On Arbitrarily Varying Bidirectional Broadcast Channels with Constraints on Input and States

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***Abstract*—The concept of bidirectional relaying is a key technique to improve the performance in future wireless networks. It applies to three-node networks, where a relay node establishes a bidirectional communication between two other nodes using a decode-and-forward protocol. It divides the whole communication into two phases, namely the multiple access and bidirectional broadcast phase. Here, we concentrate on the second phase, which is also known as the *bidirectional broadcast channel*, and assume that the transmission is affected by arbitrarily varying channels. We impose constraints on the permissible input and state sequences and derive the capacity regions for random and deterministic coding.**

I. Introduction

The recent research progress shows that the use of relays has the potential to significantly increase the performance of wireless networks such as sensor, ad-hoc, and even cellular systems. Due to practical constraints a relay needs orthogonal resources for transmission and reception which can be realized more efficiently, if bidirectional communication is considered [1]. In this work, we consider *bidirectional relaying* in a three-node network, where a relay node establishes a bidirectional communication between two other nodes using a decode-and-forward protocol as depicted in Figure 1.

In the initial multiple access phase both nodes transmit their messages to the relay node which decodes them so that we end up with the classical multiple access channel. Since each node knows its own transmitted message, in the following phase it only remains for the relay to broadcast a re-encoded composition which allows each node to decode the message it is intended to receive using the message that it transmitted in the previous phase as side information. Due to the available side information at the receiving nodes, this channel differs from the classical broadcast channel and is therefore called *bidirectional broadcast channel* (BBC). Capacity achieving strategies for perfect channel state information at all nodes can be found in [2, 3].

Due to the nature of the wireless channel, uncertainty in the channel state information is a ubiquitous phenomenon in practical systems. A well accepted model for channel uncertainty is the concept of *arbitrarily varying channels* (AVC) [4–6] which models channels with unknown parameters that may vary in an arbitrary manner during the whole transmission of a codeword. Bidirectional relaying in the arbitrarily varying scenario is analyzed in [7, 8] and [9, 10], where it is shown that it displays a similar behavior as the single-user AVC: the deterministic code capacity either equals its random code capacity or else is zero [5]. Unfortunately, many channels of practical importance are *symmetrizable* [11] and such channels fall in the category of zero capacity.

The work of H. Boche was partly supported by the German Research Foundation (DFG) under Grant BO 1734/25-1.

(a) MAC phase (b) BBC phase

Fig. 1. Multiple access (MAC) and bidirectional broadcast (BBC) phase of a bidirectional relay communication.

The situation changes significantly, if constraints on the permissible codewords and channel states are imposed. This is a reasonable assumption, since in real communication systems the transmitter as well as possible interferers are usually limited in their transmit power. For the single-user AVC under input and state constraints it is shown that the deterministic code capacity may be positive even for symmetrizable channels, but may be less than its random code capacity [11, 12].

The first phase of the bidirectional relaying protocol, namely the arbitrarily varying MAC with constraints on input and states, is considered in [13, 14] so that we concentrate on the second phase in this work. We introduce the *arbitrarily varying bidirectional broadcast channel* (AVBBC) and some preliminaries in Section II. The AVBBC under random coding and its random code capacity region is analyzed in Section III, while Section IV deals with deterministic coding and the corresponding deterministic code capacity region. Finally, a conclusion is given in Section V.[1]

II. Bidirectional Broadcast Phase

For the bidirectional broadcast phase we assume that the relay has successfully decoded both messages in the previous phase. Now, the relay broadcasts an optimal re-encoded message in such a way that both nodes can decode the intended message using their own message as side information.

[1] *Notation:* Discrete random variables are denoted by capital letters and their corresponding realizations and ranges by lower case letters and calligraphic letters respectively; all logarithms, exponentials, and information quantities are taken to the basis 2; $\mathbb{E}[\cdot]$ is the expectation; $\mathcal{P}(\cdot)$ denotes the set of all probability distributions and $(\cdot)^c$ is the complement of a set.

978–1–4244–6017–5/10/$26.00

A. Arbitrarily Varying Bidirectional Broadcast Channel

The transmission is affected by a channel which varies arbitrarily in an unknown manner from symbol to symbol during the whole transmission of a codeword. We model this behavior with the help of a finite state set $\mathcal{S}$. Further, let $\mathcal{X}$ and $\mathcal{Y}_k$, $k=1,2$, be finite input and output sets. Then, for a fixed state sequence $s^n \in \mathcal{S}^n$ of length n and input and output sequences $x^n \in \mathcal{X}^n$ and $y_k^n \in \mathcal{Y}_k^n$, $k=1,2$, the discrete memoryless broadcast channel is given by $W^{\otimes n}(y_1^n, y_2^n|x^n, s^n) := \prod_{i=1}^n W(y_{1,i}, y_{2,i}|x_i, s_i)$.

Definition 1: The discrete memoryless *arbitrarily varying broadcast channel* is the family

$$\mathcal{W} := \big\{W^{\otimes n} : \mathcal{X}^n \times \mathcal{S}^n \to \mathcal{P}(\mathcal{Y}_1^n \times \mathcal{Y}_2^n)\big\}_{n\in\mathbb{N}, s^n\in\mathcal{S}^n}.$$

Since we do not allow any cooperation between the receiving nodes, it is sufficient to consider the marginal transition probabilities $W_k^{\otimes n}(y_k^n|x^n, s^n)$, $k=1,2$, only. Further, for any probability distribution $q \in \mathcal{P}(\mathcal{S})$ we denote the averaged broadcast channel by

$$\overline{W}_q(y_1, y_2|x) := \sum_{s\in\mathcal{S}} W(y_1, y_2|x, s)q(s) \tag{1}$$

and the corresponding averaged marginal channels by $\overline{W}_{1,q}(y_1|x)$ and $\overline{W}_{2,q}(y_2|x)$.

