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Valuation of multi-dimensional derivatives in a stochastic covariance framework

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Abstract

Die Finanzkrise hat gezeigt, dass die Annahme von konstanten Kovarianzen nicht gültig ist. Diese Arbeit beschäftigt sich daher eingehend mit der Modellierung von stochastischer Kovarianz in den Finanzmärkten und entwickelt Techniken, die es erlauben die stochastische Kovarianz als Risikotreiber in die Bewertung von Derivaten miteinzubeziehen. Wir behandeln zwei Modelle näher, eines mit stochastischer Varianz und deterministischer Korrelation, ein anderes mit stochastischer Varianz und Korrelation.
Für das erste Modell können wir mit Hilfe von partiellen Differentialgleichungen und der Separationsmethode semi-analytische Lösungsformeln für Barrier-Optionen herleiten. Im zweiten Modell wenden wir Approximationstechniken der Störungstheorie an, um leicht zu berechnende und gut konvergierende Approximationen für Nicht-Vanillaprodukte mit mehr als einem Basistitel zu finden.

Abstract

The financial crisis has shown that constant covariances are an assumption which is not valid as correlations tend to increase in extreme market events. This thesis covers approaches to model stochastic covariance risk in financial markets, and develops techniques to incorporate stochastic covariance as a risk driver in the pricing of derivatives. We treat two models, one with stochastic variance and deterministic correlation, one with stochastic variance and stochastic correlation, more closely.

For the first model we find a semi-analytic pricing formula for double-barrier options using PDEs and the method of separation. In the second model we apply the techniques of perturbation theory to find simply computable and well converging approximations for non-vanilla products with more than one underlying.

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Part I

Introduction

Chapter 1

Introduction

1.1 Literature overview and purpose of the thesis

In 1973 Black and Scholes published their famous paper on the pricing and hedging of contingent claims [13]. And even though Dupire [37], Heston [65] or Stein and Stein [111] – just to name some of the papers – extended the model to relax its most rigorous assumptions, such as a constant volatility, the basic Black Scholes framework is still used as a standard to quote implicit volatility. Another assumption – the one underlying – has first been relaxed in literature by Margrabe in his paper on exchange options [86] where he found a closed-form formula for $(\max(S_1 - S_2, 0))$ by a handy choice of the numeraire, where S_i denotes the price of the stock i. Stulz [112] and Johnson [73] priced options depending on the minimum or maximum of two and more underlyings at maturity time T. He et al. [64] found a closed-form solution for the joint distribution of the maximum/minimum and maturity values of two assets in a two-dimensional Black-Scholes framework [64] and priced barrier and lookback options with two underlyings.

During the last two decades the popularity of structured derivatives on several underlyings has increased, e.g. as a component of certificates. In the late 1990s the Société Générale has marketed the so-called mountain range options. Annapurna (barrier option), Altas (call on average performance with worst and best performer removed), Everest (payoff dependent on worst performer in basket) and Himalayan (payoff dependent on best performer in basket) are basket products and depend on the performance of the best/worst performing asset in the basket [16]. Those products have been sold in certificate structures to retail investors as well.
The Bank of International Settlements conducts semi-annual and more comprehensive triennial surveys [10]. The years between the 2007 and 2010 BIS surveys are characterised

by an extreme growth in OTC derivatives that peaked in the first half of 2008, and a subsequent reduction in positions. The decline in amounts outstanding of derivatives on all types of risks is partly due to trade compression following the bankruptcy of Lehman Brothers in September 2008. But even though the decline in stock prices during the current crisis resulted in much smaller positions in the equity segment of the OTC derivatives market (notional amounts outstanding of equity-linked contracts fell by 28% to 7 trillion, whereas gross market values dropped by 23% (forwards and swaps) and 37% (options) [10]) and the sophistication of payoffs decreased, the crisis showed a need for more realistic models. Instead of improving pricing and hedging algorithms for more and more complex payoffs the interest has now shifted to relax long believed assumptions, like the one in small neglectable interest basis spreads between different payment frequencies, or the determination of the credit value adjustments (CVA) to the price of OTC derivatives. The importance of the management of counterparty risk by applying bilateral netting and collateral arrangements has increased. In the aftermath of the Lehman bankruptcy many banks have founded CVA desks to actively manage the counterparty risk. The exposure from unrealised P&L towards a certain counterparty can be treated as a complex hybrid derivative. The accurate modelling of relationship patterns has, thus, become even more important.

The growth of market volume and increase of sophistication in the late 1990s and the first years of the new millennium induced a growing interest in the literature to relax the most rigorous assumptions of the Black-Scholes framework, e.g. deterministic volatility. And since recently – also fuelled by the crisis which clearly showed that correlations increase in extreme market events – the assumption of constant covariance and correlation has been tackled in more detail. The performance of a portfolio or any multi-dimensional derivative greatly depends on the joint behaviour of the underlyings, i.e. the variances and correlations. In multi-dimensional econometrics the authors have put their effort in accurately modelling the volatility/covariance dynamics. ARCH-GARCH models (e.g. Bollerslev et al. [14]) and multi-dimensional stochastic volatility models (e.g. [61], [77], [94], [2]) have been applied to explain the dynamics in the markets. Chib et al. [22] observes in his multi-dimensional stochastic volatility framework that the correlation patterns change over time.

Ramchand and Susmel [99] fit a switching ARCH model to weekly international stock market returns and find evidence of different correlations across regimes. In particular correlations between the U.S. and other world markets are on average higher when the U.S. market is in a high variance state as compared to a low variance regime. Ball and Torous [9] model the correlation as a latent variable and find evidence that the estimated correlation structure is dynamically changing over time. Andersen et al. [5] uses model-free

estimators and observes that volatilities and correlations move dynamically. Moreover, the correlations among different stocks tend to be high/low when the variances for the underlying stocks are high/low, and when the correlations among the other stocks are also high/low [5]. Engle [40] developed the Dynamic Conditional Correlation (DCC) model, which allows the conditional correlation matrix to vary over time. Skintzi and Refenes summarised some stylised facts about implied correlation and its dynamics studying an implied correlation index [108]. They confirm an effect already observed before: There is a systematic tendency for the implied correlation to increase when the market index returns decrease and/or the market volatility increases, which indicates limited diversification when it is needed most. The authors also observe a long-run dependence in correlation.
In continuous-time finance literature Bakshi and Madan suggested a stochastic covariance model for two underlyings in [8], which is also applied by Dempster and Hong [33] to price spread options. In this thesis we work in that framework to extend the Fourier pricing formulas to price barrier options and extend the results mentioned before by He et al. [64] to derive the joint probability of hitting times.
There are some problems with the modelling of correlation as a risk driver: One is the model to choose to keep the correlation between -1 and 1 and the other is the intractability because the number of parameters grows exponentially when the dimensions are increased. Emmerich [114] tries to model correlation directly by a process which stays between -1 and 1. However, this model is analytically not tractable and difficult to expand to more than two dimensions. Gourieroux et al. [58], Philipov and Glickman [92] and da Fonseca et al. [27], [26] propose the use of Wishart processes to model stochastic multivariate covariance matrices. This approach is rather cumbersome when it comes to estimation and simulation. Pigorsch and Stelzer [93] and Muhle-Karbe et al. [89] present a multivariate stochastic volatility model of OU-Wishart type, which is analytically tractable, however the dimensions also increase.

In the framework we propose in [42] we try to tackle particularly the problem of tractability (also see [41]). We use principal component analysis to reduce the dimensionality of the framework to make it more tractable. This approach has been used before by Alexander (see [3], [4]) for the orthogonal GARCH model. Due to the affine structure of the here proposed model, vanilla options can be easily priced. Path-dependent options like barrier options can be approximated applying perturbation techniques (see e.g. [121] for an introduction). In finance this method has been applied to option pricing under a stochastic volatility model by Fouque et al. (see e.g. [47], [49], [101], [52], [50], [66], [24]). The approach has also been applied to various option types, e.g. exotic options in [70], Asian options in [48], defaultable bonds in [53] and has been extended to multi-dimensions [55].

1.2 Summary of the results and contributions to the literature

The main objective of the here presented thesis is the incorporation of stochastic covariance as a risk driver into pricing models. We treat two models, one with stochastic variance and deterministic correlation, one with stochastic variance and stochastic correlation, more closely. The first model has been proposed by Bakshi and Madan [8]. In that model we show how barrier options can be valued by combining Fourier techniques with the method of images when we assume a correlation $\rho = -\cos(\frac{\pi}{n})$, where n is a natural number and $n > 1$ between the two underlying assets. We, thus, extend the Generalized Fourier framework of Lewis [82] to two dimensions and show that the Fourier transform can also be used for path-dependent options. Furthermore, we show how different Fourier techniques (e.g. Lipton [83], Dempster and Hong [33], Lewis [82], see Schmelzle for a summary [104]) are related to each other. It can be numerically shown that the prices converge to the Black-Scholes formula counterparts when we assume a degenerated stochastic covariance model, which tends to a two-dimensional Black-Scholes model.

Using PDEs and the method of separation we finally also find a semi-analytic pricing formula for double-barrier options in this framework for any $-1 \leq \rho \leq 1$. These results, however, make it also clear that an analytic solution for barrier options in more involved models, i.e. when the covariance is driven by more than one common factor, can be excluded. Concluding, we extend the findings of He et al. [64] and Zhou [123], [124] by allowing for a third factor in the model which governs the covariance of the two underlyings. We derive prices for double-barrier options, and derive the joint probabilities of the survival time. The solution for the pricing formula is easy to implement and the pricing algorithm performs quite well. The pricing is implemented and compared to the Fourier technique pricing.

The pricing formulas are then used to value certificates under issuer risk. Issuer risk is the risk of loss on securities and other tradeable obligations because the issuer does not fulfil his contractual obligations due to his insolvency. So far, most of the time the prices of certificates have not been adjusted for the issuer risk, which means that many investors might actually pay too much for the risk they acquire. Pricing securities under counterparty risk can be traced back to Merton [87]. Johnson and Stulz [74] analysed the counterparty risk in option pricing. They used a firm-value model and assumed that the vulnerable option presents the single debt of the company. A huge increase in the derivative's value, thus, rises the risk of default of the company. This approach is only appropriate when the derivative is the only or the predominant source of funding of the counterparty. Klein [78] as well as Klein and Inglis [79] choose a firm-value model to account for the issuer risk and to model the dependencies between the issuing firm and the underlying. We follow their approach in that regard, and condition the payoff of the certificate on the survival of the issuer: The certificate only pays the total investment and gains back as long as the issuer has not defaulted, i.e. its asset value has not fallen under a certain barrier.

In a next step we propose a model with stochastic correlation. To reduce complexity we do not model the covariance or correlation as such but the eigenvalues and eigenvectors. For tractability we set the eigenvectors constant but assume the eigenvalues stochastic. An empirical analysis shows that the eigenvalues are driven by a time-scale which varies in the order of days. Thus, our model allows that the eigenvalues are driven by a fast mean-reverting Cox-Ingersoll-Ross process. Our model easily extends the Heston model to more underlyings: We allow for stochastic volatility and at the same time for stochastic correlation among assets and between variance and assets as well as between assets and correlation. The basic stochastic principal component model is an affine model for which the characteristic function is available and allows for easy calibration to plain vanilla instruments. Even some parametrisations of the extension to the stochastic principal component model which is presented here feature an affine characteristic function. As stated before, a closed-form solution for more involved payoffs cannot be found using PDE techniques. Thus, we show how perturbation theory can help to find easy and well converging approximations to non-vanilla options on two correlated underlyings. Furthermore, we give a proof and some test calculations for the convergence. Hence, we extend the results of Ilhan et al. [70] to price by the means of perturbation theory two-asset barrier options.

Hence, in this line of development, our work improves previous literature on correlation risk: The here presented model assumes stochastic correlation between the assets, and pricing stays feasible.

1.3 Structure of the thesis

In the following we give some guidance on the structure of the chapters. The thesis is split in three main parts: the introduction with the mathematical preliminaries in Chapter 2, the first main part in Chapter 3, which consists of all results for the stochastic covariance model, and the second main part in Chapter 4, which deals with the stochastic

correlation model.
The chapter about the mathematical preliminaries (see Chapter 2) is on its part divided in seven sections. Section 2.1 introduces the probability space, Section 2.2 familiarises the reader with the concepts of distribution and characteristic functions, which are needed in Chapter 3 for the Fourier pricing technique. In Section 2.3 we cover not only Itô's Lemma, but also the martingale representation and the Novikov condition which we require in Chapter 4. Section 2.4 explains diffusion processes which lead an important role throughout the main part of the thesis. Another important result is the Feynman-Kac theorem which is presented in Section 2.5. In that section we explain the connections between stochastic differential equations (SDEs) and partial differential equations (PDEs). In Section 2.6 we illuminate the basic assumption of the Black-Scholes framework, the risk-neutral valuation. Section 2.7 introduces the concepts and transformations we apply in Chapter 3 to solve PDEs.

Chapter 3 is composed of four main parts. In Sections 3.1, 3.2 and 3.3 we introduce the reader to the framework and give rational for our model choice. The second part is dedicated to pricing derivatives with Fourier techniques (see 3.4), where we extend the Fourier technique of Lewis [82] to price barrier options and find a solution for certain correlation values. We extend our findings in Section 3.5 by using PDE techniques. The findings of the previous sections is applied to pricing certificates under issuer risk in Section 3.6.

Chapter 4 consists of four major parts, in the first one (4.1-4.2) we introduce the framework and give some basic results of the instantaneous volatilities and correlations. Section 4.3 comprehends the empirical analysis of stock data for the components driving the eigenvalues. In Section 4.4 we price single-barrier options. And, finally in Section 4.5 we price double-barrier options by approximating the price by means of the perturbation theory.

Chapter 2

Mathematical preliminaries and definitions

2.1 Probability spaces and stochastic processes

In this chapter we want to provide the mathematical preliminaries for the models introduced later in this thesis. We limit the illustration here to definitions and propositions, which are important in this thesis. For the description of the theory we use Zagst [119], Bingham and Kiesel [12], Feller [45], Øksendal [91], and Karatzas and Shreve [76] in particular.

Definition 1. *(Øksendal [91], Definition 2.1.1, p. 7f, Measurable space, probability space)* *If Ω is a given non-empty set, then a σ-algebra $\mathcal{F}$ on Ω is a family $\mathcal{F}$ of subsets of Ω with the following properties:*

i. $\emptyset \in \mathcal{F}$,

ii. $\mathfrak{A} \in \mathcal{F} \Rightarrow \mathfrak{A}^C \in \mathcal{F}$, *where* $\mathfrak{A}^C = \Omega \backslash \mathfrak{A}$ *is the complement of* $\mathfrak{A} \in \Omega$,

iii. $\mathfrak{A}_1, \mathfrak{A}_2, \ldots \in \mathcal{F} \Rightarrow \mathfrak{A} := \bigcup_{i=1}^{\infty} \mathfrak{A}_i \in \mathcal{F}$.

The pair $(\Omega, \mathcal{F})$ is called a measurable space. A probability measure $\mathcal{Q}$ on a measurable space $(\Omega, \mathcal{F})$ is a function $\mathcal{Q} : \mathcal{F} \longrightarrow [0,1]$ such that

i. $\mathcal{Q}(\emptyset) = 0, \mathcal{Q}(\Omega) = 1$, *ii. If* $\mathfrak{A}_1, \mathfrak{A}_2, \ldots \in \mathcal{F}$ *are pairwise disjoint (i.e.* $\mathfrak{A}_i \bigcap \mathfrak{A}_j = \emptyset$ *if* $i \neq j$*), then*

$$\mathcal{Q}\left(\bigcup_{i=1}^{\infty} \mathfrak{A}_i\right) = \sum_{i=1}^{\infty} \mathcal{Q}\left(\mathfrak{A}_i\right).$$

The triple $(\Omega, \mathcal{F}, \mathcal{Q})$ *is called a probability space. A set* $\mathfrak{A}_0 \in \mathcal{F}$ *with* $\mathcal{Q}(\mathfrak{A}_0) = 0$ *is called a* $(\mathcal{Q}-)$ *null set.* $(\Omega, \mathcal{F}, \mathcal{Q})$ *is called a complete probability space if* $\mathcal{F}$ *contains all subsets of the* $(\mathcal{Q}-)$ *null sets.*

If a measure space $(\Omega, \mathcal{F}, \mathcal{Q})$ is not complete it can be easily completed by adjoining the set A_0 of all subsets of the $(\mathcal{Q}-)$ null sets. To do this we extend $\mathcal{F}$ to $\overline{\mathcal{F}}^{\mathcal{Q}}$, which contains all sets of the form $\mathfrak{A} \cup \mathfrak{A}_0$ with $\mathfrak{A} \in \mathcal{F}$ and $\mathfrak{A}_0 \in A_0$, and we extend the measure $\mathcal{Q}$ to the measure $\overline{\mathcal{Q}}$ by setting $\overline{\mathcal{Q}}(\mathfrak{A} \cup \mathfrak{A}_0) = \mathcal{Q}(\mathfrak{A})$ for all $\mathfrak{A} \in \mathcal{F}, \mathfrak{A}_0 \in A_0$. This process is called completion. We additionally introduce the notion of filtered probability spaces.

Definition 2. *(Zagst, [119], Definition 2.8, p. 15, Filtration)*

A filtration $\mathbb{F}$ *is a non-decreasing family of sub-sigma-algebras* $(\mathcal{F}_t)_{t\geq 0}$ *with* $\mathcal{F}_t \subset \mathcal{F}$ *and* $\mathcal{F}_s \subset \mathcal{F}_t$ *for all* $0 \leq s < t < \infty$*. We call* $(\Omega, \mathcal{F}, \mathcal{Q}, \mathbb{F})$ *a filtered probability space, and require that*

i. $\mathcal{F}_0$ *contains all subsets of the* $(\mathcal{Q}-)$ *null sets of* $\mathcal{F}$*,*

ii. $\mathbb{F}$ *is right-continuous, i.e.* $\mathcal{F}_t = \mathcal{F}_{t+} := \cap_{s>t}\mathcal{F}_s$*.*

$(\Omega, \mathcal{F}, \mathcal{Q}, \mathbb{F})$ is a complete filtered probability space, if $\mathcal{F}$ as well as each $\mathcal{F}_t$, $0 \leq s < t < \infty$, is complete. We require complete filtrations only and Definition 2 expresses this: For the completion of the filtration it is sufficient to complete the sigma-algebra $\mathcal{F}_0$ only, due to Assumption (ii) of Definition 2. However, if Assumption (ii) is not fulfilled, the ($\mathcal{Q}-$completed) filtration may be adjusted by setting $\hat{\mathcal{F}}_t := \mathcal{F}_{t+}$ for all $0 \leq t < \infty$. The process of making a filtration complete and right-continuous is called $(\mathcal{Q}-)$ augmentation of $\mathbb{F}$.

One can think of $\mathcal{F}_t$ as the information available at time t, and $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ describes the complete flow of information over time assuming that no information is lost in the course of time.

Definition 3. *(Øksendal [91], p. 9, Random vector, distribution function)*

A random vector X *is a real function* $\Omega \longrightarrow \mathbb{R}^d, d \in \mathbb{N}$*, which is measurable with respect to its underlying* σ*-algebra* $\mathcal{F}$*. For* $d = 1$ X *is a random variable. The function* F *defined by* $F(x) = \mathcal{Q}(X \leq x)$ *is called the distribution function of* X*.*

To describe the behaviour of the financial instruments, their volatility and correlation we will use stochastic processes.

Definition 4. *(Zagst [119], Definition 2.9, p. 15f, Stochastic process)*

A stochastic process is a family $X = (X_t)_{t\geq 0} = (X(t))_{t\geq 0}$ *of random vectors* $X(t)$ *defined on the filtered probability space* $(\Omega, \mathcal{F}, \mathcal{Q}, \mathbb{F})$*. The stochastic process* X *is called*

i. adapted to the filtration $\mathbb{F}$ *if* $X_t = X(t)$ *is* $(\mathcal{F}_t-)$ *measurable for all* $t \geq 0$*,*

ii. measurable if the mapping $X : [0,\infty) \times \Omega \longrightarrow \mathbb{R}^d, d \in \mathbb{N}$*, is* $\left(\mathcal{B}\left([0,\ \infty\ )\right) \otimes \mathcal{F} - \mathcal{B}\left(\mathbb{R}^d\right) -\right)$ *measurable with* $\mathcal{B}\left([0,\ \infty\ )\right) \otimes \mathcal{F}$ *denoting the product sigma-algebra created by* $\mathcal{B}\left([\ 0,\infty\ )\right)$ *and* $\mathcal{F}$*, where* $\mathcal{B}\left(\mathfrak{A}\right)$ *denotes the Borel sigma-algebra of* $\mathfrak{A}$*,*

iii. progressively measurable if the mapping $X : [0,t] \times \Omega \longrightarrow \mathbb{R}^d, d \in \mathbb{N}$*, is* $\left(\mathcal{B}\left([0,t]\right) \otimes \mathcal{F}_t - \mathcal{B}\left(\mathbb{R}^d\right)\right)$ *measurable for each* $t \geq 0$*.*

Note that for each t fixed we have a random variable

$$\tilde{\omega} \longmapsto X\left(t, \tilde{\omega}\right),$$

with $\tilde{\omega} \in \Omega$*.*
When fixing $\tilde{\omega} \in \Omega$ *we have a function in* t*, i.e.*

$$t \longmapsto X\left(t, \tilde{\omega}\right),$$

which is called a path of $X(t)$*. If the paths are continuous, i.e.* $t \longmapsto X\left(t, \tilde{\omega}\right)$ *is a continuous function for* $\mathcal{Q}$*-almost all* $\tilde{\omega}$*,* $X(t)$ *is a continuous process.*

Definition 5. *(Zagst [119], Definition 2.13, p. 17, Natural filtration)*

Let $(\Omega, \mathcal{F}, \mathcal{Q}, \mathbb{F})$ *be a filtered probability space and* X *be a stochastic process adapted to the filtration* $\mathbb{F}$*. The natural filtration* $\mathbb{F}(X)$ *is defined by the set of sigma-algebras*

$$\mathcal{F} := \mathcal{F}\left(X(s) : 0 \leq s \leq t\right), 0 \leq t < \infty,$$

with $\mathcal{F}\left(X(s) : 0 \leq s \leq t\right)$ *being the smallest sigma-algebra which contains all sets* $X(s)^{-1}\left(\mathfrak{A}\right) = \left\{\tilde{\omega} \in \Omega : X\left(s, \tilde{\omega}\right) \in \mathfrak{A}\right\}, 0 \leq s \leq t$ *where* $\mathfrak{A}$ *runs through the Borel sigma-algebra* $\mathcal{B}\left(\mathbb{R}^d\right), d \in \mathbb{N}$*. Again, we claim that* $\mathbb{F}(X)$ *has undergone a* $(\mathcal{Q}-)$ *augmentation, if necessary, to ensure that Conditions 1. and 2. of Definition 2 are satisfied.*

An important example for a stochastic process is the Wiener process.

Definition 6. *(Karatzas and Shreve [76], Definition 2.1.1, p. 47f, d-dimensional Wiener process, Brownian motion)*

Let $d \in \mathbb{N}$ *and* χ *be a probability measure on* $\left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right)$. *Let* $W = (W_t)_{t\geq 0}$ *be a continuous, adapted process with values in* $\mathbb{R}^d$, *defined on some filtered probability space* $(\Omega, \mathcal{F}, \mathcal{Q}^\chi, \mathbb{F}(W))$. *This process is called d-dimensional Brownian motion with initial distribution* χ, *if*

i. $\mathcal{Q}^\chi(W(0) \in \mathfrak{A}) = \chi(\mathfrak{A}),\ \forall\, \mathfrak{A} \in \mathcal{B}\left(\mathbb{R}^d\right)$,

ii. the increment $W(t) - W(s)$ *is independent of* $W(t'') - W(s'')$ *for all* $0 \leq s'' \leq t'' \leq s \leq t < \infty$, *and is normally distributed with mean zero and covariance matrix equal to* $(t-s)I_d$, *where* I_d *is the* $(d \times d)$ *identity matrix,*

iii. W *has continuous paths* $\mathcal{Q}$*- a.s.*

If χ *assigns measure one to some singleton* $\{\mathbf{x}\}$, *we say that* W *is a d-dimensional Brownian motion starting at* $\mathbf{x}$.

It is notationally and conceptually helpful to have a whole family of probability measures, rather than just one. Thus, we want to define the concept of a so-called Brownian family. For that introduction we need the following concept.

Definition 7. *(Karatzas and Shreve [76], Definition 2.5.6, p. 73, Universally measurable)*

Given a measurable space $(\Omega, \mathcal{F})$, *we denote by* $\overline{\mathcal{B}(\mathcal{F})}^\chi$ *the completion of the Borel* σ*-field* $\mathcal{B}(\mathcal{F})$ *with respect to the finite measure* χ *on* $(\Omega, \mathbb{F})$. *The universal* σ*-field is* $\mathcal{U}(\mathcal{F}) := \bigcap_\chi \overline{\mathcal{B}(\mathcal{F})}^\chi$, *where the intersection is over all finite measures (or, equivalently, all probability measures)* χ. *A* $\mathcal{U}(\mathcal{F}) - \mathcal{B}(\mathbb{R})$*-measurable, real-valued function is said to be universally measurable.*

Definition 8. *(Karatzas and Shreve [76], Definition 2.5.8, p. 73, Brownian family)*

A d-dimensional Brownian family is an adapted, d-dimensional process $W = (W(t))_{t\geq 0}$ *on a measurable space* $(\Omega, \mathcal{F})$ *with filtration* $\mathbb{F}$ *and a family of probability measures* $\{\mathcal{Q}^\mathbf{x}\}_{\mathbf{x}\in\mathbb{R}^d}$ *such that*

i. for each $\mathfrak{A} \in \mathcal{F}$, *the mapping* $\mathbf{x} \to \mathcal{Q}^\mathbf{x}(\mathfrak{A})$ *is universally measurable,*

ii. for each $\mathbf{x} \in \mathbb{R}^d$, $\mathcal{Q}^\mathbf{x}(W(0) = \mathbf{x}) = 1$,

iii. under each $\mathcal{Q}^\mathbf{x}$, *the process* W *is a d-dimensional Brownian motion starting at* $\mathbf{x}$.

A basic concept we will need later is the so-called martingale.

Definition 9. *(Zagst [119], Definition 2.16, p. 18, Martingale)*

Let $(\Omega, \mathcal{F}, \mathcal{Q}, \mathbb{F})$ *be a filtered probability space. A stochastic process* $X = (X(t))_{t\geq 0}$ *is called a martingale relative to* $(\mathcal{Q}, \mathbb{F})$ *if* X *is adapted,* $\mathbb{E}_{\mathcal{Q}}[||X(t)||] < \infty$ *for all* $t \geq 0$*, and*

$$\mathbb{E}_{\mathcal{Q}} = [X(t) \,|\, \mathcal{F}_s] = X(s) \quad \mathcal{Q} - a.s. \text{ for all } \quad 0 \leq s \leq t < \infty.$$

We have seen that the Brownian motion has independent increments, thus, for $W(t) = W(s) + (W(t) - W(s))$ the knowledge of the whole past up to time s provides no more useful information about $W(t)$ than knowing the value of $W(s)$. This is known as the concept of a Markov process.

Definition 10. *(Karatzas and Shreve [76], Definition 2.5.10, p. 74, Markov process)*

Let $d \in \mathbb{N}$ *and* χ *be a probability measure on* $\left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right)$*. An adapted d-dimensional process* $(X(t))_{t\geq 0}$ *on some probability space* $(\Omega, \mathcal{F}, \mathcal{Q}^\chi)$ *with filtration* $\mathbb{F}$ *is said to be a Markov process with initial distribution* χ *if*

i. $\mathcal{Q}^\chi(X(0) \in \mathfrak{A}) = \chi(\mathfrak{A}), \forall\, \mathfrak{A} \in \mathcal{B}\left(\mathbb{R}^d\right)$,

ii. for $s, t \geq 0$ *and* $\mathfrak{A} \in \mathcal{B}\left(\mathbb{R}^d\right)$,

$$\mathcal{Q}^\chi\left(X\left(t+s\right) \in \Phi \,|\, \mathcal{F}_s\right) = \mathcal{Q}^\chi\left(X\left(t+s\right) \in \Phi \,|X(s)\right), \; \mathcal{Q}^\chi - a.s. \tag{2.1}$$

Analogously to the Brownian family we define the Markov family.

Definition 11. *(Karatzas and Shreve [76], Definition 2.5.11, p. 74, Markov family)*

A d-dimensional Markov family is an adapted, d-dimensional process $X = (X(t))_{t\geq 0}$ *on a measurable space* $(\Omega, \mathcal{F})$ *with filtration* $\mathbb{F}$ *and a family of probability measures* $\{\mathcal{Q}^{\mathbf{x}}\}_{\mathbf{x}\in\mathbb{R}^d}$ *such that*

i. for each $\mathfrak{A} \in \mathcal{F}$*, the mapping* $\mathbf{x} \to \mathcal{Q}^{\mathbf{x}}(\mathfrak{A})$ *is universally measurable,*

ii. for each $\mathbf{x} \in \mathbb{R}^d$, $\mathcal{Q}^{\mathbf{x}}(X(0) = \mathbf{x}) = 1$,

iii. for each $\mathbf{x} \in \mathbb{R}^d$, $s, t \geq 0$ *and* $\mathfrak{A} \in \mathcal{B}\left(\mathbb{R}^d\right)$,

$$\mathcal{Q}^{\mathbf{x}}\left(X\left(t+s\right) \in \mathfrak{A} \,|\, \mathcal{F}_s\right) = \mathcal{Q}^{\mathbf{x}}\left(X\left(t+s\right) \in \mathfrak{A} \,|\, X(s)\right), \; \mathcal{Q}^{\mathbf{x}} - a.s. \tag{2.2}$$

iv. for each $\mathbf{x} \in \mathbb{R}^d$, $s, t \geq 0$ *and* $\mathfrak{A} \in \mathcal{B}\left(\mathbb{R}^d\right)$,

$$\mathcal{Q}^{\mathbf{x}}\left(X\left(t+s\right) \in \mathfrak{A} \,|\, X(s) = y\right) = \mathcal{Q}^{\mathbf{y}}\left(X\left(t\right) \in \mathfrak{A}\right), \; \mathcal{Q}^{\mathbf{x}} X(s)^{-1} - a.e.\; y, \tag{2.3}$$

where $\mathcal{Q}^{\mathbf{x}} X(s)^{-1} = \mathcal{Q}^{\mathbf{x}}\left\{\tilde{\omega} \in \Omega : X(s, \omega) \in \mathcal{B}(\mathbb{R}^d)\right\}$.

And as we have already seen:

Theorem 1. *(Karatzas and Shreve [76], Theorem 2.5.12, p. 75)*

A d-dimensional Brownian motion is a Markov process. A d-dimensional Brownian family is a Markov family.

The Brownian family is even strongly Markovian. To explain this concept we first need to introduce stopping and optional times.

Definition 12. *(Zagst [119], Definition 2.18, p. 20, Karatzas and Shreve [76], Definition 1.2.1, p. 6, Stopping time, optional time)*

Let $(\Omega, \mathcal{F}, \mathcal{Q}, \mathbb{F})$ *be a filtered probability space. A stopping time with respect to the filtration* $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ *is a* $(\mathcal{F} - \mathcal{B}([0, \infty]) -)$ *measurable function* $\iota : \Omega \to [\,0, \infty\,)$ *with*

$$\{\iota \leq t\} := \{\tilde{\omega} \in \Omega : \iota(\tilde{\omega}) \leq t\} \in \mathcal{F}_t \text{ for all } t \in [0,\ \infty\,). \tag{2.4}$$

A $(\mathcal{F} - \mathcal{B}([0, \infty]) -)$ *measurable function* $\iota^* : \Omega \to [\,0, \infty\,)$*, satisfying*

$$\{\iota^* < t\} := \{\tilde{\omega} \in \Omega : \iota^*(\tilde{\omega}) < t\} \in \mathcal{F}_t \text{ for all } t \in [0,\ \infty\,). \tag{2.5}$$

is called an optional time with respect to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$.

Definition 13. *(Karatzas and Shreve [76], Definition 2.6.2, p. 81, Strong Markov process)*

Let $d \in \mathbb{N}$ *and* χ *be a probability measure on* $\left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right)$*. A progressively measurable, d-dimensional process* $(X(t))_{t \geq 0}$ *on some probability space* $(\Omega, \mathcal{F}, \mathcal{Q}^{\chi})$ *with filtration* $\mathbb{F}$ *is said to be a strong Markov process with initial distribution* χ *if*

i. $\mathcal{Q}^{\chi}(X(0) \in \mathfrak{A}) = \chi(\mathfrak{A}), \forall\, \mathfrak{A} \in \mathcal{B}\left(\mathbb{R}^d\right)$,

ii. for any optional time ι^* *with respect to* $\mathbb{F} = \{\mathcal{F}\}_{t \geq 0}$ *and* $\mathfrak{A} \in \mathcal{B}\left(\mathbb{R}^d\right)$,

$$\mathcal{Q}^{\chi}\left(X\left(\iota^* + t\right) \in \mathfrak{A} \,|\, \mathcal{F}_{\iota^*+}\right) = \mathcal{Q}^{\chi}\left(X\left(\iota^* + t\right) \in \mathfrak{A} \,|X\left(\iota^*\right)\right), \; \mathcal{Q}^{\chi} - a.s.\ on\ (\iota^* < \infty)\,. \tag{2.6}$$

Accordingly, we define the strong Markov family.

Definition 14. *(Karatzas and Shreve [76], Definition 2.6.3, p. 81, Strong Markov family)*

A d-dimensional strong Markov family is a progressively measurable, d-dimensional process $X = (X(t))_{t \geq 0}$ on a measurable space $(\Omega, \mathcal{F})$ with filtration $\mathbb{F}$ and a family of probability measures $\{\mathcal{Q}^{\mathbf{x}}\}_{\mathbf{x} \in \mathbb{R}^d}$ such that

i. for each $\mathfrak{A} \in \mathcal{F}$, the mapping $\mathbf{x} \to \mathcal{Q}^{\mathbf{x}}(\mathfrak{A})$ is universally measurable,

ii. for each $\mathbf{x} \in \mathbb{R}^d$, $\mathcal{Q}^{\mathbf{x}}(X(0) = \mathbf{x}) = 1$,

iii. for each $\mathbf{x} \in \mathbb{R}^d$, $t \geq 0$, $\mathfrak{A} \in \mathcal{B}(\mathbb{R}^d)$ and any optional time ι^* of $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$,

$$\mathcal{Q}^{\mathbf{x}}\left(X\left(\iota^* + t\right) \in \mathfrak{A} \,|\, \mathcal{F}_{\iota^*+}\right) = \mathcal{Q}^{\mathbf{x}}\left(X\left(\iota^* + t\right) \in \mathfrak{A} \,|\, X\left(\iota^*\right)\right), \ \mathcal{Q}^{\mathbf{x}} - a.s. \text{ on } \left(\iota^* < \infty\right), \tag{2.7}$$

iv. for each $\mathbf{x} \in \mathbb{R}^d$, $t \geq 0$, $\mathfrak{A} \in \mathcal{B}(\mathbb{R}^d)$ and any optional time ι^* of $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$,

$$\mathcal{Q}^{\mathbf{x}}\left(X\left(\iota^* + t\right) \in \mathfrak{A} \,|\, X(\iota^*) = y\right) = \mathcal{Q}^{\mathbf{y}}\left(X\left(t\right) \in \mathfrak{A}\right), \ \mathcal{Q}^{\mathbf{x}} X(\iota^*)^{-1} - a.e.\ y. \tag{2.8}$$

Definition 15. *(Zagst [119], Definition 2.23, p. 22, Local martingale)*

Let $(\Omega, \mathcal{F}, \mathcal{Q}, \mathbb{F})$ be a filtered probability space and $X = (X_t)_{t \geq 0}$ be a stochastic process with $X(0) = 0$. If there is a sequence $(\iota_n)_{n \in \mathbb{N}}$ of non-decreasing stopping times with

$$\mathcal{Q}(\lim_{n \to \infty} \iota_n = \infty) = 1 \tag{2.9}$$

such that

$$X^n = (X^n_t)_{t \geq 0} := (X_{t \wedge \iota_n})_{t \geq 0}, \ t \wedge \iota_n := \min\{t; \iota_n\} \tag{2.10}$$

is a martingale relative to $(\mathcal{Q}, \mathbb{F})$ for all $n \in \mathbb{N}$, then we call X a local martingale. The sequence $(\iota_n)_{n \in \mathbb{N}}$ is called localizing sequence. If X is a local martingale with continuous paths, we call X a continuous local martingale.

2.2 Distribution functions and characteristic function

2.2.1 Notations

First, we introduce some concepts and notions.

Definition 16. *([63], p. 7, $\mathbf{L}^p[a,b]$- spaces and their norm)*

$\mathbf{L}^p$ is the space of Lebesgue-measurable functions f on $[\mathfrak{a},\mathfrak{b}]$, summable of degree p, with the norm

$$\|f\|_p := (\int_{\mathfrak{a}}^{\mathfrak{b}} |f(x)|^p)^{\frac{1}{p}} dx \; (\mathbf{L}^p - Norm). \tag{2.11}$$

Definition 17. *([63], Space of continuous functions)*

$C[\mathfrak{a},\mathfrak{b}]$ is the space of continuous functions f defined on a segment $[\mathfrak{a},\mathfrak{b}]$, with the norm

$$\|f\| := \sup\left\{|f(x)| \mid x \in [\mathfrak{a},\mathfrak{b}]\right\}. \tag{2.12}$$

$C^k[\mathfrak{a},\mathfrak{b}]$ is the space of functions f with continuous derivatives up to and including order $k, k \in \mathbb{N}$, on the interval $(\mathfrak{a},\mathfrak{b})$, with the norm

$$\|f\| := \sum_{n=0}^{k} \sup\left\{|f(x)^n| \mid x \in [\mathfrak{a},\mathfrak{b}]\right\}. \tag{2.13}$$

Definition 18. *([63], Absolute continuity)*

A function f defined on a segment $[\mathfrak{a},\mathfrak{b}]$ is called absolute continuous if for any ϵ, there exists $\delta > 0$ such that for any finite system of pairwise non-intersecting intervals $(\mathfrak{a}_k,\mathfrak{b}_k) \subset (\mathfrak{a},\mathfrak{b})$, $k = 1,\ldots,n$ for which

$$\sum_{k=1}^{n} (\mathfrak{b}_k - \mathfrak{a}_k) < \delta, \tag{2.14}$$

the inequality

$$\sum_{k=1}^{n} |f(\mathfrak{b}_k) - f(\mathfrak{a}_k)| < \epsilon \tag{2.15}$$

holds.

Definition 19. *([63], Hölder continuous)*

A function f defined on an open domain D, $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, is called (uniformly) Hölder continuous to the exponent $\alpha \in (0,1\,]$ iff there exists a positive real number Ξ such *that for all x, $x' \in D$,*

$$|f(x) - f(x')| \leq \Xi\, |x - x'|^{\alpha}\,. \tag{2.16}$$

Definition 20. *([63], Analytic function in a domain)*

A function $f(u)$, defined in a domain D, is said to be holomorphic (analytic) at a point $u_0 \in D$ if there exists a neighbourhood of this point in which f may be represented by a power series:

$$f(u) = \sum_{n=0}^{\infty} \mathfrak{a}_n (u - u_0)^n. \tag{2.17}$$

If this requirement is satisfied at every point u_0 of D, the function f is said to be analytic (holomorphic) in the domain D.

Definition 21. *([63], Lukacs [85], p.12, Singular function)*

A non-constant function f which is continuous on the interval $(\mathfrak{a}, \mathfrak{b})$ and non-decreasing with $f(\mathfrak{a}) < f(\mathfrak{b})$ whose derivative $\frac{df(x)}{dx}$ is almost-everywhere zero on the interval on which it is defined is called singular.

Definition 22. *([63], Reed [102], p. 37, Scalar product for vectors and complex-valued continuous functions)*

The inner product of two d-dimensional vectors $a = (a_1, \ldots, a_d)$ and $b = (b_1, \ldots, b_d)$ over the complex numbers is given by

$$\langle a, b \rangle = \sum_{i=1}^{d} a_i \overline{b}_i, \tag{2.18}$$

where $\overline{b}_i$ describes the complex conjugate. The scalar product $\langle f, g \rangle$ of complex-valued continuous functions on the interval $[a, b]$ is defined by

$$\langle f, g \rangle := \int_a^b f \overline{g} dx. \tag{2.19}$$

Definition 23. *([63], Even and odd functions)*

An even real-valued function f does not change sign when the sign of the independent variable is changed, i.e. satisfying the condition $f(x) = f(-x)$. A real-valued function that does change sign when the sign of the independent variable is changed, i.e. satisfying the condition $f(x) = -f(-x)$, is called an odd function.

Definition 24. *(Königsberger [80], p. 52, Gradient)*

Let D be an open set in $\mathbb{R}^d$. If the function $f : D \to \mathbb{R}$ is differentiable, then ∇f is a function defined by

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - \nabla f(x) \cdot h\|}{\|h\|} = 0 \tag{2.20}$$

where $\cdot$ describes the scalar product.

Definition 25. *(Königsberger [80], p. 61, Laplace operator, Poisson equation)*

The Laplace operator $\triangle f := \sum_{i=1}^{n} \partial_i^2 f$ is given as the sum of all (unmixed) second partial derivatives of a function $f : D \to \mathbb{R}$, where D is an open set, $D \subset \mathbb{R}^d$. The equation $\triangle f = 0$ is known as Laplace equation. The inhomogeneous form of the Laplace equation, i.e. $\triangle f = c$, is known as Poisson equation.

2.2.2 Distribution functions

In this chapter we give an overview of distribution functions and their characteristic function. We start by introducing the theory in $\mathbb{R}^1$. For the description of the theory we use Lukacs [84] and Feller [45].

Definition 26. *(Lukacs [84], p. 10, Distribution function)*

A point function F on a line is a distribution function if

i. *F is non-decreasing, that is, $\mathfrak{a} < \mathfrak{b}$ implies $F(\mathfrak{a}) \leq F(\mathfrak{b})$,*

ii. *F is right-continuous, that is $F(\mathfrak{a}) = F(\mathfrak{a}+)$,*

iii. *$F(-\infty) = 0$ and $F(\infty) < \infty$.*

F is a probability distribution function if it is a distribution function and $F(\infty) = 1$.

Theorem 2. *(Lukacs [84], Theorem 1.1.3, p. 12)*

Every distribution function $F(x)$ can be decomposed uniquely according to

$$F(x) = \varsigma_1 F_d(x) + \varsigma_2 F_{ac}(x) + \varsigma_3 F_s(x). \tag{2.21}$$

Here F_d, F_{ac}, F_s are three distribution functions. The points of increase of F_d are all discontinuity points. The functions F_{ac} and F_s are both continuous; however F_{ac} is absolutely continuous, while F_s is singular. The coefficients $\varsigma_1, \varsigma_2, \varsigma_3$ satisfy the relations $\varsigma_1 \geq 0$, $\varsigma_2 \geq 0$, $\varsigma_3 \geq 0$ and $\varsigma_1 + \varsigma_2 + \varsigma_3 = 1$.

For a proof see Lukacs [84], p. 11f. A distribution function is called pure if one of the coefficients in the Representation (2.21) equals one. For pure distribution functions we use the expression discrete distribution function if $\varsigma_1 = 1$, absolutely continuous distribution function if $\varsigma_2 = 1$, and singular distribution function if $\varsigma_3 = 1$.

2.2.3 Definitions and properties of the characteristic function

Integral transforms, defined by $\int_{-\infty}^{\infty} \bar{G}(u,x)dF(x)$ (provided that the integral exists as a Lebesgue integral), with suitable kernels $\bar{G}(u,x)$ are a useful tool for the analysis of distribution functions. In the following we will cover the kernels: x^k, $|x|^k$, e^{ux}, e^{iux}. The first two transform $F(x)$ into sequences, the latter into functions of the real variable u.

Definition 27. *(Lukacs [84], p. 17, Algebraic and absolute moments)*

Let X be a random variable with probability distribution F. The algebraic moment of order k of $F(x), x \in \mathbb{R}$, is then given by

$$\alpha_k = \int_{-\infty}^{\infty} x^k dF(x). \tag{2.22}$$

Similarly, the absolute moment of order k of $F(x)$ is defined by

$$\beta_k = \int_{-\infty}^{\infty} |x|^k \, dF(x). \tag{2.23}$$

Theorem 3. *(Lukacs [84], Theorems 1.4.1 and 1.4.2, p. 19)*

The algebraic moment of order k of a distribution function $F(x)$ exists if and only if its absolute moment of order k exists. Suppose that the algebraic moment of order k of $F(x)$ exists then the moments α_n and β_n for all orders $n \leq k$ exist.

For a proof see [84], p. 19.

Definition 28. *(Lukacs [84], p. 18f, Feller [45], p. 499f, Moment generating function and characteristic function)*

Let X be a random variable with probability distribution $F(x)$. The moment generating function of $F(x), x \in \mathbb{R}$, (or of X) is the function $\bar{M}$ defined for real u by

$$\bar{M}(u) = \int_{-\infty}^{\infty} e^{ux} dF(x). \tag{2.24}$$

For absolutely continuous distributions with a density f,

$$\bar{M}(u) = \mathbb{E}\left[e^{ux}\right] = \int_{-\infty}^{\infty} e^{ux} f(x)dx. \tag{2.25}$$

The characteristic function of $F(x)$ is defined by

$$\varphi(u) = \mathbb{E}\left[e^{iux}\right] = \int_{-\infty}^{\infty} e^{iux} dF(x) = \hat{w}(u) + i\hat{\varpi}(u), \tag{2.26}$$

where

$$\hat{w}(u) = \int_{-\infty}^{\infty} cos(ux)dF(x), \quad \hat{\varpi}(u) = \int_{-\infty}^{\infty} sin(ux)dF(x). \tag{2.27}$$

We see that $\bar{M}(iu) = \varphi(u)$. φ is the Fourier transform of dF.

Definition 29. *(Königsberger [80], p. 325, Fourier transform)*

Let f be a Lebesgue-integrable function on $\mathbb{R}$. Then, the Fourier transform of f, the function $\hat{f} : \mathbb{R} \to \mathbb{C}$ is defined by

$$\hat{f}(\mathbf{u}) := \int_{-\infty}^{\infty} f(\mathbf{x})e^{iux}dx, \quad u \in \mathbb{R}. \tag{2.28}$$

$\hat{f}$ is continuous and bounded by $\|f\|_1$.

The following properties of the characteristic function can be derived from the characteristics of the Fourier transform.

Theorem 4. *(Lukacs [84], Theorems 2.1.1 and 2.1.2, p. 22, Feller [45], Lemma 1, p. 499)*

Let $\varphi(u) = \hat{w}(u) + i\hat{\varpi}(u)$ be the characteristic function of a random variable X with distribution F. Then

i. φ is continuous,

ii. $\varphi(0)$=1, $|\varphi(u)| \leq 1$ for all u,

iii. $\mathfrak{a}X + \mathfrak{b}$ has the characteristic function $e^{i\mathfrak{b}u}\varphi(\mathfrak{a}u)$,

iv. $\varphi(-u) = \overline{\varphi(u)}$, where the horizontal bar atop of φ denotes the complex conjugate of φ,

v. $\hat{w}(u)$ is even and $\hat{\varpi}(u)$ is odd. The characteristic function is real if and only if F is symmetric, *vi. for all u:* $0 < 1 - \hat{w}(2u) \leq 4\,(1 - \hat{w}(u))$.

For a proof see Lukacs [84], p. 22f, and Feller [45], p. 500.

Theorem 5. *(Lukacs [84], Theorem 2.1.3, p. 23)*

Suppose that the real numbers $\varsigma_1, \varsigma_2, \ldots, \varsigma_n$ *satisfy the conditions* $\varsigma_k \geq 0$, $\sum_{k=1}^{n} \varsigma_k = 1$ *and that* $\varphi_1, \ldots, \varphi_n$ *are characteristic functions. Then*

$$\varphi(u) = \sum_{k=1}^{n} \varsigma_k \varphi_k(u) \tag{2.29}$$

is also a characteristic function.

For a proof refer to Lukacs [84], p. 23.

2.2.4 Characteristic function and the moments of the distribution

There is a close connection between the characteristic function and moments. Let us first define the first and higher (central) differences with respect to an increment u by

$$\Delta_1^u f(y) = f(y+u) - f(y-u) \tag{2.30}$$

and

$$\Delta_{k+1}^u f(y) = \Delta_1^u \Delta_k^u f(y). \tag{2.31}$$

Theorem 6. *(Lukacs [84], Theorem 2.3.1, p. 27f)*

Let $\varphi(u)$ *be the characteristic function of a distribution function* $F(x)$*, and let*

$$\frac{\Delta_{2k}^u \varphi(0)}{(2u)^{2k}} \tag{2.32}$$

be the 2nd (central) difference quotient of $\varphi(u)$ *at the origin. Assume that*

$$\liminf_{u \to 0} \left| \frac{\Delta_{2k}^u \varphi(0)}{(2u)^{2k}} \right| < \infty. \tag{2.33}$$

Then the $2k$*th moment* α_{2k} *of* $F(x)$ *exists, as do all moments of order* n, $n \leq 2k$*. Moreover, the derivatives* $\varphi^{(n)}(u)$ *exist for all* u *and for* $n = 1, 2, \ldots, 2k$ *and*

$$\varphi^{(n)}(u) = i^n \int_{-\infty}^{\infty} x^n e^{iux} dF(x) \quad (n = 1, 2, \ldots, 2k), \tag{2.34}$$

so that $\alpha_n = i^{-n}\varphi^{(n)}(0)$.

For a proof see Lukacs [84], p. 27f. The following corollary follows directly.

Corollary 1. *(Lukacs [84], Corollary 1 to Theorem 2.3.1, p. 29)*

If the characteristic function of a distribution $F(x)$ *has a derivative of order* k *at* $u = 0$*, then all the moments of* $F(x)$ *up to order* k *exist if* k *is even, respectively up to order* $k-1$ *if* k *is odd.*

2.2.5 Uniqueness and inversion

The uniqueness of characteristic functions is laid out in the following theorem.

Theorem 7. *(Lukacs [84], Theorem 3.1.1, p. 35, Uniqueness theorem)*

Two distribution functions $F_1(x)$ *and* $F_2(x)$ *are identical if and only if their characteristic functions* $\varphi_1(u)$ *and* $\varphi_2(u)$ *are identical.*

For a proof refer to Lukacs [84], p. 35f. The density can be computed by inverting the characteristic function provided that the assumptions of the following theorem are valid.

Theorem 8. *(Feller [45], Theorem 3, p. 509, Inversion theorem)*

Let φ *be the characteristic function of the distribution* F *and suppose* $\varphi \in \mathbf{L}^1$*. Then* F *has a bounded continuous density* $f(x), x \in \mathbb{R}$ *given by*

$$f(x) = F'(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\varphi(u)du. \tag{2.35}$$

The proof is given in [45], p. 509f.

For the convolution of the distribution function the following is true.

Theorem 9. *(Lukacs [84], Theorem 3.3.1, p. 45, Convolution theorem)*
A distribution function F *is the convolution of two distributions* F_1 *and* F_2*, that is*

$$F(y) = \int_{-\infty}^{\infty} F_1(y-x)dF_2(x) = \int_{-\infty}^{\infty} F_2(y-x)dF_1(x) = (F_1 * F_2)(y), \tag{2.36}$$

if and only if the corresponding characteristic functions satisfy the relationship

$$\varphi(u) = \varphi_1(u)\varphi_2(u). \tag{2.37}$$

For the inversion of a product of characteristic functions the following is valid.

Theorem 10. *(Titchmarsh [113], Theorem 40, p. 59)*

Let $\varphi_1(u)$ *be the characteristic function of* f_1, $\varphi_1, f_1 \in \mathbf{L}^1$, *and let* $f_2(x)$ *belong to* $\mathbf{L}^1$ *(so that its Fourier transform* $\varphi_2(u)$ *is bounded). Then* $\sqrt{2\pi}\varphi_1(u)\varphi_2(u)$ *belongs to* $\mathbf{L}^1$ *and the Fourier transform of the latter expression is* $\int_{-\infty}^{\infty} f_1(x-y)f_2(y)dy$.

For characteristic functions of the class $\mathbf{L}^2$ the Plancherel theorem indicates the convergence.

Theorem 11. *(Titchmarsh [113], Theorem 48, p. 69, Plancherel's theorem)*

Let f(x) be a density function of the class $\mathbf{L}^2$, *and let*

$$\varphi(u,\mathfrak{a}) = \int_{-\mathfrak{a}}^{\mathfrak{a}} f(x)e^{ixu}dx. \tag{2.38}$$

Then, as $\mathfrak{a} \to \infty$ φ *converges in mean over* $(-\infty,\infty)$ *to a function* $\varphi(u)$ *of* $\mathbf{L}^2$; *and reciprocally*

$$f(x,\mathfrak{a}) = \frac{1}{2\pi}\int_{-\mathfrak{a}}^{\mathfrak{a}} \varphi(u)e^{-ixu}du \tag{2.39}$$

converges in mean to $f(x)$.

Theorem 12. *(Titchmarsh [113], Theorem 49, p. 70, Parseval's formula)*

If $f_1(x)$, $\varphi_1(u)$, $f_2(x)$, $\varphi_2(u)$ *are Fourier transforms as in the above theorem, the following equations hold:*

$$\int_{-\infty}^{\infty} \varphi_1(u)\varphi_2(u)du = \int_{-\infty}^{\infty} f_1(x)f_2(-x)dx, \tag{2.40}$$

$$\int_{-\infty}^{\infty} \varphi_1(u)\overline{\varphi_2(u)}du = \int_{-\infty}^{\infty} f_1(x)\overline{f_2(-x)}dx, \tag{2.41}$$

$$\int_{-\infty}^{\infty} |\varphi_1(u)|^2\, du = \int_{-\infty}^{\infty} |f_1(x)|^2\, dx, \tag{2.42}$$

where the horizontal bar atop of φ *(*$f_2(x)$ *respectively) denotes the complex conjugate of* φ *(*$f_2(x)$ *respectively).*

2.2.6 Characteristic functions in higher dimensions

The theory of characteristic functions in higher dimensions is closely parallel to the theory in $\mathbb{R}^1$.

Definition 30. *(Feller [45], p. 521f, Characteristic function in higher dimensions)*

Let X be a vector of random variables $X_1, X_2, \ldots, X_n$ with probability distribution $F(X)$. The characteristic function of F (or of X) is the function φ defined for real $\mathbf{u}$

$$\varphi(\mathbf{u}) = \mathbb{E}\left[e^{i\langle \mathbf{ux}\rangle}\right] = \int_{\mathbb{R}^d} e^{i\langle \mathbf{ux}\rangle} dF(\mathbf{u}). \tag{2.43}$$

The Fourier transform can also be formulated in $\mathbb{R}^d$.

Definition 31. *(Königsberger [80], p. 325, Fourier transform)*

Let f be a Lebesgue-integrable function on $\mathbb{R}^d$. Then, the Fourier transform of f, the function $\hat{f} : \mathbb{R}^d \to \mathbb{C}$ is defined by

$$\hat{f}(\mathbf{u}) := \int_{\mathbb{R}^d} f(\mathbf{x}) e^{i\langle \mathbf{u},\mathbf{x}\rangle} d\mathbf{x}, \quad \mathbf{u} \in \mathbb{R}^d. \tag{2.44}$$

$\hat{f}$ is continuous and bounded by $\|f\|_1$ and $\langle \mathbf{u}, \mathbf{x}\rangle$ is the scalar product with $\mathbf{u} = (u_1, u_2, \ldots, u_d)$.

One of the main theorems, the inversion theorem still holds true.

Theorem 13. *(Feller [45], p. 524, Inversion theorem in higher dimensions)*

Let $\varphi(\mathbf{u})$ be the characteristic function of the distribution $F(\mathbf{x})$ and suppose $\varphi(\mathbf{u}) \in \mathbf{L}^1$. Then $F(\mathbf{x})$ has a bounded continuous density $p(\mathbf{x})$ given by

$$p(\mathbf{x}) = F'(\mathbf{x}) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle \mathbf{ux}\rangle} \varphi(\mathbf{u})\, d\mathbf{u}. \tag{2.45}$$

2.2.7 Analytic characteristic functions

We introduce now the class of analytic characteristic functions. In the following we denote by w and ϖ real variables and by $u = w + i\varpi$ a complex variable with $w, \varpi \in \mathbb{R}^1$.

Definition 32. *(Lukacs [84], p. 130, Analytic characteristic function)*

A characteristic function $\varphi(u)$ is said to be an analytic characteristic function if there exists a function $E(u)$ of the complex variable u which is regular in a circle $|u| \leq \hat{c}$ ($\hat{c} > 0$) and a constant $\Xi > 0$ such that $E(w) = \varphi(w)$ for $|w| < \Xi$.

This can be expressed in an informal manner (see [84], p. 130) by saying that an analytic characteristic function is a characteristic function which coincides with a holomorphic function in some neighbourhood of the origin in the complex u-plane.

Theorem 14. *(Lukacs [84], Theorem 7.1.1, p. 132)*

If a characteristic function $\varphi(u)$ is regular in a neighbourhood of the origin, then it is also regular in a horizontal strip and can be represented in this strip by a Fourier integral. This strip is either the whole plane, or it has one or two horizontal lines. The purely imaginary points on the boundary of the strip of regularity (if this strip is not the whole plane) are singular points of $\varphi(u)$.

A proof can be found in [84], p. 130ff. The following example is based on an example in [85].

Example 1. *Take the characteristic function*

$$\varphi(u) = (1 - \frac{iu}{a})^{-1}, \tag{2.46}$$

with $a \in \mathbb{R}$. This function satisfies the elementary necessary conditions for characteristic functions, namely $\varphi(-u) = \overline{\varphi(u)}, \varphi(0) = 1, |\varphi(u)| \leq 1$ for real u. It has a singularity on the imaginary axis in $u = -ia$, i.e. the function is regular near the origin in the strip $-a \leq \Im(u) \leq \infty$, where $\Im(u)$ denotes the imaginary part of u (see Figure 2.1). In this example we find that the strip of regularity is bounded by one horizontal line in $\Im(u) = a$.

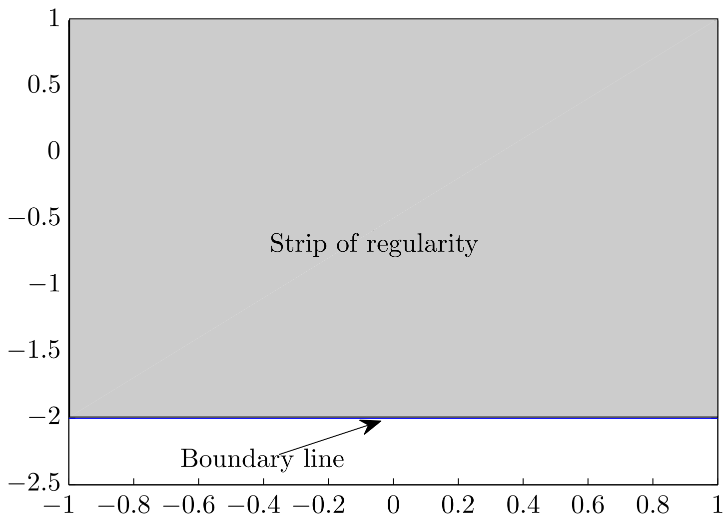

Figure 2.1: Strip of analyticity

Thus, for $\mathfrak{a} < \Im(u) < \mathfrak{b}$, where $\Im(u)$ denotes the imaginary part of u, the characteristic function of the process $X(t)$ is identical to the generalized Fourier transform of the transition density. We use here the concept of the complex Fourier transform by Titchmarsh [113], p. 4ff and 42ff: The existence of the integral defining $\hat{f}$ implies a certain restriction on $f(x)$ at infinity. But even if $\hat{f}$ does not exist, the functions

$$\hat{f}_+(u) = \int_0^\infty f(x)e^{iux}dx, \tag{2.47}$$

$$\hat{f}_-(u) = \int_{-\infty}^0 f(x)e^{iux}dx, \tag{2.48}$$

where $u = w + i\varpi$, may exist, the former for sufficiently large positive ϖ, the latter for sufficiently large negative ϖ. For

$$\hat{f}_+(u) = \int_0^\infty f(x)e^{-\varpi x}e^{iwx}dx, \tag{2.49}$$

so that $\hat{f}_+$ is the transform of the function equal to $f(x)e^{-\varpi x}$ for $x > 0$, and to 0 for $x < 0$. For the inversion we may write

$$f(x) = \frac{1}{2\pi}\left(\int_{ia_1-\infty}^{ia_1+\infty} \hat{f}_+(u)e^{ixu}du + \int_{ib_1-\infty}^{ib_1+\infty} \hat{f}_-(u)e^{-ixu}du\right), \tag{2.50}$$

where a_1 is a sufficiently large positive number, b_1 a sufficiently large negative number.

In this context the next three theorems, the Cauchy integral theorem, the Cauchy integral formula and the Residue theorem, are quite helpful.

Definition 33. *([63], Simply-connected domain)*

In a simply-connected domain D any closed path can be continuously deformed into a point, remaining the whole time in the simply-connected domain D.

Theorem 15. *([63], Cauchy integral theorem)*

If $f(u)$ is a holomorphic function of a complex variable u in a simply-connected domain D in the complex plane $\mathbb{C}$, then the integral of $f(u)$ along any closed rectifiable (i.e. having finite length) curve γ in D vanishes:

$$\int_\gamma f(u)du = 0. \tag{2.51}$$

An equivalent version of Cauchy's integral theorem states that the integral

$$\int_{\mathfrak{a}}^{\mathfrak{b}} f(u)du, \ \ \mathfrak{a}, \mathfrak{b} \in D, \tag{2.52}$$

is independent of the choice of the path of integration (contour) between the fixed points $\mathfrak{a}$ *and* $\mathfrak{b}$ *in* D.

Theorem 16. *(Königsberger [80], p. 203, [100], Cauchy integral formula)*

If f *is a holomorphic function in an open domain* $\mathcal{D}$ *which includes the closure of the disk* $K_r(\mathfrak{a}) = \{u : |u - \mathfrak{a}| \leq r\}$ *then for every point* $u_0 \in K_r(\mathfrak{a})$

$$f(u_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(u)}{u - u_0} du, \tag{2.53}$$

where γ *denotes the circle which forms the boundary of* K_r. *This formula can be extended to two dimensions. If* f *is a function which is holomorphic in its variables* u_1 *and* u_2 *in an open domain* D *which includes* $K_r(\mathfrak{a}_1, \mathfrak{a}_2) = \{u_i : |u_i - \mathfrak{a}_i| \leq r_i, i = 1, 2\}$ *then for every point* $(u_{10}, u_{20}) \in K_r(\mathfrak{a}_1, \mathfrak{a}_2)$

$$f(u_{10}, u_{20}) = \frac{1}{(2\pi i)^2} \int_\Gamma \frac{f(u_1, u_2)}{(u_1 - u_{10})(u_2 - u_{20})} du_1 du_2, \tag{2.54}$$

where $\Gamma = \{u_i : |u_i - \mathfrak{a}_i| = r_i, i = 1, 2\}$.

Definition 34. *([63], Residue)*

Let $f(u)$ *be a function of one complex variable and* f *has a finite isolated singular point at* $\mathfrak{a}$. *The integral*

$$Res_{\mathfrak{a}} f := \frac{1}{2\pi i} \int_\gamma f(u) du, \tag{2.55}$$

where γ *is a counter-clockwise oriented circle of sufficiently small radius with centre at* $\mathfrak{a}$, *is called residue of* f *in* $\mathfrak{a}$.

Remark 1. *If* f *has a simple pole at* $\mathfrak{a}$ *the residue of* f *is given by*

$$Res_{\mathfrak{a}} f = \lim_{u \to \mathfrak{a}} \left((u - \mathfrak{a}) f(u) \right). \tag{2.56}$$

See [63].

Theorem 17. *([63], Residue theorem)*

Let f *be a single-valued analytic function everywhere in a simply-connected domain* D, *except for isolated singular points; then the integral of* $f(u)$ *over any counter-clockwise oriented, simple (i.e. injective) closed rectifiable curve* γ *lying in* D *and not passing through the singular points of* $f(u)$ *can be computed by the formula*

$$\int_\gamma f(u) du = 2\pi i \sum_{k=1}^{n} Res_{\mathfrak{a}_k} f(u), \tag{2.57}$$

where $\mathfrak{a}_k$, $k = 1, \ldots, n$, are the singular points of $f(u)$ inside γ.

Corollary 2. *(Contour variation in the strip of regularity)*

Assume a function $f(u)$, $u = w + i\varpi$, analytic in a strip $\mathfrak{a} < \Im(u) < \mathfrak{b}$ which decays at $\pm\infty$ like $e^{-\tilde{c}_1|u|}$. Then it follows from Cauchy's theorem that the integral over $f(u)$ extending from $-\infty$ to $+\infty$ can be taken along any line in the strip of regularity parallel to the real axis.
If $f(u) = g(u)h(u)$, where $g(u), h(u)$ are analytic in a strip $\mathfrak{a} < \Im(u) < \mathfrak{b}$, $g(u)$ bounded and $h(u)$ decays at $\pm\infty$ like $e^{-\tilde{c}_1|u|}$ then it also follows from Cauchy's theorem that an integral over $f(u) = g(u)h(u)$ extending from $-\infty$ to $+\infty$ can be taken along any line of the strip of regularity parallel to the real axis.

See also Titchmarsh [113], p. 44f.

Proof.

Assume that we take the integral along a line in the strip of regularity at $\Im(u) = \bar{c}_1$ from $-\infty$ to $+\infty$:

$$\Gamma_1 = \int_{-\infty+i\bar{c}_1}^{\infty+i\bar{c}_1} f(u)du \tag{2.58}$$

Assume a second line $\Im(u) = \bar{c}_2$ parallel to $\Im(u) = \bar{c}_1$, which also must be within the strip of regularity.

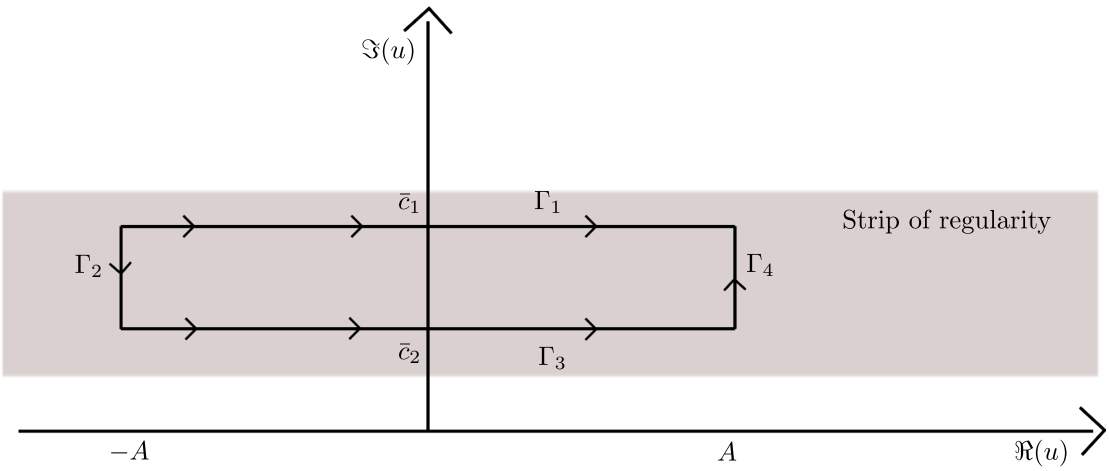

Figure 2.2: Contour variation within strip of regularity

By Cauchy's theorem this integral is equal to the sum of three integrals, which build with $-\Gamma_1$ a closed curve,

$$\Gamma_1 = \underbrace{\int_{-\infty+i\bar{c}_1}^{-\infty+i\bar{c}_2} f(u)du}_{=\Gamma_2} + \underbrace{\int_{-\infty+i\bar{c}_2}^{\infty+i\bar{c}_2} f(u)du}_{=\Gamma_3} + \underbrace{\int_{\infty+i\bar{c}_2}^{\infty+i\bar{c}_1} f(u)du}_{=\Gamma_4}. \tag{2.59}$$

For Γ_2 and Γ_4 we derive due to the decay $e^{-\tilde{c}_1|u|}$ of $f(u)$ at $\pm\infty$

$$
\begin{aligned}
\lim_{A\to-\infty}\int_{A+i\bar{c}_1}^{A+i\bar{c}_2} f(u)du &\approx \lim_{A\to-\infty}\int_{A+i\bar{c}_1}^{A+i\bar{c}_2} \underbrace{e^{-\tilde{c}_1|u|}}_{\leq e^{-\tilde{c}_1 w}} du \\
&\leq \lim_{A\to-\infty} e^{-\tilde{c}_1|A|}dw \\
&= 0, \\
\lim_{A\to\infty}\int_{A+i\bar{c}_2}^{A+i\bar{c}_1} f(u)du &= 0.
\end{aligned} \tag{2.60}
$$

Thus, $\Gamma_1 = \Gamma_3$ and the first part of the corollary follows. If $f(u) = g(u)h(u)$, with g(u) bounded, then Γ_2 and Γ_3 are given by

$$
\begin{aligned}
\lim_{A\to-\infty}\int_{A+i\bar{c}_1}^{A+i\bar{c}_2} g(u)h(u)du &\leq \lim_{A\to-\infty}\int_{A+i\bar{c}_1}^{A+i\bar{c}_2} \tilde{c}_3 e^{-\tilde{c}_1|u|}du \\
&= 0, \\
\lim_{A\to\infty}\int_{A+i\bar{c}_2}^{A+i\bar{c}_1} f(u)du &= 0.
\end{aligned} \tag{2.61}
$$

□

Corollary 3. *(Residue calculus, contour variation in a strip with simple poles)*

Assume a function $f(u)$ which decays at $\pm\infty$ like $e^{-\tilde{c}_1|u|}$. Furthermore, $f(u)$ regular in a strip $\mathfrak{a} < \Im(u) < \mathfrak{b}$ and except for simple poles at $\mathfrak{a}$ and $\mathfrak{b}$ it is regular in an even larger strip S_f^*. Then it follows from Cauchy's theorem and the Residue theorem that the integral over $f(u)$ extending from $-\infty$ to $+\infty$ can be taken along any line $\bar{c}_2$ in S_f^* parallel to the real axis taking into account the contribution of the residue the contour has been moved across or along. The residue contribution is given by

$$
\begin{cases}
-2\pi i Res_{\mathfrak{a}} f \ \textit{for}\ \bar{c}_2 < \mathfrak{a}, \\
-2\pi i Res_{\mathfrak{b}} f \ \textit{for}\ \bar{c}_2 > \mathfrak{b}, \\
-\pi i Res_{\mathfrak{a}} f \ \textit{for}\ \bar{c}_2 = \mathfrak{a}, \\
-\pi i Res_{\mathfrak{b}} f \ \textit{for}\ \bar{c}_2 = \mathfrak{b}.
\end{cases} \tag{2.62}
$$

Assume $f(u) = g(u)h(u)$ with $g(u)$ bounded and $h(u)$ decays at $\pm\infty$ like $e^{-\tilde{c}_1|u|}$. Moreover, $g(u), h(u)$ are regular in a strip $\mathfrak{a} < \Im(u) < \mathfrak{b}$ and except for simple poles at $\mathfrak{a}$ and $\mathfrak{b}$ they are regular in an even larger strip S_f^*. Then it follows from Cauchy's theorem and the Residue theorem that the integral over $f(u) = g(u)h(u)$ extending from $-\infty$ to $+\infty$ can be taken along any line in S_f^* parallel to the real axis taking into account the residue contribution.

Proof.

Assume that we take the integral along a line in the strip of regularity at $\Im(u) = \bar{c}_1$ from $-\infty$ to $+\infty$:

$$\Gamma_1 = \int_{-\infty+i\bar{c}_1}^{\infty+i\bar{c}_1} f(u)du \tag{2.63}$$

Assume a second line $\Im(u) = \bar{c}_2$ parallel to $\Im(u) = \bar{c}_1$ with $\bar{c}_2 \overset{<}{(=)} \mathfrak{a}$.

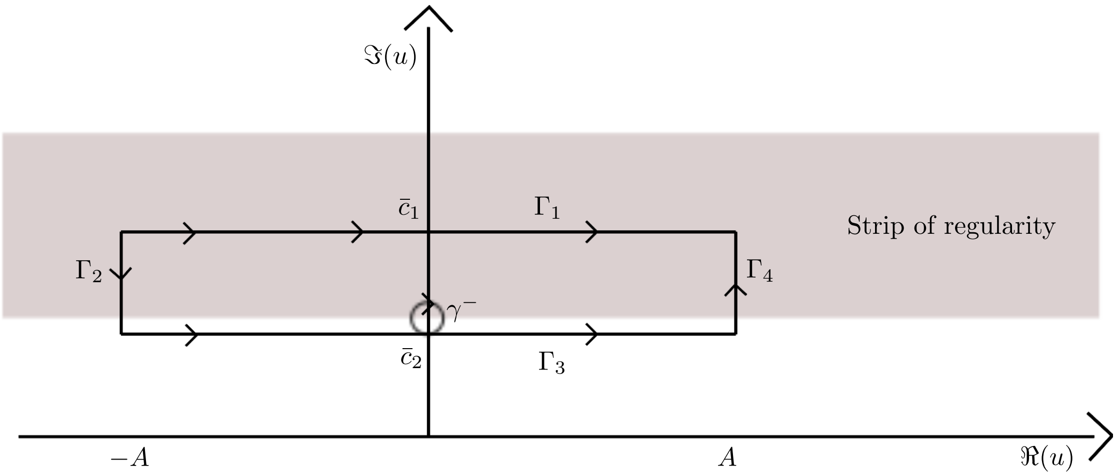

Figure 2.3: Contour variation with simple poles

By Cauchy's theorem this integral is equal to the sum of four integrals, which build with $-\Gamma_1$ a closed curve,

$$\Gamma_1 = \underbrace{\int_{-\infty+i\bar{c}_1}^{-\infty+i\bar{c}_2} f(u)du}_{=\Gamma_2} + \underbrace{\int_{-\infty+i\bar{c}_2}^{\infty+i\bar{c}_2} f(u)du}_{=\Gamma_3} + \underbrace{\int_{\infty+i\bar{c}_2}^{\infty+i\bar{c}_1} f(u)du}_{=\Gamma_4} + \underbrace{\left(\frac{1}{2}\right)}_{\text{for } \bar{c}_2=\mathfrak{a}} \int_{\gamma^-} f(u), \tag{2.64}$$

where γ^- describes a circle, in clockwise direction, of sufficiently small radius with centre at $\mathfrak{a}$. According to the Residue theorem this integral is then given by $-2\pi i Res_{\mathfrak{a}} f$ ($-\pi i Res_{\mathfrak{a}} f$ respectively). The results for $\bar{c}_2 > \mathfrak{b}$ follow accordingly. From the above proof we also know that $\Gamma_2 = 0$ and $\Gamma_4 = 0$.

The second part of the corollary follows analogously with Corollary 2. □

2.3 The Itô formula and the martingale representation theorem

In our context we work with so called Itô processes.

Definition 35. *(Zagst [119], Definition 2.32, p. 27f, Itô process)*

Let $W(t)$ be a d-dimensional Wiener process, $d \in \mathbb{N}$. A stochastic process $X = (X(t))_{t\geq 0}$ is called an Itô process if for all $t \geq 0$ we have

$$\begin{aligned} X(t) &= X(0) + \int_0^t \mu(s, X(s))\, ds + \int_0^t \sigma(s, X(s))\, dW(s) \\ &= X(0) + \int_0^t \mu(s, X(s))\, ds + \sum_{k=1}^{d} \int_0^t \sigma_k(s, X(s))\, dW_k(s), \end{aligned} \tag{2.65}$$

where $X(0)$ is ($\mathcal{F}_0-$) measurable and $\mu = (\mu(t))_{t\geq 0}$ and $\sigma = (\sigma(t))_{t\geq 0}$ with $\sigma(t) = (\sigma_1(t), \ldots, \sigma_d(t))_{t\geq 0}$ are a one and a d-dimensional progressively measurable stochastic process with

$$\int_0^t |\mu(s, X(s))|\, ds < \infty, \tag{2.66}$$

$$\int_0^t \sigma_k^2(s, X(s))\, ds < \infty, \qquad \mathcal{Q} - a.s. \text{ for all } t \geq 0,\ k = 1, \ldots, d. \tag{2.67}$$

A d-dimensional Itô process is given by a vector $X = (X_1, \ldots, X_d)$, $d \in \mathbb{N}$, with each X_i being an Itô process, $i = 1, \ldots, d$.

Remark 2. *For convenience we write (2.65) symbolically*

$$\begin{aligned} dX(t) &= \mu(t, X(t))\, dt + \sigma(t, X(t))\, dW(t) \\ &= \mu(t, X(t))\, dt + \sum_{k=1}^{d} \sigma_k(t, X(t))\, dW_k(t), \end{aligned} \tag{2.68}$$

and call this stochastic differential equation (SDE) with drift parameter μ and diffusion parameter σ.

To use Itô's Lemma we have to define the quadratic covariance process.

Definition 36. *(Zagst [119], Definition 2.33, p. 28, Quadratic covariance process)*

Let $d \in \mathbb{N}$ and $W = (W_1(t), \ldots, W_d(t))_{t\geq 0}$ and $X_i = (X_i(t))_{t\geq 0}$ with $i = 1, 2$ be two Itô processes with

$$\begin{aligned} dX_i(t) &= \mu_i\left(t, X(t)\right) dt + \sigma_i\left(t, X(t)\right) dW(t) \\ &= \mu_i\left(t, X(t)\right) dt + \sum_{k=1}^{d} \sigma_{ik}\left(t, X(t)\right) dW_k(t),\ i = 1, 2. \end{aligned}$$

Then we call the stochastic process $\langle X_1, X_2\rangle = (\langle X_1, X_2\rangle_t)_{t\geq 0}$ defined by

$$\langle X_1, X_2\rangle_t := \sum_{k=1}^{d} \int_0^t \sigma_{1k}\left(s, X(s)\right) \cdot \sigma_{2k}(s, X(s))ds$$

the quadratic covariance (process) of X_1 and X_2. If $X_1 = X_2 =: X$ we call the stochastic process $\langle X\rangle := \langle X, X\rangle$ the quadratic variation (process) of X, i.e.

$$\begin{aligned} \langle X, X\rangle_t &:= \sum_{k=1}^{d} \int_0^t \sigma_j^2\left(s, X(s)\right) ds \\ &= \int_0^t \left\|\sigma\left(s, X(s)\right)\right\|^2 ds, \end{aligned}$$

where $\|\sigma\left(t, X(t)\right)\| := \sqrt{\sum_{k=1}^{d} \sigma_k^2}$, $t \in [\,0, \infty\,)$ denotes the Euclidean norm in $\mathbb{R}^d$ and $\sigma := \sigma_1$.

Theorem 18. *(Zagst [119], Theorem 2.34, p. 29, Itô's Lemma in higher dimensions)*

Let $W = (W(t))_{t\geq 0}$ be a d-dimensional Wiener process, $d \in \mathbb{N}$, and $X = (X_1, \ldots, X_n) = (X_1(t), \ldots, X_n(t))_{t\geq 0}$ be a n-dimensional Itô process, $n \in \mathbb{N}$, with

$$dX_i(t) = \mu_i(t)dt + \sigma_i(t)dW(t) = \mu_i(t)dt + \sum_{k=1}^{d} \sigma_{ik}(t)dW_k(t),\ i = 1, \ldots, n. \tag{2.69}$$

Furthermore, let $G : \mathbb{R}^n \times [0, \infty) \longrightarrow \mathbb{R}$ be twice continuously differentiable in the first n variables, and once continuously differentiable in the last variable (t). Then we have for all $t \in [\,0, \infty\,)$

$$\begin{aligned} dG(X(t), t) &= \frac{\partial G\left(t, X(t)\right)}{\partial t} + \sum_{i=1}^{n} \frac{\partial G\left(t, X(t)\right)}{\partial x_i} dX_i(t) \\ &\quad + \frac{1}{2} \sum_{\mathfrak{k}=1}^{n} \sum_{\mathfrak{i}=1}^{n} \frac{\partial^2 G\left(t, X(t)\right)}{\partial x_i \partial x_{\mathfrak{k}}} d\left\langle X_{\mathfrak{k}}, X_{\mathfrak{i}}\right\rangle (t). \end{aligned}$$

The following theorem shows that each continuous local martingale relative to $(\mathcal{Q}, \mathbb{F}(W))$, i.e. relative to $\mathcal{Q}$ and the natural filtration $\mathbb{F}(W)$, can be written as an Itô process. In the following we denote these martingales briefly $(\mathcal{Q}-)$ martingales.

Theorem 19. *(Zagst [119], Theorem 2.38, p. 31, Martingale representation I)*

Let $d \in \mathbb{N}$, $W = (W_1(t), \ldots, W_d(t))_{t\geq 0}$ be a d-dimensional Wiener process, and $\mathfrak{M} = (\mathfrak{M}(t))_{t\in[0,T]}$ be a continuous local $(\mathcal{Q}-)$ martingale. Then there is a progressively measurable process $\phi = (\phi(t))_{t\in[0,T]}$, $\phi : [0,T] \times \Omega \longrightarrow \mathbb{R}^d$ such that

i. $\int_0^T \|\phi(t)\|^2 dt < \infty$ $\mathcal{Q}-$ *a.s.,*

ii. $\mathfrak{M}(t) = \mathfrak{M}(0) + \int_0^t \phi(s)' dW(s)$ *or briefly* $d\mathfrak{M}(t) = \phi(t)' dW(t)$ $\mathcal{Q}-$ *a.s. for all* $t \in [0,T]$*, where* $\phi(s)'$ *is the transposed of* $\phi(s)$*.*

If $\mathfrak{M} = (\mathfrak{M}(t))_{t\in[0,T]}$ is a continuous $(\mathcal{Q}-)$ martingale with $\mathbb{E}_{\mathcal{Q}}[\mathfrak{M}^2(t)] < \infty$ for all $t \in [0,T]$, then 1. is strengthened to

$$\mathbb{E}_{\mathcal{Q}}\left[\int_0^T \|\phi(t)\|^2 dt\right] < \infty,$$

while 2. still holds.

For a proof see Øksendal [91], p. 53-54.

Lemma 1. *(Zagst [119], Lemma 2.40, p. 33, Novikov condition)*

Let $\tilde{\gamma} = (\tilde{\gamma}(t))_{t>0}$ be a d-dimensional progressively measurable stochastic process, $d \in \mathbb{N}$, with

$$\int_0^t \tilde{\gamma}_k^2(s) ds < \infty, \ \mathcal{Q} - a.s. \text{ for all } t \geq 0, \ k = 1, \ldots, d$$

and let the stochastic process $\mathcal{E}(\tilde{\gamma}) = (\mathcal{E}(t, \tilde{\gamma}))_{t\geq 0} = (\mathcal{E}(t, \tilde{\gamma}(t)))_{t\geq 0}$ for all $t \geq 0$ be defined by

$$\mathcal{E}(t, \tilde{\gamma}) = e^{-\int_0^t \tilde{\gamma}(s)' dW(s) - \frac{1}{2}\int_0^t \|\tilde{\gamma}(s)\|^2 ds}.$$

Then $\mathcal{E}(\tilde{\gamma})$ is a continuous $(Q-)$ martingale if

$$\mathbb{E}_{\mathcal{Q}}\left[e^{\frac{1}{2}\int_0^T \|\tilde{\gamma}(s)\|^2 ds}\right] < \infty. \quad \textit{(Novikov condition)}$$

For a proof see Karatzas and Shreve [76], p. 198-199.

Remark 3. *Under Novikov's condition it is true that*

$$\int_0^T \|\tilde{\gamma}(s)\|^2 ds < \infty, \ \mathcal{Q} - a.s.$$

and

$$\int_0^t \|\tilde{\gamma}(s)\|^2 \, ds < \infty, \; \mathcal{Q} - a.s. \text{ for all } t \in [0,T].$$

Remark 4. *For each $T \geq 0$ we define the measure $\tilde{\mathcal{Q}} = \tilde{\mathcal{Q}}_{\mathcal{E}(T,\tilde{\gamma})}$ on the measure space $(\Omega, \mathcal{F}_T)$ by*

$$\tilde{\mathcal{Q}}(\mathfrak{A}) := \mathbb{E}_{\mathcal{Q}}\left[1_{\mathfrak{A}} \cdot \mathcal{E}(T,\tilde{\gamma})\right] = \int_{\mathfrak{A}} \mathcal{E}(T,\tilde{\gamma}) \, d\mathcal{Q} \quad \text{for all} \quad \mathfrak{A} \in \mathcal{F}_T,$$

which is a probability measure if $\mathcal{E}(T,\tilde{\gamma})$ is a $\mathcal{Q}$-martingale. In this case, $\mathcal{E}(T,\tilde{\gamma})$ is the $\mathcal{Q}$-density of $\tilde{\mathcal{Q}}$, i.e. $\mathcal{E}(T,\tilde{\gamma}) = \frac{d\tilde{\mathcal{Q}}}{d\mathcal{Q}}$ on $(\Omega, \mathcal{F}_T)$.

We provide the Girsanov theorem, which shows, how a $\left(\tilde{\mathcal{Q}}-\right)$ Wiener process $\tilde{W} = \left(\tilde{W}(t)\right)_{t\in[0,T]}$ starting with a $(\mathcal{Q}-)$ Wiener process $W = (W(t))_{t\geq 0}$ can be constructed.

Theorem 20. *(Zagst [119], Theorem 2.41, p. 34f, Girsanov theorem)*

Let $W = (W_1(t), \ldots, W_d(t))_{t\geq 0}$ be a d-dimensional $(\mathcal{Q}-)$ Wiener process, $d \in \mathbb{N}, \tilde{\gamma}, \mathcal{E}(\tilde{\gamma}), \tilde{\mathcal{Q}}$, and $T \in [\,0,\infty\,)$ be as defined above, and the d-dimensional stochastic process $\tilde{W} = \left(\tilde{W}_1, \ldots, \tilde{W}_d\right) = \left(\tilde{W}_1(t), \ldots, \tilde{W}_d(t)\right)_{t\in[0,T]}$ be defined by

$$\tilde{W}_k(t) := W_k(t) + \int_0^t \tilde{\gamma}_k(s)ds, \; t \in [0,T], \; k = 1, \ldots, d,$$

i.e.

$$d\tilde{W}(t) := \tilde{\gamma}(t)dt + dW(t), t \in [0,T].$$

If the stochastic process $\mathcal{E}(\tilde{\gamma}) = (\mathcal{E}(t,\tilde{\gamma}))_{t\in[0,T]}$ is a $(\mathcal{Q}-)$ martingale, then the stochastic process $\tilde{W}$ is a d-dimensional $\left(\tilde{\mathcal{Q}}-\right)$ Wiener process on the measure space $(\Omega, \mathcal{F}_T)$.

For $\tilde{\gamma}(t)$ constant the change of measure corresponds to a change of the drift from μ to $\mu - \tilde{\gamma}$. For a proof see Øksendal [91], p. 163-164.

2.4 Diffusions and stochastic differential equations

In this section we want to analyse the question of existence and uniqueness for solutions to stochastic differential equations. According to Karatzas and Shreve [76], p. 281ff, this endeavour is really a study of diffusion processes. Loosely speaking, a diffusion is a Markov process which has continuous sample paths and can be characterised in terms of its infinitesimal generator.

Definition 37. *(Karatzas and Shreve [76], Definition 5.1.1, p. 281f, Diffusion process)*

Let $X = (X(t))_{t \geq 0}$, $(\Omega, \mathcal{F})$, $\{\mathcal{Q}^{\mathbf{x}}\}_{\mathbf{x} \in \mathbb{R}^d}$ *be a d-dimensional Markov family, such that*

i. X has continuous sample paths,

ii. for every $\xi \in \mathbf{C}^2(\mathbb{R}^d)$ *which is bounded and has bounded first- and second-order derivatives:*

$$\lim_{t \to 0} \frac{1}{t} \left(\mathbb{E}^{\mathbf{x}} \left[\xi \left(X(t) \right) \right] - \xi(\mathbf{x}) \right) = \left(\mathcal{A} \xi \right)(\mathbf{x}); \ \forall \ \mathbf{x} \in \mathbb{R}^d, \tag{2.70}$$

where $\mathbb{E}^{\mathbf{x}}$ *denotes the expectation with respect to* $\mathcal{Q}^{\mathbf{x}}$*, and*

$$\left(\mathcal{A} \xi \right)(\mathbf{x}) := \frac{1}{2} \sum_{i=1}^{d} \sum_{k=1}^{d} a_{ik}(\mathbf{x}) \frac{\partial^2 \xi(\mathbf{x})}{\partial x_i \partial x_k} + \sum_{i=1}^{d} \mu_i(\mathbf{x}) \frac{\partial \xi(\mathbf{x})}{\partial x_i}, \tag{2.71}$$

with $a_{ik} = \sum_{j=1}^{n} \sigma_{ij} \sigma_{kj}$*. The left-hand side of (2.70) is called infinitesimal generator of the Markov family, applied to* ξ*. The operator in (2.71) is called second-order differential operator with drift vector* $\mu = (\mu_1, \ldots, \mu_d)$ *and diffusion matrix* $a = \{a_{ik}\}_{1 \leq i,k \leq d}$*.*

iii. for every $\mathbf{x} \in \mathbb{R}^d$

$$\mathbb{E}^{\mathbf{x}} \left[X_i(t) - x_i \right] = t \mu_i(\mathbf{x}) + o(t), \tag{2.72}$$

$$\mathbb{E}^{\mathbf{x}} \left[(X_i(t) - x_i)(X_k(t) - x_k) \right] = t a_{ik}(\mathbf{x}) + o(t), \tag{2.73}$$

iv. 1)-4) of Definition 14 are satisfied, but only for stopping times ι*.*

Then X is called a (Kolmogorov-Feller) diffusion process.

