in conjunction with optimum antenna excitation (beamforming), makes it possible to obtain a large amount of super-gain while still maintaining a reasonably large array efficiency. Consequently, compact antenna arrays deserve more attention in future research.

## References


[1] S. A. Schelkunoff, "A mathematical theory of linear arrays," *Bell Systems Tech. Journal*, vol. 22, pp. 80–87, jan 1943.

[2] M. T. Ivrlač and J. A. Nossek, "The Maximum Achievable Array Gain under Physical Transmit Power Constraint," in *Proc. IEEE International Symposium on Information Theory and its Applications*, dec 2008, pp. 1338–1343.

[3] E. E. Altshuler, T. H. O'Donnell, A. D. Yaghjian, and S. R. Best, "A Monopole Superdirective Array," *IEEE Transactions on Antennas and Propagation*, vol. 53(8), pp. 2653–2661, aug 2005.

[4] N. Yaru, "A note on super-gain arrays," *Proc. IRE*, vol. 39, pp. 1081–1085, sep 1951.

[5] A. Balanis, *Antenna Theory*. Second Edition, John Wiley & Sons, 1997.

[6] A. D. Yaghjian, T. H. O'Donnell, E. E. Altshuler, and S. R. Best, "Electrically Small Supergain End-Fire Arrays," *Radio Science*, vol. 43, no. RS3002, doi:10.1029/2007RS003747, 2008.

[7] H. Yordanov, M. T. Ivrlač, P. Russer, and J. A. Nossek, "Arrays of Isotropic Radiators — A Field-theoretic Justification," in *Proc. of the IEEE/ITG International Workshop on Smart Antennas, 2009*, Berlin, Germany, feb 2009.