Further, we will need the concept of symmetrizability for the AVBBC which is an extension of the one for the single-user AVC given in [11].

Definition 2: An AVBBC is $\mathcal{Y}_k$*-symmetrizable* if for some channel $U_k : \mathcal{X} \to \mathcal{P}(\mathcal{S})$

$$\sum_{s\in\mathcal{S}} W_k(y_k|x, s)U_k(s|x') = \sum_{s\in\mathcal{S}} W_k(y_k|x', s)U_k(s|x) \tag{2}$$

holds for every $x, x' \in \mathcal{X}$ and $y_k \in \mathcal{Y}_k$, $k=1,2$.

B. Input and State Constraints

In this work, we denote the mutual information [6, p. 21] between the input random variable X and the output random variable Y by $I(X;Y)$. To emphasize the dependency on the input distribution $P_X \in \mathcal{P}(\mathcal{X})$ and the channel $W : \mathcal{X} \to \mathcal{P}(\mathcal{Y})$, we also write $I(X;Y) = I(P_X, W)$ interchangeably. Furthermore, we use the concept of *types* from Csiszár and Körner. Due to the lack of space we refer to [6] for a detailed survey.

We impose constraints on the input and state sequences. We follow [11] and define cost functions $g(x)$ and $l(s)$ on $\mathcal{X}$ and $\mathcal{S}$, respectively. For convenience, we assume that $\min_{x\in\mathcal{X}} g(x) = \min_{s\in\mathcal{S}} l(s) = 0$ and define $g_{\max} := \max_{x\in\mathcal{X}} g(x)$ and $l_{\max} := \max_{s\in\mathcal{S}} l(s)$. For given $x^n = (x_1, ..., x_n)$ and $s^n = (s_1, ..., s_n)$ we set

$$g(x^n) := \frac{1}{n}\sum_{i=1}^n g(x_i), \tag{3a}$$

$$l(s^n) := \frac{1}{n}\sum_{i=1}^n l(s_i). \tag{3b}$$

Further, for notational convenience we define the costs caused by given probability distributions $p \in \mathcal{P}(\mathcal{X})$ and $q \in \mathcal{P}(\mathcal{S})$ as

$$g(p) = \sum_{x\in\mathcal{X}} p(x)g(x), \quad l(q) = \sum_{s\in\mathcal{S}} q(s)l(s)$$

and observe that, if we consider types, these definitions immediately yield

$$g(x^n) = g(P_{x^n}) \quad \text{and} \quad l(s^n) = l(P_{s^n})$$

for every $x^n \in \mathcal{X}^n$ and every $s^n \in \mathcal{S}^n$, respectively. Thereby, P_{x^n} and P_{s^n} denote types which are induced by x^n and s^n respectively.

This allows us to define the set of all state sequences of length n that satisfy a given state constraint Λ by

$$\mathcal{S}_\Lambda^n := \Big\{s^n \in \mathcal{S}^n : \frac{1}{n}\sum_{i=1}^n l(s_i) = \mathbb{E}_{P_{s^n}}[l(s^n)] \le \Lambda\Big\}.$$

Furthermore, the set of all probability distributions $q \in \mathcal{P}(\mathcal{S})$ that satisfy $\mathbb{E}_q[l(q)] \le \Lambda$ is given by

$$\mathcal{P}(\mathcal{S}, \Lambda) := \big\{q : q \in \mathcal{P}(\mathcal{S}), \mathbb{E}_q[l(q)] \le \Lambda\big\}.$$

In [10] it is shown that an AVBBC (without state constraint) has a capacity region whose interior is empty if the AVBBC is $\mathcal{Y}_1$-symmetrizable or $\mathcal{Y}_2$-symmetrizable. If we impose a state constraint, the situation changes significantly. Now, it is possible that the interior of the capacity region is non-empty even if the AVBBC is $\mathcal{Y}_k$-symmetrizable in the sense of Definition 2. Rather, $\mathcal{Y}_k$-symmetrizability enters the picture via

$$\Lambda_k(P_X) = \begin{cases} \min\limits_{U_k\in\mathcal{U}_k} \sum\limits_{x\in\mathcal{X}} \sum\limits_{s\in\mathcal{S}} P_X(x)U_k(s|x)l(s) & \text{if } \mathcal{U}_k \neq \emptyset \\ \infty & \text{if } \mathcal{U}_k = \emptyset \end{cases} \tag{4}$$

$k=1,2$, which indicates whether the symmetrization violates the imposed state constraint or not. Thereby, $\mathcal{U}_k$ is the set of all channels $U_k : \mathcal{X} \to \mathcal{P}(\mathcal{S})$ which satisfy (2). For given type P_X the quantity $\Lambda_k(P_X)$ is called *symmetrizability costs* and can be interpreted as the minimum costs which are needed to symmetrize the AVBBC. Clearly, if the AVBBC is $\mathcal{Y}_k$-symmetrizable, then $\mathcal{U}_k \neq \emptyset$ and $\Lambda_k(P_X)$ is finite. Further, if the AVBBC is non-$\mathcal{Y}_k$-symmetrizable, then $\mathcal{U}_k = \emptyset$, and we set $\Lambda_k(P_X) = \infty$ for convenience.