Definition 38. *(Karatzas and Shreve [76], p. 282f, Kolmogorov forward and backward equation)*

Assume that the Markov family of Definition 37 has a transition density function $\mathcal{Q}^x(X(\tau') \in \mathfrak{A}) = p(\tau', \mathbf{x}', \tau, \mathbf{x})$, $\mathfrak{A} \in \mathcal{B}(\mathbb{R}^d)$*.* $p(\tau', \mathbf{x}') := p(\tau', \mathbf{x}', \tau, \mathbf{x})$ *satisfies the Kolmogorov forward equation for every fixed* $\tau \in (0, \infty), \mathbf{x} \in \mathbb{R}^d$*, given by*

$$\frac{\partial}{\partial \tau'} p(\tau', \mathbf{x}') = \mathcal{A}^* p(\tau', \mathbf{x}'), p(\tau, \mathbf{x}') = g(\mathbf{x}') \text{ for } (\tau', \mathbf{x}') \in (0, \infty) \times \mathbb{R}^d, \tag{2.74}$$

and $p(\tau, \mathbf{x}) := p(\tau', \mathbf{x}', \tau, \mathbf{x})$ *for every fixed* $\tau' \in (0, \infty), \mathbf{x}' \in \mathbb{R}^d$ *the backward Kolmogorov equation, given by*

$$\frac{\partial}{\partial \tau} p(\tau, \mathbf{x}) = \mathcal{A} p(\tau, \mathbf{x}), p(\tau', \mathbf{x}) = g(\mathbf{x}) \text{ for } (\tau, \mathbf{x}) \in (0, \infty) \times \mathbb{R}^d, \tag{2.75}$$

where the adjoint operator $\mathcal{A}^*$ is given by

$$(\mathcal{A}^* p)(\tau', \mathbf{x}') := \frac{1}{2} \sum_{i=1}^{d} \sum_{k=1}^{d} \frac{\partial^2}{\partial x_i' \partial x_k'} \left(a_{ik}(\tau', \mathbf{x}') \, p(\tau', \mathbf{x}') \right) - \sum_{i=1}^{d} \frac{\partial}{\partial x_i'} \left(\mu_i(\tau', \mathbf{x}') \, p(\tau', \mathbf{x}') \right), \tag{2.76}$$

if the derivatives are bounded and Hölder-continuous. $(\tau', \mathbf{x}')$ describe the forward variables, while $(\tau, \mathbf{x})$ are the backward variables, i.e. $\tau' \geq \tau$.

Remark 5. *The following equation with terminal condition*

$$-\frac{\partial}{\partial t} p(t, \mathbf{x}) = \mathcal{A} p(t, \mathbf{x});\ t', \mathbf{x}' \text{ fixed, for } (t, \mathbf{x}) \in (0, \infty) \times \mathbb{R}^d;\ p(t', \mathbf{x}) = g(\mathbf{x}), \tag{2.77}$$

can be easily transformed into the backward Kolmogorov equation presented before with initial condition by the transformation $\tau = t' - t$ (time reversal):

$$\frac{\partial}{\partial \tau} p(\tau, \mathbf{x}) = \mathcal{A} p(\tau, \mathbf{x}); \tau', \mathbf{x}' \text{ fixed, for } (\tau, \mathbf{x}) \in (0, \infty) \times \mathbb{R}^d;\ p(0, \mathbf{x}) = g(\mathbf{x}). \tag{2.78}$$

For the specific Kolmogorov equations of the models we treat in the main part refer to Appendix A.1.1.

Theorem 21. *(Øksendal [91], Theorem 8.1.1, p. 139f, Kolmogorov's backward equation) Let $g \in \mathbf{C}_0^2(\mathbb{R}^d)$. Define*[1]

$$\xi(\tau, \mathbf{x}) = \mathbb{E}^{\tau, x} \left[g(X(0)) \right]. \tag{2.79}$$

Then

$$\frac{\partial \xi}{\partial \tau} = \mathcal{A}\xi,\ \tau > 0, \mathbf{x} \in \mathbb{R}^d, \tag{2.80}$$

$$\xi(0, \mathbf{x}) = g(\mathbf{x}),\ \mathbf{x} \in \mathbb{R}^d. \tag{2.81}$$

Moreover, if $h(\tau, \mathbf{x}) \in \mathbf{C}^{1,2}\left(\mathbb{R} \times \mathbb{R}^d\right)$ is a bounded function satisfying (2.80) and (2.81), then $h(\tau, \mathbf{x}) = \xi(\tau, \mathbf{x})$, given by (2.79).

Definition 39. *(Fouque et al. [49], p. 62f, Invariant probability density)*

The existence of an invariant probability density means that $p(t', x', t, x)$ does not depend on t' or t and consequently satisfies

$$\mathcal{A}^* p = 0. \tag{2.82}$$

[1]Note that we have reversed the time by setting $\tau = T - t$, i.e. $\tau = 0$ means $t = T$.

Definition 40. *([63], Symmetric difference)*

The symmetric difference of the sets A and B is denoted by

$$A \triangle B. \tag{2.83}$$

The symmetric difference is equivalent to the union of both relative complements, i.e.

$$A \triangle B = (A \setminus B) \cup (B \setminus A). \tag{2.84}$$

Definition 41. *([63], Measure-preserving transformation)*

A measurable mapping $\mathfrak{T} : \Omega^1 \to \Omega^2$, i.e. a mapping $\mathfrak{T}$ from measure space $(\Omega^1, \mathcal{F}^1, \mathcal{Q}^1)$ to $(\Omega^2, \mathcal{F}^2, \mathcal{Q}^2)$ with $\mathfrak{T}^{-1}(\mathfrak{A}^2) = \{x : \mathfrak{T}(x) \in \mathfrak{A}^2\} \in \mathcal{F}_1$ for each $\mathfrak{A}^2 \in \mathcal{F}_2$, such that $\mathcal{Q}^1(\mathfrak{T}^{-1}(\mathfrak{A}^2)) = \mathcal{Q}^2(\mathfrak{A}^2)$ for every $\mathfrak{A}^2 \in \mathcal{F}^2$ is called a measure-preserving transformation.

Theorem 22. *(Walters [117], Theorem 1.5, p. 27, Ergodicity)*

If $\mathfrak{T} : \Omega \to \Omega$ is a measure-preserving transformation of the probability space $(\Omega, \mathcal{F}, \mathcal{Q})$, then the following statements are equivalent:

i. $\mathfrak{T}$ is ergodic.

ii. The only members $\mathfrak{A}_1$ of $\mathcal{F}$ with $\mathcal{Q}\left(\mathfrak{T}^{-1}(\mathfrak{A}_1) \triangle \mathfrak{A}_1\right) = 0$ are those with $\mathcal{Q}\left(\mathfrak{A}_1\right) = 0$ or $\mathcal{Q}\left(\mathfrak{A}_1\right) = 1$.

iii. For every $\mathfrak{A}_1 \in \mathcal{F}$ with $\mathcal{Q}\left(\mathfrak{A}_1\right) > 0$ we have $\mathcal{Q}\left(\bigcup_{n=1}^{\infty} \mathfrak{T}^{-n}(\mathfrak{A}_1)\right) = 1$, where $\mathfrak{T}^n$ denotes the nth iterate of the transformation $\mathfrak{T}$ and $\mathfrak{T}^{-n}$ its inverse.

iv. For every $\mathfrak{A}_1, \mathfrak{A}_2 \in \mathcal{F}$ with $\mathcal{Q}\left(\mathfrak{A}_1\right)$, $\mathcal{Q}\left(\mathfrak{A}_2\right) > 0$ there exists $n > 0$ with $\mathcal{Q}\left(\mathfrak{T}^{-n}(\mathfrak{A}_1) \bigcap \mathfrak{A}_2\right) > 0$.

For a proof see [117], p. 27f.

Theorem 23. *(Fouque [54], p. 115, Ergodicity of Markov processes)*

Let us assume a Markov process, which is irreducible, i.e. the process can visit any neighbourhood of the state space with positive probability, in finite time and from any starting point, and which possesses an invariant probability density function. Then the process is ergodic.

Remark 6. *Every ergodic Markov process has a unique invariant probability distribution* p^{inv} *and the distribution converges to the invariant distribution as* $t \to \infty$ *for any initial distribution. This probability distribution belongs to the null space of the adjoint generator* $\mathcal{A}^*$ *of its infinitesimal generator* $\mathcal{A}$*, i.e. that is the invariant distribution solves the adjoint equation:* $\mathcal{A}^*\ p^{inv}=0$.

See [54], p. 116.

Instead of proving the existence of a Markov process X satisfying the Kolmogorov backward and forward equation, the methodology of stochastic differential equations was suggested. Thus, in the following we introduce the concept of stochastic differential equations with respect to Brownian motions and their solutions in the so-called strong sense.

Definition 42. *(Zagst [119], Definition 2.44, p. 36, Strong solution)*

If there exists a d-dimensional stochastic process $X = (X(t))_{t\geq 0}$ *on the filtered probability space* $(\Omega, \mathcal{F}, \mathcal{Q}, \mathbb{F})$*, where each satisfies (2.66) and (2.67), i.e. an Itô process, such that for all* $t \geq 0$

$$\begin{aligned} X(t) &= \mathbf{x} + \int_0^t \mu(s, X(s))ds + \int_0^t \sigma(s, X(s))dW(s)\ \mathcal{Q} - a.s., \\ X(0) &= \mathbf{x} \in \mathbb{R}^d, \textit{fixed}, \end{aligned} \tag{2.85}$$

we call X *a strong solution of the stochastic differential equation*

$$\begin{aligned} dX(t) &= \mu(t, X(t))dt + \sigma(t, X(t))dW(t),\ \textit{for all}\ t \geq 0, \\ X(0) &= \mathbf{x}. \end{aligned} \tag{2.86}$$

The following theorem answers the existence and uniqueness question for some stochastic differential equations.

Theorem 24. *(Zagst [119], Theorem 2.45, p. 36f, Existence and Uniqueness)*

Let μ *and* σ *of the stochastic differential equation be continuous functions such that for all* $t \geq 0, \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *and for some constant* $\Xi_1 > 0$ *the following conditions hold:*

i. (Lipschitz condition)

$$\|\mu(t,\mathbf{x}) - \mu(t,\mathbf{y})\| + \|\sigma(t,\mathbf{x}) - \sigma(t,\mathbf{y})\| \leq \Xi_1 \|\mathbf{x} - \mathbf{y}\|, \tag{2.87}$$

ii. (Growth condition)

$$\|\mu(t,\mathbf{x})\|^2 + \|\sigma(t,\mathbf{x})\|^2 \leq \Xi_1^2(1 + \|\mathbf{x}\|^2). \tag{2.88}$$

Then there exists a unique, continuous strong solution $X = (X(t))_{t\leq 0}$ of the SDE and a constant Ξ_2, depending only on Ξ_1 and $T > 0$, such that

$$\mathbb{E}_{\mathcal{Q}}\left[\|X(t)\|^2\right] \leq \Xi_2(1+\|\mathbf{x}\|^2)e^{\Xi_2 t} \quad \text{for all} \quad t \in [0,T]. \tag{2.89}$$

Moreover,

$$\mathbb{E}_{\mathcal{Q}}\left[\sup_{0\leq t\leq T}\|X(t)\|^2\right] < \infty.$$

For a proof see Øksendal [91], p. 69-71.

In the one dimensional case, the Lipschitz condition on the diffusion coefficient can be considerably relaxed.

Proposition 1. *(Karatzas and Shreve [76], Proposition 5.2.13, p. 291, Yamada and Watanabe)*
Let $d = 1$ and let us suppose that the coefficients of the one-dimensional equation

$$dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dW(t) \tag{2.90}$$

satisfy the conditions

$$\begin{aligned} |\mu(t,x) - \mu(t,y)| &\leq \Xi_1 |x-y|, \\ |\sigma(t,x) - \sigma(t,y)| &\leq h|x-y|, \end{aligned}$$

for every $0 \leq t < \infty$ and $x \in \mathbb{R},\ y \in \mathbb{R}$, where Ξ_1 is a positive constant and $h : [0,\infty) \longmapsto [0,\infty)$ is a strictly increasing function with $h(0) = 0$ and

$$\int_{0,\Xi_2} h^{-2}(u)du = \infty; \quad \forall\ \Xi_2 > 0. \tag{2.91}$$

Then strong uniqueness holds for Equation (2.90).

For a proof see Karatzas and Shreve[76], p. 291-292.

Definition 43. *(Karatzas and Shreve [76], Definition 5.3.1, p. 300, Weak solution)*
A weak solution of Equation (2.65) is a tuple (X, W), $(\Omega, \mathcal{F}, \mathcal{Q}, \mathbb{F})$, where

i. $(\Omega, \mathcal{F}, \mathcal{Q})$ *is a probability space, and $\mathbb{F}$ is a filtration of sub-σ-fields of $\mathcal{F}$ satisfying the usual conditions,*

ii. $X = \{X(t), \mathcal{F}_t, 0 \leq t < \infty\}$ *is a continuous, adapted $\mathbb{R}^d$-valued process,*
$W = \{W(t), \mathcal{F}_t, 0 \leq t < \infty\}$ *is a d-dimensional Brownian motion,*

iii. $\mathcal{Q}\left(\int_0^t \left(|\mu\left(s, X(s)\right)| + \sigma_k^2\left(s, X(s)\right) ds\right) < \infty\right) = 1$ *holds for every* $1 \leq k \leq d$, *and* $0 \leq t < \infty$, *and*

iv. the integral version of (2.65)

$$X(t) = X(0) + \int_0^t \mu\left(s, X(s)\right) ds + \int_0^t \sigma\left(s, X(s)\right) dW(s),\ 0 \leq t < \infty. \tag{2.92}$$

holds almost surely.

2.4.1 Important examples of SDE's in $\mathbb{R}^1$

Example 2. *(Geometric Brownian Motion)*

We model the stock prices with geometric Brownian motions with the following SDE

$$dX(t) = X(t)\left(\mu dt + \sigma dW(t)\right), \quad X(0) > 0, \tag{2.93}$$

where μ *and* σ *are fixed constants.*

This differential equation has the unique solution

$$X(t) = X(0)e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)}. \tag{2.94}$$

For a proof see Bingham and Kiesel [12], p. 215-216.

Furthermore, the expectation, variance, and covariance functions are (see Arcones [6], p. 178)

$$\begin{aligned} \mathbb{E}\left[X(t)\right] &= X(0)e^{\mu t}, & (2.95)\\ \mathbf{Var}\left[X(t)\right] &= X(0)^2 e^{2\mu t}(e^{\sigma^2 t} - 1), & (2.96)\\ \mathbf{Cov}\left[X(s), X(t)\right] &= X(0)^2 e^{\mu(t+s)}(e^{\sigma^2(t\wedge s)} - 1). & (2.97) \end{aligned}$$

Example 3. *(Ornstein-Uhlenbeck Process)*

The oldest example of a stochastic differential equation is the Ornstein-Uhlenbeck (OU) Process (see Bingham and Kiesel [12], p. 204)

$$dX(t) = -\kappa X(t)dt + \sigma dW(t), \quad X(0) > 0, \tag{2.98}$$

The solution of this equation is (see Karatzas and Shreve [76], p. 358)

$$X(t) = X(0)e^{-\kappa t} + \sigma \int_0^t e^{-\kappa(t-s)} dW(s); 0 \leq t < \infty. \tag{2.99}$$

The expectation, variance, and covariance functions are (see Karatzas and Shreve [76], p. 358)

$$\begin{aligned} \mathbb{E}\left[X(t)\right] &= X(0)e^{-\kappa t}, &(2.100)\\ \mathbf{Var}\left[X(t)\right] &= \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t}), &(2.101)\\ \mathbf{Cov}\left[X(s), X(t)\right] &= \frac{\sigma^2}{2\kappa} e^{-\kappa(t+s)}(e^{2\kappa(t\wedge s)} - 1). &(2.102)\end{aligned}$$

The invariant distribution of X is a zero-mean Gaussian distribution with covariance function $\mathbf{Cov}(X(s), X(t)) = (\sigma^2/2\kappa)\, e^{-\kappa|t-s|}$.

A special case is the mean-reverting Ornstein-Uhlenbeck process (see Øksendal [91], p. 75)

$$dX(t) = \kappa\left(\zeta - X\left(t\right)\right) dt + \sigma dW(t), \tag{2.103}$$

which can be solved for

$$X(t) = \zeta\left(1 - e^{-\kappa t}\right) + X(0)e^{-\kappa t} + \sigma \int_0^t e^{-\kappa(t-s)} dW(s); 0 \leq t < \infty. \tag{2.104}$$

See Øksendal [91], p. 75. The expectation, variance, and covariance functions are given by

$$\begin{aligned} \mathbb{E}\left[X(t)\right] &= \zeta\left(1 - e^{-\kappa t}\right) + X(0)e^{-\kappa t}, &(2.105)\\ \mathbf{Var}\left[X(t)\right] &= \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t}), &(2.106)\\ \mathbf{Cov}\left[X(s), X(t)\right] &= \frac{\sigma^2}{2\kappa} e^{-\kappa(t+s)}(e^{2\kappa(t\wedge s)} - 1). &(2.107)\end{aligned}$$

$X(t)$ *is mean-reverting, i.e.*

$$\mathbb{E}\left[X(t)\right] = \zeta\left(1 - e^{-\kappa t}\right) + X(0)e^{-\kappa t} \longrightarrow \zeta, \tag{2.108}$$

as $t \longrightarrow \infty$. *The invariant distribution is, thus, a Gaussian with mean* ζ *and covariance function* $\mathbf{Cov}(X(s), X(t)) = (\sigma^2/2\kappa)\, e^{-\kappa|t-s|}$.

Example 4. *(Cox-Ingersoll-Ross Process (CIR))*

$$dX(t) = \kappa\left(\zeta - X(t)\right)dt + \sigma\sqrt{X(t)}dW(t). \tag{2.109}$$

Cox, Ingersoll, Ross applied this process to model interest rates, Heston used it to describe the volatility in his model, because $X(t)$ remains positive (if $X(0) \geq 0$) provided (see Feller [43])

$$\zeta > \frac{\sigma^2}{2\kappa}. \tag{2.110}$$

Its existence and uniqueness can be proved with the Yamada-Watanabe proposition (Proposition 1). For this process there is no explicit formula for $X(t)$ in terms of $W(t)$. However, there exists an explicit formula for the transition probability density

$$p\left((t,x) \longrightarrow (s,y)\right) = p\left(X(s) = y \mid X(t) = x\right). \tag{2.111}$$

See Cox, Ingersoll and Ross [25].
The expectation and the variance are given by (see [30])

$$\mathbb{E}\left[X(t)\right] = X(0)e^{-\kappa t} + \zeta\left(1 - e^{-\kappa t}\right), \tag{2.112}$$

$$\mathbf{Var}\left[X(t)\right] = X(0)\frac{\sigma^2}{\kappa}\left(e^{-\kappa t} - e^{-2\kappa t}\right) + \zeta\frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa t}\right)^2. \tag{2.113}$$

$$\begin{aligned}\mathbf{Cov}\left[X(s), X(t)\right] &= X(0)\frac{\sigma^2}{\kappa}(e^{-\kappa t} - e^{-\kappa(s+t)}) \\ &\quad + \zeta\frac{\sigma^2}{2\kappa}(e^{-\kappa(t-s)} - 2e^{-\kappa t} + e^{-\kappa(t+s)}),\ s < t\end{aligned} \tag{2.114}$$

As the CIR process is mean-reverting:

$$\mathbb{E}\left[X(t)\right] = X(0)e^{-\kappa t} + \zeta\left(1 - e^{-\kappa t}\right) \longrightarrow \zeta. \tag{2.115}$$

The invariant distribution of the CIR process is the gamma distribution with expectation ζ and variance $\zeta\frac{\sigma^2}{2\kappa}$ [36].

2.5 Connections between stochastic differential equations and partial differential equations

A direct link is given by the Feynman-Kac representation. In the following we consider a solution to the stochastic differential Equation (2.65), under the assumptions that

i. the coefficients $\mu_i(t,\mathbf{x})$ and $\sigma_{ik}(t,\mathbf{x})$ are continuous and satisfy the linear growth condition (2.88),

ii. the Equation (2.65) has a weak solution $\left(X^{(t,\mathbf{x})}, W\right)$, $(\Omega, \mathcal{F}, \mathcal{Q}, \mathbf{F})$ for every pair $(t,\mathbf{x})$ with $X(t) = x$, and that

iii. the solution is unique in the sense of probability law.

Closely related to the Stochastic Differential Equation (2.65) is the second-order differential operator

$$(\mathcal{A}_t \xi)(\mathbf{x}) := \frac{1}{2}\sum_{i=1}^{d}\sum_{k=1}^{d} a_{ik}(t,\mathbf{x})\frac{\partial^2 \xi(\mathbf{x})}{\partial x_i \partial x_k} + \sum_{i=1}^{d}\mu_i(t,\mathbf{x})\frac{\partial \xi(\mathbf{x})}{\partial x_i},\ \xi \in \mathbf{C}^2\left(\mathbb{R}^d\right). \tag{2.116}$$

With an arbitrary but fixed $T > 0$ and appropriate constants $\Xi > 0, \hat{\lambda} \geq 1$, we consider functions $g(\mathbf{x}) : \mathbb{R}^d \to \mathbb{R}$ and $r(t,x) : [0,T] \times \mathbb{R}^d \to [0,\infty\ )$ which are continuous and $g(\mathbf{x})$ satisfies

$$|g(\mathbf{x})| \leq \Xi\left(1 + \|\mathbf{x}\|^{2\hat{\lambda}}\right) \quad \text{or} \quad g(\mathbf{x}) \geq 0,\ \forall\ \mathbf{x} \in \mathbb{R}^d. \tag{2.117}$$

Theorem 25. *(Karatzas and Shreve [76], Theorem 5.7.6, p. 366, Feynman-Kac theorem)*

Under the preceding assumptions suppose that $\xi(t,\mathbf{x}) : [\,0,T\,] \times \mathbb{R}^d \to \mathbb{R}$ is continuous, is of class $\mathbf{C}^{1,2}\left([\,0,T\,] \times \mathbb{R}^d\right)$, and satisfies the Cauchy problem

$$-\frac{\partial \xi}{\partial t} = \mathcal{A}_t\xi - r\xi,\ t > 0,\ \mathbf{x} \in \mathbb{R}^d, \tag{2.118}$$

$$\xi(T,\mathbf{x}) = g(\mathbf{x}),\ \mathbf{x} \in \mathbb{R}^d, \tag{2.119}$$

where we simplified the terminal condition by foregoing the limes-formulation, r is called killing rate of ξ, as well as the polynomial growth condition

$$\max_{0 \leq t \leq T} |\xi(t,\mathbf{x})| \leq \Xi\left(1 + \|\mathbf{x}\|^{2\hat{\lambda}}\right),\ \mathbf{x} \in \mathbb{R}^d, \tag{2.120}$$

for some $\Xi > 0$, $\hat{\lambda} \geq 1$. Then $\xi(t,\mathbf{x})$ admits on $[0,T] \times \mathbb{R}^d$ the stochastic representation

$$\xi(t,\mathbf{x}) = \mathbb{E}^{(t,\mathbf{x})}\left[g\left(X(T)\right)\exp\left\{-\int_t^T r(s,X)ds\right\}\right], \tag{2.121}$$

where $\mathbb{E}^{(t,\mathbf{x})}$ denotes the expected value with respect to $\mathcal{Q}^{t,\mathbf{x}}$ with $\mathcal{Q}^{t,x}(X(t)=\mathbf{x})=1$, on $[0,T]\times\mathbb{R}^d$. In particular, such a solution is unique.

Definition 44. *(Karatzas and Shreve [76], Definition 5.7.9, p. 368, Fundamental solution)*

A fundamental solution of the second-order partial differential Equation (2.118) is a non-negative function $\bar{G}(t',\mathbf{x}',t,\mathbf{x})$ defined for $0\le t<t'\le T$, $\mathbf{x}\in\mathbb{R}^d$, $\mathbf{x}'\in\mathbb{R}^d$, with the property that for every $g\in\mathbf{C}_0(\mathbb{R}^d)$, $t'\in(0,\ T]$, the function

$$\xi(t,\mathbf{x}):=\int_{\mathbb{R}^d}\bar{G}(t',\mathbf{x}',t,\mathbf{x})\,g(\mathbf{x}')\,d\mathbf{x}',\ 0\le t<t',\ \mathbf{x}\in\mathbb{R}^d \tag{2.122}$$

is bounded, of class $\mathbf{C}^{1,2}$, satisfies (2.118) and (2.119).

Theorem 26. *(Karatzas and Shreve [76], p. 368 Existence of a fundamental solution)*

A fundamental solution $\bar{G}$ exists if the following conditions imposed on $\mu_i(t,\mathbf{x})$, $a_{ik}(t,\mathbf{x})$, and $r(t,\mathbf{x})$ are satisfied,

i. Uniform ellipticity: There exists a positive constant Ξ such that

$$\sum_{i=1}^{d}\sum_{k=1}^{d}a_{ik}(t,\mathbf{x})x_i'x_k'\ge\Xi\left\|\mathbf{x}'\right\|^2, \tag{2.123}$$

holds for every $\mathbf{x}'\in\mathbb{R}^d$ and $(t,\mathbf{x})\in[\,0,\infty)\times\mathbb{R}^d$,

ii. Boundedness: The functions $\mu_i(t,\mathbf{x})$, $a_{ik}(t,\mathbf{x})$, and $r(t,\mathbf{x})$ are bounded in $[\,0,T)\times\mathbb{R}^d$,

iii. Hölder continuity: The functions $\mu_i(t,\mathbf{x})$, $a_{ik}(t,\mathbf{x})$, and $r(t,x)$ are (uniformly) Hölder-continuous in $[\,0,T)\times\mathbb{R}^d$.

Remark 7. *For fixed $(t',\mathbf{x}')\in(\,0,T\,]\times\mathbb{R}^d$ the function*

$$\phi(t,\mathbf{x}):=\bar{G}(t',\mathbf{x}',t,\mathbf{x}) \tag{2.124}$$

is of class $\mathbf{C}^{1,2}\left([\,0,T)\times\mathbb{R}^d\right)$ and satisfies the backward Kolmogorov equation in the backward variables $(t,\mathbf{x})$ with killing rate r and terminal conditions (exactly as in (2.118)). See Karatzas and Shreve [76], p. 368f.

If in addition the functions $\frac{\partial}{\partial x_i}\mu_i(t,\mathbf{x})$, $\frac{\partial}{\partial x_i}a_{ik}(t,\mathbf{x})$ *and* $\frac{\partial^2}{\partial x_i \partial x_k}a_{ik}(t,\mathbf{x})$ *are bounded and Hölder-continuous, then for fixed* $(t,\mathbf{x}) \in (\,0,T\,] \times \mathbb{R}^d$ *the function*

$$\psi\left(t',\mathbf{x}'\right) := \bar{G}(t',\mathbf{x}',t,\mathbf{x}) \tag{2.125}$$

is of class $\mathbf{C}^{1,2}\left([\,0,T) \times \mathbb{R}^d\right)$ *and satisfies the forward Kolmogorov equation in the forward variables* $(t',\mathbf{x}')$ *with killing rate* r*:*

$$\frac{\partial \psi}{\partial t'} = \mathcal{A}^*_{t'}\psi - r\psi,\ t>0,\ \mathbf{x} \in \mathbb{R}^d. \tag{2.126}$$

See Karatzas and Shreve [76], p. 368f.
With Theorem 25 the solution for ξ *is given by*

$$\xi(t,\mathbf{x}) = \mathbb{E}^{(\mathbf{x},t)}\left[e^{-\int_t^{t'} r(s,x)ds} g\left(X(t')\right)\right],\ g \in \mathbf{C}_0\left(\mathbb{R}^d\right), t' \in [t,T]. \tag{2.127}$$

Comparing (2.127) and (2.122) we can deduce that any fundamental solution $\bar{G}\left(t',\mathbf{x}',t,\mathbf{x}\right)$ *is also the transition probability density for the discounted process* $X^{(t,\mathbf{x})}$ *determined by*

$$\mathcal{Q}^{(t,\mathbf{x})}\left(X(t') \in \mathfrak{A}\right) = \int_{\mathfrak{A}} \bar{G}\left(t',\mathbf{x}',t,\mathbf{x}\right)d\mathbf{x}',\ \mathfrak{A} \in \mathcal{B}\left(\mathbb{R}^d\right),\ 0 \le t < t' \le T. \tag{2.128}$$

In the following our descriptions are based on Zauderer [121], p. 415ff. The here so called fundamental solution or transition probability is closely related to the Green functions. This relationship is presented in the following.
Assume that $\mu_i(t,\mathbf{x}) = \frac{1}{2}\sum_{k=1}^d \frac{\partial a_{ik}(\mathbf{x})}{\partial x_k}$ then (2.116) can be simplified. In those cases we call the second-order operator $\mathcal{A}_t^{self}$ and reformulate (2.116) to

$$\begin{aligned}
\mathcal{A}_t^{self}\xi(\mathbf{x}) &= \frac{1}{2}\sum_{i=1}^d\sum_{k=1}^d a_{ik}(\mathbf{x})\frac{\partial^2\xi(\mathbf{x})}{\partial x_i \partial x_k} + \sum_{i=1}^d\left(\frac{1}{2}\sum_{k=1}^d \frac{\partial a_{ik}(\mathbf{x})}{\partial x_k}\right)\frac{\partial \xi(\mathbf{x})}{\partial x_i} \\
&= \frac{1}{2}\sum_{i=1}^d\sum_{k=1}^d\left(a_{ik}(\mathbf{x})\frac{\partial^2\xi(\mathbf{x})}{\partial x_i \partial x_k} + \frac{\partial a_{ik}(\mathbf{x})}{\partial x_k}\frac{\partial \xi(\mathbf{x})}{\partial x_i}\right) \\
&= \sum_{i=1}^d\sum_{k=1}^d \frac{\partial}{\partial x_k}\left(a_{ik}(\mathbf{x})\frac{\partial \xi}{\partial x_i}\right).
\end{aligned} \tag{2.129}$$

Definition 45. *(Reed and Barry [102], p. 186, Adjoint operator)*

The adjoint operator $\mathcal{A}^*$ *satisfies*

$$\langle \mathcal{A}\xi, \phi \rangle = \langle \xi, \mathcal{A}^*\phi \rangle. \tag{2.130}$$

An operator for which $\mathcal{A} = \mathcal{A}^*$ is called self-adjoint.

We have already given the explicit form of the adjoint operator for a second-order differential operator in (2.76). Hence,

$$
\begin{aligned}
\mathcal{A}_t^{self*}\xi(\mathbf{x}) &= \frac{1}{2}\sum_{i=1}^{d}\sum_{k=1}^{d}\frac{\partial^2}{\partial x_k \partial x_i}\left(a_{ik}(\mathbf{x})\xi(\mathbf{x})\right) - \sum_{i=1}^{d}\frac{\partial}{\partial x_i}\left(\frac{1}{2}\sum_{k=1}^{d}\frac{\partial a_{ik}(\mathbf{x})}{\partial x_k}\xi(\mathbf{x})\right) \\
&= \frac{1}{2}\sum_{i=1}^{d}\sum_{k=1}^{d}\left(a_{ik}(\mathbf{x})\frac{\partial^2 \xi(\mathbf{x})}{\partial x_i \partial x_k} + \frac{\partial a_{ik}(\mathbf{x})}{\partial x_k}\frac{\partial \xi(\mathbf{x})}{\partial x_i}\right) \\
&\quad + \frac{1}{2}\sum_{i=1}^{d}\sum_{k=1}^{d}\left(\xi(\mathbf{x})\frac{\partial^2 a_{ik}(\mathbf{x})}{\partial x_i \partial x_k} + \frac{\partial a_{ik}(\mathbf{x})}{\partial x_i}\frac{\partial \xi(\mathbf{x})}{\partial x_k}\right) \\
&\quad - \frac{1}{2}\sum_{i=1}^{d}\sum_{k=1}^{d}\left(\xi(\mathbf{x})\frac{\partial^2 a_{ik}(\mathbf{x})}{\partial x_i \partial x_k} + \frac{\partial a_{ik}(\mathbf{x})}{\partial x_i}\frac{\partial \xi(\mathbf{x})}{\partial x_k}\right) \\
&= \mathcal{A}_t^{self}\xi(\mathbf{x}) \qquad (2.131)
\end{aligned}
$$

Hence, operators $\mathcal{A}$ of the form $\mathcal{A} = \sum_{i=1}^{d}\sum_{k=1}^{d}\frac{\partial}{\partial x_k}\left(a_{ik}(\mathbf{x})\frac{\partial \xi}{\partial x_i}\right)$ are self-adjoint. In the following we work with self-adjoint operators.

Theorem 27. *(Zauderer [121], p. 415f, Green function for parabolic equation with Dirichlet boundary conditions)*

For a parabolic equation

$$
\frac{\partial \xi\left(t', \mathbf{x}'\right)}{\partial t'} - \mathcal{A}^{self}\xi\left(t', \mathbf{x}'\right) + r(\mathbf{x}')\xi(t', \mathbf{x}') = \mathfrak{K}\left(t', \mathbf{x}'\right), \ \mathbf{x}' \in D, \ 0 < t', \qquad (2.132)
$$

with initial and Dirichlet boundary conditions

$$
\xi\left(0, \mathbf{x}'\right) = g\left(\mathbf{x}'\right), \ \mathbf{x}' \in D, \ \xi(t', \mathbf{x}') \mid_{\mathbf{x}' \in \partial D} = \mathfrak{B}\left(t', \mathbf{x}'\right), \ 0 < t', \qquad (2.133)
$$

where $r(\mathbf{x}')$ is a non-negative function, $\mathfrak{K}(t', \mathbf{x}')$ is a real-valued time-dependent function, D is a bounded region in two or three dimensions and ∂D its boundary. $\mathfrak{B}\left(t', \mathbf{x}'\right)$ is a given function evaluated on ∂D. The integral theorem for the solution of ξ is then in $(t', \mathbf{x}')$-space:

$$
\begin{aligned}
\xi\left(t', \mathbf{x}'\right) &= \int_0^T \int_D \bar{G}\left(t, \mathbf{x}, t', \mathbf{x}'\right)\mathfrak{K}\left(t, \mathbf{x}\right) d\mathbf{x}dt + \int_D \left(\bar{G}\left(t, \mathbf{x}, t', \mathbf{x}'\right) g(\mathbf{x})\right)\mid_{t=0} d\mathbf{x} \\
&\quad - \int_0^T \int_{\partial D} a(t, \mathbf{x})\mathfrak{B}(t, \mathbf{x})\frac{\partial \bar{G}\left(t, \mathbf{x}, t', \mathbf{x}'\right)}{\partial \mathbf{n}} ds dt. \qquad (2.134)
\end{aligned}
$$

where $a = \{a_{ik}\}$, ds is an element of the respective boundary area and $\mathbf{n}$ denotes the unit normal on ∂D pointing to the exterior of D. The Green function $\bar{G}(t, \mathbf{x}, t', \mathbf{x}')$ fulfils the *following problem*

$$-\frac{\partial\bar{G}\left(t,\mathbf{x},t',\mathbf{x}'\right)}{\partial t}-\mathcal{A}^{self}\bar{G}\left(t,\mathbf{x},t',\mathbf{x}'\right)+r(\mathbf{x})\bar{G}\left(t,\mathbf{x},t',\mathbf{x}'\right) = \delta\left(\mathbf{x}'-\mathbf{x}\right)\delta\left(t'-t\right),$$
$$\mathbf{x},\ \mathbf{x}'\in D,\ 0<t,t'<T, \quad (2.135)$$

with the end and boundary conditions

$$\bar{G}\left(T,\mathbf{x},t',\mathbf{x}'\right)=0,\ \bar{G}\left(t,\mathbf{x},t',\mathbf{x}'\right)|_{\mathbf{x}\in\partial D}=0,\ t\leq T. \quad (2.136)$$

The equation (2.135) satisfied by the Green's function $\bar{G}\left(t,\mathbf{x},t',\mathbf{x}'\right)$ is a backward parabolic equation that results on reversing the direction of time in the forward parabolic Equation (2.132). It can be shown that $\bar{G}\left(t,\mathbf{x},t',\mathbf{x}'\right)=\bar{G}\left(-t',\mathbf{x}',-t,\mathbf{x},\right)$. Thus, as a function of $\mathbf{x}'$ and t', $\bar{G}$ satisfies a backward parabolic differential equation, but with time now running forwards instead of backwards. [121], p. 416.

Remark 8. *When the time is reversed in the forward parabolic Equation (2.132) by a change in variables, i.e.* $t'\to t$ *and we set now* $\mathfrak{K}=0$*, remove the boundary conditions in (2.139), then the equation can be reformulated to*

$$-\frac{\partial\xi\left(t,\mathbf{x}\right)}{\partial t}-\mathcal{A}^{self}\xi\left(t,\mathbf{x}\right)+r(\mathbf{x})\xi(t,\mathbf{x})=0,\ \mathbf{x}\in D,\ 0<t<T, \quad (2.137)$$

with the end conditions

$$\xi\left(T,\mathbf{x}\right)=g\left(\mathbf{x}\right),\ \mathbf{x}\in D, \quad (2.138)$$

ξ *is then given by*

$$\xi\left(t,\mathbf{x}\right) = \int_D\left(\bar{G}\left(t',\mathbf{x}',t,\mathbf{x}\right)g(\mathbf{x}')\right)|_{t=T}\,d\mathbf{x}', \quad (2.139)$$

When we assume that the operator in (2.118) is self-adjoint the two problems are identical. Thus, (2.139) and (2.122) must agree with each other, which means that the fundamental solution is equal to the Green function in the backward variables.

The problem of the Green Function (2.135) can be reformulated to be consistent with the problem the fundamental solution satisfies in the backward variables which we show in the following. The connecting link between the two formulations lies in Duhamel's principle which is now introduced.

Theorem 28. *(Zauderer [121], p. 218ff, Duhamel's principle)*

Let $\xi\left(t',\mathbf{x}'\right)$ satisfy a PDE of parabolic case

$$\frac{\partial\xi\left(t',\mathbf{x}'\right)}{\partial t'} - \mathcal{A}^{self}\xi\left(t',\mathbf{x}'\right) + r(\mathbf{x}')\xi(t',\mathbf{x}') \quad = \quad \mathfrak{K}\left(t',\mathbf{x}'\right), \tag{2.140}$$

with the following initial conditions

$$\xi\left(0,\mathbf{x}\right) = 0, \tag{2.141}$$

and homogeneous boundary conditions (if available). Then there exists a function $\vartheta(\mathbf{x},t;s)$ with parameter s (initial time) such that

$$\xi\left(t',\mathbf{x}'\right) = \int_0^t \vartheta\left(t',\mathbf{x}';s\right)ds, \tag{2.142}$$

and ϑ satisfies the following equations

$$\frac{\partial\vartheta(t',\mathbf{x}')}{\partial t'} - \mathcal{A}^{self}\vartheta\left(t',\mathbf{x}'\right) + r(\mathbf{x}')\vartheta\left(t',\mathbf{x}'\right) = 0, \tag{2.143}$$

with the same boundary condition (if any) as ξ and the following initial conditions, given at $t = s$, where $s \geq 0$:

$$\vartheta\left(s,\mathbf{x}\right) \quad = \mathfrak{K}\left(s,\mathbf{x}\right). \tag{2.144}$$

Remark 9. *It can be shown that in the parabolic case the Green function can be alternatively constructed as follows (see [121], p. 417f)*

$$-\frac{\partial\bar{G}\left(t,\mathbf{x},t',\mathbf{x}'\right)}{\partial t} - \mathcal{A}^{self}\bar{G}\left(t,\mathbf{x},t',\mathbf{x}'\right) + r(\mathbf{x})\bar{G}\left(t,\mathbf{x},t',\mathbf{x}'\right) = 0,\ \ \mathbf{x},\mathbf{x}' \in D,\ \ t < t', \tag{2.145}$$

with the boundary conditions as of before and the initial condition substituted by

$$\bar{G}\left(t',\mathbf{x},t',\mathbf{x}'\right) = \delta\left(\mathbf{x}'-\mathbf{x}\right),\ \ \mathbf{x}' \in D. \tag{2.146}$$

The only difference is that the domain of integration in the original formulation of (2.134) extends from 0 to T whereas in the present formulation it extends from 0 to t'. However, Zauderer [121], p. 417f finds that the domains of integration are, in effect, identical for both formulations.

2.6 Pricing contingent claims

In the following we provide necessary tools to price contingent claims in financial markets. We assume the existence of a risk-neutral measure $\tilde{\mathcal{Q}}$. Then, a European contingent claim with maturity T is given by the following formula.

Theorem 29. *(Bingham and Kiesel [12], Theorem 6.2.3, p. 250, Risk-neutral valuation formula)*

Assume the existence of a risk-neutral measure $\tilde{\mathcal{Q}}$, and a European contingent claim $C(t,x)$ with maturity T. Let P_0 be an asset without systematic risk, the bank account, with a price process given by $dP_0(t) = P_0(t)r(t)dt,\ P_0(0) = 1$, where $r(t)$ is the risk-free deterministic instantaneous interest rate. The price of $C(t,x)$, the arbitrage price process of X, is then given by the risk-neutral valuation formula

$$C(t,x) = P_0(t)\mathbb{E}_{\tilde{\mathcal{Q}}}\left[\frac{X}{P_0(T)}\middle|\mathcal{F}_t\right] = \mathbb{E}_{\tilde{\mathcal{Q}}}\left[Xe^{-\int_t^T r(s)ds}\middle|\mathcal{F}_t\right].$$

In the classical Black-Scholes model the price of a contingent claim X is given by the risk-neutral valuation principle with

$$C(t,x) = e^{-r(T-t)}\mathbb{E}_{\tilde{\mathcal{Q}}}\left[X \mid \mathcal{F}_t\right],$$

where the unique martingale measure $\tilde{\mathcal{Q}}$ is given by the Girsanov transformation

$$\mathcal{E}(t,x) = e^{-\left(\frac{\mu-r}{\sigma}\right)W(t)-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 t}.$$

For the following explanations in the main part we introduce some more notations.

Definition 46. *(Operators)*

$\mathcal{L}$ denotes in the following the parabolic operator of a differential equation, $\mathcal{L}^*$ is its adjoint according to Definition 45, e.g. the adjoint operator of the differential operator $\mathcal{L} = \frac{\partial}{\partial t} + \mathcal{A} - rf(t,\mathbf{x})$ is given by $\mathcal{L}^* = -\frac{\partial}{\partial t'} + \mathcal{A}^* - rf(t,\mathbf{x})$. Thus, $\mathcal{L}_{BS}f$ denotes the Black-Scholes operator in two dimensions applied on the test function $f(t,\mathbf{x})$, given by

$$\mathcal{L}_{BS}f(t,\mathbf{x}) = \frac{\partial f(t,\mathbf{x})}{\partial t} + \mathcal{A}f(t,\mathbf{x}) - rf(t,\mathbf{x}), \tag{2.147}$$

where $\mathcal{A}$ is given in (2.71) with drift r. The adjoint operator $\mathcal{L}^*_{BS}$ is hence given by

$$\mathcal{L}^*_{BS}f(t,\mathbf{x}) = -\frac{\partial f(t,\mathbf{x})}{\partial t'} + \mathcal{A}^*f(t,\mathbf{x}) - rf(t,\mathbf{x}). \tag{2.148}$$

2.7 Solution of PDE

In the following we will loosely state some concepts on which we base some of our proofs later.

Definition 47. *(Zauderer [121], p. 182f, Method of separation of variables)*

Consider the following parabolic problem with the restrictions as defined in Definition 27

$$\frac{\partial \xi(t',\mathbf{x}')}{\partial t'} - \mathcal{A}^{self}\xi(t',\mathbf{x}') + r(\mathbf{x}')\xi(t',\mathbf{x}') = 0,\ \mathbf{x}' \in D,\ t' > 0, \tag{2.149}$$

where $0 \leq \mathbf{x}' \leq \mathfrak{b}$ *and* ξ *satisfies the boundary conditions*

$$\xi(t',\mathbf{x}')\mid_{\mathbf{x}' \in \partial D} = 0,\ t' > 0, \tag{2.150}$$

and the initial condition $\xi(0,\mathbf{x}') = g(\mathbf{x}'),\ \mathbf{x} \in D$. *The method of separation of variables requires a solution of (2.149) in the form*

$$\xi(t',\mathbf{x}') = H(\mathbf{x}')T(t'), \tag{2.151}$$

with the function $H(\mathbf{x}')$ *satisfying the boundary conditions 2.150. Substituting (2.151) into (2.149) and dividing by* HT *yields*

$$\frac{\frac{\partial T(t')}{\partial t'}}{T(t')} = \frac{\mathcal{A}^{self}H(\mathbf{x}') - r(\mathbf{x}')H(\mathbf{x}')}{H(\mathbf{x}')}. \tag{2.152}$$

For more details see [121], p. 182ff.

Definition 48. *(Zauderer [121], p. 476, Free space Green function)*

The Green function $\bar{G}$ *can be expressed in the form (see [121], p. 476ff)*

$$\bar{G} = \bar{G}^F + \bar{G}^G, \tag{2.153}$$

where $\bar{G}^F$ *is the free space Green function.* $\bar{G}^F$ *satisfies the same PDE as the Green function* $\bar{G}$. *In addition,* $\bar{G}^F$ *satisfies the terminal condition in the parabolic case. Hence,* $\bar{G}^G$ *satisfies a homogeneous PDE with homogeneous end conditions at* $t = T$. *With respect to the boundary conditions, we require* $\bar{G}^G = -\bar{G}^F$ *on* ∂D. *The method of images is used to determine* $\bar{G}^G$.

Theorem 30. *(Harrell and Herod [60], Theorem 15.12, Fredholm alternative theorems for ordinary differential operators)*

Suppose that $\mathcal{A}^d$ is a dth-order differential operator. The problem is posed as follows: Given f find ξ such that $\mathcal{A}^d\xi = f$ with $\xi(B_k) = 0$, with $k = 1, \ldots, d$, where $B_1, \ldots, B_d \in \partial D$, the bounded region.

i. *Exactly one of the following two alternatives holds:*
   (a) *(First alternative) If f is continuous then $\mathcal{A}^d\xi = f$ with $\xi(B_k) = 0$, with $k = 1, \ldots, d$, has one and only one solution.*
   (b) *(Second alternative) $\mathcal{A}^d\xi = 0$ with $\xi(B_k) = 0$, with $k = 1, \ldots, d$, has a non-trivial solution.*

ii. (a) *If $\mathcal{A}^d\xi = f$ with $\xi(B_k) = 0$ has exactly one solution, then so does $\mathcal{A}^{d^*}\xi = f$ with $\overline{\xi}(B_k) = 0$ has exactly one solution where $\overline{\xi}$ is the conjugate function of ξ.*
   (b) *$\mathcal{A}^d\xi = 0$ with $\xi(B_k) = 0$ has the same number of linearly independent solutions as $\mathcal{A}^{d^*}\xi = 0$ with $\overline{\xi}(B_k) = 0$.*

iii. *Suppose the second alternative (i(b)) holds. Then $\mathcal{A}^d\xi = f$ with $\xi(B_k) = 0$, with $k = 1, \ldots, d$, has a solution if and only if $\langle f, \vartheta\rangle = 0$ for each ϑ that is a solution for $\mathcal{A}^{d^*}\vartheta = 0$ with $\overline{\vartheta}(B_k) = 0$,*

where $\langle f, \vartheta\rangle$ denotes the scalar product and $\mathcal{A}^{d^*}$ is the adjoint operator as defined in Definition (45).

The following theorem is important when a solution of a PDE gained by transformations is in form of integrals and those integrals shall be stated in the original variables.

Theorem 31. *(Königsberger [80], p. 300, Transformation theorem)*

Let $\mathfrak{T} : \mathbb{R}^d \to \mathbb{R}^d, \mathfrak{T}\mathbf{x} = A\mathbf{x} + b$, $A \in \mathbb{R}^{d\times d}, b \in \mathbb{R}^d$, be a regular affine transformation. If $f : \mathbb{R}^d \to \mathbb{R}^d$ is integrable over some $\mathfrak{A} \subset \mathbb{R}^d$, then $f \circ \mathfrak{T}$ is integrable over $\mathfrak{T}^{-1}(\mathfrak{A})$, and

$$\int_{\mathfrak{T}^{-1}(\mathfrak{A})} f(A\mathbf{x} + b)d\mathbf{x} = \frac{1}{|detA|} \int_{\mathfrak{A}} f(y)dy, \tag{2.154}$$

where $det(A)$ is called Jacobi determinant.

Part II

Main Part

Chapter 3

Pricing of barrier options within stochastic covariance model

3.1 Introduction

Empirical studies are in support of the fact that asset price volatility is random and not deterministic as assumed by the models originally proposed by Black and Scholes [13] and Merton [87]. Although, the volume of trades in exotic options has tremendously increased in the last decade, this assumption is still very popular in practice, especially in plain vanilla option pricing context, due to its simplicity. Multi-dimensional products depend to a great extent on the volatility and covariance structure of the underlying. The financial crisis shows that the assumption that the covariance structure is constant is not valid. For options on one underlying models which do no longer assume volatility of the underlying to be constant have been introduced, e.g. the local volatility model by Dupire [37] or stochastic volatility models, e.g. the ones by Stein and Stein [111] or Heston [65]. In such models closed-form solutions for lookback and barrier options on a single underlying are still available, e.g. in [83].

Recently, multivariate models, like for example the Wishart model, have been developed by Gourieroux in [58], [57] and [56] as well as by da Fonseca and others in [27], [26]. However, these models are rather complex so that analytical expressions especially for path-dependent options are rare. Hence, most of the exotic options, particularly those involving barriers, are priced using techniques like Monte Carlo simulations, finite element or difference methods when the strict assumptions of a Brownian motion framework are released. However, numerical issues and convergence problems arise particularly in the case of path-dependent options and refinements of the grid or simulations have to be imposed to model the assets at price levels close to the barriers.

In a two-underlying geometric Brownian model an analytical expression of the joint distribution of the maximum/minimum and maturity values of two assets exists and has been derived by He et al. [64] and Zhou [123], [124].

In this chapter we can extend their analytic result to a model which allows for a third factor, which governs the stochastic volatility of the two underlying processes, and price options with time-dependent barriers. We will show that the pricing function can be derived in two ways, via Fourier transformation and inversion, as well as via Fourier series.

In Section 3.2 we introduce the framework of models, which are considered in the remainder of the chapter. Some basic transformations to bring the PDE in a form we can work with are presented in Section 3.3. In Section 3.4 we concentrate on the first method, i.e. pricing options using Fourier integration techniques. We build upon methods first introduced by Lipton [83] and Lewis [82]. One of the models we consider has been first introduced by Bakshi and Madan [8] and has been applied by Dempster and Hong [32] to value correlation and spread options using Fourier transform. As far as we know Borovkov and Novikov [15] as well as Lipton [83] have been the only ones to price barrier options on one underlying using Fourier transforms. Borovok and Novikov use a change of measure with the normalised payoff without barriers as Radon-Nikodym derivative. We follow Lipton's approach in Section 3.4.1 using the reflection principle and the method of images to price options with a time-varying barrier on each underlying in a three-factor framework. A manageable closed-form solution is attained when the correlation structure of the two Brownian motions is assumed to be of the form $\rho = -\cos\frac{\pi}{n}$, or takes those values randomly. We will show that the formulas converge point-wise. The Lipton and Lewis approach is then compared to the approach of Bakshi and Madan [8] as well as Dempster [32] in Section 3.4.5. The formulas are implemented for double-digital barrier options and correlation barrier options in 3.4.3 and 3.4.4. When we degenerate the three-factor model it converges to the two-dimensional Black-Scholes framework. We compare prices obtained in the degenerated model applying Fourier techniques with prices which have been derived using the standard Black-Scholes formulas and find the results to be quite close.

The assumed dependence structure might at first glance seem restrictive because the correlations are negative for $n > 2$. However, the assumption of random correlations makes a positive expected value for the correlation attainable (see Section 3.4.6) and adds additional randomness to the covariance dynamics. Furthermore, we believe that a closed-form solution is helpful to test various numerical methods like Monte Carlo simulations.

In the second part we deal with solutions (derived via PDE and Fourier series) which relieve the restrictions made on ρ. The solutions are a direct extension of the analytical expression of the joint distribution of the maximum/minimum and maturity values of two assets each governed by a geometric Brownian motion derived by He et al. [64]. We exploit the affine structure of the characteristic function and find an easy attainable solution (see Section 3.5.1). The general pricing formula is again applied to double-digital and correlation barrier options (see 3.5.2 and 3.5.3) and the results from the formulas derived using PDE techniques are compared to the ones using the Fourier technique. They are consistent. Finally, in Section 3.6, we apply those formulas to the pricing of certificates under issuer risk and find that neglecting issuer risk and/or stochastic covariance leads to tremendous price differences.

The focus and innovation of this chapter is the pricing of barrier options like double-digital barrier options or product barrier options in a stochastic covariance framework, but the transition-probability functions/characteristic functions presented can also be used to value other barrier options based on two assets.

3.2 Model framework

The system of processes is defined on a filtered probability space $(\Omega, \mathcal{F}, \tilde{\mathcal{Q}}, \mathbb{F})$ where $\mathcal{F}_0$ contains all subsets of the ($\tilde{\mathcal{Q}}-$) null sets of $\mathcal{F}$ and $\mathbb{F}$ is right-continuous. We define the processes under the risk-neutral measure $\tilde{\mathcal{Q}}$. We start from a simple two-dimensional geometric Brownian model (GBM) to describe the dynamics of the two underlyings of the derivatives.

$$\begin{aligned} dS_i &= S_i r dt + S_i \sigma_i dW_i, \quad \text{for } i \in I = \{1, 2\} \\ \langle dW_1, dW_2 \rangle &= \rho dt. \end{aligned} \tag{3.1}$$

This model is extended and made more flexible by allowing for a third factor governing the stochastic covariance:

$$\begin{aligned} dS_i &= S_i r dt + S_i \sigma_i v^{\nu} dW_i \ \text{ for } i \in I = \{1, 2\}, \\ dv &= \kappa(\zeta - v)dt + \sigma_v v^{\gamma} dZ, \\ \langle dW_i, dZ \rangle &= 0, \\ \langle dW_1, dW_2 \rangle &= \rho dt, \end{aligned} \tag{3.2}$$

where κ, ζ, σ_v, and γ as well as ν are constants. The model includes well-known cases, like for example the case of constant volatility, which follows from setting $\kappa = \sigma_v = 0$ and $v(0) = 1$. It also contains several known uni- and bi-dimensional stochastic volatility models, for instance:

- The model with $\gamma = \nu = \frac{1}{2}$ has been considered in the case of correlation options by Bakshi and Madan [8] and spread options in the work of Dempster and Hong [32]. Note that when $|I| = 1$, i.e. $S_1 = S_2$, as well as $\gamma = \nu = \frac{1}{2}$ the model is known as the Heston model. The process which the volatility follows in this model has been introduced in finance by Cox et al. for short rate modelling and is therefore referred to as Cox-Ingersoll-Ross process (CIR process) (see Cox et al. [25]). The popularity of the CIR process is due to the positive value of v, which is guaranteed as long as (see Feller [44])
$$2\kappa\zeta > \sigma_v^2. \tag{3.3}$$
- The case $|I| = 1$, $\gamma = 0$ and $\nu = 1$ has been introduced by Stein and Stein (see [111]). The volatility follows a Gaussian mean-reverting process, which is also called Ornstein-Uhlenbeck process. This process has been introduced by Vasicek [115] to term structure modelling and has been used by Hull and White [68], Scott [106] and Schöbel and Zhu [105].
- The case $|I| = 1$, $\gamma = 1$ and $\nu = \frac{1}{2}$ gives Wiggins' log-normal model (see [118]).
- The case $|I| = 1$, $\gamma = \frac{3}{2}$ and $\nu = \frac{1}{2}$ gives Lewis' model (see [81]).

In this chapter, we focus on the two diffusions with parameter sets $\gamma = \nu = \frac{1}{2}$ and $\gamma = 0$, $\nu = 1$, which possess affine-type characteristic functions, to derive closed-form expressions for the price of barrier derivatives.

3.3 Pricing of two-asset barrier options

A two-asset knock-out barrier option has a payout $g(S_1, S_2)$, which may depend on $S_1(T)$ and $S_2(T)$, at maturity time T provided that not any of the two assets has crossed a predefined time-dependent barrier $B_1(t) = B_1 e^{\int_0^t r(s)ds}$ or $B_2(t) = B_2 e^{\int_0^t r(s)ds}$. The popularity of barrier options has increased in the last years (see Walmsley [116], p. 220). Adding barriers is a convenient method for reducing an option's cost (see Pooley et al. [95]). Often they are part of complex structured notes, e.g. certificates. Barrier options are also used to account for default in some credit models, e.g. in the CreditGrades model (see Sepp [107]). When we assume risk-neutral valuation the value of a general two-asset barrier option with time-dependent barriers on each of the underlyings is given by

$$
\begin{aligned}
C(t, S_1, S_2, B_1, B_2) &= \mathbb{E}_{\tilde{\mathcal{Q}}}\left[e^{-\int_t^T r(s)ds} g\left(S_1(T), S_2(T)\right) \mathbb{1}_{\{\iota_1 > T,\ \iota_2 > T\}} |\mathcal{F}_t\right], \\
\text{where} & \\
\iota_1 &= \inf\left(t' \in (t, T] : S_1(t') \leq B_1(t')\right), \\
\iota_2 &= \inf\left(t' \in (t, T] : S_2(t') \leq B_2(t')\right), \qquad (3.4)
\end{aligned}
$$

where the expectation is taken with respect to the pricing measure $\tilde{\mathcal{Q}}$.
$g\left(S_1(T), S_2(T)\right)$ describes the part of the payoff which depends on the values of S_1 and S_2 in T. The following PDE can be derived for the general model (in 3.2). It is clear that the PDE and the initial conditions are equal to the case without barriers. Additionally, we introduce boundary conditions. See also Appendix A.1.1 for the transformations.

$$
\begin{cases}
\frac{1}{2}v^{2\nu}\sigma_1^2 S_1^2 \frac{\partial^2 C}{\partial S_1^2} + \frac{1}{2}v^{2\nu}\sigma_2^2 S_2^2 \frac{\partial^2 C}{\partial S_2^2} + \rho v^{2\nu}\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 C}{\partial S_1 \partial S_2} + \\
\frac{1}{2}\sigma_v^2 v^{2\gamma}\frac{\partial^2 C}{\partial v^2} + rS_1\frac{\partial C}{\partial S_1} + rS_2\frac{\partial C}{\partial S_2} + \kappa(\zeta - v)\frac{\partial C}{\partial v} + \frac{\partial C}{\partial t} - rC = 0, \\
\\
C(t, B_1(t), S_2, B_1(t), B_2(t), v) = 0,\ C(t, S_1, B_2(t), B_1(t), B_2(t), v) = 0, \\
C(T, S_1, S_2, B_1(t), B_2(t), v) = g\left(S_1, S_2\right) \mathbb{1}_{\{\iota_1 > T,\ \iota_2 > T\}}.
\end{cases}
\qquad (3.5)
$$

The PDE can be reduced to

$$
\begin{cases}
\frac{\partial G}{\partial t} + \frac{1}{2}\sigma_1^2 v^{2\nu}\frac{\partial^2 G}{\partial x_1^2} + \frac{1}{2}\sigma_2^2 v^{2\nu}\frac{\partial^2 G}{\partial x_2^2} - \frac{1}{2}\sigma_1^2 v^{2\nu}\frac{\partial G}{\partial x_1} - \frac{1}{2}\sigma_2^2 v^{2\nu}\frac{\partial G}{\partial x_2} + \\
\rho\sigma_1\sigma_2 v^{2\nu}\frac{\partial^2 G}{\partial x_1 \partial x_2} + \frac{1}{2}\sigma_v^2 v^{2\gamma}\frac{\partial^2 G}{\partial v^2} + \kappa(\zeta - v)\frac{\partial G}{\partial v} = 0, \\
\\
G(t, b_1, x_2, b_1, b_2, v) = 0,\ G(t, x_1, b_2, b_1, b_2, v) = 0, \\
G(T, x_1, x_2, b_1, b_2, v) = g\left(x_1, x_2\right) \mathbb{1}_{\{\iota_1 > T,\ \iota_2 > T\}},
\end{cases}
\qquad (3.6)
$$

where we use the transformations $x_i(t) := \ln\left(\frac{S_i(t)e^{\int_t^T r(s)ds}}{K_i}\right)$, $b_i := \ln\left(\frac{B_i(T)}{K_i}\right)$ for $i \in \{1, 2\}$, and $G(t, x_1, x_2, b_1, b_2, v) := e^{\int_t^T r(s)ds} C(t, S_1, S_2, B_1(t), B_2(t), v)$. Equation (3.6) is the Kolmogorov backward equation (see Karatzas and Shreve [76], p. 282) for the following system of SDEs

$$
\begin{aligned}
dx_i &= -\frac{1}{2}\sigma_i^2 v^{2\nu} dt + v^{\nu}\sigma_i dW_i, \\
dv &= \kappa(\zeta - v)dt + \sigma_v v^{\gamma} dZ, \qquad (3.7)
\end{aligned}
$$

with r implicitly set to zero.

3.4 Pricing of two-asset barrier options with Fourier techniques

3.4.1 General pricing formulas for two-asset barrier options with Fourier techniques

The system in (3.6) can be solved for a group of payoffs and models included in the general framework (3.2), i.e. for specific values of the parameters γ and ν. A necessary condition for this group is the existence of an affine and analytic characteristic function (see Definition 32). A characteristic function, which is regular in a neighbourhood of 0, is also regular in a strip $\mathfrak{a} < \Im(u) < \mathfrak{b}$, where $\mathfrak{a}$ and $\mathfrak{b}$ are horizontal lines to the real axis and $\Im(u)$ describes the imaginary part of u, or in the whole plane, and there it can be represented by a Fourier integral (see Theorem 14). Thus, for $\mathfrak{a} < \Im(u) < \mathfrak{b}$ the characteristic function of the process $X(t)$ is identical to the generalized Fourier transform of the transition density (see Section 2.2.7). Generalized Fourier transforms are inverted by integrating along a contour in the complex u-plane, parallel to the real axis, with u in the strip of regularity (see Equation (2.50)). Our work extends Lewis' approach to two dimensions (see [81],[82]).
These concepts can be extended to Fourier transforms in two variables.

Theorem 32. *(Extension of Theorem 7.1.1 of [84] (see Theorem 14) to $\mathbb{R}^2$)*

If a characteristic function $\varphi(\mathbf{u})$, where $\mathbf{u}$ represents a vector (u_1, u_2), is regular in a neighbourhood of the origin, then it is also regular in a horizontal strip to the real axes and can be represented in this space by a Fourier integral.

Proof.

We follow the lines of Lukacs [84], p. 130ff. Assume that $\varphi(u_1, u_2)$ is an analytic characteristic function. We know then that all moments of the corresponding distribution exist and that it admits a MacLaurin expansion

$$\varphi(u_1, u_2) = \sum_{\mathfrak{k}=0}^{\infty} \sum_{k=0}^{\infty} \frac{i^{(\mathfrak{k}+k)} u_1^{\mathfrak{k}} u_2^{k} \alpha_{\mathfrak{k},k}}{\mathfrak{k}! k!} \quad \text{for } |\mathbf{u}| \leq \tilde{\mathbf{c}}_0, \tag{3.8}$$

where $\tilde{\mathbf{c}}_0 = (\tilde{c}_{0,1}, \tilde{c}_{0,2}) > 0$ is the radius of convergence of the series. $\alpha_{\mathfrak{k},k} = \int_{\mathbb{R}^2} x_1^{\mathfrak{k}} x_2^{k} p(x_1, x_2) dx_1 dx_2$ describes the algebraic moment. This series can be decomposed in an even and an odd part:

$$\begin{aligned}
\text{even: } \varphi_1(u_1,u_2) &= \frac{1}{2}(\varphi(u_1,u_2)+\varphi(-u_1,-u_2)), && (3.9)\\
\text{odd: } \varphi_2(u_1,u_2) &= \frac{1}{2}(\varphi(u_1,u_2)-\varphi(-u_1,-u_2)). && (3.10)
\end{aligned}$$

It can be easily seen that

$$\begin{aligned}
\varphi_1(u_1,u_2) &= \varphi_{11}(u_1,u_2)+\varphi_{12}(u_1,u_2), && (3.11)\\
\varphi_2(u_1,u_2) &= \varphi_{21}(u_1,u_2)+\varphi_{22}(u_1,u_2), && (3.12)
\end{aligned}$$

where

$$\begin{aligned}
\varphi_{11}(u_1,u_2) &= \sum_{\mathfrak{k}=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{\mathfrak{k}+k}u_1^{2\mathfrak{k}}u_2^{2k}\alpha_{2\mathfrak{k},2k}}{(2\mathfrak{k})!(2k)!}, && (3.13)\\
\varphi_{12}(u_1,u_2) &= \sum_{\mathfrak{k}=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{\mathfrak{k}+k-1}u_1^{2\mathfrak{k}-1}u_2^{2k-1}\alpha_{2\mathfrak{k}-1,2k-1}}{(2\mathfrak{k}-1)!(2\mathfrak{k}-1)!}, && (3.14)\\
\varphi_{21}(u_1,u_2) &= \sum_{\mathfrak{k}=0}^{\infty}\sum_{k=1}^{\infty}\frac{i^{2\mathfrak{k}+2k-1}u_1^{2\mathfrak{k}}u_2^{2k-1}\alpha_{2\mathfrak{k},2k-1}}{(2\mathfrak{k})!(2k-1)!}, && (3.15)\\
\varphi_{22}(u_1,u_2) &= \sum_{\mathfrak{k}=1}^{\infty}\sum_{k=0}^{\infty}\frac{i^{2\mathfrak{k}+2k-1}u_1^{2\mathfrak{k}-1}u_2^{2k}\alpha_{2\mathfrak{k}-1,2k}}{(2\mathfrak{k}-1)!(2\mathfrak{k})!}. && (3.16)
\end{aligned}$$

The radii of convergence of these series are denoted by $\tilde{\mathbf{c}}_1$, $\tilde{\mathbf{c}}_2$, $\tilde{\mathbf{c}}_{11}$, $\tilde{\mathbf{c}}_{12}$, $\tilde{\mathbf{c}}_{21}$, and $\tilde{\mathbf{c}}_{22}$ respectively. From the inequality (derived from the Binomial theorem)

$$\begin{aligned}
\left|x_1^{2\mathfrak{k}-1}x_2^{2k-1}\right| &\le \frac{1}{4}\left(x_1^{2\mathfrak{k}}+x_1^{2\mathfrak{k}-2}\right)\left(x_2^{2k}+x_2^{2k-2}\right)\\
&= \frac{1}{4}\left(x_1^{2\mathfrak{k}}x_2^{2k}+x_1^{2\mathfrak{k}}x_2^{2k-2}+x_1^{2\mathfrak{k}-2}x_2^{2k}+x_1^{2\mathfrak{k}-2}x_2^{2k-2}\right), && (3.17)
\end{aligned}$$

it follows that

$$\begin{aligned}
&\frac{\alpha_{2\mathfrak{k}-1,2k-1}}{(2\mathfrak{k}-1)!(2k-1)!} \le \frac{\beta_{2\mathfrak{k}-1,2k-1}}{(2\mathfrak{k}-1)!(2k-1)!}\\
\le\ &\frac{1}{4}\Bigg(\frac{\alpha_{2\mathfrak{k},2k}}{(2\mathfrak{k})!(2k)!}(2\mathfrak{k})(2k)+\frac{\alpha_{2\mathfrak{k},2k-2}}{(2\mathfrak{k})!(2k-2)!}(2\mathfrak{k})\\
&+\frac{\alpha_{2\mathfrak{k}-2,2k}}{(2\mathfrak{k}-2)!(2k)!}(2k)+\frac{\alpha_{2\mathfrak{k}-2,2k-2}}{(2\mathfrak{k}-2)!(2k-2)!}\Bigg), && (3.18)
\end{aligned}$$

where $\beta_{\mathfrak{k},k} = \int_{\mathbb{R}^2} |x_1|^{\mathfrak{k}} |x_2|^k p(x_1,x_2)dx_1dx_2$ describes the absolute moment. Thus, $\tilde{\mathbf{c}}_{12} \geq \tilde{\mathbf{c}}_{11} \geq \tilde{\mathbf{c}}_0$, as

$$\lim_{\mathfrak{k}\to\infty,k\to\infty} \left| \underbrace{\frac{\frac{(-1)^{\mathfrak{k}+k+2}u_1^{2(\mathfrak{k}+1)}u_2^{2(k+1)}\alpha_{2(\mathfrak{k}+1),2(k+1)}}{(2(\mathfrak{k}+1))!(2(k+1))!}}{\frac{(-1)^{\mathfrak{k}+k}u_1^{2\mathfrak{k}}u_2^{2k}\alpha_{2\mathfrak{k},2k}}{(2\mathfrak{k})!(2k)!}}}_{=\frac{1}{\tilde{\mathbf{c}}_{11}}} \frac{(2(\mathfrak{k}+1))(2(k+1))}{(2\mathfrak{k})(2k)} \right| = \frac{1}{\tilde{\mathbf{c}}_{11}}. \tag{3.19}$$

We further see that the series $\sum_{\mathfrak{k}=1}^{\infty}\sum_{k=1}^{\infty} \frac{u_1^{2\mathfrak{k}-1}u_2^{2k-1}\beta_{2\mathfrak{k}-1,2k-1}}{(2\mathfrak{k}-1)!(2k-1)!}$ also converges for $|u| < \tilde{\mathbf{c}}_{11}$. Now,

$$\begin{aligned} \left|x_1^{2\mathfrak{k}}x_2^{2k-1}\right| &\leq \frac{1}{2}\left(x_1^{2\mathfrak{k}}x_2^{2k} + x_1^{2\mathfrak{k}}x_2^{2k-2}\right), \\ &\text{and} \\ \left|x_1^{2\mathfrak{k}-1}x_2^{2k}\right| &\leq \frac{1}{2}\left(x_1^{2\mathfrak{k}}x_2^{2k} + x_1^{2\mathfrak{k}-2}x_2^{2k}\right). \end{aligned}$$

Thus,

$$\begin{aligned} & \frac{\alpha_{2\mathfrak{k},2k-1}}{(2\mathfrak{k})!(2k-1)!} \leq \frac{\beta_{2\mathfrak{k},2k-1}}{(2\mathfrak{k})!(2k-1)!} \\ \leq\ & \frac{1}{2}\left(\frac{\alpha_{2\mathfrak{k},2k}}{(2\mathfrak{k})!(2k)!}(2\mathfrak{k})(2k) + \frac{\alpha_{2\mathfrak{k},2k-2}}{(2\mathfrak{k})!(2k-2)!}(2\mathfrak{k})\right), \\ \text{and}\ & \\ & \frac{\alpha_{2\mathfrak{k}-1,2k}}{(2\mathfrak{k}-1)!(2k)!} \leq \frac{\beta_{2\mathfrak{k}-1,2k}}{(2\mathfrak{k}-1)!(2k)!} \\ \leq\ & \frac{1}{2}\left(\frac{\alpha_{2\mathfrak{k},2k}}{(2\mathfrak{k})!(2k)!}(2\mathfrak{k})(2k) + \frac{\alpha_{2\mathfrak{k}-2,2k}}{(2\mathfrak{k}-2)!(2k)!}(2k)\right). \end{aligned} \tag{3.20}$$

We derive that $\tilde{\mathbf{c}}_{21} \geq \tilde{\mathbf{c}}_{11} \geq \tilde{\mathbf{c}}_0$ and $\tilde{\mathbf{c}}_{22} \geq \tilde{\mathbf{c}}_{11} \geq \tilde{\mathbf{c}}_0$. Thus, $\tilde{\mathbf{c}}_2 \geq \tilde{\mathbf{c}}_1$. Furthermore, the series $\sum_{\mathfrak{k}=0}^{\infty}\sum_{k=0}^{\infty} \frac{u_1^{\mathfrak{k}}u_2^{k}\beta_{\mathfrak{k},k}}{(\mathfrak{k})!(k)!}$ converges due to $\beta_{2\mathfrak{k},2k} = \alpha_{2\mathfrak{k},k}$ as well as (3.18) and (3.20) for $|u| < \tilde{\mathbf{c}}_{11}$.

Let $\mathbf{P}$ be a real vector and denote the radius of convergence of the Taylor series of $\varphi_1(u_1,u_2)$ (φ_2) around $\mathbf{P}$ by $\tilde{\mathbf{c}}_1(\mathbf{P})$ ($\tilde{\mathbf{c}}_2(\mathbf{P})$). Define the $\mathfrak{k}$th derivative with respect to u_1 and the kth derivative with respect to u_2 with

$$\varphi^{(\mathfrak{k},k)}(u_1,u_2) = i^{\mathfrak{k}+k}\int_{\mathbb{R}} e^{i(u_1x_1+u_2x_2)}x_1^{\mathfrak{k}}x_2^{k}p(x_1,x_2)dx_1dx_2. \tag{3.21}$$

Note that $\alpha_{\mathfrak{k},k} = i^{-(\mathfrak{k}+k)}\varphi^{\mathfrak{k},k}(0,0)$. We see

$$\left|\varphi^{(2\mathfrak{k},2k)}(\mathbf{P})\right| \leq \alpha_{2\mathfrak{k},2k} \quad \text{and} \quad \left|\varphi^{(2\mathfrak{k}-1,2k-1)}(\mathbf{P})\right| \leq \beta_{2\mathfrak{k}-1,2k-1}, \tag{3.22}$$

$$\left|\varphi^{(2\mathfrak{k}-1,2k)}(\mathbf{P})\right| \leq \beta_{2\mathfrak{k}-1,2k} \quad \text{and} \quad \left|\varphi^{(2\mathfrak{k},2k-1)}(\mathbf{P})\right| \leq \beta_{2\mathfrak{k},2k-1}. \tag{3.23}$$

Hence,

$$\tilde{\mathbf{c}}_1(\mathbf{P}) \geq \tilde{\mathbf{c}}_1(0,0) \geq \tilde{\mathbf{c}}_0 \text{ and } \tilde{\mathbf{c}}_2(\mathbf{P}) \geq \tilde{\mathbf{c}}_2(0,0) \geq \tilde{\mathbf{c}}_0, \tag{3.24}$$

and the Taylor series of $\varphi_1(u_1,u_2)$ and $\varphi_2(u_1,u_2)$ around $\mathbf{P}$ converge therefore in circles of radii at least equal to $\tilde{\mathbf{c}}_0$. The same is true for the expansion of $\varphi(u_1,u_2)$ around $\mathbf{P}$ so that $\varphi(u_1,u_2)$ is regular at least in the strip $|\Im(\mathbf{u})| \leq \tilde{\mathbf{c}}_0$.
We have already shown that $\sum_{\mathfrak{k}=0}^{\infty}\sum_{k=0}^{\infty}\frac{|\varpi_1|^{\mathfrak{k}}|\varpi_2|^{k}\beta_{\mathfrak{k}i,k}}{\mathfrak{k}!k!}$ converges for $|\boldsymbol{\varpi}| < \tilde{\mathbf{c}}_0$. Thus,

$$\begin{aligned}
& \sum_{\mathfrak{k}=0}^{\infty}\sum_{k=0}^{\infty}\frac{|\varpi_1|^{\mathfrak{k}}|\varpi_2|^{k}\beta_{\mathfrak{k},k}}{\mathfrak{k}!k!} \\
\geq & \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{|\varpi_1|^{\mathfrak{k}}|\varpi_2|^{k}}{\mathfrak{k}!k!}\int_{-\mathfrak{A}}^{\mathfrak{A}}\int_{-\mathfrak{A}}^{\mathfrak{A}}|x_1|^{\mathfrak{k}}|x_2|^{k}p(x_1,x_2)dx_1dx_2 \\
= & \int_{-\mathfrak{A}}^{\mathfrak{A}}\int_{-\mathfrak{A}}^{\mathfrak{A}} e^{|x_1\varpi_1|+|\varpi_2 x_2|}p(x_1,x_2)dx_1dx_2,
\end{aligned} \tag{3.25}$$

for any $\mathfrak{A}$ and $|\boldsymbol{\varpi}| < \tilde{\mathbf{c}}_0$. Therefore, the integral $\int_{\mathbb{R}^2} e^{i(u_1x_1+u_2x_2)}p(x_1,x_2)dx_1dx_2$, where $\mathbf{u} = \mathbf{w} + i\boldsymbol{\varpi}$, is convergent for any $|\boldsymbol{\varpi}| < \tilde{\mathbf{c}}_0$ and any $\mathbf{w}$. This integral is a regular function in its strip of convergence and agrees with $\varphi(\mathbf{u})$ for real $\mathbf{u}$. Therefore, it must agree with $\varphi(u_1,u_2)$ also for complex values $\mathbf{u}$, provided $|\Im(\mathbf{u})| < \tilde{\mathbf{c}}_0$ (see Lukacs [84], p. 131f). The integral converges in a strip $\mathfrak{a} < \Im(\mathbf{u}) < \mathfrak{b}$, where $|\mathfrak{a}| \geq \tilde{\mathbf{c}}_0, \mathfrak{b} \geq \tilde{\mathbf{c}}_0$ and is regular inside this strip. □

Remark 10. *As the moment generating function $\bar{M}(\mathbf{w})$ is given by $\bar{M}(\mathbf{w}) = \varphi(i\mathbf{w})$ we can derive from Theorem 32 that if $\varphi(\mathbf{u})$, $\mathbf{u} = \mathbf{w} + i\boldsymbol{\varpi}$, is regular in a neighbourhood of the origin the moment generating function exists in a (real) neighbourhood of 0 as long as $\boldsymbol{\varpi}$ is in the strip of convergence of φ. Furthermore, there exists a complex analytic extension $\bar{M}(\boldsymbol{\varpi} - i\mathbf{w})$ to an open set $\mathcal{D} \subset \mathbb{C}$ in the neighbourhood of the origin. See also [31].*

Thus, for $\mathfrak{a} < \Im(\mathbf{u}) < \mathfrak{b}$ the characteristic function of the processes $X(t)$ is identical to the multi-dimensional generalized Fourier transform of the transition density. To apply these concepts to option pricing we need the following conclusions about the convergence of the inversion of the Fourier transform of a convolution.

Theorem 33. *(Fourier inversion of a product of functions with $\Re(\mathbf{u}) \in \mathbb{R}^d$)*

Let $\hat{f}(\mathbf{u}) \in \mathbf{L}^1(\mathbb{R}^d)$ be the Fourier transform of a function $f(\mathbf{x})$, $f \in \mathbf{L}^1(\mathbb{R}^d)$, and let $g(\mathbf{x})$ and its Fourier transform $\hat{g}(\mathbf{u})$ belong to $\mathbf{L}^1(\mathbb{R}^d)$. Then $\hat{f}(\mathbf{u}) \cdot \hat{g}(\mathbf{u})$ belongs to $\mathbf{L}^1(\mathbb{R}^d)$ and its Fourier inversion is $\int_{\mathbb{R}^d} g(\mathbf{x}')f(\mathbf{x}-\mathbf{x}')d\mathbf{x}'$. The Fourier inversion is defined by

$$\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\langle \mathbf{x},\mathbf{u}\rangle}\hat{f}(\mathbf{u})\hat{g}(\mathbf{u})d\mathbf{u}, \tag{3.26}$$

where $\langle \mathbf{x},\mathbf{u}\rangle$ is the scalar product of $\mathbf{u}$ and $\mathbf{x}$. If the map $x_1, x_2 \to \int_{\mathbb{R}^d} g(\mathbf{x}')f(\mathbf{x}-\mathbf{x}')d\mathbf{x}'$ is continuous the convergence is point-wise.

Proof.

Let $g(\mathbf{x})$ be an integrable function in $\mathbb{R}^d$. The Fourier transform in $\mathbb{R}^d$ of g, $\hat{g}$, is bounded by $\frac{1}{(2\pi)^{\frac{d}{2}}}\|g\|_1$ (see [80], p. 325). Furthermore, we also assume $\hat{f}(\mathbf{u}), \hat{g}(\mathbf{u}) \in \mathbf{L}^1$. Hence, the product $\hat{f}\hat{g}$ is Lebesgue integrable (see [80], p. 243). Thus,

$$\begin{aligned}
\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\langle \mathbf{x},\mathbf{u}\rangle}\hat{f}(\mathbf{u})\hat{g}(\mathbf{u})d\mathbf{u} &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\langle \mathbf{x},\mathbf{u}\rangle}\hat{f}(\mathbf{u})\int_{\mathbb{R}^d} g(\mathbf{x}')e^{i\langle \mathbf{x}',\mathbf{u}\rangle}d\mathbf{x}'d\mathbf{u}\\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} g(\mathbf{x}')\int_{\mathbb{R}^d} e^{-i\langle \mathbf{x}-\mathbf{x}',\mathbf{u}\rangle}\hat{f}(\mathbf{u})d\mathbf{u}d\mathbf{x}'\\
&= \int_{\mathbb{R}^d} g(\mathbf{x}')f(\mathbf{x}-\mathbf{x}')d\mathbf{x}'.
\end{aligned} \tag{3.27}$$

The inversion is justified by absolute convergence. See also [113], p. 51. If the map $x_1, x_2 \to \int_{\mathbb{R}^d} g(\mathbf{x}')f(\mathbf{x}-\mathbf{x}')d\mathbf{x}'$ is continuous the convergence is point-wise (see [80], p. 327f): Let $\int_{\mathbb{R}^d} g(\mathbf{x}')f(\mathbf{x}-\mathbf{x}')d\mathbf{x}'$ be continuous in $\mathbf{P}_0$. As (3.26) is valid nearly everywhere there exists a sequence (S_n) with $S_n \to \mathbf{P}_0$ in such a way that (3.26) is true in the points of S_n. The integral in (3.26) is continuous (see [80], p. 282) because the Fourier transform is a continuous and bounded function. Thus, the identity is also true in $\mathbf{P}_0$. □

Adding these pieces together we can make the following statement about pricing of barrier options:

Theorem 34. *(Barrier option pricing in $\mathbb{R}^2$)*

Let us assume

i. the setting described in Equation (3.2),

ii. the existence of an affine analytic characteristic function $\varphi(\tau, \mathbf{u}, \mathbf{z})$ of the respective model in the variables $z_1 := \frac{1}{\sqrt{1-\rho^2}}\left(\frac{\ln\frac{S_1 e^{\int_t^T r(s)ds}}{K_1} - \ln\frac{B_1(T)}{K_1}}{\sigma_1} - \rho z_2\right)$

and $z_2 := \frac{\ln \frac{S_2 e^{\int_t^T r(s)ds}}{K_2} - \ln \frac{B_2(T)}{K_2}}{\sigma_2}$, *which is regular in a neighbourhood* $S_\varphi = \{\mathbf{u} = \mathbf{w} + i\boldsymbol{\varpi} : \boldsymbol{\varpi} \in (\mathfrak{a}_\varphi, \mathfrak{b}_\varphi)\}$, $\mathfrak{a}_\varphi < 0, \mathfrak{b}_\varphi > 0$ *of the origin, and integrable,*

iii. *the generalized Fourier transform* $\hat{\mathfrak{h}}(\mathbf{x})$ *of the transformed payoff function* $e^{-c_1 x_1 - c_2 x_2} g(\mathbf{x})$ *exists in a space* $S_g = \{\mathbf{u} = \mathbf{w} + i\boldsymbol{\varpi} : \boldsymbol{\varpi} \in (\mathfrak{a}_g, \mathfrak{b}_g)\}$, *is integrable for* $|\mathbf{x}| < \infty$, *and*

iv. $\rho = -\cos(\frac{\pi}{n})$, *where* n *is a natural number and* $n > 1$.

If the space $S_C \equiv S_\varphi \cap S_g$ *is not empty, then the barrier option value (3.4) is given by*

$$\begin{aligned} C_B\left(t, S_1, S_2, B_1, B_2, v\right) &= \frac{e^{x_1 c_1 + x_2 c_2 - \int_t^T r(s)ds}}{4\pi^2 \sigma_1 \sigma_2 \sqrt{1-\rho^2}} \sum_{k=0}^{n-1} \int_{i\varpi_2 - \infty}^{i\varpi_2 + \infty} \int_{i\varpi_1 - \infty}^{i\varpi_1 + \infty} \Big(\hat{\mathfrak{h}}(u_1, u_2) \\ &\quad \big(\varphi(\tau, \mathbf{u}, -\mathbf{z}_k, v) - \varphi(\tau, \mathbf{u}, -\mathbf{z}_k{}^-), v \big) \\ &\quad e^{\left(iu_1\left(-\frac{b_1}{\sigma_1\sqrt{1-\rho^2}} + \frac{b_2 \rho}{\sigma_2 \sqrt{1-\rho^2}}\right) - iu_2 \frac{b_2}{\sigma_2}\right)} \Big) du_1 du_2, \; \mathbf{u} \in S_C, \end{aligned} \tag{3.28}$$

where

$$\begin{aligned} z_{k1}^{(-)} &= r_p \cos\left(\frac{2k\pi}{n} \overset{+}{(-)} \theta_p\right), & z_{k2}^{(-)} &= r_p \sin\left(\frac{2k\pi}{n} \overset{+}{(-)} \theta_p\right), \\ x_i &= \ln \frac{S_i e^{\int_t^T r(s)ds}}{K_i}, & b_i &= \ln \frac{B_i(T)}{K_i}, \\ c_1 &= \frac{\sigma_1 - \sigma_2 \rho}{2\sigma_1(1-\rho^2)}, & c_2 &= \frac{\sigma_2 - \sigma_1 \rho}{2\sigma_2(1-\rho^2)}, \end{aligned}$$

$$\begin{aligned} \hat{\mathfrak{h}}(u_1, u_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{+\infty} e^{-x_1' c_1 - x_2' c_2} g(x_1', x_2') \\ &\quad e^{\left(iu_1 \frac{x_1'}{\sigma_1 \sqrt{1-\rho^2}} + \frac{x_2'}{\sigma_2}\left(-\frac{\rho}{\sqrt{1-\rho^2}} iu_1 + iu_2\right)\right)} dx_1' dx_2', \; \mathbf{u} \in S_g, \\ \varphi(\tau, \mathbf{u}, \mathbf{z}, v) &= \exp\left\{ iu_1 z_1 + iu_2 z_2 + V(\tau, \mathbf{u}, v) \right\}, \end{aligned}$$

where $V(\tau, \mathbf{u})$ *satisfies the System (3.36). The price converges point-wise if the map* $S_1, S_2 \to C_B(t, S_1, S_2, B_1, B_2, v)$ *is continuous.*

Proof.

We assume the settings described in (3.2) (see Theorem 34, i.). By introducing the following transformations (see also Appendix A.1.1)

$$\begin{aligned} Z(t, x_1, x_2, b_1, b_2, v) &:= e^{-c_1 x_1 - c_2 x_2} G(t, x_1, x_2, b_1, b_2, v), \\ c_1 &:= \frac{\sigma_1 - \sigma_2 \rho}{2\sigma_1(1-\rho^2)}, \\ c_2 &:= \frac{\sigma_2 - \sigma_1 \rho}{2\sigma_2(1-\rho^2)}, \end{aligned}$$

we can reduce the PDE problem (3.6) to

$$\begin{cases} \frac{\partial Z}{\partial t} + \frac{1}{2}\sigma_1^2 v^{2\nu} \frac{\partial^2 Z}{\partial x_1^2} + \frac{1}{2}\sigma_2^2 v^{2\nu} \frac{\partial^2 Z}{\partial x_2^2} - v^{2\nu} \frac{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}{8(1-\rho^2)} Z \\ +\rho\sigma_1\sigma_2 v^{2\nu} \frac{\partial^2 Z}{\partial x_1 \partial x_2} + \frac{1}{2}\sigma_v^2 v^{2\gamma} \frac{\partial^2 Z}{\partial v^2} + \kappa(\zeta - v)\frac{\partial Z}{\partial v} = 0, \\ \\ Z(t, b_1, x_2, b_1, b_2, v) = 0, \; Z(t, x_1, b_2, b_1, b_2, v) = 0, \\ Z(T, x_1, x_2, b_1, b_2, v) = e^{-c_1 x_1 - c_2 x_2} g(x_1, x_2) \mathbb{1}_{\{\iota_1 > T, \; \iota_2 > T\}}. \end{cases} \tag{3.29}$$

The transition probability density function $p(t', \mathbf{x}', v', t, \mathbf{x}, v)$ is governed by the following Kolmogorov backward equation and boundary conditions in the backward variables

$$\begin{aligned} \frac{\partial}{\partial \tau} p(\tau, x_1', x_2', v', x_1, x_2, v) &= \frac{1}{2}\sigma_1^2 v^{2\nu} \frac{\partial^2}{\partial x_1^2} p(\tau, x_1', x_2', v', x_1, x_2, v) \\ &+ \frac{1}{2}\sigma_2^2 v^{2\nu} \frac{\partial^2}{\partial x_2^2} p(\tau, x_1', x_2', v', x_1, x_2, v) \\ &+ \rho\sigma_1\sigma_2 v^{2\nu} \frac{\partial^2}{\partial x_1 \partial x_2} p(\tau, x_1', x_2', v', x_1, x_2, v) \\ &- v^{2\nu} \frac{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}{8(1-\rho^2)} p(\tau, x_1', x_2', v', x_1, x_2, v) \\ &+ \frac{1}{2}\sigma_v^2 v^{2\gamma} \frac{\partial^2 p(\tau, x_1', x_2', v', x_1, x_2, v)}{\partial v^2} \\ &+ \kappa(\zeta - v) \frac{\partial p(\tau, x_1', x_2', v', x_1, x_2, v)}{\partial v}, \end{aligned} \tag{3.30}$$

where $\tau = t' - t$, $t' > t$. We apply the following initial and boundary conditions

$$\begin{aligned} p(0, x_1', x_2', v', x_1, x_2, v) &= \delta(x_1' - x_1)\delta(x_2' - x_2)\delta(v' - v), \\ p(\tau, x_2', v', b_1, x_2, v) &= 0, \\ p(\tau, x_1', x_2', v', x_1, b_2, v) &= 0. \end{aligned}$$

The payoff of the derivative does not depend on the volatility, therefore we can proceed with

$$q(\tau, \mathbf{x}', \mathbf{x}, v) = \int_0^\infty p(\tau, \mathbf{x}', v', \mathbf{x}, v) dv'. \tag{3.31}$$

q solves the Kolmogorov Equation (3.30) (see Definition 38) supplied with the initial and boundary conditions

$$\begin{aligned} q(0, x_1', x_2', x_1, x_2, v) &= \delta(x_1' - x_1)\delta(x_2' - x_2), \\ q(\tau, x_1', x_2', b_1, x_2, v) &= 0, \\ q(\tau, x_1', x_2', x_1, b_2, v) &= 0. \end{aligned}$$

In the following we reduce this PDE following the lines of He et al. [64]. We eliminate the mixing term by a transformation of coordinates. By using the transformations $z_1 := \frac{1}{\sqrt{1-\rho^2}}(\frac{x_1-b_1}{\sigma_1} - \rho\frac{x_2-b_2}{\sigma_2})$ and $z_2 := \frac{x_2-b_2}{\sigma_2}$ we map the vertical axis to the line $z_2 = -\frac{\sqrt{1-\rho^2}}{\rho} z_1$, while the second (horizontal) boundary is only translated to $z_2 = 0$. Thus, $q(\tau, z_1', z_2', z_1, z_2, v)$ satisfies the following PDE

$$\begin{aligned} \frac{\partial q}{\partial \tau} &= \frac{1}{2} v^{2\nu} \frac{\partial^2 q}{\partial z_1^2} + \frac{1}{2} v^{2\nu} \frac{\partial^2 q}{\partial z_2^2} - v^{2\nu} \frac{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}{8(1-\rho^2)} q \\ &\quad + \frac{1}{2}\sigma_v^2 v^{2\gamma} \frac{\partial^2 q}{\partial v^2} + \kappa(\zeta - v)\frac{\partial q}{\partial v}, \end{aligned} \tag{3.32}$$

with the following initial and boundary conditions

$$q(0, z_1', z_2', z_1, z_2, v) = \delta(z_1' - z_1)\delta(z_2' - z_2), \tag{3.33}$$
$$q(\tau, z_1', z_2', z_1, 0, v) = 0,$$
$$q(\tau, z_1', z_2', z_1, -\frac{\sqrt{1-\rho^2}}{\rho} z_1, v) = 0. \tag{3.34}$$

We derive a solution of this PDE for particular values of the correlation, $\rho = -\cos\left(\frac{\pi}{n}\right)$ where n is any natural number (see Theorem 34, iv.). The idea is to find a solution $\bar{G}^F$ for (3.32) and (3.33) in the whole plane first and restrict it to the actual space (3.34) it is defined for by using symmetries.