C. Preliminaries

We consider the standard model with a block code of arbitrary but sufficient fixed length n. Let $\mathcal{M}_k := \{1, ..., M_k^{(n)}\}$ be the message set of node k, $k=1,2$, which is also known at the relay node. Further, we use the abbreviation $\mathcal{M} := \mathcal{M}_1 \times \mathcal{M}_2$.

First, we introduce the deterministic strategy, where the relay and the receivers use pre-specified encoder and decoders.

Definition 3: A *deterministic* $(M_1^{(n)}, M_2^{(n)}, n)$*-code* $\mathcal{C}_{\mathcal{W},\text{det}}$ of length n for the AVBBC $\mathcal{W}$ under input constraint Γ and state constraint Λ is a family

$$\mathcal{C}_{\mathcal{W},\text{det}} := \big\{(x_m^n, D_{m_2|m_1}^{(1)}, D_{m_1|m_2}^{(2)}) : m_1 \in \mathcal{M}_1, m_2 \in \mathcal{M}_2\big\}$$

with codewords

$$x_m^n \in \mathcal{X}^n \qquad \text{with } g(x_m^n) \leq \Gamma,$$

one for each message $m = (m_1, m_2)$, satisfying the input constraint Γ, and decoding sets at nodes 1 and 2

$$D_{m_2|m_1}^{(1)} \subseteq \mathcal{Y}_1^n \quad \text{and} \quad D_{m_1|m_2}^{(2)} \subseteq \mathcal{Y}_2^n$$

for all $m_1 \in \mathcal{M}_1$ and $m_2 \in \mathcal{M}_2$. For given m_1 at node 1 the decoding sets must be disjoint, i.e., $D_{m_2|m_1}^{(1)} \cap D_{m_2'|m_1}^{(1)} = \emptyset$ for $m_2' \neq m_2$, and similarly for given m_2 at node 2 the decoding sets must satisfy $D_{m_1|m_2}^{(2)} \cap D_{m_1'|m_2}^{(2)} = \emptyset$ for $m_1' \neq m_1$.

When x_m^n with $m = (m_1, m_2)$ and $g(x_m^n) \leq \Gamma$ has been sent, and y_1^n and y_2^n have been received at nodes 1 and 2, the decoder at node 1 is in error if y_1^n is not in $D_{m_2|m_1}^{(1)}$. Accordingly, the decoder at node 2 is in error if y_2^n is not in $D_{m_1|m_2}^{(2)}$. This allows us to define the probability of error for the deterministic code $\mathcal{C}_{\mathcal{W},\text{det}}$ for given message $m = (m_1, m_2)$ and state sequence $s^n \in \mathcal{S}_\Lambda^n$ as

$$e(m, s^n|\mathcal{C}_{\mathcal{W},\text{det}}) := W^{\otimes n}\big((D_{m_2|m_1}^{(1)} \times D_{m_1|m_2}^{(2)})^c | x_m^n, s^n\big)$$

and the corresponding marginal probabilities of error at nodes 1 and 2 as $e_1(m, s^n|\mathcal{C}_{\mathcal{W},\text{det}}) := W_1^{\otimes n}((D_{m_2|m_1}^{(1)})^c | x_m^n, s^n)$ and $e_2(m, s^n|\mathcal{C}_{\mathcal{W},\text{det}}) := W_2^{\otimes n}((D_{m_1|m_2}^{(2)})^c | x_m^n, s^n)$, respectively. Thus, the *average probability of error* for state sequence $s^n \in \mathcal{S}_\Lambda^n$, i.e., it satisfies the state constraint Λ, is given by

$$\bar{e}(s^n|\mathcal{C}_{\mathcal{W},\text{det}}) := \frac{1}{|\mathcal{M}|} \sum_{m \in \mathcal{M}} e(m, s^n|\mathcal{C}_{\mathcal{W},\text{det}})$$

and the corresponding marginal average probability of error at node k by $\bar{e}_k(s^n|\mathcal{C}_{\mathcal{W},\text{det}}) := \frac{1}{|\mathcal{M}|} \sum_{m \in \mathcal{M}} e_k(m, s^n|\mathcal{C}_{\mathcal{W},\text{det}})$, $k = 1, 2$.

Definition 4: A rate pair $(R_{\text{R}1}, R_{\text{R}2}) \in \mathbb{R}_+^2$ is said to be *deterministically achievable for the AVBBC $\mathcal{W}$ under input constraint* Γ *and state constraint* Λ if for any $\delta > 0$ there exists an $n(\delta) \in \mathbb{N}$ and a sequence of deterministic $(M_1^{(n)}, M_2^{(n)}, n)$-codes $(\mathcal{C}_{\mathcal{W},\text{det}}^{(n)})_{n \in \mathbb{N}}$ with codewords x_{m_1,m_2}^n, $m_1 = 1, ..., M_1^{(n)}$, $m_2 = 1, ..., M_2^{(n)}$, each satisfying $g(x_{m_1,m_2}^n) \leq \Gamma$, such that for all $n \geq n(\delta)$ we have

$$\frac{\log M_1^{(n)}}{n} \geq R_{\text{R}2} - \delta \quad \text{and} \quad \frac{\log M_2^{(n)}}{n} \geq R_{\text{R}1} - \delta$$

while

$$\max_{s^n : l(s^n) \leq \Lambda} \bar{e}(s^n|\mathcal{C}_{\mathcal{W},\text{det}}) \to 0 \quad \text{as } n \to \infty,$$

$k = 1, 2$. The set of all achievable rate pairs is the *deterministic code capacity region of the AVBBC $\mathcal{W}$ under input constraint* Γ *and state constraint* Λ and is denoted by $\mathcal{R}_{\mathcal{W},\text{det}}(\Gamma, \Lambda)$.

Next, we introduce the random strategy, where the encoder and the decoders are chosen according to a common random experiment whose outcome has to be known at all nodes in advance.

Definition 5: A *random* $(M_1^{(n)}, M_2^{(n)}, n, \mathcal{Z}, \mu)$*-code* $\mathcal{C}_{\mathcal{W},\text{ran}}$ of length n for the AVBBC $\mathcal{W}$ under input constraint Γ and state constraint Λ is a collection of deterministic $(M_1^{(n)}, M_2^{(n)}, n)$-codes $\mathcal{C}(Z)$,

$$\mathcal{C}(Z) := \big\{\big(x_m^n(Z), D_{m_2|m_1}^{(1)}(Z), D_{m_1|m_2}^{(2)}(Z)\big) : m_1 \in \mathcal{M}_1, m_2 \in \mathcal{M}_2\big\}$$

where $Z \in \mathcal{Z}$, $|\mathcal{Z}| < \infty$, and $Z \sim \mu$. Thereby, each $\mathcal{C}(Z)$ is a deterministic code in the sense of Definition 3 which means that each $\mathcal{C}(Z)$ satisfies the constraints individually.

Then, the average probability of error of the random code $\mathcal{C}_{\mathcal{W},\text{ran}}$ for given state sequence $s^n \in \mathcal{S}_\Lambda^n$ is given by

$$\bar{e}(s^n|\mathcal{C}_{\mathcal{W},\text{ran}}) := \mathbb{E}_Z[\bar{e}(s^n|\mathcal{C}(Z))]$$

and accordingly the corresponding marginal average probability of error at node k by $\bar{e}_k(s^n|\mathcal{C}_{\mathcal{W},\text{ran}}) := \mathbb{E}_Z[\bar{e}_k(s^n|\mathcal{C}(Z))]$, $k = 1, 2$.

Clearly, the definitions of a *randomly achievable rate pair under input and state constraint* and the *random code capacity region under input and state constraint* follow accordingly.

III. Random Code Capacity Region

Here, we derive the random code capacity region of the AVBBC $\mathcal{W}$ under input constraint Γ and state constraint Λ. For this purpose we define the region

$$\mathcal{R}_{\overline{\mathcal{W}}}(P_X) := \big\{(R_{\text{R}1}, R_{\text{R}2}) \in \mathbb{R}_+^2 : R_{\text{R}k} \leq \inf_{q \in \mathcal{P}(\mathcal{S},\Lambda)} I(P_X, \overline{W}_{k,q}),\ k = 1, 2\big\} \tag{5}$$

for probability distributions $\{P_X(x)\overline{W}_q(y_1, y_2|x)\}_{q \in \mathcal{P}(\mathcal{S},\Lambda)}$.

Theorem 1: The random code capacity region $\mathcal{R}_{\mathcal{W},\text{ran}}(\Gamma, \Lambda)$ of the AVBBC $\mathcal{W}$ under input constraint Γ and state constraint Λ is

$$\mathcal{R}_{\mathcal{W},\text{ran}}(\Gamma, \Lambda) = \bigcup_{P_X : g(P_X) \leq \Gamma} \mathcal{R}_{\overline{\mathcal{W}}}(P_X).$$

In the following we prove the random code capacity region which is mainly based on the idea of *Ahlswede's robustification technique* [15, 16].

A. Compound Bidirectional Broadcast Channel

As in [9] for the AVBBC without constraints on input and states, we start with a construction of a suitable compound BBC, where the key idea is to restrict it in an appropriate way. Having the state constraint Λ in mind, it is reasonable to restrict our attention to all probability distributions $q \in \mathcal{P}(\mathcal{S}, \Lambda)$. Let us consider the family of averaged broadcast channels, cf. (1),

$$\overline{\mathcal{W}} := \big\{\overline{W}_q(y_1, y_2|x)\big\}_{q \in \mathcal{P}(\mathcal{S},\Lambda)} \tag{6}$$

and observe that $\overline{\mathcal{W}}$ already corresponds to a compound BBC where each permissible probability distribution $q \in \mathcal{P}(\mathcal{S}, \Lambda)$ parametrizes one element of the compound channel $\overline{\mathcal{W}}$. The capacity region of the compound BBC is known and can be found in [17, 18]. It is shown that for given input distribution P_X all rate pairs $(R_{\text{R}1}, R_{\text{R}2})$ satisfying $(R_{\text{R}1}, R_{\text{R}2}) \in \mathcal{R}_{\overline{\mathcal{W}}}(P_X)$, cf. (5), are deterministically achievable.