Due to the fact that for certain values of ν and γ the PDE is linear in v, we can guess affine solutions for $\bar{G}^F$, i.e. the free space probability density, for those models.

$$\begin{aligned} \bar{G}^F(\tau, z_1', z_2', z_1, z_2, v) &= \frac{1}{4\pi^2} \int_{i\varpi_2-\infty}^{i\varpi_2+\infty} \int_{i\varpi_1-\infty}^{i\varpi_1+\infty} \exp\{iu_1(z_1' - z_1) + iu_2(z_2' - z_2) \\ &\quad + V(\tau, \mathbf{u}, v)\} du_1 du_2, \quad \mathbf{u} = \mathbf{w} + i\boldsymbol{\varpi} \in S_\varphi, \end{aligned} \tag{3.35}$$

where $\mathbf{u} = (u_1, u_2)$ and S_φ describes a space in a neighbourhood of the origin, parallel to the real axis, in which the integrand is regular. Inserting (3.35) in (3.32) and (3.33) we see that $V(\tau, \mathbf{u}, v)$ has to satisfy the following system.

$$\begin{aligned}
\frac{\partial V}{\partial \tau} &= -\frac{1}{2}v^{2\nu}u_1^2 - \frac{1}{2}v^{2\nu}u_2^2 - v^{2\nu}\frac{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}{8(1-\rho^2)} \\
&\quad + \frac{1}{2}\sigma_v^2 v^{2\gamma}\left(\frac{\partial^2 V}{\partial v^2} + \left(\frac{\partial V}{\partial v}\right)^2\right) + \kappa(\zeta - v)\frac{\partial V}{\partial v}, \\
V(0, \mathbf{u}, v) &= 0.
\end{aligned} \tag{3.36}$$

Since the process is affine $V(\tau, \mathbf{u}, v)$ can be denoted by $\exp\{A_0(\tau, u) + \sum_{i=1}^k A_i(\tau, u)v^i\}$, where in the sum $A_i(\tau, u)$ are multiplied by v raised to the power of i, $i \leq k$ and (3.36) breaks then down into a system of Riccati equations (see [75]). This solution is closely related to the concept of characteristic functions: To see this, we Fourier transform PDE (3.32) and the initial conditions (3.33), i.e. $\varphi(\tau, \mathbf{u}, v) := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(u_1z_1+u_2z_2)}q(\tau, \mathbf{z}, v)dz_1dz_2$ or respectively $q(\tau, \mathbf{z}, v) := \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(u_1z_1+u_2z_2)}\varphi(\tau, \mathbf{u}, v)du_1du_2$. Thus, with (A.21) we get

$$\begin{aligned}
\frac{\partial \varphi}{\partial \tau} &= -\frac{1}{2}v^{2\nu}u_1^2\varphi - \frac{1}{2}v^{2\nu}u_2^2\varphi - v^{2\nu}\frac{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}{8(1-\rho^2)}\varphi \\
&\quad + \frac{1}{2}\sigma_v^2 v^{2\gamma}\frac{\partial^2 \varphi}{\partial v^2} + \kappa(\zeta - v)\frac{\partial \varphi}{\partial v},
\end{aligned} \tag{3.37}$$

with the following initial condition

$$\varphi(0, u_1, u_2, v) = e^{iu_1z_1' + iu_2z_2'}. \tag{3.38}$$

Inserting

$$\varphi(\tau, \mathbf{u}, v) = \exp\{iu_1z_1' + iu_2z_2' + V(\tau, \mathbf{u}, v)\} \tag{3.39}$$

in Equation (3.37) and initial Condition (3.38) we see that both are satisfied. Thus, φ is the characteristic function of z_1' and z_2'. According to Theorem 32 an analytic characteristic function, which is regular in a neighbourhood of the origin, is also regular in a horizontal space $S_\varphi = \{\mathbf{u} = \mathbf{w} + i\boldsymbol{\varpi} : \boldsymbol{\varpi} \in (\mathfrak{a}_\varphi, \mathfrak{b}_\varphi)\}$ and can there be represented by a Fourier integral (see ii.). Hence, in this case φ is also the Fourier transform of the transition density of z_1 and z_2 (see Theorem 31). The free space solution (3.35) to the Kolmogorov backward Equation (3.32) can, thus, be interpreted as the Fourier inversion in the backward variables (z_1, z_2) of the characteristic function/Fourier transform in the forward variables (z_1', z_2'). $\bar{G}^F$ is a bounded continuous density if $\varphi \in \mathbf{L}^1$ (see Theorem 13) (see ii.), and is, thus, the transition density of z_1', z_2' starting in z_1 and z_2. The solution for q, satisfying the boundary conditions, can be found from $\bar{G}^F$ using the method of images as described in [64]. By using the transformations $z_1 := \frac{1}{\sqrt{1-\rho^2}}(\frac{x_1-b_1}{\sigma_1} - \frac{x_2-b_2}{\sigma_2})$ and $z_2 := \frac{x_2-b_2}{\sigma_2}$ we mapped the vertical axis to the line $z_2 = -\frac{\sqrt{1-\rho^2}}{\rho}z_1$, while the first (horizontal) boundary is translated to $z_2 = 0$. By transforming these to polar coordinates the vertical boundary is described by the angle $\tan\theta_p = -\frac{\sqrt{1-\rho^2}}{\rho}$ and the horizontal boundary by $\theta_p = 0$. When $\rho = -\cos\left(\frac{\pi}{n}\right)$ (see Theorem 34, iv.), the angles take the special values $\beta_{p,n} = \frac{\pi}{n}$; n = 1; 2,... For these angles, a method of images solution to the PDE is possible. Some $\bar{G}_{\mathbf{k}}$

$$\bar{G}^{\pm}_{\mathbf{k}}(\tau, z_1', z_2', z_1, z_2, v) = \pm\bar{G}^F\left(\tau, z_1', z_2', z_{k1}, z_{k2}, v\right)$$

satisfy the PDE (3.32) with initial condition for arbitrary z_{k1}, z_{k2}:

$$\bar{G}^{\pm}_{\mathbf{k}}(0, z_1', z_2', z_1, z_2, v) = \pm\delta\left(z_1' - z_{k1}\right)\delta\left(z_2' - z_{k2}\right).$$

As the PDE is linear in $\bar{G}^F$, any linear combination of these $\bar{G}^{\pm}_k$'s, with different (starting) values of (z_{k1}, z_{k2}) also satisfies the PDE. For the particular solution we take a combination of n $\bar{G}^{\pm}_k$. We have to find this particular solution that also satisfies the boundary and initial condition.

Consider the case with $\beta = \frac{\pi}{3}$, i.e. $\rho = -\frac{1}{2}$. The first hextant of Figure 3.1 is the region we want to solve the PDE for, i.e. $\theta_p \in [0, \frac{\pi}{3}]$. In this region a plus symbol is positioned at $P_0 = (z_{01} = r_p\cos(\theta_p), z_{02} = r_p\sin(\theta_p))$. This point makes an angle θ_p with respect to the z_1-axis and is located at a distance r_p. Let us denote the lines, which bound the first hextant by L_1 ($\theta_p = 0$) and L_2 ($\theta_p = \frac{\pi}{3}$). For the other functions we choose z_{k1}, z_{k2} at a distance r_p from the origin and with the angles $\theta_p + \frac{2\pi}{3}, \theta_p + \frac{4\pi}{3}$ and $-\theta_p, -\theta_p + \frac{2\pi}{3}, -\theta_p + \frac{4\pi}{3}$. As $\bar{G}^+_0$ is the only function with (z_{k1}, z_{k2}) positioned in the first hextant the initial condition is satisfied. We claim that q is given by the combination of those 6 functions.

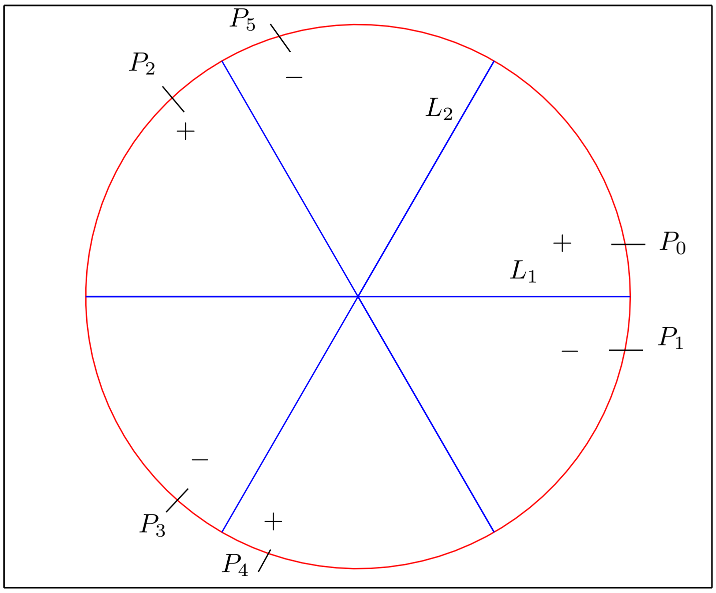

Figure 3.1: Method of images in a circle (with $\beta = \frac{\pi}{3}$).

As already seen this combination fulfils the initial condition and it also satisfies the PDE. Additionally, we know that:

$$
\begin{aligned}
r_p \cos(\overset{+}{-}\, \theta_p + \frac{2k\pi}{n}) &= r_p(\cos(\theta_p)\cos(\frac{2\pi k}{n}) \overset{-}{+} \sin(\theta_p)\sin(\frac{2\pi k}{n})) \\
&= z_1 \cos(\frac{2\pi k}{n}) \overset{-}{+} z_2 \sin(\frac{2\pi k}{n}), \\
r_p \sin(\overset{+}{-}\, \theta_p + \frac{2k\pi}{n}) &= r_p(\overset{+}{-} \sin(\theta_p)\cos(\frac{2\pi k}{n}) + \cos(\theta_p)\sin(\frac{2\pi k}{n})) \\
&= \overset{+}{-}\, z_2 \cos(\frac{2\pi k}{n}) + z_1 \sin(\frac{2\pi k}{n}),
\end{aligned}
$$

We can see from the symmetry of Figure 3.1 that the functions $\bar{G}_k^{\pm}$ cancel pairwise along the solid lines: To obtain zero-value at L_1 we add to our first function $\bar{G}_0^{+}$ the function $\bar{G}_0^{-}$ with the image point of $P_0 = (z_{01}, z_{02})$ in L_1 as starting point: $P_1 = (r_p \cos(-\theta_p), r_p \sin(-\theta_p)) = (z_{01}^{-}, z_{02}^{-})$. For example one can show for any function f in L_1, i.e. $\theta_p = 0$,

$$
\begin{aligned}
& f(z_1' - r_p\cos(\theta_p), z_2' - r_p \sin(\theta_p)) - f(z_1' - r_p \cos(-\theta_p), z_2' - r_p \sin(-\theta_p)) \\
= \; & f(z_1' - r_p \cos(\theta_p), z_2') - f(z_1' - r_p \cos(\theta_p), z_2') \\
= \; & 0.
\end{aligned}
\tag{3.40}
$$

To balance the point P_1 in L_2, the solid line with angle $\beta = \frac{\pi}{3}$, we add the function $\bar{G}_1^{+}$

with starting point $P_2 = (r_p \cos(\theta_p + 2\frac{\pi}{3}), r_p \sin(\theta_p + 2\frac{\pi}{3})) = (z_{11}, z_{12})$.
Furthermore to balance P_2 in L_1 we add the function $\bar{G}_2^-$ with starting value $P_3 = (r_p \cos(-\theta_p + 4\frac{\pi}{3}), r_p \sin(-\theta_p + 4\frac{\pi}{3})) = (z_{11}^-, z_{12}^-)$ and so on. See [19], p. 277f. Hence, the boundary conditions are also satisfied, and the combination of the six functions is the unique solution to the problem.
So in general, we slice a circle in $2n$ wedges with the same angles $\beta_{p,n} = \frac{\pi}{n}; n = 1; 2, \ldots$ The first wedge with $z_{01} = r_p \cos(\theta_p)$ and $z_{02} = r_p \sin(\theta_p)$ always relates to the space where we want to find a solution to the PDE. For this space the initial condition (3.33) is satisfied, as $z_1 = r_p \cos(\theta_p)$ and $z_2 = r_p \sin(\theta_p)$. Thus, the combination in fact fulfils the PDE and the initial condition. So we only need to show that the absorbing conditions are satisfied. By the symmetry of the wedges in the circle the $2n$ density functions cancel along the lines $\theta_p = 0$ and $\theta_p = \beta_{p,n}$ in pairs. Thus, the solution is given by

$$q(\tau, x_1', x_2', x_1, x_2, v) = \frac{\sum_{k=0}^{n-1} \left(\bar{G}_k^+(\tau, z_1', z_2', z_{k1}, z_{k2}, v) + \bar{G}_k^-(\tau, z_1', z_2', z_{k1}^-, z_{k2}^-, v) \right)}{\sigma_1 \sigma_2 \sqrt{1-\rho^2}},$$

where $\frac{1}{\sigma_1 \sigma_2 \sqrt{1-\rho^2}}$ is the Jacobi matrix of the transformation from $\mathbf{x} \to \mathbf{z}$ which needs to be applied if we carry out the integration in the original variable $\mathbf{x}$ (see Theorem 31),

$$\bar{G}_k^{\pm}(\tau, z_1', z_2', z_{k1}^{(-)}, z_{k2}^{(-)}, v) = \overset{+}{(-)}\, \bar{G}^F \left(\tau, z_1', z_2', z_1 = z_{k1}{}^{(-)}, z_2 = z_{k2}{}^{(-)}, v \right),$$

$$z_{k1}{}^{(-)} = r_p \cos\left(\frac{2k\pi}{n} \overset{+}{(-)} \theta_p \right), \qquad z_{k2}{}^{(-)} = r_p \sin\left(\frac{2k\pi}{n} \overset{+}{(-)} \theta_p \right),$$
$$r_p = \sqrt{z_1^2 + z_2^2}, \qquad \tan\theta_p = \frac{z_2}{z_1},$$
$$z_1 = \frac{1}{\sqrt{1-\rho^2}} \left[\frac{x_1 - b_1}{\sigma_1} - \rho \left(\frac{x_2 - b_2}{\sigma_2} \right) \right] = z_{01}, \qquad z_2 = \frac{x_2 - b_2}{\sigma_2} = z_{02}. \tag{3.41}$$

From risk-neutral pricing, it follows

$$\begin{aligned}
C_B\left(t, S_1, S_2, B_1, B_2, v\right) &= e^{x_1 c_1 + x_2 c_2 - \int_t^T r(s)ds} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Big(e^{-x_1' c_1 - x_2' c_2} g(x_1', x_2') \\
& \quad q(\tau, x_1', x_2', v, x_1, x_2) \Big) dx_1' dx_2' \\
&\overset{(3.41)}{=} \frac{e^{x_1 c_1 + x_2 c_2 - \int_t^T r(s)ds}}{4\pi^2 \sigma_1 \sigma_2 \sqrt{1-\rho^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x_1' c_1 - x_2' c_2} g(x_1', x_2') \\
& \quad \sum_{k=0}^{n-1} \int_{i\varpi_2 - \infty}^{i\varpi_2 + \infty}\int_{i\varpi_1 - \infty}^{i\varpi_1 + \infty} \Big(\varphi(\tau, \mathbf{u}, \mathbf{z}' - \mathbf{z}_k, v) \\
& \quad -\varphi(\tau, \mathbf{u}, \mathbf{z}' - \mathbf{z}_k{}^{-}, v) \Big) du_1 du_2 dx_1' dx_2'. \qquad (3.42)
\end{aligned}$$

We have required that the payoff function and the characteristic function are Lebesgue integrable in the generalized sense in a space S_C (see ii. and iii.). Moreover, we know that the transformed payoff function is Lebesgue integrable (see iii.). Hence, $g(x_1', x_2')$ is bounded and the integrand is Lebesgue integrable (see [80], p. 243), and we can apply Fubini's theorem and change the sum and integrals if there exists a space $S_C = S_\varphi \cap S_g$. Then, the above expression can be simplified. We denote the Fourier transform of the complete payoff function by $\hat{\mathfrak{h}}(u_1, u_2)$

$$\begin{aligned}
\hat{\mathfrak{h}}(u_1, u_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{+\infty} e^{-x_1' c_1 - x_2' c_2} g(x_1', x_2') \\
& \quad e^{\left(iu_1 \frac{x_1'}{\sigma_1\sqrt{1-\rho^2}} + \frac{x_2'}{\sigma_2}\left(-\frac{\rho}{\sqrt{1-\rho^2}} iu_1 + iu_2\right)\right)} dx_1' dx_2', \ \mathbf{u} \in S_g. \qquad (3.43)
\end{aligned}$$

Then, using (3.39) and (3.35),

$$\begin{aligned}
C_B\left(t, S_1, S_2, B_1, B_2, v\right) &= \frac{e^{x_1 c_1 + x_2 c_2 - \int_t^T r(s)ds}}{4\pi^2 \sigma_1 \sigma_2 \sqrt{1-\rho^2}} \sum_{k=0}^{n-1} \int_{i\varpi_2 - \infty}^{i\varpi_2 + \infty}\int_{i\varpi_1 - \infty}^{i\varpi_1 + \infty} \Big(\hat{\mathfrak{h}}(u_1, u_2) \\
& \quad \left(\varphi(\tau, \mathbf{u}, -\mathbf{z}_k, v) - \varphi(\tau, \mathbf{u}, -\mathbf{z}_k{}^{-}, v) \right) \qquad (3.44) \\
& \quad e^{\left(iu_1\left(-\frac{b_1}{\sigma_1\sqrt{1-\rho^2}} + \frac{b_2 \rho}{\sigma_2\sqrt{1-\rho^2}}\right) - iu_2 \frac{b_2}{\sigma_2}\right)} \Big) du_1 du_2, \\
& \quad \mathbf{u} \in S_C = S_\varphi \cap S_g.
\end{aligned}$$

According to Theorem 33 together with (ii. and iii.) $C_B(t, S_1, S_2, B_1, B_2, v)$ converges point-wise to the actual solution if the function $S_1, S_2 \to C_B\left(t, S_1, S_2, B_1, B_2, v\right)$ is continuous in S_1 and S_2. □

Remark 11. *We know that*

$$q(\tau, x_1' - x_1, x_2' - x_2, v) = \frac{1}{4\pi^2\sigma_1\sigma_2\sqrt{1-\rho^2}} \sum_{k=0}^{n-1} \int_{i\varpi_2-\infty}^{i\varpi_2+\infty} \int_{i\varpi_1-\infty}^{i\varpi_1+\infty} e^{iu_1z_1'+iu_2z_2'} \big(\varphi(\tau, \mathbf{u}, -\mathbf{z}_k, v) - \varphi(\tau, \mathbf{u}, -\mathbf{z}_k{}^{-}, v)\big)\, du_1 du_2, \qquad (3.45)$$

and (3.43). Hence, (3.44) can be transformed to

$$\begin{aligned} C_B(t, S_1, S_2, B_1, B_2, v) &= \frac{e^{x_1c_1+x_2c_2-\int_t^T r(s)ds}}{4\pi^2\sigma_1\sigma_2\sqrt{1-\rho^2}} \sum_{k=0}^{n-1} \int_{i\varpi_2-\infty}^{i\varpi_2+\infty} \int_{i\varpi_1-\infty}^{i\varpi_1+\infty} \\ &\quad \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (e^{-c_1x_1'-c_2x_2'} g(x_1', x_2') e^{iu_1z_1'+iu_2z_2'} dx_1' dx_2') \right. \\ &\quad \left. \big(\varphi(\tau, \mathbf{u}, -\mathbf{z}_k, v) - \varphi(\tau, \mathbf{u}, -\mathbf{z}_k{}^{-}, v)\big) \right) du_1 du_2 \\ &= \frac{e^{x_1c_1+x_2c_2-\int_t^T r(s)ds}}{4\pi^2\sigma_1\sigma_2\sqrt{1-\rho^2}} \sum_{k=0}^{n-1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-c_1x_1'-c_2x_2'} g(x_1', x_2') \\ &\quad \int_{i\varpi_2-\infty}^{i\varpi_2+\infty} \int_{i\varpi_1-\infty}^{i\varpi_1+\infty} e^{iu_1z_1'+iu_2z_2'} \\ &\quad (\varphi(\tau, \mathbf{u}, -\mathbf{z}_k, v) - \varphi(\tau, \mathbf{u}, -\mathbf{z}_k{}^{-}, v)) du_1 du_2 dx_1' dx_2' \\ &= e^{x_1c_1+x_2c_2-\int_t^T r(s)ds} \qquad (3.46) \\ &\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-c_1x_1'-c_2x_2'} g(x_1', x_2') q(\tau, x_1'-x_1, x_2'-x_2, v) dx_1' dx_2'. \end{aligned}$$

3.4.2 Properties of selected two-dimensional affine characteristic functions

In this section we want to treat properties of two characteristic functions which are included in the general set-up of (3.2) and which fulfil the PDE (3.37), i.e. have an affine characteristic function. The first one is a Heston-type characteristic function with $\gamma = \nu = \frac{1}{2}$, the other model, in which the covariance is governed by an Ornstein-Uhlenbeck process, has actually no affine characteristics for (S_i, v), however, a characteristic function can be derived (see Kallsen [75]). In the following we analyse properties like the regularity at the origin in S_φ and integrability in this space for the Heston- and the Stein and Stein-type two-dimensional models.

Proposition 2. *(Heston-type characteristic function)*

The Heston-type characteristic function is defined by

$$\varphi_H(\tau, \mathbf{u}) = \exp\left\{ iu_1z_1 + iu_2z_2 + \frac{1}{\sigma_v^2}(A_H(\tau, \mathbf{u}) + B_H(\tau, \mathbf{u})v) \right\}, \qquad (3.47)$$

where

$$
\begin{aligned}
B_H(\tau,\mathbf{u}) &= \frac{(\kappa-\mathfrak{d})\left(1-\exp\left(-\mathfrak{d}\tau\right)\right)}{1-\frac{\kappa-\mathfrak{d}}{\kappa+\mathfrak{d}}\exp\left(-\mathfrak{d}\tau\right)} \\
&= \left(\kappa-\mathfrak{d}\frac{\sinh\left(\frac{\mathfrak{d}}{2}\tau\right)+\frac{\kappa}{\mathfrak{d}}\cosh\left(\frac{\mathfrak{d}}{2}\tau\right)}{\cosh\left(\frac{\mathfrak{d}}{2}\tau\right)+\frac{\kappa}{\mathfrak{d}}\sinh\left(\frac{\mathfrak{d}}{2}\tau\right)}\right), \qquad (3.48) \\
A_H(\tau,\mathbf{u}) &= \zeta\kappa\left(\left(\kappa-\mathfrak{d}\right)\tau-2\ln\left(\frac{1-\frac{\kappa-\mathfrak{d}}{\kappa+\mathfrak{d}}\exp\left(-\mathfrak{d}\tau\right)}{1-\frac{\kappa-\mathfrak{d}}{\kappa+\mathfrak{d}}}\right)\right) \\
&= \kappa\zeta\left(\kappa\tau-2\ln\left(\frac{\kappa}{\mathfrak{d}}\sinh\left(\frac{\mathfrak{d}}{2}\tau\right)+\cosh\left(\frac{\mathfrak{d}}{2}\tau\right)\right)\right), \qquad (3.49) \\
\mathfrak{d}=\mathfrak{d}(\mathbf{u}) &= \sqrt{\kappa^2+\sigma_v^2\left(\mathbf{u}^2+\frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}{4(1-\rho^2)}\right)}, \qquad (3.50)
\end{aligned}
$$

where $\mathbf{u}^2=u_1^2+u_2^2$. *For* $\frac{4\kappa^2}{\epsilon^2}>-\frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}{1-\rho^2}$ *the characteristic function,* φ_H*, is integrable and regular in a neighborhood of the origin* $[\mathbf{w}^-,\mathbf{w}^+]$ *with* $w_1^-<0<w_1^+$ *(*$w_2^-<0<w_2^+$ *respectively).*

Proof.

For the derivation of the characteristic function see Appendix A.1.2. For the analysis of the regularity in a space S_{φ_H} in the neighbourhood of the origin we follow the lines of del Bano Rollin et al. [31]. Define $\mathfrak{D}(\mathbf{u})=\mathfrak{d}(\mathbf{u})^2$. If we assume that $\frac{4\kappa^2}{\sigma_v^2}>-\frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}{1-\rho^2}$, we see that φ_H is well-defined and regular in a neighbourhood of the origin according to Cauchy's integral theorem (see Theorem 15), i.e. according to Remark 10 the moment generating function $\bar{M}(\varpi)$ exists in a (real) neighbourhood of 0 and there exists a complex analytic extension of $\bar{M}_H(\mathbf{w})=\varphi(i\mathbf{w})$ to an open set $\mathcal{D}\subset\mathbb{C}$ in the neighbourhood of the origin. Observing the components of the moment generating function we see that the main and most critical ingredient of the moment generating function is

$$
f(\mathbf{w}) = \frac{\kappa}{\bar{\mathfrak{d}}(\mathbf{w})}\sinh\left(\frac{\bar{\mathfrak{d}}(\mathbf{w})}{2}\tau\right)+\cosh\left(\frac{\bar{\mathfrak{d}}(\mathbf{w})}{2}\tau\right), \qquad (3.51)
$$

with

$$
\bar{\mathfrak{d}}(\mathbf{w})^2=\mathfrak{d}(i\mathbf{w})^2=\mathfrak{D}(-\mathbf{w})=\kappa^2+\sigma_v^2\left(-w_1^2-w_2^2+\frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}{4(1-\rho^2)}\right), \qquad (3.52)
$$

because this part is in the denominator of B_H and in the log-part of A_H. $\mathfrak{D}(\mathbf{w})$ with $\mathbf{w}\in\mathbb{R}^2$ is a cup-shaped inverted parabola with leading coefficient $-\sigma_v^2$ and real roots

$w_1^- < 0 < w_1^+$ ($w_2^- < 0 < w_2^+$ respectively) given by

$$w_{1,2}^{\pm} = \pm\sqrt{\frac{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}{4(1-\rho^2)} + \frac{\kappa^2}{\sigma_v^2}}, \tag{3.53}$$

i.e. $\bar{\mathfrak{d}}(\mathbf{w}) \geq 0$ on $[\mathbf{w}^-, \mathbf{w}^+]$. Hence, $f(\mathbf{z})$ is well-defined and analytic in such an interval. As f has no zeroes in $[\mathbf{w}^-, \mathbf{w}^+]$ the moment generating function exists there. We define

$$\mathrm{Def}_H(\mathbf{z}) = \left\{\mathbf{w} \in \mathbb{R}^2 : M_H(\mathbf{w}) = \mathbb{E}\left[e^{\langle \mathbf{wz} \rangle}\right] < \infty\right\}. \tag{3.54}$$

And thus, $[\mathbf{w}^-, \mathbf{w}^+] \subset \mathrm{Def}_H(\mathbf{z})$ and $S_{\varphi_H} = \{\mathbf{u} = \mathbf{w} + i\boldsymbol{\varpi} : \boldsymbol{\varpi} \in (\mathfrak{a}_{\varphi_H}, \mathfrak{b}_{\varphi_H})\}$, where $\mathfrak{a}_{\varphi_H} = \mathbf{w}^-$ and $\mathfrak{b}_{\varphi_H} = \mathbf{w}^+$. Please note that $\mathbb{E}\left(e^{z_1+z_2}\right)$ might not exist. We prove integrability of the characteristic function $\varphi_H(\mathbf{w} + i\boldsymbol{\varpi})$ with $\mathbf{u} = \mathbf{w} + i\boldsymbol{\varpi} \subset S_{\varphi_H}$. It suffices to show that the real part of the exponent decays like $|\mathbf{w}|$ (see [38]). In the following the real part of x is denoted $\Re(x)$. The real part of a square root in $\mathbb{R}^2$ is given by $\Re(\sqrt{w + i\varpi}) = \sqrt{\frac{|w+i\varpi|+w}{2}}$ (see [1], 3.7.27). Thus, for $-\Re(\mathfrak{d})$ we see

$$\begin{aligned} -\Re(\mathfrak{d}) &= -\frac{1}{\sqrt{2}}\Bigg(\Big|\kappa^2 + (w_1^2 + w_2^2)\sigma_v^2 - (\varpi_1^2 + \varpi_2^2)\sigma_v^2 + 2iw_1\varpi_1\sigma_v^2 + 2iw_2\varpi_2\sigma_v^2 \\ &\quad + \sigma_v^2\frac{\sigma_1 + \sigma_2 - 2\rho\sigma_1\sigma_2}{4(1-\rho^2)}\Big| \\ &\quad + \kappa^2 + (w_1^2 + w_2^2)\sigma_v^2 - (\varpi_1^2 + \varpi_2^2)\sigma_v^2 + \sigma_v^2\frac{\sigma_1 + \sigma_2 - 2\rho\sigma_1\sigma_2}{4(1-\rho^2)}\Bigg)^{\frac{1}{2}} \\ &\leq -\frac{1}{\sqrt{2}}\sqrt{\kappa^2 + (w_1^2 + w_2^2)\sigma_v^2 - (\varpi_1^2 + \varpi_2^2)\sigma_v^2 + \sigma_v^2\frac{\sigma_1 + \sigma_2 - 2\rho\sigma_1\sigma_2}{4(1-\rho^2)}} \\ &\leq -\frac{\tilde{c}}{\sqrt{2}}\,|\mathbf{w}|\,\sigma_v, \end{aligned} \tag{3.55}$$

where $\tilde{c}$ is a constant. Next, we take a closer look at the real part of B_H. Note that as long as $\kappa < \mathfrak{d}$ (which is always the case for $\mathbf{w} \to \infty$ the denominator of the fraction is bigger than the nominator. The fraction, however, approaches 1 as $\mathbf{w}$ grows due to the graphs of the cosh- and sinh- functions. Thus, we find

$$-\Re\left(\mathfrak{d}\frac{\sinh\left(\frac{\mathfrak{d}}{2}\tau\right) + \frac{\kappa}{\mathfrak{d}}\cosh\left(\frac{\mathfrak{d}}{2}\tau\right)}{\cosh\left(\frac{\mathfrak{d}}{2}\tau\right) + \frac{\kappa}{\mathfrak{d}}\sinh\left(\frac{\mathfrak{d}}{2}\tau\right)}\right) \leq -\frac{\tilde{c}_1}{\sqrt{2}}\,|\mathbf{w}|\,\sigma_v, \tag{3.56}$$

where $\tilde{c}_1$ is a constant. Another important ingredient is the log part. The following is true as w_1 and w_2 increase, i.e. we can assume $\kappa < \mathfrak{d}$:

$$\begin{aligned} \Re\left(-2\ln\left(\frac{\kappa}{\mathfrak{d}}\sinh\left(\frac{\mathfrak{d}}{2}\tau\right)+\cosh\left(\frac{\mathfrak{d}}{2}\tau\right)\right)\right) &\leq \Re\left(-2\ln\left(e^{\frac{\mathfrak{d}}{2}\tau}\right)\right) \\ &\leq -\frac{\tilde{c}\tau}{\sqrt{2}}\left|\mathbf{w}\right|\sigma_v. \end{aligned} \tag{3.57}$$

Hence, the exponent decays component-wise like $|\mathbf{w}|$ and the characteristic function is integrable in the space S_{φ_H}. □

Proposition 3. *(Stein and Stein-type characteristic function)*

The Stein and Stein-type characteristic function is defined by

$$\varphi_{S2}(\tau,\mathbf{u}) = \exp\left\{iu_1z_1 + iu_2z_2 + \frac{1}{\sigma_v^2}\left(A_{S2}(\tau,\mathbf{u}) + B_{S2}(\tau,\mathbf{u})v + C_{S2}(\tau,\mathbf{u})v^2\right)\right\}, \tag{3.58}$$

where

$$C_{S2}(\tau,\mathbf{u}) = \frac{1}{2}\left(\kappa - \mathfrak{d}\frac{\sinh(\mathfrak{d}\tau)+\frac{\kappa}{\mathfrak{d}}\cosh(d\tau)}{\cosh(\mathfrak{d}\tau)+\frac{\kappa}{\mathfrak{d}}\sinh(d\tau)}\right), \tag{3.59}$$

$$B_{S2}(\tau,\mathbf{u}) = \frac{1}{\mathfrak{d}}\left(\frac{\left(\kappa\zeta\mathfrak{d} - \frac{\kappa^3\zeta}{\mathfrak{d}}\right) + \kappa^2\zeta\left(\sinh(\mathfrak{d}\tau)+\frac{\kappa}{\mathfrak{d}}\cosh(\mathfrak{d}\tau)\right)}{\cosh(\mathfrak{d}\tau)+\frac{\kappa}{\mathfrak{d}}\sinh(\mathfrak{d}\tau)} - \kappa\zeta\mathfrak{d}\right), \tag{3.60}$$

$$\begin{aligned} A_{S2}(\tau,\mathbf{u}) = & -\frac{\sigma_v^2}{2}\ln\left(\cosh(\mathfrak{d}\tau)+\frac{\kappa}{\mathfrak{d}}\sinh(\mathfrak{d}\tau)\right) + \frac{\sigma_v^2}{2}\kappa\tau \\ & + \frac{\kappa^2\zeta^2\mathfrak{d}^2 - \kappa^4\zeta^2}{2\mathfrak{d}^3}\left(\frac{\sinh(\mathfrak{d}\tau)}{\cosh(\mathfrak{d}\tau)+\frac{\kappa}{\mathfrak{d}}\sinh(\mathfrak{d}\tau)} - \mathfrak{d}\tau\right) \\ & + \frac{\kappa^2\zeta\left(\kappa\zeta\mathfrak{d} - \frac{\kappa^3\zeta}{\mathfrak{d}}\right)}{\mathfrak{d}^3}\left(\frac{\cosh(\mathfrak{d}\tau)-1}{\cosh(\mathfrak{d}\tau)+\frac{\kappa}{\mathfrak{d}}\sinh(\mathfrak{d}\tau)}\right), \end{aligned} \tag{3.61}$$

with $\mathfrak{d}(\mathbf{u})$ given in (3.50). For $\frac{4\kappa^2}{\epsilon^2} > -\frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}{1-\rho^2}$ the characteristic function, φ_H, is integrable and regular in a neighborhood of the origin $[\mathbf{w}^-,\mathbf{w}^+]$ with $w_1^- < 0 < w_1^+$ ($w_2^- < 0 < w_2^+$ respectively).

Proof.

For a proof of the characteristic function see Appendix A.1.2. We proceed analogously to before with the analysis of the regularity in a space S_φ in the neighbourhood of the origin. Again, if we assume that $\frac{4\kappa^2}{\sigma_v^2} > -\frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}{1-\rho^2}$, we see that φ is well-defined and regular in a neighbourhood of the origin. The main ingredient of the moment generating function is

$$f(\mathbf{w}) \;=\; \frac{\kappa}{\bar{\mathfrak{d}}(\mathbf{w})}\sinh\left(\frac{\bar{\mathfrak{d}}(\mathbf{w})}{2}\tau\right)+\cosh\left(\frac{\bar{\mathfrak{d}}(\mathbf{w})}{2}\tau\right). \tag{3.62}$$

Thus, we can transfer the results from above and conclude that $[\mathbf{w}^-,\mathbf{w}^+]\subset \mathrm{Def}_{S2}(\mathbf{z})$ and $S_{\varphi_{S2}}=\{\mathbf{u}=\mathbf{w}+i\boldsymbol{\varpi}:\boldsymbol{\varpi}\in(\mathfrak{a}_{\varphi_{S2}},\mathfrak{b}_{\varphi_{S2}})\}$, where $\mathfrak{a}_{\varphi_{S2}}=\mathbf{w}^-$ and $\mathfrak{b}_{\varphi_{S2}}=\mathbf{w}^+$. With respect to the integrability we can see that φ_{S2} is composed of similar expressions as φ_H. Performing similar techniques one can easily see that the exponent decays like $|\mathbf{w}|$ and the characteristic function is integrable in the space $S_{\varphi_{S2}}$. □

Remark 12. *If the three-factor model degenerates to a two-factor GBM model then the characteristic function is given by*

$$\varphi_{GBM}(\mathbf{u}) \;=\; e^{\left(iu_1z_1+iu_2z_2-\frac{1}{2}\tau\left(u_1^2+u_2^2+(\frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}{4(1-\rho^2)})\right)\right)}. \tag{3.63}$$

Proof.

For $\kappa,\zeta,\sigma_v\to 0, v(0)\to 1$ we can show that the three-factor stochastic volatility Heston-type and the Stein and Stein-type model degenerate to the two-factor geometric Brownian motion model. First, we take the limits of $A_H(\tau,\mathbf{u})$, $B_H(\tau,\mathbf{u})$, $A_{S2}(\tau,\mathbf{u})$, $B_{S2}(\tau,\mathbf{u})$, and $C_{S2}(\tau,\mathbf{u})$ when $\kappa,\zeta,\sigma_v\to 0, v(0)\to 1$:

$$\begin{aligned}
\lim_{\kappa,\sigma_v,\zeta\to 0,v(0)\to 1}\frac{A_H(\tau,\mathbf{u})}{\sigma_v^2} &= 0,\\
\lim_{\kappa,\sigma_v,\zeta\to 0,v(0)\to 1}\frac{B_H(\tau,\mathbf{u})v(0)}{\sigma_v^2} &= \lim_{\kappa,\sigma_v,\zeta\to 0,v(0)\to 1}\frac{1}{\sigma_v^2}\frac{(\kappa^2-\mathfrak{d}^2)\left(1-\exp\left(-\mathfrak{d}\tau\right)\right)}{(\kappa+\mathfrak{d})-(\kappa-\mathfrak{d})\exp\left(-\mathfrak{d}\tau\right)}\\
&= \lim_{\kappa,\sigma_v,\zeta\to 0,v(0)\to 1}-\frac{\left(u_1^2+u_2^2+\frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}{4(1-\rho^2)}\right)(1-1+\mathfrak{d}\tau)}{(\kappa+\mathfrak{d})-(\kappa-\mathfrak{d})\left(1-\mathfrak{d}\tau\right)}\\
&= \lim_{\kappa,\sigma_v,\zeta\to 0,v(0)\to 1}-\frac{\left(u_1^2+u_2^2+\frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}{4(1-\rho^2)}\right)\mathfrak{d}\tau}{2\mathfrak{d}+\mathfrak{d}\tau\left(\kappa-\mathfrak{d}\right)}\\
&= -\frac{\left(u_1^2+u_2^2+\frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}{4(1-\rho^2)}\right)\tau}{2},\\
\lim_{\kappa,\sigma_v,\zeta\to 0,v(0)\to 1}\frac{A_{S2}(\tau,\mathbf{u})}{\sigma_v^2} &= 0,\\
\lim_{\kappa,\sigma_v,\zeta\to 0,v(0)\to 1}\frac{B_{S2}(\tau,\mathbf{u})v(0)}{\sigma_v^2} &= 0,\\
\lim_{\kappa,\sigma_v,\zeta\to 0,v(0)\to 1}\frac{C_{S2}(\tau,\mathbf{u})v(0)^2}{\sigma_v^2} &= -\frac{\left(u_1^2+u_2^2+\frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}{4(1-\rho^2)}\right)\tau}{2},
\end{aligned}$$

where the limit of $\frac{C_{S2}v(0)^2}{\sigma_v^2}$ is calculated analogously to the limes of $\frac{B_H v(0)}{\sigma_v^2}$. Thus, both characteristic functions, $\varphi_H(\tau, \mathbf{u})\ \varphi_{S2}(\tau, \mathbf{u})$, approach in the limit

$$\lim_{\kappa,\sigma_v,\zeta\to 0, v(0)\to 1} \varphi_{H,S2}(\mathbf{u}) = e^{iu_1 z_1 + iu_2 z_2 - \frac{\left(u_1^2+u_2^2+\frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}{4(1-\rho^2)}\right)\tau}{2}}. \tag{3.64}$$

It can be shown that this limit is consistent with the characteristic function of the two-factor GBM model. The characteristic function of the GBM model satisfies the following PDE

$$\frac{\partial q}{\partial \tau} = \frac{1}{2}\frac{\partial^2 q}{\partial z_1^2} + \frac{1}{2}\frac{\partial^2 q}{\partial z_2^2} - \frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}{8(1-\rho^2)}q, \tag{3.65}$$

with the initial condition $\varphi_{GBM}(0, \mathbf{u}) = e^{iu_1 z_1 + iu_2 z_2}$. It can be easily seen that (3.64) satisfies the terminal condition and also (3.65). □

This function is obviously regular in the whole space, and the moment-generating function exists everywhere. Furthermore, it is integrable in the whole space.

3.4.3 Pricing of two-asset double-digital options with Fourier techniques

Double-digital options pay out one unit provided that both underlyings are at maturity time T above their predefined strike price K_1 and K_2. In the case of a double-digital barrier option the payout is only guaranteed if both underlyings do not cross the given barriers $B_1(t) = B_1 e^{\int_0^t r(s)ds}$ and $B_2(t) = B_2 e^{\int_0^t r(s)ds}$ during the lifetime of the option.
The value of a digital option with time-dependent barriers on each of the underlyings is given as the expectation of two discounted indicator functions:

$$\begin{aligned} C_{2D}(t, S_1, S_2, B_1, B_2) &= \mathbb{E}_{\tilde{Q}}\big[e^{-\int_t^T r(s)ds}\mathbb{1}_{\{S_1(T)>K_1,\ S_2(T)>K_2\}} \\ &\quad \mathbb{1}_{\{\iota_1>T,\ \iota_2>T\}}\,|\mathcal{F}_t\,\big], \end{aligned} \tag{3.66}$$

where

$$\begin{aligned} \iota_1 &= \inf\left(t' \in (t,T] : S_1(t') \le B_1(t')\right), \\ \iota_2 &= \inf\left(t' \in (t,T] : S_2(t') \le B_2(t')\right). \end{aligned}$$

Thus, we have to Fourier transform $e^{-c_1x_1-c_2x_2}g_D(x_1,x_2) = e^{-c_1x_1-c_2x_2}\mathbb{1}_{\{x_1(T)>0\ \wedge\ x_2(T)>0\}}$. For $c_1, c_2 < 0$ the integrals are unbounded and $e^{-c_1x_1-c_2x_2}g_D$ does not belong to $\mathbf{L}^1$. Thus, the Fourier transform does not exist (see also Schmelzle [104] for a summary). This could be circumvented by introducing a damping factor like in Carr and Madan [18], Dempster and Hong [33], or Eberlein et al. [38]. The introduction of a damping factor has the same effect as the generalized Fourier transform along a contour in the complex space, parallel to the real axis, i.e. loosely speaking it renders an actual non-Lebesgue integrable payoff integrable (see also [82] and [104]):

$$\begin{aligned}\hat{\mathfrak{h}}_D(u_1,u_2) &= \int_0^\infty\int_0^\infty e^{-x_1'c_1-x_2'c_2}e^{\left(iu_1\frac{x_1'}{\sigma_1\sqrt{1-\rho^2}}+\frac{x_2'}{\sigma_2}\left(-\frac{\rho}{\sqrt{1-\rho^2}}iu_1+iu_2\right)\right)}dx_1'dx_2' \\ &= \frac{1}{i\frac{u_1}{\sigma_1\sqrt{1-\rho^2}}-c_1}\frac{1}{i\left(\frac{u_2}{\sigma_2}-\frac{\rho u_1}{\sigma_2\sqrt{1-\rho^2}}\right)-c_2}. \qquad (3.67)\end{aligned}$$

The integral only exists if $\mathbf{u}\in S_{g_D}=\Big\{\mathbf{u}=\mathbf{w}+i\boldsymbol{\varpi}:\varpi_1>-c_1\sigma_1\sqrt{1-\rho^2}\wedge$ $\varpi_2>\frac{\rho}{\sqrt{1-\rho^2}}\varpi_1-c_2\sigma_2\Big\}$. The same payoff function dampened by $e^{-\alpha_1^d z_1'-\alpha_2^d z_2'}$, where z_1' and z_2' are given in 3.41, can also be transformed

$$\begin{aligned}\hat{\mathfrak{h}}_D^*(u_1,u_2) &= \int_0^\infty\int_0^\infty e^{-\alpha_1^d\frac{x_1'}{\sigma_1\sqrt{1-\rho^2}}-\frac{x_2'}{\sigma_2}\left(-\frac{\rho}{\sqrt{1-\rho^2}}\alpha_1^d+\alpha_2^d\right)}e^{-x_1'c_1-x_2'c_2} \\ &\quad e^{\left(iu_1\frac{x_1'}{\sigma_1\sqrt{1-\rho^2}}+\frac{x_2'}{\sigma_2}(-\frac{\rho}{\sqrt{1-\rho^2}}iu_1+iu_2)\right)}dx_1'dx_2' \\ &= \frac{1}{i\frac{u_1}{\sigma_1\sqrt{1-\rho^2}}-\frac{\alpha_1^d}{\sigma_1\sqrt{1-\rho^2}}-c_1}\frac{1}{i\left(\frac{u_2}{\sigma_2}-\frac{\rho u_1}{\sigma_2\sqrt{1-\rho^2}}\right)+\frac{\rho\alpha_1^d}{\sigma_2\sqrt{1-\rho^2}}-\frac{\alpha_2^d}{\sigma_2}-c_2},\end{aligned}$$

where we have to choose α_i^d accordingly with $\alpha_1^d>-c_1\sigma_1\sqrt{1-\rho^2}$ and $\alpha_2^d>\frac{\rho}{\sqrt{1-\rho^2}}\alpha_1^d-c_2\sigma_2$ assuming $\mathbf{u}$ real. In Section 3.4.5 we show one of the possibilities to derive the price of the derivative with the dampened payoff function. The price of the option in the generalized Fourier framework can be given in the following corollary.

Corollary 4. *(Double-digital barrier option price)*

Let us assume

i. the setting described in Equation (3.2),

ii. the existence of an affine analytic characteristic function $\varphi(\tau,\mathbf{u},\mathbf{z})$, which is regular in a neighbourhood $S_\varphi=\{\mathbf{u}=\mathbf{w}+i\boldsymbol{\varpi}:\boldsymbol{\varpi}\in(\mathfrak{a}_\varphi,\mathfrak{b}_\varphi)\}$, $\mathfrak{a}_\varphi<0,\mathfrak{b}_\varphi>0$ of the origin, and integrable,

iii. that the generalized Fourier transform $\hat{\mathfrak{h}}_D(\mathbf{x})$ of the transformed payoff function $e^{-c_1x_1-c_2x_2}g_D(\mathbf{x})\mathbb{1}_{\{\iota_1>T,\iota_2>T\}}$ exists in a space $S_{g_D}=\{\mathbf{u}=\mathbf{w}+i\boldsymbol{\varpi}:$ $\varpi_1>-c_1\sigma_1\sqrt{1-\rho^2}\wedge\varpi_2>\frac{\rho}{\sqrt{1-\rho^2}}\varpi_1-c_2\sigma_2\}$, is there integrable for $|\mathbf{x}|<\infty$, and

iv. $\rho=-\cos(\frac{\pi}{n})$, where n is a natural number and $n>1$.

If the space $S_{C_D} \equiv S_\varphi \cap S_{g_D}$ is not empty, the double-digital barrier option value is given by

$$\begin{aligned} C_{2D}\left(t, S_1, S_2, B_1, B_2, v\right) &= \frac{e^{x_1 c_1 + x_2 c_2 - \int_t^T r(s)ds}}{4\pi^2 \sigma_1 \sigma_2 \sqrt{1-\rho^2}} \sum_{k=0}^{n-1} \int_{i\varpi_2-\infty}^{i\varpi_2+\infty} \int_{i\varpi_1-\infty}^{i\varpi_1+\infty} \hat{\mathfrak{h}}_D(\mathbf{u}) \\ &\quad \big(\varphi(\tau, \mathbf{u}, -\mathbf{z}_k, v) - \varphi(\tau, \mathbf{u}, -\mathbf{z}_k{}^-, v)\big) \\ &\quad e^{(iu_1(-\frac{b_1}{\sigma_1\sqrt{1-\rho^2}} + \frac{b_2\rho}{\sigma_2\sqrt{1-\rho^2}}) - iu_2\frac{b_2}{\sigma_2})} du_1 du_2, \; \mathbf{u} \in S_{C_D}, \end{aligned} \tag{3.68}$$

where φ, $\mathbf{z}_i^{(-)}$, x_i, b_i, and c_i are given in Theorem 34. $\hat{\mathfrak{h}}_D(u_1, u_2)$ is indicated in (3.67).

The corollary directly follows from Theorem 34 and Equation (3.67). Note that if $\rho < 0$, $c_1, c_2 > 0$. Thus, in those cases we can choose $\boldsymbol{\varpi} = 0$, i.e. we integrate on the real space. By using the method of images in the plane we can derive prices for double-digitals with a barrier on S_2 only: $C_{1D}(t, S_1, S_2, B_2) = \mathbb{E}_{\tilde{\mathbb{Q}}}\big[e^{-\int_t^T r(s)ds} \mathbb{1}_{\{S_1(T)>K_1 \,\wedge\, S_2(T)>K_2\}} \mathbb{1}_{\{\iota_2 > T\}} \,|\mathcal{F}_t\,\big]$. The solution is found by reflecting the characteristic function in the plane $(z_2(0) \to -z_2(0))$.

Corollary 5. *(Double-digital single-barrier option price)*

Let us assume the setting described in Equation (3.2) and the assumptions of Corollary 4, then the double-digital single-barrier (in S_2) option value is given for any ρ by

$$\begin{aligned} C_{1D}(t, S_1, S_2, B_2, v) &= \frac{e^{x_1 c_1 + x_2 c_2 - \int_t^T r(s)ds}}{4\pi^2 \sigma_1 \sigma_2 \sqrt{1-\rho^2}} \int_{i\varpi_2-\infty}^{i\varpi_2+\infty} \int_{i\varpi_1-\infty}^{i\varpi_1+\infty} \hat{\mathfrak{h}}_D(\mathbf{u}) \\ &\quad (\varphi(\tau, \mathbf{u}, -z_1, -z_2, v) - \varphi(\tau, \mathbf{u}, -z_1, z_2, v)) \\ &\quad e^{\left(iu_1 \frac{b_2\rho}{\sigma_2\sqrt{1-\rho^2}} - iu_2 \frac{b_2}{\sigma_2}\right)} du_1 du_2, \; \mathbf{u} \in S_{C_D}, \end{aligned} \tag{3.69}$$

where φ, x_i, b_2, c_i, are given in Theorem 34, and $\hat{\mathfrak{h}}_D(u_1, u_2)$ is given in (3.67).

The corollary directly follows from Theorem 34 and Equation (3.67). For details on the method of images in a half-space please refer to Appendix A.1.3.

We implement the above proposed models and pricing formulas. First, we compute prices arising from the three-factor stochastic volatility Model (3.2) using the formula in Theorem 4 with the Heston-type characteristic function while we set $\zeta, \sigma_v, \kappa \to 0$, $v(0) \to 1$. Thus, the three-factor model degenerates to the two-factor GBM Model (3.1). The resulting prices are compared in Table 3.1 to prices for the two-asset digital option, which we compute in the GBM Model (3.1) with a formula derived by He et al. [64]. The figures show the accuracy of the Fourier transform method in the degenerated case.

Table 3.1: Prices of the two-asset double-digital barrier option computed in the limit to bivariate normal distribution.

	GBM model			Degenerated 3-factor model		
	Correlation ρ					
Strike K_1	0	-0.5	-0.71	0	-0.5	-0.71
80	0.1049	0.0507	0.0288	0.1049	0.0507	0.0288
85	0.1032	0.0493	0.0277	0.1032	0.0493	0.0277
90	0.1001	0.0469	0.0258	0.1001	0.0469	0.0258
95	0.0960	0.0438	0.0234	0.0960	0.0438	0.0234
100	0.0912	0.0403	0.0208	0.0912	0.0403	0.0208
105	0.0860	0.0367	0.0181	0.0860	0.0367	0.0181
110	0.0805	0.0331	0.0156	0.0805	0.0331	0.0156

$S_1(0) = S_2(0) = 100$, $r = 0.04$, $T = 1.0$, $\sigma_1 = \sigma_2 = 0.5$, $v(0) = 1.0$,
$B_1(0) = B_2(0) = 75$, $\zeta = 0.0004$, $\kappa = 0.0004$, $\sigma_v = 0.0004$.

For the following scenario computations for the two-asset barrier option in the three-factor Model (3.2) applying Theorem 4 with the Heston-type characteristic function we choose the parameters: $S_1 = S_2 = 100$, $K_1 = K_2 = 100$, $r = 0.04$, $T = 1$, $\sigma_1 = \sigma_2 = 0.5$, $\rho = 0$, $\sigma_v = 0.4$, $\zeta = 0.9$, $\kappa = 0.4$, $v(0) = 0.9$. For the double-digital options, a downward sliding graph is drawn when the barriers $B_1(0)$, $B_2(0)$ are increased (see Figure 3.2). The slope of the graph is less pronounced for smaller barriers and more distinct for higher ones. We have repeated the computations for correlation $\rho = -0.5$. The same appearance of the graphs can be observed as for $\rho = 0$ in Figure 3.3.

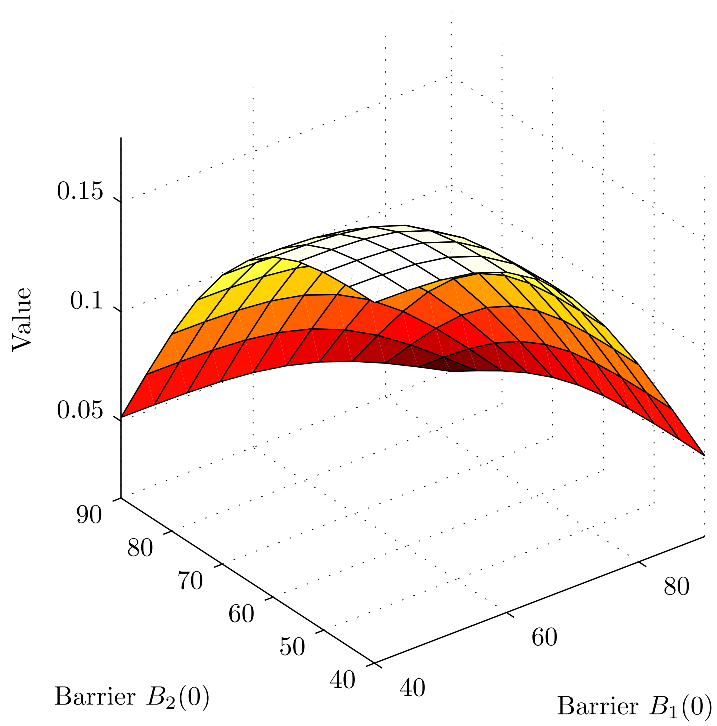

Figure 3.2: Double-Digital Option with Barriers (ρ=0).

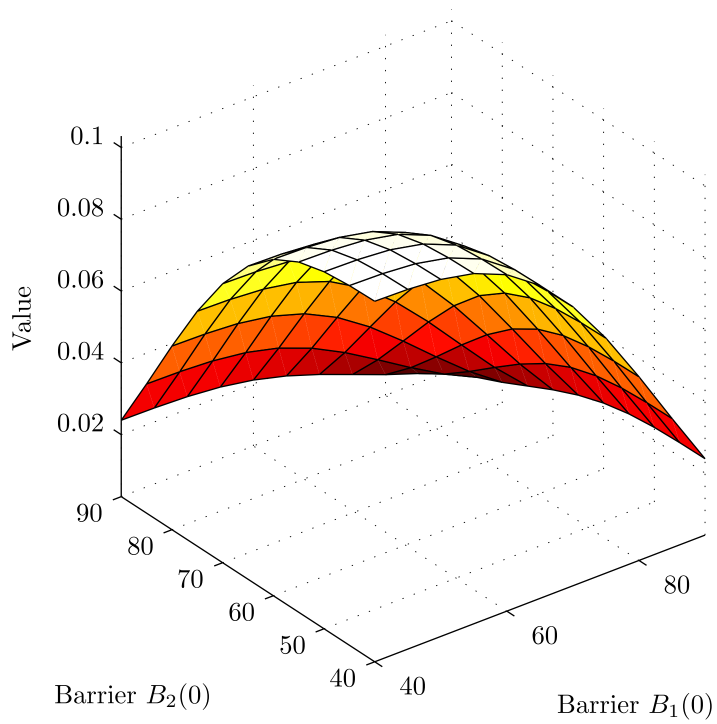

Figure 3.3: Double-Digital Option with Barriers (ρ=-0.5).

In Table 3.2 we provide more results showing the impact of the various parameters of the covariance process on the price of the derivatives. The barrier levels $B_1(0)$ and $B_2(0)$ are set at 75. The scenario is described in detail in the respective table. The digital barrier option increases in value when the volatility of the third process σ_v is raised and it falls when the mean-reversion level is incremented. The characteristics towards κ, the mean-reversion speed, are ambiguous: For a mean-reversion level of $\zeta = 0.6$ an increase in κ leads to a higher value, however, for higher mean-reversion levels, such as 0.9 or 1.2, while setting $\sigma_v = 0.8$, an increase in κ reduces the value of the derivative. The value of the digital barrier option reflects, in fact, the probability that both assets stay above the barrier levels during the lifetime of the option and are above the strike levels at maturity. In the scenario we have chosen the probability for that is higher when the overall volatility in the system is low because then the derivative stays above the barriers and ends in the money.

To analyse the influence of the volatility and the various parameters we change the scenario and choose a barrier level of $B_1(0) = B_2(0) = 60$ and strike levels $K_1 = K_2 = 110$. σ_1 and σ_2 are set to 0.2 (see Table 3.3). The barriers and strikes are set at such levels to make more volatility a necessary component to raise the probability that the options finish in the money at maturity. Thus, the value of the digital barrier options increases with higher mean-reversion levels as the barrier correlation option does.

Table 3.2: Prices of double-digital options with barriers in Heston-type model (Fourier technique).

	$\kappa = 0.6$			$\kappa = 0.9$			$\kappa = 1.2$		
	$\zeta =$								
σ_v	0.6	0.9	1.2	0.6	0.9	1.2	0.6	0.9	1.2
0.4	0.1041	0.0962	0.0891	0.1078	0.0966	0.0871	0.1111	0.0970	0.0855
0.6	0.1077	0.0994	0.0920	0.1109	0.0993	0.0894	0.1138	0.0993	0.0874
0.8	0.1125	0.1037	0.0960	0.1151	0.1029	0.0926	0.1174	0.1023	0.0899

$S_1(0) = S_2(0) = 100$, $T = 1.0$, $r = 0.04$, $\sigma_1 = \sigma_2 = 0.5$,
$B_1(0) = B_2(0) = 75$, $K_1 = K_2 = 100$, $v(0) = 1.0$, $\rho = 0$.

Table 3.3: Prices of double-digital options with barriers in Heston-type model (Fourier technique) with low σ-values.

	$\kappa = 0.6$			$\kappa = 0.9$			$\kappa = 1.2$		
	$\zeta =$								
σ_v	0.6	0.9	1.2	0.6	0.9	1.2	0.6	0.9	1.2
0.4	0.1165	0.1185	0.1203	0.1156	0.1184	0.1208	0.1148	0.1184	0.1212
0.6	0.1154	0.1175	0.1194	0.1146	0.1176	0.1201	0.1139	0.1177	0.1206
0.8	0.1137	0.1161	0.1181	0.1131	0.1164	0.1191	0.1126	0.1167	0.1198

$S_1(0) = S_2(0) = 100$, $T = 1.0$, $r = 0.04$, $\sigma_1 = \sigma_2 = 0.2$,
$B_1(0) = B_2(0) = 60$, $K_1 = K_2 = 110$, $v(0) = 1.0$, $\rho = 0$.

3.4.4 Pricing of correlation barrier options with Fourier techniques

In the following we derive formulas to price two-asset and single-asset barrier correlation options with time-dependent barriers. Bakshi and Madan price correlation options in [8]. Correlation derivatives are desirable for coping with cross-market or cross-currency (commodity) risks (see [122], [8]). As Bakshi and Madan point out for equity markets they even allow investors to position on a stock/sector relative to a market index. The correlation option is actually a product of two call options. Each security can be interpreted as the expectation of a unity payout conditional on both calls expiring in the money. The option is closely related to the digital presented in the previous section. Barriers are again introduced to lower the price of the derivative. Assuming risk-neutral valuation we can compute the value of a correlation option with time-dependent barriers on each of the underlyings by

$$\begin{aligned} C_{2C}(t,S_1,S_2,B_1,B_2) &= \mathbb{E}_{\tilde{Q}}\Big[e^{-\int_t^T r(s)ds}\max\left(S_1(T)-K_1,0\right) \\ &\quad \max\left(S_2(T)-K_2,0\right)\mathbb{1}_{\{\iota_1>T,\ \iota_2>T\}}\,|\mathcal{F}_t\Big]. \end{aligned} \tag{3.70}$$

Thus, we have to Fourier transform $e^{-c_1x_1-c_2x_2}g_C(x_1,x_2) = e^{-c_1x_1-c_2x_2}\left(e^{x_1}K_1-K_1\right)^+\left(e^{x_2}K_2-K_2\right)^+ = K_1K_2\left(e^{(1-c_1)x_1}-e^{-c_1x_1}\right)^+\left(e^{(1-c_2)x_2}-e^{-c_2x_2}\right)^+$ (see A.14), where $x_i = \ln\frac{S_ie^{\int_t^T r(s)ds}}{K_i}$. $e^{-c_1x_1-c_2x_2}g_C$ does not belong to $\mathbf{L}^1$. The ordinary Fourier transform does not exist and we have to apply the generalized Fourier transform:

$$\begin{aligned} \hat{\mathfrak{h}}_C(u_1,u_2) &= K_1K_2\int_0^\infty\int_0^\infty\left(e^{(1-c_1)x_1'}-e^{-c_1x_1'}\right)\left(e^{(1-c_2)x_2'}-e^{-c_2x_2'}\right) \\ &\quad e^{\left(iu_1\frac{x_1'}{\sigma_1\sqrt{1-\rho^2}}+\frac{x_2'}{\sigma_2}\left(-\frac{\rho}{\sqrt{1-\rho^2}}iu_1+iu_2\right)\right)}dx_1'dx_2' \\ &= K_1K_2\left(-\frac{1}{i\frac{u_1}{\sigma_1\sqrt{1-\rho^2}}+(1-c_1)}+\frac{1}{i\frac{u_1}{\sigma_1\sqrt{1-\rho^2}}-c_1}\right) \\ &\quad \left(-\frac{1}{i\left(\frac{u_2}{\sigma_2}-\frac{\rho u_1}{\sigma_2\sqrt{1-\rho^2}}\right)+(1-c_2)}+\frac{1}{i\left(\frac{u_2}{\sigma_2}-\frac{\rho u_1}{\sigma_2\sqrt{1-\rho^2}}\right)-c_2}\right). \end{aligned} \tag{3.71}$$

Note that we choose $\Im(u_1) > \sigma_1\sqrt{1-\rho^2}(1-c_1)$ and $\Im(u_2) > \frac{\rho}{\sqrt{1-\rho^2}}\Im(u_1)+\sigma_2(1-c_2)$, i.e. $S_{g_C} = \left\{\mathbf{u}=\mathbf{w}+i\boldsymbol{\varpi} : \varpi_1 > \sigma_1\sqrt{1-\rho^2}(1-c_1)\wedge\varpi_2 > \frac{\rho}{\sqrt{1-\rho^2}}\varpi_1+\sigma_2(1-c_2)\right\}$. The price of the option can be given in the following corollary.

Corollary 6. *(Correlation barrier option price)*

Let us assume

i. *the setting described in Equation (3.2),*

ii. *the existence of an affine analytic characteristic function* $\varphi(\tau, \mathbf{u}, \mathbf{z})$*, which is regular in a neighbourhood* $S_\varphi = \{\mathbf{u} = \mathbf{w} + i\boldsymbol{\varpi} : \boldsymbol{\varpi} \in (\mathfrak{a}_\varphi, \mathfrak{b}_\varphi)\}$, $\mathfrak{a}_\varphi < 0, \mathfrak{b}_\varphi > 0$ *of the origin, and integrable,*

iii. *that the generalized Fourier transform* $\hat{\mathfrak{h}}_C(\mathbf{x})$ *of the payoff function* $e^{-c_1x_1-c_2x_2}g_C(\mathbf{x})$ *at maturity exists in a space* $S_{g_C} = \left\{\mathbf{u} = \mathbf{w} + i\boldsymbol{\varpi} : \varpi_1 > \sigma_1\sqrt{1-\rho^2}(1-c_1) \wedge \varpi_2 > \frac{\rho}{\sqrt{1-\rho^2}}\varpi_1 + \sigma_2(1-c_2)\right\}$*, is there integrable for* $|\mathbf{x}| < \infty$*, and*

iv. $\rho = -\cos(\frac{\pi}{n})$*, where* n *is a natural number and* $n > 1$*.*

If the space $S_{C_C} \equiv S_\varphi \cap S_{g_C}$ *is not empty, the correlation barrier option (see (3.70) for the payoff profile) value is given by*

$$\begin{aligned} C_{2C}(t, S_1, S_2, B_1, B_2, v) = & \frac{e^{x_1c_1+x_2c_2-\int_t^T r(s)ds}}{4\pi^2\sigma_1\sigma_2\sqrt{1-\rho^2}} \sum_{k=0}^{n-1} \int_{i\varpi_2-\infty}^{i\varpi_2+\infty} \int_{i\varpi_1-\infty}^{i\varpi_1+\infty} \hat{\mathfrak{h}}_C(\mathbf{u}) \\ & \left(\varphi(\tau, \mathbf{u}, -\mathbf{z}_k, v) - \varphi(\tau, \mathbf{u}, -\mathbf{z}_k{}^-, v)\right) \\ & e^{(iu_1(-\frac{b_1}{\sigma_1\sqrt{1-\rho^2}}+\frac{b_2\rho}{\sigma_2\sqrt{1-\rho^2}})-iu_2\frac{b_2}{\sigma_2})} du_1 du_2, \\ & \mathbf{u} \in S_{C_C}, \end{aligned} \tag{3.72}$$

where φ, $\mathbf{z}_i^{(-)}$, x_i, b_i*, and* c_i *as in Theorem 34.* $\hat{\mathfrak{h}}_C(u_1, u_2)$ *as defined in 3.71*

The corollary directly follows from Theorem 34 and Equation (3.71).

Excursus: Contour variation

The general formula (Theorem 34) with S_g has many variations. Those variations can be obtained by applying residue calculus (see Corollary 3 and [81]). This is best shown with the example of the correlation option. The integrand in Formula (3.72) is regular in u_1 throughout S_φ except for simple poles in $\breve{u}_1 = i(1-c_1)\sigma_1\sqrt{1-\rho^2}$ and $\breve{u}_1 = -ic_1\sigma_1\sqrt{1-\rho^2}$. In u_2 we have poles at $\breve{u}_2 = \frac{\rho}{\sqrt{1-\rho^2}}u_1 + i\sigma_2(1-c_2)$ and

$\breve{u}_2 = \frac{\rho}{\sqrt{1-\rho^2}} u_1 - i\sigma_2 c_2$. Thus, if we move the contour (path of integration) to $\varpi_1 = 0$ while keeping $\varpi_2 > \frac{\rho}{\sqrt{1-\rho^2}}\varpi_1 + \sigma_2(1-c_2)$ we only cross a single singularity at $\breve{u}_1 = i(1-c_1)\sigma_1\sqrt{1-\rho^2}$. This can be seen in the example ($\rho = 0.5,\ \sigma_1 = \sigma_2 = 0.3$) pictured in Figure 3.4. Hence, we can apply Corollary 3.

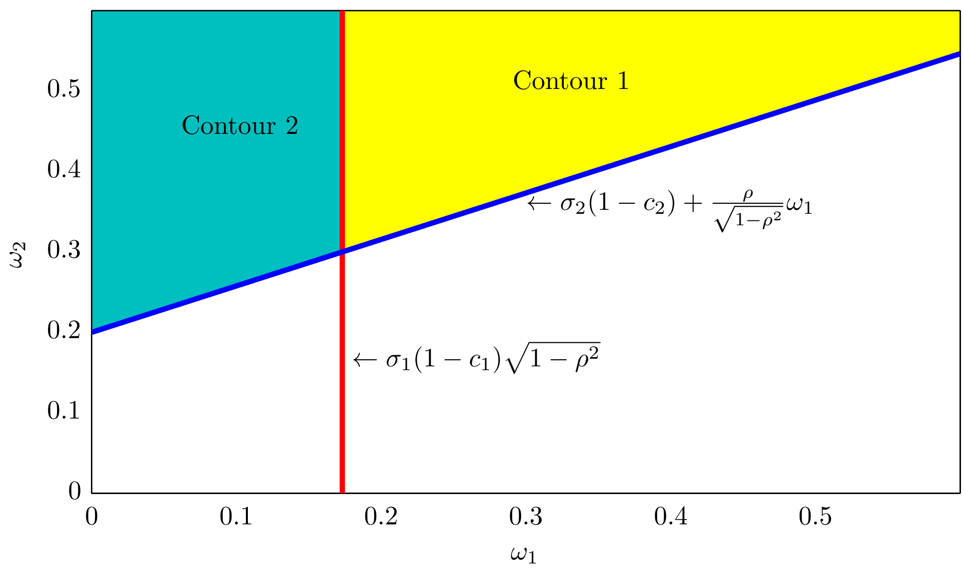

Figure 3.4: Contour variation.

Then by Corollary 3 the correlation option price must also equal the integral along the real axis of u_1 minus $2\pi i$ times the residue at $\breve{u}_1 = i(1-c_1)\sigma_1\sqrt{1-\rho^2}$. According to (2.56) the residue at $\breve{u}_1 = i(1-c_1)\sigma_1\sqrt{1-\rho^2}$ is given by

$$\begin{aligned} Res_{\breve{u}_1} f \;=\; & K_1 K_2 \lim_{u_1 \to \breve{u}_1} \left(u_1 - i\sigma_1\sqrt{1-\rho^2}(1-c_1)\right) \frac{e^{x_1c_1 + x_2c_2 - \int_t^T r(s)ds}}{4\pi^2\sigma_1\sigma_2\sqrt{1-\rho^2}} \\ & \sum_{k=0}^{n-1} \frac{\sigma_1\sqrt{1-\rho^2}}{i\left(u_1 - i\sigma_1\sqrt{1-\rho^2}(1-c_1)\right)\left(i\frac{u_1}{\sigma_1\sqrt{1-\rho^2}} - c_1\right)} \\ & \frac{1}{\left(i\left(\frac{u_2}{\sigma_2} - \frac{\rho u_1}{\sigma_2\sqrt{1-\rho^2}}\right) + (1-c_2)\right)\left(i\left(\frac{u_2}{\sigma_2} - \frac{\rho u_1}{\sigma_2\sqrt{1-\rho^2}}\right) - c_2\right)} \\ & \left(\varphi(\tau, u_1, u_2, -\mathbf{z}_k, v) - \varphi(\tau, u_1, u_2, -\mathbf{z}_k^{-}, v)\right) \\ & e^{\left((1-c_1)\left(\frac{b_1}{\sigma_1} - \frac{b_2\sigma_1\rho}{\sigma_2}\right) - iu_2\frac{b_2}{\sigma_2}\right)} \end{aligned} \qquad (3.73)$$

$$
\begin{aligned}
= \quad & K_1 K_2 \frac{e^{x_1 c_1 + x_2 c_2 - \int_t^T r(s)ds}}{4\pi^2 \sigma_1 \sigma_2 \sqrt{1-\rho^2}} \sum_{k=0}^{n-1} i\sigma_1 \sqrt{1-\rho^2} \\
& \frac{1}{\left(i\left(\frac{u_2}{\sigma_2} - \frac{\rho \check{u}_1}{\sigma_2\sqrt{1-\rho^2}}\right) + (1-c_2)\right)\left(i\left(\frac{u_2}{\sigma_2} - \frac{\rho \check{u}_1}{\sigma_2\sqrt{1-\rho^2}}\right) - c_2\right)} \\
& \left(\varphi(\tau, \check{u}_1, u_2, -\mathbf{z}_k, v) - \varphi(\tau, \check{u}_1, u_2, -\mathbf{z}_k{}^{-}, v)\right) \qquad (3.74) \\
& e^{\left((1-c_1)\left(\frac{b_1}{\sigma_1} - \frac{b_2\sigma_1\rho}{\sigma_2}\right) - iu_2\frac{b_2}{\sigma_2}\right)},
\end{aligned}
$$

where f is the integrand in (3.72). Hence, the new pricing formula is indicated by

$$
\begin{aligned}
C_{2C}(t, S_1, S_2, B_1, B_2, v) \quad = \quad & \sum_{k=0}^{n-1} \Bigg(-2\pi i \int_{i\varpi_2-\infty}^{i\varpi_2+\infty} Res_{\check{u}_1} du_2 \\
& + \frac{e^{x_1 c_1 + x_2 c_2 - \int_t^T r(s)ds}}{4\pi^2 \sigma_1 \sigma_2 \sqrt{1-\rho^2}} \\
& \int_{i\varpi_2-\infty}^{i\varpi_2+\infty} \int_{-\infty}^{\infty} \hat{\mathfrak{h}}_C(\mathbf{u}) \\
& \left(\varphi(\tau, \mathbf{u}, -\mathbf{z}_k, v) - \varphi(\tau, \mathbf{u}, -\mathbf{z}_k{}^{-}, v)\right) \qquad (3.75) \\
& e^{(iu_1(-\frac{b_1}{\sigma_1\sqrt{1-\rho^2}} + \frac{b_2\rho}{\sigma_2\sqrt{1-\rho^2}}) - iu_2\frac{b_2}{\sigma_2})} du_1 du_2 \Bigg), \\
& \varpi_1 = 0, \ \varpi_2 > \sigma_2(1-c_2).
\end{aligned}
$$

In a next step we want to move ϖ_2 to 0. This time we pass singularities at $\check{u}_2 = \frac{\rho u_1}{\sqrt{1-\rho^2}} + i\sigma_2(1-c_2)$. The residue for the second integral in (3.75) is given by

$$
\begin{aligned}
Res^{(2)}_{\check{u}_2} f_2 \quad = \quad & K_1 K_2 \lim_{u_2 \to \check{u}_2} (u_2 - \frac{\rho u_1}{\sqrt{1-\rho^2}} - i\sigma_2(1-c_2)) \frac{e^{x_1 c_1 + x_2 c_2 - \int_t^T r(s)ds}}{4\pi^2 \sigma_1 \sigma_2 \sqrt{1-\rho^2}} \\
& \sum_{k=0}^{n-1} \frac{1}{(i\frac{u_1}{\sigma_1\sqrt{1-\rho^2}} + (1-c_1))(i\frac{u_1}{\sigma_1\sqrt{1-\rho^2}} - c_1)} \\
& \frac{\sigma_2}{i\left(u_2 - \frac{\rho u_1}{\sqrt{1-\rho^2}} - i\sigma_2(1-c_2)\right)\left(i\left(\frac{u_2}{\sigma_2} - \frac{\rho u_1}{\sigma_2\sqrt{1-\rho^2}}\right) - c_2\right)} \\
& \left(\varphi(\tau, u_1, u_2, -\mathbf{z}_k, v) - \varphi(\tau, u_1, u_2, -\mathbf{z}_k{}^{-}, v)\right) \qquad (3.76) \\
& e^{\left(iu_1\left(-\frac{b_1}{\sigma_1\sqrt{1-\rho^2}} + \frac{b_2\rho}{\sigma_2\sqrt{1-\rho^2}}\right) - iu_2\frac{b_2}{\sigma_2}\right)}
\end{aligned}
$$

$$
\begin{aligned}
&= K_1 K_2 \frac{e^{x_1 c_1 + x_2 c_2 - \int_t^T r(s)ds}}{4\pi^2 \sigma_1 \sigma_2 \sqrt{1-\rho^2}} \sum_{k=0}^{n-1} i\sigma_2 \\
&\quad \frac{1}{\left(i\frac{u_1}{\sigma_1\sqrt{1-\rho^2}} + (1-c_1)\right)\left(i\frac{u_1}{\sigma_1\sqrt{1-\rho^2}} - c_1\right)} \\
&\quad \left(\varphi(\tau, u_1, \check{u}_2, -\mathbf{z}_k, v) - \varphi(\tau, u_1, \check{u}_2, -\mathbf{z}_k^{-}, v)\right) \\
&\quad e^{\left(iu_1\left(-\frac{b_1}{\sigma_1\sqrt{1-\rho^2}} + \frac{b_2\rho}{\sigma_2\sqrt{1-\rho^2}}\right) - i\check{u}_2\frac{b_2}{\sigma_2}\right)},
\end{aligned}
\tag{3.77}
$$

where f_2 is the integrand of the second integral in (3.75). The residue for the first integral is given by

$$
\begin{aligned}
Res_{\check{u}_2}^{(1)} f_1 &= -2\pi i K_1 K_2 \lim_{u_2 \to \check{u}_2} \left(u_2 - \frac{\rho u_1}{\sqrt{1-\rho^2}} - i\sigma_2(1-c_2)\right) \frac{e^{x_1 c_1 + x_2 c_2 - \int_t^T r(s)ds}}{4\pi^2 \sigma_1 \sigma_2 \sqrt{1-\rho^2}} \\
&\quad \sum_{k=0}^{n-1} i\sigma_1 \sqrt{1-\rho^2} \\
&\quad \frac{\sigma_2}{i\left(u_2 - \frac{\rho u_1}{\sqrt{1-\rho^2}} - i\sigma_2(1-c_2)\right)\left(i\left(\frac{u_2}{\sigma_2} - \frac{\rho u_1}{\sigma_2\sqrt{1-\rho^2}}\right) - c_2\right)} \\
&\quad \left(\varphi(\tau, u_1, u_2, -\mathbf{z}_k, v) - \varphi(\tau, u_1, u_2, -\mathbf{z}_k^{-}, v)\right) \\
&\quad e^{\left(iu_1\left(-\frac{b_1}{\sigma_1\sqrt{1-\rho^2}} + \frac{b_2\rho}{\sigma_2\sqrt{1-\rho^2}}\right) - iu_2\frac{b_2}{\sigma_2}\right)} \\
&= K_1 K_2 \frac{e^{x_1 c_1 + x_2 c_2 - \int_t^T r(s)ds}}{2\pi\sigma_2} \\
&\quad \sum_{k=0}^{n-1} i\sigma_2 \left(\varphi(\tau, \check{u}_1, \check{u}_2, -\mathbf{z}_k, v) - \varphi(\tau, \check{u}_1, \check{u}_2, -\mathbf{z}_k^{-}, v)\right) \\
&\quad e^{\left((1-c_1)\left(\frac{b_1}{\sigma_1} - \frac{b_2\sigma_1\rho}{\sigma_2}\right) - i\check{u}_2\frac{b_2}{\sigma_2}\right)},
\end{aligned}
\tag{3.78}
$$

where f_1 is the integrand of the first integral in (3.75). Thus, the pricing formula with integration on the real axis is indicated by

$$\begin{aligned}
C_{2C}(t, S_1, S_2, B_1, B_2, v) &= \sum_{k=0}^{n-1} \Bigg(-2\pi i \int_{-\infty}^{\infty} Res_{\breve{u}_1} du_2 \\
&-2\pi i \int_{-\infty}^{\infty} Res_{\breve{u}_2}^{(2)} du_1 \\
&-2\pi i Res_{\breve{u}_2}^{(1)} + \frac{e^{x_1 c_1 + x_2 c_2 - \int_t^T r(s)ds}}{4\pi^2 \sigma_1 \sigma_2 \sqrt{1-\rho^2}} \\
&\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\mathfrak{h}}_C(\mathbf{u}) \\
&\left(\varphi(\tau, \mathbf{u}, -\mathbf{z}_k, v) - \varphi\left(\tau, \mathbf{u}, -\mathbf{z}_k^{-}, v\right)\right) \qquad (3.79)\\
&e^{\left(iu_1\left(-\frac{b_1}{\sigma_1\sqrt{1-\rho^2}} + \frac{b_2\rho}{\sigma_2\sqrt{1-\rho^2}}\right) - iu_2 \frac{b_2}{\sigma_2}\right)} du_1 du_2 \Bigg).
\end{aligned}$$

By using the method of images in the plane we can derive prices for correlation options with a barrier on S_2 only: $C_{1C}(t, S_1, S_2, B_2) = \mathbb{E}_{\tilde{\mathbb{Q}}}\Big[e^{-\int_t^T r(s)ds} \max(S_1(T) - K_1, 0)$ $\max(S_2(T) - K_2, 0) \mathbb{1}_{\{\iota_2 > T\}} | \mathcal{F}_t\Big]$. The solution is found by reflecting the characteristic function in the plane ($z_2(0) \to -z_2(0)$).

Corollary 7. *(Correlation single-barrier option price)*

Let us assume the setting described in Equation (3.2) and the assumptions of Corollary 6, then the correlation single-barrier option value is given for any ρ by

$$\begin{aligned}
C_{1C}(t, S_1, S_2, B_2, v) &= \frac{e^{x_1 c_1 + x_2 c_2 - \int_t^T r(s)ds}}{4\pi^2 \sigma_1 \sigma_2 \sqrt{1-\rho^2}} \int_{i\varpi_2 - \infty}^{i\varpi_2 + \infty} \int_{i\varpi_1 - \infty}^{i\varpi_1 + \infty} \hat{\mathfrak{h}}_C(\mathbf{u}) \\
&(\varphi(\tau, \mathbf{u}, -z_1, -z_2, v) - \varphi(\tau, \mathbf{u}, -z_1, z_2, v)) \qquad (3.80)\\
&e^{(iu_1 \frac{b_2\rho}{\sigma_2\sqrt{1-\rho^2}} - iu_2 \frac{b_2}{\sigma_2})} du_1 du_2, \ \mathbf{u} \in S_{g_C},
\end{aligned}$$

where φ, x_i, b_i, and c_i are as defined in Theorem 34, as well as $\hat{\mathfrak{h}}_C(u_1, u_2)$ in 3.71.

The corollary directly follows from Theorem 34 and Equation (3.71). For details on the method of images in half-space please refer to Appendix A.1.3. In the following we compute prices of a two-asset correlation option in the three-factor Model (3.2) using Theorem 6 with the Heston-type characteristic function, which we degenerate to the two-factor GBM Model (3.1). We compare the resulting values to prices computed in the GBM model with a formula derived by He et al. [64]. The exactness of the prices computed using Fourier transform can be seen in Table 3.4. For the following scenario

Table 3.4: Prices of the two-asset barrier correlation option computed in the limit to bivariate normal distribution.

	GBM model			Degenerated 3-factor model		
	Correlation ρ					
strike K_1	0	-0.5	-0.71	0	-0.5	-0.71
80	453.06	99.59	31.58	453.06	99.60	31.58
85	421.59	88.71	26.92	421.59	88.71	26.92
90	390.85	78.36	22.61	390.85	78.36	22.62
95	361.20	68.78	18.77	361.20	68.78	18.79
100	332.91	60.05	15.47	332.90	60.05	15.47
105	306.14	52.22	12.65	306.14	52.22	12.65
110	280.99	45.27	10.29	280.98	45.27	10.29

$S_1(0) = S_2(0) = 100$, $T = 1.0$, $r = 0.04$, $\sigma_1 = \sigma_2 = 0.5$, $v(0) = 1.0$,
$B_1(0) = B_2(0) = 75$, $\zeta = 0.0004$, $\kappa = 0.0004$, $\sigma_v = 0.0004$.

computations for Theorem 6 we set again $S_1 = S_2 = 100$, $K_1 = K_2 = 100$, $r = 0.04$, $T = 1$, $\sigma_1 = \sigma_2 = 0.5$, $\rho = 0$, $\sigma_v = 0.4$, $\zeta = 0.9$, $\kappa = 0.4$, $v(0) = 0.9$. When the barriers $B_1(0)$, $B_2(0)$ are increased we also observe in Figure 3.5 for correlation options with two barriers a downward sliding graph.

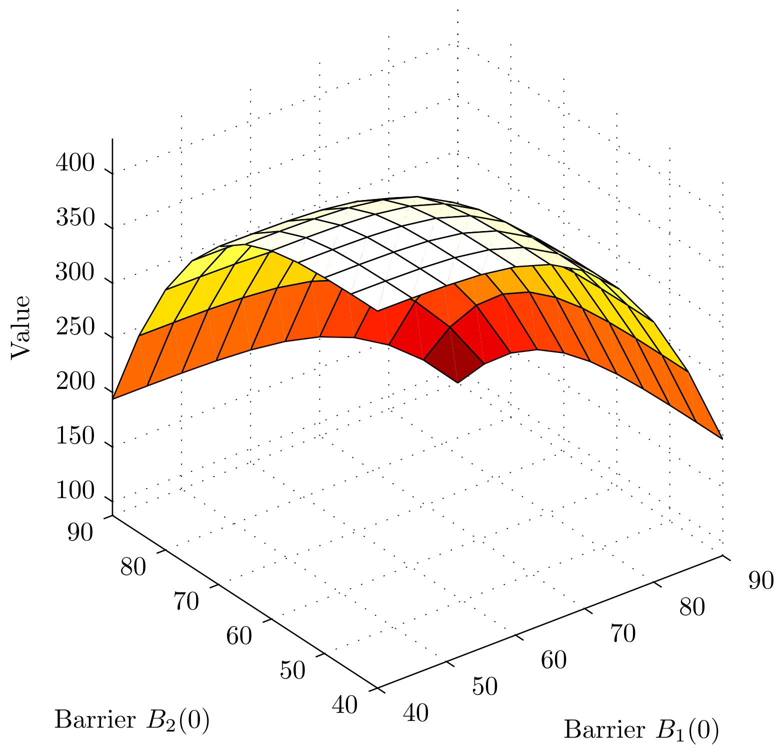

Figure 3.5: Correlation Option with Barriers (ρ=0).

We repeat the computations again for correlation $\rho = -0.5$. For the two-asset barrier correlation option the features of the graph do not change in comparison to $\rho = 0$ in Figure 3.6.

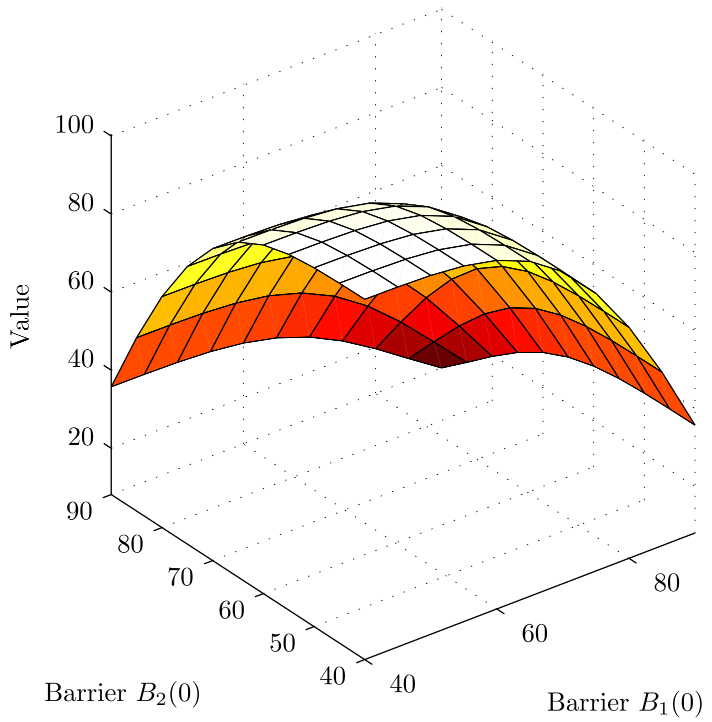

Figure 3.6: Correlation Option with Barriers (ρ=-0.5).

In Table 3.5 we provide more results which show the impact of the various parameters of the covariance process on the price of the two-asset barrier correlation option. The barrier levels $B_1(0)$ and $B_2(0)$ are set at 75 as before. One notes the – at first – striking different behaviour of digital and correlation options towards the parameters of the model: For the correlation option the picture is different to before. The payoff of this derivative is not only influenced by the probability that the underlyings stay above the barriers during lifetime and above the strike levels at maturity but also by the actual level of the underlyings at maturity. Thus, in the here presented scenario there are two contrary effects: A higher volatility increases the probability that the underlyings are far above the strike levels at maturity but it also increases the probability that the underlyings fall below the barrier levels during the lifetime. The former effect seems to be dominant in this scenario as the value of the barrier correlation option raises when the mean-reversion level is increased. If the volatility of the volatility process is incremented the value of the option falls. Therefore, the probability that both underlyings stay above the barrier and far above the strike levels at maturity is lowered. As the opposite effect is visible with digital options it seems that a higher σ_v leads to a higher probability of lower volatility values in the whole system. When κ is augmented the overall effect depends again on the mean-reversion level: In the case of a low mean-reversion level of $\zeta = 0.6$ the value of the derivative falls. Obviously, lower volatility values below and at the mean-reversion level become more probable. The opposite is true for the higher mean-reversion levels as the fast mean reversion ensures high values of volatilities.

To analyse the influence of the volatility and the various parameters in more detail we proceed analogously to the digital options and choose now a barrier level of $B_1(0) = B_2(0) = 60$ and strike levels $K_1 = K_2 = 110$. σ_1 and σ_2 are set to 0.2. Comparing Tables (3.3) and (3.6) we can observe that digital and correlation options behave in the same way to a change in the different parameter values.

Table 3.5: Prices of correlation options with barriers barriers in Heston-type model (Fourier technique).

	$\kappa = 0.6$			$\kappa = 0.9$			$\kappa = 1.2$		
	$\zeta =$								
σ_v	0.6	0.9	1.2	0.6	0.9	1.2	0.6	0.9	1.2
0.4	313.86	326.25	337.69	307.88	325.27	340.86	302.70	324.43	343.41
0.6	310.11	322.79	334.49	304.69	322.40	338.27	299.98	322.03	341.29
0.8	305.05	318.09	330.12	300.39	318.51	334.73	296.30	318.77	338.38

$S_1(0) = S_2(0) = 100$, $T = 1.0$, $r = 0.04$, $\sigma_1 = \sigma_2 = 0.5$,
$B_1(0) = B_2(0) = 75$, $K_1 = K_2 = 100$, $v(0) = 1.0$, $\rho = 0$.