In more detail, in [18] it is shown that there exist a sequence of deterministic codes $(\mathcal{C}_{\overline{\mathcal{W}},\text{det}}^{(n)})_{n\in\mathbb{N}}$ for the compound BBC $\overline{\mathcal{W}}$ such that all rate pairs $(R_{\text{R1}}, R_{\text{R2}}) \in \mathcal{R}_{\overline{\mathcal{W}}}(P_X)$ are achievable while the average probability of error can be bounded from above by

$$\bar{e}(q|\mathcal{C}_{\overline{\mathcal{W}},\text{det}}) := \frac{1}{|\mathcal{M}|}\sum_{m\in\mathcal{M}} \overline{W}_q^{\otimes n}\big((D_{m_2|m_1}^{(1)} \times D_{m_1|m_2}^{(2)})^c|x_m^n\big) \leq \lambda_{\overline{\mathcal{W}}}^{(n)}$$

for all $q \in \mathcal{P}(\mathcal{S},\Lambda)$ with $\lambda_{\overline{\mathcal{W}}}^{(n)} = \lambda_{\overline{\mathcal{W}},1}^{(n)} + \lambda_{\overline{\mathcal{W}},2}^{(n)}$ where $\lambda_{\overline{\mathcal{W}},k}^{(n)}$ is the average probability of error at node k, $k = 1,2$. Moreover, for n large enough, we have

$$\lambda_{\overline{\mathcal{W}},k}^{(n)} = (n+1)^{|\mathcal{X}||\mathcal{Y}_k|}2^{-n\frac{c\delta^2}{2}} + \frac{(n+1)^{|\mathcal{X}||\mathcal{Y}_k|}}{1-(n+1)^{|\mathcal{X}|}2^{-nc\delta^2}}2^{-n\frac{\tau}{8}}$$

which decreases exponentially fast for increasing block length n. Thereby, $\delta > 0$, $\tau > 0$, and $c > 0$ are constants, cf. [18].

Together with (1) this immediately implies that for $\mathcal{C}_{\overline{\mathcal{W}},\text{det}}$ the average probability of a successful transmission over the compound BBC $\overline{\mathcal{W}}$ can be bounded from below by

$$\frac{1}{|\mathcal{M}|}\sum_{m\in\mathcal{M}} \overline{W}_q^{\otimes n}(D_{m_2|m_1}^{(1)} \times D_{m_1|m_2}^{(2)}|x_m^n) > 1 - \lambda_{\overline{\mathcal{W}}}^{(n)}$$

or equivalently by

$$\frac{1}{|\mathcal{M}|}\sum_{m\in\mathcal{M}}\sum_{s^n\in\mathcal{S}^n} W^{\otimes n}(D_{m_2|m_1}^{(1)} \times D_{m_1|m_2}^{(2)}|x_m^n, s^n)q^{\otimes n}(s^n)$$
$$> 1 - \lambda_{\overline{\mathcal{W}}}^{(n)} \quad \text{for all } q^{\otimes n} = \prod_{i=1}^n q \text{ and } q \in \mathcal{P}(\mathcal{S},\Lambda).$$

B. Robustification

As in [9] for the AVBBC without constraints, we use the deterministic code $\mathcal{C}_{\overline{\mathcal{W}},\text{det}}$ for the compound BBC $\overline{\mathcal{W}}$ to construct a random code $\mathcal{C}_{\mathcal{W},\text{ran}}$ for the AVBBC $\mathcal{W}$ under input constraint Γ and state constraint Λ.

Let Π_n be the group of permutations acting on $\{1,2,...,n\}$. For given sequence $s^n = (s_1, s_2, ..., s_n) \in \mathcal{S}^n$ and permutation $\pi \in \Pi_n : \mathcal{S}^n \to \mathcal{S}^n$, we denote the permuted sequence $(s_{\pi(1)}, s_{\pi(2)}, ..., s_{\pi(n)}) \in \mathcal{S}^n$ by $\pi(s^n)$. Further, we denote the inverse permutation by π^{-1} so that $\pi^{-1}(\pi(s^n)) = s^n$ since π is bijective.

Theorem 2 (Robustification technique): Let $f : \mathcal{S}^n \to [0,1]$ be a function such that for some $\alpha \in (0,1)$ the inequality

$$\sum_{s^n\in\mathcal{S}^n} f(s^n)q^{\otimes n}(s^n) > 1-\alpha \qquad \text{for all } q \in \mathcal{P}_0(n,\mathcal{S},\Lambda)$$

holds where $\mathcal{P}_0(n,\mathcal{S},\Lambda) := \{q \in \mathcal{P}_0(n,\mathcal{S}) : \mathbb{E}_q[l(q)] \leq \Lambda\}$ and $\mathcal{P}_0(n,\mathcal{S})$ is the set of all types of sequences in $\mathcal{S}^n$. Then

$$\frac{1}{n!}\sum_{\pi\in\Pi_n} f\big(\pi(s^n)\big) > 1-(n+1)^{|\mathcal{S}|}\alpha \qquad \text{for all } s^n \in \mathcal{S}_\Lambda^n.$$

Proof: The proof is just a slight modification of the corresponding proof in [16], where a similar result is given without constraints on the sequences of states. We omit the details due to the lack of space. ■

With the robustification technique and

$$f\big(\pi(s^n)\big) = \frac{1}{|\mathcal{M}|}\sum_{m\in\mathcal{M}} W^{\otimes n}\big(D_{m_2|m_1}^{(1)} \times D_{m_1|m_2}^{(2)}|x_m^n, \pi(s^n)\big)$$

we immediately obtain a random $(M_1^{(n)}, M_2^{(n)}, n, \Pi_n, \mu)$-code $\mathcal{C}_{\mathcal{W},\text{ran}}$ for the AVBBC $\mathcal{W}$ under input constraint Γ and state constraint Λ which is given by the family

$$\mathcal{C}_{\mathcal{W},\text{ran}} = \big\{(\pi^{-1}(x_m^n), \pi^{-1}(D_{m_2|m_1}^{(1)}), \pi^{-1}(D_{m_1|m_2}^{(2)})) : m_1 \in \mathcal{M}_1, m_2 \in \mathcal{M}_2, \pi \in \Pi_n, \mu\big\}$$

with $\pi^{-1}(D_{m_2|m_1}^{(1)}) = \bigcup_{y_1^n\in D_{m_2|m_1}^{(1)}} \pi^{-1}(y_1^n)$, $\pi^{-1}(D_{m_1|m_2}^{(2)}) = \bigcup_{y_2^n\in D_{m_1|m_2}^{(2)}} \pi^{-1}(y_2^n)$ and μ the uniform distribution on Π_n.