Table 3.6: Prices of correlation options with barriers in Heston-type model (Fourier technique) with low σ-values.

	$\kappa = 0.6$			$\kappa = 0.9$			$\kappa = 1.2$		
	$\zeta =$								
σ_v	0.6	0.9	1.2	0.6	0.9	1.2	0.6	0.9	1.2
0.4	28.99	32.37	35.81	27.30	31.91	36.65	25.90	31.53	37.35
0.6	29.24	32.60	36.03	27.51	32.10	36.82	26.08	31.69	37.49
0.8	29.58	32.91	36.32	27.79	32.36	37.06	26.32	31.91	37.69

$S_1(0) = S_2(0) = 100$, $T = 1.0$, $r = 0.04$, $\sigma_1 = \sigma_2 = 0.2$,
$B_1(0) = B_2(0) = 60$, $K_1 = K_2 = 110$, $v(0) = 1.0$, $\rho = 0$.

3.4.5 Alternative Fourier Technique

As already mentioned, alternatively, the methods derived by Carr and Madan [18] and Dempster and Hong [32] could be extended to allow for barrier option pricing. They Fourier transform the option price directly. As many of those integrands are not Lebesgue integrable and singular, they transform a modified (dampened) call price, which assures integrability. We price any terminal payoff $g(S_1, S_2)$ without barriers. Starting from

Equation (3.6) and for $K_i = 1$ we define the characteristic function of the two variables x_1' and x_2' at time t starting from x_1 and x_2 in $t = 0$ in terms of the two independent variables z_1 and z_2 (see transformations for (3.32)) by

$$\begin{aligned}\varphi(t, w_1, w_2) &= \mathbb{E}_{\tilde{\mathbb{Q}}}\left[\exp\left\{iw_1 z_1'(t) + iw_2 z_2'(t)\right\}\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{iw_1 z_1' + iw_1 z_2'} \hat{q}(t, z_1', z_2', z_1, z_2, v) dz_1' dz_2',\ \mathbf{w} \in \mathbb{R}^2, \end{aligned} \quad (3.81)$$

where

$$\begin{aligned} z_1' &= \frac{1}{\sqrt{1-\rho^2}}\left(\frac{x_1' - b_1}{\sigma_1} - \rho\frac{x_2' - b_2}{\sigma_2}\right), \\ z_2' &= \frac{x_2' - b_2}{\sigma_2}, \\ x_i &= \ln\left(S_i e^{\int_t^T r(s)ds}\right), \\ b_i &= \ln B_i(T), \\ \tau &= T - t. \end{aligned}$$

$\hat{q}$ is the joint density function of x_1', x_2' in T conditional on x_1 and x_2 in t. As already shown, we can reformulate any option for which the payoff only depends on the terminal values of the stocks by

$$\begin{aligned} C(T-t, x_1, x_2, k_1, k_2) &= e^{-\int_t^T r(s)ds + c_1 x_1 + c_2 x_2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1, x_2, k_1, k_2) \\ & \quad e^{-c_1 x_1' - c_2 x_2'} \hat{q}(\tau, x_1', x_2', x_1, x_2, v) dx_2' dx_1', \end{aligned} \quad (3.82)$$

where $k_i = \ln(K_i)$. In many cases the joint density function is not known in closed form but the characteristic function so that we cannot use the formulation in (3.82). Therefore we want to Fourier transform the above problem. However, if $C(T-t, x_1, x_2, k_1, k_2)$ is not $\mathbf{L}^1$ (see also Schmelzle [104] for a summary) the Fourier transform does not exist. As mentioned above we can circumvent this by introducing a damping factor α_i^d. Following Carr and Madan [18] and Dempster and Hong [32] we multiply the option price by:

$$c(t, x_1, x_2, k_1, k_2) := e^{\alpha_1^d k_1 + \alpha_2^d k_2} C(t, x_1, x_2, k_1, k_2).$$

For suitable α_1^d, α_2^d, $c(t, x_1, x_2, k_1, k_2)$ is an integrable function, since then

$$\int_{\mathbb{R}^2} \left| e^{\alpha_1^d k_1 + \alpha_2^d k_2} C(t, x_1, x_2, k_1, k_2)\right| dk_1 dk_2 < \infty. \quad (3.83)$$

Along the lines of Dempster and Hong [32] we derive the Fourier transform of the modified double-digital option price (without barriers), i.e. $g(x_1, x_2) = \mathbb{1}_{\{x_1>k_1, x_2>k_2\}}$, where $k_i = \ln K_i$.

$$\begin{aligned}
\hat{c}_D(w_1, w_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{iw_1k_1+iw_2k_2} c_D(t, x_1, x_2, k_1, k_2) dk_2 dk_1 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{(\alpha_1^d+iw_1)k_1+(\alpha_2^d+iw_2)k_2} e^{-\int_t^T r(s)ds+c_1x_1+c_2x_2} \\
&\quad \int_{k_1}^{\infty}\int_{k_2}^{\infty} e^{-c_1x_1'-c_2x_2'} \hat{q}(\tau, x_1', x_2', x_1, x_2, v) dx_2' dx_1' dk_2 dk_1 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\int_t^T r(s)ds+c_1x_1+c_2x_2} \hat{q}(\tau, x_1', x_2', x_1, x_2, v) \\
&\quad \int_{-\infty}^{x_1'}\int_{-\infty}^{x_2'} e^{(\alpha_1^d+iw_1)k_1+(\alpha_2^d+iw_2)k_2} e^{-c_1x_1'-c_2x_2'} dk_2 dk_1 dx_2' dx_1' \\
&= e^{-\int_t^T r(s)ds+c_1x_1+c_2x_2} \\
&\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{(\alpha_1^d-c_1+iw_1)x_1'+(\alpha_2^d-c_2+iw_2)x_2'}\hat{q}(\tau, x_1', x_2', x_1, x_2, v)}{(\alpha_1^d+iw_1)(\alpha_2^d+iw_2)} dx_2' dx_1' \\
&= e^{-\int_t^T r(s)ds+c_1x_1+c_2x_2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{\left(\alpha_1^d-c_1+iw_1\right)\left(\sqrt{1-\rho^2}\sigma_1 z_1'+b_1+\rho\sigma_1 z_2'\right)}}{(\alpha_1^d+iw_1)(\alpha_2^d+iw_2)} \\
&\quad e^{(\alpha_2^d-c_2+iw_2)(z_2'\sigma_2+b_2)} \hat{q}(\tau, z_1', z_2', z_1, z_2, v) dz_2' dz_1' \\
&= e^{-\int_t^T r(s)ds+c_1x_1+c_2x_2} \frac{e^{(\alpha_1^d-c_1+iw_1)b_1+(\alpha_2^d-c_2+iw_2)b_2}}{(\alpha_1^d+iw_1)(\alpha_2^d+iw_2)} \varphi(\tau, u_1, u_2, v),
\end{aligned}$$

where

$$\begin{aligned}
u_1 &= \sigma_1\sqrt{1-\rho^2}\left(w_1 - i\left(\alpha_1^d - c_1\right)\right), \\
u_2 &= \sigma_2\left(u_2 - i\left(\alpha_2^d - c_2\right)\right) + \rho\sigma_1\left(w_1 - i\left(\alpha_1^d - c_1\right)\right).
\end{aligned}$$

We can apply the Theorem of Fubini in the forth line due to 3.83.
The damping factor improves the integrability on the negative real axis. A sufficient condition for c_D to be square-integrable is the finiteness of $\hat{c}_D(0,0)$, which can be achieved by choosing α_i^d accordingly. Then the inversion of $\hat{c}_D$ converges to c_D in $\mathbf{L}^2$- norm in line with Plancherel's Theorem (see Theorem 11). Comparing u_1 to (3.67) and the there defined strip of regularity for u_1 we can observe that the introduction of the damping factor α_i^d ensures in a similar manner as the application of the generalized Fourier inversion (integration in the complex plane) integrability.

The price of a digital option without barriers is given by

$$C_D(\tau, S_1, S_2, K_1, K_2, v) = \frac{e^{-\alpha_1 k_1 - \alpha_2 k_2}}{4\pi^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(w_1 k_1 + w_2 k_2)} \hat{c}_D(\tau, w_1, w_2) dw_2 dw_1. \quad (3.84)$$

The value of the barrier option in the three-factor model can be found when we apply the method of images in a wedge (see He et al. [64], Carslaw and Jaeger [19], p. 277f, Sommerfeld [109]) to the characteristic function in accordance with the proof of Theorem 34.

Corollary 8. *(Double-digital barrier option price with alternative method)*

Let us assume

i. the setting described in Equation (3.2),

ii. the existence of an affine characteristic function,

iii. $\rho = -\cos\left(\frac{\pi}{n}\right)$, *where* n *is a natural number and* $n > 1$, *and that*

iv. there are α_1^d, α_2^d *so that* $\exp\{\alpha_1 x_1 + \alpha_2 x_2\} C_D \in \mathbf{L}^2$.

Then the price of a two-asset digital barrier option (see (3.66) for the payoff profile) is:

$$C_{2D}(t, S_1, S_2, K_1, K_2, B_1, B_2, v) = \frac{e^{-\alpha_1 k_1 - \alpha_2 k_2}}{4\pi^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(w_1 k_1 + w_2 k_2)} \hat{c}_{2D}(w_1, w_2) dw_2 dw_1, \quad (3.85)$$

where

$$\hat{c}_{2D}(\mathbf{w}) = \frac{e^{-\int_t^T r(s)ds + c_1 x_1 + c_2 x_2} e^{(\alpha_1^d - c_1 + iw_1)b_1 + (\alpha_2^d - c_2 + iw_2)b_2}}{(\alpha_1^d + iw_1)(\alpha_2^d + iw_2)} \sum_{k=0}^{n-1} \left(\varphi\left(\tau, \mathbf{u}, -\mathbf{z}_k, v\right) - \varphi\left(\tau, \mathbf{u}, -\mathbf{z}_k^-, v\right)\right),$$

where φ, $\mathbf{z}_{k1}^{(-)}$ *and* $\mathbf{z}_{k2}^{(-)}$ *are given Theorem 34. The convergence is in* $\mathbf{L}^2$*-norm.*

Similarly, the price of the correlation barrier option can be derived. The transform of the dampened correlation option price is given by

$$
\begin{aligned}
\hat{c}_C(w_1,w_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{(iw_1+\alpha_1^d)k_1+(iw_2+\alpha_2^d)k_2} e^{-\int_t^T r(s)ds+c_1x_1+c_2x_2} \\
&\quad \int_{k_1}^{\infty}\int_{k_2}^{\infty} e^{-c_1x_1'-c_2x_2'}\left(e^{x_1'}-e^{k_1}\right)\left(e^{x_2'}-e^{k_2}\right) \\
&\quad \hat{q}(\tau,x_1',x_2',x_1,x_2,v)dx_2'dx_1'dk_2dk_1 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\int_t^T r(s)ds+c_1x_1+c_2x_2}\hat{q}(\tau,x_1',x_2',x_1,x_2,v)\int_{-\infty}^{x_1'}\int_{-\infty}^{x_2'} e^{(iw_1+\alpha_1^d)k_1} \\
&\quad e^{(iw_2+\alpha_2^d)k_2}e^{-c_1x_1'-c_2x_2'}\left(e^{x_1'}-e^{k_1}\right)\left(e^{x_2'}-e^{k_2}\right)dk_2dk_1dx_2'dx_1' \\
&= e^{-\int_t^T r(s)ds+c_1x_1+c_2x_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{q}_T(x_1',x_2',v) \\
&\quad \frac{e^{x_1'(\alpha_1^d+1-c_1+iw_1)}(\alpha_1^d+1+iw_1)-e^{x_1'(\alpha_1^d+1-c_1+iw_1)}(\alpha_1^d+iw_1)}{(\alpha_1^d+iw_1)(\alpha_1^d+1+iw_1)} \\
&\quad \frac{e^{x_2'(\alpha_2^d+iw_2+1-c_2)}(\alpha_2^d+1+iw_2)-e^{x_2'(\alpha_2^d+iw_2+1-c_2)}(\alpha_2^d+iw_2)}{(\alpha_2^d+iw_2+1)(\alpha_2^d+iw_2)}dx_2'dx_1' \\
&= e^{-\int_t^T r(s)ds+c_1x_1+c_2x_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{x_1'(\alpha_1^d+1-c_1+iw_1)}}{(\alpha_1^d+iw_1)(\alpha_1^d+1+iw_1)} \\
&\quad \frac{e^{x_2'(\alpha_2^d+iw_2+1-c_2)}}{(\alpha_2^d+iw_2+1)(\alpha_2^d+iw_2)}\hat{q}_T(x_1',x_2',v)dx_2'dx_1' \\
&= e^{-\int_t^T r(s)ds+c_1x_1+c_2x_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{q}_T(z_1',z_2',v) \\
&\quad \frac{e^{(\sqrt{1-\rho^2}\sigma_1z_1'+b_1+\rho\sigma_1z_2')(\alpha_1^d+1-c_1+iw_1)}e^{(z_2'\sigma_2+b_2)(\alpha_2^d+1-c_2+iw_2)}}{(\alpha_1^d+iu_1)(\alpha_1^d+1+iw_1)(\alpha_2^d+1+iw_2)(\alpha_2^d+iw_2)}dz_2'dz_1' \\
&= e^{-\int_t^T r(s)ds+c_1x_1+c_2x_2}\frac{e^{b_1(\alpha_1^d+1-c_1+iw_1)}e^{b_2(\alpha_2^d+1-c_2+iw_2)}\varphi(\tau,u_1',u_2',v)}{(\alpha_1^d+iw_1)(\alpha_1^d+1+iw_1)(\alpha_2^d+iw_2)(\alpha_2^d+1+iw_2)},
\end{aligned}
$$

where

$$
\begin{aligned}
u_1' &= \sigma_1\sqrt{1-\rho^2}\left(w_1-i\left(\alpha_1^d+1-c_1\right)\right), \\
u_2' &= \sigma_2\left(w_2-i\left(\alpha_2^d+1-c_2\right)\right)+\rho\sigma_1\left(w_1-\left(\alpha_1^d+1-c_1\right)\right). \qquad (3.86)
\end{aligned}
$$

Thus, the price of the correlation option is given by

$$
\begin{aligned}
C_C(t,S_1,S_2,K_1,K_2,v) &= \frac{e^{-\alpha_1k_1-\alpha_2k_2}}{4\pi^2} \\
&\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(w_1k_1+w_2k_2)}\hat{c}_C(u_1,u_2)dw_2dw_1, \qquad (3.87)
\end{aligned}
$$

where

$$\begin{aligned}
\hat{c}_C(w_1,w_2) &= \frac{e^{-\int_t^T r(s)ds+c_1x_1+c_2x_2}e^{\left(\alpha_1^d+1-c_1+iw_1\right)b_1+\left(\alpha_2^d+1-c_2+iw_2\right)b_2}}{\left(\alpha_1^d+iw_1\right)\left(\alpha_1^d+1+iw_1\right)\left(\alpha_2^d+iw_2\right)\left(\alpha_2^d+1+iw_2\right)}\varphi\left(\tau,u_1',u_2',v\right),\\
u_1' &= \sigma_1\sqrt{1-\rho^2}\left(w_1-i\left(\alpha_1^d+1-c_1\right)\right),\\
u_2' &= \sigma_2(w_2-i(\alpha_2^d+1-c_2))+\rho\sigma_1\left(w_1-i\left(\alpha_1^d+1-c_1\right)\right).
\end{aligned}$$

Corollary 9. *(Correlation barrier option price with alternative method)*

Let us assume

i. the setting described in Equation (3.2),

ii. the existence of an affine characteristic function,

iii. $\rho=-\cos\left(\frac{\pi}{n}\right)$*, where* n *is a natural number and* $n>1$*, and that*

iv. there are α_1^d*,* α_2^d *so that* $\exp\left\{\alpha_1x_1+\alpha_2x_2\right\}C_D\in\mathbf{L}^2$*.*

Then the price of a two-asset barrier correlation option (see (3.70) for the payoff profile) is given by:

$$\begin{aligned}
C_{2C}(t,S_1,S_2,K_1,K_2,B_1,B_2,v) &= \frac{e^{-\alpha_1k_1-\alpha_2k_2}}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-i(w_1k_1+w_2k_2)}\\
&\quad\hat{c}_{2C}(u_1,u_2)dw_2dw_1, \qquad (3.88)
\end{aligned}$$

where

$$\begin{aligned}
\hat{c}_{2C}(w_1,w_2) &= \frac{e^{-\int_t^T r(s)ds+c_1x_1+c_2x_2}e^{\left(\alpha_1^d+1-c_1+iw_1\right)b_1+\left(\alpha_2^d+1-c_2+iw_2\right)b_2}}{\left(\alpha_1^d+iw_1\right)\left(\alpha_1^d+1+iw_1\right)\left(\alpha_2^d+iw_2\right)\left(\alpha_2^d+1+iw_2\right)}\\
&\quad\sum_{k=0}^{n-1}\left(\varphi\left(\tau,\mathbf{u},-\mathbf{z}_k+,v\right)-\varphi\left(\tau,\mathbf{u},-\mathbf{z}_k^-,v\right)\right),\\
u_1' &= \sigma_1\sqrt{1-\rho^2}\left(w_1-i\left(\alpha_1^d+1-c_1\right)\right),\\
u_2' &= \sigma_2(w_2-i(\alpha_2^d+1-c_2))+\rho\sigma_1\left(w_1-i\left(\alpha_1^d+1-c_1\right)\right),
\end{aligned}$$

where φ*,* $\mathbf{z}_{k1}^{(-)}$ *and* $\mathbf{z}_{k2}^{(-)}$ *are given Theorem 34. The convergence is in* $\mathbf{L}^2$*-norm.*

3.4.6 Random correlations

The restriction of the solution for the two-asset barrier options to correlations $\rho = -\cos\left(\frac{\pi}{n}\right)$ with $n \in \mathbb{N}$ might seem rather restrictive because all possible correlations are negative for $n > 2$. We can, however, loosen this constraint. The assumption that the correlation takes the attainable values ρ_n randomly with a probability p_n makes positive expected correlations achievable. For example, if we assume that $p_1 = p_2 = \frac{1}{2}$ and $p_i = 0$, for $i > 2$ then the expected correlation is $\frac{1}{2}$ with variance $\frac{1}{4}$.

Theorem 35. *(Random correlation)*

Let us assume that the correlation ρ can take any value $\rho_n = -\cos\frac{\pi}{n}$ with a positive probability p_n and that this probability p_n is known. Then, the value of any derivative is given as the weighted sum

$$C(t, S_1, S_2, v) = \sum_{i=1}^{\infty} p_n C_{(\rho=-cos\frac{\pi}{n})}(t, S_1, S_2, v). \tag{3.89}$$

Remark 13. *Given this assumption $E(\rho) > 0$ is attainable, although most of the possible values for ρ are actually negative.*

3.4.7 Conclusion

In this section we have shown how to derive easily attainable and quickly computable solutions to a range of two-asset barrier options in a stochastic covariance framework for a special correlation structure. This framework is extended to allow for a random correlation structure. The formulas derived can be computed via fast Fourier transform following the transformations done by Dempster and Hong [33] but even without the adoption of the fast Fourier transform the formulas can be quickly evaluated.

3.5 Pricing of two-asset barrier options with PDE techniques

In the following we use the method of separation and the solution to well-known ODEs to find pricing techniques for valuing options for any ρ. In Subsection 3.5.1 we use our knowledge about affine characteristic functions and the PDEs which they solve to find a simple extension to the solution He et al. proposed for two correlated geometric Brownian motions for our three-factor stochastic covariance Model (3.2). Moreover, we derive the survival probabilities.

3.5.1 General pricing formulas for two-asset barrier options exploiting the affine form in v

The System (3.6) can be solved for a group of models included in the general framework (3.2), i.e. for specific values of the parameters γ and ν. This group is characterised by the existence of an affine characteristic function. For these models a feasible and easily manageable solution for the pricing of two-asset barrier options can be found:

Theorem 36. *(Barrier option pricing in $\mathbb{R}^2$ with PDE technique)*

Let us assume the setting described in Equation (3.2) and the existence of an affine characteristic function. Then the price of a two-asset barrier option (see (3.4) for the payoff profile) in the three-factor stochastic volatility model is given by

$$\begin{aligned} C_B\left(t,S_1,S_2,B_1,B_2,v\right) &= \frac{e^{x_1c_1+x_2c_2-\int_t^T r(s)ds}}{\sigma_1\sigma_2\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1',x_2') \\ &\quad e^{-x_1'c_1-x_2'c_2} q\left(\tau,x_1,x_2,x_1',x_2',v\right)dx_1'dx_2', \end{aligned} \tag{3.90}$$

where

$$\begin{aligned} x_i &= \ln\frac{S_i e^{\int_t^T r(s)ds}}{K_i}, \\ q\left(\tau,x_1,x_2,x_1',x_2',v\right) &= \frac{2}{\beta_p}\int_0^{\infty}\lambda V(\tau,\lambda) \\ &\quad \sum_{n=1}^{\infty}\sin\left(\frac{n\pi\theta_p'}{\beta_p}\right)\sin\left(\frac{n\pi\theta_p}{\beta_p}\right)J_{\frac{n\pi}{\beta_p}}\left(\lambda r_p\right)J_{\frac{n\pi}{\beta_p}}\left(\lambda r_p'\right)d\lambda, \end{aligned} \tag{3.91}$$

$$\begin{aligned} \tan\beta_p &= -\frac{\sqrt{1-\rho^2}}{\rho},\ \beta\in[0,\pi], \\ r_p &= \sqrt{\frac{1}{(1-\rho^2)}\left(\frac{x_1-b_1}{\sigma_1}-\rho\frac{x_2-b_2}{\sigma_2}\right)^2+\left(\frac{x_2-b_2}{\sigma_2}\right)^2}, \\ \tan(\theta_p) &= \frac{\frac{x_2-b_2}{\sigma_2}}{\frac{1}{\sqrt{1-\rho^2}}\left(\frac{x_1-b_1}{\sigma_1}-\rho\frac{x_2-b_2}{\sigma_2}\right)},\ \theta_p\in[0,\beta_p], \end{aligned}$$

$$c_1=\frac{\sigma_1-\sigma_2\rho}{2\sigma_1(1-\rho^2)},\qquad c_2=\frac{\sigma_2-\sigma_1\rho}{2\sigma_2(1-\rho^2)},$$

where $V(\tau,\lambda)$ solves the PDE given in (3.100) and $J_\varsigma(x)$ denotes the Bessel function of the first kind.

Proof.

We start from (3.32) and separate $q(\tau,z_1,z_2,v)=H(z_1,z_2)V(\tau,v)$. We get

$$\begin{aligned}\frac{\partial V}{\partial \tau}H &= \frac{1}{2}v^{2\nu}V\frac{\partial^2 H}{\partial z_1^2}+\frac{1}{2}v^{2\nu}V\frac{\partial^2 H}{\partial z_2^2}-v^{2\nu}\frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}{8(1-\rho^2)}HV\\ &+\frac{1}{2}\sigma_v^2v^{2\gamma}\frac{\partial^2 V}{\partial v^2}H+\kappa(\zeta-v)\frac{\partial V}{\partial v}H. \end{aligned}\tag{3.92}$$

Dividing by H we obtain

$$\begin{aligned}\frac{\partial V}{\partial \tau} &= \frac{1}{2}v^{2\nu}\frac{V}{H}\left(\frac{\partial^2 H}{\partial z_1^2}+\frac{\partial^2 H}{\partial z_2^2}\right)-v^{2\nu}\frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}{8(1-\rho^2)}V\\ &+\frac{1}{2}\sigma_v^2v^{2\gamma}\frac{\partial^2 V}{\partial v^2}+\kappa(\zeta-v)\frac{\partial V}{\partial v}. \end{aligned}\tag{3.93}$$

Furthermore, we set

$$\frac{1}{2}\left(\frac{\partial^2 H}{\partial z_1^2}+\frac{\partial^2 H}{\partial z_2^2}\right)=-\frac{\lambda^2}{2}H. \tag{3.94}$$

By transforming z_1 and z_2 in Equation (3.94) to polar coordinates the vertical boundary is described by the angle $\tan\theta_p=-\frac{\sqrt{1-\rho^2}}{\rho}$ and the horizontal boundary by $\theta_p=0$. Thus, the bounded area for which the PDE is defined is a wedge $Y=\{(r_p\cos(\theta_p),r_p\sin(\theta_p)):r_p>0,\ 0<\theta_p<\beta_p\}\subset\mathbb{R}^2$, where $\tan\beta_p=-\frac{\sqrt{1-\rho^2}}{\rho}$, $\beta_p\in[0,\pi]$. The boundary of the wedge is described by $\partial Y=\{(r_p\cos(\theta_p),r_p\sin(\theta_p)):r_p\geq 0,\ \theta_p\in\{0,\beta_p\}\}\subset\mathbb{R}^2$. Choosing a separable solution of the form $R(r_p)\Theta(\theta_p)$ we get the following relationship

$$\left(r_p^2\frac{\frac{d^2R}{dr_p^2}}{R}+r_p\frac{\frac{dR}{dr_p}}{R}+\lambda^2r_p^2\right)+\left(\frac{\frac{d^2\Theta}{d\theta_p^2}}{\Theta}\right)=0. \tag{3.95}$$

We define $\frac{\frac{d^2\Theta}{d\theta_p^2}}{\Theta}=-k^2$ and find

$$\Theta(\theta_p)\sim A\sin(k\theta_p)+B\cos(k\theta_p). \tag{3.96}$$

Θ has to fulfil the boundary conditions $\Theta(0)=\Theta(\beta_p)=0$. Thus, $B=0$ as k is real. Hence,

$$k_n=\frac{n\pi}{\beta_p},\ n=1,2\ldots \tag{3.97}$$

The radial part is given by

$$(\lambda r_p)^2 \frac{d^2 R}{d(\lambda r_p)^2} + (\lambda r_p) \frac{dR}{d(\lambda r_p)} + \left((\lambda r_p)^2 - {k_n}^2\right) R = 0. \tag{3.98}$$

As $R(0)$ has to be well-behaved, the general solution of the radial part is

$$R(r_p) \sim J_{k_n}(\lambda r_p), \tag{3.99}$$

where $J_\varsigma(x)$ is the Bessel function of the first kind. We insert (3.94) in (3.93) and get

$$\begin{aligned} \frac{\partial V}{\partial \tau} &= -\frac{1}{2} v^{2\nu} \lambda^2 V - v^{2\nu} \frac{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}{8(1-\rho^2)} V \\ &\quad + \frac{1}{2}\sigma_v^2 v^{2\gamma} \frac{\partial^2 V}{\partial v^2} + \kappa(\zeta - v)\frac{\partial V}{\partial v}, \qquad (3.100) \\ V(0, v) &= 1. \qquad (3.101) \end{aligned}$$

If the process possesses an affine-type characteristic function, (3.100) collapses to a system of ODEs and the solution can be given by

$$\begin{aligned} q\left(\tau, r_p, \theta_p, r_p', \theta_p', v\right) &= \int_0^\infty V(\tau, \lambda) \sum_{n=1}^\infty c_n(\lambda) \\ &\quad \sin\left(\frac{n\pi\theta_p}{\beta_p}\right) J_{\frac{n\pi}{\beta_p}}(\lambda r_p)\, d\lambda. \end{aligned} \tag{3.102}$$

To determine $c_n(\lambda)$ we use the initial condition $q(0, r_p, \theta_p, v) = \frac{1}{r_p'}\delta\left(r_p - r_p'\right)\delta\left(\theta_p - \theta_p'\right)$. Multiply (3.102) at $\tau = 0$ by $\sin\left(\frac{m\pi\theta_p}{\beta_p}\right)$ and integrate over θ_p.

$$\frac{1}{r_p'}\delta\left(r_p - r_p'\right) \sin\left(\frac{m\pi\theta_p'}{\beta_p}\right) = \frac{\beta_p}{2}\int_0^\infty c_m(\lambda) J_{\frac{m\pi}{\beta_p}}(\lambda r_p)\, d\lambda, \tag{3.103}$$

where we use the integral identity $\int_0^\pi \sin(mx)\sin(nx)\, dx = \frac{\pi}{2}\delta(m-n)$ (see Arfken [7], p. 632) and, thus, $\int_0^{\beta_p} \sin\left(\frac{m\pi\theta_p}{\beta_p}\right)\sin\left(\frac{n\pi\theta_p}{\beta_p}\right) d\theta = \frac{\beta_p}{2}\delta(m-n)$.

Finally, multiplying by $r_p J_{\frac{m\pi}{\beta_p}}(\lambda' r_p)$ and integrating over r_p results in the following solution for $c_n(\lambda)$

$$c_m(\lambda') = \frac{2\lambda'}{\beta_p} \sin\left(\frac{m\pi\theta_p'}{\beta_p}\right) J_{\frac{m\pi}{\beta_p}}\left(\lambda' r_p'\right), \tag{3.104}$$

where we use the Bessel function closure equation $\int_0^\infty x J_\varsigma(ax) J_\varsigma(bx) dx = \frac{1}{\mathfrak{a}}\delta(a-b)$ (see

Arfken [7], p. 648). Thus,

$$q(\tau, r_p, \theta_p, r'_p, \theta'_p, v) = \frac{2}{\beta_p}\int_0^\infty \lambda V(\tau,\lambda) \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta'_p}{\beta_p}\right)\sin\left(\frac{n\pi\theta_p}{\beta_p}\right) J_{\frac{n\pi}{\beta_p}}\left(\lambda r_p\right) J_{\frac{n\pi}{\beta_p}}\left(\lambda r'_p\right) d\lambda.$$

and,

$$p(x_1(\tau)\in dx_1, x_2(\tau)\in dx_2, \underline{x}_1(\tau) > b_1, \underline{x}_2(\tau) > b_2, v) = \\ = \frac{e^{c_1x_1+c_2x_2-\int_t^T r(s)ds}}{\sigma_1\sigma_2\sqrt{1-\rho^2}} q\left(\tau, x_1, x_2, x'_1, x'_2, v\right) dx'_1 dx'_2, \tag{3.105}$$

where $\underline{x}_i$ denotes the minimum value x_i takes in τ and $\frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}}$ is the Jacobi determinant. □

Remark 14. *If the three-factor model degenerates to a two-factor GBM model then (3.91) is consistent with the formula He et al. [64] found.*

$$q_{GBM}\left(\tau, x_1, x_2, x'_1, x'_2, v\right) = \frac{2}{\beta_p\tau} e^{-\frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}{8(1-\rho^2)}\tau} \sum_{n=1}^{\infty} e^{-\frac{r_p^2+{r'_p}^2}{2\tau}} \sin\left(\frac{n\pi\theta'_p}{\beta_p}\right)\sin\left(\frac{n\pi\theta_p}{\beta_p}\right) I_{\frac{n\pi}{\beta_p}}\left(\frac{r'_p r_p}{\tau}\right), \tag{3.106}$$

where $I_\varsigma(x)$ denotes the modified Bessel function of the first kind.

Proof.

If the three-factor model degenerates to a two-factor GBM model then (3.91) is consistent with the formula He et al. [64] found:

$$\begin{aligned} q_{GBM}(\tau, x_1, x_2, v) &= \frac{2}{\beta_p}\int_0^\infty \lambda e^{-\frac{1}{2}\left(\lambda^2\tau + \frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}{4(1-\rho^2)}\tau\right)} \\ &\quad \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta'_p}{\beta_p}\right)\sin\left(\frac{n\pi\theta_p}{\beta_p}\right) J_{\frac{n\pi}{\beta_p}}\left(\lambda r_p\right) J_{\frac{n\pi}{\beta_p}}\left(\lambda r'_p\right) d\lambda \\ &= \frac{2}{\beta_p\tau} e^{-\frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}{8(1-\rho^2)}\tau} \\ &\quad \sum_{n=1}^{\infty} e^{-\frac{r_p^2+{r'_p}^2}{2\tau}} \sin\left(\frac{n\pi\theta'_p}{\beta_p}\right)\sin\left(\frac{n\pi\theta_p}{\beta_p}\right) I_{\frac{n\pi}{\beta_p}}\left(\frac{r'_p r_p}{\tau}\right), \end{aligned} \tag{3.107}$$

where we used the formula (see Gradshtein [59], 6.633)

$$\int_0^\infty xe^{-c^2x}J_\varsigma(ax)J_\varsigma(bx)dx = \frac{1}{2c^2}e^{-\frac{(a^2+b^2)}{4c^2}}I_\varsigma\left(\frac{ab}{2c^2}\right), \tag{3.108}$$

with $I_\varsigma\left(\frac{ab}{2c^2}\right)$, the modified Bessel function of the first kind. □

One of the models for which the affine characteristic function exists is the model with the Heston-type third factor. The price of a two-asset barrier option is presented in the following corollary.

Corollary 10. *(Barrier option pricing in Heston-type model in $\mathbb{R}^2$ with PDE technique) Let us assume the setting described in Equations (3.2) and (3.3) with $\nu = \gamma = \frac{1}{2}$. Then the price of a two-asset barrier option (see (3.4) for the payoff profile) in the three-factor stochastic volatility model is*

$$\begin{aligned} C_B(t, S_1, S_2, B_1, B_2, v) &= \frac{e^{x_1c_1+x_2c_2-\int_t^T r(s)ds}}{\sigma_1\sigma_2\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1', x_2') \\ &\quad e^{-x_1'c_1-x_2'c_2}q(\tau, x_1, x_2, x_1', x_2', v)\,dx_1'dx_2', \end{aligned} \tag{3.109}$$

where

$$\begin{aligned} x_i &= \ln\left(\frac{S_ie^{\int_t^T r(s)ds}}{K_i}\right), \\ q(\tau, x_1, x_2, x_1', x_2', v) &= \frac{2}{\beta_p}\int_0^\infty \lambda e^{\frac{1}{\sigma_v^2}(A_H(\tau,\lambda)+B_H(\tau,\lambda)v)} \\ &\quad \sum_{n=1}^{\infty}\sin\left(\frac{n\pi\theta_p'}{\beta_p}\right)\sin\left(\frac{n\pi\theta_p}{\beta_p}\right)J_{\frac{n\pi}{\beta_p}}(\lambda r_p)J_{\frac{n\pi}{\beta_p}}(\lambda r_p')\,d\lambda, \\ \tan\beta_p &= -\frac{\sqrt{1-\rho^2}}{\rho},\ \beta\in[0,\pi], \\ r_p &= \sqrt{\frac{1}{(1-\rho^2)}\left(\frac{x_1-b_1}{\sigma_1}-\rho\frac{x_2-b_2}{\sigma_2}\right)^2+\left(\frac{x_2-b_2}{\sigma_2}\right)^2}, \\ \tan(\theta_p) &= \frac{\frac{x_2-b_2}{\sigma_2}}{\frac{1}{\sqrt{1-\rho^2}}\left(\frac{x_1-b_1}{\sigma_1}-\rho\frac{x_2-b_2}{\sigma_2}\right)},\ \theta_p\in[0,\beta_p], \\ \mathfrak{d} = \mathfrak{d}(\lambda) &= \sqrt{\kappa^2+\sigma_v^2\left(\lambda^2+\frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}{4(1-\rho^2)}\right)}, \\ c_1 = \frac{\sigma_1-\sigma_2\rho}{2\sigma_1(1-\rho^2)}, &\qquad c_2 = \frac{\sigma_2-\sigma_1\rho}{2\sigma_2(1-\rho^2)}, \end{aligned} \tag{3.110}$$

where A_H and B_H are given in (3.49) and (3.48).

Proof.

Inserting (A.24) in (3.100) we find the same ODEs as before in (A.1.2). The solutions follow. □

We have seen before that the Ornstein-Uhlenbeck process also possesses a characteristic function. The pricing formula for the two-asset barrier option can, thus, be indicated in the same way as for the Heston-type model:

Corollary 11. *(Barrier option pricing in a Stein and Stein-type model in $\mathbb{R}^2$ with PDE technique)*

Let us assume the setting described in Equation (3.2) with $\nu = 1$ and $\gamma = 0$. Then the price of a two-asset barrier option (see (3.4) for the payoff profile) in the three-factor stochastic volatility model is given by

$$\begin{aligned} C_B(t, S_1, S_2, B_1, B_2, v) &= \frac{e^{x_1c_1+x_2c_2-\int_t^T r(s)ds}}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1', x_2') \\ &\quad e^{-x_1'c_1-x_2'c_2} q(\tau, x_1, x_2, x_1', x_2', v)\, dx_1' dx_2', \end{aligned} \tag{3.111}$$

where

$$\begin{aligned} x_i &= \ln\left(\frac{S_i e^{\int_t^T r(s)ds}}{K_i}\right), \\ q(\tau, x_1, x_2, x_1', x_2', v) &= \frac{2}{\beta_p}\int_0^\infty \lambda e^{\frac{1}{\sigma_v^2}\left(A_{S2}(\tau,\lambda)+B_{S2}(\tau,\lambda)v+C_{S2}(\tau,\lambda)v^2\right)} \\ &\quad \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta_p'}{\beta_p}\right)\sin\left(\frac{n\pi\theta_p}{\beta_p}\right) J_{\frac{n\pi}{\beta_p}}(\lambda r_p)\, J_{\frac{n\pi}{\beta_p}}(\lambda r_p')\, d\lambda, \\ \tan\beta_p &= -\frac{\sqrt{1-\rho^2}}{\rho},\ \beta \in [0, \pi], \\ r_p &= \sqrt{\frac{1}{(1-\rho^2)}\left(\frac{x_1-b_1}{\sigma_1} - \rho\frac{x_2-b_2}{\sigma_2}\right)^2 + \left(\frac{x_2-b_2}{\sigma_2}\right)^2}, \\ \tan(\theta_p) &= \frac{\frac{x_2-b_2}{\sigma_2}}{\frac{1}{\sqrt{1-\rho^2}}\left(\frac{x_1-b_1}{\sigma_1} - \rho\frac{x_2-b_2}{\sigma_2}\right)},\ \theta_p \in [0, \beta], \end{aligned} \tag{3.112}$$

where A_{S2}, B_{S2} and C_{S2} are given in (3.59-3.61).

Proof.

Inserting (A.37) in (3.100) we find the same ODEs as before in (A.1.2). The solutions follow. □

Using the Result (3.91) we can derive some more densities along the lines of Iyengar [71], i.e. the density of the first hitting time ι' and $x_i(\iota')$ and the survival probability, i.e. $p^{\mathbf{x}}(\iota' \in dt', \mathbf{x}(\iota') \in \partial D)$ and $p^{\mathbf{x}}(\iota' < t')$ with $0 \leq t \leq t' \leq T$.
By using the same arguments as Daniels [28], Iyengar [71] and Metzler [88] we derive the probability density $p^{\mathbf{x}}(\iota' \in dt', \mathbf{x}(\iota') \in \partial D)$ of the hitting time ι', where $\iota' = \min(\iota_1, \iota_2)$, and the density of $\mathbf{x}(\iota') = (x_1(\iota'), x_2(\iota'))$ in ι'. ∂D describes the boundaries with $(b_1, x_2(\iota')) \bigcup (x_1(\iota'), b_2) \subset \partial D$.

Theorem 37. *($p^{\mathbf{x}}(\iota' \in dt', \mathbf{x}(\iota') \in \partial D)$)*

The probability density function $p^{\mathbf{x}}(\iota' \in dt', \mathbf{x}(\iota') \in \partial D)$ is given by

$$\begin{aligned} p^{\mathbf{x}}\left(\iota' \in dt', \mathbf{x}(\iota') \in \partial D\right) &= e^{c_1 x_1 + c_2 x_2 - \int_t^{t'} r(s)ds} \frac{\pi}{\beta_p^2 r_p} \int_0^\infty \lambda V\left(t', \lambda\right) \\ &\quad \sum_{n=1}^{\infty} n\delta_n \sin\left(\frac{n\pi\theta_p}{\beta_p}\right) \\ &\quad J_{\frac{n\pi}{\beta_p}}\left(\lambda r_p\right) J_{\frac{n\pi}{\beta_p}}\left(\lambda r_p'\right) d\lambda \partial D dt', \end{aligned} \tag{3.113}$$

where $\delta_n = (-1)^{n+1}$ if $\theta_p' = \beta_p$ and $\delta_n = 1$ if $\theta_p' = 0$,

$$\begin{aligned} x_i &= \ln\left(\frac{S_i e^{\int_t^T r(s)ds}}{K_i}\right), \\ r_p &= \sqrt{\frac{1}{(1-\rho^2)}\left(\frac{x_1 - b_1}{\sigma_1} - \rho\frac{x_2 - b_2}{\sigma_2}\right)^2 + \left(\frac{x_2 - b_2}{\sigma_2}\right)^2}, \\ \tan\left(\theta_p\right) &= \frac{\frac{x_2 - b_2}{\sigma_2}}{\frac{1}{\sqrt{1-\rho^2}}\left(\frac{x_1 - b_1}{\sigma_1} - \rho\frac{x_2 - b_2}{\sigma_2}\right)}. \end{aligned}$$

Proof.

Using a similar argument to Daniels [28], Iyengar [71] and Metzler [88] and transforming to polar coordinates it can be shown that

$$\begin{aligned} p^{\mathbf{z}}\left(\iota' \in dt', \mathbf{z}(\iota') \in \partial Y\right) &= e^{c_1 x_1 + c_2 x_2 - \int_t^{t'} r(s)ds} \\ &\quad q^{\mathbf{z}}\left(\iota' \in dt', \mathbf{z}(\iota') \in \partial Y\right) dz_1 dz_2 \\ &= \frac{e^{c_1 x_1 + c_2 x_2 - \int_t^{t'} r(s)ds}}{2} \\ &\quad \left[\frac{\partial}{\partial \mathbf{n}} q\left(t, r_p, \theta_p, t', r_p', \theta_p'\right) r_p' \Big|_{\mathbf{z} \in \partial Y}\right] dr_p' d\theta_p', \end{aligned}$$

where $(b_1, x_2(\iota')) \bigcup (x_1(\iota'), b_2) \subset \partial D$, ∂Y describes ∂D in polar coordinates. $\frac{\partial}{\partial \mathbf{n}}$ denotes the normal derivative, i.e. a directional derivative taken in the inward normal (orthogonal) direction, to the boundary ∂Y in point $\mathbf{z}$ ($\mathbf{z} \in \partial Y$). $\mathbf{x} = \left(\sigma_1\left(\sqrt{1-\rho^2}z_1 + \rho z_2\right) + b_1, \sigma_2 z_2 + b_2\right)$ and $(z_1, z_2) = r_p\left(\cos\left(\theta'_p\right), \sin\left(\theta'_p\right)\right)$. In our case the (unit) normal vector $\mathbf{n}$ on the boundary is either $(0,1)$ for $\mathbf{z} = (r_p, 0)$, i.e. $z_2 = 0$, or $(\sin(\beta_p), -\cos(\beta_p))$ for $\mathbf{z} = r_p(\cos(\beta_p), \sin(\beta_p))$, i.e. $z_2 = -\frac{\sqrt{1-\rho^2}}{\rho}z_1$. With $\frac{\partial q}{\partial \mathbf{n}} = \nabla q \cdot \mathbf{n} = \left(\frac{\partial q}{\partial r'_p}, \frac{\partial q}{r'_p \partial \theta'_p}\right) \cdot (0,1) = \frac{1}{r'_p}\frac{\partial q}{\partial \theta'_p}$ we obtain for $\mathbf{z} = (r_p, 0)$

$$
\begin{aligned}
q^{\mathbf{z}}\left(\iota' \in dt', \mathbf{z}(\iota') \in \partial Y\right) &= \Bigg[\frac{1}{2r'_p}\frac{\partial}{\partial \theta'_p}\frac{2r'_p}{\beta_p}\int_0^\infty \lambda V\left(t', \lambda\right) \\
&\quad \sum_{n=1}^\infty \sin\left(\frac{n\pi\theta'_p}{\beta_p}\right)\sin\left(\frac{n\pi\theta_p}{\beta_p}\right) \\
&\quad J_{\frac{n\pi}{\beta_p}}\left(\lambda r_p\right) J_{\frac{n\pi}{\beta_p}}\left(\lambda r'_p\right) d\lambda dr'_p d\theta'_p\Bigg]_{\theta'_p=0, r'_p \geq 0} \\
&= \Bigg[\frac{1}{\beta_p}\int_0^\infty \lambda V\left(t', \lambda\right) \\
&\quad \sum_{n=1}^\infty \frac{n\pi}{\beta_p}\cos\left(\frac{n\pi\theta'_p}{\beta_p}\right)\sin\left(\frac{n\pi\theta_p}{\beta_p}\right) \\
&\quad J_{\frac{n\pi}{\beta_p}}\left(\lambda r_p\right) J_{\frac{n\pi}{\beta_p}}\left(\lambda r'_p\right) d\lambda dr'_p d\theta'_p\Bigg]_{\theta'_p=0, r'_p \geq 0} \\
&= \frac{\pi}{\beta_p^2}\int_0^\infty \lambda V\left(t', \lambda\right) \\
&\quad \sum_{n=1}^\infty n \sin\left(\frac{n\pi\theta_p}{\beta_p}\right) J_{\frac{n\pi}{\beta_p}}\left(\lambda r_p\right) J_{\frac{n\pi}{\beta_p}}\left(\lambda r'_p\right) d\lambda dr'_p d\theta'_p \\
&= \frac{\pi}{r'_p \beta_p^2}\int_0^\infty \lambda V\left(t', \lambda\right) \sum_{n=1}^\infty n \sin\left(\frac{n\pi\theta_p}{\beta_p}\right) \\
&\quad J_{\frac{n\pi}{\beta_p}}\left(\lambda r_p\right) J_{\frac{n\pi}{\beta_p}}\left(\lambda r'_p\right) d\lambda dz'_1 dz'_2. \qquad (3.114)
\end{aligned}
$$

The respective probability for $\mathbf{z} = r_p(\cos(\beta_p), \sin(\beta_p))$ can be derived by reflecting $\mathbf{z}$ about the line $z_2 = \tan\left(\frac{\beta_p}{2}\right) z_1$, i.e. the reflected $\tilde{\theta_p} = \beta_p - \theta_p$.

We insert the reflection in (3.114) and compute

$$\begin{aligned} q^{\mathbf{z}}\left(\iota' \in dt', \mathbf{z}(\iota') \in \partial Y\right) &= \frac{\pi}{r'_p \beta_p^2} \int_0^\infty \lambda V\left(t',\lambda\right) \sum_{n=1}^{\infty} n \sin\left(\frac{n\pi\tilde{\theta}_p}{\beta_p}\right) \\ &\quad J_{\frac{n\pi}{\beta_p}}\left(\lambda r_p\right) J_{\frac{n\pi}{\beta_p}}\left(\lambda r'_p\right) d\lambda dz'_1 dz'_2 \\ &= -\frac{\pi}{r'_p \beta_p^2} \int_0^\infty \lambda V\left(t',\lambda\right) \sum_{n=1}^{\infty} n \sin\left(\frac{n\pi\theta_p}{\beta_p}\right) \\ &\quad \cos\left(n\pi\right) J_{\frac{n\pi}{\beta_p}}\left(\lambda r_p\right) J_{\frac{n\pi}{\beta_p}}\left(\lambda r'_p\right) d\lambda dz'_1 dz'_2. \end{aligned} \tag{3.115}$$

□

The survival probability can be obtained by integrating (3.91) over the wedge (see Iyengar [71] and Metzler [88]).

Theorem 38. *(Survival probability)*

The survival probability $p^{\mathbf{x}}(\iota' > t')$ *is given by the following expression in polar coordinates.*

$$\begin{aligned} p^{\mathbf{x}}(\iota' > t') &= e^{c_1 x_1 + c_2 x_2 - \int_t^{t'} r(s) ds} \sum_{n=1,3,5...} \frac{4}{\pi n} \sin\left(\frac{n\pi\theta_p}{\beta_p}\right) \int_0^\infty V\left(t',\lambda\right) J_{\frac{n\pi}{\beta_p}}\left(\lambda r_p\right) \\ &\quad \int_0^\infty r'_p J_{\frac{n\pi}{\beta_p}}\left(\lambda r'_p\right) dr'_p d\lambda, \end{aligned} \tag{3.116}$$

where

$$\begin{aligned} x_i &= \ln\left(\frac{S_i e^{\int_t^T r(s) ds}}{K_i}\right), \\ r_p &= \sqrt{\frac{1}{(1-\rho^2)}\left(\frac{x_1 - b_1}{\sigma_1} - \rho\frac{x_2 - b_2}{\sigma_2}\right)^2 + \left(\frac{x_2 - b_2}{\sigma_2}\right)^2}, \\ \tan(\theta_p) &= \frac{\frac{x_2 - b_2}{\sigma_2}}{\frac{1}{\sqrt{1-\rho^2}}\left(\frac{x_1 - b_1}{\sigma_1} - \rho\frac{x_2 - b_2}{\sigma_2}\right)}, \ \theta_p \in [0, \beta_p]. \end{aligned}$$

Proof.

The probability that x_1 and x_2 do not hit the barrier during lifetime in $0 \le t \le t' \le T$ can be obtained by integrating (3.91) over the wedge. This is facilitated by converting to polar coordinates.

$$\begin{aligned}
p^{\mathsf{x}}(\iota' > t') &= e^{c_1x_1+c_2x_2-\int_t^{t'} r(s)ds} \int_0^{\beta_p}\int_0^{\infty} q\left(t, r_p, \theta_p, r_p', \theta_p', v\right) r_p' dr_p' d\theta_p' \\
&= e^{c_1x_1+c_2x_2-\int_t^{t'} r(s)ds} \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin\left(\frac{n\pi\theta_p}{\beta_p}\right) \int_0^{\beta_p} \frac{n\pi}{\beta_p} \sin\left(\frac{n\pi\theta_p'}{\beta_p}\right) d\theta_p' \\
&\quad \int_0^{\infty} \lambda V\left(t', \lambda\right) J_{\frac{n\pi}{\beta_p}}\left(\lambda r_p\right) \int_0^{\infty} r_p' J_{\frac{n\pi}{\beta_p}}\left(\lambda r_p'\right) dr_p' d\lambda \\
&= e^{c_1x_1+c_2x_2-\int_t^{t'} r(s)ds} \sum_{n=1,3,5\ldots} \frac{4}{\pi n} \sin\left(\frac{n\pi\theta_p}{\beta_p}\right) \int_0^{\infty} \lambda V\left(t', \lambda\right) J_{\frac{n\pi}{\beta_p}}\left(\lambda r_p\right) \\
&\quad \int_0^{\infty} r_p' J_{\frac{n\pi}{\beta_p}}\left(\lambda r_p'\right) dr_p' d\lambda.
\end{aligned}$$

If the three-factor model is degenerated to the two-factor model the probability is given by (see Metzler [88])

$$\begin{aligned}
p_{GBM}^{\mathsf{x}}(\iota' > t') &= e^{c_1x_1+c_2x_2-\int_t^{t'} r(s)ds} \sum_{n=1,3,5\ldots} \frac{4}{\pi n t'} \sin\left(\frac{n\pi\theta_p}{\beta_p}\right) e^{-\frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}{8(1-\rho^2)}t'} \\
&\quad \int_0^{\infty} r_p' e^{-\frac{\left(r_p^2+{r_p'}^2\right)}{2t'}} I_{\frac{n\pi}{\beta_p}}\left(\frac{r_p r_p'}{t'}\right) dr_p' \\
&= e^{c_1x_1+c_2x_2-\int_t^{t'} r(s)ds} \sum_{n=1,3,5\ldots} \frac{4}{\pi n t'} \sin\left(\frac{n\pi\theta_p}{\beta_p}\right) e^{-\frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}{8(1-\rho^2)}t'} \\
&\quad r_p \int_0^{\infty} e^{-\frac{(r_p^2+{r_p'}^2)}{2t'}} \frac{1}{2}\left(I_{\frac{n\pi}{\beta_p}-1}\left(\frac{r_p r_p'}{t'}\right) + I_{\frac{n\pi}{\beta_p}+1}\left(\frac{r_p r_p'}{t'}\right)\right) dr_p' \\
&= e^{c_1x_1+c_2x_2-\int_t^{t'} r(s)ds} \frac{2r_p}{\sqrt{2\pi t'}} e^{-\frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}{8(1-\rho^2)}t'} e^{-\frac{r_p^2}{4t'}} \\
&\quad \sum_{n=1,3,5\ldots} \frac{1}{n} \sin\left(\frac{n\pi\theta_p}{\beta_p}\right) \left(I_{\frac{\varpi-1}{2}}\left(\frac{r_p^2}{4t'}\right) + I_{\frac{\varpi+1}{2}}\left(\frac{r_p^2}{4t'}\right)\right),
\end{aligned}$$

where we used the identities $2I_\varsigma'(x) = I_{\varsigma-1}(x) + I_{\varsigma+1}(x)$ (see Abramowitz and Stegun [1], 9.6.26) and $\int_0^\infty e^{-\mathfrak{b}t^2} I_\varsigma(\mathfrak{a}t)dt = \frac{1}{2}\sqrt{\frac{\pi}{\mathfrak{b}}} e^{\frac{\mathfrak{a}^2}{8\mathfrak{b}}} I_{\frac{\varsigma}{2}}(\frac{\mathfrak{a}^2}{8\mathfrak{b}})$ (see Gradshtein [59], 6.618). □

3.5.2 Pricing of two-asset double-digital barrier options with PDE techniques

In the following we are going to derive formulas to price two-asset barrier double-digital options with two time-dependent barriers.

Corollary 12. *(Double-digital barrier option in Heston-type model with PDE technique)*

Let us assume the setting described in Equations (3.2) and (3.3) with $\nu = \gamma = \frac{1}{2}$. *Then the price of a two-asset barrier double-digital option (see (3.66) for the payoff profile) is given by*

$$C_{2D}(t, S_1, S_2, K_1, K_2, B_1, B_2, v) = \frac{e^{x_1 c_1 + x_2 c_2 - \int_t^T r(s) ds}}{\sigma_1 \sigma_2 \sqrt{1-\rho^2}} \int_0^\infty \int_0^\infty \left(e^{-c_1 x_1' - c_2 x_2'} \right) q\left(\tau, x_1, x_2, x_1', x_2', v\right) dx_1' dx_2', \qquad (3.117)$$

where $q\left(\tau, x_1, x_2, x_1', x_2', v\right)$ *is given in Formula (3.110).*

Table 3.7: Prices of double-digital options with barriers in Heston-type model (PDE technique).

	$\kappa = 0.6$			$\kappa = 0.9$			$\kappa = 1.2$		
				ζ					
σ_v	0.6	0.9	1.2	0.6	0.9	1.2	0.6	0.9	1.2
0.4	0.1054	0.0973	0.0901	0.1092	0.0978	0.0881	0.1126	0.0982	0.0864
0.6	0.1091	0.1006	0.0931	0.1124	0.1005	0.0904	0.1154	0.1005	0.0883
0.8	0.1142	0.1052	0.0972	0.1168	0.1043	0.0937	0.1191	0.1036	0.0910

$S_1(0) = S_2(0) = 100$, $T = 1.0$, $r = 0.04$, $\sigma_1 = \sigma_2 = 0.5$,
$B_1(0) = B_2(0) = 75$, $K_1 = K_2 = 100$, $v(0) = 1.0$, $\rho = 0$.

Table 3.8: Prices of double-digital options with barriers in Heston-type model (PDE technique).

	$\kappa = 0.6$			$\kappa = 0.9$			$\kappa = 1.2$		
				ζ					
σ_v	0.6	0.9	1.2	0.6	0.9	1.2	0.6	0.9	1.2
0.4	0.1417	0.1325	0.1243	0.1461	0.1331	0.1219	0.1499	0.1336	0.1200
0.6	0.1456	0.1360	0.1274	0.1494	0.1360	0.1244	0.1528	0.1360	0.1221
0.8	0.1507	0.1407	0.1317	0.1539	0.1399	0.1278	0.1566	0.1393	0.1249

$S_1(0) = S_2(0) = 100$, $T = 1.0$, $r = 0.04$, $\sigma_1 = \sigma_2 = 0.5$,
$B_1(0) = B_2(0) = 75$, $K_1 = K_2 = 100$, $v(0) = 1.0$, $\rho = 0.3$.

In a first step we want to compare the prices of the formulas in this chapter with the formulas derived using Fourier techniques (see Section 3.4.3). Thus, we reprice the scenario in Table 3.2 using the PDE based formulas (see Table 3.7). For the following scenario computations we have chosen the parameters: $S_1 = S_2 = 100$, $K_1 = K_2 = 100$, $B_1 = B_2 = 75$, $r = 0.04$, $T = 1$, $\sigma_1 = \sigma_2 = 0.5$, $\rho = 0.3$, $v(0) = 1.0$. In Table 3.8 one can find several scenario computations showing the impact of the various parameters of the covariance process on the price of the derivatives. For the Heston-type model the value of a two-asset digital option decreases for rising mean-reversion levels, i.e. when the overall volatility of the two assets rises. The option increases in value when the volatility of the third process σ_v is raised. The characteristics towards κ, the mean-reversion speed, is ambiguous: When we increase κ from 0.6 to 0.9 for a mean-reversion level of $\zeta = 0.6$ as well as in the case $\zeta = 0.9$ and $\sigma_v = 0.3$ an increase in κ leads to a higher value, however, for higher mean-reversion levels, such as 1.2, an increase in κ reduces the value of the derivative. The value of the digital barrier option reflects, in fact, the probability that both assets stay above the barrier levels during the lifetime of the option and are above the strike levels at maturity. In the scenario we have chosen the probability for that is higher when the overall volatility in the system is low because then the derivative stays above the barriers and ends in the money.

Corollary 13. *(Double-digital barrier option in Stein and Stein-type model with PDE technique)*

Let us assume the setting described in Equation (3.2) with $\nu = 1$ *and* $\gamma = 0$. *Then the price of a two-asset barrier double-digital option (see (3.66) for the payoff profile) is given by*

$$C_{2D}(t, S_1, S_2, K_1, K_2, B_1, B_2, v) = \frac{e^{x_1c_1+x_2c_2-\int_t^T r(s)ds}}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \int_0^\infty \int_0^\infty \left(e^{-c_1x_1'-c_2x_2'}\right) q\left(\tau, x_1, x_2, x_1', x_2', v\right) dx_1' dx_2', \tag{3.118}$$

where $q\left(\tau, x_1, x_2, x_1', x_2', v\right)$ *is given in Formula (3.112).*

Table 3.9: Prices of double-digital options with barriers in Stein-type model (PDE technique).

	$\kappa = 0.6$			$\kappa = 0.9$			$\kappa = 1.2$		
				ζ					
σ_v	0.6	0.9	1.2	0.6	0.9	1.2	0.6	0.9	1.2
0.4	0.1571	0.1429	0.1304	0.1692	0.1503	0.1340	0.1812	0.1585	0.1392
0.6	0.1580	0.1442	0.1316	0.1681	0.1493	0.1324	0.1780	0.1548	0.1344
0.8	0.1564	0.1434	0.1311	0.1647	0.1463	0.1290	0.1725	0.1492	0.1277

$S_1(0) = S_2(0) = 100$, $T = 1.0$, $r = 0.04$, $\sigma_1 = \sigma_2 = 0.5$,
$B_1(0) = B_2(0) = 75$, $K_1 = K_2 = 100$, $v(0) = 1.0$, $\rho = 0.3$.

For the Stein-type model the behaviour towards ζ is comparable to the Heston-type model: The value of the option falls with an increase of the mean-reversion level. Incrementing σ_v we can observe a fall in value except for the scenario $\kappa = 0.6$. The value of the option increases in the Stein-type model when we increase the value of κ and only falls when both σ_v and ζ are very high ($\sigma_v = 0.8$ and $\zeta = 1.2$). The different characteristics of the

Heston and the Stein model can be explained by the different form of the diffusion term of the third factor process: The diffusion in the former model depends on $\sqrt{v}\sigma_v$ while the diffusion of the later only depends on σ_v.

3.5.3 Pricing of two-asset barrier correlation options with PDE techniques

In the following we derive formulas to price two-asset barrier correlation options with time-dependent barriers.

Corollary 14. *(Correlation barrier option in Heston-type model with PDE technique)*

Let us assume the setting described in Equations (3.2) and (3.3) with $\nu = \gamma = \frac{1}{2}$. Then the price of a two-asset barrier correlation option (see (3.70) for the payoff profile) is given by

$$\begin{aligned} C_{2C}(t, S_1, S_2, K_1, K_2, B_1, B_2, v) \quad = \quad & \frac{e^{x_1 c_1 + x_2 c_2 - \int_t^T r(s)ds}}{\sigma_1 \sigma_2 \sqrt{1-\rho^2}} K_1 K_2 \\ & \int_0^\infty \int_0^\infty \left(e^{x_1'(1-c_1)} - e^{-x_1' c_1} \right) \\ & \left(e^{x_2'(1-c_2)} - e^{-x_2' c_2} \right) \\ & q\left(\tau, x_1, x_2, x_1', x_2', v\right) dx_1' dx_2', \end{aligned} \quad (3.119)$$

where $q\left(\tau, x_1, x_2, x_1', x_2', v\right)$ is given in Formula (3.110).

Table 3.10: Prices of correlation options with barriers in Heston-type model (PDE technique).

	$\kappa = 0.6$			$\kappa = 0.9$			$\kappa = 1.2$		
				ζ					
σ_v	0.6	0.9	1.2	0.6	0.9	1.2	0.6	0.9	1.2
0.4	313.87	326.25	337.69	307.90	325.27	340.85	302.73	324.43	343.40
0.6	310.23	322.84	334.50	304.80	322.43	338.27	300.08	322.05	341.28
0.8	305.52	318.31	330.21	300.76	318.64	334.76	296.62	318.85	338.38

$S_1(0) = S_2(0) = 100$, $T = 1.0$, $r = 0.04$, $\sigma_1 = \sigma_2 = 0.5$,
$B_1(0) = B_2(0) = 75$, $K_1 = K_2 = 100$, $v(0) = 1.0$, $\rho = 0$.

Again, we first compare prices computed using the Fourier technique (see Section 3.4.4) with the prices computed with PDE-based formulas. In Table 3.10 it can be seen that the results are quite close to Table 3.5. For the correlation option the picture is different. The payoff of this derivative is not only influenced by the probability that the underlyings stay above the barriers during lifetime and above the strike levels at maturity but also by

Table 3.11: Prices of correlation options with barriers in Heston-type model (PDE technique).

	$\kappa = 0.6$			$\kappa = 0.9$			$\kappa = 1.2$		
				ζ					
σ_v	0.6	0.9	1.2	0.6	0.9	1.2	0.6	0.9	1.2
0.4	590.26	623.48	655.53	573.94	620.06	663.92	560.10	617.18	670.81
0.6	585.78	619.16	651.40	570.16	616.46	660.55	556.91	614.15	668.02
0.8	580.17	613.55	645.89	565.39	611.72	655.99	552.84	610.13	664.22

$S_1(0) = S_2(0) = 100$, $T = 1.0$, $r = 0.04$, $\sigma_1 = \sigma_2 = 0.5$,
$B_1(0) = B_2(0) = 75$, $K_1 = K_2 = 100$, $v(0) = 1.0$, $\rho = 0.3$.

the actual level of the underlyings at maturity. Thus, in the here presented scenario there are two contrary effects: A higher volatility increases the probability that the underlyings are far above the strike levels at maturity but it also increases the probability that the underlyings fall below the barrier levels during the lifetime. The former effect seems to be dominant in this scenario for the Heston-type model as the value of the barrier correlation option raises when the mean-reversion level is increased. If the volatility of the covariance process is incremented the value of the option falls. Therefore, the probability that both underlyings stay above the barrier and far above the strike levels at maturity is lowered. As the opposite effect is visible with digital options it seems that a higher σ_v leads to a higher probability of lower volatility values in the whole system in the Heston-type model. When κ is augmented the overall effect depends again on the mean-reversion level: In the case of a low or medium mean-reversion level of $\zeta = 0.6$ or 0.9 the value of the derivative falls. Obviously, lower volatility values below and at the mean-reversion level become more probable. The opposite is true for the higher mean-reversion levels as the fast mean reversion ensures high values of volatilities.

Corollary 15. *(Correlation barrier option in Stein and Stein-type model with PDE technique)*

Let us assume the setting described in Equation (3.2) with $\nu = 1$ and $\gamma = 0$. Then the price of a two-asset barrier correlation option (see (3.70) for the payoff profile) is given by

$$C_{2C}(t, S_1, S_2, K_1, K_2, B_1, B_2, v) = \frac{e^{x_1 c_1 + x_2 c_2 - \int_t^T r(s)ds}}{\sigma_1 \sigma_2 \sqrt{1-\rho^2}} K_1 K_2 \int_0^\infty \int_0^\infty \left(e^{x_1'(1-c_1)} - e^{-x_1' c_1}\right) \left(e^{x_2'(1-c_2)} - e^{-x_2' c_2}\right) q\left(\tau, x_1, x_2, x_1', x_2', v\right) dx_1' dx_2', \tag{3.120}$$

where $q\left(\tau, x_1, x_2, x_1', x_2', v\right)$ is given in Formula (3.112).

Table 3.12: Prices of correlation options with barriers in Stein-type model (PDE technique).

	$\kappa = 0.6$			$\kappa = 0.9$			$\kappa = 1.2$		
	ζ								
σ_v	0.6	0.9	1.2	0.6	0.9	1.2	0.6	0.9	1.2
0.4	556.82	604.77	651.57	515.67	575.07	633.05	478.92	545.26	610.33
0.6	579.87	627.79	675.83	540.39	601.81	664.02	505.65	576.50	649.18
0.8	611.56	659.42	708.59	574.0	638.28	705.47	541.37	618.56	700.60

$S_1(0) = S_2(0) = 100$, $T = 1.0$, $r = 0.04$, $\sigma_1 = \sigma_2 = 0.5$,
$B_1(0) = B_2(0) = 75$, $K_1 = K_2 = 100$, $v(0) = 1.0$, $\rho = 0.3$.

In Table 3.12 we note that the behaviour of both models is similar with respect to the mean-reversion level ζ: The values of a two-asset correlation option in the Stein-type and the Heston-type model raise with an increase of ζ. Thus, the effect that higher volatility increases the probability that the underlyings are far above the strike levels dominates the second effect in this case in both models. In the Stein-type model an increase in σ_v leads to a higher value of the option, i.e. the sensitivity towards σ_v is reversed in most of the cases in comparison to the valuation of the digital option. A higher value of σ_v, thus, increases the probability that both underlyings take on values in the money at maturity. This effect even compensates the decrease in the probability that both underlyings stay above the barrier during the lifetime of the option. The sensitivity towards κ in the Stein-type model is also reversed: With an increase in κ the value of the option is lowered.

3.5.4 Conclusion

We have found a closed-form expression to price two-asset barrier options. Those options have a payout at maturity time T, which may depend on $S_1(T)$ and $S_2(T)$ provided that not any of the two assets crossed a predefined barrier. Closed-form expressions for barrier options on two assets are rare. He et al. [64] and Zhou [123], [124] presented closed-form pricing of those derivatives in the context of constant covariance. We extend their result by allowing for a third factor in the model which governs the covariance of the two underlyings. The solution found in Theorem 36 is true for any correlation $-1 \leq \rho \leq 1$ between the underlyings, and the implementation is numerically stable. Moreover, we derive the joint survival probability of the two assets. In some scenario calculations we analyse the impact of the model parameters on the price of two-asset barrier correlation and digital options. From our proof of the general pricing formula with PDE techniques, it is also clear that the assumption that the covariance process and the underlying processes are independent cannot be relaxed. For those frameworks a closed-form solution is not possible. Thus, we deal with approximation techniques in the next chapter. There we also relax the assumption that the covariance is only driven by one common factor. We rather assume several drivers and even set the correlation stochastic.

In the following we apply the techniques derived in this chapter to the pricing of certificates under issuer risk. Certificates are retail products which consist of simpler internal hedging derivatives.

3.6 Pricing certificates under issuer risk

3.6.1 Introduction

Since market introduction in 1989, certificates have become very popular with retail customers in Germany. In the years 2002 to 2007, the market volume doubled every year [120], peaking in an all time high of 139 billion EUR at the end of the 3rd quarter of 2007. In the financial market crisis, the market volume began to decline. Following the filing of Chapter 11 of Lehman brothers in September 2008, the market for certificates dropped by 30 billion EUR in one quarter to 80 billion EUR. Until the end of March 2010, the market in Germany recovered to 106 billion EUR [34].

Certificate are also referred to as structured products. The simplest structures are index certificates which allow a retail customer to directly invest in an index like the German stock index Deutscher Aktien Index (DAX) or the Dow Jones Euro Stoxx 50. From a legal point of view those products are bonds and the investors are, thus, creditors of the respective issuer. This rather technical aspect has been disregarded in the investment decision by many retail investors. Even the Value at Risk figures which are sometimes given by banks as an indicator of the risk involved in single certificates are based on the assumption of a non-defaultable issuer. However, in the case of an insolvency of the issuer the investor may lose his total investment regardless of the performance of the underlyings of the certificate. The case of Lehman Brothers shows that this risk can in fact materialise. Certificates differ in this feature from an investment in funds. The investment in a fund is a so-called special property and is not affected at all by the creditworthiness of the issuing company because the fund is protected against the bankruptcy of the issuer.

Issuer risk is the risk of loss on securities and other tradeable obligations because the issuer does not fulfil his contractual obligations due to his insolvency. Up to today this kind of risk has hardly been addressed in the pricing of exotic securities and especially not from a retailer's perspective but only in connection with regulatory capital requirements, e.g. Basel II. More details about modelling and evaluating counterparty/issuer risk under an economic or regulatory perspective can be found in [17], [97], [98].

Pricing securities under counterparty risk can be traced back to Merton [87]. Johnson and Stulz [74] analysed the counterparty risk in option pricing. They used a firm value model and assumed that the vulnerable option presents the single debt of the company. A huge increase in the derivative's value, thus, rises the risk of default of the company. This approach is only appropriate when the derivative is the only or the predominant source of funding of the counterparty. Hull and White [69] as well as Jarrow and Turnbull [72] value so-called vulnerable options, options on a bond written by a defaultable party, in a reduced form model. They assume independence between the credit risk of the counterparty and the asset underlying the derivative. Cherubini and Luciano (see [20], [21]) suggest to use a copula approach to value the counterparty risk in an investment. This allows to model a dependence structure between term-structure movements and a default of one of the parties.

Klein [78] as well as Klein and Inglis [79] choose a firm-value model to account for the issuer risk and to model the dependencies between the issuing firm and the underlying. We follow their approach in that regard, and condition the payoff of the certificate on the survival of the issuer: The certificate only pays the total investment and gains back as long as the issuer has not defaulted, i.e. its asset value has not fallen under a certain barrier.

Like Klein [78] and Klein and Inglis [79] we model the correlation between the assets of the issuer and the asset underlying the derivative explicitly. The barrier is exponentially increasing in time. The issuer can default any time before maturity. In the case of default the investor recovers a constant fraction of the market value of his investment. As we deal with retail products we furthermore assume that the exotic structures are fully hedged, i.e. all debt owed to the investor has to be seen alongside assets which the company owns. In the case of default, however, these assets do not cover the claims of the investors but are part of the insolvency estate. This allows us to assume the boundary as deterministically increasing rather than stochastic (see [79]).

3.6.2 The model

The model formulation is influenced by the CreditGrades framework (for details see [110] and [107]). This approach allows us to derive closed-form expressions for index, participation guarantee, bonus guarantee, discount, and bonus certificates under issuer risk in a Black-Scholes and a stochastic covariance framework.

The system of processes is defined on a filtered probability space $(\Omega, \mathcal{F}, \tilde{\mathcal{Q}}, \mathbb{F})$. We assume the existence of an equivalent martingale measure. As an immediate consequence the market is arbitrage-free. The processes are directly formulated under the martingale measure $\tilde{\mathcal{Q}}$. We consider an issuing company $i = 1$ with an asset value per share at time t of $V_1(t)$, an equity price $S_1(t)$, and a total debt per share of $D_1(t)$. The firm's asset value dynamics follow a geometric Brownian motion

$$\frac{dV_1}{V_1} = (r - d_1)dt + \sigma_1 dW_1, \; V_1(0) = S_1(0) + D_1(0), \tag{3.121}$$

or a process with stochastic volatility, respectively

$$\frac{dV_1}{V_1} = (r - d_1)dt + \sigma_1 v^{\nu} dW_1,\ V_1(0) = S_1(0) + D_1(0), \tag{3.122}$$

where r is the risk-free interest rate and d_1 is the dividend yield on the company's assets. Both processes, r and d_1, are assumed deterministic. v is driven by a stochastic process

$$\begin{aligned} dv &= \kappa(\zeta - v)dt + \sigma_v v^{\gamma} dZ, \\ \langle dW_1, dZ \rangle &= 0, \end{aligned} \tag{3.123}$$

and ν, κ, ζ, σ_v, and γ are constants. The company's debt is deterministic and yields a continuous interest of $r(s) - d_1(s)$.

$$D_1(t) = D_1(0)e^{\int_0^t (r(s)-d_1(s))ds}. \tag{3.124}$$

The issuing company defaults if its asset value falls below that barrier $D_1(t)$. Thus, the time of default is defined by the stopping time ι_1'

$$\iota_1' = \inf\left\{t' \in (t_0, T] : V_1(t') < D_1(t')\right\}. \tag{3.125}$$

We denote the equity per share by

$$\begin{cases} S_1(t) = V_1(t) - D_1(t), \text{ if } \iota_1 > t \text{ and,} \\ S_1(t) = 0 \text{ otherwise.} \end{cases} \tag{3.126}$$

This implies that default occurs whenever the stock price $S_1(t)$ falls to zero. As soon as $S_1(t)$ reaches zero it remains there. In this framework an European option is seen as the corresponding down-and-out barrier option with an absorbing barrier for $S_1(t)$ set at 0. Hence, an equivalent time to default to ι_1' is

$$\iota_1 = \inf\left\{t' \in (t_0, T] : S_1(t') \le 0\right\}. \tag{3.127}$$

In the GBM framework the dynamics for $S_1(t)$ are, prior to default, found by applying Itô to (3.126) and are given by

$$dS_1 = S_1(r - d_1)dt + (D_1(t) + S_1)\sigma_1 dW_1. \tag{3.128}$$

The stock price of the issuer follows – prior to default – a shifted log-normal distribution. This distribution implies negative stock prices with positive probability (see [107] for more details): The higher the leverage (debt-to-equity ratio) the higher is the probability of default and vice versa.

The dynamics of the underlying assets of a certificate are modelled by geometric Brownian motions. The following system is considered to describe the assets of the issuer and the underlying assets of the derivatives:

$$\begin{aligned} dS_1 &= S_1(r-d_1)dt + (D_1(t)+S_1)\sigma_1 dW_1, \qquad (3.129)\\ dS_i &= S_i(r-d_i)dt + S_i\sigma_i dW_i \;\text{ for } i=2,\dots,d,\\ \langle dW_i, dW_k\rangle &= \rho_{ik}dt, \end{aligned}$$

where σ_i and ρ are constants (see [67]).

This framework is compared to a stochastic covariance framework, where S_1 is modelled by

$$dS_1 = S_1(r-d_1)dt + (D_1(t)+S_1)\sigma_1 v^{\nu} dW_1, \qquad (3.130)$$

where v is driven by (3.131). Hence, the framework is given by

$$\begin{aligned} dS_1 &= S_1(r-d_1)dt + (D_1(t)+S_1)\sigma_1 v^{\nu} dW_1, \qquad (3.131)\\ dS_i &= S_i(r-d_i)dt + S_i\sigma_i v^{\nu} dW_i \;\text{ for } i=2,\dots,d,\\ dv &= \kappa(\zeta - v)dt + \sigma_v v^{\gamma} dZ, \qquad (3.132)\\ \langle dW_i, dZ\rangle &= 0,\\ \langle dW_i, dW_k\rangle &= \rho_{ik}dt. \end{aligned}$$

In the following we consider the Heston-type model with $\nu = \gamma = \frac{1}{2}$ and the Stein-type model with $\nu = 1$ and $\gamma = 0$.

3.6.3 Pricing of certificates under issuer risk

Building blocks

The most popular certificates are composed from simple building blocks such as zero-coupon bonds $C_Z(t)$, investments in an underlying $S_2(t)$, call options $C_{Call}(t, S_2, K_2)$, and digital options $C_D(t, S_2, K_2)$ as well as knock-out put options $C_{1P}(t, S_2, K_2, B_2)$. The formulas for these components are derived for the case that the issuer can default and has a recovery rate of zero (for a similar model see [83], p. 635ff). In the case of no default the valuation of the building blocks is well-known and will be provided for the purpose of completeness. By means of these building blocks defaultable index, guarantee, and bonus certificates can be valued for any assumption of the recovery rate. The proofs

for the results in this section will be provided in the Appendix A.2.
A main component of index certificates and discount certificates is the investment in the underlying S_2, which is actually worth $S_2(t)$ at time t, when we assume that the issuer cannot default. However, when we suppose that the investor does not obtain the dividends paid out the value is given by $S_2(t)e^{-\int_t^T d_2(s)ds}$.

Proposition 4. *(Investment in a stock)*

In the case when the investment in the equity is guaranteed by an issuer who is defaultable with recovery rate zero, the price $S_{2,t}^D$ of this investment in Model (3.129) at time t is given by

$$\begin{aligned} C_S^D &= S_2 e^{-\int_t^T d_2(s)ds}\Big(\mathcal{N}(\mathbf{d}_1) \\ &\quad - \exp\left\{2x_1^*(c_1 - \rho\frac{\sigma_2}{\sigma_1}(1-c_2))\right\}\mathcal{N}\left(\tilde{\mathbf{d}}_1\right)\Big), \end{aligned}$$

where $\mathcal{N}_2(x,y,\rho)$ is the standard bivariate normal distribution function with correlation ρ,

$$\begin{aligned} x_1^* &= \ln(\frac{S_1 + D_1(t)}{D_1(t)}), & x_2 &= \ln\left(S_2 e^{\int_t^T (r(s)-d_2(s))ds}\right), \\ \tau &= T - t, \\ c_1 &= \frac{\sigma_1 - \rho\sigma_2}{2\sigma_1(1-\rho^2)}, & c_2 &= \frac{\sigma_2 - \rho\sigma_1}{2\sigma_2(1-\rho^2)}, \\ \mathbf{d}_1 &= \frac{x_1^*}{\sigma_1\sqrt{\tau}} - \frac{1}{2}\sigma_1\sqrt{\tau} + \rho\sigma_2\sqrt{\tau}, & \tilde{\mathbf{d}}_1 &= -\frac{x_1^*}{\sigma_1\sqrt{\tau}} - \frac{1}{2}\sigma_1\sqrt{\tau} + \rho\sigma_2\sqrt{\tau}. \end{aligned}$$

In the stochastic covariance Framework (3.131) the price is given by

$$\begin{aligned} C_S^D(t, S_1, S_2, v) &= \frac{e^{x^*{}_1 c_1 + x_2 c_2 - \int_t^T r(s)ds}}{2\pi\sigma_1} \int_{i\varpi_1 - \infty}^{i\varpi_1 + \infty} \tilde{\hat{\mathfrak{h}}}_{C_S^D}(u_1) \\ &\quad (\varphi(\tau, \mathbf{u}, -z_1, -z_2, v) - \varphi(\tau, \mathbf{u}, z_1, -z_2, v))\, du_1, \\ &\quad \mathbf{u} \in S_{C_S^D} = S_\varphi \cap S_{g_{C_S^D}}, \end{aligned}$$

where

$$\tilde{\hat{\mathfrak{h}}}_S^D(u_1, u_2) = \left(\frac{-1}{\frac{iu_1}{\sigma_1} + \frac{\rho\sigma_2(1-c_2)}{\sigma_1} - c_1}\right),$$

with $\Im(u_1) = \varpi_1 > -\sigma_1 c_1 + (1 - c_2)\sigma_2\rho$, i.e. $S_{g_{C_S^D}} = \{\mathbf{u} = \mathbf{w} + i\boldsymbol{\varpi} : \varpi_2 = 0 \wedge \varpi_1 > -\sigma_1 c_1 + (1-c_2)\sigma_2\rho\}$ and the characteristic functions

$\varphi(\tau, \mathbf{u}, \mathbf{z}, v)$ for the Heston-type and the Stein and Stein-type models are given by

$$\varphi_H(\tau, \mathbf{u}, \mathbf{z}, v) = \exp\left\{iu_1 z_1 + iu_2 z_2 + \frac{1}{\sigma_v^2}\left(A_H(\tau, \mathbf{u}) + B_H(\tau, \mathbf{u})v\right)\right\}, \tag{3.133}$$

$$\varphi_{S2}(\tau, \mathbf{u}, \mathbf{z}, v) = \exp\left\{iu_1 z_1 + iu_2 z_2 + \frac{1}{\sigma_v^2}\left(A_{S2}(\tau, \mathbf{u}) + B_{S2}(\tau, \mathbf{u})v + C_{S2}(\tau, \mathbf{u})v^2\right)\right\},$$

with A_H, B_H, A_{S2}, B_{S2}, and C_{S2} as defined in (3.48)-(3.49) and (3.59)-(3.61).

For a proof see A.2.2.
Products, which ensure a repayment of an investment with notional 1 at maturity, are internally stripped into a zero-coupon bond and several derivatives. As known, a non-defaultable zero-coupon bond is priced by $C_Z(t) = e^{-\int_t^T r(s)ds}$.

Proposition 5. *(Zero-coupon bond)*

When we assume that the issuer can default and – if he defaults – he does so with no recovery, the value of the zero-coupon bond in the GBM framework can be computed by (see [107])

$$C_Z^D(t) = e^{-\int_t^T r(s)ds}\Big(\mathcal{N}\left(\mathbf{d}_1^*\right) - e^{x_1^*}\mathcal{N}\left(\tilde{\mathbf{d}}_1^*\right)\Big), \tag{3.134}$$

with $\mathcal{N}(.)$ being the standard cumulative normal distribution function, and

$$\mathbf{d}_1^* = \frac{x_1^*}{\sigma_1\sqrt{\tau}} - \frac{1}{2}\sigma_1\sqrt{\tau},\ \tilde{\mathbf{d}}_1^* = -\frac{x_1^*}{\sigma_1\sqrt{\tau}} - \frac{1}{2}\sigma_1\sqrt{\tau}.$$

When the issuer additionally promises a fixed interest r_I on the investment this is valued, assuming no default, by $C_Z^D(t, I) = (1 + r_I)\, e^{-\int_t^T r(s)ds}$ and accordingly by $C_Z^D(t, I) = e^{-\int_t^T r(s)ds}\,(1 + r_I)\left(\mathcal{N}\left(\mathbf{d}_1^*\right) - e^{x_1^*}\mathcal{N}\left(\tilde{\mathbf{d}}_1^*\right)\right)$ in a world with defaults and zero recovery.
In the stochastic covariance framework with defaults the zero bond is priced by (see [107])

$$C_Z^D(t, S_1, v) = \frac{e^{\frac{1}{2}x_1^* - \int_t^T r(s)ds}}{2\pi}\int_{-\infty}^{\infty}\frac{\varphi(\tau, u_1, x_1^*, v) - \varphi(\tau, u_1, -x_1^*, v)}{iu_1 - \frac{1}{2}}du_1, \tag{3.135}$$

where the one-dimensional characteristic functions for the Heston-type model and the Stein-type model are given by

$$\varphi_H(\tau, u, v) = \exp\left\{iu_1 x_1^* + \frac{1}{\sigma_v^2}\left(A_H(\tau, u_1) + B_H(\tau, u_1)v\right)\right\}, \tag{3.136}$$

$$\varphi_{S2}(\tau, u, v) = \exp\left\{iu_1 x_1^* + \frac{1}{\sigma_v^2}\left(A_{S2}(\tau, u_1) + B_{S2}(\tau, u_1)v + C_{S2}(\tau, u_1)v^2\right)\right\},$$

where

$$\mathfrak{d} = \mathfrak{d}(u_1) \;=\; \sqrt{\kappa^2 + \sigma_v^2\sigma^2\left({u_1}^2 + \frac{1}{4}\right)},$$

and with A_H, B_H, A_{S2}, B_{S2}, and C_{S2} as defined in (3.48)-(3.49) and (3.59)-(3.61).

Call options are introduced in the structured products to leverage the investment and to grant the investor a high participation rate in the returns of the underlying. In the GBM framework with a non-defaultable issuer the call option with strike K_2 is priced by the well-known Black-Scholes formula $C_{Call}\left(t, S_2, K_2\right) = S_2 e^{\int_t^T r(s)ds}\mathcal{N}\left(\mathbf{d}_2\right) - K_2 e^{-\int_t^T r(s)ds}\mathcal{N}\left(\mathbf{d}_2^*\right)$, where $\mathbf{d}_2 = \frac{x_2}{\sigma_2\sqrt{\tau}} + \frac{1}{2}\sigma_2\sqrt{\tau}$, $x_2 = \frac{S_2 e^{\int_t^T (r(s)-d_2(s))ds}}{K_2}$ and $\mathbf{d}_2^* = \mathbf{d}_2 - \sigma_2\sqrt{\tau}$. In the stochastic covariance framework the call price is given by $C_{Call}(t, S_2, K_2, v) = \frac{K_2 e^{\frac{1}{2}x_2 - \int_t^T r(s)ds}}{2\pi}\left(2\pi e^{\frac{1}{2}x_2} - \int_{-\infty}^{\infty}\frac{\varphi(\tau,u_2,-x_2,v)}{u_2^2+\frac{1}{4}}\right)$ (see [107]).

Proposition 6. *(Call option)*

In the GBM Framework (3.129) at time t the price $C_C^D(t, S_1, S_2, K_2)$ of a defaultable call option, when the payment of the call option is guaranteed by an issuer who is defaultable with recovery rate zero, is given by

$$\begin{aligned} C_{Call}^D(t, S_1, S_2, K_2) \;=\; & \Bigg(\left(S_2 e^{-\int_t^T d_2(s)ds}\mathcal{N}_2\left(\mathbf{d}_1, \mathbf{d}_2, \rho\right) - K_2 e^{-\int_t^T r(s)ds}\mathcal{N}_2\left(\mathbf{d}_1^*, \mathbf{d}_2^*, \rho\right)\right) \\ & -\Bigg(S_2 e^{-\int_t^T d_2(s)ds}\exp\left\{2x_1^*(c_1 - (1-c_2)\frac{\sigma_2\rho}{\sigma_1})\right\}\mathcal{N}_2\left(\tilde{\mathbf{d}}_1, \tilde{\mathbf{d}}_2, \rho\right) \\ & -K_2 e^{-\int_t^T r(s)ds}\exp\left\{2x_1^*(c_1 + c_2\frac{\sigma_2\rho}{\sigma_1})\right\}\mathcal{N}_2\left(\tilde{\mathbf{d}}_1^*, \tilde{\mathbf{d}}_2^*, \rho\right)\Bigg)\Bigg), \end{aligned}$$

where x_1^*, x_2, c_1, c_2, as well as $\mathbf{d}_1$, and $\tilde{\mathbf{d}}_1$ are given in Proposition 4,

$$\begin{aligned} \mathbf{d}_2 &= \tfrac{x_2}{\sigma_2\sqrt{\tau}} + \tfrac{1}{2}\sigma_2\sqrt{\tau}, & \tilde{\mathbf{d}}_2 &= \frac{x_2}{\sigma_2\sqrt{\tau}} - 2\rho\frac{x_1^*}{\sigma_1\sqrt{\tau}} + \frac{1}{2}\sigma_2\sqrt{\tau}, \\ \mathbf{d}_1^* &= \mathbf{d}_1 - \rho\sigma_2\sqrt{\tau}, & \mathbf{d}_2^* &= \mathbf{d}_2 - \sigma_2\sqrt{\tau}, \\ \tilde{\mathbf{d}}_1^* &= \tilde{\mathbf{d}}_1 - \rho\sigma_2\sqrt{\tau}, & \tilde{\mathbf{d}}_2^* &= \tilde{\mathbf{d}}_2 - \sigma_2\sqrt{\tau}. \end{aligned}$$

In the stochastic covariance Framework (3.131) the price is given by

$$\begin{aligned} C_{Call}^D(t, S_1, S_2, v) \;=\; & \frac{e^{x^*{}_1 c_1 + x_2 c_2 - \int_t^T r(s)ds}}{4\pi^2\sigma_1\sigma_2\sqrt{1-\rho^2}}\int_{i\varpi_2-\infty}^{i\varpi_2+\infty}\int_{i\varpi_1-\infty}^{i\varpi_1+\infty}\hat{\mathfrak{h}}_{C_{Call}^D}(u_1) \\ & \left(\varphi\left(\tau, \mathbf{u}, -z_1, -z_2, v\right) - \varphi\left(\tau, \mathbf{u}, z_1, -z_2, v\right)\right)du_1 du_2, \\ & \mathbf{u} \in S_{C_{Call}^D} = S_\varphi \cap S_{g_{C_{Call}^D}}, \end{aligned}$$

where

$$\hat{\mathfrak{h}}_C^D(u_1,u_2) = K_2 \frac{1}{i\left(\frac{u_1}{\sigma_1}-\frac{u_2\rho}{\sigma_1\sqrt{1-\rho^2}}\right)-c_1} \frac{1}{\left(i\frac{u_2}{\sigma_2\sqrt{1-\rho^2}}+1-c_2\right)\left(i\frac{u_2}{\sigma_2\sqrt{1-\rho^2}}-c_2\right)}.$$

Note that we have to set $\Im(u_2) > \sigma_2\sqrt{1-\rho^2}(1-c_2)$ *and* $\Im(u_1) > \frac{\rho}{\sqrt{1-\rho^2}}\Im(u_2)-\sigma_1 c_1$, *i.e.* $S_{g_C^D} = \left\{\mathbf{u}=\mathbf{w}+i\boldsymbol{\varpi} : \varpi_2 > \sigma_2\sqrt{1-\rho^2}(1-c_2) \wedge \varpi_1 > \frac{\rho}{\sqrt{1-\rho^2}}\varpi_2 - \sigma_1 c_1\right\}$. *The characteristic functions for the Heston-type and the Stein and Stein-type models are as given in Proposition 4.*

For a proof see A.2.2.

Similarly in the GBM framework, the non-defaultable digital option with strike K_2 is priced by $C_D(t,S_2,K_2) = e^{-\int_t^T r(s)ds}N(\mathbf{d}_2^*)$ and in the stochastic covariance model by $C_D(t,S_2,K_2,v) = \frac{e^{\frac{1}{2}x_2-\int_t^T r(s)ds}}{2\pi}\int_{-\infty}^{\infty}\frac{\varphi(\tau,u_2,-x_2,v)}{\frac{1}{2}-iu_2}$.