From the robustification technique follows that the average probability of error of $\mathcal{C}_{\mathcal{W},\text{ran}}$ is bounded from above by

$$\bar{e}(s^n|\mathcal{C}_{\mathcal{W},\text{ran}}) \leq (n+1)^{|\mathcal{S}|}\lambda_{\overline{\mathcal{W}}}^{(n)} =: \lambda_{\mathcal{W},\text{ran}}^{(n)} \quad \text{for all } s^n \in \mathcal{S}_\Lambda^n.$$

Moreover, from the construction it is clear that for given input P_X, the random code $\mathcal{C}_{\mathcal{W},\text{ran}}$ achieves for the AVBBC $\mathcal{W}$ the same rate pairs as $\mathcal{C}_{\overline{\mathcal{W}},\text{det}}$ for the compound BBC $\overline{\mathcal{W}}$. Finally, taking the union over all input distributions P_X that satisfy the input constraint $g(P_X) \leq \Gamma$ establishes the achievability of the random code capacity region $\mathcal{R}_{\mathcal{W},\text{ran}}(\Gamma,\Lambda)$.

C. Converse

It remains to show that the presented random coding strategy actually achieves all possible rate pairs so that it is optimal in the sense that no other rate pairs are achievable.

As a first step, it is easy to show that the average probability of error for the random code $\mathcal{C}_{\mathcal{W},\text{ran}}$ for the AVBBC $\mathcal{W}$ equals the average probability of error for the random code for the compound BBC $\overline{\mathcal{W}}$. Hence, it is clear that we cannot achieve higher rates as for the constructed compound BBC $\overline{\mathcal{W}}$ with random codes. The deterministic rates of the compound channel can be found in [18]. As in [19] for the single-user compound channel, it can be easily shown that for the compound BBC the achievable rates for deterministic and random coding are equal. Since the constructed random code for the AVBBC $\mathcal{W}$ already achieves these rates, the converse is proved and therewith also Theorem 1. ■

IV. Deterministic Code Capacity Region

A random coding strategy as given by $\mathcal{C}_{\mathcal{W},\text{ran}}$ for the AVBBC $\mathcal{W}$ requires *common randomness* between the transmitter and the receivers [6]. If this kind of resource is not available, one is interested in deterministic strategies. Here, we derive the deterministic code capacity region of the AVBBC $\mathcal{W}$ under input constraint Γ and state constraint Λ.

Theorem 3: If $\max_{P_X: g(P_X)\leq\Gamma} \Lambda_k(P_X) > \Lambda$, $k = 1,2$, then the deterministic code capacity region $\mathcal{R}_{\mathcal{W},\text{det}}(\Gamma,\Lambda)$ of the AVBBC $\mathcal{W}$ under input constraint Γ and state constraint Λ is

$$\mathcal{R}_{\mathcal{W},\text{det}}(\Gamma,\Lambda) = \bigcup_{\substack{P_X:\, g(P_X)\leq\Gamma \\ \Lambda_k(P_X)>\Lambda,\, k=1,2}} \mathcal{R}_{\overline{\mathcal{W}}}(P_X).$$

If $\max_{P_X: g(P_X) \leq \Gamma} \Lambda_k(P_X) < \Lambda$, $k \in \{1,2\}$, then interior$(\mathcal{R}_{\mathcal{W},\text{det}}(\Gamma,\Lambda)) = \emptyset$.

As already observed in [11] for the single-user AVC *Ahlswede's elimination technique* [5] does not work anymore if constraints are imposed. Consequently, for the proof of Theorem 3 we need a proof which does not rely on this technique. In the following we sketch the proof idea which is mainly based on an extension of [11].

A. Symmetrizability

The following lemma shows that under state constraint Λ no code with codewords of type P_X satisfying $\Lambda_1(P_X) < \Lambda$ or $\Lambda_2(P_X) < \Lambda$ can be good.

Lemma 1: For a $\mathcal{Y}_1$-symmetrizable AVBBC any code of block length n with $M_2^{(n)} \geq 2$ codewords, each of type P_X, with $\Lambda_1(P_X) < \Lambda$, has

$$\max_{s^n: l(s^n) \leq \Lambda} \bar{e}_1(s^n|\mathcal{C}_{\mathcal{W},\text{det}}) \geq \frac{M_2^{(n)} - 1}{2M_2^{(n)}} - \frac{1}{n}\frac{l_{\max}^2}{(\Lambda - \Lambda_1(P_X))^2}.$$

Similarly, for a $\mathcal{Y}_2$-symmetrizable AVBBC any code of block length n with $M_1^{(n)} \geq 2$ codewords, each of type P_X, with $\Lambda_2(P_X) < \Lambda$, has

$$\max_{s^n: l(s^n) \leq \Lambda} \bar{e}_2(s^n|\mathcal{C}_{\mathcal{W},\text{det}}) \geq \frac{M_1^{(n)} - 1}{2M_1^{(n)}} - \frac{1}{n}\frac{l_{\max}^2}{(\Lambda - \Lambda_2(P_X))^2}.$$

Proof: In [11, Lemma 1] a similar result is proved for the single-user case. The proof of Lemma 1 is almost identical to it so that it is omitted here for brevity. ■

Remark 1: Lemma 1 indicates that for a successful transmission using codewords of type P_X the symmetrizability costs $\Lambda_k(P_X)$, $k = 1,2$, have to exceed the permissible (or available) costs Λ since otherwise the AVBBC can be symmetrized which prohibits any reliable communication. This establishes the second part of Theorem 3 and therewith characterizes when interior$(\mathcal{R}_{\mathcal{W},\text{det}}(\Gamma,\Lambda)) = \emptyset$.

B. Positive Rates

Next, we present a coding strategy with codewords of type P_X which achieves the desired rates if the symmetrizability costs exceed the permissible costs, i.e., $\Lambda_k(P_X) > \Lambda$, $k = 1,2$, as specified in Theorem 3. Fortunately, we are in the same position as in the single-user AVC [11]: the coding strategy for the AVBBC without state constraint [10] must only be slightly modified to apply also to the AVBBC with state constraint Λ.

1) Codewords: The set of codewords is the same as for the unconstrained case [10] and is restated here for completeness. We need codewords x_{m_1,m_2}^n, $m_1 = 1,...,M_1^{(n)}$, $m_2 = 1,...,M_2^{(n)}$ with the following properties.

Lemma 2: For any $\epsilon > 0$, $n \geq n_0(\epsilon)$, $M_k^{(n)} \geq 2^{n\epsilon}$, $k = 1,2$, and given type P_X, there exist codewords $x_{m_1,m_2}^n \in \mathcal{X}^n$, $m_1 = 1,...,M_1^{(n)}$, $m_2 = 1,...,M_2^{(n)}$, each of type P_X, such that for every $x^n \in \mathcal{X}^n$, $s^n \in \mathcal{S}_\Lambda^n$, and every joint type $P_{XX'S}$, with $R_{\text{R1}} = \frac{1}{n}\log M_2^{(n)}$ and $R_{\text{R2}} = \frac{1}{n}\log M_1^{(n)}$, we have for each fixed $m_1 \in \mathcal{M}_1$ the following properties

$$\big|\{m_2' : (x^n, x_{m_1,m_2'}^n, s^n) \in \tau_{XX'S}\}\big| \leq 2^{n(|R_{\text{R1}} - I(X';XS)|^+ + \epsilon)}$$

$$\frac{1}{M_2^{(n)}}\big|\{m_2 : (x_{m_1,m_2}^n, s^n) \in \tau_{XS}\}\big| \leq 2^{-n\frac{\epsilon}{2}} \;\text{ if } I(X;S) > \epsilon$$

$$\frac{1}{M_2^{(n)}}\big|\{m_2 : (x_{m_1,m_2}^n, x_{m_1,m_2'}^n, s^n) \in \tau_{XX'S} \text{ for some } m_2' \neq m_2\}\big| \leq 2^{-n\frac{\epsilon}{2}} \;\text{ if } I(X;X'S) - |R_{\text{R1}} - I(X';S)|^+ > \epsilon$$

and, of course, similar properties for each fixed $m_2 \in \mathcal{M}_2$. Thereby, τ_X denotes the set of all sequences of type P_X with its natural extensions to joint types. ■

2) Decoding Sets: The decoding sets as given in [10] must slightly be modified to include the state constraint Λ as similarly done in [11] for the single-user AVC. Therefore, we define the set

$$\mathcal{C}_{\eta_k}(\Lambda) = \big\{P_{XSY_k} : D(P_{XSY_k}\|P_X \times P_S \times W_k) \leq \eta_k, l(P_S) \leq \Lambda\big\}.$$

Then, the decoding sets at node 1 are defined as follows.

Definition 6: For given codewords $x_{m_1,m_2}^n \in \mathcal{X}^n$, $m_1 = 1,...,M_1^{(n)}$, $m_2 = 1,...,M_2^{(n)}$, each of type P_X, and $\eta_1 > 0$ we have $y_1^n \in D_{m_2|m_1}^{(1)}$ if and only if

I) There exists an $s^n \in \mathcal{S}_\Lambda^n$ such that $P_{x_{m_1,m_2}^n, s^n, y_1^n} \in \mathcal{C}_{\eta_1}(\Lambda)$

II) For each codeword $x_{m_1,m_2'}^n$ with $m_2' \neq m_2$ which satisfies $P_{x_{m_1,m_2'}^n, s'^n, y_1^n} \in \mathcal{C}_{\eta_1}(\Lambda)$ for some $s'^n \in \mathcal{S}_\Lambda^n$, we have $I(XY_1;X'|S) \leq \eta_1$ where X, X', S, Y_1 are dummy random variables such that $P_{XX'SY_1}$ equals the joint type of $(x_{m_1,m_2}^n, x_{m_1,m_2'}^n, s^n, y_1^n)$.

The decoding sets at node 2 are defined accordingly with $\eta_2 > 0$. A key part is now to ensure that these decoding sets are disjoint for small enough η_1 and η_2 which can be shown analogously to [11]. Here is where the conditions on the symmetrizability costs, i.e., $\Lambda_k(P_X) > \Lambda$, come in.