Proposition 7. *(Digital option)*

In the GBM Model (3.129) at time t the price $C_D^D(t,S_1,S_2,K_2)$ *of a defaultable digital call option, when the payment of one is guaranteed by an issuer who is defaultable with recovery rate zero, is given by*

$$C_D^D(t,S_1,S_2,K_2) = e^{-\int_t^T r(s)ds}\Bigg(\mathcal{N}_2\left(\mathbf{d}_1^*,\mathbf{d}_2^*,\rho\right) - \exp\left\{2x_1^*(c_1+c_2\frac{\sigma_2\rho}{\sigma_1})\right\} \mathcal{N}_2\left(\tilde{\mathbf{d}}_1^*,\tilde{\mathbf{d}}_2^*,\rho\right)\Bigg),$$

where x_1^*, x_2, c_1, c_2, $\mathbf{d}_1^*$, *and* $\tilde{\mathbf{d}}_1^*$ *are given in Proposition 4,* $\mathbf{d}_2^*$ *and* $\tilde{\mathbf{d}}_2^*$ *are defined in Proposition 6.*

In the stochastic covariance Framework (3.131) with defaults the price is given by

$$C_D^D(t,S_1,S_2,K_2,v) = \frac{e^{x^*{}_1c_1+x_2c_2-\int_t^T r(s)ds}}{4\pi^2\sigma_1\sigma_2\sqrt{1-\rho^2}}\int_{i\varpi_2-\infty}^{i\varpi_2+\infty}\int_{i\varpi_1-\infty}^{i\varpi_1+\infty}\hat{\mathfrak{h}}_{C_D^D}(u_1) \left(\varphi(\tau,\mathbf{u},-z_1,-z_2,v)-\varphi(\tau,\mathbf{u},z_1,-z_2,v)\right)du_1du_2,$$
$$\mathbf{u}\in S_{C_D^D} = S_\varphi \cap S_{g_{C_D^D}},$$

where

$$\hat{\mathfrak{h}}_D^D(u_1,u_2) = \frac{1}{i\frac{u_2}{\sigma_2}\sqrt{1-\rho^2}-c_2}\frac{1}{i\left(\frac{u_1}{\sigma_1}-\rho\frac{u_2}{\sigma_2\sqrt{1-\rho^2}}\right)-c_1},$$

with $\Im(u_2) > -\sigma_2\sqrt{1-\rho^2}c_2$ *and* $\Im(u_1) > \frac{\rho}{\sqrt{1-\rho^2}}\Im(u_2) - \sigma_1 c_1$*, i.e.* $S_{g_C^D} = \{\mathbf{u} = \mathbf{w} + i\boldsymbol{\varpi} : \varpi_2 > -\sigma_2\sqrt{1-\rho^2}c_2 \wedge \varpi_1 > \frac{\rho}{\sqrt{1-\rho^2}}\varpi_2 - \sigma_1 c_1\ \}$*. The characteristic functions for the Heston-type and the Stein and Stein-type models are as given in Proposition 4.*

For a proof see A.2.2.

The bonus certificate includes, beside other derivatives, a knock-out put option. The price of a non-defaultable knock-out put option with barrier $B_2(t) = B_2 e^{\int_0^t (r(s)-d_2(s))ds}$ and strike K_2 is given by (see also [64])

$$\begin{aligned} C_{1P}\left(t, S_1, S_2, K_2, B_2\right) &= C_P(t, S_2, K_2) \\ &\quad - e^{-\int_t^T d_2(s)ds} B_2 \\ &\quad \mathcal{N}\left(\frac{2b_2 - x_2}{\sigma_2\sqrt{\tau}} - \frac{1}{2}\sigma_2\sqrt{\tau}\right) \\ &\quad + S e^{-\int_t^T r(s)ds} \frac{K_2}{B_2} \\ &\quad \mathcal{N}\left(\frac{2b_2 - x_2}{\sigma_2\sqrt{T-t}} + \frac{1}{2}\sigma_2\sqrt{\tau}\right), \end{aligned}$$

where

$$\begin{aligned} C_P(t, S_2, K_2) &= K_2 e^{-\int_t^T r(s)ds} \mathcal{N}\left(-\hat{\mathbf{d}}_2\right) - S_2 e^{-\int_t^T d_2(s)ds} \mathcal{N}\left(-\hat{\mathbf{d}}_1\right), \\ x_2 &= \ln\left(\frac{S_2 e^{\int_t^T (r(s)-d_2(s))ds}}{K_2}\right),\ b_2 = \ln\left(\frac{B_2(T)}{K_2}\right). \end{aligned}$$

In the stochastic covariance framework the default-free price is given by

$$\begin{aligned} C_{1P}(t, S_2, K_2, B_2, v) &= -\frac{K_2 e^{-\frac{1}{2}x_2 - \int_t^T r(s)ds}}{2\pi} \int_{-\infty}^{\infty} \left(\varphi(\tau, u_2, -x_2, v) - \varphi(\tau, u_2, x_2 - 2b, v)\right) \\ &\quad \frac{1 + e^{ib_2 u_2}\left(\left(iu_2 - \frac{1}{2}\right)e^{\frac{b_2}{2}} - \left(iu_2 - \frac{1}{2}\right)e^{-\frac{b_2}{2}}\right)}{u_2^2 + \frac{1}{4}} du_2, \end{aligned} \tag{3.137}$$

where the one-dimensional characteristic functions are given in (3.136). For a proof see [39].

Proposition 8. *(Barrier option)*

In the GBM Model (3.129) the price $C^D_{1P}(t, S_1, S_2, K_2, B_2)$ of a defaultable knock-out put option at time t, when the payoff is guaranteed by an issuer who is defaultable with recovery rate zero, is given by

$$C^D_{1P}(t, S_1, S_2, K_2, B_2) = \frac{K_2 e^{c_1 x_1^* + c_2 x_2 - \int_t^T r(s)ds}}{\sigma_1 \sigma_2 \sqrt{1-\rho^2}} \int_0^\infty \int_{-\infty}^0 e^{-c_1 x_1^{*\prime}} \left(e^{-c_2 x_2'} - e^{(1-c_2)x_2} \right) p_{GBM}(\tau, x_1^{*\prime}, x_2', x_1^*, x_2) dx_1^{*\prime} dx_2', \quad (3.138)$$

where $y_1 = \frac{x_1^*}{\sigma_1}$, $y_2 = \frac{x_2}{\sigma_2}$, and

$$\begin{aligned} p_{GBM}(\tau, x_1^{*\prime}, x_2', x_1^*, x_2) &= \frac{2e^{-\alpha\tau}}{\beta_p \tau} \sum_{n=1}^\infty e^{-\frac{(r_p^2 + r_p'^2)}{2\tau}} \sin\frac{n\pi\theta_p'}{\beta_p} \sin\frac{n\pi\theta_p}{\beta_p} I_{\frac{n\pi}{\beta_p}} \left(\frac{r_p' r_p}{\tau} \right), \\ \tan\beta_p &= -\frac{\sqrt{1-\rho^2}}{\rho}, \ \beta \in [0, \pi], \\ r_p &= \sqrt{\frac{1}{(1-\rho^2)} \left(\frac{x_1^*}{\sigma_1} - \rho\frac{x_2}{\sigma_2} \right)^2 + \left(\frac{x_2}{\sigma_2} \right)^2}, \\ \tan(\theta_p) &= \frac{\frac{x_2}{\sigma_2}}{\frac{1}{\sqrt{1-\rho^2}} \left(\frac{x_1^*}{\sigma_1} - \rho\frac{x_2}{\sigma_2} \right)}, \ \theta_p \in [0, \beta_p]. \end{aligned} \quad (3.139)$$

In the stochastic covariance framework with defaults the respective formulas were derived in Section 3.5.1. Hence,

$$C^D_{1P}(t, S_1, S_2, K_2, B_2, v) = \frac{K_2 e^{c_1 x_1^* + c_2 x_2 - \int_t^T r(s)ds}}{\sigma_1 \sigma_2 \sqrt{1-\rho^2}} \int_0^\infty \int_{-\infty}^0 e^{-c_1 x_1^{*\prime}} (e^{-c_2 x_2'} - e^{(1-c_2)x_2}) q(\tau, x_1^{*\prime}, x_2', x_1^*, x_2, v) dx_2' dx_1^{*\prime}, \quad (3.140)$$

where $q(\tau, x_1^{*\prime}, x_2', x_1^*, x_2, v)$ is given in (3.91) for the general case, for the Heston-type model in (3.110) and for the Stein-type model in (3.112).

Index certificates

In this section we price an index certificate with price $IC(t)$ at time t under issuer risk. As stated before, the index certificate allows a retail investor to invest in a single stock or stock index with price $S_2(t)$ at time t. He fully participates in any movement of the underlying. The index certificate does not provide any protection against a decline of the underlying. Internally this certificate is hedged by buying the respective index or single stock. Taking into consideration the issuer risk and assuming a constant recovery rate $\mathfrak{R}$

in the case of default of the issuer, the index certificate can be valued by

$$IC\left(t,S_1,S_2\right) \;=\; S_2 e^{-\int_t^T d_2(s)ds} - \left(1-\mathfrak{R}\right) e^{-\int_t^T r(s)ds} \mathbb{E}_{\tilde{\mathcal{Q}}}\left[S_2(T)\mathbb{1}_{\{\iota_1 \le T\}} \mid \mathcal{F}_t\right], \tag{3.141}$$

where ι_1 is given in (3.127). The index certificate can be valued by using the building blocks of Proposition 4. Thus, the certificate is priced by the following formula

$$IC\left(t,S_1,S_2\right) \;=\; \mathfrak{R} S_2(t) e^{-\int_t^T d_2(s)ds} + \left(1-\mathfrak{R}\right) C_S^D(t,S_1,S_2). \tag{3.142}$$

The number of parameters rises considerably if the notion of issuer risk is incorporated in the pricing of derivatives. In the following we show the results of the valuation of the index certificate in different scenarios. Our analysis is two-fold. In a first step we compare the issuer-risk adjusted prices to values which neglect issuer risk in the GBM framework. Secondly, we analyse the influences of stochastic volatility and compare the Heston-type and the Stein-type model. We focus in this analysis on the effect of the volatility of the covariance process.[1] For the scenario computations in this chapter we have chosen the following instrument parametrisations if not stated differently in the respective examples:

- Initial stock prices of the issuer and the underlying: $S_1(0) = S_2(0) = 100$,
- Initial debt endowment: $D_1(0) = 50$,
- Maturity: $T = 2$,
- Volatility of the issuer and the underlying: $\sigma_1 = \sigma_2 = 0.4$,
- Correlation between the underlying and the issuer: $\rho = 0.3$,
- Risk-free rate of return: $r = 0.04$,
- Dividend yield of the underlying and the issuer: $d_1 = d_2 = 0$,
- Stochastic volatility:
  - Heston-type framework: $\zeta = v(0) = 1$, $\kappa = 2$, $\sigma_v = 1$, $\nu = \gamma = \frac{1}{2}$,[2]
  - Stein and Stein-type framework: $\zeta = v(0) = 1$, $\kappa = 2$, $\sigma_v = 1$, $\nu = 1$, $\gamma = 0$.

[1] The scenario analysis in the stochastic covariance model has been prepared during a master thesis project in cooperation with Kolja Einig. See [39].

[2] In contrast to the previous examples the volatility level of S_1 and S_2 is determined by σ_1 or σ_2 respectively. The covariance process governs the stochasticity only.

Figure 3.7 shows that for increasing debt of the issuing company the value of the index certificate declines and finally approaches $\mathfrak{R}S_2(t)$. The price of the non-defaultable certificate stays at par regardless of the debt level. This shows that for a retail investor a simple comparison of the prices of index certificates of different issuers is not appropriate in order to find the security with the best price-performance ratio. For his investment decision the investor has to take the financial soundness of the issuer into consideration. Figure 3.7 also analyses the impact of the volatility of the issuer's asset (the volatility of the underlying is kept to 0.4) and the correlation between the issuer's asset and the underlying. For that reason we choose a relatively low debt scenario. The negative relationship between the volatility of the issuer's assets and the price is clearly visible. The probability of a default considerably increases in high volatility scenarios and, thus, the price decreases.

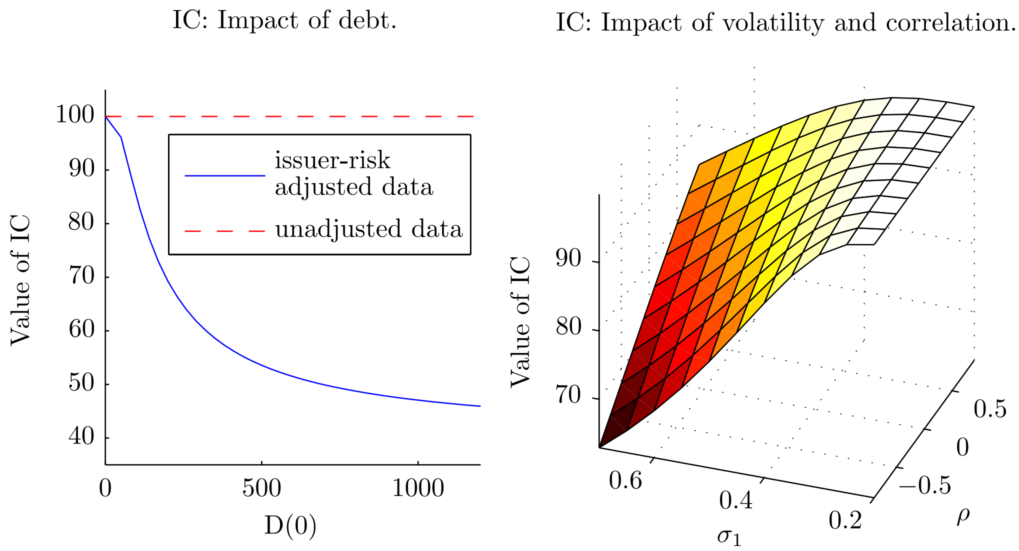

Figure 3.7: IC: Analysis of issuer risk in GBM framework.

It strikes that the impact of the correlation on the price grows with the volatility: For a very high volatility a correlation near 1 leads to a considerably higher value than a correlation near −1. In a low volatility and low debt scenario with $D_1(0) = 50$ the price is not much affected by the level of the correlation as the probability of default is relatively low. However, in a high volatility and low debt scenario with, thus, higher probability of default of the issuer, both assets tend to move in the same direction if the correlation rises. Thus, in contrast to a negative correlation the probability that the issuer survives and the underlying asset features a positive return goes up. A negative correlation results in a different performance of the issuer's assets and the underlying. Thus, the probability increases that the certificate matures at a low value or worthless. This effect is known as wrong way risk. According to the ISDA wrong way risk occurs when exposure to a counterparty is adversely correlated with the credit quality of that counterparty [35]. Now we turn to the analysis of the influence of stochastic covariance in the model framework (see 3.131). In Figure 3.8, we show the impact of stochastic volatility in both, the Heston-type as well as the Stein and Stein-type framework, for the three $t = 0$ levels of debt, $D_1(0) = 20, 50, 100$.

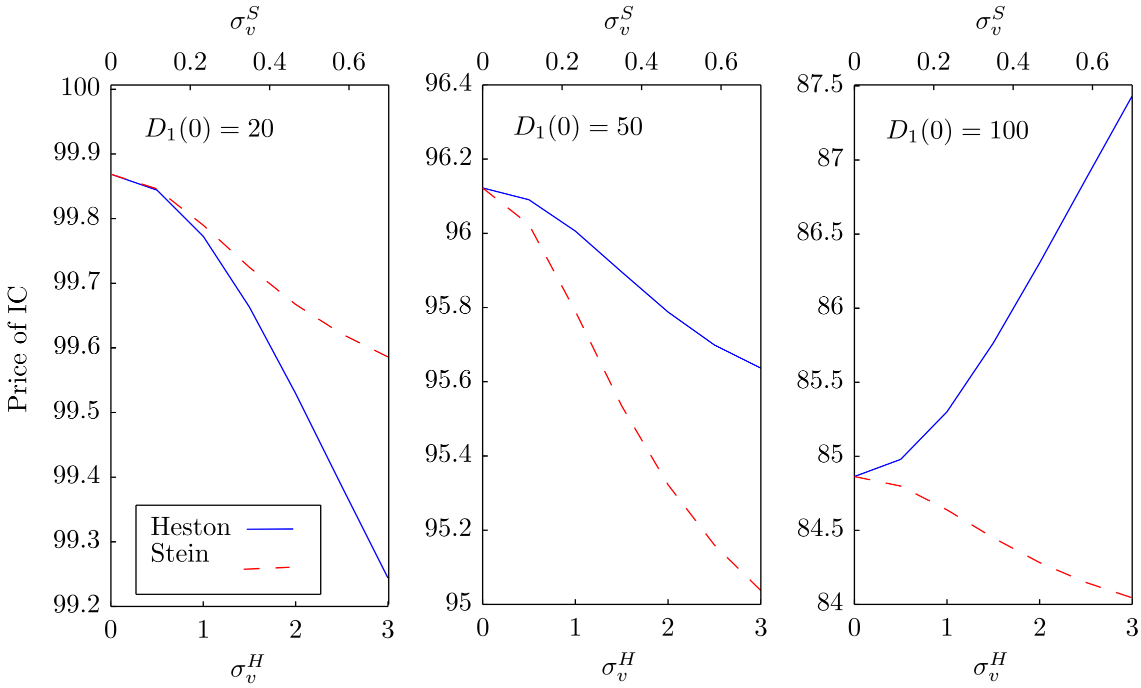

Figure 3.8: IC: Impact of σ_v in stochastic covariance framework.

In the case $\sigma_v = 0$ both models degenerate to a simple two-factor geometric Brownian motion model. When σ_v is now increased we observe that the prices of the index certificate decrease for the lower initial debt levels $D_1(0) = 20$ and $D_1(0) = 50$ in both models. In the Heston-type model the high debt case $D_1(0) = 100$ can be differentiated from the other cases in terms of that the price increases with rising volatility of volatility. It seems that the probability of default decreases with higher σ_v in the Heston model. This suggestion is supported by Figure 3.9, which shows the reaction of the probability of default with respect to a rise in σ_v in the same debt scenarios $D_1(0) = 20$, $D_1(0) = 50$, and $D_1(0) = 100$. As suggested we see that the probability of default decreases with increased σ_v in the high debt scenario $D_1(0) = 100$.

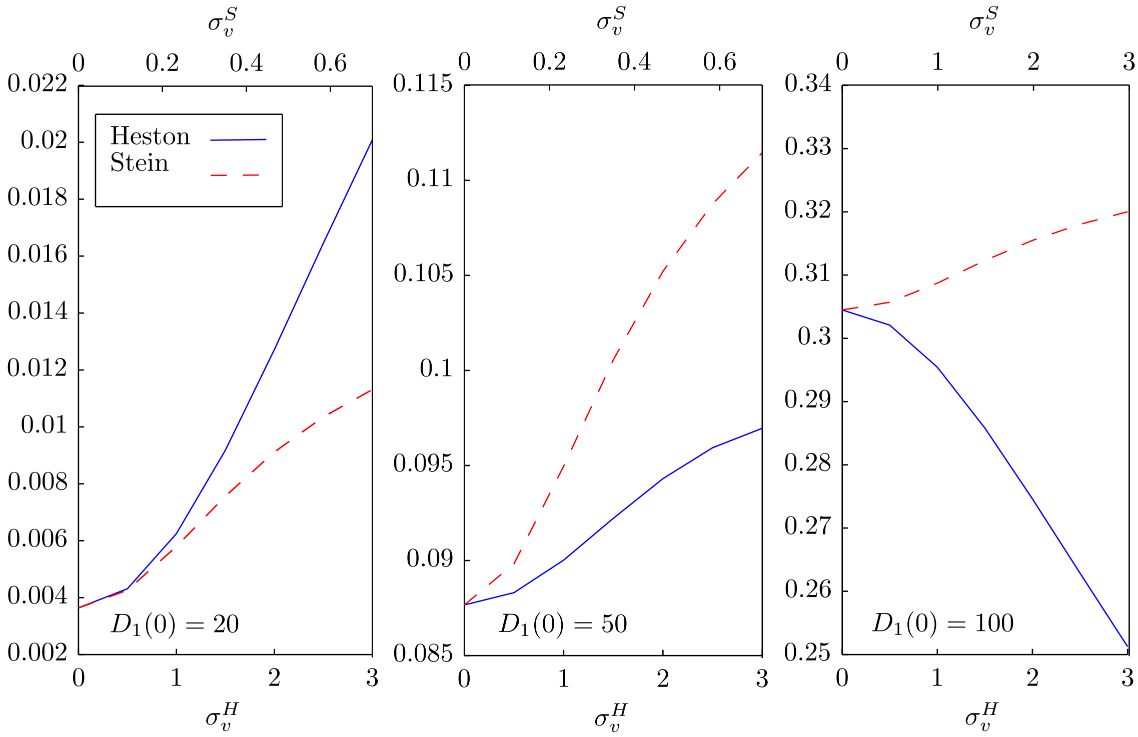

Figure 3.9: Default probability: Impact of σ_v.

This behaviour might seem rather peculiar. Hence, in Figure 3.10 the term structure of default probabilities for two different levels of debt and volatility, $D_1(0) = 50, 100$ and $\sigma_v = 0, 2$ is analysed.

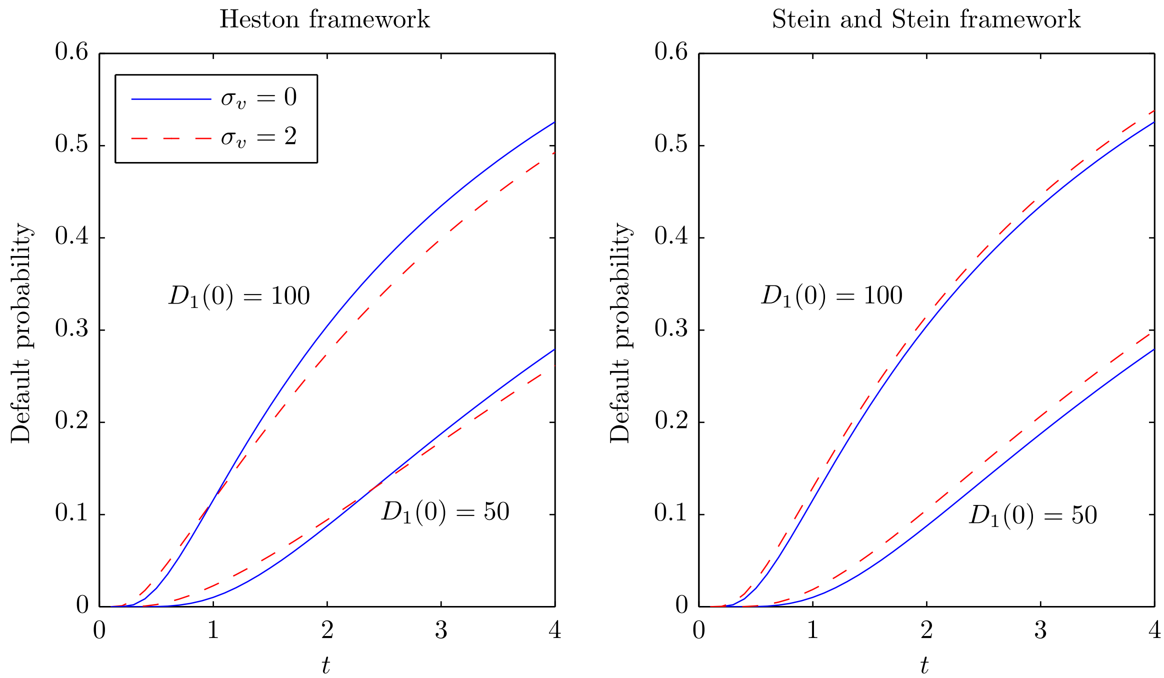

Figure 3.10: Default probabilities for $\sigma_v = 0, 2$ and $D_1(0) = 50, 100$.

The figure shows that for large t, a higher σ_v causes the default probabilities in the Heston-type model to decrease for all levels of debt. This effect is comparable to what we found for double-digital options when we increased the mean-reversion speed in Section 3.4.3.

Participation guarantee certificates

In contrast to the index certificate the participation guarantee certificate offers the investor some risk protection against a decrease of the underlying. However, this protection is financed by a limited profit in cases when the underlying increases. The investor participates in any positive performance of the underlying on the basis of the so-called participation rate $\mathfrak{p}$. This structure is built up by a long position in a zero-coupon bond with a standardized notional of 1 and stock options whereby the number of stock options is determined by the level of the participation rate. Taking into consideration the issuer risk, the price is indicated by

$$\begin{aligned} PG\left(t, S_1, S_2, K_2, p\right) &= e^{-\int_t^T r(s)ds}\Big(\mathbb{E}_{\tilde{Q}}\left[\left(1+\mathfrak{p}\max\left[S_2(T)-K_2, 0\right]\right) \mid \mathcal{F}_t\right] \\ &\quad - (1-\mathfrak{R})\,\mathbb{E}_{\tilde{Q}}\left[\left(1+\mathfrak{p}\max\left[S_2(T)-K_2, 0\right]\right)\mathbb{1}_{\{\iota_1 \le T\}} \mid \mathcal{F}_t\right]\Big), \end{aligned} \tag{3.143}$$

where ι_1 is as described in (3.127), $\mathfrak{p}$ is the participation rate of the contract, and K_2 describes the price level where the participation starts. This means that below this point the investor does not profit from any increase in the stock price. A defaultable participation guarantee certificate of an issuer with recovery rate $\mathfrak{R}$ is, thus, priced by:

$$\begin{aligned} PG\left(t, S_1, S_2, K_2, p\right) &= \mathfrak{R}\left(C_Z(t) + \mathfrak{p}C_{Call}\left(t, S_2, K_2\right)\right) \\ &\quad +(1-\mathfrak{R})\left(C_Z^D(t, S_1) + \mathfrak{p}C_{Call}^D\left(t, S_1, S_2, K_2\right)\right). \end{aligned} \tag{3.144}$$

For the participation guarantee certificate we compute some scenario values in the GBM model to analyse the effect of the issuer risk. For these computations we choose a participation level of $K_2 = 100$ and a participation rate of $\mathfrak{p} = 50\%$.

In the first plot of Figure 3.11 we find the impact of the debt level on the participation guarantee certificate similar to that of the index certificate: As before the price falls especially steeply for smaller $D_1(0)$ levels.

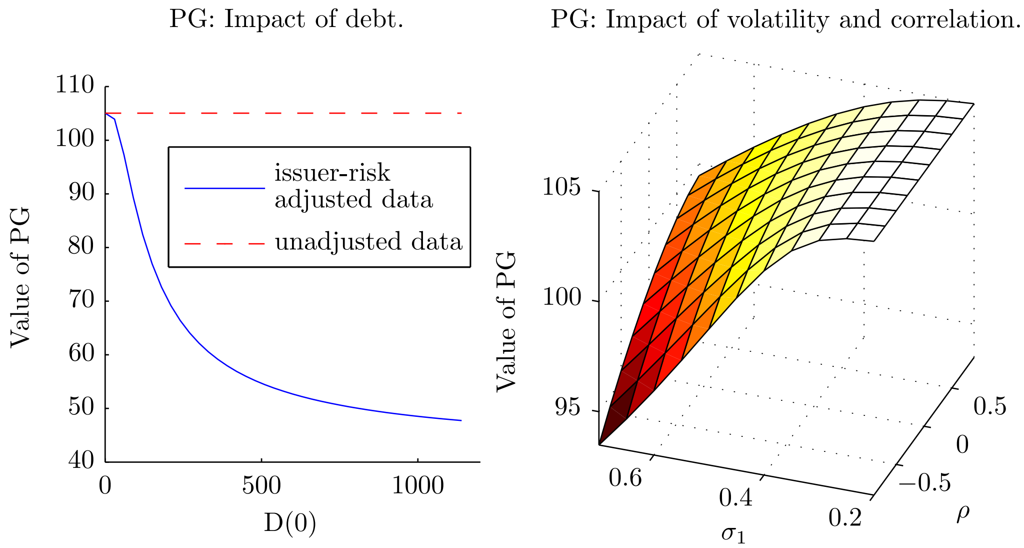

Figure 3.11: PG: Analysis of issuer risk in GBM framework.

In the second plot in Figure 3.11 the joint impact of the issuer's volatility and the correlation is shown. The form of the graphs resembles the graph of Figure 3.7 and the same explanations apply to explain the form. The effect of the correlation on the price of the participation guarantee certificate is, however, clearly less distinct because the total price of the certificate considerably depends on the value of the zero-coupon bond for which the correlation is irrelevant. In Figure 3.12, we show the impact of stochastic covariance in the Heston-type and the Stein and Stein-type model, for the initial debt values $D_1(0) = 20, 50, 100$. In the stochastic covariance framework the behaviour of the participation guarantee certificate resembles again the one of the index certificate: In the Heston-type model the negative relationship between the volatility of volatility and the price of the certificate for lower levels of debt, and the positive relationship between both for higher levels of debt are again clearly visible. However, due to the call option component, which allows an investor to participate in value increases of the underlying of the certificate, the volatility of the underlying S_2 becomes now an important impact factor for the value of the total certificate. Comparing the graphs with the ones of the index certificate we see that this additional impact factor amplifies/dampens the slopes in the Heston-type model: The value of the participation guarantee certificate decreases faster for lower levels of debt and increases slower for higher levels of debt. It seems that for higher debt level the positive effect of an increased σ_v on the default probability is diluted by the negative effects of a rising σ_v on the price of the call option. This is comparable to what we observed for the double-barrier correlation options in Section 3.4.4.

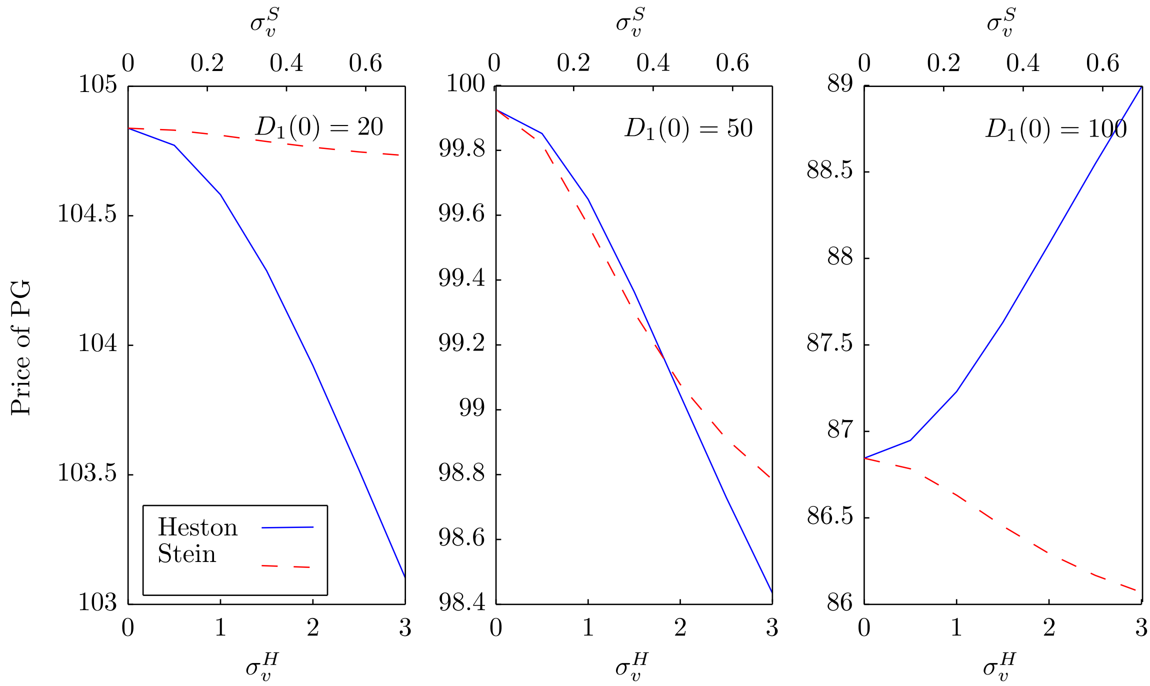

Figure 3.12: PG: Impact of σ_v.

Bonus guarantee certificates

When investing in a bonus guarantee certificate one does not directly participate in fluctuations of the value of the underlying. Rather, one receives a fixed interest rate and additionally a bonus payment if the underlying is above the bonus barrier at maturity. The investor could compose this payoff by buying a zero-coupon bond and a digital option:

$$\begin{aligned} BG\left(t, S_1, S_2, K_2, r_I, \Psi\right) &= e^{-\int_t^T r(s)ds}\Big(\mathbb{E}_{\tilde{\mathcal{Q}}}\left[\left(1 + r_I + \Psi\, \mathbb{1}_{\{S_2(T) > K_2\}}\right) \mid \mathcal{F}_t\right] \\ &\quad -(1-\mathfrak{R})\mathbb{E}_{\tilde{\mathcal{Q}}}\left[\left(1 + r_I + \Psi\, \mathbb{1}_{\{S_2(T) > K_2\}}\right) \mathbb{1}_{\{\iota_1 \leq T\}} \mid \mathcal{F}_t\right]\Big), \end{aligned} \tag{3.145}$$

where ι_1 is defined in (3.127), r_I the basic interest, Ψ the bonus payment rate, and K_2 the bonus barrier. In our framework the risk-neutral price of the bonus guarantee certificate is given by

$$\begin{aligned} BG\left(t, S_1, S_2, K_2, r_I, \Psi\right) &= \mathfrak{R}\left(C_Z(t, I) + \Psi\, C_D\left(t, S_2, K_2\right)\right) \\ &\quad +(1-\mathfrak{R})\left(C_Z^D(t, I, S_1) + \Psi\, C_D^D\left(t, S_1, S_2, K_2\right)\right). \end{aligned} \tag{3.146}$$

To show the impact of debt, volatility, and correlation we structure a bonus guarantee certificate with bonus barrier $K_2 = 120$, basic interest $r_I = 3.0\%$, and bonus payment rate $= 5.0\%$. For the bonus guarantee certificate we observe the same typical feature of the graph, which shows prices for different debt levels (see first plot of Figure 3.13): The graph decreases sharply for lower debt levels and approaches $\mathfrak{R}$-times the price of the analogous certificate of a non-defaultable issuer.

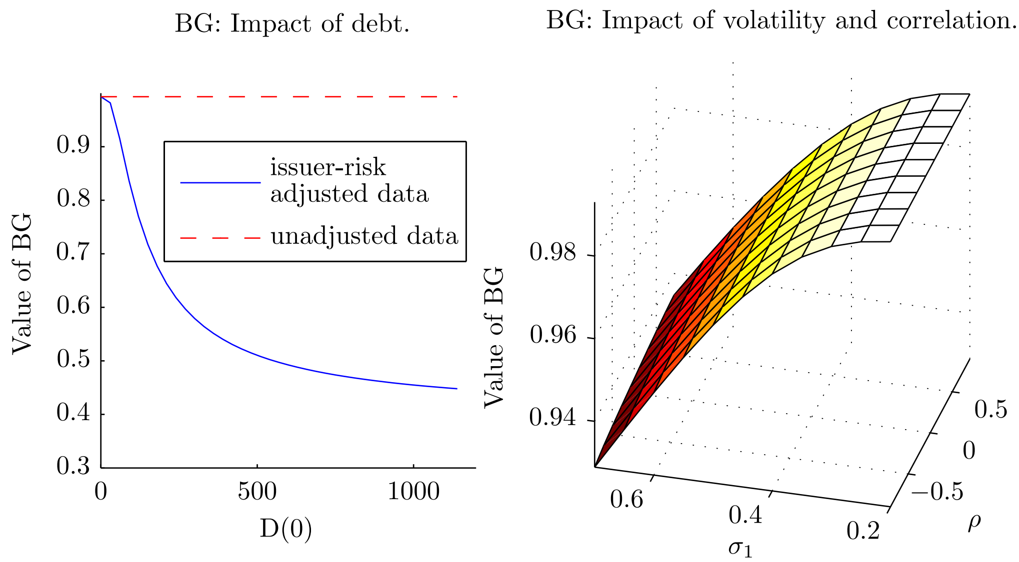

Figure 3.13: BG: Analysis of issuer risk in GBM framework.

In the second plot in Figure 3.13 the relationship between price and volatility is of nearly linear kind: The higher the volatility the lower the price the investor has to pay. The impact of the correlation is less distinct as in the case of index certificates due to the fact that a major part of the value of the bonus guarantee depends on the zero-coupon bond. This fact is also reflected in the stochastic covariance framework. We see here clearly that the value of the zero bond is mainly driven by the quality of the underlying credit: Hence, the graphs in Figure 3.14 resemble a mirrored image of the ones in Figure 3.9.

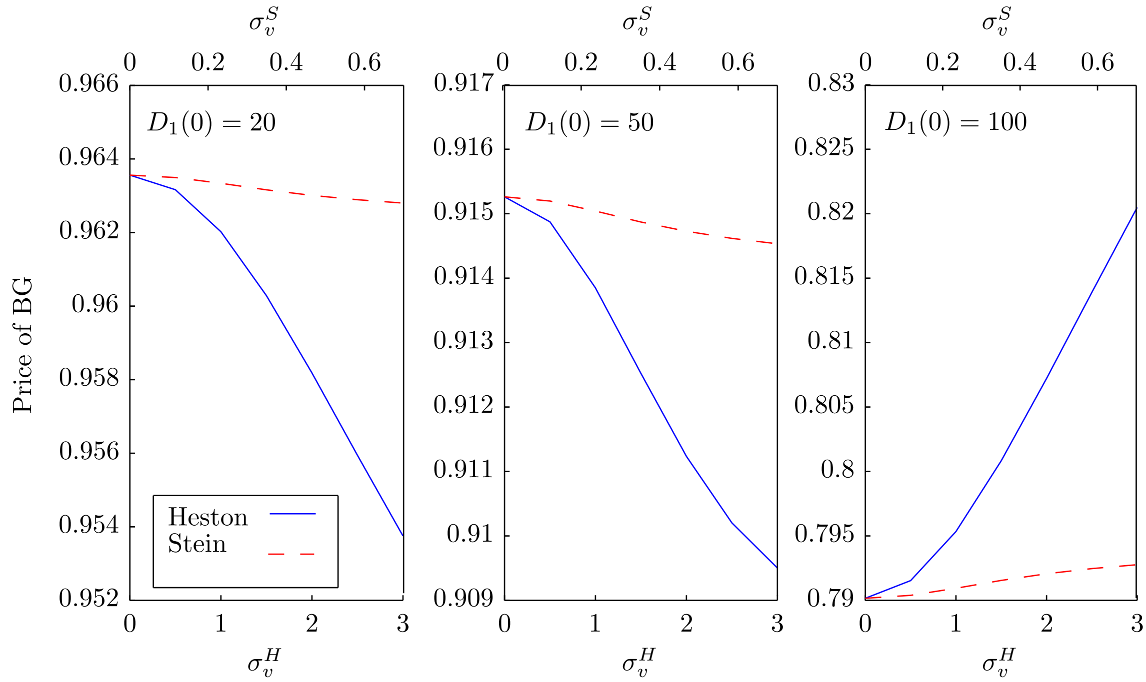

Figure 3.14: BG: Impact of σ_v.

Discount certificates

The risk protection of a discount certificate consists in a risk buffer: The investor buys the certificate at a discount on the actual value of the underlying. This risk limitation is again financed by a gain limit. The structure can be hedged by investing in the underlying and writing a call option, i.e. the value is specified by

$$\begin{aligned} DC\left(t, S_1, S_2, K_2\right) &= S_2(t) e^{-\int_t^T d_2(s)ds} - C_{Call}\left(t, S_2, K\right) - (1 - \mathfrak{R})\, e^{-\int_t^T r(s)ds} \\ &\quad \mathbb{E}_{\tilde{Q}}\left[\left(S_2(T) - \max\left[S_2(T) - K_2, 0\right]\right) \mathbb{1}_{\{\iota_1 \leq T\}} \left|\mathcal{F}_t\right.\right], \end{aligned} \tag{3.147}$$

where ι_1 is as in (3.127). The discount certificate can be valued by the following formula

$$\begin{aligned} DC\left(t, S_1, S_2, K_2\right) &= \mathfrak{R}\left(S_2(t) e^{-\int_t^T d_2(s)ds} - C_{Call}\left(t, S_2, K_2\right)\right) \\ &\quad +(1 - \mathfrak{R})\left(C_S^D\left(t, S_1, S_2\right) - C_{Call}^D\left(t, S_1, S_2, K_2\right)\right). \end{aligned} \tag{3.148}$$

We value a discount certificate with $K_2 = 120$ in different scenarios. In respect to the debt level the value of the discount certificate does not differ in its characteristics from the certificates analysed before (see Figure 3.15). In Figure 3.15 we look into the dependence of the price on the issuer's volatility and the correlation. Regarding the volatility we see the known structure, i.e. the price falls when the volatility increases. The slopes are similar for all correlation scenarios. As the certificate consists of building blocks clearly dependent on the correlation the impact of the correlation on the value of the certificate is similar to the index certificate example.

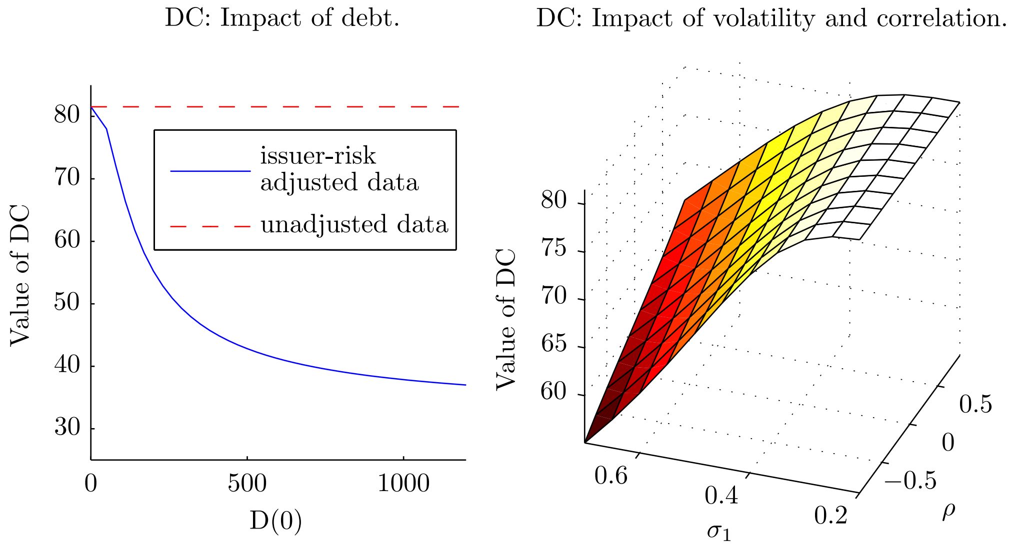

Figure 3.15: DC: Analysis of issuer risk in GBM framework.

In the analysis of the discount certificate in the stochastic covariance framework (see Figure 3.16) we see in the Heston-type model that by increasing σ_v we do not only lower the default probability but also increase the price of the short call (in contrast to the long call in the participation guarantee certificate). Both effects lead to increases in the price of the certificate in the Heston-type model.

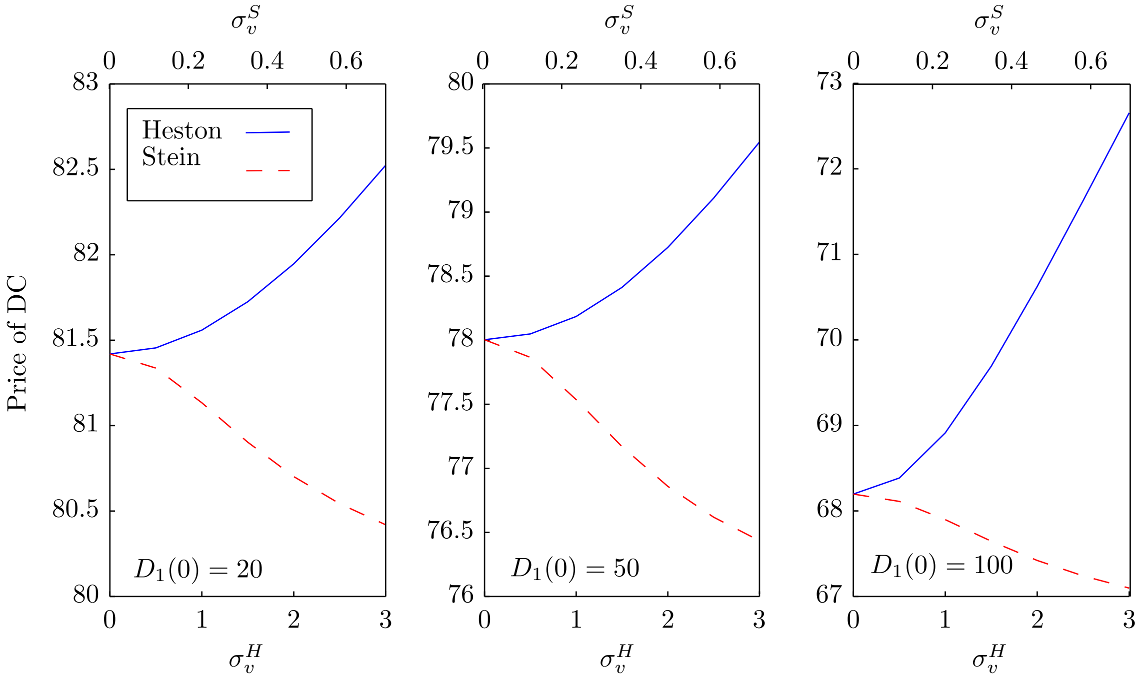

Figure 3.16: DC: Impact of σ_v.

Bonus certificates

The investor in this certificate is protected from a decline of the underlying up to a certain point, the protection barrier $B_2(t) = B_2 e^{\int_0^t r(s)ds}$. Below this point the investor fully participates in any fluctuation of the underlying. The same is true for the performance of the underlying beyond the bonus barrier K_2. The certificate can be fully hedged by an investment in the underlying and by buying a knock-out put option with barrier $B_2(t) = B_2 e^{\int_0^t r(s)ds}$. When the issuer risk is incorporated in the pricing model the price is specified by

$$\begin{aligned} BC\left(t, S_1, S_2, K_2, B_2(t)\right) &= S_2(t) e^{-\int_t^T d_2(s)ds} + e^{-\int_t^T r(s)ds} \\ &\quad \mathbb{E}_{\tilde{Q}}\left[\max\left[K_2 - S_{2,T}, 0\right] \mathbb{1}_{\{\iota_2 > T\}} \mid \mathcal{F}_t\right] - (1 - \mathfrak{R})\, e^{-\int_t^T r(s)ds} \\ &\quad \mathbb{E}_{\tilde{Q}}\left[\left(S_2(T) + \max\left[K_2 - S_2(T), 0\right] \mathbb{1}_{\{\iota_2 > T\}}\right) \mathbb{1}_{\{\iota_1 \le T\}} \mid \mathcal{F}_t\right], \end{aligned} \tag{3.149}$$

where

$$\iota_2 = \inf\left(t' \in (t_0, T] : S_2(t') \le B_2(t')\right), \tag{3.150}$$

where ι_1 is as in (3.127). The following formula evaluates the payoff:

$$\begin{aligned} BC\left(t,S_1,S_2\right) &= \Re\left(S_2(t)e^{-\int_t^T d_2(s)ds} + C_{1P}\left(t,S_2,K_2,B_2(t)\right)\right) \\ &\quad + (1-\Re)\left(C_S^D(t,S_1,S_2) + C_{1P}^D\left(t,S_1,S_2,K_2,B_2(t)\right)\right). \quad (3.151) \end{aligned}$$

Finally, we show some exemplary computations of the bonus certificate. We assumed a protection barrier $B_2(0) = 70$ and a bonus barrier $K_2 = 130$ for our computations.

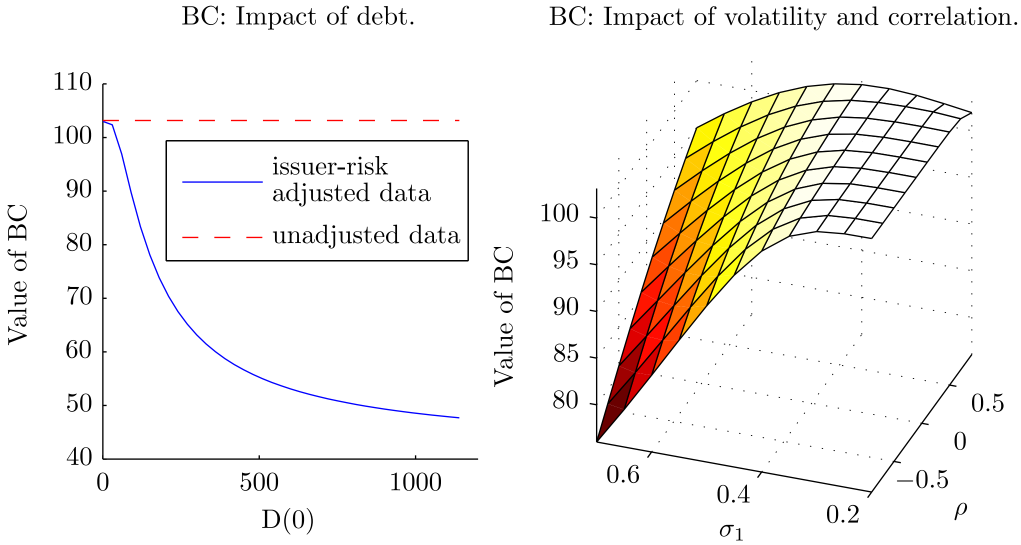

Figure 3.17: BC: Analysis of issuer risk in GBM framework.

Not surprisingly, the graph (Figure 3.17) plotting the debt level of the issuer against the value of the bonus certificates shows the same features as in the examples above. The bonus certificate is constructed by combining the investment in the underlying and a knock-out put option. Thus, it is not surprising that the second plot in Figure 3.17 resembles strongly the respective graph for the index certificate.

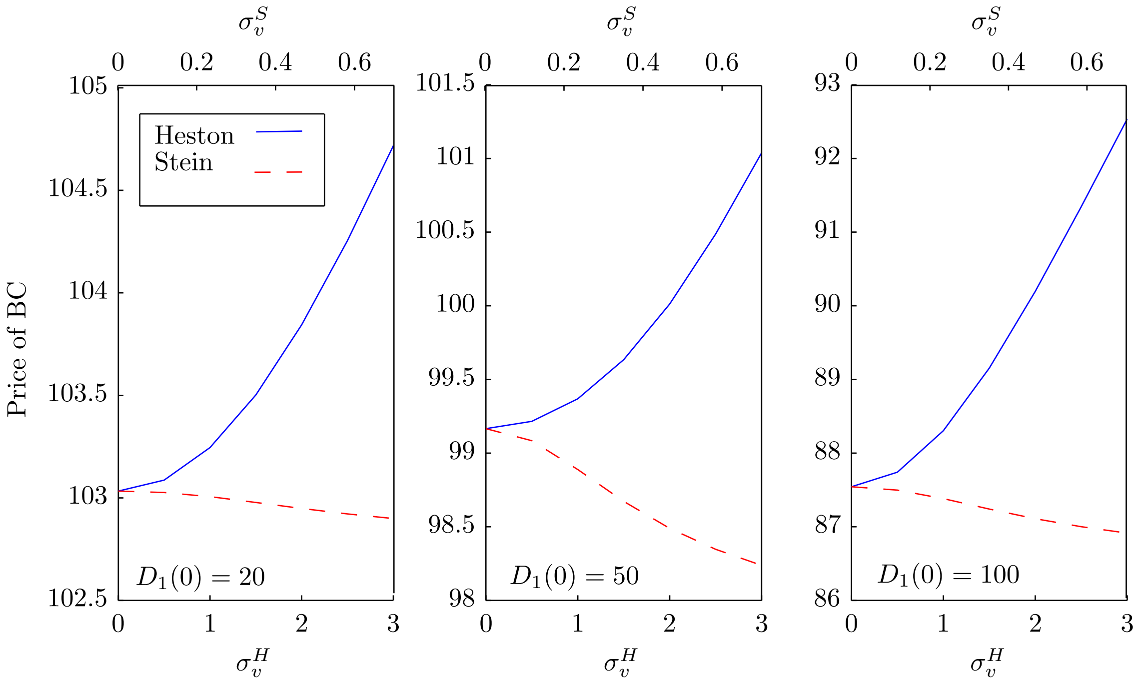

Figure 3.18: BC: Impact of σ_v.

The impact of the index components is, however, not so visible in the analysis of the stochastic covariance framework: In contrast to the graphs of the index certificate where in nearly all scenarios the prices decreased with increasing σ_v we see here all graphs of the Heston-type model increasing. Thus, we deduce from Figure 3.18 that the impact of stochastic volatility on the value of the down-and-out put is stronger than on the index component. The large impact of the down-and-out put option on the price can be traced back to its moneyness. In the Stein and Stein framework this picture is less uniform.

3.6.4 Conclusion

We have derived closed-form expressions for index, discount, participation guarantee, and bonus certificates under issuer risk in a Black-Scholes model and a stochastic covariance model framework. Our scenario computations clearly depict that, depending on the issuer's capital soundness, a pricing formula which neglects issuer risk considerably overprices the value of the singular certificate. Thus, for a retail investor a simple comparison of the prices of analogous certificates of different issuers is not appropriate in order to find the security with the best price-performance ratio. For his investment decision the investor has to take the financial soundness of the issuer into consideration. Furthermore, the introduction of a third process which governs the covariance matrix of the issuer and the underlying leads to big differences in the valuation compared to a Black-Scholes framework. We have analysed the behaviour of the certificate prices for rising stochasticity in the covariance in detail. The choice of the assumed stochastic covariance model considerable influences the price dynamics, e.g. the price in the Heston-type model reacts completely different to a rise in the volatility of the covariance process than the value in the Stein-type model.

Chapter 4

Pricing of barrier options within stochastic correlation model

4.1 Introduction

As already mentioned, local and stochastic volatility models have been in place for some years now, e.g. Dupire [37] or Stein and Stein [111] and Heston [65]. A natural extension of the latter ones is a multivariate model with stochastic correlation, not only stochastic covariance like the model presented in Section 3. And indeed, the performance of a portfolio or a multi-dimensional derivative depends very much on the joint behaviour of the underlyings, i.e. the covariances. Correlations are not constant over time. The popularity of discrete stochastic correlation models like the Dynamic Conditional Correlation (DCC) model proposed by Engle [40] supports this idea. Stylised facts featured by historical data as well as by implicit market prices, e.g. the smile and skew of volatilities and correlations, can be captured by the assumption of a stochastic behaviour of correlation (besides volatility) (see e.g. da Fonseca et al. [26], Christoffersen et al. [23]). There are two main problems with the modelling of correlation: One is the model to choose to keep the correlation between -1 and 1 and the other is intractability because the number of parameters grows exponentially when the dimensions are increased.

Gourieroux et al. [58], Philipov and Glickman [92] and da Fonseca et al. [27], [26] propose the use of Wishart processes to model stochastic multivariate covariance matrices. However, this approach is rather cumbersome when it comes to estimation and simulation. Pigorsch and Stelzer [93] and Muhle-Karbe et al. [89] present a multivariate stochastic volatility model of OU-Wishart type, which is analytically tractable, however the dimensions increase which renders a calibration rather complicated.

In their discrete modelling works Kim et al. [77], Aguilar and West [2], Pitt and Shephard [94] Chib et al. [22] suggest a multivariate factor stochastic volatility model to represent stochastic dependencies between the underlyings. The factor models as well as our principal component model reduce the dimension of the original problem. Our paper is based on a stochastic principal component model introduced earlier by works of Escobar et al. [42] and Escobar et al. [41]. This framework introduces stochastic eigenvalues and, thus, avoids proliferation of parameters as the number of eigenvalues can be diminished to a suitable number. Other publications applying principal component analysis are in support of the fact that two to three eigenvalues are sufficient to describe most of the variation in the portfolio (see Alexander [3], [4]).

Moreover, our model easily extends the Heston model to more underlyings: We allow for stochastic volatility and at the same time for stochastic correlation among assets and between variance and assets as well as between assets and correlation. The basic stochastic principal component model is an affine model for which the characteristic function is available and allows for easy calibration to plain vanilla instruments. Even some parametrisations of the extension to the stochastic principal component model which is presented here feature an affine characteristic function.

We continue in this chapter to price barrier derivatives in one and two dimensions as in Chapter 3 but now within the context of the principal component model. By means of the affine characteristic function in some parametrisations of the model we are able to find analytical expressions for single-barrier options. However, as mentioned in Chapter 3 we are not able to find a closed-form solution for two barriers in two dimensions and we hence follow a perturbation theory approximation. The accuracy of the approximation is analytically proven.

Hence, in this line of development, our work improves previous literature on correlation risk and default dependencies: The here presented model assumes stochastic correlation between the assets, and the pricing stays feasible. The simplicity of the approximative pricing scheme is a key element of a good pricing performance.
This chapter is structured as follows. In Section 4.2 we give a short insight in the stochastic principal component model and we empirically motivate the structure of the model (see Section 4.3). We derive closed-form solutions for single-asset instruments such as single-barrier call options in this framework in Section 4.4. To solve for barrier options on two underlyings we apply an approximation, which is motivated by perturbation theory in Section 4.5.

4.2 Model framework

The system of processes is defined on a filtered probability space $(\Omega, \mathcal{F}, \tilde{\mathcal{Q}}, \mathbb{F})$ where $\mathcal{F}_0$ contains all subsets of the ($\tilde{\mathcal{Q}}-$) null sets of $\mathcal{F}$ and $\mathbb{F}$ is right-continuous. We define the processes under the risk-neutral measure $\tilde{\mathcal{Q}}$.

We introduce a fast mean-reverting stochastic principal component model. In the next section we motivate the choice of a fast mean-reverting component to the eigenvalue empirically. One can think of the fast mean-reverting component as influenced by the fast mean reversion of the volatility which has been found by several authors (see [125], [46]). In our analysis we limit ourselves for demonstration purposes to two underlyings. However, an extension to d underlyings is straight forward.

$$\begin{aligned} dS_i &= rS_i dt + S_i \sum_{j=1}^{\bar{p}} a_{ij} f(v_j(t)) dW_j, \; i \in \{1,2\}, \; \bar{p} = 2, && (4.1) \\ dv_j &= \frac{\kappa_{v_j}}{\delta_j^2}(\zeta_{v_j} - v_j)dt + \frac{\sigma_{v_j}}{\delta_j}\sqrt{v_j} dZ_j, && (4.2) \\ \langle dW_i, dZ_j \rangle &= 0, \\ \langle dW_j, dZ_j \rangle &= \rho_j^v dt, \\ \langle dW_i, dW_j \rangle &= 0, \text{ for } j \in \{1,2\}. \end{aligned}$$

$f(v_j)$ represents the jth eigenvalue of the instantaneous covariance matrix Σ of the underlying process S_i. The eigenvalues are driven by a fast mean-reverting Cox-Ingersoll-Ross process with mean-reversion rate $\frac{\kappa_j}{\delta_j}$, where we assume δ_j very small. We further suppose that the eigenvalues $f(v_j)$ are positive and bounded. We assume that there are constants $\mathfrak{o}_{11}, \mathfrak{o}_{12}, \mathfrak{o}_{21}, \mathfrak{o}_{22}$ such that $0 < \mathfrak{o}_{j1} \leq f(v_j) \leq \mathfrak{o}_{j2}$, for all $j \leq \bar{p}$. (a_{ij}) represents the (2×2) matrix of eigenvectors. For any fixed time t the process decomposes into two orthogonal directions given by the eigenvectors a_{ij}, which are not dependent on time. We presume that the Feller condition is satisfied for all processes.
In the following we denote the instantaneous variance of the log-asset $\ln S_i$ by $\sigma^2_{S_i}$ and the instantaneous correlation between the two assets $\ln S_i$ and $\ln S_k$ by ρ_{S_i,S_k}.

Remark 15. *The log-assets exhibit stochastic variance and correlation given by*

$$\begin{aligned} \sigma^2_{S_i}(t) &= \sum_{j=1}^{2} a_{ij}^2 f\left(v_j(t)\right)^2, \\ \rho_{S_i,S_k}(t) &= \frac{\sum_{j=1}^2 a_{ij} a_{kj} f\left(v_j(t)\right)^2}{\sqrt{\sum_{j=1}^2 a_{ij}^2 f\left(v_j(t)\right)^2}\sqrt{\sum_{j=1}^{\bar{p}} a_{kj}^2 f\left(v_j(t)\right)^2}}. && (4.3) \end{aligned}$$

Proof.

The expression(4.3) can be obtained by applying Itô's formula (see Theorem 18) and quadratic variation (see Definition 36). Thus, with

$$\begin{aligned} df\left(v_j(t)\right)^2 &= \left(\frac{\partial f\left(v_j(t)\right)^2}{\partial v_j}\frac{\kappa_{v_j}}{\delta_j^2}\left(\zeta_{v_j}-v_j\right)+\frac{1}{2}\frac{\partial^2 f\left(v_j(t)\right)^2}{\partial v_j^2}\frac{\sigma_{v_j}^2}{\delta_j^2}v_j\right)dt \\ &\quad +\frac{\partial f\left(v_j(t)\right)^2}{\partial v_j}\frac{\sigma_{v_j}}{\delta_j}\sqrt{v_j}dZ_{v_j} \end{aligned} \tag{4.4}$$

and

$$\sigma^2_{f(v_j(t))} = \left(\frac{\partial f\left(v_j(t)\right)^2}{\partial v_j}\right)^2\frac{\sigma_{v_j}^2}{\delta_j^2}v_j, \tag{4.5}$$

the results follow. ☐

Theorem 39. *(Characteristic function)*

For any time t the joint conditional characteristic function $\varphi(t'-t,\mathbf{u}) = \mathbb{E}\left[\exp\left\{i\left\langle \mathbf{u}\mathbf{x}(t')\right\rangle\right\}|\mathcal{F}_t\right]$ of the log-assets $x_1=\ln\left(S_1e^{\int_t^{t'}r(s)ds}\right)$ and $x_2=\ln\left(S_2e^{\int_t^{t'}r(s)ds}\right)$, where S_1 and S_2 are given by Model (4.1), satisfies the following PDE

$$\begin{aligned} &-\frac{\partial\varphi(\tau,\mathbf{u})}{\partial\tau}+\sum_{i=1}^{2}\left(r-\frac{1}{2}\sum_{j=1}^{2}a_{ij}^2 f\left(v_j(t)\right)^2\right)\frac{\partial\varphi(\tau,\mathbf{u})}{\partial x_i} \\ &\qquad +\frac{1}{2}\sum_{i,k=1}^{2}\frac{\partial^2\varphi(\tau,\mathbf{u})}{\partial x_i\partial x_k}\sum_{j=1}^{2}a_{ij}a_{kj}f\left(v_j(t)\right)^2 \\ &+\sum_{j=1}^{2}\left(\frac{\kappa_{v_j}}{\delta_j^2}\left(\zeta_{v_j}-v_j\right)\frac{\partial\varphi(\tau,\mathbf{u})}{\partial v_j}+\frac{1}{2}\frac{\sigma_{v_j}^2}{\delta_j^2}v_j\frac{\partial^2\varphi(\tau,\mathbf{u})}{\partial v_j^2}\right) \\ &\qquad +\sum_{i=1}^{2}\sum_{j=1}^{2}a_{ij}f\left(v_j(t)\right)\left(\rho_j^v\sqrt{v_j}\frac{\sigma_{v_j}}{\delta_j}\frac{\partial^2\varphi(\tau,\mathbf{u})}{\partial x_i\partial v_j}\right) = 0, \end{aligned} \tag{4.6}$$

where $\tau=t'-t$, $t'>t$, with initial condition $\varphi(0,\mathbf{u})=\exp\left\{i\left\langle\mathbf{u}\mathbf{x}\right\rangle\right\}$.

Proof. The theorem trivially follows from Itô's Lemma. ☐

Corollary 16. *(Special case)*

In the case $\rho_j^v = 0$, $\forall j$, *and* $f(v_j) = \sqrt{v_j}$ *the characteristic function has an affine solution given by*

$$\varphi(\tau, \mathbf{u}) = \exp\left\{ i \langle \mathbf{ux} \rangle + A_H^*(\tau, \mathbf{u}) + \sum_{j=1}^{2} \frac{\delta_j^2}{\sigma_{v_j}^2} B_{H1j}^*(\tau, \mathbf{u}) v_j \right\}, \tag{4.7}$$

where

$$B_{H1j}^*(\tau, \mathbf{u}) = \frac{\kappa_{v_j}}{\delta_j^2} - \mathfrak{d}_{v_j} \frac{\sinh\left(\frac{\mathfrak{d}_{v_j}}{2}\tau\right) + \frac{\kappa_{v_j}}{\delta_j^2 \mathfrak{d}_{v_j}} \cosh\left(\frac{\mathfrak{d}_{v_j}}{2}\tau\right)}{\cosh\left(\frac{\mathfrak{d}_{v_j}}{2}\tau\right) + \frac{\kappa_{v_j}}{\delta_j^2 \mathfrak{d}_{v_j}} \sinh\left(\frac{\mathfrak{d}_{v_j}}{2}\tau\right)}, \tag{4.8}$$

$$\begin{aligned} A_H^*(\tau, \mathbf{u}) &= \tau \left(ir(u_1 + u_2) + \sum_{j=1}^{2} \frac{\kappa_{v_j}^2}{\delta_j^2 \sigma_{v_j}^2} \zeta_{v_j} \right) \\ &\quad -2 \sum_{j=1}^{2} \left(\frac{\kappa_{v_j}}{\sigma_{v_j}^2} \zeta_{v_j} \ln\left(\frac{\kappa_{v_j}}{\delta_j^2 \mathfrak{d}_{v_j}} \sinh\left(\frac{\mathfrak{d}_{v_j}}{2}\tau\right) + \cosh\left(\frac{\mathfrak{d}_{v_j}}{2}\tau\right) \right) \right), \end{aligned} \tag{4.9}$$

$$\mathfrak{d}_{v_j} = \mathfrak{d}_{v_j}(\mathbf{u}) = \sqrt{\frac{\kappa_{v_j}^2}{\delta_j^4} + \frac{\sigma_{v_j}^2}{\delta_j^2}\left(i\left(u_1 a_{1j}^2 + u_2 a_{2j}^2\right) + 2u_1 u_2 a_{1j} a_{2j} + u_1^2 a_{1j}^2 + u_2^2 a_{2j}^2\right)}.$$

This characteristic function is well defined and analytic in a neighbourhood of 0.

For a proof refer to Appendix B.1.

4.3 Data analysis for mean-reversion scales

In the following[1] we study if there exists a fast mean-reversion scale of the eigenvalues and/or covariances of the assets as suggested by the model. In our analysis we follow the lines of Fouque et al. [51], who analyse variograms of the log absolute returns for the mean-reversion speed, and extend their approach to two dimensions from an underlying perspective. As proposed by our model we assume constant orthonormal eigenvectors $(a_{11}, a_{21})'$ and $(a_{12}, a_{22})'$ and focus on the estimation of the mean-reversion speed parameters $\kappa_{v_j}^* = \frac{\kappa_{v_j}}{\delta_j^2}$, $j = 1, 2$, of the dynamics of the eigenvalues $f(v_1) := f_1$ and $f(v_2) := f_2$. In order to facilitate the estimation we set the correlation between the Brownian motions

[1] The following chapter has been prepared in cooperation with Daniela Neykova during her master thesis project. See [90].

for the assets and the volatility processes ρ_j^v, $j = 1, 2$, to zero. An analysis by Fouque et al. (see [49], p. 81ff) showed in a one-dimensional setting that the rate of mean-reversion was insensitive to the correlation parameter.
Our data set[2] comprises stock prices of the three companies IBM, Apple and Dell. We use their high-frequency intra-day data over one year (from October 1, 2008 to September 30, 2009) to proof our hypothesis of the existence of a fast mean-reverting factor. To estimate the covariance matrices[3] we apply daily stock-price data over 15 years (from June 30, 1995 to June 30, 2010). The large data set should ensure the stability of the estimators. The high-frequency data points are averaged over intervals of 5 minutes. Intervals during working days without any observations are collapsed. So we finally end up with 78 matching observations per company per day, thus a total of $19,593$ data points per company. Furthermore, the high-frequency data must be adjusted by the day effect, i.e. the systematic effect that the volatility is the highest at the beginning of the day when trading starts and at the end of the day. We follow here Fouque et al. [51] who model the day effect by fitting a periodical function to the 78 observations.

The steps which are described in the following are performed for both time series, the daily and the intra-day data, separately for a combination of two companies at a time. We adjust the stock price returns by their means and normalise the data:

$$\begin{aligned} \bar{S}_1^d(n) &:= \frac{2(\overline{S}_1(n) - \overline{S}_1(n-1))}{\sqrt{\Delta t}(\overline{S}_1(n) + \overline{S}_1(n-1))}, \\ \bar{S}_2^d(n) &:= \frac{2(\overline{S}_2(n) - \overline{S}_2(n-1))}{\sqrt{\Delta t}(\overline{S}_2(n) + \overline{S}_2(n-1))}, \ n = 1, 2, \dots, \end{aligned} \tag{4.10}$$

where n describes the nth observation, Δt is either 5 minutes in the case of the high-frequency data or 1 day in the case of the daily data, and $\overline{S}_i$ denotes the mean-adjusted data point. When we theoretically discretise the stock price processes (4.1) by applying Euler approximation[4]

$$\begin{aligned} \frac{\Delta S_1(n)}{S_1(n)\sqrt{\Delta t}} &= \mu_1\sqrt{\Delta t} + a_{11} f_1(v_1(n))\frac{\Delta W_1(n)}{\sqrt{\Delta t}} + a_{12} f_2(v_2(n))\frac{\Delta W_2(n)}{\sqrt{\Delta t}}, \\ \frac{\Delta S_2(n)}{S_2(n)\sqrt{\Delta t}} &= \mu_2\sqrt{\Delta t} + a_{21} f_1(v_1(n))\frac{\Delta W_1(n)}{\sqrt{\Delta t}} + a_{22} f_2(v_2(n))\frac{\Delta W_2(n)}{\sqrt{\Delta t}}, \\ n &= 1, 2, \dots \end{aligned} \tag{4.11}$$

where $(a_{i1}, a_{i2})'$ describe the eigenvalues of asset S_i, $f(v_j(n)) := f_j(n)$ is the positive

[2]The data has been downloaded from The Trade and Quote (TAQ) database.
[3]Note that we assume constant eigenvectors.
[4]Note that we work now under the pricing measure. μ_i are r in the risk-neutral world (see (4.1)).

eigenfunction process, and ΔW_j is the respective increment of the Brownian motion W_j, we see that

$$\begin{aligned}\bar{S}_1^d(n) &= a_{11} f_1(n) \frac{\Delta W_1(n)}{\sqrt{\Delta t}} + a_{12} f_2(n) \frac{\Delta W_2(n)}{\sqrt{\Delta t}}, \\ \bar{S}_2^d(n) &= a_{21} f_1(n) \frac{\Delta W_1(n)}{\sqrt{\Delta t}} + a_{22} f_2(n) \frac{\Delta W_2(n)}{\sqrt{\Delta t}}, \ n = 1, 2, \ldots. \end{aligned} \tag{4.12}$$

To extract the behaviour of f_1 and f_2 from the data we de-correlate the two-dimensional discrete process $(\bar{S}_1^d, \bar{S}_2^d)'$ by multiplying it with the eigenvectors of the covariance matrix for $(\ln(S_1), \ln(S_2))'$. As already mentioned our assumptions of constant eigenvectors allows us to estimate the covariance matrix from the daily data set over its whole period from 1995 to 2010. From there we gain the two eigenvectors $\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$ and $\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$.

$$\begin{pmatrix} \tilde{S}_1^d(n) \\ \tilde{S}_2^d(n) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} \bar{S}_1^d(n) \\ \bar{S}_2^d(n) \end{pmatrix}, \ n = 1, 2, \ldots. \tag{4.13}$$

We, thus, obtain the two independent time series $(\tilde{S}_1^d, \tilde{S}_2^d)'$ for our high-frequency data.

$$\tilde{S}_1^d(n) = f_1(n) \frac{\Delta W_1(n)}{\sqrt{\Delta t}} \text{ and } \tilde{S}_2^d(n) = f_2(n) \frac{\Delta W_2(n)}{\sqrt{\Delta t}}, \ n = 1, 2, \ldots. \tag{4.14}$$

As mentioned above the day effect in the intra-day data, i.e. the fact that the volatility of stocks is especially high when the stock exchanges open and decreases to a lower level in the course of time over the day before it rises again, has to be accounted for. We include this effect into our model by replacing (4.14) with

$$\hat{S}_1^d(n) = f_1(n) f_1^d(n) \frac{\Delta W_1(n)}{\sqrt{\Delta t}} \text{ and } \hat{S}_2^d(n) = f_2(n) f_2^d(n) \frac{\Delta W_2(n)}{\sqrt{\Delta t}}, n = 1, 2, \ldots, \tag{4.15}$$

where $f_j^d(n),\ j = 1, 2$ is a periodic function of day-time with period 78 (which corresponds to the 78 observation points per day). For the estimation of the periodic function we treat each data point $\hat{S}_1^d(n)$ as influenced by the realisation of the function $f_j^d(n)$ at the time point n, $n = 1, \ldots, 78$. We compute the root-mean square over all data points $\hat{S}_1^d(n)$ realised at a certain time n, $n = 1, \ldots, 78$, e.g. for the calculation of the first root mean-square at $n = 1$ we take all those data points into account which were realised at 9am. The result of all 78 root-mean squares is presented in Figure 4.1 for the examples Dell and Apple. The graph falls until a certain point in time before it increases again. This form could be reversed engineered with the sum of two exponential functions, one with a component $\exp\left\{-n/l_1^{(j)}\right\}$, which causes a decrease in n, and one with a component

$\exp\left\{(n-1)/l_2^{(j)}\right\}$, which is an increasing function for $n > 1$. $l_1^{(j)}$ and $l_2^{(j)}$ govern the form and determine the inflexion point of the graph. Hence, we fit the root-mean square results to the the sum of two exponentials, $f_j^d(n) = f_j^{d^{(1)}}(n) + f_j^{d^{(2)}}(n)$ with $f_j^{d^{(1)}}(n) = \mathfrak{a}_1^{(j)} + \mathfrak{a}_2^{(j)} \exp\left\{-n/l_1^{(j)}\right\}$ and $f_j^{d^{(2)}}(n) = \mathfrak{a}_3^{(j)} + \mathfrak{a}_4^{(j)} \exp\left\{(n-1)/l_2^{(j)}\right\}$, $j = 1, 2$, to the 78 root-mean squares with a least-square approach. The fitted functions are marked in Figure 4.1 with a dotted line. In our analysis we obtained the following estimates for the parameters: $\mathfrak{a}_1^{(1)} = -0.1491, \mathfrak{a}_2^{(1)} = 0.0083, \mathfrak{a}_3^{(1)} = 0.1439, \mathfrak{a}_4^{(1)} = 0.1509, l_1^{(1)} = 0.0042, l_2^{(1)} = 0.1732$ and $\mathfrak{a}_1^{(2)} = -0.1478, \mathfrak{a}_2^{(2)} = 0.0328, \mathfrak{a}_3^{(2)} = 0.1706, \mathfrak{a}_4^{(2)} = 0.1522, l_1^{(2)} = 0.0168, l_2^{(2)} = 0.2530$.

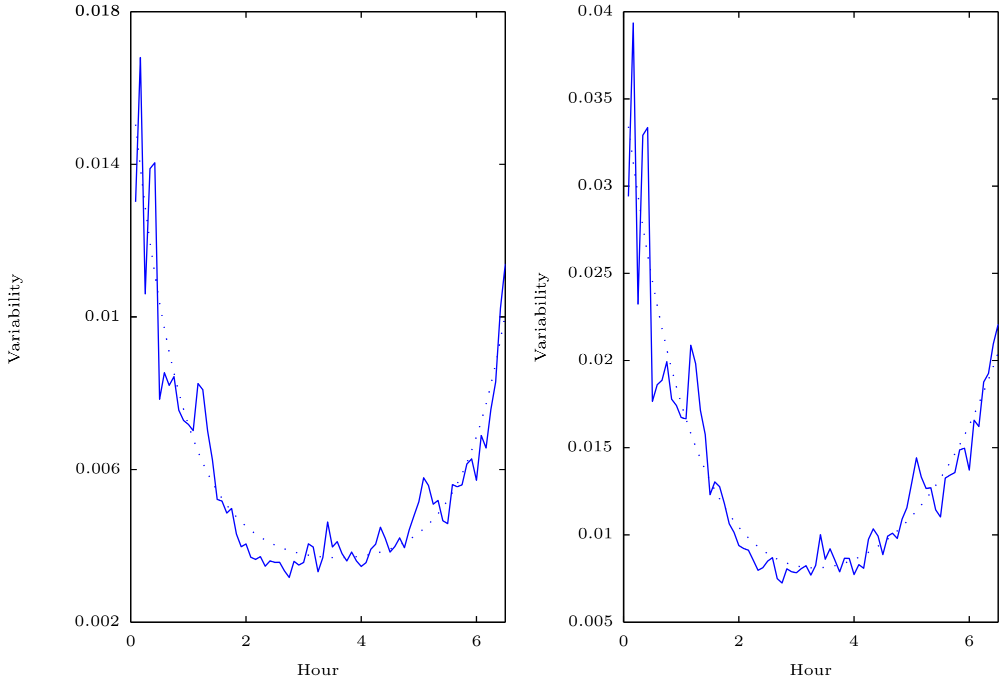

Figure 4.1: Observed data (solid line) and estimated functions (dotted line) for the systematic intra-day behaviour of variability for the eigenfunction $j = 1$ in the left figure and for $j = 2$ in the right one.

To clean the actual data set we transform $\tilde{S}_1^d(n) := \hat{S}_1^d(n)/f_1^d(n), \tilde{S}_2^d(n) := \hat{S}_2^d(n)/f_2^d(n)$. In the following we consider the log of the absolute high-frequency data of $\tilde{S}_i^d(n)$,

$$\begin{aligned} \bar{L}_1^d(n) &:= \ln\left|\tilde{S}_1^d(n)\right| = \ln(f_1(n)) + \ln\left(\left|\frac{\Delta W_1(n)}{\sqrt{\Delta t}}\right|\right), \\ \bar{L}_2^d(n) &:= \ln\left|\tilde{S}_2^d(n)\right| = \ln(f_2(n)) + \ln\left(\left|\frac{\Delta W_2(n)}{\sqrt{\Delta t}}\right|\right). \end{aligned} \tag{4.16}$$

and apply a 10-points median filter to compensate for the noise in the data.[5] Finally, we

[5]In one-dimensional form, a median filter uses a sliding window with respect to the data series. When

can compute the variogram functions for the the high-frequency data which allows us to analyse the autocorrelations in our data series in closer detail.

$$\begin{aligned} V_1^{d,N}(\mathfrak{k}) &= \frac{1}{N-\mathfrak{k}} \sum_{n=1}^{N-\mathfrak{k}} \left(\bar{L}_1^d(n+\mathfrak{k}) - \bar{L}_1^d(n)\right)^2, \\ V_2^{d,N}(k) &= \frac{1}{N-\mathfrak{k}} \sum_{n=1}^{N-\mathfrak{k}} \left(\bar{L}_2^d(n+\mathfrak{k}) - \bar{L}_2^d(n)\right)^2, \end{aligned} \tag{4.17}$$

where N is the total number of observations, and $\mathfrak{k}$, the lag. If the graphs of those variogram functions for the intra-day data were now flat we would have to conclude that there is no mean-reversion effect in the respective time scale. To estimate the extent of the curvature and, thus, the respective mean-reversion scale, we fit the variogram to

$$V_j^{d,N}(\mathfrak{k}) \approx 2\vartheta_{v_j}^2 \left(1 - e^{-\kappa_{v_j}^* \mathfrak{k}\Delta t}\right) + c_j, \tag{4.18}$$
$$\text{with } \vartheta_{v_j}^2 = \frac{(\sigma_{v_j}^*)^2 \zeta_{v_j}}{2\kappa_{v_j}^*}, \sigma_{v_j}^* = \frac{\sigma_{v_j}}{\delta_j},\ j = 1, 2,$$

where $\kappa_{v_j}^* = \frac{\kappa_{v_j}}{\delta_j^2}$, $j = 1, 2$. As shown in the Appendix B.4 $V_j^{d,N}(\mathfrak{k})$ is an estimator of the variogram function in a mean-reversion model. $\frac{1}{\kappa_{v_j}^*}$ represents the time-scale of the fast mean-reversion in the data, and is, thus, the figure of interest. Finally, we fit the functions defined in (4.18) to the transformed intra-day variogram data $(V_1^{d,N}(\mathfrak{k}), V_2^{d,N}(\mathfrak{k}))'$. The parameters c_j and $\vartheta_{v_j}^2$, $j = 1, 2$, are determined by averaging the first and the last values of the variogram function. From there we carry out a least-square estimation for a non-linear regression to find the mean-reverting speed parameters $\kappa_{v_j}^*$, $j = 1, 2$. The Figure 4.2 pictures the empirical and fitted variograms of the eigenvalues of the Dell and Apple time series for the intra-day data.

the number is odd the centre data point in the respective window is replaced by the median of the data points in the window, e.g. if the values of the data points within a window are $1, 2, 9, 4, 5$ the centre data point would be replaced by the value 4, which is the median value of the sorted sequence $1, 2, 4, 5, 9$. When, like in our case, the number of elements n in the window is even, the median filter sorts the numbers and then takes the average of the $n/2$ and $n/2 + 1$ elements, e.g. if the values of the data points within a window are $1, 2, 9, 4$ the data point 9 would be replaced by the value 3, which is the average of 2 and 4.[96], p. 271f.

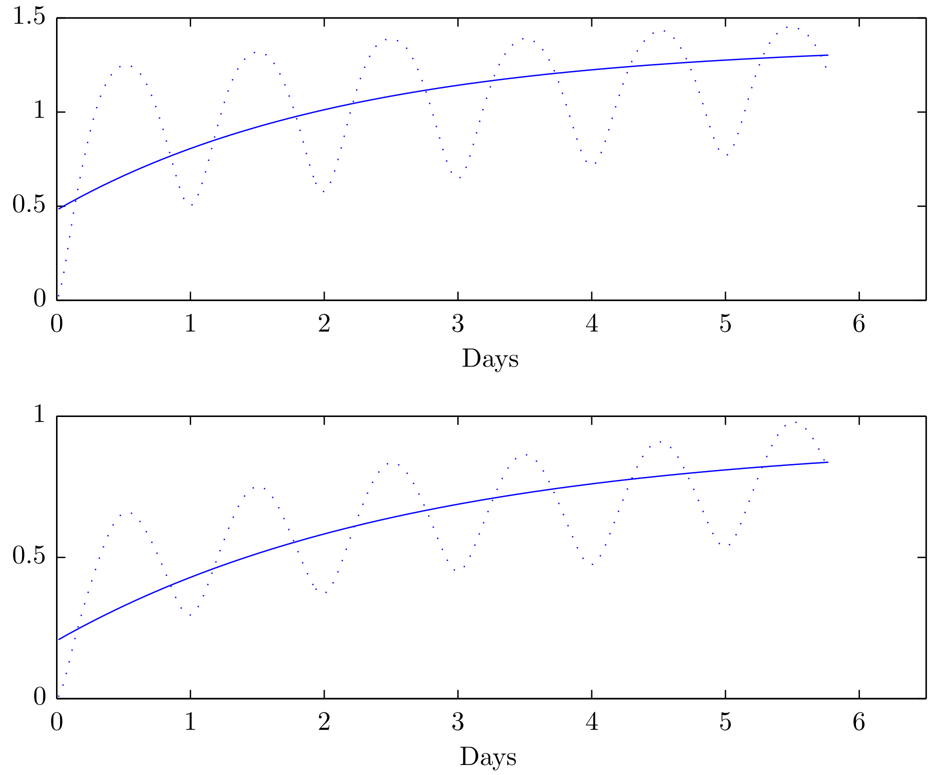

Figure 4.2: Empirical variogram functions (dotted lines) and fitted curves (solid lines) for the two series of normalised eigenvalues $\{\hat{S}_1^d(n)\}$ (top plot) and $\{\hat{S}_2^d(n)\}$ (bottom plot) obtained from the intra-day data analysis for Dell and Apple without considering the day effect.

The already discussed day effect is reflected in the oscillating graph of the empirical variograms of the high-frequency data. The adjusted variogram function is plotted in Figure 4.3. The slope of the fitted curves for adjusted and unadjusted data are very similar.

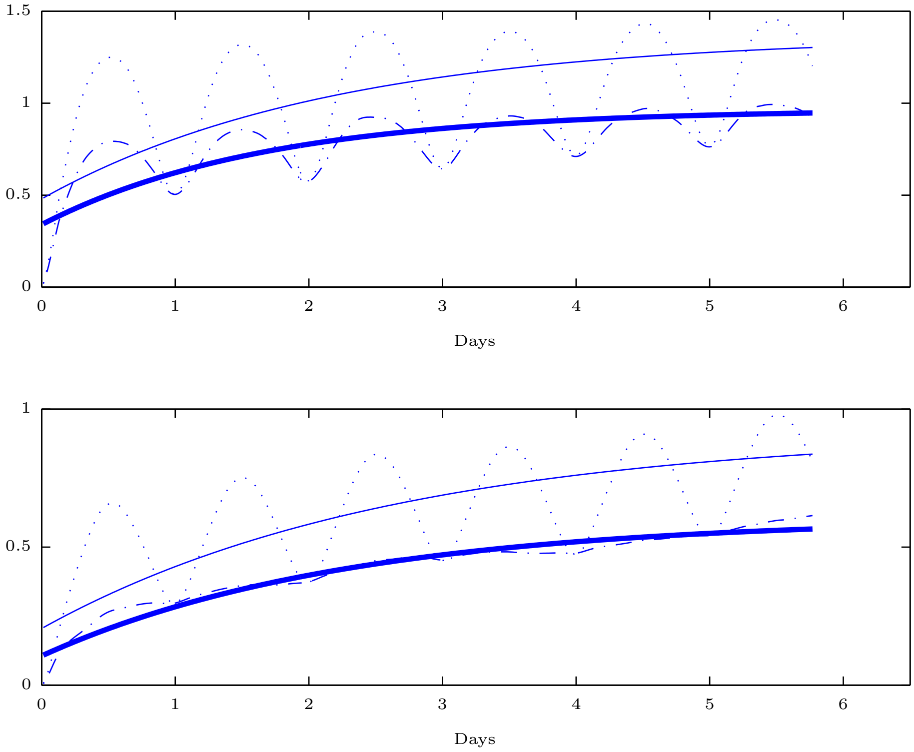

Figure 4.3: Empirical variograms (dashed lines, i.e. $\{\hat{S}_1^d(n)\}$ (thinner dashed line with higher amplitude), $\{\tilde{S}_1^d(n)\}$ (thicker dashed graph with lower amplitude)) and the respective fitted curves (thinner solid line is the fit for $\{\hat{S}_1^d(n)\}$, the thicker solid line the respective fit for $\{\tilde{S}_1^d(n)\}$) after compensating for the systematic intra-day variability behaviour, compared to the case without considering the day effect (dotted and solid lines respectively).

This analysis has been carried out for each of the three tuples of the three companies Dell, Apple and IBM. In Table 4.1 we summarise the results. The results show clearly the existence of a fast mean-reverting scale for each eigenvalue.

Data	$\kappa_{v_1}^*$	$\kappa_{v_2}^*$	$\frac{1}{\kappa_{v_1}^*}$ (day)	$\frac{1}{\kappa_{v_2}^*}$ (day)
$IBM, Dell$	257.70	361.66	0.98	0.70
$Dell, Apple$	150.82	111.04	1.67	2.27
$Apple, IBM$	115.10	100.53	2.19	2.51

Table 4.1: Estimated mean-reverting speeds ($\kappa_{v_j}^*,\ j = 1, 2$) and typical times for mean-reversion ($\frac{1}{\kappa_{v_j}^*},\ j = 1, 2$).

4.4 Pricing of single-barrier options

The marginal knock-out barrier option has a payout $g(S_1)$, which may depend on $S_1(T)$ at maturity time T provided that $S_1(t)$ has not crossed a predefined time-dependent barrier $B_1(t) = B_1 e^{\int_0^t r(s)ds}$ prior to maturity. When we assume risk-neutral valuation the value of a general barrier option with a time-dependent barrier on its underlying is given by

$$\begin{aligned} C(t, S_1, B_1, \mathbf{v}) &= \mathbb{E}_{\tilde{\mathcal{Q}}}\left[e^{-\int_t^T r(s)ds} g\left(S_1(T)\right) \mathbb{1}_{\{\iota_1 > T\}} \,|\mathcal{F}_t\right], \\ \text{where} & \\ \iota_1 &= \inf\left(t' \in (t, T] : S_1(t') \leq B_1(t')\right), \end{aligned} \tag{4.19}$$

where the expectation is taken with respect to the pricing measure $\tilde{\mathcal{Q}}$.
$g\left(S_1(T)\right)$ describes the part of the payoff which depends on the value of S_1 in T. The following PDE can be derived for the Model (4.1) with $\rho_j^v = 0$. The restriction is necessary to apply the method of images.

$$\begin{cases} \sum_{j=1}^2 \left(\frac{1}{2} a_{1j}^2 f(v_j)^2 S_1^2 \frac{\partial^2 C}{\partial S_1^2}\right) + r S_1 \frac{\partial C}{\partial S_1} + \frac{\partial C}{\partial t} - rC \\ + \sum_{j=1}^2 \left(\frac{\kappa_{v_j}}{\delta_j^2}(\zeta_{v_j} - v_j)\frac{\partial C}{\partial v_j} + \frac{1}{2}\frac{\sigma_{v_j}^2}{\delta_j^2} v_j \frac{\partial^2 C}{\partial v_j^2}\right) = 0, \\ C\left(t, B_1(t), B_1(t), \mathbf{v}\right) = 0,\ C\left(T, S_1, B_1(t), \mathbf{v}\right) = g\left(S_1\right) \mathbb{1}_{\{\iota_1 > T\}}. \end{cases}$$

The PDE can be reduced to

$$\begin{cases} \sum_{j=1}^2 \left(\frac{1}{2} a_{1j}^2 f(v_j)^2 \frac{\partial^2 G}{\partial x_1^2} - \frac{1}{2} a_{1j}^2 f(v_j)^2 \frac{\partial G}{\partial x_1}\right) + \frac{\partial G}{\partial t} \\ + \sum_{j=1}^2 \left(\frac{\kappa_{v_j}}{\delta_j^2}(\zeta_{v_j} - v_j)\frac{\partial G}{\partial v_j} + \frac{1}{2}\frac{\sigma_{v_j}^2}{\delta_j^2} v_j \frac{\partial^2 G}{\partial v_j^2}\right) = 0, \\ G\left(t, b_1, b_1, \mathbf{v}\right) = 0,\ G\left(T, x_1, b_1, \mathbf{v}\right) = g\left(x_1\right) \mathbb{1}_{\{\iota_1 > T\}}, \end{cases}$$

where we use the transformations $x_1(t) := \ln\left(\frac{S_1(t) e^{\int_t^T r(s)ds}}{K_1}\right)$, $b_1 := \ln\left(\frac{B_1(T)}{K_1}\right)$, and $G\left(t, x_1, b_1, \mathbf{v}\right) := e^{\int_t^T r(s)ds} C\left(t, S_1, B_1, \mathbf{v}\right)$. See also Appendix A.1.1. Following the lines of the derivation in Section 3.4 we can then derive the following theorem on pricing a single-barrier option.

Theorem 40. *(Single-barrier option pricing in Model (4.1))*

Let us assume

i. the setting described in Equation (4.1) as well as ρ_j^v=0*,*

ii. the existence of an affine analytic characteristic function $\varphi(\tau, u_1)$ *in* z_1*, which is* *regular in a neighbourhood* $S_\varphi = \{u_1 = w_1 + i\omega_1 : \omega_1 \in (\mathfrak{a}_\varphi, \mathfrak{b}_\varphi)\}$, $\mathfrak{a}_\varphi < 0, \mathfrak{b}_\varphi > 0$ *of the origin, and integrable, and*

iii. *that the generalized Fourier transform* $\hat{\mathfrak{h}}(x_1)$ *of the payoff function* $e^{-\frac{x_1}{2}}g(x_1)$ *at maturity exists in a space* $S_g = \{u_1 = w_1 + i\omega_1 : \omega_1 \in (\mathfrak{a}_g, \mathfrak{b}_g)\}$, *is integrable for* $|x_1| < \infty$.

If the space $S_C \equiv S_\varphi \cap S_g$ *is not empty, then the barrier option value (4.19) is given by*

$$\begin{aligned} C_B(t, S_1, B_1, \mathbf{v}) &= \frac{e^{\frac{x_1}{2} - \int_t^T r(s)ds}}{2\pi} \int_{i\omega_1 - \infty}^{i\omega_1 + \infty} \hat{\mathfrak{h}}(u_1) \\ &\quad (\varphi(\tau, u_1, -x_1) - \varphi(\tau, u_1, x_1 - 2b_1))\, du_1, \\ &\quad u_1 \in S_C, \end{aligned} \tag{4.20}$$

where

$$\begin{aligned} \varphi(\tau, u_1, x_1) &= \exp\left\{ iu_1 x_1 + A^*(\tau, u_1) + \sum_{j=1}^{2} \left(\frac{\delta_j^2}{\sigma_{v_j}^2} B_{1j}^*(\tau, u_1) v_j \right) \right\}, \\ \hat{\mathfrak{h}}(u_1) &= \int_{-\infty}^{+\infty} e^{-\frac{x_1'}{2}} g(x_1') e^{iu_1 x_1'} dx_1', \; u_1 \in S_g, \end{aligned}$$

$$x_1 = \ln \frac{S_1 e^{\int_t^T r(s)ds}}{K_1}, \qquad b_1 = \ln \frac{B_1(T)}{K_1},$$

where $\varphi(\tau, u_1, x_1)$ *satisfies the PDE 4.6. The price converges point-wise if the map* $S_1 \to C_B(t, S_1, B_1, \mathbf{v})$ *is continuous.*

Proof.

By introducing the following transformations

$$\begin{aligned} G(t, x_1, b_1, \mathbf{v}) &:= e^{\frac{x_1}{2}} Z(t, x_1, b_1, \mathbf{v}), \\ \frac{\partial G}{\partial x_1} &= \frac{1}{2} G + e^{\frac{x_1}{2}} \frac{\partial Z}{\partial x_1}, \\ \frac{\partial^2 G}{\partial x_1^2} &= \frac{1}{4} G + e^{\frac{x_1}{2}} \frac{\partial Z}{\partial x_1} + e^{\frac{x_1}{2}} \frac{\partial^2 Z}{\partial x_1^2}, \end{aligned}$$

we can reduce the PDE problem (4.20) to

$$\begin{cases} \sum_{j=1}^2 \left(\frac{1}{2} a_{1j}^2 f(v_j)^2 \frac{\partial^2 Z}{\partial x_1^2} + \frac{\partial G}{\partial t} - \frac{1}{8} \sum_{j=1}^2 a_{1j}^2 f(v_j)^2 Z \right) \\ + \sum_{j=1}^2 \left(\frac{\kappa_{v_j}}{\delta_j^2} (\zeta_{v_j} - v_j) \frac{\partial Z}{\partial v_j} + \frac{1}{2} \frac{\sigma_{v_j}^2}{\delta_j^2} v_j \frac{\partial^2 Z}{\partial v_j^2} \right) = 0, \\ Z(t, b_1, b_1, \mathbf{v}) = 0, \; Z(T, x_1, b_1, \mathbf{v}) = e^{-\frac{x_1}{2}} g(x_1) \mathbb{1}_{\{\iota_1 > T\}}. \end{cases}$$

As in Section 3.4.1 we can specify the Kolmogorov backward equation and boundary conditions of the transition probability density function $p(t', x_1', \mathbf{v}', t, x_1, \mathbf{v})$. Since the payoff of the derivative does not depend on the volatility, we pursue with

$$q(\tau, x_1, \mathbf{v}, x_1') = \int_0^\infty p(\tau, {x_1}', \mathbf{v}', x_1, \mathbf{v}) d\mathbf{v}'. \tag{4.21}$$

Here q solves the Kolmogorov Equation (see Definition 38) supplied with the initial and boundary conditions

$$\begin{aligned} q(0, x_1, b_1, \mathbf{v}) &= \delta(x_1 - x_1'), \\ q(\tau, b_1, b_1, \mathbf{v}) &= 0. \end{aligned} \tag{4.22}$$

In the following we apply the reflection principle (see [76], p. 79f) (the counterpart to the method of images in a half space in two dimensions (see A.1.3)), i.e. we first derive a solution $\bar{G}^F$ in the whole plane and restrict it to the space (see (4.22)) it is defined for by using symmetries.
Assume an affine solution can be found[6]:

$$\begin{aligned} \bar{G}^F(\tau, x_1', \mathbf{v}; x_1) &= \frac{1}{2\pi} \int_{i\omega_1 - \infty}^{i\omega_1 + \infty} \exp\Bigg\{ iu_1(x_1' - x_1) + A^*(\tau, u_1) \\ &\quad + \sum_{j=1}^{2} \Big(\frac{\delta_j^2}{\sigma_{v_j}^2} B_{1j}^*(\tau, u_1) v_j \Big) \Bigg\} du_1, \\ &\quad u_1 = v_1 + i\omega_1 \in S_\varphi, \end{aligned} \tag{4.23}$$

where S_φ describes a space in a neighbourhood to the origin, parallel to the real axis, in which the integrand is regular.

The solution for q, satisfying the boundary conditions, can be found by the approach $q = \bar{G}^F + \bar{G}^G$ by using the symmetry of $\bar{G}^F$ in x_1 (see [83]). Note that the point $2b_1 - x_1$ is symmetric to x_1 in b_1. Thus, we set $\bar{G}^G(\tau, x_1' - x_1, \mathbf{v}) = -\bar{G}^F(\tau, x_1' - 2b_1 + x_1, \mathbf{v})$. In $x_1 = b_1$, $\bar{G}^G = -\bar{G}^F$ and the boundary condition is, thus, satisfied. The PDE is also

[6] e.g. with $f(v_j) = \sqrt{v_j}$ with $B_{1j}^* = B_{H1j}^*(\tau, u_1)$ and $A^* = A_H^*(\tau, u)$ given in (4.8)-(4.9),

$$\mathfrak{d}_{v_j} = \mathfrak{d}_{v_j}(\mathbf{u}) = \sqrt{\frac{\kappa_{v_j}^2}{\delta_j^4} + \frac{\sigma_{v_j}^2}{\delta_j^2} \left(u_1^2 a_{1j}^2 + \frac{1}{4} a_{1j}^2 \right)},$$

satisfied by $\bar{G}^G$. Hence, the solution is given by

$$\begin{aligned} q(\tau, x_1', \mathbf{v}, x_1) &= \bar{G}^F(\tau, x_1' - x_1, \mathbf{v}) - \bar{G}^F(\tau, x_1' + x_1 - 2b_1, \mathbf{v}) \\ &= \frac{1}{2\pi} \int_{i\omega_1 - \infty}^{i\omega_1 + \infty} \exp\left\{ A^*(\tau, u_1) + \sum_{j=1}^{2} \left(\frac{\delta_j^2}{\sigma_{v_j}^2} B_{1j}^*(\tau, u_1) v_j \right) \right\} \\ &\quad (\exp\{iu_1(x_1' - x_1)\} - \exp\{iu_1(x_1' + x_1 - 2b_1)\})\, du_1. \end{aligned} \tag{4.24}$$

From risk-neutral pricing, then

$$C_B(t, S_1, B_1, \mathbf{v}) = e^{\frac{x_1}{2} - \int_t^T r(s)ds} \int_{-\infty}^{\infty} \left(e^{-\frac{x_1'}{2}} g(x_1') q(\tau, x_1', \mathbf{v}, x_1) \right) dx_1'. \tag{4.25}$$

We have required that the payoff function, its transform and the characteristic function are Lebesgue integrable in a space S_C. Hence, we can apply Fubini's theorem and change the integrals if there exists a space $S_C = S_\varphi \cap S_g$. Then, the above expression can be simplified. We denote the Fourier transform of the payoff by $\hat{\mathfrak{h}}(u_1)$

$$\hat{\mathfrak{h}}(u_1) = \int_{-\infty}^{+\infty} e^{-\frac{x_1'}{2}} g(x_1') e^{iu_1 x_1'} dx_1', \; u_1 \in S_g.$$

Then,

$$\begin{aligned} C_B(t, S_1, B_1, \mathbf{v}) &= \frac{e^{\frac{x_1}{2} - \int_t^T r(s)ds}}{2\pi} \int_{i\omega_1 - \infty}^{i\omega_1 + \infty} \hat{\mathfrak{h}}(u_1) \\ &\quad (\varphi(\tau, u_1, -x_1) - \varphi(\tau, u_1, x_1 - 2b_1))\, du_1, \\ &\quad u_1 \in S_C = S_\varphi \cap S_g, \end{aligned}$$

where

$$\varphi(\tau, u_1, x_1) = \exp\left\{ iu_1 x_1 + A^*(\tau, u_1) + \sum_{j=1}^{2} \left(\frac{\delta_j^2}{\sigma_{v_j}^2} B_{1j}^*(\tau, u_1) v_j \right) \right\}. \tag{4.26}$$

If $S_1 \to C_B(t, S_1, B_1, \mathbf{v})$ is continuous the above relationship converges point-wise with Theorem 33. □

Exemplarily, we price single-barrier call options with $g_{Call}(S_1) = \max(S_1 - K_1, 0)$, i.e. $g_{Call}(x_1) = K_1 \max(e^{x_1} - 1, 0)$. Thus, we have to Fourier transform $e^{-\frac{1}{2}x_1} g_{Call}(x_1)$. $e^{-\frac{1}{2}x_1} g_{Call}$ does not belong to $\mathbf{L}^1$. The ordinary Fourier transform does not exist and we apply the generalized Fourier transform:

$$\begin{aligned}\hat{\mathfrak{h}}_{Call}(u_1) &= K_1 \int_0^{\infty} (e^{\frac{x_1'}{2}} - e^{-\frac{x_1'}{2}}) e^{iu_1 x_1'} dx_1' \\ &= K_1 \left(-\frac{1}{iu_1 + \frac{1}{2}} + \frac{1}{iu_1 - \frac{1}{2}} \right). \end{aligned} \tag{4.27}$$

Note that we choose $\Im(u_1) > \frac{1}{2}$, i.e. $S_{g_{Call}} = \left\{u_1 = w_1 + i\omega_1 : \omega_1 > \frac{1}{2}\right\}$. Then, the price of the barrier call option can be specified with

$$\begin{aligned}C_{1Call}\left(t, S_1, B_1, \mathbf{v}\right) &= \frac{e^{\frac{x_1}{2} - \int_t^T r(s)ds}}{2\pi} \int_{i\omega_1 - \infty}^{i\omega_1 + \infty} \hat{\mathfrak{h}}_{Call}(u_1) \\ &\quad (\varphi(\tau, u_1, -x_1) - \varphi(\tau, u_1, x_1 - 2b_1))\, du_1, \\ &\quad u_1 \in S_{C_{Call}} = S_{\varphi} \cap S_{g_{Call}}. \end{aligned} \tag{4.28}$$

$\hat{\mathfrak{h}}_{Call}$ has a simple singularity in $\omega_1 = \frac{1}{2}$. We can, thus, again apply residue calculus and move the contour of integration to the real axis (see Corollary 3). With (2.56) we derive the residue of $\hat{\mathfrak{h}}_{Call}$:

$$\begin{aligned}Res_{Call} &= \lim_{u_1 \to i\frac{1}{2}} \left(u_1 - i\frac{1}{2}\right) K_1 \left(\frac{1}{i(u_1 - i\frac{1}{2})(iu_1 - \frac{1}{2})} \right) \\ &= K_1(-i)(-1) = K_1 i. \end{aligned} \tag{4.29}$$

Then by Corollary 3 the option price also equals the integral along the real axis of u_1 minus $2\pi i$ times the residue of the Call price at $\check{u}_1 = i\frac{1}{2}$, i.e. $2K_1\pi \int_{-\infty}^{\infty} \delta(u_1 - i\frac{1}{2}) \Bigg(\varphi(\tau, u_1, -x_1) - \varphi(\tau, u_1, x_1 - 2b_1) \Bigg) du_1$. See also [82]. The pricing formula with integration on the real axis is then given by

$$\begin{aligned}C_{1Call}(t, S_1, B_1, \mathbf{v}) &= e^{\frac{x_1}{2} - \int_t^T r(s)ds} K_1 \Big(\varphi(\tau, i\frac{1}{2}, -x_1, \mathbf{v}) \\ &\quad - \varphi(\tau, i\frac{1}{2}, x_1 - 2b_1) \Big) + \frac{e^{\frac{x_1}{2} - \int_t^T r(s)ds}}{2\pi} \int_{-\infty}^{\infty} \hat{\mathfrak{h}}_{Call}(u_1) \\ &\quad (\varphi(\tau, u_1, -x_1) - \varphi(\tau, u_1, x_1 - 2b_1)\, du_1, \\ &\quad u_1 \text{ real}. \end{aligned} \tag{4.30}$$

4.5 Pricing of two-asset barrier options with perturbation theory

Options with barriers on more than one underlying cannot be priced in closed form but have to be approximated. Due to the form of our model an approximation using perturbation techniques seems to be the right choice. In finance this method has been applied to option pricing under a stochastic volatility model by Fouque et al. (see e.g. [47], [49]). They theoretically prove the convergence of their approach in [52]. The approximation is correct to an order of $\mathcal{O}\left(\delta^{\frac{1}{2}}\right)$ and it is valid everywhere except in a boundary layer near expiry. This group applied the approach to various option types, e.g. exotic options in [70], Asian options in [48], defaultable bonds in [53]. Their approach has been extended to additionally allow a slow mean-reverting component [50] and to multi-dimensions [55]. Howison [66] applies perturbation theory to price vanilla options near expiry, options in illiquid markets etc. Howison [66], Rasmussen [101] as well as Conlon and Sullivan [24] extend the expansion and show the convergence.

We shortly introduce perturbation theory. The introduction to perturbation theory is based on the explications of Zauderer [121], p. 572ff.

Consider a differential equation

$$\mathcal{L}(C, \epsilon) = 0, \tag{4.31}$$

that depends on the small positive parameter ϵ and is given over a spatial region D. For a parabolic or hyperbolic problem the boundary conditions are assigned on ∂D and initial data are given in D at the time $t = 0$. The so-called reduced or unperturbed problem associated with (4.31) results when we set $\epsilon = 0$ in (4.31), i.e. $\mathcal{L}(C, 0) = 0$. The given problem is called regular if the reduced problem has a unique solution. If this is not the case we have to deal with a so called singular perturbation problem. When we set $\epsilon = 0$ in a singular problem we obtain an equation for which either the order or the type of the differential equation has changed.

For regular perturbation problems, the solution C is expanded in the perturbation series

$$C = \sum_{n=0}^{\infty} C_n \epsilon^n. \tag{4.32}$$

The difference between C and C_0 is called a perturbation on the solution C_0 of the reduced or unperturbed problem. Inserting the expansion into (4.31) yields

$$\mathcal{L}(C,\epsilon) = \mathcal{L}\left(\sum_{n=0}^{\infty} C_n \epsilon^n, \epsilon\right) = 0. \tag{4.33}$$

The assumption is taken that $\mathcal{L}(C,\epsilon)$ can be expanded in a power series in C and ϵ. Thus, (4.31) can be written as a series

$$\mathcal{L}(C,\epsilon) = \sum_{n=0}^{\infty} \mathcal{L}_n(C_n, C_{n-1}, \dots, C_1, C_0)\epsilon^n = 0, \tag{4.34}$$

where $\mathcal{L}_n$ describe differential operators which may be linear or non-linear, and which act on the functions $C_0, C_1, \dots, C_n$. To solve the problem we equate the coefficients of ϵ^n in (4.34) to zero (i.e. equating like powers of ϵ) and get

$$\mathcal{L}_n(C_n, C_{n-1}, \dots, C_1, C_0) = 0, \quad n = 0, 1, \dots. \tag{4.35}$$

The same is true for any initial and/or boundary conditions. Hence, we obtain a system of equations which we can solve recursively.

In the following sections we exemplary price two-asset options with and without barriers in the Model (4.1) with perturbation theory, describe the convergence analytically and give some numerical examples. Furthermore, we present a possible extension to Model (4.1) and indicate how perturbation theory could also be applied in this model.

4.5.1 Approximation of Model (4.1)

We will see that the PDE for the valuation in fast mean-reversion models is a singular perturbation problem. We will show that the singular perturbation can be applied and converges everywhere except in a boundary layer near expiry. We also provide some numerical results which back the theoretical convergence results.

By means of the perturbation technique we want to price options which depend on two underlyings with and without barriers on both of the underlyings in the Model (4.1). The payoff in T is indicated by $g(S_1, S_2)$ in the following. Exemplarily, we show the computations of a two-asset option with and without barriers (see [62] for two-asset digital options without barriers). In the case of the option without barriers the value can be determined by

$$C(t, S_1, S_2) = \mathbb{E}_{\tilde{Q}}\Big[\max(S_2 - K_2, 0)\mathbb{1}_{S_1 > K_1}\Big], \tag{4.36}$$

the value of the barrier option is given by

$$C_B(t,S_1,S_2,B_1,B_2) = \mathbb{E}_{\tilde{\mathcal{Q}}}\Big[\max(S_2-K_2,0)\mathbb{1}_{S_1>K_1}\mathbb{1}_{\{\iota_1>T,\iota_2>T\}}\Big], \tag{4.37}$$

where $\iota_1 = \inf\Big(t' \in (\,t,T\,] : S_1(t') \le B_1(t')\Big)$ and $\iota_2 = \inf\Big(t' \in (\,t,T\,] : S_2(t') \le B_2(t')\Big)$. $B_1(t)$ and $B_2(t)$ describe the barriers on S_1 and S_2 respectively. As we presume fast mean reversion for all the eigenvalues, i.e. $\delta_j \to 0$, $C(S_1,S_2)$ and $C_B(S_1,S_2,B_1,B_2)$ can be asymptotically approximated (see [49]). We pursue the expansion for the two-asset option $C\,(t,S_1,S_2)$ and only provide the expressions for $C_B(t,S_1,S_2,B_1,B_2)$ explicitly when there are clear differences in the solutions.

First, we expand the problem in δ_1 (see (4.32)), after that in δ_2.

$$C^\delta = \sum_{n=0}^{\infty} C_n^{\delta_2}\delta_1^n, \tag{4.38}$$

$$C_n^{\delta_2} = \sum_{k=0}^{\infty} C_{n,k}\delta_2^k. \tag{4.39}$$

The infinitesimal generator $\mathcal{L}^\delta$ is given by

$$\mathcal{L}^\delta := \frac{1}{\delta_1^2}\mathcal{L}_0^1 + \frac{1}{\delta_2^2}\mathcal{L}_0^2 + \frac{1}{\delta_1}\mathcal{L}_1^1 + \frac{1}{\delta_2}\mathcal{L}_1^2 + \mathcal{L}_2. \tag{4.40}$$

Note that it is expressed in the form of a power series in δ_j. Thus, the problem to be solved for the two-asset option becomes

$$\begin{aligned} \mathcal{L}^\delta C^\delta &= 0, \\ C^\delta\,(T,S_1,S_2) &= g(S_1,S_2), \end{aligned} \tag{4.41}$$

and for the barrier option, respectively

$$\begin{aligned} \mathcal{L}^\delta C_B^\delta &= 0, \\ C_B^\delta(t,B_1(t),S_2) &= 0, \\ C_B^\delta(t,S_1,B_2(t)) &= 0, \\ C_B^\delta\,(T,S_1,S_2) &= g(S_1,S_2), \end{aligned} \tag{4.42}$$

where

$$\mathcal{L}_0^1 = \kappa_{v_1}\left(\zeta_{v_1} - v_1\right)\frac{\partial}{\partial v_1} + \frac{\sigma_{v_1}^2}{2} v_1 \frac{\partial^2}{\partial v_1^2}, \quad (4.43)$$

$$\mathcal{L}_0^2 = \kappa_{v_2}\left(\zeta_{v_2} - v_2\right)\frac{\partial}{\partial v_2} + \frac{\sigma_{v_2}^2}{2} v_2 \frac{\partial^2}{\partial v_2^2}, \quad (4.44)$$

$$\mathcal{L}_1^1 = \rho_1^v S_1 \sigma_{v_1} a_{11}\sqrt{v_1} f(v_1)\frac{\partial^2}{\partial S_1 \partial v_1} + \rho_1^v S_2 \sigma_{v_1} a_{21}\sqrt{v_1} f(v_1)\frac{\partial^2}{\partial S_2 \partial v_1}, \quad (4.45)$$

$$\mathcal{L}_1^2 = \rho_2^v S_1 \sigma_{v_2} a_{12}\sqrt{v_2} f(v_2)\frac{\partial^2}{\partial S_1 \partial v_2} + \rho_2^v S_2 \sigma_{v_2} a_{22}\sqrt{v_2} f(v_2)\frac{\partial^2}{\partial S_2 \partial v_2}, \quad (4.46)$$

$$\begin{aligned}\mathcal{L}_2 = {} & \frac{\partial}{\partial t} + \frac{1}{2}S_1^2 \sum_{j=1}^{2} a_{1j}^2 f(v_j)^2 \frac{\partial^2}{\partial S_1^2} + \frac{1}{2}S_2^2 \sum_{j=1}^{2} a_{2j}^2 f(v_j)^2 \frac{\partial^2}{\partial S_2^2} \\ & + S_1 S_2 \sum_{j=1}^{2} a_{1j} a_{2j} f(v_j)^2 \frac{\partial^2}{\partial S_1 \partial S_2} + r\left(S_1 \frac{\partial}{\partial S_1} + S_2 \frac{\partial}{\partial S_2} - 1\right). \quad (4.47)\end{aligned}$$

Theorem 41. *(Barrier option pricing in $\mathbb{R}^2$)*

Let us assume the setting described in Equation (4.1). Then the price of a two-asset option without barriers (4.36) (and with barriers (4.37) respectively) can be approximated by

$$C^\delta(t, S_1, S_2, \mathbf{v}) \approx Q^\delta = C_{0,0} + \delta_1 C_{1,0} + \delta_2 C_{0,1} + \delta_1\delta_2 C_{1,1}, \quad (4.48)$$

$$\begin{aligned}C_B^\delta(t, S_1, S_2, B_1(t), B_2(t), \mathbf{v}) \approx Q_B^\delta = {} & C_{B,0,0} + \delta_1 C_{B,1,0} + \delta_2 C_{B,0,1} \\ & + \delta_1\delta_2 C_{B,1,1}, \quad (4.49)\end{aligned}$$

where $C_{0,0}$ and $C_{B,0,0}$ are given in (4.66) and (B.6), $C_{1,0}$ and $C_{B,1,0}$ in (4.84) and (4.96), $C_{0,1}$ and $C_{B,0,1}$ in (4.105) and (4.112), $C_{1,1}$ and $C_{B,1,1}$ in (4.116) and (4.123).

Proof.

We insert (4.38) in (4.41) and (4.42) respectively. To solve the problem we equate like powers of δ_1. This results in several PDE problems which can be solved. We restrict the asymptotic solution to the first expansion terms:

$$\begin{aligned} & \frac{1}{\delta_1^2}\mathcal{L}_0^1 C_0^{\delta_2} + \frac{1}{\delta_1}\left(\mathcal{L}_0^1 C_1^{\delta_2} + \mathcal{L}_1^1 C_0^{\delta_2}\right) + \left(\left(\frac{1}{\delta_2^2}\mathcal{L}_0^2 + \frac{1}{\delta_2}\mathcal{L}_1^2 + \mathcal{L}_2\right) C_0^{\delta_2}\right. \quad (4.50) \\ & \left. + \mathcal{L}_1^1 C_1^{\delta_2} + \mathcal{L}_0^1 C_2^{\delta_2}\right) + \delta_1\left(\left(\frac{1}{\delta_2^2}\mathcal{L}_0^2 + \frac{1}{\delta_2}\mathcal{L}_1^2 + \mathcal{L}_2\right) C_1^{\delta_2} + \mathcal{L}_1^1 C_2^{\delta_2} + \mathcal{L}_0^1 C_3^{\delta_2}\right) \ldots \quad = \quad 0. \end{aligned}$$

We set the first four leading order terms to zero and find the following relationships

$$\begin{aligned} \delta_1^{-2}: &\quad \mathcal{L}_0^1 C_0^{\delta_2} &= 0, \quad (4.51) \\ \delta_1^{-1}: &\quad \mathcal{L}_0^1 C_1^{\delta_2} + \mathcal{L}_1^1 C_0^{\delta_2} &= 0, \quad (4.52) \\ \delta_1^{0}: &\quad \left(\tfrac{1}{\delta_2^2}\mathcal{L}_0^2 + \tfrac{1}{\delta_2}\mathcal{L}_1^2 + \mathcal{L}_2\right) C_0^{\delta_2} + \mathcal{L}_1^1 C_1^{\delta_2} + \mathcal{L}_0^1 C_2^{\delta_2} &= 0, \quad (4.53) \\ \delta_1^{1}: &\quad \left(\tfrac{1}{\delta_2^2}\mathcal{L}_0^2 + \tfrac{1}{\delta_2}\mathcal{L}_1^2 + \mathcal{L}_2\right) C_1^{\delta_2} + \mathcal{L}_1^1 C_2^{\delta_2} + \mathcal{L}_0^1 C_3^{\delta_2} &= 0. \quad (4.54) \end{aligned}$$

Regarding Equation (4.51) one observes that $\mathcal{L}_0^1$ only involves derivatives in v_1. Thus, any function independent of v_1 is a solution of (4.51). On the other hand v_1-dependent solutions show the unreasonable growth $e^{\frac{2\kappa_{v_1}}{\sigma_{v_1}} v_1}$ at infinity[7]. We will draw on that argument repeatedly in the course of this proof. Thus, we search for a solution of $C_0^{\delta_2}$ which does not depend on v_1.

In Equation (4.52) $\mathcal{L}_1^1$ also takes a derivative in v_1, thus, $\mathcal{L}_1^1 C_0^{\delta_2} = 0$. It follows due to the same reasons as for Equation (4.51) $C_1^{\delta_2} = C_1^{\delta_2}(t, S_1, S_2, v_2)$. Hence, the approximation $C_0^{\delta_2} + \delta_1 C_1^{\delta_2}$ does not depend on the current level of the eigenvalue v_1.
As $C_1^{\delta_2}$ does not depend on v_1 Equation (4.53) can be reformulated to

$$\left(\frac{1}{\delta_2^2}\mathcal{L}_0^2 + \frac{1}{\delta_2}\mathcal{L}_1^2 + \mathcal{L}_2\right) C_0^{\delta_2} + \mathcal{L}_0^1 C_2^{\delta_2} = 0. \quad (4.55)$$

[7] $\mathcal{L}_0^1 C_0^{\delta_2}$ can be solved by dividing the equation by $\frac{1}{2}\sigma_{v_1}^2 v_1$ and by multiplying the term by the integrating factor $e^{\int_0^v \frac{2\kappa_{v_1}(\zeta_{v_1} - w_1)}{\sigma_{v_1}^2 w_1} dw_1}$. The ODE can then be transformed to $\frac{\partial}{\partial v_1}\left(e^{\int_0^v \frac{2\kappa_{v_1}(\zeta_{v_1} - w_1)}{\sigma_{v_1}^2 w_1} dw_1} \frac{\partial C_0^{\delta_2}}{\partial v_1}\right) = 0$. We, thus, see that $\frac{\partial C_0^{\delta_2}}{\partial v} = \tilde{c} e^{-\frac{2\kappa_{v_1}\zeta_{v_1}}{\sigma_{v_1}^2}\ln(v_1) + \frac{2\kappa_{v_1}}{\sigma_{v_1}^2}}$. See also [90].

Then, we expand $C_i^{\delta_2}$, $i \geq 0$ in δ_2

$$C_i^{\delta_2} = \sum_{n=0}^{\infty} C_{i,n}\delta_2^n. \tag{4.56}$$

Inserting this expansion in (4.55) and grouping terms of equal order in δ_2, and setting the first four leading terms to zero we get

$$\delta_2^{-2}: \quad \mathcal{L}_0^2 C_{0,0} = 0, \tag{4.57}$$

$$\delta_2^{-1}: \quad \mathcal{L}_0^2 C_{0,1} + \mathcal{L}_1^2 C_{0,0} = 0, \tag{4.58}$$

$$\delta_2^{0}: \quad \mathcal{L}_0^2 C_{0,2} + \mathcal{L}_1^2 C_{0,1} + \mathcal{L}_2 C_{0,0} + \mathcal{L}_0^1 C_{2,0} = 0, \tag{4.59}$$

$$\delta_2^{1}: \quad \mathcal{L}_0^2 C_{0,3} + \mathcal{L}_1^2 C_{0,2} + \mathcal{L}_2 C_{0,1} + \mathcal{L}_0^1 C_{2,1} = 0. \tag{4.60}$$

Solving the PDE for the leading terms $C_{0,0}$ and $C_{B,0,0}$

Similarly to before, we can derive from (4.57) and (4.58) that $C_{0,0}$ and $C_{0,1}$ are both independent from v_2. Equation (4.59) can be simplified by the fact that $\mathcal{L}_1^2 C_{0,1} = 0$ to $\mathcal{L}_0^2 C_{0,2} + \mathcal{L}_2 C_{0,0} + \mathcal{L}_0^1 C_{2,0} = 0$. This is a Poisson equation (see Definition (25)) in $C_{0,2}$ and $C_{2,0}$ with respect to v_1 and v_2.
The Fredholm alternatives (see Theorem 30, i(b) with iii) state that there exists only a solution if $\mathcal{L}_2 C_{0,0}$ is orthogonal to the null space of the adjoint generator $\mathcal{L}_0^*$ of $\mathcal{L}_0 = \mathcal{L}_0^1 + \mathcal{L}_0^2$. According to Remark 6 an ergodic Markov process has a unique invariant probability function, which solves the adjoint equation. The CIR process is ergodic and has, thus, an invariant distribution (see Example (4)). Hence, $\mathcal{L}_2 C_{0,0}$ must feature mean zero (be orthogonal) with respect to the invariant measures of v_1 and v_2. See also Equation (B.29).

$$\int\int \mathcal{L}_2 C_{0,0} p^{inv}(v_1, v_2) dv_1 dv_2 = \langle\langle \mathcal{L}_2 C_{0,0} \rangle\rangle = \langle\langle \mathcal{L}_2 \rangle\rangle\, C_{0,0} = 0, \tag{4.61}$$

where $p^{inv}(v_1, v_2)$ is the invariant distribution of the CIR processes v_1, v_2. The second equality in (4.61) is true because $C_{0,0}$ is independent from v_1 and v_2. We use here the notation of Fouque (see [50], [52], [53], [55], [46]): The bracket notation means integration with respect to the invariant distributions of the CIR processes for v_1 and v_2, i.e. the inner product with respect to the invariant measure.
Note if we set $f(v_1) = \sqrt{v_1}$ and $f(v_2) = \sqrt{v_2}$ respectively that $\langle f(v_1)^2 \rangle = \int_0^\infty p^{inv}(v_1) v_1 dv_1 = \zeta_{v_1}$ and $\langle f(v_2)^2 \rangle = \int_0^\infty p^{inv}(v_2) v_2 dv_2 = \zeta_{v_2}$, where $p^{inv}(v_j)$ denotes the invariant distribution of the CIR process, the Gamma distribution (see Appendix B.3). $\langle\langle \mathcal{L}_2 \rangle\rangle$ is equal to the two-dimensional Black-Scholes operator $\mathcal{L}_{BS}$ (see (2.147))

with eigenvalues $\langle f(v_1)^2\rangle = \overline{f(v_1)^2}$ and $\langle f(v_2)^2\rangle = \overline{f(v_2)^2}$.

$$\begin{aligned}
\langle\langle \mathcal{L}_2\rangle\rangle &= \mathcal{L}_{BS} \\
&= \frac{\partial}{\partial t} + \frac{1}{2}S_1^2\sum_{j=1}^{2} a_{1j}^2\langle f(v_j)^2\rangle \frac{\partial^2}{\partial S_1^2} + \frac{1}{2}S_2^2\sum_{j=1}^{2} a_{2j}^2\langle f(v_j)^2\rangle \frac{\partial^2}{\partial S_2^2} \\
&\quad + S_1S_2\sum_{j=1}^{2} a_{1j}a_{2j}\langle f(v_j)^2\rangle \frac{\partial^2}{\partial S_1 \partial S_2} + r\left(S_1\frac{\partial}{\partial S_1} + S_2\frac{\partial}{\partial S_2} - 1\right). \qquad (4.62)
\end{aligned}$$

Hence, we can also write (4.61) as

$$\mathcal{L}_{BS}(\overline{\sigma}_1, \overline{\sigma}_2, \overline{\rho})C_{0,0} = 0, \qquad (4.63)$$

where the centred volatilities of S_1 and S_2, $\overline{\sigma}_1$ and $\overline{\sigma}_2$, and the centred correlation $\overline{\rho}$ between the two assets can be indicated by

$$\begin{aligned}
\overline{\sigma}_1^2 &= a_{11}^2\langle f(v_1)^2\rangle + a_{12}^2\langle f(v_2)^2\rangle, \\
\overline{\sigma}_2^2 &= a_{21}^2\langle f(v_1)^2\rangle + a_{22}^2\langle f(v_2)^2\rangle, \\
\overline{\rho} &= \frac{a_{11}a_{21}\langle f(v_1)^2\rangle + a_{12}a_{22}\langle f(v_2)^2\rangle}{\overline{\sigma}_1\overline{\sigma}_2}. \qquad (4.64)
\end{aligned}$$

Corollary 17. *($C_{0,0}$)*

The term $C_{0,0}$ is the solution of the following problem

$$\begin{aligned}
\mathcal{L}_{BS}(\overline{\sigma}_1, \overline{\sigma}_2, \overline{\rho})C_{0,0} &= 0, \\
C_{0,0}(T, S_1, S_2) &= \max(S_2 - K_2, 0)\mathbb{1}_{S_1 > K_1}. \qquad (4.65)
\end{aligned}$$

$C_{0,0}$ can be evaluated by the following formula

$$C_{0,0}(t, S_1, S_2) = S_2\mathcal{N}_2(\mathbf{d}_2, \mathbf{d}_1, \overline{\rho}) - K_2e^{-r\tau}\mathcal{N}_2(\mathbf{d}_2^*; \mathbf{d}_1^*, \overline{\rho}), \qquad (4.66)$$

where

$$\begin{aligned}
\tau = T - t, &\qquad x_i = \ln\frac{S_ie^{-\int_t^T r(s)ds}}{K_i}, \\
\mathbf{d}_1^* = \mathbf{d}_1 - \overline{\rho\sigma_2}\sqrt{\tau}, &\qquad \mathbf{d}_1 = \frac{x_1}{\overline{\sigma}_1\sqrt{\tau}} - \frac{1}{2}\overline{\sigma}_1\sqrt{\tau} + \overline{\rho\sigma_2}\sqrt{\tau}, \\
\mathbf{d}_2^* = \mathbf{d}_2 - \overline{\sigma}_2\sqrt{\tau}, &\qquad \mathbf{d}_2 = \frac{x_2}{\overline{\sigma}_2\sqrt{\tau}} + \frac{1}{2}\overline{\sigma}_2\sqrt{\tau}.
\end{aligned}$$

For a proof see Appendix B.2.

Corollary 18. *($C_{B,0,0}$)*

$C_{B,0,0}$, the respective expansion term for the two-asset option with two barriers, is given by (see [62])

$$\begin{aligned}
\mathcal{L}_{BS}(\overline{\sigma}_1,\overline{\sigma}_2,\overline{\rho})C_{B,0,0} &= 0,\\
C_{B,0,0}\left(T,S_1,S_2\right) &= \max\left(S_2-K_2,0\right)\mathbb{1}_{S_1>K_1},\\
C_{B,0,0}(t,B_1(t),S_2) &= 0,\\
C_{B,0,0}(t,S_1,B_2(t)) &= 0. \qquad (4.67)
\end{aligned}$$

(4.67) can be solved for $\overline{\rho}=-\cos\left(\frac{2\pi k}{n}\right)$:

$$\begin{aligned}
C_{B,0,0}\left(t,S_1,S_2,B_1(t),B_2(t)\right) &= \sum_{k=0}^{n-1} e^{y_1\left(c_1\overline{\sigma}_1-c_1\overline{\sigma}_1\cos\left(\frac{2\pi k}{n}\right)+\frac{1}{\sqrt{1-\overline{\rho}^2}}\left(-c_2\overline{\sigma}_2-\overline{\rho}c_1\overline{\sigma}_1\right)\sin\left(\frac{2\pi k}{n}\right)\right)}\\
&\quad \left(B_2\left(H_1^+-H_1^-\right)-e^{-r\tau}K_2\left(H_2^+-H_2^-\right)\right), \qquad (4.68)
\end{aligned}$$

where

$$\begin{aligned}
y_1 &= \frac{\ln\left(\frac{S_1e^{r\tau}}{K_1}\right)-b_1}{\overline{\sigma}_1},\ y_2=\frac{\ln\left(\frac{S_2e^{r\tau}}{K_2}\right)-b_2}{\overline{\sigma}_2},\\
c_1 &= \frac{\overline{\sigma}_1-\overline{\rho}\overline{\sigma}_2}{2\overline{\sigma}_1\left(1-\overline{\rho}^2\right)},\ c_2=\frac{\overline{\sigma}_2-\overline{\rho}\overline{\sigma}_1}{2\overline{\sigma}_2\left(1-\overline{\rho}^2\right)}\\
H_1^+ &= e^{y_2\left(c_2\overline{\sigma}_2+(1-c_2)\overline{\sigma}_2\cos\left(\frac{2\pi k}{n}\right)+\frac{1}{\sqrt{1-\overline{\rho}^2}}\left(c_1\overline{\sigma}_1-\overline{\rho}(1-c_2)\overline{\sigma}_2\right)\sin\left(\frac{2\pi k}{n}\right)\right)}\\
&\quad e^{y_1\left(\frac{\overline{\sigma}_2}{\sqrt{1-\overline{\rho}^2}}\sin\left(\frac{2\pi k}{n}\right)\right)}\mathcal{N}_2\left(l_1\left(\gamma_1^+,\eta_2^+\right),l_2\left(\gamma_1^+,\eta_2^+\right),\overline{\rho}\right), \qquad (4.69)\\
H_1^- &= e^{y_2\left(c_2\overline{\sigma}_2+\left(-(1-c_2)\overline{\sigma}_2+2\overline{\sigma}_1c_1\overline{\rho}\right)\cos\left(\frac{2\pi k}{n}\right)+\frac{1}{\sqrt{1-\overline{\rho}^2}}\left(-c_1\overline{\sigma}_1\left(1-2\overline{\rho}^2\right)-\overline{\rho}(1-c_2)\overline{\sigma}_2\right)\sin\left(\frac{2\pi k}{n}\right)\right)}\\
&\quad e^{y_1\left(\frac{\overline{\sigma}_2}{\sqrt{1-\overline{\rho}^2}}\sin\left(\frac{2\pi k}{n}\right)\right)}\mathcal{N}_2\left(l_1\left(\gamma_1^-,\eta_2^-\right),l_2\left(\gamma_1^-,\eta_2^-\right),\overline{\rho}\right), \qquad (4.70)\\
H_2^+ &= e^{y_2\left(c_2\overline{\sigma}_2-c_2\overline{\sigma}_2\cos\left(\frac{2\pi k}{n}\right)+\frac{1}{\sqrt{1-\overline{\rho}^2}}\left(c_1\overline{\sigma}_1+\overline{\rho}c_2\overline{\sigma}_2\right)\sin\left(\frac{2\pi k}{n}\right)\right)}\\
&\quad \mathcal{N}_2\left(l_1\left(\gamma_1^+,\gamma_2^+\right),l_2\left(\gamma_1^+,\gamma_2^+\right),\overline{\rho}\right), \qquad (4.71)\\
H_2^- &= e^{y_2\left(c_2\overline{\sigma}_2+\left(c_2\overline{\sigma}_2+2\overline{\sigma}_1c_1\overline{\rho}\right)\cos\left(\frac{2\pi k}{n}\right)+\frac{1}{\sqrt{1-\overline{\rho}^2}}\left(-c_1\overline{\sigma}_1\left(1-2\overline{\rho}^2\right)+\overline{\rho}c_2\overline{\sigma}_2\right)\sin\left(\frac{2\pi k}{n}\right)\right)}\\
&\quad \mathcal{N}_2\left(l_1\left(\gamma_1^-,\gamma_2^-\right),l_2\left(\gamma_1^-,\gamma_2^-\right),\overline{\rho}\right),
\end{aligned}$$

$$\begin{aligned}
l_1(k_1,k_2) &= \sqrt{T-t}\,(k_1+\overline{\rho}k_2)+\frac{b_1}{\overline{\sigma}_1\sqrt{T-t}},\\
l_2(k_1,k_2) &= \sqrt{T-t}\,(\overline{\rho}k_1+k_2)+\frac{b_2}{\overline{\sigma}_2\sqrt{T-t}},\\
\gamma_1^+ &= -c_1\overline{\sigma}_1+y_1\frac{\cos\left(\frac{2\pi k}{n}\right)}{(T-t)\left(1-\overline{\rho}^2\right)}\\
&\quad +y_2\left(-\frac{\sin\left(\frac{2\pi k}{n}\right)}{(T-t)\sqrt{1-\overline{\rho}^2}}-\frac{\overline{\rho}\cos\left(\frac{2\pi k}{n}\right)}{(T-t)\left(1-\overline{\rho}^2\right)}\right),\\
\gamma_1^- &= -c_1\overline{\sigma}_1+y_1\frac{\cos\left(\frac{2\pi k}{n}\right)}{(T-t)\left(1-\overline{\rho}^2\right)}\\
&\quad +y_2\left(\frac{\sin\left(\frac{2\pi k}{n}\right)}{(T-t)\sqrt{1-\overline{\rho}^2}}-\frac{\overline{\rho}\cos\left(\frac{2\pi k}{n}\right)}{(T-t)\left(1-\overline{\rho}^2\right)}\right),\\
\gamma_2^+ &= -c_2\overline{\sigma}_2+y_1\left(-\frac{\overline{\rho}\cos\left(\frac{2\pi k}{n}\right)}{(T-t)\left(1-\overline{\rho}^2\right)}+\frac{\sin\left(\frac{2\pi k}{n}\right)}{\sqrt{1-\overline{\rho}^2}(T-t)}\right)\\
&\quad +y_2\frac{\cos\left(\frac{2\pi k}{n}\right)}{(1-\overline{\rho}^2)(T-t)},\\
\gamma_2^- &= -c_2\overline{\sigma}_2+y_1\left(-\frac{\overline{\rho}\cos\left(\frac{2\pi k}{n}\right)}{(T-t)\left(1-\overline{\rho}^2\right)}+\frac{1}{\sqrt{1-\overline{\rho}^2}(T-t)}\sin\left(\frac{2\pi k}{n}\right)\right)\\
&\quad +y_2\left(-\frac{1-2\overline{\rho}^2}{(1-\overline{\rho}^2)(T-t)}\cos\left(\frac{2\pi k}{n}\right)-2\frac{\overline{\rho}\sin\left(\frac{2\pi k}{n}\right)}{\sqrt{1-\overline{\rho}^2}(T-t)}\right),\\
\eta_2^+ &= (1-c_2)\overline{\sigma}_2+y_1\left(-\frac{\overline{\rho}\cos\left(\frac{2\pi k}{n}\right)}{(T-t)\left(1-\overline{\rho}^2\right)}+\frac{\sin\left(\frac{2\pi k}{n}\right)}{\sqrt{1-\overline{\rho}^2}(T-t)}\right)\\
&\quad +y_2\frac{\cos\left(\frac{2\pi k}{n}\right)}{(1-\overline{\rho}^2)(T-t)},\\
\eta_2^- &= (1-c_2)\overline{\sigma}_2+y_1\left(-\frac{\overline{\rho}\cos\left(\frac{2\pi k}{n}\right)}{(T-t)\left(1-\overline{\rho}^2\right)}+\frac{\sin\left(\frac{2\pi k}{n}\right)}{\sqrt{1-\overline{\rho}^2}(T-t)}\right)\\
&\quad +y_2\left(-\frac{1-2\overline{\rho}^2}{\left(1-\overline{\rho}^2\right)(T-t)}\cos\left(\frac{2\pi k}{n}\right)-2\frac{\overline{\rho}\sin\left(\frac{2\pi k}{n}\right)}{\sqrt{1-\overline{\rho}^2}(T-t)}\right).
\end{aligned}$$

For a proof refer to Appendix B.2.

Solving the PDE for the first-order corrections $C_{0,1}$ and $C_{B,0,1}$

Equation (4.59) can be interpreted as a Poisson equation in v_2 only (in v_1 respectively) for $C_{0,2}$ ($C_{2,0}$). Thus, the following solvability conditions which have to be imposed due to the Fredholm alternatives for the two differential equations, are given by

$$\begin{aligned}\left\langle \mathcal{L}_0^1 C_{2,0} + \mathcal{L}_2 C_{0,0} \right\rangle_{v_2} &= 0,\\ \left\langle \mathcal{L}_0^2 C_{0,2} + \mathcal{L}_2 C_{0,0} \right\rangle_{v_1} &= 0,\end{aligned} \tag{4.72}$$

where we indicate by $\langle h \rangle_{v_j}$ that h is centred with respect to the invariant distribution of v_j.[8] As $\mathcal{L}_0^2$ and $C_{0,2}$ are independent from v_1 (due to (4.51) and explanations all $C_{0,i}$ are independent from v_1) and analogously $\mathcal{L}_0^1$ and $C_{2,0}$[9] are independent from v_2 and we find

$$\begin{aligned}\mathcal{L}_0^1 C_{2,0} &= -(\langle \mathcal{L}_2 \rangle_{v_2} C_{0,0}) + f_{2,0}^e(t, S_1, S_2),\\ \mathcal{L}_0^2 C_{0,2} &= -(\langle \mathcal{L}_2 \rangle_{v_1} C_{0,0}) + f_{0,2}^e(t, S_1, S_2),\end{aligned} \tag{4.73}$$

where $f_{2,0}^e$ ($f_{0,2}^e$) is another eigenfunction, which does not depend on v_1 (or v_2 respectively). These expressions can be simplified by the use of the following relationship

$$\begin{aligned}-\mathcal{L}_0^1 C_{2,0} \overset{4.73}{=} \langle \mathcal{L}_2 \rangle_{v_2} C_{0,0} &= \langle \mathcal{L}_2 \rangle_{v_2} C_{0,0} - \underbrace{\langle \langle \mathcal{L}_2 C_{0,0} \rangle \rangle}_{=0,\text{ see (4.61)}}\\ &= \frac{1}{2} a_{11}^2 (f(v_1)^2 - \overline{f(v_1)^2}) S_1^2 \frac{\partial^2 C_{0,0}}{\partial S_1^2}\\ &\quad + \frac{1}{2} a_{21}^2 (f(v_1)^2 - \overline{f(v_1)^2}) S_2^2 \frac{\partial^2 C_{0,0}}{\partial S_2^2}\\ &\quad + a_{11} a_{21} (f(v_1)^2 - \overline{f(v_1)^2}) S_1 S_2 \frac{\partial^2 C_{0,0}}{\partial S_2 \partial S_1}\\ &= (f(v_1)^2 - \overline{f(v_1)^2}) \Big(\frac{1}{2} a_{11}^2 S_1^2 \frac{\partial^2 C_{0,0}}{\partial S_1^2} + \frac{1}{2} a_{21}^2 S_2^2 \frac{\partial^2 C_{0,0}}{\partial S_2^2}\\ &\quad + a_{11} a_{21} S_1 S_2 \frac{\partial^2 C_{0,0}}{\partial S_1 \partial S_2} \Big),\end{aligned} \tag{4.74}$$

and

$$\begin{aligned}-\mathcal{L}_0^2 C_{0,2} = \langle \mathcal{L}_2 \rangle_{v_1} C_{0,0} &= (f(v_2)^2 - \overline{f(v_2)^2}) \Big(\frac{1}{2} a_{12}^2 S_1^2 \frac{\partial^2 C_{0,0}}{\partial S_1^2} + \frac{1}{2} a_{22}^2 S_2^2 \frac{\partial^2 C_{0,0}}{\partial S_2^2}\\ &\quad + a_{12} a_{22} S_1 S_2 \frac{\partial^2 C_{0,0}}{\partial S_1 \partial S_2} \Big).\end{aligned} \tag{4.75}$$

[8] Note that both conditions are necessary and do not contradict the results we get when we regard (4.59) as Poisson equation in v_1 and v_2. This can be seen when we solve $\left\langle \mathcal{L}_0^1 C_{2,0} + \mathcal{L}_2 C_{0,0} \right\rangle_{v_2} = \left\langle \mathcal{L}_0^1 \right\rangle_{v_2} C_{2,0} + \mathcal{L}_2 C_{0,0} = 0$. When $\mathcal{L}_2 C_{0,0}$ is unknown the last expression is again a Poisson equation, this time in v_1. Thus, according to the Fredholm alternatives $\left\langle \left\langle \mathcal{L}_0^1 \right\rangle_{v_2} \right\rangle_{v_1}$. The same follows for $\left\langle \mathcal{L}_0^2 C_{0,2} + \mathcal{L}_2 C_{0,0} \right\rangle_{v_1} = 0$.

[9] The perturbation could also analogously be started in δ_2. Thus, all $C_{i,0}$ are independent from v_2.

Further, we define ϕ as the solution of Poisson equations

$$\begin{aligned}(\mathcal{L}_0^1+\mathcal{L}_0^2)\phi &= \sum_{j=1}^{2}(f(v_j)^2-\overline{f(v_j)^2}), \\ \mathcal{L}_0^1\phi_1 &= f(v_1)^2-\overline{f(v_1)^2}, \\ \mathcal{L}_0^2\phi_2 &= f(v_2)^2-\overline{f(v_2)^2}.\end{aligned} \tag{4.76}$$

Hence, we can write with (4.76) and (4.73)

$$\begin{aligned}C_{2,0} &= -\frac{1}{2}\phi_1\left(a_{11}^2S_1^2\frac{\partial^2}{\partial S_1^2}+a_{21}^2S_2^2\frac{\partial^2}{\partial S_2^2}+2a_{11}a_{21}S_1S_2\frac{\partial^2}{\partial S_1\partial S_2}\right)C_{0,0} \\ &\quad +f_{2,0}^e\left(t,S_1,S_2\right),\end{aligned} \tag{4.77}$$

$$\begin{aligned}C_{0,2} &= -\frac{1}{2}\phi_2\left(a_{12}^2S_1^2\frac{\partial^2}{\partial S_1^2}+a_{22}^2S_2^2\frac{\partial^2}{\partial S_2^2}+2a_{12}a_{22}S_1S_2\frac{\partial^2}{\partial S_1\partial S_2}\right)C_{0,0} \\ &\quad +f_{0,2}^e\left(t,S_1,S_2\right).\end{aligned} \tag{4.78}$$

Equation (4.60) is a Poisson equation in v_1, v_2. As $C_{0,3}$ is independent from v_1 (due to (4.51) and explanations all $C_{0,i}$ are independent from v_1) and $C_{2,1}$ is independent from v_2[10] we can reformulate (4.60):

$$(\mathcal{L}_0^2+\mathcal{L}_0^1)(C_{0,3}+C_{2,1})=-(\mathcal{L}_1^2C_{0,2}+\mathcal{L}_2C_{0,1}). \tag{4.79}$$

Hence, the following statement must be imposed to guarantee the solvability of the Poisson equation

$$\begin{aligned}\langle\langle\mathcal{L}_2\rangle\rangle\, C_{0,1} &= -\left\langle\left\langle\mathcal{L}_1^2C_{0,2}\right\rangle\right\rangle=-\left\langle\mathcal{L}_1^2C_{0,2}\right\rangle_{v_2} \qquad (4.80)\\ &\overset{(4.78)}{=} \left\langle\mathcal{L}_1^2\frac{1}{2}\phi_2\left(a_{12}^2S_1^2\frac{\partial^2}{\partial S_1^2}+a_{22}^2S_2^2\frac{\partial^2}{\partial S_2^2}+2a_{12}a_{22}S_1S_2\frac{\partial^2}{\partial S_1\partial S_2}\right)C_{0,0}\right\rangle_{v_2}.\end{aligned}$$

Applying (4.76) the problem for $C_{0,1}$ can be simplified.

Corollary 19. *($C_{0,1}$)*

The problem for $C_{0,1}$ is given by

$$\begin{aligned}\mathcal{L}_{BS}(\overline{\sigma}_1,\overline{\sigma}_2,\overline{\rho})C_{0,1} &= \mathcal{A}_2C_{0,0}, \\ C_{0,1}\left(T,S_1,S_2\right) &= 0,\end{aligned} \tag{4.81}$$

[10] Due to (4.58) and the following explanations all $C_{1,i}$ are independent from v_1. We get the analogous result for $C_{i,1}$ being independent from v_2 when we start the perturbation in δ_2.

with

$$\mathcal{A}_2 = \left\langle \sqrt{v_2} f(v_2) \frac{\partial \phi_2}{\partial v_2} \right\rangle_{v_2} \rho_2^v \sigma_{v_2} \left(\frac{a_{12}}{2} S_1 \frac{\partial}{\partial S_1} + \frac{a_{22}}{2} S_2 \frac{\partial}{\partial S_2} \right) \left(a_{12}^2 S_1^2 \frac{\partial^2}{\partial S_1^2} + a_{22}^2 S_2^2 \frac{\partial^2}{\partial S_2^2} + 2a_{12}a_{22}S_1S_2 \frac{\partial^2}{\partial S_1 \partial S_2} \right), \tag{4.82}$$

where $\frac{\partial \phi_2}{\partial v_2} = -\frac{2}{\sigma_{v_2}^2 v_2 p(v_2)} \int_0^{v_2} (f(z_2)^2 - \overline{f(v_2)^2}) p(z_2) dz_2$ *(see Equation (B.32)). Thus,*

$$\begin{aligned} \left\langle \sqrt{v_2} f(v_2) \frac{\partial \phi_2}{\partial v_2} \right\rangle_{v_2} &= \int_0^\infty \sqrt{v_2} f(v_2) \frac{\partial \phi_2}{\partial v_2} p(v_2) dv_2 \\ &= -\frac{2}{\sigma_{2,v}^2} \int_0^\infty \int_0^{v_2} \frac{f(v_2)(f(z_2)^2 - \overline{f(v_2)^2})}{\sqrt{v_2}} p(z_2) dz_2 dv_2. \end{aligned}$$

If $f(v_2) = \sqrt{v_2}$

$$\begin{aligned} \left\langle \sqrt{v_2} f(v_2) \frac{\partial \phi_2}{\partial v_2} \right\rangle_{v_2} &= -\frac{2}{\sigma_{2,v}^2} \int_0^\infty \int_0^{v_2} (z_2 - \overline{v}_2) p(z_2) dz_2 dv_2 \\ &\overset{(B.28)}{=} -\frac{2}{\sigma_{2,v}^2} \int_0^\infty \left(\int_0^{v_2} \frac{\mu^{CIR^{a^{CIR}}}}{\Gamma(a^{CIR})} z_2^{a^{CIR}} e^{-\mu^{CIR} z_2} dz_2 - \overline{v}_2 \right) dv_2 \\ &= -\frac{2}{\sigma_{2,v}^2} \int_0^\infty \left(\frac{1}{\mu^{CIR}} \int_0^{v_2} \frac{x_2^{(a^{CIR}+1)-1}}{\Gamma(a^{CIR})} e^{-x_2} dx_2 - \overline{v}_2 \right) dv_2 \\ &= -\frac{2}{\sigma_{2,v}^2} \int_0^\infty \left(\frac{\gamma(a^{CIR}+1, \mu^{CIR} v_2)}{\Gamma(a^{CIR})\mu^{CIR}} - \overline{v}_2 \right) dv_2, \end{aligned} \tag{4.83}$$

where $\gamma(\tilde{c}, x) = \int_0^x e^{-t} t^{\tilde{c}-1} dt$ *is the lower incomplete gamma function (see [59], 8.350 and 8.351).*

It can be easily shown that

$$C_{0,1} = -(T-t)\mathcal{A}_2 C_{0,0}, \tag{4.84}$$

as $\mathcal{L}_{BS}$ *commutes with* $S_i \frac{\partial^k}{\partial S_i^k}$ *for* $k = 1, 2$[11] *and* $\mathcal{L}_{BS} C_{0,0} = 0$, *i.e.* $\mathcal{L}_{BS}\left(-(T-t)\mathcal{A}_2 C_{0,0}\right) = \mathcal{A}_2 C_{0,0} - (T-t)\mathcal{A}_2 \mathcal{L}_{BS} C_{0,0} \overset{(4.61)}{=} \mathcal{A}_2 C_{0,0}$.

[11] For $k=1$: $S_i \frac{\partial}{\partial S_i}\left(S_i^2 \frac{\partial^2 C_{0,0}}{\partial S_i^2}\right) = S_i \left(2S_i \frac{\partial^2 C_{0,0}}{\partial S_i^2} + S_i^2 \frac{\partial^3 C_{0,0}}{\partial S_i^3}\right) = S_i^2 \frac{\partial^2}{\partial S_i^2}\left(S_i \frac{\partial C_{0,0}}{\partial S_i}\right)$

The PDE in the case of the barrier option is given by

$$\begin{aligned}\mathcal{L}_{BS}(\overline{\sigma}_1,\overline{\sigma}_2,\overline{\rho})C_{B,0,1} &= \mathcal{A}_2 C_{B,0,0},\\ C_{B,0,1}\,(T,S_1,S_2) &= 0,\\ C_{B,0,1}(t,B_1(t),S_2) &= 0,\\ C_{B,0,1}(t,S_1,B_2(t)) &= 0.\end{aligned}$$

We simplify the problem by defining (see [70])

$$\hat{C}_{B,0,1} = C_{B,0,1} + \tilde{V}_{12}S_1\frac{\partial^2 C_{B,0,0}}{\partial\overline{f(v_2)^2}\partial S_1} + \tilde{V}_{22}S_2\frac{\partial^2 C_{B,0,0}}{\partial\overline{f(v_2)^2}\partial S_2}. \tag{4.85}$$

When we differentiate $\mathcal{L}_{BS}C_{B,0,0}$ with respect to $\overline{f(v_j)^2}$ we see

$$\begin{aligned}\frac{\partial}{\partial\overline{f(v_j)^2}}\left(\mathcal{L}_{BS}C_{B,0,0}\right) &= \frac{\partial\frac{\partial C_{B,0,0}}{\partial\overline{f(v_j)^2}}}{\partial t} + \frac{1}{2}S_1^2\sum_{j=1}^{2}a_{1j}^2\overline{f(v_j)^2}\frac{\partial^2\frac{\partial C_{B,0,0}}{\partial\overline{f(v_j)^2}}}{\partial S_1^2}\\ &+\frac{1}{2}S_2^2\sum_{j=1}^{2}a_{2j}^2\overline{f(v_j)^2}\frac{\partial^2\frac{\partial C_{B,0,0}}{\partial\overline{f(v_j)^2}}}{\partial S_2^2} + S_1S_2\sum_{j=1}^{2}a_{1j}a_{2j}\overline{f(v_j)^2}\frac{\partial^2\frac{\partial C_{B,0,0}}{\partial\overline{f(v_j)^2}}}{\partial S_1\partial S_2}\\ &+r\left(S_1\frac{\partial\frac{\partial C_{B,0,0}}{\partial\overline{f(v_j)^2}}}{\partial S_1} + S_2\frac{\partial\frac{\partial C_{B,0,0}}{\partial\overline{f(v_j)^2}}}{\partial S_2} - \frac{\partial C_{B,0,0}}{\partial\overline{f(v_j)^2}}\right)\\ &+\frac{1}{2}\left(S_1^2a_{1j}^2\frac{\partial^2 C_{B,0,0}}{\partial S_1^2} + S_2^2a_{2j}^2\frac{\partial^2 C_{B,0,0}}{\partial S_2^2} + 2S_1S_2a_{1j}a_{2j}\frac{\partial^2 C_{B,0,0}}{\partial S_1\partial S_2}\right)\\ &= \mathcal{L}_{BS}\frac{\partial C_{B,0,0}}{\partial\overline{f(v_j)^2}} + \qquad (4.86)\\ &+\frac{1}{2}\left(S_1^2a_{1j}^2\frac{\partial^2 C_{B,0,0}}{\partial S_1^2} + S_2^2a_{2j}^2\frac{\partial^2 C_{B,0,0}}{\partial S_2^2} + 2S_1S_2a_{1j}a_{2j}\frac{\partial^2 C_{B,0,0}}{\partial S_1\partial S_2}\right) = 0,\end{aligned}$$

due to Equation (4.67).[12] Hence, $\mathcal{L}_{BS}(\overline{\sigma}_1, \overline{\sigma}_2, \overline{\rho})\frac{\partial C_{B,0,0}}{\partial \overline{f(v_j)^2}}$ for $j = 1, 2$ solves the PDE problem

$$\begin{aligned}
\mathcal{L}_{BS}\frac{\partial C_{B,0,0}}{\partial \overline{f(v_j)^2}} &= -\frac{1}{2}\Big(S_1^2 a_{1j}^2 \frac{\partial^2 C_{B,0,0}}{\partial S_1^2} + S_2^2 a_{2j}^2 \frac{\partial^2 C_{B,0,0}}{\partial S_2^2} \\
&\quad +2S_1S_2a_{1j}a_{2j}\frac{\partial^2 C_{B,0,0}}{\partial S_1 \partial S_2}\Big), \\
\frac{\partial C_{B,0,0}}{\partial \overline{f(v_j)^2}}(T, S_1, S_2) &= 0, \\
\frac{\partial C_{B,0,0}}{\partial \overline{f(v_j)^2}}(t, B_1(t), S_2) &= 0, \\
\frac{\partial C_{B,0,0}}{\partial \overline{f(v_j)^2}}(t, S_1, B_2(t)) &= 0. \qquad (4.87)
\end{aligned}$$

Differentiating this with respect to S_i with $i = 1, 2$ we see that due to the fact that $\mathcal{L}_{BS}$ commutes with $S_i\frac{\partial^k}{\partial S_i^k}$ for $k = 1, 2$,

$$\begin{aligned}
\mathcal{L}_{BS}(S_i\frac{\partial^2 C_{B,0,0}}{\partial S_i \partial \overline{f(v_j)^2}}) &= -\frac{1}{2}S_i\frac{\partial}{\partial S_i}\Big(S_1^2 a_{1j}^2 \frac{\partial^2 C_{B,0,0}}{\partial S_1^2} + S_2^2 a_{2j}^2 \frac{\partial^2 C_{B,0,0}}{\partial S_2^2} \\
&\quad +2S_1S_2a_{1j}a_{2j}\frac{\partial^2 C_{B,0,0}}{\partial S_1 \partial S_2}\Big), \\
\frac{\partial^2 C_{B,0,0}}{\partial S_i \partial \overline{f(v_j)^2}}(T, S_i, S_k) &= 0, \\
\frac{\partial^2 C_{B,0,0}}{\partial S_i \partial \overline{f(v_j)^2}}(t, B_i(t), S_k) & \quad \text{in general not } 0, \\
\frac{\partial^2 C_{B,0,0}}{\partial S_i \partial \overline{f(v_j)^2}}(t, S_1, B_k(t)) &= 0. \qquad (4.88)
\end{aligned}$$

$\hat{C}_{B,0,1}$ is therefore the solution to the problem

$$\begin{aligned}
\mathcal{L}_{BS}(\overline{\sigma}_1, \overline{\sigma}_2, \overline{\rho})\hat{C}_{B,0,1} &= 0, \\
\hat{C}_{B,0,1}(T, S_1, S_2) &= 0, \\
\hat{C}_{B,0,1}(t, B_1(t), S_2) &= \tilde{V}_{12}\mathfrak{g}_{12}(t, B_1, S_2(t)), \\
\hat{C}_{B,0,1}(t, S_1, B_2(t)) &= \tilde{V}_{22}\mathfrak{g}_{22}(t, S_1(t), B_2), \qquad (4.89)
\end{aligned}$$

[12] Note that with (4.62) $\mathcal{L}_{BS}C_{B,0,0} = \frac{\partial C_{B,0,0}}{\partial t} + \frac{1}{2}S_1^2\sum_{j=1}^2 a_{1j}^2\overline{f(v_j)^2}\frac{\partial^2 C_{B,0,0}}{\partial S_1^2} + \frac{1}{2}S_2^2\sum_{j=1}^2 a_{2j}^2\overline{f(v_j)^2}\frac{\partial^2 C_{B,0,0}}{\partial S_2^2} + S_1S_2\sum_{j=1}^2 a_{1j}a_{2j}\overline{f(v_j)^2}\frac{\partial^2 C_{B,0,0}}{\partial S_1 \partial S_2} + r\left(S_1\frac{\partial C_{B,0,0}}{\partial S_1} + S_2\frac{\partial C_{B,0,0}}{\partial S_2} - C_{B,0,0}\right) = 0.$

where

$$\tilde{V}_{12} = \left\langle \sqrt{v_2} f(v_2) \frac{\partial \phi_2}{\partial v_2} \right\rangle_{v_2} \rho_2^v \sigma_{v_2} a_{12}, \tag{4.90}$$

$$\tilde{V}_{22} = \left\langle \sqrt{v_2} f(v_2) \frac{\partial \phi_2}{\partial v_2} \right\rangle_{v_2} \rho_2^v \sigma_{v_2} a_{22}, \tag{4.91}$$

$$\mathfrak{g}_{12}(t, B_1, S_2(t)) = S_1 \frac{\partial^2 C_{B,0,0}}{\partial f(v_2)^2 \partial S_1}\bigg|_{S_1=B_1(\iota_1), S_2=S_2(\iota_1)},$$

$$\mathfrak{g}_{22}(t, S_1(t), B_2) = S_2 \frac{\partial^2 C_{B,0,0}}{\partial f(v_2)^2 \partial S_2}\bigg|_{S_1=S_1(\iota_2), S_2=B_2(\iota_2)}.$$

The solution can be found by (for the following results see [71].)

$$\begin{aligned} \hat{C}_{B,0,1}(t, S_1, S_2, B_1, B_2) &= \int_0^T \int_0^\infty e^{-c_1 b_1 - c_2 b_2} e^{-c_2(\sigma_2 a'_p \sin \beta_p)} \tilde{V}_{12} \mathfrak{g}_{12}(t', B_1, B_2 e^{\sigma_2 a'_p \sin \beta_p}) \\ & p^x_{GBM}\left(\iota' \in dt', \theta'_p = \beta_p\right) da'_p dt' \\ &+ \int_0^T \int_0^\infty e^{-c_1 b_1 - c_2 b_2} e^{-c_1(\sigma_1 a'_p \sqrt{1-\rho^2})} \tilde{V}_{22} \mathfrak{g}_{22}(t', B_1 e^{\sigma_1 \sqrt{1-\rho^2} a'_p}, B_2) \\ & p^x_{GBM}\left(\iota' \in dt', \theta'_p = 0\right) da'_p dt', \end{aligned} \tag{4.92}$$

where $r'_p = a'_p$ and $\theta'_p = (0, \beta_p)$.

$$\begin{aligned} p^x_{GBM}(\iota' \in dt', \mathbf{z}(\iota') \in \partial Y) &= \frac{\pi e^{c_1 x_1 + c_2 x_2 - \int_t^{t'} r(s) ds - \alpha(t'-t)}}{\beta_p^2 t' a'_p} e^{\frac{{a'_p}^2 + r_p^2}{2t'}} \sum_{n=1}^\infty \delta_n n \\ & \sin\left(\frac{n\pi\theta_p}{\beta_p}\right) I_{\frac{n\pi}{\beta_p}}\left(\frac{a'_p r_p}{t'}\right), \end{aligned} \tag{4.93}$$

where $\mathbf{z}$ describes the vector of transformed variables in polar coordinates, ∂Y describes the boundary, i.e. the wedge with $\delta_n = 1$ if $\theta'_p = 0$ (i.e. $S_2 = B_2(t)$, $S_1 = e^{x'_1} K_1 = B_1 e^{\sigma_1 \sqrt{1-\rho^2} z'_1} = B_1 e^{\sigma_1 \sqrt{1-\rho^2} r'_p \cos(\theta'_p)} = B_1 e^{\sigma_1 \sqrt{1-\rho^2} r'_p}$) and $\delta_n = (-1)^{n+1}$ if $\theta'_p = \beta_p$ (i.e. $S_1 = B_1(t)$,$S_2 = K_2 e^{x'_2} = B_2 e^{\sigma_2 z'_2} = B_2 e^{\sigma_2 r'_p \sin \theta'_p}$). $\tan \beta_p = -\frac{\sqrt{1-\overline{\rho}^2}}{\overline{\rho}}$, $\beta_p \in [0, \pi]$. For $\overline{\rho} = -\cos\frac{\pi}{n}$, $p^x_{GBM}(\iota' \in dt', \mathbf{z}(\iota') \in \partial Y)$ can be attained in an easier way.
For $\theta'_p = 0$

$$\begin{aligned} p^x_{GBM}(\iota' \in dt', \mathbf{z}(\iota') \in \partial Y) &= e^{c_1 x_1 + c_2 x_2 - \int_t^{t'} r(s) ds - \alpha(t'-t)} \\ & \frac{1}{t'^2} \sum_{k=0}^{2n-1} (-1)^k \Phi\left(\frac{\mathbf{z}' - \mathfrak{T}_k \mathbf{z}}{\sqrt{t}}\right)\bigg|_{z'_2=0} (e_2 \mathfrak{T}_k \mathbf{z}'), \end{aligned}$$

where e_2 is the vector $(0, 1)$, Φ is the standard normal distribution with $\frac{1}{2\pi} e^{-\frac{(z'_1 - z_1)^2 + (z'_2 - z_2)^2}{2}}$,

$\mathbf{z}' = (z_1', z_2')$, $\mathbf{z}_0 = (z_1, z_2)$,

$$\mathfrak{T}_k\mathbf{z} = \begin{cases} (r_p\cos(\frac{2k\pi}{n} + \theta_p), r_p\sin(\frac{2k\pi}{n} + \theta_p)) & \text{for } k \text{ even,} \\ (r_p\cos(\frac{2(k-1)\pi}{n} - \theta_p), r_p\sin(\frac{2(k-1)\pi}{n} - \theta_p)) & \text{for } k \text{ odd,} \end{cases} \tag{4.94}$$

$$\begin{aligned} x_i &= \ln\left(\frac{S_i e^{\int_t^T r ds}}{K_i}\right), \\ z_1 &= \frac{1}{\sqrt{1-\rho^2}}\left(\frac{x_1 - b_1}{\sigma_1} - \rho\left(\frac{x_2 - b_2}{\sigma_2}\right)\right), \\ z_2 &= \frac{x_2 - b_2}{\sigma_2}. \end{aligned}$$

For $\theta_p' = \beta_p$

$$\begin{aligned} p^x_{GBM}(\iota' \in dt', \mathbf{z}(\iota') \in \partial Y) &= e^{c_1x_1 + c_2x_2 - \int_t^{t'} r(s)ds - \alpha(t'-t)} \\ & \frac{1}{t'^2}\sum_{k=0}^{2n-1}(-1)^k\Phi(\frac{\mathbf{z}' - \mathfrak{T}_k\mathbf{z}}{\sqrt{t}})\Big|_{z_2'=0}(e_2\mathfrak{T}_k\mathbf{z}'), \end{aligned}$$

with $\tilde{\theta}_p = \beta_p - \theta_p$ and

$$\mathfrak{T}_k\mathbf{z} = \begin{cases} (r_p\cos(\frac{2k\pi}{n} + \tilde{\theta}_p), r_p\sin(\frac{2k\pi}{n} + \tilde{\theta}_p)) & \text{for } k \text{ even,} \\ (r_p\cos(\frac{2(k-1)\pi}{n} - \tilde{\theta}_p), r_p\sin(\frac{2(k-1)\pi}{n} - \tilde{\theta}_p)) & \text{for } k \text{ odd.} \end{cases} \tag{4.95}$$

Corollary 20. *($C_{B,0,1}$)*

The solution for $C_{B,0,1}$ is given by

$$C_{B,0,1} = \hat{C}_{B,0,1} - \tilde{V}_{12}S_1\frac{\partial^2 C_{B,0,0}}{\partial \overline{f(v_2)^2}\partial S_1} - \tilde{V}_{22}S_2\frac{\partial^2 C_{B,0,0}}{\partial \overline{f(v_2)^2}\partial S_2}. \tag{4.96}$$

Solving the PDE for the first-order corrections $C_{1,0}$ and $C_{B,1,0}$

Inserting expression (4.56) in (4.54) and collecting terms of the same order of δ_2 we obtain the following relationships

$$\begin{aligned} \delta_2^{-2}: &\quad \mathcal{L}_0^2 C_{1,0} &= 0, \quad (4.97) \\ \delta_2^{-1}: &\quad \mathcal{L}_0^2 C_{1,1} + \mathcal{L}_1^2 C_{1,0} &= 0, \quad (4.98) \\ \delta_2^{0}: &\quad \mathcal{L}_0^2 C_{1,2} + \mathcal{L}_1^2 C_{1,1} + \mathcal{L}_2 C_{1,0} + \mathcal{L}_1^1 C_{2,0} + \mathcal{L}_0^1 C_{3,0} &= 0, \quad (4.99) \\ \delta_2^{1}: &\quad \mathcal{L}_0^2 C_{1,3} + \mathcal{L}_1^2 C_{1,2} + \mathcal{L}_2 C_{1,1} + \mathcal{L}_1^1 C_{2,1} + \mathcal{L}_0^1 C_{3,1} &= 0. \quad (4.100) \end{aligned}$$

We notice that $C_{1,0}$ and $C_{1,1}$ are both independent from v_2. Equation (4.99) is a Poisson equation in v_1, v_2 while $\mathcal{L}_1^2 C_{1,1} = 0$.

As $\mathcal{L}_0^2 C_{3,0} = 0$ due to the independence of $C_{3,0}$ from v_2 (due to (4.51) and explanations all $C_{0,i}$ are independent from v_1. The same follows analogously for $C_{i,0}$ as the perturbation could be also started with δ_2.) and $\mathcal{L}_0^1 C_{1,2} = 0$ because $C_{1,2}$ is independent from v_1 (due to (4.52) and the respective explanations) we can reformulate (4.99)

$$\left(\mathcal{L}_0^2 + \mathcal{L}_0^1\right)\left(C_{1,2} + C_{3,0}\right) = -\left(\mathcal{L}_1^1 C_{2,0} + \mathcal{L}_2 C_{1,0}\right). \tag{4.101}$$

The following condition has to be fulfilled so that this Poisson equation is solvable:

$$\langle\langle \mathcal{L}_2 \rangle\rangle\, C_{1,0} = -\left\langle\left\langle \mathcal{L}_1^1 C_{2,0}\right\rangle\right\rangle \overset{(4.77)}{=} -\left\langle \mathcal{L}_1^1 C_{2,0}\right\rangle_{v_1}. \tag{4.102}$$

Using (4.76) and (4.77) the respective problem for $C_{1,0}$ can be found.

Corollary 21. *($C_{1,0}$)*

$C_{1,0}$ solves the following problem

$$\begin{aligned} \mathcal{L}_{BS}(\overline{\sigma}_1, \overline{\sigma}_2) C_{1,0} &= \mathcal{A}_1 C_{0,0}, \\ C_{1,0}\left(T, S_1, S_2\right) &= 0, \end{aligned} \tag{4.103}$$

where

$$\begin{aligned} \mathcal{A}_1 = & \left\langle \sqrt{v_1} f(v_1) \frac{\partial \phi_1}{\partial v_1} \right\rangle_{v_1} \rho_1^v \sigma_{v_1} \left(\frac{a_{11}}{2} S_1 \frac{\partial}{\partial S_1} + \frac{a_{21}}{2} S_2 \frac{\partial}{\partial S_2} \right) \\ & \left(a_{11}^2 S_1^2 \frac{\partial^2}{\partial S_1^2} + a_{21}^2 S_2^2 \frac{\partial^2}{\partial S_2^2} + 2 a_{11} a_{21} S_1 S_2 \frac{\partial^2}{\partial S_1 \partial S_2} \right). \end{aligned} \tag{4.104}$$

Analogously to (4.84) the solution for the two-asset option without barriers can be easily indicated by

$$C_{1,0} = -(T-t)\mathcal{A}_1 C_{0,0}. \tag{4.105}$$

Accordingly, the problem for the barrier option is given by

$$\begin{aligned} \mathcal{L}_{BS}(\overline{\sigma}_1, \overline{\sigma}_2, \overline{\rho}) C_{B,1,0} &= \mathcal{A}_1 C_{B,0,0}, \\ C_{B,1,0}\left(T, S_1, S_2\right) &= 0, \\ C_{B,1,0}(t, B_1(t), S_2) &= 0, \\ C_{B,1,0}(t, S_1, B_2(t)) &= 0. \end{aligned} \tag{4.106}$$

We redefine the problem analogously to (4.87)-(4.88) by

$$\hat{C}_{B,1,0} = C_{B,1,0} + \tilde{V}_{11} S_1 \frac{\partial^2 C_{B,0,0}}{\partial \overline{f(v_1)^2} \partial S_1} + \tilde{V}_{21} S_2 \frac{\partial^2 C_{B,0,0}}{\partial \overline{f(v_1)^2} \partial S_2}. \tag{4.107}$$

$\hat{C}_{B,1,0}$ is the solution to the problem

$$\begin{aligned} \mathcal{L}_{BS}(\overline{\sigma}_1, \overline{\sigma}_2)\hat{C}_{B,1,0} &= 0, \\ \hat{C}_{B,1,0}(T, S_1, S_2) &= 0, \\ \hat{C}_{B,1,0}(t, B_1(t), S_2) &= \tilde{V}_{11} \mathfrak{g}_{11}(t, B_1, S_2(t)), \\ \hat{C}_{B,1,0}(t, S_1, B_2(t)) &= \tilde{V}_{21} \mathfrak{g}_{21}(t, S_1, B_2(t)), \end{aligned} \tag{4.108}$$

where

$$\tilde{V}_{11} = \frac{1}{2} \left\langle \sqrt{v_1} f(v_1) \frac{\partial \phi_1}{\partial v_1} \right\rangle_{v_1} \rho_1^v \sigma_{1,v} a_{11}, \tag{4.109}$$

$$\tilde{V}_{21} = \frac{1}{2} \left\langle \sqrt{v_1} f(v_1) \frac{\partial \phi_1}{\partial v_1} \right\rangle_{v_1} \rho_1^v \sigma_{1,v} a_{21}, \tag{4.110}$$

$$\mathfrak{g}_{11}(t, B_1, S_2(t)) = S_1 \frac{\partial^2 C_{B,0,0}}{\partial \overline{f(v_1)^2} \partial S_1} \bigg|_{S_1 = B_1(\iota_1), S_2 = S_2(\iota_1)},$$

$$\mathfrak{g}_{21}(t, S_1, B_2(t)) = S_2 \frac{\partial^2 C_{B,0,0}}{\partial \overline{f(v_1)^2} \partial S_2} \bigg|_{S_1 = S_1(\iota_2), S_2 = B_2(\iota_2)}.$$

Analogously to (4.92) the solution can be found by

$$\begin{aligned} \hat{C}_{B,1,0}(t, S_1, S_2, B_1, B_2) = & \int_0^T \int_0^\infty e^{-c_1 b_1 - c_2 b_2} e^{-c_2(\sigma_2 a_p' \sin \beta_p)} \tilde{V}_{12} \mathfrak{g}_{11}(t', B_1, B_2 e^{\sigma_2 a_p' \sin \beta_p}) \\ & p_{GBM}^x \left(\iota' \in dt', \theta_p' = \beta_p \right) da_p' dt' \\ & + \int_0^T \int_0^\infty e^{-c_1 b_1 - c_2 b_2} e^{-c_1(\sigma_1 a_p' \sqrt{1-\rho^2})} \tilde{V}_{22} \mathfrak{g}_{21}(t', B_1 e^{\sigma_1 \sqrt{1-\rho^2} a_p'}, B_2) \\ & p_{GBM}^x \left(\iota' \in dt', \theta_p' = 0 \right) da_p' dt', \end{aligned} \tag{4.111}$$

where the probability is given in (4.93).

Corollary 22. $C_{B,1,0}$

The solution to $C_{B,1,0}$ *is given by*

$$C_{B,1,0} = \hat{C}_{B,1,0} - \tilde{V}_{11} S_1 \frac{\partial^2 C_{B,0,0}}{\partial \overline{f(v_1)^2} \partial S_1} - \tilde{V}_{21} S_2 \frac{\partial^2 C_{B,0,0}}{\partial \overline{f(v_1)^2} \partial S_2}. \tag{4.112}$$

Solving the PDE for the first-order corrections $C_{1,1}$ and $C_{B,1,1}$

Equation (4.99) is a Poisson equation with respect to v_1 for $C_{3,0}$. Thus, we deduce with (4.102)

$$\begin{aligned}
\mathcal{L}_0^2 C_{1,2} &= -\left\langle \mathcal{L}_1^1 C_{2,0} + \mathcal{L}_2 C_{1,0} \right\rangle_{v_1} \\
&\overset{(4.102)}{=} -\left(-\langle\langle \mathcal{L}_2 \rangle\rangle C_{1,0} + \langle \mathcal{L}_2 \rangle_{v_1} C_{1,0}\right) \\
&\overset{(4.105)}{=} \frac{1}{2}(T-t)(f(v_2)^2 - \overline{f(v_2)^2}) \\
&\quad \left(a_{12}^2 S_1^2 \frac{\partial^2}{\partial S_1^2} + a_{22}^2 S_2^2 \frac{\partial^2}{\partial S_2^2} + 2a_{12}a_{22}S_1S_2 \frac{\partial^2}{\partial S_1 \partial S_2}\right) \mathcal{A}_1 C_{0,0}, \qquad (4.113) \\
C_{1,2} &= \frac{1}{2}(T-t)\phi_2 \left(a_{12}^2 S_1^2 \frac{\partial^2}{\partial S_1^2} + a_{22}^2 S_2^2 \frac{\partial^2}{\partial S_2^2} + 2a_{12}a_{22}S_1S_2 \frac{\partial^2}{\partial S_1 \partial S_2}\right) \mathcal{A}_1 C_{0,0} \\
&\quad + f_{1,2}^e(t, S_1, S_2),
\end{aligned}$$

where $f_{1,2}^e(t, S_1, S_2)$ is another eigenfunction, which does not depend on v_1. In this way, Equation (4.60) is a Poisson equation in v_2 for $C_{0,3}$. Hence, from the solvability condition together with (4.80) follows, analogue to (4.113):

$$\begin{aligned}
\mathcal{L}_0^1 C_{2,1} &= -\left\langle \mathcal{L}_1^2 C_{0,2} + \mathcal{L}_2 C_{0,1} \right\rangle_{v_2} \\
&= \frac{1}{2}(T-t)(f(v_1)^2 - \overline{f(v_1)^2}) \\
&\quad \left(a_{11}^2 S_1^2 \frac{\partial^2}{\partial S_1^2} + a_{21}^2 S_2^2 \frac{\partial^2}{\partial S_2^2} + 2a_{11}a_{21}S_1S_2 \frac{\partial^2}{\partial S_1 \partial S_2}\right) \mathcal{A}_2 C_{0,0}, \qquad (4.114) \\
C_{2,1} &= \frac{1}{2}(T-t)\phi_1 \left(a_{11}^2 S_1^2 \frac{\partial^2}{\partial S_1^2} + a_{21}^2 S_2^2 \frac{\partial^2}{\partial S_2^2} + 2a_{11}a_{21}S_1S_2 \frac{\partial^2}{\partial S_1 \partial S_2}\right) \mathcal{A}_2 C_{0,0} \\
&\quad + f_{2,1}^e(t, S_1, S_2),
\end{aligned}$$

where $f_{2,1}^e(t, S_1, S_2)$ is another eigenfunction, which does not depend on v_2. Finally, Equation (4.100) is a Poisson equation in v_1 and v_2 for $C_{3,1}$ and $C_{1,3}$. To ensure its solvability we have to enforce the following condition

$$\langle\langle \mathcal{L}_2 \rangle\rangle C_{1,1} = -\left\langle\left\langle \mathcal{L}_1^2 C_{1,2} + \mathcal{L}_1^1 C_{2,1} \right\rangle\right\rangle.$$

Corollary 23. *($C_{1,1}$)*

$C_{1,1}$ satisfies the following problem

$$\langle\langle \mathcal{L}_2 \rangle\rangle C_{1,1} = 2(t-T)\mathcal{A}_1 \mathcal{A}_2 C_{0,0}. \qquad (4.115)$$

Concluding, as in (4.84) and (4.105),

$$C_{1,1} = (T-t)^2 \mathcal{A}_1 \mathcal{A}_2 C_{0,0}. \tag{4.116}$$

This result can be easily checked by inserting (4.116) in (4.115).

For the barrier option the problem is given by

$$\begin{aligned}
\mathcal{L}_{BS}(\overline{\sigma}_1, \overline{\sigma}_2, \overline{\rho}) C_{B,1,1} &= 2(T-t)\mathcal{A}_1 \mathcal{A}_2 C_{B,0,0}, \\
C_{B,1,1}(T, S_1, S_2) &= 0, \\
C_{B,1,1}(t, B_1(t), S_2) &= 0, \\
C_{B,1,1}(t, S_1, B_2(t)) &= 0.
\end{aligned} \tag{4.117}$$

In order to simplify the problem we deduce that

$$\mathcal{L}_{BS}\left((T-t)^2 \mathcal{A}_1 \mathcal{A}_2 C_{B,0,0}\right) = -2(T-t)\mathcal{A}_1 \mathcal{A}_2 C_{B,0,0} + \underbrace{(T-t)^2 \mathcal{L}_{BS} \mathcal{A}_1 \mathcal{A}_2 C_{B,0,0}}_{=0}, \tag{4.118}$$

the last term equals zero as $\mathcal{A}_1$ and $\mathcal{A}_2$ commutes with $\mathcal{L}_{BS}$. Hence, we define

$$\hat{C}_{B,1,1} = C_{B,1,1} + (T-t)^2 \mathcal{A}_1 \mathcal{A}_2 C_{B,0,0} \tag{4.119}$$

$\hat{C}_{B,1,1}$ is the solution to the problem

$$\begin{aligned}
\mathcal{L}_{BS}(\overline{\sigma}_1, \overline{\sigma}_2) \hat{C}_{B,1,1} &= 0, \\
\hat{C}_{B,1,1}(T, S_1, S_2) &= 0, \\
\hat{C}_{B,1,1}(t, B_1(t), S_2) &= \tilde{\mathfrak{g}}_1(t, B_1, S_2), \\
\hat{C}_{B,1,1}(t, S_1, B_2(t)) &= \tilde{\mathfrak{g}}_2(t, S_1, B_2),
\end{aligned} \tag{4.120}$$

where

$$\begin{aligned}
\tilde{\mathfrak{g}}_1(t, B_1, S_1) &= (T-t)^2 \mathcal{A}_1 \mathcal{A}_2 C_{B,0,0}\Big|_{S_1=B_1, S_2=S_2(\iota_1)}, \\
\tilde{\mathfrak{g}}_2(t, S_1, B_2) &= (T-t)^2 \mathcal{A}_1 \mathcal{A}_2 C_{B,0,0}\Big|_{S_1=S_1(\iota_2), S_2=B_2}.
\end{aligned} \tag{4.121}$$

The solution can be found by

$$\begin{aligned}\hat{C}_{B,1,1}\left(t,S_1,S_2,B_1,B_2\right) &= \int_0^T\int_0^\infty e^{-c_1b_1-c_2b_2}e^{-c_2(\sigma_2 a'_p\sin\beta_p)}\tilde{\mathfrak{g}}_1(t',B_1,B_2e^{\sigma_2 a'_p\sin\beta_p})\\ &\quad p^x_{GBM}\left(\iota'\in dt',\theta'_p=\beta_p\right)da'_pdt'\\ &+\int_0^T\int_0^\infty e^{-c_1b_1-c_2b_2}e^{-c_1(\sigma_1 a'_p\sqrt{1-\rho^2})}\tilde{\mathfrak{g}}_2(t',B_1e^{\sigma_1\sqrt{1-\rho^2}a'_p},B_2)\\ &\quad p^x_{GBM}\left(\iota'\in dt',\theta'_p=0\right)da'_pdt', \end{aligned} \tag{4.122}$$

where the probability is given in (4.93).

Corollary 24. *($C_{B,1,1}$)*

The solution for $C_{B,1,1}$ given by

$$C_{B,1,1}=\hat{C}_{B,1,1}-(T-t)^2\mathcal{A}_1\mathcal{A}_2C_{B,0,0}. \tag{4.123}$$

□

Accuracy of the approximation

We derive the accuracy of the price approximation along the lines of Fouque et al [52], [49]. Each call or digital option has a payoff function which is only $\mathbf{C}^0$ smooth with a discontinuous first derivative at the kink $S_i=K_i$. However, for the proof we require a smooth payoff function and smooth derivatives.

Thus, in order to proceed we regularise the payoff, which is a two-asset option, by replacing it with the Black-Scholes price C_{BS} of the two-asset option with time to maturity $\tilde{\epsilon}$ with volatilities $\overline{\sigma}_1$, $\overline{\sigma}_2$ and correlation $\overline{\rho}$, i.e. deterministic covariance because at $t<T$ the Black-Scholes price C_{BS} is smooth, and the derivatives are well-defined.