Lemma 3: Given $\Lambda > 0$ and arbitrarily small $\alpha > 0$, for any type P_X with $\min_x P_X(x) \geq \beta > 0$ and $\Lambda_k(P_X) \geq \Lambda + \alpha$, $k = 1,2$, one can choose $\eta_k > 0$ sufficiently small such that for any set of codewords $x_{m_1,m_2}^n \in \mathcal{X}^n$, $m_1 = 1,...,M_1^{(n)}$, $m_2 = 1,...,M_2^{(n)}$, each of type P_X, the decoding sets as given in Definition 6 above are disjoint. ■

3) Rates: Next, we show that codewords of type P_X with properties as given in Lemma 2 and decoding sets as given in Definition 6 suffices to achieve all rate pairs as specified by the region $\mathcal{R}_{\overline{\mathcal{W}}}(P_X)$.

Lemma 4: Given $\Lambda > 0$ and arbitrarily small $\alpha > 0$, $\beta > 0$, $\delta > 0$, for any type P_X with $\min_x P_X(x) \geq \beta > 0$ and

$$\Lambda_k(P_X) \geq \Lambda + \alpha, \qquad k = 1,2,$$

there are codewords $x_{m_1,m_2}^n \in \mathcal{X}^n$, $m_1 = 1,...,M_1^{(n)}$, $m_2 = 1,...,M_2^{(n)}$, each of type P_X, such that

$$\frac{\log M_1^{(n)}}{n} > \inf_{q \in \mathcal{P}(\mathcal{S},\Lambda)} I(P_X, \overline{W}_{2,q}) - \delta$$

$$\frac{\log M_2^{(n)}}{n} > \inf_{q \in \mathcal{P}(\mathcal{S},\Lambda)} I(P_X, \overline{W}_{1,q}) - \delta$$

while

$$\max_{s^n: l(s^n)\leq\Lambda} \bar{e}(s^n|\mathcal{C}_{\mathcal{W},\text{det}}) \to 0 \quad \text{as } n\to\infty.$$

Using codewords with properties as stated in Lemma 2 and decoding sets as given in Definition 6, the proof of Lemma 4 is a straightforward extension of a similar result for single-user AVC given in [11, Theorem 2]. Due to the lack of space the details are omitted.

4) Converse: It remains to show that $\mathcal{R}_{\overline{\mathcal{W}}}(P_X)$, cf. (5), characterizes already the achievable rate pairs for given P_X. This is done by the following lemma.

Lemma 5: For any $\Lambda > 0$, $\delta > 0$, and $\epsilon < 1$, there exists an n_0 such that for any deterministic code $\mathcal{C}_{\mathcal{W},\text{det}}$ of block length $n \geq n_0$ with $M_1^{(n)} M_2^{(n)}$ codewords, each of type P_X, satisfying

$$\frac{\log M_2^{(n)}}{n} \geq \inf_{q\in\mathcal{P}(\mathcal{S},\Lambda)} I(P_X, \overline{W}_{1,q}) + \delta$$

implies

$$\max_{s^n: l(s^n)\leq\Lambda} \bar{e}_1(s^n|\mathcal{C}_{\mathcal{W},\text{det}}) > \epsilon.$$

And similarly, if the codewords satisfy $\frac{1}{n}\log M_1^{(n)} \geq \inf_{q\in\mathcal{P}(\mathcal{S},\Lambda)} I(P_X, \overline{W}_{2,q}) + \delta$, then $\max_{s^n: l(s^n)\leq\Lambda} \bar{e}_2(s^n|\mathcal{C}_{\mathcal{W},\text{det}}) > \epsilon$.

The proof follows [11, Lemma 2] where a similar converse result is shown for the single-user case. The details are omitted for brevity.

5) Capacity Region: Summarizing the results obtained so far, we see that for given input distribution P_X the achievable rates for the AVBBC $\mathcal{W}$ under input constraint Γ and state constraint Λ are given by $\mathcal{R}_{\overline{\mathcal{W}}}(P_X)$, cf. (5), if $\Lambda_k(P_X) > \Lambda$, $k = 1, 2$.

Finally, the deterministic code capacity region $\mathcal{R}_{\mathcal{W},\text{det}}(\Gamma,\Lambda)$ of the AVBBC $\mathcal{W}$ under input constraint Γ and state constraint Λ is obtained in a straightforward manner by taking the union over all input types P_X which satisfy the input constraint $g(P_X) \leq \Gamma$. ■

Remark 2: The case where $\Lambda_k(P_X) = \Lambda$, $k \in \{1, 2\}$, remains unsolved in a similar way as for the single-user AVC [11]. Likewise, we expect that interior$(\mathcal{R}_{\mathcal{W},\text{det}}(\Gamma,\Lambda)) = \emptyset$ in that case.

V. Conclusion

Ahlswede's elimination technique [5] reveals the following dichotomy of the deterministic code capacity of an AVC: it either equals its random code capacity or else is zero. Unfortunately, many channels of practical interest are symmetrizable and fall in the latter category. Imposing constraints on the permissible sequence of channel states changes the situation. Now, even when the channel is symmetrizable, the deterministic code capacity under input and state constraints may be positive but less than its random code capacity.

In this work we established the random code and deterministic code capacity regions of the arbitrarily varying bidirectional broadcast channel. Thereby, we observed that the constraints on the state sequences may reduce the deterministic code capacity region so that it is strictly smaller than the corresponding random code capacity region, but they preserve the general dichotomy behavior of the deterministic code capacity region: it still equals either a non-empty region or else has an empty interior. Although the deterministic code capacity region displays a dichotomy behavior, it cannot be exploited to prove the corresponding capacity region since Ahlswede's elimination technique [5] does not work anymore in the presence of constraints on input and states. This necessitated a proof technique which does not rely on the dichotomy behavior and is based on an idea of Csiszár and Narayan [11].

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