We define therefore

$$g^{\tilde{\epsilon}}\left(S_1,S_2\right)=C_{BS}\left(T,S_1,S_2,K_1,K_2,T+\tilde{\epsilon},\overline{\sigma}_1,\overline{\sigma}_2,\overline{\rho}\right), \tag{4.124}$$

with

$$C_{BS}\left(t,S_1,S_2,K_1,K_2,T,\overline{\sigma}_1,\overline{\sigma}_2,\overline{\rho}\right) = C_{0,0}\left(t,S_1,S_2,K_1,K_2,T,\overline{\sigma}_1,\overline{\sigma}_2,\overline{\rho}\right). \tag{4.125}$$

Hence, the regularised price $C^{\tilde{\epsilon},\delta}$ solves

$$\begin{aligned} \mathcal{L}^{\delta} C^{\tilde{\epsilon},\delta} &= 0, \\ C^{\tilde{\epsilon},\delta}(T, S_1, S_2) &= g^{\tilde{\epsilon}}(S_1, S_2). \end{aligned}$$

Let $Q^{\tilde{\epsilon},\delta}$ denote the first-order approximation to the regularised option price, i.e.

$$Q^{\tilde{\epsilon},\delta} \equiv C_{0,0}^{\tilde{\epsilon}} + \delta_1 C_{1,0}^{\tilde{\epsilon}} + \delta_2 C_{0,1}^{\tilde{\epsilon}} + \delta_1 \delta_2 C_{1,1}^{\tilde{\epsilon}}, \tag{4.126}$$

where

$$C_{0,0}^{\tilde{\epsilon}}(t, S_1, S_2) = C_{BS}(t, S_1, S_2, K_1, K_2, T + \tilde{\epsilon}, \overline{\sigma}_1, \overline{\sigma}_2, \overline{\rho}), \tag{4.127}$$

$$C_{1,0}^{\tilde{\epsilon}} = -(T-t)\mathcal{A}_1 C_{0,0}^{\tilde{\epsilon}}, \tag{4.128}$$

$$C_{0,1}^{\tilde{\epsilon}} = -(T-t)\mathcal{A}_2 C_{0,0}^{\tilde{\epsilon}}, \tag{4.129}$$

$$C_{1,1}^{\tilde{\epsilon}} = (T-t)^2 \mathcal{A}_1 \mathcal{A}_2 C_{0,0}^{\tilde{\epsilon}}. \tag{4.130}$$

Concluding, the proof must involve three steps. First, we show that the regularised price, $C^{\tilde{\epsilon},\delta}$, is converging to the actual unregularised price, C^{δ} (see Lemma 2), in a next step we deduct that the regularised approximation $Q^{\tilde{\epsilon},\delta}$ is close to the approximation Q^{δ}, which is defined in Equation (4.48), (see Lemma 3). Finally, it is left to prove that $C^{\tilde{\epsilon},\delta} \sim Q^{\tilde{\epsilon},\delta}$ in Lemma 4.

Lemma 2. *($C^{\tilde{\epsilon},\delta} \sim C^{\delta}$)*

For the fixed point (t, S_1, S_2, v), where $t < T$ and $\mathbf{v} = (v_1, v_2)$, there exist constants $\overline{\delta}_1^1 > 0$, $\overline{\delta}_2^1 > 0$, $\overline{\tilde{\epsilon}}_1 > 0$, and $\mathfrak{c}_1 > 0$ such that

$$\left| C^{\delta}(t, S_1, S_2, v) - C^{\tilde{\epsilon},\delta}(t, S_1, S_2, v) \right| \leq \mathfrak{c}_1 \tilde{\epsilon}, \tag{4.131}$$

for all $0 < \delta_1 < \overline{\delta}_1^1$, $0 < \delta_2 < \overline{\delta}_2^1$, and $0 < \tilde{\epsilon} < \overline{\tilde{\epsilon}}_1$.

Proof.

In the following we show that $C^{\tilde{\epsilon},\delta}$ and C^{δ} converge to each other by conditioning the difference in the prices on the paths of the eigenvectors up to T. For that we use the probabilistic representation of the price as the risk-neutral value of the discounted payoff. For the proof of this lemma we introduce the processes for $\tilde{S}_i$, $i \in (1, 2)$ with ρ_j^v:

$$d\tilde{S}_i = r\tilde{S}_i dt + \tilde{S}_i \sum_{j=1}^{2} a_{ij} f(\tilde{v}_j(t)) \left(\sqrt{1 - \rho_j^{v2}} d\hat{W}_j + \rho_j^v dZ_j \right), \ i = 1, 2, \tag{4.132}$$

where

$$f(\tilde{v}_j(t)) = \begin{cases} f(v_j(t)) & \text{for } t \leq T \\ \overline{f(v_j)^2} & \text{for } t > T \end{cases} \tag{4.133}$$

and v_j follows the process described in (4.2). Note that $\hat{W}_j$ and Z_j are two independent Brownian motions with $Z_j = Z_j$ in (4.2) and $\hat{W}_j = \frac{1}{\sqrt{1-\rho_j^{v2}}}(W_j - \rho_j^v Z_j)$ in (4.1). The assumption of a risk-neutral equivalent martingale measure allows for the probabilistic representation of the price as the expected discounted payoff. Thus, we find

$$\begin{aligned} C^{\tilde{\epsilon},\delta} &= \mathbb{E}_{\tilde{\mathcal{Q}}}\left[e^{-r(T-t+\tilde{\epsilon})} g\left(\tilde{S}_1(T+\tilde{\epsilon}), \tilde{S}_2(T+\tilde{\epsilon})\right)\right], \\ C^{\delta} &= \mathbb{E}_{\tilde{\mathcal{Q}}}\left[e^{-r(T-t)} g\left(\tilde{S}_1(T), \tilde{S}_2(T)\right)\right]. \end{aligned}$$

In the next step we condition the difference of the regularised and the unregularised prices on the paths of the Brownian motions of the eigenvalues follow, i.e.

$$\begin{aligned} C^{\tilde{\epsilon},\delta} - C^{\delta} = \mathbb{E}_{\tilde{\mathcal{Q}}}\Big[\mathbb{E}\Big[& e^{-r(T-t+\tilde{\epsilon})} g\left(\tilde{S}_1\left(T+\tilde{\epsilon}\right), \tilde{S}_2(T+\tilde{\epsilon})\right) \\ & -e^{-r(T-t)} g\left(\tilde{S}_1(T), \tilde{S}_2(T)\right) \Big| Z_j(s)_{t \leq s \leq T},\ j = 1, 2\Big]\Big]. \end{aligned} \tag{4.134}$$

To calculate the expectations we determine the conditional joint distribution of $\tilde{S}_1$ and $\tilde{S}_2$. We start with the expectation in T and obtain the following diffusions for $\ln(\tilde{S}_1)$ and $\ln(\tilde{S}_2)$:

$$\begin{aligned} d\left(\ln(\tilde{S}_1)\right) &= rdt - \frac{1}{2}\sum_{j=1}^{2} a_{1j}^2 f(\tilde{v}_j(t))^2(1-\rho_j^{v2})dt - \frac{1}{2}\sum_{i=1}^{2} a_{1j}^2 f(\tilde{v}_j(t))^2 \rho_j^{v2} dt \\ &\quad + \sum_{j=1}^{2} a_{1j} f(\tilde{v}_j(t))\sqrt{1-\rho_j^{v2}} d\hat{W}_j + \sum_{j=1}^{2} a_{1j} f(\tilde{v}_j(t))\rho_j^v dZ_j, \end{aligned}$$

$$\begin{aligned} d\left(\ln(\tilde{S}_2)\right) &= rdt - \frac{1}{2}\sum_{j=1}^{2} a_{2j}^2 f(\tilde{v}_j(t))^2(1-\rho_j^{v2})dt - \frac{1}{2}\sum_{j=1}^{2} a_{2j}^2 f(\tilde{v}_j(t))^2 \rho_j^{v2} dt \\ &\quad + \sum_{j=1}^{2} a_{2j} f(\tilde{v}_j(t))\sqrt{1-\rho_j^{v2}} d\hat{W}_j + \sum_{j=1}^{2} a_{2j} f(\tilde{v}_j(t))\rho_j^v dZ_j, \end{aligned}$$

or

$$\begin{aligned}
\tilde{S}_1(T) &= \tilde{S}_1(t)e^{\Lambda_1}\exp\left\{r(T-t)-\frac{1}{2}\int_t^T\sum_{j=1}^{2}a_{1j}^2 f(\tilde{v}_j(s))^2(1-\rho_j^{v2})ds\right.\\
&\quad\left.+\sum_{j=1}^{2}\int_t^T a_{1j}f(\tilde{v}_j(s))\sqrt{1-\rho_j^{v2}}d\hat{W}_j\right\},\\
\tilde{S}_2(T) &= \tilde{S}_2(t)e^{\Lambda_2}\exp\left\{r(T-t)-\frac{1}{2}\int_t^T\sum_{j=1}^{2}a_{2j}^2 f(\tilde{v}_1(s))^2(1-\rho_j^{v2})ds\right.\\
&\quad\left.+\sum_{j=1}^{2}\int_t^T a_{2j}f(\tilde{v}_j(s))\sqrt{1-\rho_j^{v2}}d\hat{W}_j\right\},
\end{aligned}$$

where

$$\Lambda_i = \int_t^T\sum_{j=1}^{2}a_{ij}f(\tilde{v}_j(s))\rho_j^v dZ_j - \frac{1}{2}\int_t^T\sum_{j=1}^{2}a_{ij}^2 f(\tilde{v}_j(s))^2\rho_j^{v2}ds. \tag{4.135}$$

$v_j,\ j=1,2$, i.e. $\tilde{v}_j,\ j=1,2,t\le T$ are independent from $\left(\hat{W}_j(s)\right)_{0\le s\le T}$ under $\tilde{\mathcal{Q}}$ as we have reformulated the processes with independent Brownian motions $\left(\hat{W}_j(s)\right)_{0\le s\le T}$ and $(Z_j(s))_{0\le s\le T}$ (see (4.132)). The conditional distribution of $\tilde{S}_1(T)$ ($\tilde{S}_2(T)$ respectively) is, thus,

$$\ln\left(\frac{\tilde{S}_1(T)}{\tilde{S}_1(t)}\right)|Z_j(s)_{t\le s\le T}\sim\mathcal{N}\left(r(T-t)+\Lambda_1-\frac{1}{2}\hat{\sigma}_1^2,\hat{\sigma}_1\right), \tag{4.136}$$

with

$$\begin{aligned}
\hat{\sigma}_i^2 &= \hat{\sigma}_i^2(t,T),\\
&= \int_t^T\left(\left(1-\rho^{v2}_1\right)a_{i1}^2 f(\tilde{v}_1(s))^2+\left(1-\rho^{v2}_2\right)a_{i2}^2 f(\tilde{v}_2(s))^2\right)ds. \qquad (4.137)
\end{aligned}$$

To summarise we obtain the following conditional joint distribution $F\left(\ln\left(\tilde{S}_1\right),\ln\left(\tilde{S}_2\right)|Z_j(s)_{t\le s\le T},\ j=1,2\right)$ for $\ln(\tilde{S}_1)$ and $\ln(\tilde{S}_2)$ (for a single underlying see [103])

$$F\left(\ln\left(\tilde{S}_1(T)\right),\ln\left(\tilde{S}_2(T)\right)|Z_j(s)_{t\le s\le T},\ j=1,2\right)=\mathcal{N}_2\left(m_1,m_2,\hat{\sigma}_1^2,\hat{\sigma}_2^2,\hat{\rho}\right), \tag{4.138}$$

where

$$m_i = \ln\left(S_i(0)\right)+\Lambda_i+r(T-t)-\frac{1}{2}\hat{\sigma}_i^2, \tag{4.139}$$

$$\hat{\rho} = \frac{\int_t^T(a_{11}a_{21}(1-\rho_1^{v2})f(\tilde{v}_1(s))^2+a_{12}a_{22}(1-\rho_2^{v2})f(\tilde{v}_2(s))^2)ds}{\hat{\sigma}_1\hat{\sigma}_2}. \tag{4.140}$$

It follows that

$$\begin{aligned}&\mathbb{E}\left[e^{-r(T-t)}g(\tilde{S}_1(T),\tilde{S}_2(T))\big|Z_j(s)_{t\leq s\leq T},\ j=1,2\right]\\&=C_{BS}\left(t,S_1e^{\Lambda_1},S_2e^{\Lambda_2},K_1,K_2,T,\frac{\hat{\sigma}_1}{\sqrt{T-t}},\frac{\hat{\sigma}_2}{\sqrt{T-t}},\hat{\rho}\right).\end{aligned} \tag{4.141}$$

Accordingly, we get

$$\begin{aligned}&F\left(\ln\left(\tilde{S}_1\left(T+\tilde{\epsilon}\right)\right),\ln\left(\tilde{S}_2\left(T+\tilde{\epsilon}\right)\right)\big|Z_j(s)_{t\leq s\leq T},\ j=1,2\right)\\&\qquad=\mathcal{N}_2\left(\tilde{m}_1,\tilde{m}_2,\tilde{\sigma}_1^2,\tilde{\sigma}_2^2,\tilde{\rho}\right),\end{aligned} \tag{4.142}$$

where

$$\tilde{m}_i = \ln\left(S_i(0)\right)+\Lambda_i+r(T-t)+r\tilde{\epsilon}-\frac{1}{2}\tilde{\sigma}_i^2, \tag{4.143}$$

$$\begin{aligned}\tilde{\sigma}_i^2 &= \tilde{\sigma}_i(t,T+\tilde{\epsilon})^2\\&\overset{(4.133,4.137)}{=}\hat{\sigma}_i(t,T)^2+\int_T^{T+\epsilon}\left(a_{i1}^2\overline{f(v_1(s))^2}+a_{i2}^2\overline{f(v_2(s))^2}\right)\\&\overset{(4.64)}{=}\hat{\sigma}_i^2+\tilde{\epsilon}\overline{\sigma}_i^2,\end{aligned} \tag{4.144}$$

$$\begin{aligned}\tilde{\rho} &= \tilde{\rho}(t,T+\epsilon)\\&\overset{(4.140)}{=}\frac{\hat{\rho}(t,T)\hat{\sigma}_1(t,T)\hat{\sigma}_2(t,T)+\int_T^{T+\epsilon}(a_{11}a_{21}\overline{f(v_1(s))^2}+a_{12}a_{22}\overline{f(v_2(s))^2})ds}{\tilde{\sigma}_1(t,T+\tilde{\epsilon})\tilde{\sigma}_2(t,T+\tilde{\epsilon})}\\&\overset{(4.64)}{=}\frac{\hat{\rho}\hat{\sigma}_1\hat{\sigma}_2+\tilde{\epsilon}\overline{\rho}\ \overline{\sigma}_1\ \overline{\sigma}_2}{\tilde{\sigma}_1\tilde{\sigma}_2}.\end{aligned} \tag{4.145}$$

The conditional expectation can be computed by

$$\begin{aligned}&\mathbb{E}\left[e^{-r(T-t+\tilde{\epsilon})}g\left(\tilde{S}_1(T+\tilde{\epsilon}),\tilde{S}_2\left(T+\tilde{\epsilon}\right)\right)\big|Z_j(s)_{t\leq s\leq T},\ j=1,2\right]\\&=e^{-r\tilde{\epsilon}}C_{BS}^{\delta}\left(t,S_1e^{\Lambda_1+r\tilde{\epsilon}},S_2e^{\Lambda_2+r\tilde{\epsilon}},K_1,K_2,T,\frac{\tilde{\sigma}_1}{\sqrt{T-t}},\frac{\tilde{\sigma}_2}{\sqrt{T-t}},\tilde{\rho}\right).\end{aligned} \tag{4.146}$$

Using the explicit formulas and the assumption that $\hat{\sigma}_i,\ i=1,2$, is bounded above and below (for detailed calculations see (B.2)), we derive that

$$\begin{aligned}&\left|e^{-r\tilde{\epsilon}}C_{BS}\left(t,S_1e^{\Lambda_1+r\tilde{\epsilon}},S_2e^{\Lambda_2+r\tilde{\epsilon}},\tilde{\sigma}_1,\tilde{\sigma}_2,\tilde{\rho}\right)-C_{BS}^{\tilde{\epsilon},\delta}\left(t,S_1e^{\Lambda_1},S_2e^{\Lambda_2},\hat{\sigma}_1,\hat{\sigma}_2,\hat{\rho}\right)\right|\\&\qquad\leq\epsilon\tilde{c}_4\left(1+|\Lambda_1|+|\Lambda_2|+|\Lambda_1|\,|\Lambda_2|\right)\left(e^{\Lambda_2}+1\right).\end{aligned} \tag{4.147}$$

$\mathcal{E}(T):=e^{\Lambda_i}$ is the stochastic exponential of $\tilde{\gamma}(T)=\int_t^T\sum_{j=1}^{\overline{p}}\rho_j^v f(v_j)a_{ij}$ because $e^{\Lambda_i}=e^{\tilde{\gamma}(T)-\frac{1}{2}\langle\tilde{\gamma},\tilde{\gamma}\rangle}$ and $d\mathcal{E}=\mathcal{E}d\tilde{\gamma}$. The expectation of the exponential martingale exists if the Novikov condition is fulfilled, i.e. if $\mathbb{E}\left[e^{\frac{1}{2}\int_t^T||\sqrt{v_j}||^2ds}\right]<\infty$, which is the case for the CIR process (see [29]). We find

$$\left|C^{\delta}(t,S_1,S_2,v)-C^{\tilde{\epsilon},\delta}(t,S_1,S_2,v)\right|\leq \mathfrak{c}_1\tilde{\epsilon}, \tag{4.148}$$

for $\mathfrak{c}_1$ and $\tilde{\epsilon}$ small enough. □

Lemma 3. *($Q^{\tilde{\epsilon},\delta}\sim Q^{\delta}$)*

For the fixed point (t,S_1,S_2,v), where $t<T$ and $\mathbf{v}=(v_1,v_2)$, there exist constants $\overline{\delta}_1^2>0$, $\overline{\delta}_2^2>0$, $\bar{\bar{\epsilon}}_2>0$, and $\mathfrak{c}_2>0$ such that

$$\left|Q^{\delta}\left(t,S_1,S_2,\mathbf{v}\right)-Q^{\tilde{\epsilon},\delta}\left(t,S_1,S_2,\mathbf{v}\right)\right|\leq \mathfrak{c}_2\tilde{\epsilon}, \tag{4.149}$$

for all $0<\delta_1<\overline{\delta}_1^2$, $0<\delta_2<\overline{\delta}_2^2$, and $0<\tilde{\epsilon}<\bar{\bar{\epsilon}}_2$.

Proof.

In the following we show that the approximation $Q^{\tilde{\epsilon},\delta}$ of the regularised payoff is close to Q^{δ}, the approximation of the unregularised payoff. From the definitions of the correction terms $C_{1,0}$ (see (4.105)), $C_{0,1}$ (see (4.84)), and $C_{1,1}$ (see (4.116)) it follows that

$$Q^{\tilde{\epsilon},\delta}-Q^{\delta}=\left(1-(T-t)\left(\delta_1\mathcal{A}_1+\delta_2\mathcal{A}_2\right)+(T-t)^2\delta_1\delta_2\mathcal{A}_1\mathcal{A}_2\right)\left(C_{0,0}^{\tilde{\epsilon}}-C_{0,0}\right).$$

From the definition of $\mathcal{A}_1$ (see (4.104)) and $\mathcal{A}_2$ (see (4.82)) one can see that they are bounded due to the boundedness of the solutions of the Poisson equations in v_1 and v_2 (see (B.3)).

As $C_{0,0}^{\tilde{\epsilon}}(t,T,S_1,S_2)=C_{0,0}(t,T+\tilde{\epsilon},S_1,S_2)$ and $C_{0,0}$ as well as its successive derivatives with respect to S_1 and S_2 are differentiable in t at any $t<T$ we deduce for (t,S_1,S_2,v) with fixed $t<T$:

$$|Q^{\delta}-Q^{\delta,\tilde{\epsilon}}|\leq \mathfrak{c}_2\tilde{\epsilon}, \tag{4.150}$$

for some $\mathfrak{c}_2>0$ and $\tilde{\epsilon}$ small enough. □

Lemma 4. *($C^{\tilde{\epsilon},\delta}\sim Q^{\tilde{\epsilon},\delta}$)*

For the fixed point (t,S_1,S_2,v), where $t<T$ and $\mathbf{v}=(v_1,v_2)$, there exist constants $\overline{\delta}_1^3>0$, $\overline{\delta}_2^3>0$, $\bar{\bar{\epsilon}}_2>0$, and $\mathfrak{c}_3>0$ such that

$$\begin{aligned}\left|C^{\tilde{\epsilon},\delta}(t,S_1,S_2,v)-Q^{\tilde{\epsilon},\delta}(t,S_1,S_2,v)\right| \leq\ & \mathfrak{c}_3\Big(\delta_1^2\left(1+\tilde{\epsilon}^{-\frac{1}{2}}+\delta_1\tilde{\epsilon}^{-1}\right)\\ &+\delta_2^2\left(1+\tilde{\epsilon}^{-\frac{1}{2}}+\delta_2\tilde{\epsilon}^{-1}\right)\\ &\delta_1\delta_2\left(1+\tilde{\epsilon}^{-\frac{1}{2}}+\delta_1\tilde{\epsilon}^{-1}+\delta_2\tilde{\epsilon}^{-1}\right)\Big),\end{aligned} \tag{4.151}$$

for all $0 < \delta_1 < \overline{\delta}_1^3$, $0 < \delta_2 < \overline{\delta}_2^3$, *and* $0 < \tilde{\epsilon} < \overline{\tilde{\epsilon}}_3$.

Proof.

In the following we show that the regularised approximation $Q^{\tilde{\epsilon},\delta}$ converges to the actual regularised price $C^{\tilde{\epsilon},\delta}$ by finding a boundary for the absolute value of the error in the approximation, $R^{\tilde{\epsilon},\delta}$. We first introduce $\hat{C}^{\tilde{\epsilon},\delta}$, which we define by

$$\hat{C}^{\tilde{\epsilon},\delta} = Q^{\tilde{\epsilon},\delta} + \delta_1^2 C_{2,0}^{\tilde{\epsilon}} + \delta_2^2 C_{0,2}^{\tilde{\epsilon}} + \delta_1^2\delta_2 C_{2,1}^{\tilde{\epsilon}} + \delta_1\delta_2^2 C_{1,2}^{\tilde{\epsilon}} + \delta_1^3 C_{3,0}^{\tilde{\epsilon}} + \delta_2^3 C_{0,3}^{\tilde{\epsilon}}, \tag{4.152}$$

and, thus, define the error in the approximation for the regularised problem as

$$R^{\tilde{\epsilon},\delta} = \hat{C}^{\tilde{\epsilon},\delta} - C^{\tilde{\epsilon},\delta}. \tag{4.153}$$

By applying $\mathcal{L}^\delta$ on $R^{\tilde{\epsilon},\delta}$ and forming terms of equal power in δ_1 and δ_2 we see that

$$\begin{aligned}
\mathcal{L}^\delta R^{\tilde{\epsilon},\delta} &= \mathcal{L}^\delta(\hat{C}^{\tilde{\epsilon},\delta} - C^{\tilde{\epsilon},\delta}) \\
&= \frac{1}{\delta_1^2}\mathcal{L}_0^1 C_{0,0}^{\tilde{\epsilon}} + \frac{1}{\delta_1}\left(\mathcal{L}_0^1 C_{1,0}^{\tilde{\epsilon}} + \mathcal{L}_1^1 C_{0,0}^{\tilde{\epsilon}}\right) \\
&\quad + \frac{1}{\delta_2^2}\mathcal{L}_0^2 C_{0,0}^{\tilde{\epsilon}} + \frac{1}{\delta_2}\left(\mathcal{L}_0^2 C_{0,1}^{\tilde{\epsilon}} + \mathcal{L}_1^2 C_{0,0}^{\tilde{\epsilon}}\right) \\
&\quad + \left(\mathcal{L}_0^2 C_{0,2}^{\tilde{\epsilon}} + \mathcal{L}_1^2 C_{0,1}^{\tilde{\epsilon}} + \mathcal{L}_2 C_{0,0}^{\tilde{\epsilon}} + \mathcal{L}_0^1 C_{2,0}^{\tilde{\epsilon}}\right) \\
&\quad + \delta_2\left(\mathcal{L}_0^2 C_{0,3}^{\tilde{\epsilon}} + \mathcal{L}_1^2 C_{0,2}^{\tilde{\epsilon}} + \mathcal{L}_2 C_{0,1}^{\tilde{\epsilon}} + \mathcal{L}_0^1 C_{2,1}^{\tilde{\epsilon}}\right) \\
&\quad + \frac{\delta_1}{\delta_2^2}\mathcal{L}_0^2 C_{1,0}^{\tilde{\epsilon}} + \frac{\delta_1}{\delta_2}(\mathcal{L}_0^2 C_{1,1}^{\tilde{\epsilon}} + \mathcal{L}_1^2 C_{1,0}^{\tilde{\epsilon}}) \\
&\quad + \delta_1\left(\mathcal{L}_0^2 C_{1,2}^{\tilde{\epsilon}} + \mathcal{L}_1^2 C_{1,1}^{\tilde{\epsilon}} + \mathcal{L}_2 C_{1,0}^{\tilde{\epsilon}} + \mathcal{L}_0^1 C_{2,0}^{\tilde{\epsilon}} + \mathcal{L}_1^1 C_{2,0}^{\tilde{\epsilon}}\right) \\
&\quad + \frac{\delta_2}{\delta_1^2}\mathcal{L}_0^1 C_{0,1}^{\tilde{\epsilon}} + \frac{\delta_2}{\delta_1}\left(\mathcal{L}_0^1 C_{1,1}^{\tilde{\epsilon}} + \mathcal{L}_1^1 C_{0,1}^{\tilde{\epsilon}}\right) \\
&\quad + \delta_1\delta_2\left(\mathcal{L}_1^2 C_{1,2}^{\tilde{\epsilon}} + \mathcal{L}_2 C_{1,1}^{\tilde{\epsilon}} + \mathcal{L}_1^1 C_{2,1}^{\tilde{\epsilon}}\right) \\
&\quad + \delta_1^2\left(\mathcal{L}_2 C_{2,0}^{\tilde{\epsilon}} + \delta_2\mathcal{L}_2 C_{2,1}^{\tilde{\epsilon}} + \mathcal{L}_1^1 C_{3,0}^{\tilde{\epsilon}} + \delta_1\mathcal{L}_2 C_{3,0}^{\tilde{\epsilon}}\right) \\
&\quad + \delta_2^2\left(\mathcal{L}_2 C_{0,2}^{\tilde{\epsilon}} + \delta_1\mathcal{L}_2 C_{1,2}^{\tilde{\epsilon}} + \mathcal{L}_1^2 C_{0,3}^{\tilde{\epsilon}} + \delta_2\mathcal{L}_2 C_{0,3}^{\tilde{\epsilon}}\right) \\
&= \delta_1\delta_2 F_1^{\tilde{\epsilon}}(t, S_1, S_2, \mathbf{v}) + \delta_1^2 F_2^{\tilde{\epsilon}}(t, S_1, S_2, \mathbf{v}) \\
&\quad + \delta_2^2 F_3^{\tilde{\epsilon}}(t, S_1, S_2, \mathbf{v}),
\end{aligned} \tag{4.154}$$

because $\mathcal{L}^\delta C^{\tilde{\epsilon},\delta} = 0$ and the first approximations are chosen to cancel the first brackets (see (4.51)-(4.54) and (4.57)-(4.60)). We define

$$\begin{aligned}
F_1^{\tilde{\epsilon}}(t, S_1, S_2, \mathbf{v}) &= \mathcal{L}_1^2 C_{1,2}^{\tilde{\epsilon}} + \mathcal{L}_2 C_{1,1}^{\tilde{\epsilon}} + \mathcal{L}_1^1 C_{2,1}^{\tilde{\epsilon}} + \delta_1\mathcal{L}_2 C_{2,1}^{\tilde{\epsilon}} + \delta_2\mathcal{L}_2 C_{1,2}^{\tilde{\epsilon}}, \\
F_2^{\tilde{\epsilon}}(t, S_1, S_2, \mathbf{v}) &= \mathcal{L}_2 C_{2,0}^{\tilde{\epsilon}} + \mathcal{L}_1^1 C_{3,0}^{\tilde{\epsilon}} + \delta_1\mathcal{L}_2 C_{3,0}^{\tilde{\epsilon}}, \\
F_3^{\tilde{\epsilon}}(t, S_1, S_2, \mathbf{v}) &= \mathcal{L}_2 C_{0,2}^{\tilde{\epsilon}} + \mathcal{L}_1^2 C_{0,3}^{\tilde{\epsilon}} + \delta_2\mathcal{L}_2 C_{0,3}^{\tilde{\epsilon}}.
\end{aligned} \tag{4.155}$$

The derivation of $C_{3,0}$ and $C_{0,3}$ is shown in the Appendix B.2. From Equations (4.77, 4.78, 4.113, 4.114) we moreover deduce

$$\begin{aligned}
C_{0,2}^{\tilde{\epsilon}} &= -\frac{1}{2}\phi_2\left(a_{12}^2S_1^2\frac{\partial^2}{\partial S_1^2}+a_{22}^2S_2^2\frac{\partial^2}{\partial S_2^2}+2a_{12}a_{22}S_1S_2\frac{\partial^2}{\partial S_1\partial S_2}\right)C_{0,0}^{\tilde{\epsilon}}\\
&\quad+f_{0,2}{}^{e,\tilde{\epsilon}}\left(t,S_1,S_2\right),\\
C_{2,0}^{\tilde{\epsilon}} &= -\frac{1}{2}\phi_1\left(a_{11}^2S_1^2\frac{\partial^2}{\partial S_1^2}+a_{21}^2S_2^2\frac{\partial^2}{\partial S_2^2}+2a_{11}a_{21}S_1S_2\frac{\partial^2}{\partial S_1\partial S_2}\right)C_{0,0}^{\tilde{\epsilon}}\\
&\quad+f_{2,0}{}^{e,\tilde{\epsilon}}\left(t,S_1,S_2\right),
\end{aligned}$$

$$\begin{aligned}
C_{1,2}^{\tilde{\epsilon}} &= \frac{1}{2}(T-t)\phi_2\left(a_{12}^2S_1^2\frac{\partial^2}{\partial S_1^2}+a_{22}^2S_2^2\frac{\partial^2}{\partial S_2^2}+2a_{12}a_{22}S_1S_2\frac{\partial^2}{\partial S_1\partial S_2}\right)\mathcal{A}_1C_{0,0}^{\tilde{\epsilon}}\\
&\quad+f_{1,2}{}^{e,\tilde{\epsilon}}\left(t,S_1,S_2\right),\\
C_{2,1}^{\tilde{\epsilon}} &= \frac{1}{2}(T-t)\phi_1\left(a_{11}^2S_1^2\frac{\partial^2}{\partial S_1^2}+a_{21}^2S_2^2\frac{\partial^2}{\partial S_2^2}+2a_{11}a_{21}S_1S_2\frac{\partial^2}{\partial S_1\partial S_2}\right)\mathcal{A}_2C_{0,0}^{\tilde{\epsilon}}\\
&\quad+f_{2,1}{}^{e,\tilde{\epsilon}}\left(t,S_1,S_2\right),\\
C_{0,3}^{\tilde{\epsilon}} &= \frac{1}{2}(T-t)\phi_2\left(a_{12}^2S_1^2\frac{\partial^2}{\partial S_1^2}+a_{22}^2S_2^2\frac{\partial^2}{\partial S_2^2}+2a_{12}a_{22}S_1S_2\frac{\partial^2}{\partial S_1\partial S_2}\right)\mathcal{A}_2C_{0,0}^{\tilde{\epsilon}}\\
&\quad+\rho_2\sigma_{2v}\xi_2\left(S_1\frac{a_{12}}{2}\frac{\partial}{\partial S_1}+S_2\frac{a_{22}}{2}\frac{\partial}{\partial S_2}\right)\\
&\quad\left(a_{12}^2S_1^2\frac{\partial^2}{\partial S_1^2}+a_{22}^2S_2^2\frac{\partial^2}{\partial S_2^2}+2a_{12}a_{22}S_1S_2\frac{\partial^2}{\partial S_1\partial S_2}\right)C_{0,0}^{\tilde{\epsilon}}\\
&\quad+f_{0,3}{}^{e,\tilde{\epsilon}}\left(t,S_1,S_2\right),\\
C_{3,0}^{\tilde{\epsilon}} &= \frac{1}{2}(T-t)\phi_1\left(a_{11}^2S_1^2\frac{\partial^2}{\partial S_1^2}+a_{21}^2S_2^2\frac{\partial^2}{\partial S_2^2}+2a_{11}a_{21}S_1S_2\frac{\partial^2}{\partial S_1\partial S_2}\right)\mathcal{A}_1C_{0,0}^{\tilde{\epsilon}}\\
&\quad+\rho_1\sigma_{1v}\xi_1\left(S_1\frac{a_{11}}{2}\frac{\partial}{\partial S_1}+S_2\frac{a_{21}}{2}\frac{\partial}{\partial S_2}\right)\\
&\quad\left(a_{11}^2S_1^2\frac{\partial^2}{\partial S_1^2}+a_{21}^2S_2^2\frac{\partial^2}{\partial S_2^2}+2a_{11}a_{21}S_1S_2\frac{\partial^2}{\partial S_1\partial S_2}\right)C_{0,0}^{\tilde{\epsilon}}\\
&\quad+f_{3,0}{}^{e,\tilde{\epsilon}}\left(t,S_1,S_2\right), \qquad (4.156)
\end{aligned}$$

where

$$\begin{aligned}
(\mathcal{L}_0^1+\mathcal{L}_0^2)\xi &= \sum_{i=1}^{2}\left(\sqrt{v_i}f(v_i)\frac{\partial\phi_i}{\partial v_i}-\left\langle\sqrt{v_i}f(v_i)\frac{\partial\phi_i}{\partial v_i}\right\rangle_{v_i}\right), \qquad (4.157)\\
\mathcal{L}_0^1\xi_1 &= \sqrt{v_1}f(v_1)\frac{\partial\phi_1}{\partial v_1}-\left\langle\sqrt{v_1}f(v_1)\frac{\partial\phi_1}{\partial v_1}\right\rangle_{v_1},\\
\mathcal{L}_0^2\xi_2 &= \sqrt{v_2}f(v_2)\frac{\partial\phi_2}{\partial v_2}-\left\langle\sqrt{v_2}f(v_2)\frac{\partial\phi_2 f}{\partial v_2}\right\rangle_{v_2}.
\end{aligned}$$

and f_i^e are other eigenfunctions which are not dependent on $\mathbf{v}$.

Furthermore, we see that the value of $R^{\tilde{\epsilon},\delta}$ at maturity T is given by

$$\begin{aligned} R^{\tilde{\epsilon},\delta}(T,S_1,S_2,\mathbf{v}) &= \delta_1\delta_2 G_1^{\tilde{\epsilon}}(T,S_1,S_2,\mathbf{v}) + \delta_1^2 G_2^{\tilde{\epsilon}}(T,S_1,S_2,\mathbf{v}) \\ &\quad +\delta_2^2 G_3^{\tilde{\epsilon}}(T,S_1,S_2,\mathbf{v}), \end{aligned} \tag{4.158}$$

where

$$\begin{aligned} G_1^{\tilde{\epsilon}}(T,S_1,S_2,\mathbf{v}) &= \delta_1 C_{2,1}^{\tilde{\epsilon}}(T,S_1,S_2) + \delta_2 C_{1,2}^{\tilde{\epsilon}}(T,S_1,S_2), \\ G_2^{\tilde{\epsilon}}(T,S_1,S_2,\mathbf{v}) &= C_{2,0}^{\tilde{\epsilon}}(T,S_1,S_2) + \delta_1 C_{3,0}^{\tilde{\epsilon}}(T,S_1,S_2), \\ G_3^{\tilde{\epsilon}}(T,S_1,S_2,\mathbf{v}) &= C_{0,2}^{\tilde{\epsilon}}(T,S_1,S_2) + \delta_2 C_{0,3}^{\tilde{\epsilon}}(T,S_1,S_2). \end{aligned} \tag{4.159}$$

Here we have exploited the definition of $\hat{C}^{\tilde{\epsilon},\delta}$ (4.152), the terminal conditions $C^{\tilde{\epsilon},\delta}(T,S_1,S_2) = C_{0,0}^{\tilde{\epsilon}}(T,S_1,S_2) = g^{\tilde{\epsilon}}(S_1,S_2)$, $C_{1,0}^{\tilde{\epsilon}}(T,S_1,S_2) = C_{0,1}^{\tilde{\epsilon}}(T,S_1,S_2) = C_{1,1}^{\tilde{\epsilon}}(T,S_1,S_2) = 0$. Using (4.156) we find that $G_i^{\tilde{\epsilon}}$ can be written in the form

$$\begin{aligned} G_1^{\tilde{\epsilon}}(T,S_1,S_2) &= \tilde{q}_1(T,S_1,S_2), \\ G_2^{\tilde{\epsilon}}(T,S_1,S_2,\mathbf{v}) &= \sum_{i,k;i+k=2} q_{2,ik}^1 S_1^i S_2^k \frac{\partial^{i+k} C_{0,0}^{\tilde{\epsilon}}}{\partial S_1^i \partial S_2^k} + \delta_1 \sum_{i,k;i+k=3} q_{2,ik}^2 S_1^i S_2^k \frac{\partial^{i+k} C_{0,0}^{\tilde{\epsilon}}}{\partial S_1^i \partial S_2^k} \\ &\quad +\tilde{q}_2(T,S_1,S_2), \\ G_3^{\tilde{\epsilon}}(T,S_1,S_2,\mathbf{v}) &= \sum_{i,k;i+k=2} q_{3,ik}^1 S_1^i S_2^k \frac{\partial^{i+k} C_{0,0}^{\tilde{\epsilon}}}{\partial S_1^i \partial S_2^k} + \delta_2 \sum_{i,k;i+k=3} q_{3,ik}^2 S_1^i S_2^k \frac{\partial^{i+k} C_{0,0}^{\tilde{\epsilon}}}{\partial S_1^i \partial S_2^k} \\ &\quad +\tilde{q}_3(T,S_1,S_2), \end{aligned}$$

where $\tilde{q}_i(t,S_1,S_2)$ are functions which depend on the eigenfunctions. $\tilde{q}_{1,ik}^1$ ($\tilde{q}_{2,ik}^1$) is a function that depends on ϕ_1 (ϕ_2 respectively). Whereas, $\tilde{q}_{1,ik}^2$ ($\tilde{q}_{2,ik}^2$) is a function that depends on ξ_1 (ξ_2 respectively).

$F_i^{\tilde{\epsilon}}$ can be written in the form (see Appendix B.2 for detailed calculations):

$$\begin{aligned}
F_1^{\tilde{\epsilon}}(t,S_1,S_2,\mathbf{v}) &= (T-t)^2 \sum_{i,k;i+k=8} \tilde{r}^1_{1,ik} S_1^i S_2^k \frac{\partial^{i+k} C_{0,0}^{\tilde{\epsilon}}}{\partial S_1^i \partial S_2^k} \\
&+(T-t) \sum_{i,k;i+k=6} \tilde{r}^2_{1,ik} S_1^i S_2^k \frac{\partial^{i+k} C_{0,0}^{\tilde{\epsilon}}}{\partial S_1^i \partial S_2^k} \\
&+\delta_1 \Bigg(\sum_{i,k;i+k=5} \tilde{r}^3_{1,ik} S_1^i S_2^k \frac{\partial^{i+k} C_{0,0}^{\tilde{\epsilon}}}{\partial S_1^i \partial S_2^k} \\
&+(T-t) \sum_{i,k;i+k=7} \tilde{r}^4_{1,ik} S_1^i S_2^k \frac{\partial^{i+k} C_{0,0}^{\tilde{\epsilon}}}{\partial S_1^i \partial S_2^k} \Bigg) \\
&+\delta_2 \Bigg(\sum_{i,k;i+k=5} \tilde{r}^5_{1,ik} S_1^i S_2^k \frac{\partial^{i+k} C_{0,0}^{\tilde{\epsilon}}}{\partial S_1^i \partial S_2^k} \\
&+(T-t) \sum_{i,k;i+k=7} \tilde{r}^6_{1,ik} S_1^i S_2^k \frac{\partial^{i+k} C_{0,0}^{\tilde{\epsilon}}}{\partial S_1^i \partial S_2^k} \Bigg) + \tilde{q}_4(t,S_1,S_2),
\end{aligned}$$

$$\begin{aligned}
F_2^{\tilde{\epsilon}}(t,S_1,S_2,\mathbf{v}) &= (T-t) \sum_{i,k;i+k=6} \tilde{r}^1_{2,ik} S_1^i S_2^k \frac{\partial^{i+k} C_{0,0}^{\tilde{\epsilon}}}{\partial S_1^i \partial S_2^k} + \sum_{i,k;i+k=4} \tilde{r}^2_{2,ik} S_1^i S_2^k \frac{\partial^{i+k} C_{0,0}^{\tilde{\epsilon}}}{\partial S_1^i \partial S_2^k} \\
&+\delta_1 \Bigg((T-t) \sum_{i,k;i+k=7} \tilde{r}^3_{2,ik} S_1^i S_2^k \frac{\partial^{i+k} C_{0,0}^{\tilde{\epsilon}}}{\partial S_1^i \partial S_2^k} \\
&+ \sum_{i,k;i+k=5} \tilde{r}^4_{2,ik} S_1^i S_2^k \frac{\partial^{i+k} C_{0,0}^{\tilde{\epsilon}}}{\partial S_1^i \partial S_2^k} \Bigg) + \tilde{q}_5(t,S_1,S_2), \\
F_3^{\tilde{\epsilon}}(t,S_1,S_2,\mathbf{v}) &= (T-t) \sum_{i,k;i+k=6} \tilde{r}^1_{3,ik} S_1^i S_2^k \frac{\partial^{i+k} C_{0,0}^{\tilde{\epsilon}}}{\partial S_1^i \partial S_2^k} + \sum_{i,k;i+k=4} \tilde{r}^2_{3,ik} S_1^i S_2^k \frac{\partial^{i+k} C_{0,0}^{\tilde{\epsilon}}}{\partial S_1^i \partial S_2^k} \\
&+\delta_2 \Bigg((T-t) \sum_{i,k;i+k=7} \tilde{r}^3_{3,ik} S_1^i S_2^k \frac{\partial^{i+k} C_{0,0}^{\tilde{\epsilon}}}{\partial S_1^i \partial S_2^k} \\
&+ \sum_{i,k;i+k=5} \tilde{r}^4_{3,ik} S_1^i S_2^k \frac{\partial^{i+k} C_{0,0}^{\tilde{\epsilon}}}{\partial S_1^i \partial S_2^k} \Bigg) + \tilde{q}_6(t,S_1,S_2),
\end{aligned}$$

where $\tilde{r}_i^j$ defines functions which either depend on ϕ or ξ.

First, let us analyse those terms $\tilde{r}_i^j$ more closely with respect to their boundedness. Let $\psi = \tilde{r}_i^j$ or $\psi = \tilde{q}_i^j$ with the functions $\tilde{r}_i^j$ and $\tilde{q}_i^j$ being defined above.

Then

$$\|\psi(\mathbf{v})\| \leq \tilde{c}_5 \max(\|\phi(\mathbf{v})\|, \|\phi'(\mathbf{v})\|, \|\xi(\mathbf{v})\|, \|\xi'(\mathbf{v})\|), \tag{4.160}$$

for some constant $\tilde{c}_5$ and with $\phi(\mathbf{v})$ and $\xi(\mathbf{v})$ defined in (4.76) and (4.157), i.e. ϕ and ξ are the solutions of Poisson equations with $h = f(\mathbf{v})^2 - \overline{f(\mathbf{v})^2}$ or $h = \sqrt{\mathbf{v}} f(\mathbf{v}) \frac{\partial \phi}{\partial \mathbf{v}} - \langle \sqrt{\mathbf{v}} f(\mathbf{v}) \frac{\partial \phi}{\partial \mathbf{v}} \rangle$. Due to the results in Appendix B.3 ψ is at most logarithmically growing in $\mathbf{v}$ at infinity (see Appendix B.3). As the moments of the process of $\mathbf{v}$ are uniformly bounded in $\boldsymbol{\delta} = (\delta_1, \delta_2)$ (see Section 2.4.1) we can conclude that there exists a constant $\tilde{c}_6 > 0$, which may depend on v, such that

$$\mathbb{E}\left[|\psi(\mathbf{v}(s))|\,|\mathbf{v}(t)\right] \leq \tilde{c}_6 \leq \infty \text{ for } t \leq s \leq T. \tag{4.161}$$

Next, let us deal with the derivatives in S_i and analyse their boundedness. It can be derived that (see Appendix B.1)

$$\frac{\partial^{i+k} C_{0,0}^{\tilde{\epsilon}}}{\partial x_1^i \partial x_2^k} = \begin{cases} e^{x_2} K_2 e^{-r\tau^{\tilde{\epsilon}}} \mathcal{N}_2(\mathbf{d}_2^{\tilde{\epsilon}}, \mathbf{d}_1^{\tilde{\epsilon}}, \overline{\rho}) & \text{for } i = 0 \wedge k = 1, \\[2ex] e^{x_2} K_2 e^{-r\tau^{\tilde{\epsilon}}} \mathcal{N}_2(\mathbf{d}_2^{\tilde{\epsilon}}, \mathbf{d}_1^{\tilde{\epsilon}}, \overline{\rho}) \\ + \sum_{\mathfrak{k}=0}^{k-2} e^{x_2} \int_{-\infty}^{\mathbf{d}_1^{\tilde{\epsilon}}} \frac{\tilde{b}_{\mathfrak{k}}^1}{\sqrt{\tau^{\tilde{\epsilon}}}} \frac{\partial^{\mathfrak{k}}}{\partial x_2^{\mathfrak{k}}} e^{\left(-\frac{1}{2(1-\overline{\rho}^2)}(y_1^2 + {\mathbf{d}_2^{\tilde{\epsilon}}}^2 - 2\overline{\rho}\mathbf{d}_2^{\tilde{\epsilon}} y_1)\right)} dy_1 & \text{for } i = 0 \wedge k \geq 2, \\[2ex] e^{x_2} \int_{-\infty}^{\mathbf{d}_2^{\tilde{\epsilon}}} \frac{\tilde{b}_i^2}{\sqrt{\tau^{\tilde{\epsilon}}}} \frac{\partial^{i-1}}{\partial x_1^{i-1}} e^{\left(-\frac{1}{2(1-\overline{\rho}^2)}({\mathbf{d}_1^{\tilde{\epsilon}}}^2 + y_2^2 - 2\overline{\rho}\mathbf{d}_1^{\tilde{\epsilon}} y_2)\right)} dy_2 \\ -K e^{-r\tau} \int_{-\infty}^{{\mathbf{d}_2^{\tilde{\epsilon}}}^*} \frac{\tilde{b}_i^3}{\sqrt{\tau^{\tilde{\epsilon}}}} \frac{\partial^{i-1}}{\partial x_1^{i-1}} e^{\left(-\frac{1}{2(1-\overline{\rho}^2)}(y_2^2 + {\mathbf{d}_1^{\tilde{\epsilon}}}^{*2} - 2\overline{\rho} y_2 \mathbf{d}_1^{\tilde{\epsilon}*})\right)} dy_2 & \text{for } i \geq 1,\ k = 0, \\[2ex] e^{x_2} \int_{-\infty}^{\mathbf{d}_2^{\tilde{\epsilon}}} \frac{\tilde{b}_i^4}{\sqrt{\tau^{\tilde{\epsilon}}}} \frac{\partial^{i-1}}{\partial x_1^{i-1}} e^{\left(-\frac{1}{2(1-\overline{\rho}^2)}({\mathbf{d}_1^{\tilde{\epsilon}}}^2 + y_2^2 - 2\overline{\rho}\mathbf{d}_1^{\tilde{\epsilon}} y_2)\right)} dy_2 & \text{for } i \geq 1 \wedge k = 1, \\[2ex] e^{x_2} \int_{-\infty}^{\mathbf{d}_2^{\tilde{\epsilon}}} \frac{\tilde{b}_i^4}{\sqrt{\tau^{\tilde{\epsilon}}}} \frac{\partial^{i-1}}{\partial x_1^{i-1}} e^{\left(-\frac{1}{2(1-\overline{\rho}^2)}({\mathbf{d}_1^{\tilde{\epsilon}}}^2 + y_2^2 - 2\overline{\rho}\mathbf{d}_1^{\tilde{\epsilon}} y_2)\right)} dy_2 \\ + \sum_{\mathfrak{k}=0}^{k-2} e^{x_2} \frac{\tilde{b}_{\mathfrak{k}}^5}{\tau^{\tilde{\epsilon}}} \frac{\partial^{i+\mathfrak{k}-1}}{\partial x_1^{i-1} \partial x_2^{\mathfrak{k}}} e^{\left(-\frac{1}{2(1-\overline{\rho}^2)}({\mathbf{d}_1^{\tilde{\epsilon}}}^2 + {\mathbf{d}_2^{\tilde{\epsilon}}}^2 - 2\overline{\rho}\mathbf{d}_1^{\tilde{\epsilon}} \mathbf{d}_2^{\tilde{\epsilon}})\right)} & \text{for } i \geq 1 \wedge k \geq 2, \end{cases} \tag{4.162}$$

for some constants $\tilde{b}_i$ and $\tilde{b}_{\mathfrak{k}}$ and with

$$\begin{aligned} \mathbf{d}_2^{\tilde{\epsilon}} &= \frac{x_2}{\overline{\sigma}_2\sqrt{\tau^{\tilde{\epsilon}}}} + \tfrac{1}{2}\overline{\sigma}_2\sqrt{\tau^{\tilde{\epsilon}}}, & \mathbf{d}_2^{\tilde{\epsilon}*} &= \mathbf{d}_2^{\tilde{\epsilon}} - \overline{\sigma}_2\sqrt{\tau^{\tilde{\epsilon}}}, \\ \mathbf{d}_1^{\tilde{\epsilon}} &= \frac{x_1}{\overline{\sigma}_1\sqrt{\tau^{\tilde{\epsilon}}}} - \tfrac{1}{2}\overline{\sigma}_1\sqrt{\tau^{\tilde{\epsilon}}} + \overline{\rho}\overline{\sigma}_2\sqrt{\tau^{\tilde{\epsilon}}}, & \mathbf{d}_1^{\tilde{\epsilon}*} &= \mathbf{d}_1^{\tilde{\epsilon}} - \overline{\rho}\overline{\sigma}_2\sqrt{\tau^{\tilde{\epsilon}}}, \\ \tau^{\tilde{\epsilon}} &= T - s + \tilde{\epsilon}, & x_2 &= \ln \frac{S_2 e^{r\tau^{\tilde{\epsilon}}}}{K_2}. \end{aligned}$$

In the following we have to differentiate between options with short and long remaining maturity. For long maturities, i.e. $T - s \geq \frac{T-t}{2} > 0$, $\left|\mathbb{E}\left[\frac{\partial^{i+k} C_{0,0}^{\tilde{\epsilon}}}{\partial x_1^i \partial x_2^k}\right]\right| \leq \tilde{c}_7$, for some constant $\tilde{c}_7$, which depends on x_1 and x_2, as the derivatives are uniformly bounded in $\tilde{\epsilon}$.

For $0 < T - s < \frac{T-t}{2}$ we consider

$$\left|\mathbb{E}\left[\frac{\partial^{i+k} C^{\tilde{\epsilon}}_{0,0}}{\partial x_1^i \partial x_2^k}\right]\right| = \left|\mathbb{E}\left[\mathbb{E}\left[\frac{\partial^{i+k} C^{\tilde{\epsilon}}_{0,0}}{\partial x_1^i \partial x_2^k}\middle| Z_{j,t'}; t \leq t' \leq s\right]\right]\right|. \tag{4.163}$$

The conditional (conditioned on $Z_{j,t'}, j = 1, 2, t \leq t' \leq s$) distribution of $\ln S_1$ and $\ln S_2$ is Gaussian with means, variances and correlations as given in (4.139)-(4.140) with T replaced by s. The joint conditional density of $x_1 = \ln \frac{S_1 e^{\int_t^s r(t')dt'}}{K_1}$ and $x_2 = \ln \frac{S_2 e^{\int_t^s r(t')dt'}}{K_2}$ is denoted by $p(x_1, x_2)$ and is given by

$$\begin{aligned}
p(x_1, x_2) &= \frac{1}{2\pi\hat{\sigma}_1\hat{\sigma}_2\sqrt{1-\hat{\rho}^2}} e^{-\frac{1}{2(1-\hat{\rho})}\left(\frac{(x_1-\hat{m}_1)^2}{\hat{\sigma}_1} + \frac{(x_2-\hat{m}_2)^2}{\hat{\sigma}_2} - 2\hat{\rho}\frac{(x_1-\hat{m}_1)}{\hat{\sigma}_1}\frac{(x_2-\hat{m}_2)}{\hat{\sigma}_2}\right)}, \\
&\text{with} \\
\hat{m}_i &= x_i(0) + \Lambda_i - \frac{1}{2}\hat{\sigma}_i^2, \\
\hat{\sigma}_i^2 &= \int_t^s \left(\left(1-\rho_1^{v2}\right) a_{11}^2 f(\tilde{v}_1(t'))^2 + \left(1-\rho_2^{v2}\right) a_{12}^2 f(\tilde{v}_2(t'))^2\right) dt', \\
\hat{\rho} &= \frac{\int_t^s (a_{11}a_{21}(1-\rho_1^{v2}) f(\tilde{v}_1(t'))^2 + a_{12}a_{22}(1-\rho_2^{v2}) f(\tilde{v}_2(t'))^2) dt'}{\hat{\sigma}_1\hat{\sigma}_2}, \\
\Lambda_i &= \int_t^s \sum_{j=1}^2 a_{ij} f(\tilde{v}_j(t'))\rho_j^v dZ_j - \frac{1}{2}\int_t^s \sum_{j=1}^2 a_{ij}^2 f(\tilde{v}_j(t'))^2 \rho_j^{v2} dt'.
\end{aligned}$$

To simplify the analysis of the conditional expectation we compute the following expressions

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{w_2} p(w_1, w_2) dw_2 &= \int_{-\infty}^{\infty} e^{w_2} \frac{e^{-\frac{1}{2(1-\hat{\rho})}\left(\frac{(x_1-\hat{m}_1)^2}{\hat{\sigma}_1} + \frac{(x_2-\hat{m}_2)^2}{\hat{\sigma}_2} - 2\hat{\rho}\frac{(x_1-\hat{m}_1)}{\hat{\sigma}_1}\frac{(x_2-\hat{m}_2)}{\hat{\sigma}_2}\right)}}{2\pi\hat{\sigma}_1\hat{\sigma}_2\sqrt{1-\hat{\rho}^2}} dx_2 \\
&= \frac{1}{2\pi\hat{\sigma}_1\hat{\sigma}_2} \int_{-\infty}^{\infty} e^{z_2\hat{\sigma}_2 + \hat{m}_2} e^{-\frac{1}{2}(z_1^2+z_2^2)} dz_2^2 \\
&= \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z_2-\hat{\sigma}_2)^2} e^{\hat{m}_2 + \frac{1}{2}\hat{\sigma}_2^2} \underbrace{e^{-\frac{z_1^2}{2}}}_{\leq 1} dz_2 \\
&\leq e^{\hat{m}_2 + \frac{1}{2}\hat{\sigma}_2^2} \tilde{c}_8,
\end{aligned} \tag{4.164}$$

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |x|^i e^{-\frac{1}{2}x^2} dx &= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} (x)^i e^{-\frac{1}{2}x^2} dx \\
&= \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} 2^{\frac{i}{2}} \chi^{\frac{i}{2}} e^{-\chi} (2\chi)^{-\frac{1}{2}} d\chi \\
&= \frac{2^{\frac{i}{2}}}{\sqrt{\pi}} \Gamma(\frac{i+1}{2}).
\end{aligned} \tag{4.165}$$

See for example [11].

$$\begin{aligned}
&\left|\mathbb{E}\left[\frac{\tilde{b}_i}{\sqrt{\tau^{\tilde{\epsilon}}}}e^{x_2}\int_{-\infty}^{\mathbf{d}_2^{\tilde{\epsilon}}}\frac{\partial^i}{\partial x_1^i}e^{\left(-\frac{1}{2(1-\overline{\rho}^2)}\left(\mathbf{d}_1^{\tilde{\epsilon}\,2}+y_2^2-2\overline{\rho}\mathbf{d}_1^{\tilde{\epsilon}}y_2\right)\right)}dy_2\middle|Z_j(t');t\le t'\le s\right]\right| \\
= &\;\mathbb{E}\Bigg[\frac{\tilde{b}_i}{\sqrt{\tau^{\tilde{\epsilon}}}}e^{x_2}\Bigg(\left|\frac{\partial^i}{\partial x_1^i}e^{-\frac{\mathbf{d}_1^{\tilde{\epsilon}\,2}}{2}}\right|\underbrace{\int_{-\infty}^{\mathbf{d}_2^{\tilde{\epsilon}}}e^{-\frac{1}{2(1-\overline{\rho}^2)}(y_2-\overline{\rho}\mathbf{d}_1^{\tilde{\epsilon}})^2}dy_2}_{\le 1} \\
&+e^{-\frac{\mathbf{d}_1^{\tilde{\epsilon}\,2}}{2}}\left|\int_{-\infty}^{\mathbf{d}_2^{\tilde{\epsilon}}}\frac{\partial^i}{\partial x_1^i}e^{-\frac{1}{2(1-\overline{\rho}^2)}(y_2-\overline{\rho}\mathbf{d}_1^{\tilde{\epsilon}})^2}dy_2\right|\Bigg)\Bigg] \\
\le &\;\mathbb{E}\left[\frac{\tilde{b}_i}{\sqrt{\tau^{\tilde{\epsilon}}}}e^{x_2}\left(\left|\frac{\partial^i}{\partial x_1^i}e^{-\frac{\mathbf{d}_1^{\tilde{\epsilon}\,2}}{2}}\right|+e^{-\frac{\mathbf{d}_1^{\tilde{\epsilon}\,2}}{2}}\underbrace{\int_{-\infty}^{\mathbf{d}_2^{\tilde{\epsilon}}}\sum_{k=1}^{i}\tilde{c}_9\left|\tilde{y}_2\right|^k e^{-\frac{1}{2}\tilde{y}_2^2}d\tilde{y}_2}_{bounded}\right)\right] \\
\le &\;\mathbb{E}\left[\frac{\tilde{b}_i}{\sqrt{\tau^{\tilde{\epsilon}}}}e^{x_2}e^{-\frac{\mathbf{d}_1^{\tilde{\epsilon}\,2}}{2}}\tilde{c}_{10}\sum_{k=0}^{i}\frac{|x_1|^k}{(\tau^{\epsilon})^k}\right]\le\frac{\tilde{c}_{11}}{\tau^{\epsilon\frac{i+1}{2}}}\int\int\sum_{k=0}^{i}\frac{|x_1|^k}{\tau^{\epsilon\frac{k}{2}}}e^{w_2}e^{-\frac{\mathbf{d}_1^{\tilde{\epsilon}\,2}}{2}}p(w_1,w_2)dw_2dw_1 \\
\le &\;\frac{\tilde{c}_{12}}{\tau^{\epsilon\frac{i+1}{2}}}\int\sum_{k=0}^{i}\frac{|w_1|^k}{\tau^{\epsilon\frac{k}{2}}}e^{-\frac{\mathbf{d}_1^{\tilde{\epsilon}\,2}}{2}}dw_1=\frac{\tilde{c}_{12}}{\tau^{\epsilon\frac{i}{2}}}\int\sum_{k=0}^{i}|\overline{w}_1|^k e^{-\frac{\mathbf{d}_1^{\tilde{\epsilon},\overline{w}_1\,2}}{2}}d\overline{w}_1\le\frac{\tilde{c}_{13}}{\tau^{\epsilon\frac{i}{2}}}
\end{aligned}\tag{4.166}$$

where we apply (4.164) in the fourth last line and (4.165) in the third and the last line, $y_i=\frac{x_i}{\overline{\sigma}_i}$, $\tilde{y}_2=\frac{y_2-\overline{\rho}\mathbf{d}_1^{\tilde{\epsilon}}}{(1-\overline{\rho}^2)}$, $\overline{w}_i=\frac{x_i}{\sqrt{\tau^{\tilde{\epsilon}}}}$, $\mathbf{d}_1^{\tilde{\epsilon},\overline{w}_1}=\frac{\overline{w}_1}{\overline{\sigma}_1}-\frac{1}{2}\overline{\sigma}_1\sqrt{\tau^{\tilde{\epsilon}}}+\overline{\rho\sigma}_2\sqrt{\tau^{\tilde{\epsilon}}}$, and $\mathbf{d}_2^{\tilde{\epsilon},\overline{w}_2}=\frac{\overline{w}_2}{\overline{\sigma}_2}+\frac{1}{2}\overline{\sigma}_2\sqrt{\tau^{\tilde{\epsilon}}}$. Analogously, we obtain

$$\begin{aligned}
&\left|\mathbb{E}\left[\frac{\tilde{b}_{\mathfrak{k}}}{\sqrt{\tau^{\tilde{\epsilon}}}}e^{x_2}\int_{-\infty}^{\mathbf{d}_1^{\tilde{\epsilon}}}\frac{\partial^{\mathfrak{k}}}{\partial x_2^{\mathfrak{k}}}e^{\left(-\frac{1}{2(1-\overline{\rho}^2)}\left(\mathbf{d}_2^{\tilde{\epsilon}\,2}+y_1^2-2\overline{\rho}\mathbf{d}_2^{\tilde{\epsilon}}y_1\right)\right)}dy_1\middle|Z_j(t');t\le t'\le s\right]\right| \\
\le &\;\mathbb{E}\left[\frac{\tilde{b}_{\mathfrak{k}}}{\sqrt{\tau^{\tilde{\epsilon}}}}e^{x_2}e^{\left(-\frac{1}{2}\mathbf{d}_2^{\tilde{\epsilon}\,2}\right)}\sum_{k=0}^{\mathfrak{k}}\frac{|x_2|^k}{(\tau^{\tilde{\epsilon}})^k}\tilde{c}_{14}\middle|Z_j(t');t\le t'\le s\right] \\
= &\;\int\frac{\tilde{c}_{14}\tilde{b}_{\mathfrak{k}}}{\tau^{\tilde{\epsilon}\frac{\mathfrak{k}+1}{2}}}e^{\sqrt{\tau^{\tilde{\epsilon}}}\overline{w}_2}e^{-\frac{\mathbf{d}_2^{\tilde{\epsilon},\overline{w}_2\,2}}{2}}\sum_{k=0}^{\mathfrak{k}}\frac{|w_2|^k}{\tau^{\tilde{\epsilon}\frac{k}{2}}}\tau\underbrace{\int p(\overline{w}_1\sqrt{\tau^{\tilde{\epsilon}}},\overline{w}_2\sqrt{\tau^{\tilde{\epsilon}}})d\overline{w}_1}_{=p(\overline{w}_2\sqrt{\tau^{\tilde{\epsilon}}})}d\overline{w}_2 \\
\le &\;\int\frac{\tilde{c}_{15}\tilde{b}_{\mathfrak{k}}}{\tau^{\tilde{\epsilon}\frac{\mathfrak{k}-1}{2}}}e^{\sqrt{\tau^{\tilde{\epsilon}}}\overline{w}_2}e^{-\frac{\mathbf{d}_2^{\tilde{\epsilon},\overline{w}_2\,2}}{2}}\sum_{k=0}^{\mathfrak{k}}|\overline{w}_2|^k\underbrace{p(\overline{w}_2\sqrt{\tau^{\tilde{\epsilon}}})}_{\le 1}d\overline{w}_2 \\
\le &\;\int\frac{\tilde{c}_{15}\tilde{b}_{\mathfrak{k}}}{\tau^{\tilde{\epsilon}\frac{\mathfrak{k}-1}{2}}}e^{\sqrt{\tau^{\tilde{\epsilon}}}\overline{w}_2-\frac{1}{2}\frac{\overline{w}_2^2}{\overline{\sigma}_2^2}-\frac{1}{8}\overline{\sigma}_2^2\tau^{\tilde{\epsilon}}-\overline{w}_2\sqrt{\tau^{\tilde{\epsilon}}}}\sum_{k=0}^{\mathfrak{k}}|\overline{w}_2|^k d\overline{w}_2\le\frac{\tilde{c}_{16}}{\tau^{\tilde{\epsilon}\frac{\mathfrak{k}-1}{2}}},
\end{aligned}\tag{4.167}$$

and by integrating the prior expression multiplied by $(T-s+\tilde{\epsilon})$ we obtain

$$\begin{aligned}
&\left|\mathbb{E}\left[\frac{\tilde{b}_{\mathfrak{k}}}{\tau^{\tilde{\epsilon}}}e^{x_2}\frac{\partial^{i+\mathfrak{k}}}{\partial x_1^i\partial x_2^{\mathfrak{k}}}e^{\left(-\frac{1}{2(1-\overline{\rho}^2)}\left(\mathbf{d}_1^{\tilde{\epsilon}\,2}+\mathbf{d}_2^{\tilde{\epsilon}\,2}-2\overline{\rho}\mathbf{d}_1^{\tilde{\epsilon}}\mathbf{d}_2^{\tilde{\epsilon}}\right)\right)}\,|Z_j(t');t\leq t'\leq s\right]\right|\\
\leq\ & \mathbb{E}\left[\frac{\tilde{b}_{\mathfrak{k}}}{\tau^{\tilde{\epsilon}}}e^{x_2}e^{\left(-\frac{1}{2(1-\overline{\rho}^2)}\left(\mathbf{d}_1^{\tilde{\epsilon}\,2}+\mathbf{d}_2^{\tilde{\epsilon}\,2}-2\overline{\rho}\mathbf{d}_1^{\tilde{\epsilon}}\mathbf{d}_2^{\tilde{\epsilon}}\right)\right)}\tilde{c}_{17}\sum_{k=0}^{i}\sum_{l=0}^{\mathfrak{k}}\frac{|x_1|^k|x_2|^l}{\tau^{\tilde{\epsilon}k+l}}\,|Z_j(t');t\leq t'\leq s\right]\\
\leq\ & \int\int\frac{\tilde{c}_{18}\tilde{b}_{\mathfrak{k}}}{\tau^{\tilde{\epsilon}\frac{i+\mathfrak{k}+2}{2}}}e^{w_2}e^{\left(-\frac{1}{2(1-\overline{\rho}^2)}\left(\mathbf{d}_1^{\tilde{\epsilon}\,2}+\mathbf{d}_2^{\tilde{\epsilon}\,2}-2\overline{\rho}\mathbf{d}_1^{\tilde{\epsilon}}\mathbf{d}_2^{\tilde{\epsilon}}\right)\right)}\sum_{k=0}^{i}\sum_{l=0}^{\mathfrak{k}}\frac{|w_1|^k|w_2|^l}{\tau^{\tilde{\epsilon}\frac{k+l}{2}}}p(w_1,w_2)dw_1dw_2\\
=\ & \frac{\tilde{c}_{18}\tilde{b}_{\mathfrak{k}}}{\tau^{\tilde{\epsilon}\frac{i+\mathfrak{k}+2}{2}}}\int\int e^{\overline{w}_2\sqrt{\tau^{\tilde{\epsilon}}}}e^{\left(-\frac{1}{2(1-\overline{\rho}^2)}\left(\mathbf{d}_1^{\tilde{\epsilon},\overline{w}_1\,2}+\mathbf{d}_2^{\tilde{\epsilon},\overline{w}_2\,2}-2\overline{\rho}\mathbf{d}_1^{\tilde{\epsilon},\overline{w}_1}\mathbf{d}_2^{\tilde{\epsilon},\overline{w}_2}\right)\right)}\\
& \sum_{k=0}^{i}\sum_{l=0}^{\mathfrak{k}}|\overline{w}_1|^k|\overline{w}_2|^l p(\overline{w}_1\sqrt{\tau^{\tilde{\epsilon}}},\overline{w}_2\sqrt{\tau^{\tilde{\epsilon}}})\tau d\overline{w}_1d\overline{w}_2\\
\leq\ & \frac{\tilde{c}_{19}\tilde{b}_{\mathfrak{k}}}{\tau^{\tilde{\epsilon}\frac{i+\mathfrak{k}}{2}}}\int\int e^{\overline{\sigma}_2\sqrt{1-\overline{\rho}^2}\overline{z}_2\sqrt{\tau^{\tilde{\epsilon}}}}e^{\overline{\sigma}_2\overline{\rho}^2\overline{z}_1\sqrt{\tau^{\tilde{\epsilon}}}}e^{\left(-\frac{1}{2}\left(\mathbf{d}_1^{\tilde{\epsilon},\overline{z}_1\,2}+\mathbf{d}_2^{\tilde{\epsilon},\overline{z}_2\,2}\right)\right)}\\
& \sum_{k=0}^{i}\sum_{l=0}^{\mathfrak{k}}|\overline{\sigma}_1\overline{z}_1|^k\left|\sqrt{1-\overline{\rho}^2}\overline{\sigma}_2\overline{z}_2+\rho\overline{\sigma}_2\overline{z}_1\right|^l p(\overline{z}_1\sqrt{\tau^{\tilde{\epsilon}}},\overline{z}_2\sqrt{\tau^{\tilde{\epsilon}}})d\overline{z}_1d\overline{z}_2\\
\leq\ & \frac{\tilde{c}_{20}\tilde{b}_{\mathfrak{k}}}{\tau^{\tilde{\epsilon}\frac{i+\mathfrak{k}}{2}}}\int\int e^{\overline{\sigma}_2\sqrt{1-\overline{\rho}^2}\overline{z}_2\sqrt{\tau^{\tilde{\epsilon}}}}e^{\overline{\sigma}_2\overline{\rho}^2\overline{z}_1\sqrt{\tau^{\tilde{\epsilon}}}}e^{\left(-\frac{1}{2}\left(\mathbf{d}_1^{\tilde{\epsilon},\overline{z}_1\,2}+\mathbf{d}_2^{\tilde{\epsilon},\overline{z}_2\,2}\right)\right)}\\
& \sum_{k=0}^{i}\sum_{l=0}^{\mathfrak{k}}|\overline{\sigma}_1\overline{z}_1|^k\sum_{m=0}^{l}\left|\overline{z}_1^n\overline{z}_2^{m-n}\right|p(\overline{z}_1\sqrt{\tau^{\tilde{\epsilon}}},\overline{z}_2\sqrt{\tau^{\tilde{\epsilon}}})d\overline{z}_1d\overline{z}_2\\
\leq\ & \frac{\tilde{c}_{21}}{\tau^{\tilde{\epsilon}\frac{i+\mathfrak{k}}{2}}},
\end{aligned}\tag{4.168}$$

where $\overline{z}_2=\overline{w}_2$ and $\overline{z}_1=\frac{1}{\sqrt{1-\overline{\rho}^2}}(\overline{w}_1-\overline{\rho}\overline{w}_2)$.

Due to (4.161) we can deduce from these results that

$$\left|\mathbb{E}\left[\sum_{i,k;i+k=n}\psi(v)\frac{\partial^{i+k}C_{0,0}^{\tilde{\epsilon}}}{\partial x_1^i\partial x_2^k}\right]\right|\leq\tilde{c}_{22}(T+\tilde{\epsilon}-s)^{\min\left(0,\frac{1-n}{2}\right)},\tag{4.169}$$

and by integrating the prior expression we obtain

$$\left|\mathbb{E}\left[\int_t^T(T-s)^l\sum_{i,k;i+k=n}e^{-r(s-t)}\psi(v)\frac{\partial^{i+k}C_{0,0}^{\tilde{\epsilon}}}{\partial x_1^i\partial x_2^k}\right]ds\right|\leq\begin{cases}\tilde{c}_{23}\,|\ln(\tilde{\epsilon})| & n=3+2l\\ \tilde{c}_{24}\tilde{\epsilon}^{\min\left(0,l+\frac{3-n}{2}\right)}\end{cases}\tag{4.170}$$

Using the Green Theorem we can represent $R^{\tilde{\epsilon},\delta}$ as

$$\begin{aligned} R^{\tilde{\epsilon},\delta}(t, S_1, S_2, \mathbf{v}) &= \mathbb{E}\Big[e^{-r(T-t)}\delta_1\delta_2 G_1^{\tilde{\epsilon}}(S_1(T), S_2(T), \mathbf{v}) \\ &\quad - \int_t^T e^{-r(s-t)}\delta_1\delta_2 F_1^{\tilde{\epsilon}}(s, S_1(s), S_2(s), \mathbf{v})ds\Big] \\ &+ \mathbb{E}\Big[e^{-r(T-t)}\delta_1^2 G_2^{\tilde{\epsilon}}(S_1(T), S_2(T), \mathbf{v}) \\ &\quad - \int_t^T e^{-r(s-t)}\delta_1^2 F_2^{\tilde{\epsilon}}(s, S_1(s), S_2(s), \mathbf{v})ds\Big] \\ &+ \mathbb{E}\Big[e^{-r(T-t)}\delta_2^2 G_3^{\tilde{\epsilon}}(S_1(T), S_2(T), \mathbf{v}) \\ &\quad - \int_t^T e^{-r(s-t)}\delta_2^2 F_3^{\tilde{\epsilon}}(s, S_1(s), S_2(s), \mathbf{v})ds\Big]. \end{aligned}$$

Assuming that the eigenfunctions are bounded it follows that

$$\begin{aligned} \left|\mathbb{E}\left[\delta_1\delta_2 G_1^{\tilde{\epsilon}}(S_1(T), S_2(T), \mathbf{v})\right]\right| &\leq \hat{c}\delta_1\delta_2, \\ \left|\mathbb{E}\left[\delta_1^2 G_2^{\tilde{\epsilon}}(S_1(T), S_2(T), \mathbf{v})\right]\right| &\leq \hat{c}_2\delta_1^2\left(1 + \delta_1\tilde{\epsilon}^{-\frac{1}{2}} + \delta_1\tilde{\epsilon}^{-1}\right), \\ \left|\mathbb{E}\left[\delta_2^2 G_3^{\tilde{\epsilon}}(S_1(T), S_2(T), \mathbf{v})\right]\right| &\leq \tilde{c}_3\delta_2^2\left(1 + \delta_2\tilde{\epsilon}^{-\frac{1}{2}} + \delta_2\tilde{\epsilon}^{-1}\right), \\ \left|\mathbb{E}\left[\int_t^T e^{-r(s-t)}\delta_1\delta_2 F_1^{\tilde{\epsilon}}(S_1(T), S_2(T), \mathbf{v})\right]\right| &\leq \hat{c}_4\delta_1\delta_2(1 + \tilde{\epsilon}^{-\frac{1}{2}} + \delta_1\tilde{\epsilon}^{-1} + \delta_2\tilde{\epsilon}^{-1}), \\ \left|\mathbb{E}\left[\int_t^T e^{-r(s-t)}\delta_1^2 F_2^{\tilde{\epsilon}}(S_1(T), S_2(T), \mathbf{v})\right]\right| &\leq \hat{c}_5\delta_1^2(1 + \tilde{\epsilon}^{-\frac{1}{2}} + \delta_1\tilde{\epsilon}^{-1}), \\ \left|\mathbb{E}\left[\int_t^T e^{-r(s-t)}\delta_2^2 F_3^{\tilde{\epsilon}}(S_1(T), S_2(T), \mathbf{v})\right]\right| &\leq \hat{c}_6\delta_2^2\left(1 + \tilde{\epsilon}^{-\frac{1}{2}} + \delta_2\tilde{\epsilon}^{-1}\right). \end{aligned}$$

Concluding,

$$\begin{aligned} \left|R^{\tilde{\epsilon},\delta}\right| &\leq \hat{c}_7\Big[\delta_1^2(1 + \tilde{\epsilon}^{-\frac{1}{2}} + \delta_1\tilde{\epsilon}^{-1}) + \delta_2^2\left(1 + \tilde{\epsilon}^{-\frac{1}{2}} + \delta_2\tilde{\epsilon}^{-1}\right) \\ &\quad + \delta_1\delta_2\left(1 + \tilde{\epsilon}^{-\frac{1}{2}} + \delta_1\tilde{\epsilon}^{-1} + \delta_2\tilde{\epsilon}^{-1}\right)\Big], \end{aligned} \tag{4.171}$$

and therefore for (t, S_1, S_2, v) fixed with $t < T$ using (4.171) and the bounds from 4.170

we have

$$\begin{aligned}\left|C^{\tilde{\epsilon},\delta} - Q^{\tilde{\epsilon},\delta}\right| &= \Big| - R^{\tilde{\epsilon},\delta} + \delta_1^2 C^{\tilde{\epsilon}}_{2,0} + \delta_2^2 C^{\tilde{\epsilon}}_{0,2} + \delta_1^2\delta_2 C^{\tilde{\epsilon}}_{1,2} + \delta_1\delta_2^2 C^{\tilde{\epsilon}}_{1,2} \\ &\quad + \delta_1^3 C^{\tilde{\epsilon}}_{3,0} + \delta_2^3 C^{\tilde{\epsilon}}_{0,3}\Big| \\ &\leq \mathfrak{c}_3\Big[\delta_1^2\left(1+\tilde{\epsilon}^{-\frac{1}{2}}+\delta_1\tilde{\epsilon}^{-1}\right)+\delta_2^2(1+\tilde{\epsilon}^{-\frac{1}{2}}+\delta_2\tilde{\epsilon}^{-1}) \\ &\quad +\delta_1\delta_2\left(1+\tilde{\epsilon}^{-\frac{1}{2}}+\delta_1\tilde{\epsilon}^{-1}+\delta_2\tilde{\epsilon}^{-1}\right)\Big]. \end{aligned} \tag{4.172}$$

□

Theorem 42. *(Accuracy of the perturbation)*

At a fixed point $(t, S_1, S_2, \mathbf{v})$, $t < T$ and under the assumption that the eigenvalues are positive and bounded the accuracy of the approximation of the two-asset option is given by

$$\lim_{\delta_1\to 0,\delta_2\to 0} \frac{\left|C^\delta(t,S_1,S_2,v) - Q^\delta\left(t,S_1,S_2\right)\right|}{\left(\delta_1^{\frac{4}{3}-p}\right)} = 0, \tag{4.173}$$

for $\delta_1 > \delta_2$, $p > 0$.

Remark 16. *The price $C_{B,0,0}$ is smooth in $S_1 > B_1(t)$ and $S_2 > B_2(t)$ and its derivatives have finite limits as $S_1 \to B_1^+$ and/or $S_2 \to B_2^+$. Hence, the convergence for knock-out barrier options can be proved in just the same way.*

For single-barrier options see [70].

Proof.

Take $\overline{\delta}_1 = \min\left(\overline{\delta}_1^1, \overline{\delta}_1^2, \overline{\delta}_1^3\right)$, $\overline{\delta}_2 = \min\left(\overline{\delta}_2^1, \overline{\delta}_2^2, \overline{\delta}_2^3\right)$, and $\overline{\tilde{\epsilon}} = \min\left(\overline{\tilde{\epsilon}}_1, \overline{\tilde{\epsilon}}_2, \overline{\tilde{\epsilon}}_3\right)$. Using Lemmas 2, 3, and 4 we deduce

$$\begin{aligned}\left|C^\delta - Q^\delta\right| &\leq \left|C^\delta - C^{\tilde{\epsilon},\delta}\right| + \left|C^{\tilde{\epsilon},\delta} - Q^{\tilde{\epsilon},\delta}\right| + \left|Q^{\tilde{\epsilon},\delta} - Q^\delta\right| \\ &\leq 2\max(\mathfrak{c}_1,\mathfrak{c}_2)\tilde{\epsilon} + \mathfrak{c}_3\Big[\delta_1^2\left(1+\tilde{\epsilon}^{-\frac{1}{2}}+\delta_1\tilde{\epsilon}^{-1}\right)+\delta_2^2\left(1+\tilde{\epsilon}^{-\frac{1}{2}}+\delta_2\tilde{\epsilon}^{-1}\right) \\ &\quad +\delta_1\delta_2\left(1+\tilde{\epsilon}^{-\frac{1}{2}}+\delta_1\tilde{\epsilon}^{-1}+\delta_2\tilde{\epsilon}^{-1}\right)\Big], \end{aligned} \tag{4.174}$$

for $0 < \delta_1 < \overline{\delta}_1$, $0 < \delta_2 < \overline{\delta}_2$, and $0 < \tilde{\epsilon} < \overline{\tilde{\epsilon}}$, where the functions are evaluated at the fixed point (t, S_1, S_2, v). Taking $\tilde{\epsilon} = \max\left(\delta_1^{\frac{4}{3}}, \delta_2^{\frac{4}{3}}\right)$. Assume that $\delta_1 > \delta_2$ then

$$\left|C^\delta - Q^\delta\right| \leq \mathfrak{c}_4\left(\delta_1^2\left(1+\delta_1^{-\frac{2}{3}}+\delta_1^{-\frac{1}{3}}\right)\right), \tag{4.175}$$

for some fixed $\mathfrak{c}_4 > 0$ and Theorem 42 follows. □

Numerical accuracy of the approximation

In the following we compute the exact value of the two-asset option without barriers (4.36) with the alternative Fourier technique described in Section 3.4.5 and compare it to our approximation. We calculate some scenarios to get a deeper insight into the quality of the approximation. The basic scenario is given by $f(v_j) = \sqrt{v_j}, \sigma_{v_1} = \sigma_{v_2} = 0.1$, $\zeta_{v_j} = 0.1$, $\kappa_{v_1} = \kappa_{v_2} = 0.2$, $T = 1$, and $K_1 = K_2 = 10.5$. We set the eigenvalues to $a_{11} = 0.9$, $a_{12} = a_{21} = \sqrt{(1 - 0.9^2)}$, and $a_{22} = -0.9$. In our first calculation we set the value of $S_1 = 10.5$ and vary the value of S_2. It takes values between 8 and 12. In Table 4.2 we indicate the results of the approximation for $\delta_1 = \delta_2 = \frac{1}{20}$ and $\delta_1 = \delta_2 = \frac{1}{2}$. In the Plots 4.4 and 4.5 we show the relative differences between the exact result and the approximation for $\delta_1 = \delta_2 = \frac{1}{20}$ and $\delta_1 = \delta_2 = \frac{1}{2}$ respectively. The relative error is a decaying function in the value of S_2. The relative error seems to be the highest for out-of-the-money options.

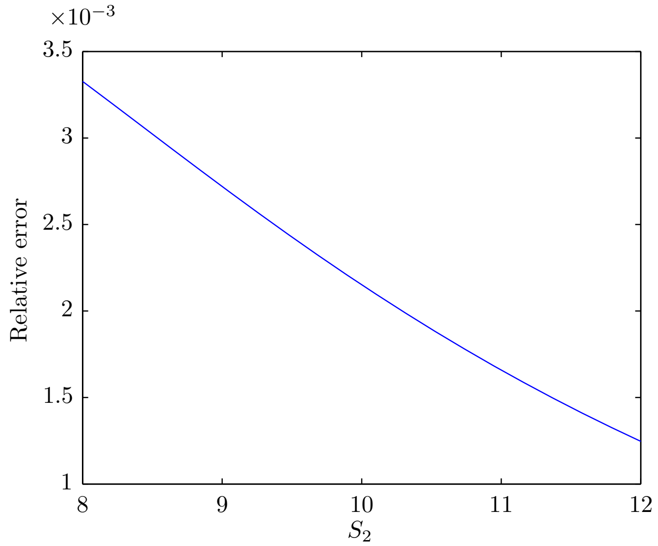

Figure 4.4: Relative difference between exact result and approximation for two-asset option without barriers for $\delta_j = \frac{1}{20}$.

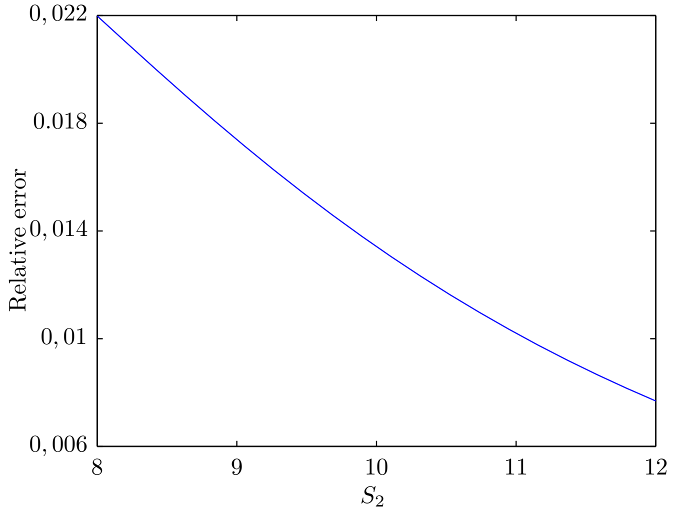

Figure 4.5: Relative difference between exact result and approximation for two-asset option without barriers for $\delta_j = \frac{1}{2}$.

Table 4.2: Prices of the two-asset option computed with Fourier technique and approximation.

	Exact result		Approximation	
	$\delta_1 = \delta_2$			
S_2	$\frac{1}{20}$	$\frac{1}{2}$	$\frac{1}{20}$	$\frac{1}{2}$
8	0.3582	0.3515	0.3594	0.3592
8,21	0.3971	0.3900	0.3983	0.3982
8,42	0.4381	0.4306	0.4394	0.4393
8,63	0.4812	0.4734	0.4826	0.4825
8,84	0.5264	0.5184	0.5279	0.5278
9,05	0.5736	0.5654	0.5752	0.5751
9,26	0.6229	0.6144	0.6245	0.6244
9,47	0.6741	0.6655	0.6758	0.6757
9,68	0.7273	0.7185	0.7289	0.7290
9,89	0.7822	0.7733	0.7840	0.7840
10,11	0.8390	0.8300	0.8408	0.8409
10,32	0.8975	0.8885	0.8993	0.8994
10,53	0.9577	0.9487	0.9595	0.9597
10,74	1.0196	1.0105	1.0214	1.0216
10,95	1.0830	1.0739	1.0848	1.0851
11,16	1.1479	1.1389	1.1497	1.1500
11,37	1.2143	1.2054	1.2161	1.2164
11,58	1.2820	1.2732	1.2838	1.2843
11,75	1.3512	1.3425	1.3530	1.3534
12	1.4216	1.4130	1.4234	1.4239

$S_1(0)$ =10.5, $K_1 = K_2$ =10.5, r =0.05,
$\rho_1 = \rho_2$ =0.05, T =1.0, ζ_{v_j} =0.1, κ_{v_j} =0.2,
σ_{v_j} =0.1, $v_j(0)$ =0.2, $a_{12} = a_{21} = \sqrt{1-0.9^2}$,
a_{11}=0.9, a_{22}=-0.9

4.5.2 Extension of Model (4.1)

We suggest here an extension to Model (4.1) and introduce – beside the fast mean-reverting component – a further, slow mean-reverting component to the stochastic dynamic of the eigenvalue. In this context we shortly want to give an insight how perturbation could be applied in the extended model as well. Further future research will have to be undertaken as far as for example convergence and the number of necessary perturbation terms are concerned. The following model is suggested:

$$\begin{aligned}
dS_i &= rS_i dt + S_i \sum_{j=1}^{\bar{p}} a_{ij} f(v_j(t), y_j(t)) dW_j, \; i \in \{1,2\}, \; \bar{p} = 2 && (4.176)\\
dv_j &= \frac{\kappa_{v_j}}{\delta_j^2}(\zeta_{v_j} - v_j)dt + \frac{\sigma_{v_j}}{\delta_j}\sqrt{v_j} dZ_{v_j}, && (4.177)\\
dy_j &= \epsilon_j^2 \kappa_{y_j}(\zeta_{y_j} - y_j)dt + \epsilon_j \sigma_{y_j}\sqrt{y_j} dZ_{y_j}, && (4.178)\\
\left\langle dW_j, dZ_{v_j} \right\rangle &= \rho_j^v dt,\\
\left\langle dW_j, dZ_{y_j} \right\rangle &= \rho_j^y dt,\\
\left\langle dZ_{v_j}, dZ_{y_j} \right\rangle &= \rho_j^{vy} dt, \text{ for } j \in \{1,2\},
\end{aligned}$$

all other correlations are set to 0. To allow for slow mean reversion we assume ϵ_j very small.

In the following we shortly indicate how an approximation applying perturbation theory could by analogously to before performed in this model.[13]

As before we price in this framework options which depend on two underlyings with and without barriers on both of the underlyings (see (4.36) and (4.37)). The payoff in T is indicated by $g(S_1, S_2)$ in the following. We asymptotically approximate the prices with singular (in δ_j) and regular (in ϵ_j) perturbation theory. The infinitesimal generator $\mathcal{L}^{\delta,\epsilon}$ is again expressed as a power series, this time in ϵ_j and δ_j

$$\begin{aligned}
\mathcal{L}^{\delta_1,\delta_2,\tilde{\epsilon}_1,\tilde{\epsilon}_2} &= \frac{1}{\delta_1^2}\mathcal{L}_0^1 + \frac{1}{\delta_2^2}\mathcal{L}_0^2 + \frac{1}{\delta_1}\mathcal{L}_1^1 + \frac{1}{\delta_2}\mathcal{L}_1^2 + \mathcal{L}_2 + \varepsilon_1^2\mathcal{M}_0^1 + \varepsilon_2^2\mathcal{M}_0^2\\
&\quad + \varepsilon_1\mathcal{M}_1^1 + \varepsilon_2\mathcal{M}_1^2 + \frac{\varepsilon_1}{\delta_1}\mathcal{M}_3^1 + \frac{\varepsilon_2}{\delta_2}\mathcal{M}_3^2, && (4.179)
\end{aligned}$$

[13] This chapter has been prepared during a master thesis project in cooperation with Daniela Neykova. See also [90]

where

$$
\begin{aligned}
\mathcal{L}_0^1 &= \kappa_{v_1}(\zeta_{v_1} - v_1)\frac{\partial}{\partial v_1} + \frac{1}{2}\sigma_{v_1}^2 v_1 \frac{\partial^2}{\partial v_1^2}, && (4.180)\\
\mathcal{L}_0^2 &= \kappa_{v_2}(\zeta_{v_2} - v_2)\frac{\partial}{\partial v_2} + \frac{1}{2}\sigma_{v_2}^2 v_2 \frac{\partial^2}{\partial v_2^2}, && (4.181)\\
\mathcal{L}_1^1 &= \rho_1^v a_{11} f_1(v_1, y_1)\sigma_{v_1}\sqrt{v_1} S_1 \frac{\partial^2}{\partial S_1 \partial v_1} + \rho_1^v a_{21} f_1(v_1, y_1)\sigma_{v_1}\sqrt{v_1} S_2 \frac{\partial^2}{\partial S_2 \partial v_1}, && (4.182)\\
\mathcal{L}_1^2 &= \rho_2^v a_{12} f_2(v_2, y_2)\sigma_{v_2}\sqrt{v_2} S_1 \frac{\partial^2}{\partial S_1 \partial v_2} + \rho_2^v a_{22} f_2(v_2, y_2)\sigma_{v_2}\sqrt{v_2} S_2 \frac{\partial^2}{\partial S_2 \partial v_2}, && (4.183)\\
\mathcal{L}_2 &= \frac{\partial}{\partial t} + r\left(S_1\frac{\partial}{\partial S_1} + S_2\frac{\partial}{\partial S_2} - \cdot\right) + \left(a_{11}a_{21}f_1^2 + a_{12}a_{22}f_2^2\right) S_1 S_2 \frac{\partial^2}{\partial S_1 \partial S_2}\\
&\quad + \frac{1}{2}\left(a_{11}^2 f_1^2 + a_{12}^2 f_2^2\right) S_1^2 \frac{\partial^2}{\partial S_1^2} + \frac{1}{2}\left(a_{21}^2 f_1^2 + a_{22}^2 f_2^2\right) S_2^2 \frac{\partial^2}{\partial S_2^2}, && (4.184)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{M}_0^1 &= \kappa_{y_1}(\zeta_{y_1} - y_1)\frac{\partial}{\partial y_1} + \frac{1}{2}\sigma_{y_1}^2 y_1 \frac{\partial^2}{\partial y_1^2}, && (4.185)\\
\mathcal{M}_0^2 &= \kappa_{y_2}(\zeta_{y_2} - y_2)\frac{\partial}{\partial y_2} + \frac{1}{2}\sigma_{y_2}^2 y_2 \frac{\partial^2}{\partial y_2^2}, && (4.186)\\
\mathcal{M}_1^1 &= \rho_1^y a_{11} f_1 \sigma_{y_1}\sqrt{y_1} S_1 \frac{\partial^2}{\partial S_1 \partial y_1} + \rho_1^y a_{21} f_1 \sigma_{y_1}\sqrt{y_1} S_2 \frac{\partial^2}{\partial S_2 \partial y_1}, && (4.187)\\
\mathcal{M}_1^2 &= \rho_2^y a_{12} f_2 \sigma_{y_2}\sqrt{y_2} S_1 \frac{\partial^2}{\partial S_1 \partial y_2} + \rho_2^y a_{22} f_2 \sigma_{y_2}\sqrt{y_2} S_2 \frac{\partial^2}{\partial S_2 \partial y_2}, && (4.188)\\
\mathcal{M}_3^1 &= \rho_1^{vy}\sigma_{v_1}\sigma_{y_1}\sqrt{v_1 y_1}\frac{\partial^2}{\partial v_1 \partial y_1}, && (4.189)\\
\mathcal{M}_3^2 &= \rho_2^{vy}\sigma_{v_2}\sigma_{y_2}\sqrt{v_2 y_2}\frac{\partial^2}{\partial v_2 \partial y_2}, && (4.190)
\end{aligned}
$$

where $f_j = (v_j, y_j)$. The problem to be solved for the two-asset option $C^{\delta,\epsilon}$, hence, becomes

$$
\begin{aligned}
\mathcal{L}^{\delta,\epsilon} C^{\delta,\epsilon} &= 0, && (4.191)\\
C^{\delta,\epsilon}(T, S_1, S_2) &= g(S_1, S_2),
\end{aligned}
$$

and for the barrier option, respectively,

$$
\begin{aligned}
\mathcal{L}^{\delta} C_B^{\delta,\epsilon} &= 0,\\
C_B^{\delta,\epsilon}(t, B_1(t), S_2) &= 0,\\
C_B^{\delta,\epsilon}(t, S_1, B_2(t)) &= 0,\\
C_B^{\delta,\epsilon}(T, S_1, S_2) &= g(S_1, S_2). && (4.192)
\end{aligned}
$$

The similarity to our problem before can be seen. Thus, an approximation of a two-asset option without barriers (4.36) (and with barriers (4.37) respectively) can be approximated by

$$
\begin{aligned}
C^{\delta,\epsilon}(t,S_1,S_1,\mathbf{v},\mathbf{y}) &\approx Q^{\delta,\epsilon} \qquad (4.193)\\
&= C_{0,0,0,0}+\epsilon_1 C_{1,0,0,0}+\epsilon_2 C_{0,1,0,0}\\
&\quad +\delta_1 C_{0,0,1,0}+\delta_2 C_{0,0,0,1}+\ldots,\\
C_B^{\delta,\epsilon}(t,S_1,S_2,B_1(t),B_2(t),\mathbf{v},\mathbf{y}) &\approx Q_B^{\delta,\epsilon} \qquad (4.194)\\
&= C_{B,0,0,0,0}+\epsilon_1 C_{B,1,0,0,0}+\epsilon_2 C_{B,0,1,0,0}\\
&\quad +\delta_1 C_{B,0,0,1,0}+\delta_2 C_{B,0,0,0,1}+\ldots, \qquad (4.195)
\end{aligned}
$$

where $C_{0,0,0,0}$ and $C_{B,0,0,0,0}$ are given in (B.66) and (B.68), $C_{1,0,0,0}$ and $C_{B,1,0,0,0}$ in (B.74) and (B.80), $C_{0,1,0,0}$ and $C_{B,0,1,0,0}$ in (B.75) and (B.82). $C_{0,0,1,0}$ and $C_{B,0,0,1,0}$ are denoted in (B.70) and (B.72) and $C_{0,0,0,1}$ and $C_{B,0,0,0,1}$ in (B.69) and (B.71). We show in the Appendix B.5 that the results of the terms indicated can be found analogously to Section 4.5.

4.6 Conclusion

In this second chapter of the main part we have presented a multivariate model with stochastic correlation which offers enough flexibility to reflect the stylised facts about correlation explained in the introduction (see 1.1). By analysing high-frequency data we can empirically show that there exists a fast mean-reverting factor (with a time-scale in the order of days) which drives the eigenvalues. Moreover, our model allows to avoid the dimensionality pitfall. In many cases the first three to five eigenvalues are sufficient to explain the dynamics of a certain basket. In some cases an affine characteristic function is available and, thus, eases the calibration of the model. With the perturbation technique we furthermore present an approximation for path-dependant options which is easy and quick to calculate and implement. Moreover, the convergence is proved and the performance is shown numerically. Thus, we contribute a feasible and also empirically valid model to literature.

Part III

Appendix

Appendix A

Appendix for Chapter 3

A.1 Appendix for Section 3.4

A.1.1 Transformations used for PDE

For constant barriers B_1 and B_2 (only GBM framework):

$$\begin{cases} \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 C}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 C}{\partial S_2^2} + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 C}{\partial S_1 \partial S_2} + \\ + r S_1 \frac{\partial C}{\partial S_1} + r S_2 \frac{\partial C}{\partial S_2} + \frac{\partial C}{\partial t} - rC = 0, \\ \\ C(t, B_1, S_2) = 0,\ C(t, S_1, B_2) = 0, \\ C(T, S_1, S_2, B_1, B_2) = g\left(S_1, S_2\right) \mathbb{1}_{\{\iota_t > T, \iota_2 > T\}}. \end{cases} \tag{A.1}$$

Transform S_i to $x_i := \ln(\frac{S_i}{K_i})$ and B_i to $b_i := \ln(\frac{B_i}{K_i})$. The new derivatives in terms of C' are given by

$$\begin{aligned} \frac{\partial C}{\partial S_i} &= \frac{1}{S_i}\frac{\partial C'}{\partial x_i}, \\ \frac{\partial^2 C}{\partial S_i^2} &= \frac{1}{S_i^2}\frac{\partial^2 C'}{\partial x_i^2} - \frac{1}{S_i^2}\frac{\partial C'}{\partial x_i}, \\ \frac{\partial^2 C}{\partial S_i S_j} &= \frac{1}{S_i S_j}\frac{\partial^2 C'}{\partial x_i \partial x_j}. \end{aligned}$$

Thus, the PDE in terms of $C'(t, x_1, x_2, b_1, b_2)$ has the form

$$\begin{cases} \frac{1}{2}\sigma_1^2\frac{\partial^2 C'}{\partial x_1^2} + \frac{1}{2}\sigma_2^2\frac{\partial^2 C'}{\partial x_2^2} + (r - \frac{1}{2}\sigma_1^2)\frac{\partial C'}{\partial x_1} + (r - \frac{1}{2}\sigma_2^2)\frac{\partial C'}{\partial x_2} + \rho\sigma_1\sigma_2\frac{\partial^2 C'}{\partial x_1 \partial x_2} + \\ + \frac{\partial C'}{\partial t} - rC' = 0, \\ \\ C'(t, b_1, x_2) = 0, \; C'(t, x_1, b_2) = 0, \\ C'(T, x_1, x_2, b_1, b_2) = g(x_1, x_2)\,\mathbb{1}_{\{\iota_1 > T, \iota_2 > T\}}. \end{cases} \tag{A.2}$$

Then transform to $G := e^{\int_t^T r(s)ds - \beta_1 x_1 - \beta_2 x_2 + \alpha(T-t)} C'$. The derivatives are given by

$$\begin{aligned}
\frac{\partial C'}{\partial t} &= e^{-\int_t^T r(s)ds + \beta_1 x_1 + \beta_2 x_2 + \alpha t}\left((r+\alpha)G + \frac{\partial G}{\partial t}\right), \\
\frac{\partial C'}{\partial x_i} &= e^{-\int_t^T r(s)ds + \beta_1 x_1 + \beta_2 x_2 + \alpha t}\left(\beta_i G + \frac{\partial G}{\partial x_i}\right), \\
\frac{\partial^2 C'}{\partial x_i^2} &= e^{-\int_t^T r(s)ds + \beta_1 x_1 + \beta_2 x_2 + \alpha t}\left(2\beta_i\frac{\partial G}{\partial x_i} + \beta_i^2 G + \frac{\partial^2 G}{\partial x_i^2}\right), \\
\frac{\partial^2 C}{\partial x_i \partial x_j} &= e^{-\int_t^T r(s)ds + \beta_1 x_1 + \beta_2 x_2 + \alpha t}\left(\beta_i\beta_j G + \beta_j\frac{\partial G}{\partial x_i} + \beta_i\frac{\partial G}{\partial x_j}\right).
\end{aligned}$$

β_1 and β_2 are determined in such a way that we get rid of the first derivatives in x_1 and x_2. Thus,

$$\begin{aligned}
\beta_1 &= \frac{\sigma_1\left(\frac{1}{2} - \frac{r}{\sigma_1^2}\right) - \rho\sigma_2(\frac{1}{2} - \frac{r}{\sigma_2^2})}{\sigma_1(1-\rho^2)}, \\
\beta_2 &= \frac{\sigma_2(\frac{1}{2} - \frac{r}{\sigma_2^2}) - \rho\sigma_1\left(\frac{1}{2} - \frac{r}{\sigma_1^2}\right)}{\sigma_2(1-\rho^2)}, \\
\alpha &= -\frac{1}{2}\sigma_1^2\beta_1^2 - \frac{1}{2}\sigma_2^2\beta_2^2 - (r - \frac{1}{2}\sigma_1^2)\beta_1^2 - (r - \frac{1}{2}\sigma_2^2)\beta_2^2 - \rho\sigma_1\sigma_2\beta_1\beta_2.
\end{aligned}$$

Inserting β_i and α and grouping the derivatives of order one and the terms in G this becomes clear

$$\begin{aligned}
& \frac{\partial G}{\partial x_1}\left(-\frac{1}{2}\sigma_1^2 + \sigma_1^2\beta_1 + \rho\sigma_1\sigma_2\beta_2 + r\right) \\
= \; & \frac{\partial G}{\partial x_1}\left(-\frac{1}{2}\sigma_1^2 + \sigma_1^2\frac{\sigma_1\left(\frac{1}{2} - \frac{r}{\sigma_1^2}\right) - \rho\sigma_2(\frac{1}{2} - \frac{r}{\sigma_2^2})}{\sigma_1(1-\rho^2)} + \rho\sigma_1\sigma_2\frac{\sigma_2(\frac{1}{2} - \frac{r}{\sigma_2^2}) - \rho\sigma_1\left(\frac{1}{2} - \frac{r}{\sigma_1^2}\right)}{\sigma_2(1-\rho^2)} + r\right) \\
= \; & \frac{\partial G}{\partial x_1}\left(\frac{-\frac{1}{2}\sigma_1^2(1-\rho^2)}{(1-\rho^2)} + \frac{\sigma_1^2\left(\frac{1}{2} - \frac{r}{\sigma_1^2}\right) - \rho^2\sigma_1^2\left(\frac{1}{2} - \frac{r}{\sigma_1^2}\right)}{(1-\rho^2)} + \frac{r(1-\rho^2)}{(1-\rho^2)}\right) \\
= \; & 0.
\end{aligned}$$

The same is true for $\frac{\partial G}{\partial t}$.

$$G\left(-r+r+\alpha+(r-\frac{1}{2}\sigma_1^2)\beta_1+(r-\frac{1}{2}\sigma_2^2)\beta_2+\rho\sigma_1\sigma_2\beta_1\beta_2+\frac{1}{2}\sigma_1^2\beta_1^2+\frac{1}{2}\sigma_2^2\beta2^2\right)=0. \tag{A.3}$$

The PDE for $G(t,x_1,x_2,b_1,b_2)$ is given by

$$\begin{cases}\frac{1}{2}\sigma_1^2\frac{\partial^2 G}{\partial x_1^2}+\frac{1}{2}\sigma_2^2\frac{\partial^2 G}{\partial x_2^2}+\rho\sigma_1\sigma_2\frac{\partial^2 G}{\partial x_1\partial x_2}++\frac{\partial G}{\partial t}=0,\\ \\ G\left(t,b_1,x_2\right)=0,\ G\left(t,x_1,b_2\right)=0,\\ G\left(T,x_1,x_2,b_1,b_2\right)=g\left(x_1,x_2\right)\mathbb{1}_{\{\iota_1>T,\iota_2>T\}}e^{-\beta_1x_1-\beta_2x_2}.\end{cases} \tag{A.4}$$

The respective Kolmogorov backward equation for the transition density $p(t,x_1',x_2',x_1,x_2)$ is now given by (see Remark 7)

$$\begin{aligned}-\frac{\partial p}{\partial t}&=\frac{1}{2}\sigma_1^2\frac{\partial^2 p}{\partial x_1^2}+\frac{1}{2}\sigma_2^2\frac{\partial^2 p}{\partial x_2^2}+\rho\sigma_1\sigma_2\frac{\partial^2 p}{\partial x_1\partial x_2} &\text{(A.5)}\\ p\left(t,x_1',x_2',b_1,x_2\right)&=0,\\ p\left(t,x_1',x_2',x_1,b_2\right)&=0,\\ p\left(T,x_1',x_2',x_1,x_2\right)&=\delta(x_1'-x_1)\delta(x_2'-x_2). &\text{(A.6)}\end{aligned}$$

This equation can be transformed in the standard form by the transformation $\tau:=T-t$:

$$\begin{aligned}\frac{\partial p}{\partial \tau}&=\frac{1}{2}\sigma_1^2\frac{\partial^2 p}{\partial x_1^2}+\frac{1}{2}\sigma_2^2\frac{\partial^2 p}{\partial x_2^2}+\rho\sigma_1\sigma_2\frac{\partial^2 p}{\partial x_1\partial x_2} &\text{(A.7)}\\ p\left(\tau,x_1',x_2',x_1',x_2',b_1,x_2\right)&=0,\\ p\left(\tau,x_1',x_2',x_1,b_2\right)&=0,\\ p\left(0,x_1',x_2',x_1,x_2\right)&=\delta(x_1'-x_1)\delta(x_2'-x_2). &\text{(A.8)}\end{aligned}$$

By Fourier transforming this PDE we find that the equation the characteristic function φ has to fulfil. We set $\varphi(\tau,u_1,u_2,x_1',x_2'):=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i(u_1x_1+u_2x_2)}p(\tau,x_1',x_2',x_1,x_2)dx_1dx_2$ or respectively $p(\tau,x_1',x_2',x_1,x_2):=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-i(u_1x_1+u_2x_2)}\varphi(\tau,u_1,u_2,x_1',x_2')du_1du_2$.

Thus,

$$\begin{aligned}
\frac{\partial p}{\partial \tau} &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(u_1x_1+u_2x_2)}\frac{\partial \varphi(\tau,u_1,u_2,x_1',x_2')}{\partial \tau}du_1du_2,\\
\frac{\partial p}{\partial x_1} &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -e^{-i(u_1x_1+u_2x_2)}iu_1\varphi(\tau,u_1,u_2,x_1',x_2')du_1du_2,\\
\frac{\partial^2 p}{\partial x_1^2} &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -e^{-i(u_1x_1+u_2x_2)}u_1^2\varphi(\tau,u_1,u_2,x_1',x_2')du_1du_2,\\
\frac{\partial^2 p}{\partial x_1\partial x_2} &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -e^{-i(u_1x_1+u_2x_2)}u_1u_2\varphi(\tau,u_1,u_2,x_1',x_2')du_1du_2,\\
\varphi(0,u_1,u_2,x_1',x_2') &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(u_1x_1+u_2x_2)}\delta(x_1'-x_1)\delta(x_2'-x_2)dx_1dx_2\\
&= e^{i(u_1x_1'+u_2xz_2')}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \varphi}{\partial \tau} &= -\frac{1}{2}\sigma_1^2u_1^2\varphi-\frac{1}{2}\sigma_2^2u_2^2\varphi-\rho\sigma_1\sigma_2u_1u_2\varphi \qquad (A.9)\\
\varphi\left(\tau,u_1,u_2,b_1,x_2'\right) &= 0,\\
\varphi\left(\tau,u_1,u_2,x_1',b_2\right) &= 0,\\
\varphi\left(0,u_1,u_2,x_1',x_2'\right) &= e^{i(u_1x_1'+u_2xz_2')}. \qquad (A.10)
\end{aligned}$$

For time-dependent barriers, which are applied particularly in the case of stochastic or local volatilities:

$$\begin{cases}
\frac{1}{2}v^{2\nu}\sigma_1^2S_1^2\frac{\partial^2 C}{\partial S_1^2}+\frac{1}{2}v^{2\nu}\sigma_2^2S_2^2\frac{\partial^2 C}{\partial S_2^2}+\rho v^{2\nu}\sigma_1\sigma_2S_1S_2\frac{\partial^2 C}{\partial S_1\partial S_2}+\\
\frac{1}{2}\epsilon^2v^{2\gamma}\frac{\partial^2 C}{\partial v^2}+rS_1\frac{\partial C}{\partial S_1}+rS_2\frac{\partial C}{\partial S_2}+\kappa(\zeta-v)\frac{\partial C}{\partial v}+\frac{\partial C}{\partial t}-rC=0,\\
\\
C\left(t,B_1(t),S_2\right)=0,\ C\left(t,S_1,B_2(t)\right)=0,\\
C\left(T,S_1,S_2,B_1(t),B_2(t)\right)=g\left(S_1,S_2\right)\mathbb{1}_{\{\iota_1>T,\iota_2>T\}}.
\end{cases} \qquad (A.11)$$

To reduce the above PDE transform S_i to $x_i := \ln(\frac{S_ie^{\int_t^T r(s)ds}}{K_i})$ and $B_i(t)$ to $b_i := \ln\left(\frac{B_i(T)}{K_i}\right)$ for $i \in \{1,2\}$. The new derivatives for $C(t,S_1,S_2,K_1,K_2,B_1(t),B_2(t))$ in terms of $C'(t, x_1, x_2, B_1(t), B_2(t))$ are given by

$$\begin{aligned}
\frac{\partial C}{\partial t} &= \frac{\partial C'}{\partial t} - r\sum_{i=1}^{2}\frac{\partial C'}{\partial x_i},\\
\frac{\partial C}{\partial S_i} &= \frac{1}{S_i}\frac{\partial C'}{\partial x_i},\\
\frac{\partial^2 C}{\partial S_i^2} &= \frac{1}{S_i^2}\frac{\partial^2 C'}{\partial x_i^2} - \frac{1}{S_i^2}\frac{\partial C'}{\partial x_i},\\
\frac{\partial^2 C}{\partial S_i S_j} &= \frac{1}{S_i S_j}\frac{\partial^2 C'}{\partial x_i \partial x_j}.
\end{aligned}$$

Thus, the PDE in terms of $C'(t, x_1, x_2, b_1, b_2)$ has the form

$$\begin{cases}
\frac{1}{2}\upsilon^{2\nu}\sigma_1^2\frac{\partial^2 C'}{\partial x_1^2} + \frac{1}{2}\upsilon^{2\nu}\sigma_2^2\frac{\partial^2 C'}{\partial x_2^2} - \frac{1}{2}\upsilon^{2\nu}\sigma_1^2\frac{\partial C'}{\partial x_1} - \frac{1}{2}\upsilon^{2\nu}\sigma_2^2\frac{\partial C'}{\partial x_2} + \rho\upsilon^{2\nu}\sigma_1\sigma_2\frac{\partial C'}{\partial x_1\partial x_2} + \\
\frac{1}{2}\epsilon^2\upsilon^{2\gamma}\frac{\partial^2 C'}{\partial \upsilon^2} + \kappa(\zeta - \upsilon)\frac{\partial C'}{\partial \upsilon} + \frac{\partial C'}{\partial t} - rC' = 0, \\
\\
C'(t, b_1, x_2) = 0,\ C'(t, x_1, b_2) = 0, \\
C'(T, x_1, x_2, b_1, b_2) = g(x_1, x_2)\,\mathbb{1}_{\{\iota_1 > T, \iota_2 > T\}}.
\end{cases} \tag{A.12}$$

Then transform to $G := e^{\int_t^T r(s)ds - c_1 x_1 - c_2 x_2} C'$. The derivatives are given by

$$\begin{aligned}
\frac{\partial C'}{\partial t} &= e^{-\int_t^T r(s)ds + c_1 x_1 + c_2 x_2}\left(rG + \frac{\partial G}{\partial t}\right),\\
\frac{\partial C'}{\partial x_i} &= e^{-\int_t^T r(s)ds + c_1 x_1 + c_2 x_2}\left(c_i G + \frac{\partial G}{\partial x_i}\right),\\
\frac{\partial^2 C'}{\partial x_i^2} &= e^{-\int_t^T r(s)ds + c_1 x_1 + c_2 x_2}\left(2c_i\frac{\partial G}{\partial x_i} + c_i^2 G + \frac{\partial^2 G}{\partial x_i^2}\right),\\
\frac{\partial^2 C'}{\partial x_i \partial x_j} &= e^{-\int_t^T r(s)ds + c_1 x_1 + c_2 x_2}\left(c_i c_j G + c_j\frac{\partial G}{\partial x_i} + c_i\frac{\partial G}{\partial x_j}\right).
\end{aligned}$$

c_1 and c_2 are determined in such a way that we get rid of the first derivatives in x_1 and x_2:

$$\begin{aligned}
c_1 &= \frac{\sigma_1 - \sigma_2\rho}{2\sigma_1(1-\rho^2)},\\
c_2 &= \frac{\sigma_2 - \sigma_1\rho}{2\sigma_2(1-\rho^2)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
& \frac{\partial G}{\partial x_1}\left(-\frac{1}{2}v^{2\nu}\sigma_1^2 + v^{2\nu}\sigma_1^2 c_1 + \rho v^{2\nu}\sigma_1\sigma_2 c_2\right) \\
= & \frac{\partial G}{\partial x_1}\left(-\frac{1}{2}v^{2\nu}\sigma_1^2 + v^{2\nu}\sigma_1^2\frac{\sigma_1-\sigma_2\rho}{2\sigma_1(1-\rho^2)} + \rho v^{2\nu}\sigma_1\sigma_2\frac{\sigma_2-\sigma_1\rho}{2\sigma_2(1-\rho^2)}\right) \\
= & \frac{\partial G}{\partial x_1}\left(v^{2\nu}\frac{-\sigma_1^2(1-\rho^2)}{2(1-\rho^2)} + v^{2\nu}\sigma_1\frac{\sigma_1-\sigma_2\rho}{2(1-\rho^2)} + \rho v^{2\nu}\sigma_1\frac{\sigma_2-\sigma_1\rho}{2(1-\rho^2)}\right) \\
= & \; 0.
\end{aligned}$$

The computation for $\frac{\partial G}{\partial x_2}$ is analogue. In the following we determine the killing rate of the PDE, i.e. the term G:

$$\begin{aligned}
& G\left(c_1c_2\rho\sigma_1\sigma_2 v^{2\nu} + \frac{1}{2}c_1^2\sigma_1^2 v^{2\nu} + \frac{1}{2}c_2^2\sigma_2^2 v^{2\nu} - \frac{1}{2}\sigma_1^2 c_1 v^{2\nu} - \frac{1}{2}\sigma_2^2 c_2 v^{2\nu}\right) \\
= & \; v^{2\nu}G\left(\frac{\sigma_1\sigma_2\rho - \rho^2\sigma_1^2 - \rho^2\sigma_2^2 + \rho^3\sigma_1\sigma_2}{4(1-\rho^2)^2} + \frac{-\sigma_1^2 - \sigma_2^2 + 2\rho\sigma_1\sigma_2}{4(1-\rho^2)}\right. \\
& \left. + \frac{(1+\rho^2)\sigma_1^2 - 4\rho\sigma_1\sigma_2 + \sigma_2^2(1+\rho^2)}{8(1-\rho^2)^2}\right. \\
= & \; -v^{2\nu}G\left(\frac{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}{8(1-\rho^2)}\right). \qquad (A.13)
\end{aligned}$$

The PDE in terms of $G(t, x_1, x_2, b_1, b_2)$ has the form

$$\begin{cases}
\frac{1}{2}v^{2\nu}\sigma_1^2\frac{\partial^2 G}{\partial x_1^2} + \frac{1}{2}v^{2\nu}\sigma_2^2\frac{\partial^2 G}{\partial x_2^2} - v^{2\nu}G\left(\frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}{8(1-\rho^2)}\right) + \rho v^{2\nu}\sigma_1\sigma_2\frac{\partial G}{\partial x_1\partial x_2} + \\
\frac{1}{2}\epsilon^2 v^{2\gamma}\frac{\partial^2 G}{\partial v^2} + \kappa(\zeta - v)\frac{\partial G}{\partial v} + \frac{\partial G}{\partial t} = 0, \\
\\
G(t, b_1, x_2) = 0,\ G(t, x_1, b_2) = 0, \\
G(T, x_1, x_2, b_1, b_2) = e^{-c_1x_1 - c_2x_2}g(x_1, x_2)\mathbb{1}_{\{\iota_1 > T, \iota_2 > T\}}.
\end{cases} \qquad (A.14)$$

In a constant volatility framework ($v^{2\nu} = 1$) it is now possible to remove the component with the killing rate, i.e $-v^{2\nu}(\frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}{8(1-\rho^2)})$, as well by the transformation $U := e^{\alpha(T-t)}G$ as

$$\frac{\partial G}{\partial t} = e^{-\alpha(T-t)}\left(\alpha U + \frac{\partial U}{\partial t}\right). \qquad (A.15)$$

Thus,

$$\begin{aligned}
\alpha &= -\left(c_1c_2\rho\sigma_1\sigma_2 + \frac{1}{2}c_1^2\sigma_1^2 + \frac{1}{2}c_2^2\sigma_2^2 - \frac{1}{2}\sigma_1^2 c_1 - \frac{1}{2}\sigma_2^2 c_2\right) \\
&= \frac{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_2\sigma_2}{8(1-\rho^2)}. \qquad (A.16)
\end{aligned}$$

By introducing

$$z_1 := \frac{1}{\sqrt{1-\rho^2}}\left(\frac{x_1}{\sigma_1} - \rho\frac{x_2}{\sigma_2}\right),$$
$$z_2 := \frac{x_2}{\sigma_2}, \tag{A.17}$$

the mixing derivative in x_1 and x_2 is removed. The derivatives are given by

$$\begin{aligned}
\frac{\partial}{\partial x_1} &= \frac{1}{\sqrt{1-\rho^2}\sigma_1}\frac{\partial}{\partial z_1},\\
\frac{\partial}{\partial x_2} &= \frac{1}{\sigma_2}\frac{\partial}{\partial z_2} - \frac{\rho}{\sqrt{1-\rho^2}\sigma_2}\frac{\partial}{\partial z_1},\\
\frac{\partial^2}{\partial x_1^2} &= \frac{1}{(1-\rho^2)\,\sigma_1^2}\frac{\partial^2}{\partial z_1^2},\\
\frac{\partial^2}{\partial x_2^2} &= \frac{1}{\sigma_2^2}\frac{\partial^2}{\partial z_2^2} + \frac{\rho^2}{(1-\rho^2)\,\sigma_2^2}\frac{\partial^2}{\partial z_1^2} - 2\frac{\rho}{\sqrt{1-\rho^2}\sigma_2^2}\frac{\partial^2}{\partial z_1 \partial z_2},\\
\frac{\partial^2}{\partial x_1 \partial x_2} &= -\frac{\rho}{(1-\rho^2)\,\sigma_1\sigma_1}\frac{\partial^2}{\partial z_1^2} + \frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}}\frac{\partial^2}{\partial z_1\partial z_2}.
\end{aligned}$$

We insert these derivatives and group terms of the same derivative:

$$\begin{aligned}
\frac{\partial^2 G}{\partial z_1^2}\left(\frac{1}{2}\upsilon^{2\nu}\sigma_1^2 + \frac{1}{2}\upsilon^{2\nu}\sigma_2^2\frac{\rho^2}{\sigma_2^2\,(1-\rho^2)} - \upsilon^{2\nu}\sigma_1\sigma_2\rho\frac{\rho}{\sigma_1\sigma_2\,(1-\rho^2)}\right) &= \frac{1}{2}\upsilon^{2\nu}\frac{\partial^2 G}{\partial z_1^2},\\
\frac{\partial^2 G}{\partial z_1 \partial z_2}\left(-2\frac{1}{2}\upsilon^{2\nu}\sigma_2^2\frac{\rho}{\sigma_2^2\sqrt{1-\rho^2}} + \upsilon^{2\nu}\sigma_1\sigma_2\rho\frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}}\right) &= 0.
\end{aligned}$$

Furthermore, we transform $t \to \tau$, where $\tau := T - t$.

$$\begin{cases}
-\frac{1}{2}\upsilon^{2\nu}\frac{\partial^2 G}{\partial z_1^2} - \frac{1}{2}\upsilon^{2\nu}\frac{\partial^2 G}{\partial z_2^2} + \upsilon^{2\nu}G\left(\frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}{8(1-\rho^2)}\right) - \\
\frac{1}{2}\epsilon^2\upsilon^{2\gamma}\frac{\partial^2 G}{\partial \upsilon^2} - \kappa(\zeta-\upsilon)\frac{\partial G}{\partial \upsilon} + \frac{\partial G}{\partial \tau} = 0,\\
\\
G\left(\tau, z_1, 0\right) = 0,\ G\left(\tau, z_1, -\frac{\sqrt{1-\rho^2}}{\rho}z_1\right) = 0,\\
G\left(0, z_1, z_2\right) = e^{-c_1x_1-c_2x_2}g\left(z_1,z_2\right)\mathbb{1}_{\{\iota_1>T,\iota_2>T\}}.
\end{cases} \tag{A.18}$$

The respective Kolmogorov backward equation for the transition density

$p(\tau, z_1', z_2', z_1, z_2, v)$ is given by

$$\begin{aligned}\frac{\partial p}{\partial \tau} &= \frac{1}{2}v^{2\nu}\frac{\partial^2 p}{\partial z_1^2} + \frac{1}{2}v^{2\nu}\frac{\partial^2 p}{\partial z_2^2} - v^{2\nu}p(\frac{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}{8\,(1-\rho^2)}) \\ &+\frac{1}{2}\epsilon^2 v^{2\gamma}\frac{\partial^2 p}{\partial v^2} + \kappa(\zeta - v)\frac{\partial p}{\partial v}, \end{aligned} \tag{A.19}$$

$$\begin{aligned} p\left(\tau, z_1', z_2', z_1, 0\right) &= 0, \\ p\left(\tau, z_1', z_2', z_1, -\frac{\sqrt{1-\rho^2}}{\rho}z_1\right) &= 0, \\ p\left(0, z_1', z_2', z_1, z_2\right) &= \delta(z_1' - z_1)\delta(z_2' - z_2). \end{aligned} \tag{A.20}$$

By Fourier transforming this PDE we find the equation the characteristic function φ has to fulfil. We set $\varphi(\tau, u_1, u_2, z_1', z_2') := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(u_1 z_1 + u_2 z_2)}p(\tau, z_1', z_2', z_1, z_2)dz_1 dz_2$ or respectively $p(\tau, z_1', z_2', z_1, z_2) := \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(u_1 x_1 + u_2 x_2)}\varphi(\tau, u_1, u_2, z_1', z_2')du_1 du_2$. Thus,

$$\begin{aligned}\frac{\partial p}{\partial \tau} &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(u_1 z_1 + u_2 z_2)}\frac{\partial \varphi(\tau, u_1, u_2, z_1', z_2')}{\partial \tau}du_1 du_2, \\ \frac{\partial p}{\partial z_1} &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -e^{-i(u_1 z_1 + u_2 z_2)}iu_1\varphi(\tau, u_1, u_2, z_1', z_2')du_1 du_2, \\ \frac{\partial^2 p}{\partial z_1^2} &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -e^{-i(u_1 z_1 + u_2 z_2)}u_1^2\varphi(\tau, u_1, u_2, z_1', z_2')du_1 du_2, \\ \frac{\partial p}{\partial v} &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(u_1 z_1 + u_2 z_2)}\frac{\partial \varphi(\tau, u_1, u_2, z_1', z_2')}{\partial v}du_1 du_2, \\ \varphi(0, u_1, u_2, z_1', z_2')) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(u_1 z_1 + u_2 z_2)}\delta(z_1' - z_1)\delta(z_2' - z_2)dz_1 dz_2 \\ &= e^{i(u_1 z_1' + u_2 z_2')} \end{aligned} \tag{A.21}$$

and we get

$$\begin{aligned}\frac{\partial \varphi}{\partial \tau} &= -\frac{1}{2}v^{2\nu}u_1^2\varphi - \frac{1}{2}v^{2\nu}u_2^2\varphi - v^{2\nu}\varphi(\frac{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}{8\,(1-\rho^2)}) \\ &+\frac{1}{2}\epsilon^2 v^{2\gamma}\frac{\partial^2 \varphi}{\partial v^2} + \kappa(\zeta - v)\frac{\partial \varphi}{\partial v}, \end{aligned} \tag{A.22}$$

$$\begin{aligned} \varphi\left(\tau, u_1, u_2, z_1', 0\right) &= 0, \\ \varphi\left(\tau, u_1, u_2, z_1', -\frac{\sqrt{1-\rho^2}}{\rho}z_1'\right) &= 0, \\ \varphi\left(0, u_1, u_2, z_1', z_2'\right) &= e^{i(u_1 z_1' + u_2 z_2')}. \end{aligned} \tag{A.23}$$

A.1.2 Characteristic functions

Proof. (Derivation of Heston-type characteristic function)

Assume the following affine form

$$\varphi_H(\tau,\mathbf{u}) = \exp\left\{ iu_1 z_1 + iu_2 z_2 + \frac{1}{\sigma_v^2}\left(A_H(\tau,\mathbf{u}) + B_H(\tau,\mathbf{u})v\right)\right\}, \tag{A.24}$$

and plug it in the PDE (3.37) with $\nu = \gamma = \frac{1}{2}$:

$$\begin{aligned}
\frac{1}{\sigma_v^2}\left(\frac{\partial A_H(\tau,\mathbf{u})}{\partial\tau} + v\frac{\partial B_H(\tau,\mathbf{u})}{\partial\tau}\right) &= -\frac{1}{2}v\left(u_1^2 + u_2^2 + \frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_1}{4(1-\rho^2)}\right)\\
&\quad + \frac{1}{2\sigma_v^2}vB_H(\tau,\mathbf{u})^2 + \frac{1}{\sigma_v^2}\kappa(\zeta - v)B_H(\tau,\mathbf{u}),\\
A_H(0,\mathbf{u}) &= 0, \qquad \text{(A.25)}\\
B_H(0,\mathbf{u}) &= 0. \qquad \text{(A.26)}
\end{aligned}$$

We find then the following ODEs for A_H and B_H.

$$\begin{aligned}
\frac{\partial A_H(\tau,\mathbf{u})}{\partial\tau} - \kappa\zeta B_H(\tau,\mathbf{u}) &= 0,\\
A_H(0,\mathbf{u}) &= 0, \qquad \text{(A.27)}\\
\frac{\partial B_H(\tau,\mathbf{u})}{\partial\tau} - \frac{1}{2}B_H(\tau,\mathbf{u})^2 + \kappa B_H(\tau,\mathbf{u}) &\\
+\frac{1}{2}\sigma_v^2\left(u_1^2+u_2^2+\frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_1}{4(1-\rho^2)}\right) &= 0,\\
B_H(0,\mathbf{u}) &= 0. \qquad \text{(A.28)}
\end{aligned}$$

For the proof of the solution see Lipton (see [83], p. 380). (A.28) is a Riccati ODE which we solve with the usual transformation $B_H = -2\frac{\frac{\partial E_H}{\partial\tau}}{E_H}$. In terms of $E_H(\tau,\mathbf{u})$, $A_H(\tau,\mathbf{u})$ can be written as $A_H = -2\kappa\zeta\ln E_H$. The ODE and appropriate initial conditions for $E_H(\tau,\mathbf{u})$ have the form

$$\begin{aligned}
-2\frac{\frac{\partial^2 E_H}{\partial\tau^2}}{E_H} + 2\left(\frac{\frac{\partial E_H}{\partial\tau}}{E_H}\right)^2 - 2\left(\frac{\frac{\partial E_H}{\partial\tau}}{E_H}\right)^2 &\\
-2\kappa\frac{\frac{\partial E_H}{\partial\tau}}{E_H} + \frac{1}{2}\sigma_v^2\left(u_1^2+u_2^2+\frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}{4(1-\rho^2)}\right) &= 0,\\
\Leftrightarrow\quad \frac{\partial^2 E_H}{\partial\tau^2} + \kappa\frac{\partial E_H}{\partial\tau} - \frac{1}{4}\sigma_v^2\left({u_1}^2+u_2^2+\frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}{4(1-\rho^2)}\right)E_H &= 0,\\
E_H(0,\mathbf{u}) = 1,\quad \frac{\partial E_H}{\partial\tau}(0,\mathbf{u}) &= 0. \qquad \text{(A.29)}
\end{aligned}$$

The general solution of (A.29) can be indicated by

$$E_H(\tau,\mathbf{u}) = e_{H+}e^{K_{H+}\tau} + e_{H-}e^{K_{H-}\tau}, \tag{A.30}$$

where $e_{H\pm}$ are arbitrary constants and $K_{H\pm}$ solve the following quadratic equation

$$K_{H\pm}^2 + \kappa K_{H\pm} - \frac{1}{4}\sigma_v^2\left(u_1^2 + u_2^2 + \frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}{4(1-\rho^2)}\right) = 0. \tag{A.31}$$

Hence,

$$K_{H\pm} = \frac{-\kappa \pm \sqrt{\kappa^2 + \sigma_v^2\left(u_1^2+u_2^2+\frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}{4(1-\rho^2)}\right)}}{2}. \tag{A.32}$$

We introduce $\mathfrak{d} = \mathfrak{d}(\mathbf{u}) = \sqrt{\kappa^2 + \sigma_v^2\left(u_1^2+u_2^2+\frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}{4(1-\rho^2)}\right)}$. $E_H(\tau,\mathbf{u})$ has, thus, the following form

$$E_H(\tau,\mathbf{u}) = e_{H+}e^{\frac{1}{2}(-\kappa+\mathfrak{d})\tau} + e_{H-}e^{\frac{1}{2}(-\kappa-\mathfrak{d})\tau}. \tag{A.33}$$

From the first initial condition follows $e_{H+} = 1 - e_{H-}$. From the second equation we obtain for $e_{H-} = \frac{-\kappa+\mathfrak{d}}{2\mathfrak{d}}$ and for $e_{H+} = \frac{\kappa+\mathfrak{d}}{2\mathfrak{d}}$. Thus,

$$\begin{aligned} E_H(\tau,\mathbf{u}) &= \frac{(\kappa+\mathfrak{d})e^{\frac{1}{2}(-\kappa+\mathfrak{d})\tau} + (-\kappa+\mathfrak{d})e^{\frac{1}{2}(-\kappa-\mathfrak{d})\tau}}{2\mathfrak{d}} \\ &= e^{\frac{1}{2}(-\kappa+\mathfrak{d})\tau}\frac{\kappa+\mathfrak{d}+(-\kappa+\mathfrak{d})e^{-\mathfrak{d}\tau}}{2\mathfrak{d}}. \end{aligned} \tag{A.34}$$

A_H and B_H have then the following solutions:

$$\begin{aligned} B_H(\tau,\mathbf{u}) &= \frac{(\kappa-\mathfrak{d})(1-\exp(-\mathfrak{d}\tau))}{1-\frac{\kappa-\mathfrak{d}}{\kappa+\mathfrak{d}}\exp(-\mathfrak{d}\tau)} \\ &= \kappa - \mathfrak{d}\frac{\sinh(\frac{\mathfrak{d}}{2}\tau)+\frac{\kappa}{\mathfrak{d}}\cosh(\frac{\mathfrak{d}}{2}\tau)}{\cosh(\frac{\mathfrak{d}}{2}\tau)+\frac{\kappa}{\mathfrak{d}}\sinh(\frac{\mathfrak{d}}{2}\tau)}, \end{aligned} \tag{A.35}$$

$$\begin{aligned} A_H(\tau,\mathbf{u}) &= \zeta\kappa\left((\kappa-\mathfrak{d})\,\tau - 2\ln\left(\frac{1-\frac{\kappa-\mathfrak{d}}{\kappa+\mathfrak{d}}\exp(-\mathfrak{d}\tau)}{1-\frac{\kappa-\mathfrak{d}}{\kappa+\mathfrak{d}}}\right)\right) \\ &= \kappa\zeta\left(\kappa\tau - 2\ln\left(\frac{\kappa}{\mathfrak{d}}\sinh\left(\frac{\mathfrak{d}}{2}\tau\right)+\cosh\left(\frac{\mathfrak{d}}{2}\tau\right)\right)\right). \end{aligned} \tag{A.36}$$

□

Proof. (Derivation of Stein and Stein-type characteristic function)

For the Stein model we guess the following affine form.

$$\varphi_{S2}(\tau,\mathbf{u}) = \exp\left\{iu_1 z_1 + iu_2 z_2 + \frac{1}{\sigma_v^2}\left(A_{S2}(\tau,\mathbf{u}) + B_{S2}(\tau,\mathbf{u})v + C_{S2}(\tau,\mathbf{u})v^2\right)\right\}, \quad \text{(A.37)}$$

where $A_{S2}(0,\mathbf{u}) = B_{S2}(0,\mathbf{u}) = C_{S2}(0,\mathbf{u}) = 0$. We plug it in the PDE (3.37) with $\nu = 1$ and $\gamma = 0$:

$$\begin{aligned}
&\frac{1}{\sigma_v^2}\left(\frac{\partial A_{S2}(\tau,\mathbf{u})}{\partial\tau} + v\frac{\partial B_{S2}(\tau,\mathbf{u})}{\partial\tau} + v^2\frac{\partial C_{S2}(\tau,\mathbf{u})}{\partial\tau}\right)\\
= & -\frac{1}{2}v^2\left(u_1^2 + u_2^2 + \frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_1}{4(1-\rho^2)}\right)\\
&+\frac{1}{2\sigma_v^2}\Big(\frac{1}{\sigma_v^4}B_{S2}(\tau,\mathbf{u})^2 + \frac{4}{\sigma_v^4}vB_{S2}(\tau,\mathbf{u})C_{S2}(\tau,\mathbf{u})\\
&+\frac{2}{\sigma_v^2}C_{S2}(\tau,\mathbf{u}) + \frac{4}{\sigma_v^4}v^2 C_{S2}(\tau,\mathbf{u})^2\Big)\\
&+\frac{1}{\sigma_v^2}\kappa(\zeta - v)\left(B_{2S}(\tau,\mathbf{u}) + 2vC_{2S}(\tau,\mathbf{u})\right),
\end{aligned}$$

$$A_{S2}(0,\mathbf{u}) = 0, \quad \text{(A.38)}$$

$$B_{S2}(0,\mathbf{u}) = 0, \quad \text{(A.39)}$$

$$C_{S2}(0,\mathbf{u}) = 0. \quad \text{(A.40)}$$

Inserting (A.37) in (3.32) we find the following ODEs

$$\frac{\partial A_{S2}(\tau,\mathbf{u})}{\partial\tau} - \kappa\zeta B_{S2}(\tau,\mathbf{u}) - \sigma_v^2 C_{S2}(\tau,\mathbf{u}) - \frac{1}{2}B_{S2}(\tau,\mathbf{u})^2 = 0,$$

$$A_{S2}(0,\mathbf{u}) = 0, \quad \text{(A.41)}$$

$$\frac{\partial B_{S2}(\tau,\mathbf{u})}{\partial\tau} + \kappa B_{S2}(\tau,\mathbf{u}) - 2B_{S2}(\tau,\mathbf{u})C_{S2}(\tau,\mathbf{u}) - 2\kappa\zeta C_{S2}(\tau,\mathbf{u}) = 0,$$

$$B_{S2}(0,\mathbf{u}) = 0, \quad \text{(A.42)}$$

$$\begin{aligned}
&\frac{\partial C_{S2}(\tau,\mathbf{u})}{\partial\tau} + 2\kappa C_{S2}(\tau,\mathbf{u}) - 2C_{S2}(\tau,\mathbf{u})^2\\
&+\frac{\sigma_v^2}{2}\left(u_1^2 + u_2^2 + \frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}{4(1-\rho^2)}\right) = 0,
\end{aligned}$$

$$C_{S2}(0,\mathbf{u}) = 0. \quad \text{(A.43)}$$

(A.43) is again a Riccati equation. We set $C_{S2} = -\frac{1}{2}\frac{\frac{\partial D_{S2}}{\partial\tau}}{D_{S2}}$. The ODE and appropriate initial conditions for $D_{S2}(\tau, \mathbf{u})$ have the form:

$$\begin{aligned}
-\frac{1}{2}\frac{\frac{\partial^2 D_{S2}}{\partial\tau^2}}{D_{S2}} + \frac{1}{2}\left(\frac{\frac{\partial D_{S2}}{\partial\tau}}{D_{S2}}\right)^2 - \frac{1}{2}\left(\frac{\frac{\partial D_{S2}}{\partial\tau}}{D_{S2}}\right)^2 - \kappa\frac{\frac{\partial D_{S2}}{\partial\tau}}{D_{S2}} & \\
+\frac{\sigma_v^2}{2}\left(u_1^2 + u_2^2 + \frac{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}{4(1-\rho^2)}\right) &= 0, \\
\frac{\partial^2 D_{S2}}{\partial\tau^2} + 2\kappa\frac{\partial D_{S2}}{\partial\tau} - \sigma_v^2\left(u_1^2 + u_2^2 + \frac{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}{4(1-\rho^2)}\right) D_{S2} &= 0, \\
D_{S2}(0, \mathbf{u}) = 1, \quad \frac{\partial D_{S2}}{\partial\tau}(0, \mathbf{u}) = 0. &
\end{aligned} \tag{A.44}$$

The general solution of (A.44) can be indicated by

$$D_{S2}(\tau, \mathbf{u}) = \mathbf{d}_{S2+}e^{L_{S2+}\tau} + \mathbf{d}_{S2-}e^{L_{S2-}\tau}, \tag{A.45}$$

where $\mathbf{d}_{S2\pm}$ are arbitrary constants and $L_{S2\pm}$ solves the following quadratic equation

$$L_{S2\pm}^2 + 2\kappa L_{S2\pm} - \sigma_v^2\left(u_1^2 + u_2^2 + \frac{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}{4(1-\rho^2)}\right) = 0. \tag{A.46}$$

Hence,

$$\begin{aligned}
L_{S2\pm} &= -\kappa \pm \sqrt{\kappa^2 + \sigma_v^2\left(u_1^2 + u_2^2 + \frac{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}{4(1-\rho^2)}\right)}, \\
\mathfrak{d} = \mathfrak{d}(\mathbf{u}) &= \sqrt{\kappa^2 + \sigma_v^2\left(u_1^2 + u_2^2 + \frac{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}{4(1-\rho^2)}\right)}.
\end{aligned}$$

$D_{S2}(\tau, \mathbf{u})$ has, thus, the following form

$$D_{S2}(\tau, \mathbf{u}) = \mathbf{d}_{S2+}e^{(-\kappa+\mathfrak{d})\tau} + \mathbf{d}_{S2-}e^{(-\kappa-\mathfrak{d})\tau}. \tag{A.47}$$

From the first initial condition follows $\mathbf{d}_{S2+} = 1 - \mathbf{d}_{S2-}$. From the second equation we obtain for $\mathbf{d}_{S2-} = \frac{-\kappa+\mathfrak{d}}{2\mathfrak{d}}$ and for $\mathbf{d}_{S2+} = \frac{\kappa+\mathfrak{d}}{2\mathfrak{d}}$.

$$\begin{aligned}
D_{S2}(\tau, \mathbf{u}) &= \frac{(\kappa+\mathfrak{d})e^{(-\kappa+\mathfrak{d})\tau} + (-\kappa+\mathfrak{d})e^{(-\kappa-\mathfrak{d})\tau}}{2\mathfrak{d}} \\
&= \frac{e^{-\kappa\tau}}{2\mathfrak{d}}\left(\kappa(e^{\mathfrak{d}\tau} - e^{-\mathfrak{d}\tau}) + \mathfrak{d}\left(e^{\mathfrak{d}\tau} + e^{-\mathfrak{d}\tau}\right)\right) \\
&= e^{-\kappa\tau}\left(\frac{\kappa}{\mathfrak{d}}\sinh(\mathfrak{d}\tau) + \cosh(\mathfrak{d}\tau)\right).
\end{aligned}$$

Similar to (A.35) we conclude

$$C_{S2}(\tau,\mathbf{u}) = \frac{1}{2}\left(\kappa - \eth\frac{\sinh(\eth\tau) + \frac{\kappa}{\eth}\cosh(\eth\tau)}{\cosh(\eth\tau) + \frac{\kappa}{\eth}\sinh(\eth\tau)}\right). \qquad \text{(A.48)}$$

(A.42) is a first-order ODE of the form $\frac{dy}{d\tau} + p(\tau)y = q(\tau)$, which can be solved by $y = \frac{\int \exp(\int^{\tau'} p(\tau'')d\tau'')q(\tau')d\tau' + c}{\exp(\int^{\tau} p(\tau')d\tau')}$. In this case $p(\tau) = \kappa - 2C_{S2}(\tau,\mathbf{u})$ and $q(\tau) = 2\kappa\zeta C_{S2}(\tau,\mathbf{u})$ (see Arfken [7], p. 465ff). Due to the initial condition B_{S2} has the form

$$\begin{aligned}
B_{S2}(\tau,\mathbf{u}) &= \left(\int_0^{\tau} 2\kappa\zeta C_{S2}(\tau',\mathbf{u})\exp\left\{\int_0^{\tau'}(\kappa - 2C_{S2}(\tau'',\mathbf{u})d\tau'')\right\}d\tau'\right) \\
&\quad \exp\left\{-\int_0^{\tau}\kappa - 2C_{S2}(\tau',\mathbf{u})d\tau'\right\} \\
&= \left(\int_0^{\tau} 2\kappa\zeta C_{S2}(\tau',\mathbf{u})\exp\left\{\kappa\tau' + \ln\left(D_{S2}(\tau',\mathbf{u})\right)\right\}d\tau'\right) \\
&\quad \exp\left\{-\kappa\tau - \ln\left(D_{S2}(\tau,\mathbf{u})\right)\right\} \\
&= \left(-\int_0^{\tau}\kappa\zeta\frac{\partial D_{S2}(\tau',\mathbf{u})}{\partial\tau'}\exp(\kappa\tau')d\tau'\right)\frac{1}{D_{S2}(\tau,\mathbf{u})}\exp\left\{-\kappa\tau\right\} \\
&= -\kappa\zeta\int_0^{\tau}\Bigg(-\kappa\left(\frac{\kappa}{\eth}\sinh(\eth\tau') + \cosh(\eth\tau')\right) \\
&\quad +\eth\left(\frac{\kappa}{\eth}\cosh(\eth\tau') + \sinh(\eth\tau')\right)\Bigg)d\tau'\frac{1}{D_{S2}(\tau,\mathbf{u})}\exp\left\{-\kappa\tau\right\} \\
&= -\frac{\kappa\zeta\exp\left\{-\kappa\tau\right\}}{D_{S2}(\tau,\mathbf{u})}\int_0^{\tau}(\eth - \frac{\kappa^2}{\eth})\sinh(\eth\tau')d\tau' \\
&= \frac{\kappa\zeta\exp\left\{-\kappa\tau\right\}}{D_{S2}(\tau,\mathbf{u})}\left((1 - \frac{\kappa^2}{\eth^2})\cosh(0) - (1 - \frac{\kappa^2}{\eth^2})\cosh(\eth\tau)\right) \\
&= \frac{\kappa\zeta\exp\left\{-\kappa\tau\right\}}{D_{S2}(\tau,\mathbf{u})}\left(\left(\frac{\kappa^2}{\eth^2} - 1\right)\cosh(\eth\tau) - \left(\frac{\kappa^2}{\eth^2} - 1\right)\right) \\
&= \frac{1}{\eth}\Bigg(\frac{\left(\kappa\zeta\eth - \frac{\kappa^3\zeta}{\eth}\right) + \kappa^2\zeta\left(\frac{\kappa}{\eth}\cosh(\eth\tau) + \sinh(\eth\tau)\right)}{\cosh(\eth\tau) + \frac{\kappa}{\eth}\sinh(\eth\tau)} \\
&\quad - \frac{\kappa\zeta\eth\left(\frac{\kappa}{\eth}\sinh(\eth\tau) + \cosh(\eth\tau)\right)}{\cosh(\eth\tau) + \frac{\kappa}{\eth}\sinh(\eth\tau)}\Bigg) \\
&= \frac{1}{\eth}\left(\frac{\left(\kappa\zeta\eth - \frac{\kappa^3\zeta}{\eth}\right) + \kappa^2\zeta\left(\sinh(\eth\tau) + \frac{\kappa}{\eth}\cosh(\eth\tau)\right)}{\cosh(\eth\tau) + \frac{\kappa}{\eth}\sinh(\eth\tau)} - \kappa\zeta\eth\right).
\end{aligned}$$

For A_{S2} we show that

$$\begin{aligned} A_{S2}(\tau,\mathbf{u}) &= -\frac{\sigma_v^2}{2}\ln\Big(\cosh(\mathfrak{d}\tau)+\frac{\kappa}{\mathfrak{d}}\sinh(\mathfrak{d}\tau)\Big)+\frac{\sigma_v^2}{2}\kappa\tau \\ &+\frac{\kappa^2\zeta^2\mathfrak{d}^2-\kappa^4\zeta^2}{2\mathfrak{d}^3}\left(\frac{\sinh(\mathfrak{d}\tau)}{\cosh(\mathfrak{d}\tau)+\frac{\kappa}{\mathfrak{d}}\sinh(\mathfrak{d}\tau)}-\mathfrak{d}\tau\right) \\ &+\frac{\kappa^2\zeta(\kappa\zeta\mathfrak{d}-\frac{\kappa^3\zeta}{\mathfrak{d}})}{\mathfrak{d}^3}\left(\frac{\cosh(\mathfrak{d}\tau)-1}{\cosh(\mathfrak{d}\tau)+\frac{\kappa}{\mathfrak{d}}\sinh(\mathfrak{d}\tau)}\right) \end{aligned}$$

fulfils (A.41). For that reason we compute $\kappa\zeta B_{S2}+\frac{1}{2}B_{S2}^2$:

$$\begin{aligned} \kappa\zeta B_{S2}+\frac{1}{2}B_{S2}^2 &= \frac{\frac{1}{2}\kappa^2\zeta^2+\frac{\kappa^6\zeta^2}{2\mathfrak{d}^4}-\frac{\kappa^4\zeta^2}{\mathfrak{d}^2}}{\big(\cosh(\mathfrak{d}\tau)+\frac{\kappa}{\mathfrak{d}}\sinh(\mathfrak{d}\tau)\big)^2} \\ &+\frac{\frac{\kappa^2\zeta}{\mathfrak{d}^2}(\kappa\zeta\mathfrak{d}-\frac{\kappa^3\zeta}{\mathfrak{d}})\big(\sinh(\mathfrak{d}\tau)+\frac{\kappa}{\mathfrak{d}}\cosh(\mathfrak{d}\tau)\big)}{\big(\cosh(\mathfrak{d}\tau)+\frac{\kappa}{\mathfrak{d}}\sinh(\mathfrak{d}\tau)\big)^2} \\ &+\frac{\frac{1}{2}\frac{\kappa^4\zeta^2}{\mathfrak{d}^2}\big(\sinh(\mathfrak{d}\tau)+\frac{\kappa}{\mathfrak{d}}\cosh(\mathfrak{d}\tau)\big)^2}{\big(\cosh(\mathfrak{d}\tau)+\frac{\kappa}{\mathfrak{d}}\sinh(\mathfrak{d}\tau)\big)^2}-\frac{1}{2}\kappa^2\zeta^2. \end{aligned} \tag{A.49}$$

This is compared to

$$\begin{aligned} \frac{\partial A_{S2}(\tau,\mathbf{u})}{\partial\tau} &= \frac{\sigma_v^2}{2}\left(\kappa-\mathfrak{d}\frac{\sinh(\mathfrak{d}\tau)+\frac{\kappa}{\mathfrak{d}}\cosh(\mathfrak{d}\tau)}{\cosh(\mathfrak{d}\tau)+\frac{\kappa}{\mathfrak{d}}\sinh(\mathfrak{d}\tau)}\right) \\ &+\left(\frac{\cosh^2(\mathfrak{d}\tau)\mathfrak{d}-\sinh^2(\mathfrak{d}\tau)\mathfrak{d}}{(\cosh(\mathfrak{d}\tau)+\frac{\kappa}{\mathfrak{d}}\sinh(\mathfrak{d}\tau))^2}-\mathfrak{d}\right)\left(\frac{\kappa^2\zeta^2}{2\mathfrak{d}}-\frac{\kappa^4\zeta^2}{2\mathfrak{d}^3}\right) \\ &+\left(\frac{\kappa\sinh^2(\mathfrak{d}\tau)-\kappa\cosh^2(\mathfrak{d}\tau)+\mathfrak{d}\sinh(\mathfrak{d}\tau)+\kappa\cosh(\mathfrak{d}\tau)}{(\cosh(\mathfrak{d}\tau)+\frac{\kappa}{\mathfrak{d}}\sinh(\mathfrak{d}\tau))^2}\right) \\ &\cdot\kappa^2\zeta\left(\frac{\kappa\zeta}{\mathfrak{d}^2}-\frac{\kappa^3\zeta}{\mathfrak{d}^4}\right) \\ &= \sigma_v^2 C_{S2}(\tau,\mathbf{u})+\frac{\frac{1}{2}\kappa^2\zeta^2-\frac{\kappa^4\zeta^2}{2\mathfrak{d}^2}-\frac{\kappa^4\zeta^2}{\mathfrak{d}^2}+\frac{\kappa^6\zeta^2}{\mathfrak{d}^4}}{\big(\cosh(\mathfrak{d}\tau)+\frac{\kappa}{\mathfrak{d}}\sinh(\mathfrak{d}\tau)\big)^2} \\ &-\frac{1}{2}\kappa^2\zeta^2+\frac{\kappa^4\zeta^2}{2\mathfrak{d}^2}+\kappa^2\zeta(\frac{\kappa\zeta}{\mathfrak{d}^2}-\frac{\kappa^3\zeta^2}{\mathfrak{d}^4})\mathfrak{d}\frac{\sinh(\mathfrak{d}\tau)+\frac{\kappa}{\mathfrak{d}}\cosh(\mathfrak{d}\tau)}{\big(\cosh(\mathfrak{d}\tau)+\frac{\kappa}{\mathfrak{d}}\sinh(\mathfrak{d}\tau)\big)^2} \end{aligned}$$

$$
\begin{aligned}
&= \sigma_v^2 C_{S2}(\tau,\mathbf{u}) + \frac{\frac{1}{2}\kappa^2\zeta^2 - \frac{\kappa^4\zeta^2}{\mathfrak{d}^2} + \frac{\kappa^6\zeta^2}{2\mathfrak{d}^4}}{\left(\cosh(\mathfrak{d}\tau) + \frac{\kappa}{\mathfrak{d}}\sinh(\mathfrak{d}\tau)\right)^2} - \frac{\kappa^2\zeta^2}{2} \\
&\quad + \frac{\frac{\kappa^2\zeta}{\mathfrak{d}^2}\left(\kappa\zeta\mathfrak{d} - \frac{\kappa^3\zeta}{\mathfrak{d}}\right)\left(\sinh(\mathfrak{d}\tau) + \frac{\kappa}{\mathfrak{d}}\cosh(\mathfrak{d}\tau)\right)}{\left(\cosh(\mathfrak{d}\tau) + \frac{\kappa}{\mathfrak{d}}\sinh(\mathfrak{d}\tau)\right)^2} \\
&\quad + \frac{\frac{\kappa^4\zeta^2}{2\mathfrak{d}^2}\left(\cosh(\mathfrak{d}\tau) + \frac{\kappa}{\mathfrak{d}}\sinh(\mathfrak{d}\tau)\right)^2}{\left(\cosh(\mathfrak{d}\tau) + \frac{\kappa}{\mathfrak{d}}\sinh(\mathfrak{d}\tau)\right)^2} \\
&\quad + \frac{\frac{\kappa^4\zeta^2}{2\mathfrak{d}^2}\left(\sinh^2(\mathfrak{d}\tau) - \cosh^2(\mathfrak{d}\tau)\right)}{\left(\cosh(\mathfrak{d}\tau) + \frac{\kappa}{\mathfrak{d}}\sinh(\mathfrak{d}\tau)\right)^2} + \frac{\frac{\kappa^6\zeta^2}{2\mathfrak{d}^4}\left(-\sinh^2(\mathfrak{d}\tau) + \cosh^2(\mathfrak{d}\tau)\right)}{\left(\cosh(\mathfrak{d}\tau) + \frac{\kappa}{\mathfrak{d}}\sinh(\mathfrak{d}\tau)\right)^2} \\
&= \sigma_v^2 C_{S2}(\tau,\mathbf{u}) + \kappa\zeta B_{S2}(\tau,\mathbf{u}) + \frac{1}{2}B_{S2}(\tau,\mathbf{u})^2. \qquad \text{(A.50)}
\end{aligned}
$$

□

A.1.3 Method of images in a half-space

For the Corollaries 5 and 7 we apply the method of images in a half-space to the characteristic function φ. Thus, according to (3.32) $q(\tau, z_1, z_2, v)$ satisfies the following PDE

$$
\begin{aligned}
\frac{\partial q}{\partial \tau} &= \frac{1}{2}v^{2\nu}\frac{\partial^2 q}{\partial z_1^2} + \frac{1}{2}v^{2\nu}\frac{\partial^2 q}{\partial z_2^2} - v^{2\nu}\frac{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}{8(1-\rho^2)}q \\
&\quad + \frac{1}{2}\sigma_v^2 v^{2\gamma}\frac{\partial^2 q}{\partial v^2} + \kappa(\zeta - v)\frac{\partial q}{\partial v}, \qquad \text{(A.51)}
\end{aligned}
$$

with the following initial and boundary conditions

$$
\begin{aligned}
q(0, z_1, z_2, v) &= \delta(z_1' - z_1)\delta(z_2' - z_2), \qquad \text{(A.52)} \\
q(\tau, z_1, 0, v) &= 0. \qquad \text{(A.53)}
\end{aligned}
$$

As before the idea is to find a solution $\bar{G}^F$ for (A.51) and (A.52) in the whole plane first and restrict it to the actual space (A.53) by the following approach $\bar{G} = \bar{G}^F + \bar{G}^G$.
$\bar{G}^F$ has been found in (3.35). $\bar{G}^G$ must satisfy (A.51) and, as $\bar{G}$ is required to vanish at $z_2 = 0$, then $\bar{G}^F = -\bar{G}^G$ in $z_2 = 0$. Due to the form of $\bar{G}^F$ we test for $\bar{G}^G(\tau, u_1, u_2, z_1' - z_1, z_2' - z_2) = -\bar{G}^F(\tau, u_1, u_2, z_1' - z_1, z_2' + z_2)$. It can be easily seen that $\bar{G}^G$ satisfies (A.51) as there are only pure second order derivatives in z_i. On the boundary $\bar{G}^G$ is given by

$$
\begin{aligned}
\bar{G}^G(\tau, z_1', z_2', v; z_1, z_2) &= \frac{1}{4\pi^2}\int_{i\varpi_2-\infty}^{i\varpi_2+\infty}\int_{i\varpi_1-\infty}^{i\varpi_1+\infty} \exp\{iu_1(z_1' - z_1) + iu_2 z_2 \\
&\quad + V(\tau,\mathbf{u})\}\, du_1 du_2 \qquad \text{(A.54)}
\end{aligned}
$$

As $V(\tau, \mathbf{u})$ is symmetric in u_2 (see (3.36)) $\bar{G}^G$ is invariant to a reflection $-u_2 \to u_2$ and, thus, also to a reflection $-z_2 \to z_2$. Hence $\bar{G}^G = -\bar{G}^F$ at the boundary $z_2 = 0$. In the bounded area, i.e. for $z_2 > 0$ the initial condition is satisfied as G^F fulfils it. The transition density function is therefore given by

$$
\begin{aligned}
q(\tau, z_1', z_2', v; z_1, z_2) \quad = \quad & \frac{1}{4\pi^2} \int_{i\varpi_2-\infty}^{i\varpi_2+\infty} \int_{i\varpi_1-\infty}^{i\varpi_1+\infty} \Big(\bar{G}(\tau, u_1, u_2, z_1' - z_1, z_2' - z_2) \\
& -\bar{G}(\tau, u_1, u_2, z_1' - z_1, z_2' + z_2) \Big) du_1 du_2, \\
& \mathbf{u} = \mathbf{w} + i\boldsymbol{\varpi} \in S_{\varphi}.
\end{aligned} \tag{A.55}
$$

A.2 Appendix for Section 3.6

A.2.1 Derivation of general pricing formula for defaultable options in GBM and stochastic volatility framework

Proof. In the following we want to price defaultable options with a payoff $g(S_2)$ in T in an extended CreditGrades model. In this section we place the general expressions. In Section A.2.2 the specific propositions are derived.
Due to the shifted form of the model we apply slightly different transformations than in Appendix A.1.1.
Hence,

$$C^D(t,S_1,S_2) \;=\; \mathbb{E}\left[e^{-\int_t^T r(s)ds} g(S_2)\mathbb{1}_{\{\iota_1>T\}}\,|\mathcal{F}_t\right], \tag{A.56}$$

where

$$\iota_1 = \inf\,(t' \in (t,T\;] : S_1(t') \leq 0)\,, \tag{A.57}$$

where the expectation is with respect to $\tilde{\mathcal{Q}}$. In the GBM framework C^D fulfils the following partial differential equation and boundary conditions

$$\begin{cases} \frac{1}{2}\sigma_1^2(S_1+D_1(t))^2\frac{\partial^2 C^D}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2\frac{\partial^2 C^D}{\partial S_2^2} + \rho\sigma_1\sigma_2(S_1+D_1(t))S_2\frac{\partial^2 C^D}{\partial S_1 \partial S_2} + \\ +(r-d_1)S_1\frac{\partial C^D}{\partial S_1} + (r-d_2)S_2\frac{\partial C^D}{\partial S_2} + \frac{\partial C^D}{\partial t} - rC^D = 0, \\ \\ C^D(t,0,S_2) = 0, \\ C^D(T,S_1,S_2) = g(S_2)\mathbb{1}_{\{\iota_1>T\}}. \end{cases}$$

In the stochastic volatility framework the PDE and boundary condition are given by

$$\begin{cases} \frac{1}{2}v^{2\nu}\sigma_1^2(S_1+D_1(t))^2\frac{\partial^2 C^D}{\partial S_1^2} + \frac{1}{2}v^{2\nu}\sigma_2^2 S_2^2\frac{\partial^2 C^D}{\partial S_2^2} \\ +\rho v^{2\nu}\sigma_1\sigma_2(S_1+D_1(t))S_2\frac{\partial^2 C^D}{\partial S_1 \partial S_2} + \frac{1}{2}\sigma_v^2 v^{2\gamma}\frac{\partial^2 C}{\partial v^2} + \kappa(\zeta - v)\frac{\partial C}{\partial v} \\ +(r-d_1)S_1\frac{\partial C^D}{\partial S_1} + (r-d_2)S_2\frac{\partial C^D}{\partial S_2} + \frac{\partial C^D}{\partial t} - rC^D = 0, \\ \\ C^D(t,0,S_2) = 0, \\ C^D(T,S_1,S_2) = g(S_2)\mathbb{1}_{\{\iota_1>T\}}. \end{cases}$$

Those PDEs can be reduced to

$$\begin{cases} \frac{\partial G^D}{\partial t} + \frac{1}{2}\sigma_1^2 \frac{\partial^2 G^D}{\partial x_1^{*2}} + \frac{1}{2}\sigma_2^2 \frac{\partial^2 G^D}{\partial x_2^2} - \frac{1}{2}\sigma_1^2 \frac{\partial G^D}{\partial x_1^*} - \frac{1}{2}\sigma_2^2 \frac{\partial G^D}{\partial x_2} + \\ \rho\sigma_1\sigma_2 \frac{\partial^2 G^D}{\partial x_1^* \partial x_2} = 0, \\ \\ G^D(t, 0, x_2) = 0, \\ G^D(T, x_1^*, x_2) = g(x_2)\mathbb{1}_{\{\iota_1 > T\}}, \end{cases} \tag{A.58}$$

or respectively,

$$\begin{cases} \frac{\partial G^D}{\partial t} + \frac{1}{2}\sigma_1^2 v^{2\nu} \frac{\partial^2 G^D}{\partial x_1^{*2}} + \frac{1}{2}\sigma_2^2 v^{2\nu} \frac{\partial^2 G^D}{\partial x_2^2} - \frac{1}{2}\sigma_1^2 v^{2\nu} \frac{\partial G^D}{\partial x_1^*} - \frac{1}{2}\sigma_2^2 v^{2\nu} \frac{\partial G^D}{\partial x_2} + \\ \rho\sigma_1\sigma_2 v^{2\nu} \frac{\partial^2 G^D}{\partial x_1^* \partial x_2} + \frac{1}{2}\sigma_v^2 v^{2\gamma} \frac{\partial^2 G^D}{\partial v^2} + \kappa(\zeta - v)\frac{\partial G^D}{\partial v} = 0, \\ \\ G^D(t, 0, x_2) = 0, \\ G^D(T, x_1^*, x_2) = g(x_2)\mathbb{1}_{\{\iota_1 > T\}}, \end{cases} \tag{A.59}$$

where we use the transformations $x_2(t) := \ln \frac{S_2(t)e^{\int_t^T (r(s)-d_2(s))ds}}{K_2}$, $x_1^*(t) := \ln\left(\frac{S_1(t)+D_1(t)}{D_1(t)}\right)$. We further apply the transformation $G^D(t, x_1^*, x_2) := e^{\int_t^T r(s)ds} C^D(t, S_1, S_2)$. The transformations for x_1^* are given below:

$$\begin{aligned}
\frac{\partial C^D}{\partial S_1} &= \frac{\partial C^D}{\partial x_1^*} \frac{1}{S_1 + D_1(t)}, \\
\frac{\partial^2 C^D}{\partial S_1^2} &= \frac{1}{(S_1 + D_1(t))^2}\left(\frac{\partial^2 C^D}{\partial x_1^{*2}} - \frac{\partial C^D}{\partial x_1^*}\right), \\
\frac{\partial^2 C^D}{\partial S_1 \partial S_2} &= \frac{1}{S_2}\frac{1}{S_1 + D_1(t)}\frac{\partial^2 C^D}{\partial x_1^* \partial x_2}, \\
\frac{\partial C^D}{\partial t} &= \frac{\partial C^D}{\partial t} + \frac{\partial C^D}{\partial x_1^*}\frac{D_1(t)}{S_1 + D_1(t)}\frac{D_1(t)^2(r - d_1) - (S_1 + D_1(t))D_1(t)(r_1 - d_1)}{D_1(t)^2} \\
&\quad -\frac{\partial C^D}{\partial x_2}(r - d_2) \\
&= \frac{\partial C^D}{\partial t} - \frac{\partial C^D}{\partial x_1^*}(r - d_1)\frac{S_1}{S_1 + D_1(t)} - \frac{\partial C^D}{\partial x_2}(r - d_2).
\end{aligned}$$

Now we can pursue the transformations suggested in (A.12)-(A.16) for the GBM model. Then choose $y_1 := \frac{x_1^*}{\sigma_1}$, and $y_2 := \frac{x_2}{\sigma_2}$. For the stochastic volatility framework we proceed as in Section 3.4.1 with only the difference that we set $z_1 := \frac{x_1^*}{\sigma_1}$ and $z_2 := \frac{1}{\sqrt{1-\rho^2}}(\frac{x_2}{\sigma_2} - \rho\frac{x_1^*}{\sigma_1})$. Finally, we end with the following Kolmogorov backward equation for the probability density $p(\tau, y'_1, y'_2, y_1, y_2)$ in the GBM model.

$$\frac{\partial p}{\partial \tau} = \frac{1}{2}\frac{\partial^2 p}{\partial y_1^2} + \frac{1}{2}\frac{\partial^2 p}{\partial y_2^2} + \rho\frac{\partial^2 p}{\partial y_1 \partial y_2}, \tag{A.60}$$

with the following initial and boundary conditions

$$\begin{aligned} p(0, y_1, y_2) &= \delta(y'_1 - y_1)\delta(y'_2 - y_2), \\ p(\tau, 0, y_2) &= 0. \end{aligned} \tag{A.61}$$

For the GBM framework the solution for the free space is (see [83], p. 511)

$$\begin{aligned} p^F(\tau, y'_1, y'_2, y_1, y_2) &= \frac{1}{2\pi\sqrt{1-\rho^2}\tau} e^{-\frac{\left((y'_1-y_1)^2+(y'_2-y_2)^2-2\rho(y'_1-y_1)(y'_2-y_2)\right)}{2(1-\rho^2)\tau}} \\ &=: \bar{p}^F(\tau, y'_1 - y_1, y'_2 - y_2). \end{aligned} \tag{A.62}$$

We apply the method of images (see [121], p. 476ff) to restrict the fundamental solution p^F to the area the problem is defined for. This method is appropriate when the region to which the solution should be bounded is highly symmetric: A solution for the free space is first derived ($p^F(\tau, y'_1, y'_2, y_1, y_2)$) and then restricted to the defined region via symmetry, i.e. the principle of reflection. The point $(-y_1, -2\rho y_1 + y_2)$ is symmetric with respect to the boundary $y'_1 = 0$ to (y_1, y_2). We test this by substituting $(-y_1, -2\rho y_1 + y_2)$ in the nominator of the exponential in (A.62):

$$\begin{aligned} & y_1^2 + (y'_2 - y_2 + 2\rho y_1)^2 - 2\rho y_1(y'_2 - y_2 + 2\rho y_1) \\ = {} & y_1^2 + (y'_2 - y_2)^2 + 4\rho^2 y_1^2 + 4\rho y_1(y'_2 - y_2) - 2\rho y_1(y'_2 - y_2) - 4\rho^2 y_1^2 \\ = {} & y_1^2 + (y'_2 - y_2)^2 + 2\rho y_1(y'_2 - y_2) \end{aligned} \tag{A.63}$$

Thus,

$$\begin{aligned} p^G(\tau, y'_1, y'_2, y_1, y_2) &= \frac{1}{2\pi\sqrt{1-\rho^2}\tau} e^{-\frac{\left((y'_1+y_1)^2+(y'_2+2\rho y_1-y_2)^2-2\rho(y'_1+y_1)(y'_2+2\rho y_1-y_2)\right)}{2(1-\rho^2)\tau}} \\ &=: \bar{p}^F(\tau, y'_1 + y_1, y'_2 + 2\rho y_1 - y_2). \end{aligned} \tag{A.64}$$

satisfies the requirement to compensate p^F on the boundary, i.e. $p^F - p^G = 0$ if $y_1 = 0$.

p^G must also satisfy the PDE (A.60):

$$
\begin{aligned}
\frac{\partial p^G}{\partial \tau} &= -\frac{p^G}{\tau} + p^G \frac{((y_1' + y_1)^2 + (y_2' + 2\rho y_1 - y_2)^2 - 2\rho(y_1' + y_1)(y_2' + 2\rho y_1 - y_2))}{2(1-\rho^2)\tau^2}, \\
\frac{\partial p^G}{\partial y_1} &= -p^G \frac{(y_1' + y_1) + (y_2' - y_2 + 2\rho y_1)2\rho - \rho((y_2' - y_2 + 2\rho y_1) + (y_1' + y_1)2\rho)}{(1-\rho^2)\tau} \\
&= -p^G \frac{(y_1' + y_1) + \rho(y_2' - y_2) - 2\rho^2 y_1'}{\tau(1-\rho^2)}, \\
\frac{\partial p^G}{\partial y_2} &= -p^G \frac{-(y_2' - y_2 + 2\rho y_1) + \rho(y_1' + y_1)}{(1-\rho^2)\tau}, \\
\frac{\partial^2 p^G}{\partial y_1^2} &= p^G \frac{((y_1' + y_1) + \rho(y_2' - y_2) - 2\rho^2 y_1')^2 - (1-\rho^2)\tau}{(1-\rho^2)^2\tau^2}, \\
\frac{\partial^2 p^G}{\partial y_2^2} &= p^G \frac{(-(y_2' - y_2 + 2\rho y_1) + \rho(y_1' + y_1))^2 - (1-\rho^2)\tau}{(1-\rho^2)^2\tau^2}, \\
\frac{\partial^2 p^G}{\partial y_1 \partial y_2} &= p^G \frac{(-(y_2' - y_2 + 2\rho y_1) + \rho(y_1' + y_1))((y_1' + y_1) + \rho(y_2' - y_2) - 2\rho^2 y_1')}{(1-\rho^2)^2\tau^2} \\
&\quad + \frac{\rho(1-\rho^2)\tau}{(1-\rho^2)^2\tau^2} p^G.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{\partial p^G}{\partial \tau} &= \frac{1}{2}\frac{\partial^2 p^G}{\partial y_1^2} + \frac{1}{2}\frac{\partial^2 p^G}{\partial y_2^2} + \rho \frac{\partial^2 p^G}{\partial y_1 \partial y_2}, \\
&\quad -2\tau(1-\rho^2) + (y_1' + y_1)^2 + (y_2' - y_2 + 2\rho y_1)^2 - 2\rho(y_1' + y_1)(y_2' - y_2 + 2\rho y_1) \\
&= \frac{((y_1' + y_1) - 2\rho^2(y_1' + y_1) + \rho(y_2' - y_2 + 2\rho y_1))^2 - (1-\rho^2)\tau}{1-\rho^2} \\
&\quad + \frac{(\rho(y_1' + y_1) - (y_2' - y_2 + 2\rho y_1))^2 - (1-\rho^2)\tau + 2\rho^2(1-\rho^2)\tau}{1-\rho^2} \\
&\quad + \frac{2\rho\left(-(y_2' - y_2 + 2\rho y_1) + \rho(y_1' + y_1)\right)\left((y_1' + y_1) + \rho(y_2' - y_2 + 2\rho y_1) - 2\rho^2(y_1' + y_1)\right)}{1-\rho^2} \\
&= \frac{-(1-\rho^2)2\tau(1-\rho^2) + (y_2' - y_2 + 2\rho y_1)^2(1 + \rho^2 - 2\rho^2)}{1-\rho^2} \\
&\quad + \frac{(y_1' + y_1)^2(\rho^2 + 1 + 4\rho^4 - 4\rho^2 + 2\rho^2 - 4\rho^4)}{1-\rho^2} \\
&\quad - \frac{2\rho(y_1' + y_1)(y_2' - y_2 + 2\rho y_1)(1 + (-1 + 2\rho^2) + 1 - 2\rho^2 - \rho^2)}{1-\rho^2} \\
&= -(1-\rho^2)2\tau + (y_1' + y_1)^2 + (y_2' - y_2 + 2\rho y_1)^2 - 2\rho(y_1' + y_1)(y_2' - y_2 + 2\rho y_1).
\end{aligned}
$$

Hence, with α as in (A.16) and with y_1, y_2 as defined above the solution for general payoff functions $g(y_2)$ is found by

$$
\begin{aligned}
C^D(\tau, y_1, y_2) &= e^{x_1^* c_1 + x_2 c_2 - \alpha\tau - \int_t^T r(s)ds} \Bigg(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{p}^F(\tau, y_1' - y_1, y_2' - y_2) \\
&\quad e^{-c_1 x_1^{*\prime} - c_2 x_2'} g(y_2') dy_1' dy_2' \\
&- \int_{-\infty}^{\infty} \int_0^{\infty} \bar{p}^F(\tau, y_1 + y_1', y_2' + 2\rho y_1 - y_2) \\
&\quad e^{-c_1 x_1^{*\prime} - c_2 x_2'} g(y_2') dy_1' dy_2' \Bigg). \qquad (A.65)
\end{aligned}
$$

For further reference we compute the following double integral

$$
\begin{aligned}
L &= \frac{1}{2\pi\sqrt{1-\rho^2}\tau} \int_{b_1}^{\infty} \int_{b_2}^{\infty} e^{\gamma_1 y_1 + \gamma_2 y_2} e^{-\frac{y_1^2 + y_2^2 - 2\rho y_1 y_2}{2(1-\rho^2)\tau}} dy_2 dy_1 \\
&= \frac{1}{2\pi\sqrt{1-\rho^2}\tau} \int_{b_1}^{\infty} \int_{b_2}^{\infty} e^{\gamma_1 y_1 + \gamma_2 y_2} e^{-\frac{\tilde{y}_1^2 + \tilde{y}_2^2 - 2\rho \tilde{y}_1 \tilde{y}_2}{2(1-\rho^2)\tau}} \\
&\quad e^{-\frac{-\tau^2(\gamma_1+\rho\gamma_2)^2 - \tau^2(\gamma_2+\rho\gamma_1)^2 + 2y_1\tau(\gamma_1+\rho\gamma_2) + 2y_2\tau(\gamma_2+\rho\gamma_1)}{2(1-\rho^2)\tau}} \\
&\quad e^{-\frac{-2\rho y_1 \tau(\gamma_2+\rho\gamma_1) - 2\rho y_2 \tau(\gamma_1+\rho\gamma_2) + 2\rho\tau^2(\gamma_1+\rho\gamma_2)(\gamma_2+\rho\gamma_1)}{2(1-\rho^2)\tau}} dy_2 dy_1 \\
&= \frac{1}{2\pi\sqrt{1-\rho^2}\tau} \int_{b_1}^{\infty} \int_{b_2}^{\infty} e^{-\frac{-\tau(\gamma_1^2(1+\rho^2) + \gamma_2^2(1+\rho^2) + 4\rho\gamma_1\gamma_2) + 2\gamma_1 y_1(1-\rho^2) + 2\gamma_2 y_2(1-\rho^2)}{2(1-\rho^2)}} \\
&\quad e^{-\frac{\tau(2\rho^2\gamma_1^2 + 2\rho^2\gamma_2^2 + 2\rho\gamma_1\gamma_2 + 2\rho^3\gamma_1\gamma_2)}{2(1-\rho^2)}} e^{-\frac{\tilde{y}_1^2 + \tilde{y}_2^2 - 2\rho\tilde{y}_1\tilde{y}_2}{2(1-\rho^2)\tau}} e^{\gamma_1 y_1 + \gamma_2 y_2} dy_2 dy_1 \\
&= \frac{1}{2\pi\sqrt{1-\rho^2}\tau} \int_{b_1 - \tau(\gamma_1+\rho\gamma_2)}^{\infty} \int_{b_2 - \tau(\gamma_2+\rho\gamma_1)}^{\infty} e^{\tau\frac{\gamma_1^2+\gamma_2^2+2\rho\gamma_1\gamma_2}{2}} e^{-\frac{\tilde{y}_1^2 + \tilde{y}_2^2 - 2\rho\tilde{y}_1\tilde{y}_2}{2(1-\rho^2)\tau}} d\tilde{y}_2 d\tilde{y}_1 \\
&= \frac{e^{\tau\frac{\gamma_1^2+\gamma_2^2+2\rho\gamma_1\gamma_2}{2}}}{2\pi\sqrt{1-\rho^2}} \int_{\frac{b_1}{\sqrt{\tau}} - \sqrt{\tau}(\gamma_1+\rho\gamma_2)}^{\infty} \int_{\frac{b_2}{\sqrt{\tau}} - \sqrt{\tau}(\gamma_2+\rho\gamma_1)}^{\infty} e^{-\frac{\hat{y}_1^2 + \hat{y}_2^2 - 2\rho\hat{y}_1\hat{y}_2}{2(1-\rho^2)}} d\hat{y}_2 d\hat{y}_1 \\
&= e^{\tau\frac{\gamma_1^2+\gamma_2^2+2\rho\gamma_1\gamma_2}{2}} \mathcal{N}_2\left(\sqrt{\tau}(\gamma_1+\rho\gamma_2) - \frac{b_1}{\sqrt{\tau}}, \sqrt{\tau}(\gamma_2+\rho\gamma_1) - \frac{b_2}{\sqrt{\tau}}; \rho\right), \qquad (A.66)
\end{aligned}
$$

where $\tilde{y}_1 = y_1 - \tau(\gamma_1 + \rho\gamma_2)$, $\tilde{y}_2 = y_2 - \tau(\gamma_2 + \rho\gamma_1)$, and $\hat{y}_i = \frac{\tilde{y}_i}{\sqrt{\tau}}$.
Furthermore, note that

$$
\begin{aligned}
&\frac{1}{2\pi\sqrt{1-\rho^2}\tau} e^{-\frac{\left((y_1'-y_1)^2 + (y_2'-y_2)^2 - 2\rho(y_1'-y_1)(y_2'-y_2)\right)}{2(1-\rho^2)\tau}} \\
&= \frac{1}{2\pi\sqrt{1-\rho^2}\tau} e^{-\frac{{y_1'}^2 + {y_2'}^2 - 2\rho y_1' y_2' + y_1^2 + y_2^2 - 2\rho y_1 y_2 - 2y_1'(y_1 - \rho y_2) - 2y_2'(y_2 - \rho y_1)}{2(1-\rho^2)\tau}} \qquad (A.67)
\end{aligned}
$$

and

$$\begin{aligned} & \frac{1}{2\pi\sqrt{1-\rho^2}\tau} e^{-\frac{\left((y_1'+y_1)^2+(y_2'-y_2+2\rho y_1)^2-2\rho(y_1'+y_1)(y_2'-y_2+2\rho y_1)\right)}{2(1-\rho^2)\tau}} \\ = & \frac{1}{2\pi\sqrt{1-\rho^2}\tau} e^{-\frac{{y_1'}^2+{y_2'}^2-2\rho y_1' y_2'+y_1^2+y_2^2-2\rho y_1 y_2+2y_1'(y_1-2\rho^2 y_1+\rho y_2)-2y_2'(y_2-\rho y_1)}{2(1-\rho^2)\tau}} \end{aligned} \tag{A.68}$$

For the stochastic covariance framework we proceed from (A.59) exactly as in 3.4.1. With Corollary 5 we can, thus, derive

$$\begin{aligned} C^D(t,S_1,S_2,v) &= \frac{e^{x_1^* c_1 + x_2 c_2 - \int_t^T r(s)ds}}{4\pi^2\sigma_1\sigma_2\sqrt{1-\rho^2}} \int_{i\varpi_2-\infty}^{i\varpi_2+\infty} \int_{i\varpi_1-\infty}^{i\varpi_1+\infty} \hat{g}_{C^D}(u_1) \\ & \Big(\varphi(\tau,\mathbf{u},z_1,z_2)-\varphi(\tau,\mathbf{u},z_1,-z_2)\Big) du_1 du_2, \\ & \mathbf{u} \in S_{C^D} = S_\varphi \cap S_{g_{C^D}}. \end{aligned}$$

□

A.2.2 Proof of propositions 4-8

Proof. (Proof of proposition 4)

Thus, with A.65 and $g(S_2(T)) = S_2(T) = e^{x_2'}$ in the GBM framework the value of the index in t is given by

$$\begin{aligned} C_S^D(\tau,y_1,y_2) &= e^{x_1^* c_1 + x_2 c_2 - \alpha\tau - \int_t^T r(s)ds} \Bigg(\int_{-\infty}^{\infty}\int_0^{\infty} p^F(\tau, y_1'-y_1, y_2'-y_2) \\ & e^{-c_1 {x_1^*}' + (1-c_2)x_2'} dy_1' dy_2' \\ &- \int_{-\infty}^{\infty}\int_0^{\infty} p^F(\tau, y_1'+y_1, y_2'+2\rho y_1 - y_2) \\ & e^{-c_1 {x_1^*}' + (1-c_2)x_2'} dy_1' dy_2' \Bigg), \end{aligned}$$

where $x_2 = \ln(S_2 e^{\int_t^T (r(s)-d_2(s))ds})$. With (A.66)-(A.68) follows

$$
\begin{aligned}
C_S^D(\tau, y_1, y_2) &= e^{x_1^* c_1 + x_2 c_2 - \alpha\tau - \int_t^T r(s)ds} \exp\left\{-\frac{y_1^2 + y_2^2 - 2y_1 y_2 \rho}{2(1-\rho^2)\tau}\right\} \frac{1}{2\pi\sqrt{1-\rho^2}\tau} \\
&\left(\int_{-\infty}^{\infty}\int_0^{\infty} \exp\left\{-\frac{{y_1'}^2 + {y_2'}^2 - 2y_1' y_2' \rho}{2(1-\rho^2)\tau}\right\}\right. \\
&\left(\exp\left\{\left(-c_1\sigma_1 + \frac{y_1 - \rho y_2}{(1-\rho^2)\tau}\right) y_1' + \left((1-c_2)\sigma_2 + \frac{y_2 - \rho y_1}{(1-\rho^2)\tau}\right) y_2'\right\}\right. \\
&- \exp\left\{\left(-c_1\sigma_1 - \frac{y_1 - 2\rho^2 y_1 + \rho y_2}{(1-\rho^2)\tau}\right) y_1'\right\} \\
&\left.\left.\exp\left\{\left((1-c_2)\sigma_2 + \frac{y_2 - \rho y_1}{(1-\rho^2)\tau}\right) y_2'\right\}\right) dy_1' dy_2'\right) \\
&= e^{x_1^* c_1 + x_2 c_2 - \alpha\tau - \int_t^T r(s)ds} \exp\left\{-\frac{y_1^2 + y_2^2 - 2y_1 y_2 \rho}{2(1-\rho^2)\tau}\right\} \\
&\left(\frac{e^{\frac{\tau}{2}({\gamma_1^+}^2 + {\eta_2^+}^2 + 2\rho\gamma_1^+\eta_2^+)}}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty}\int_{-\sqrt{\tau}(\gamma_1^+ + \rho\eta_2^+)}^{\infty} \exp\left\{-\frac{{y_1'}^2 + {y_2'}^2 - 2y_1' y_2' \rho}{2(1-\rho^2)\tau}\right\} d\hat{y}_1 d\hat{y}_2\right. \\
&\left. -e^{\frac{\tau}{2}({\gamma_1^-}^2 + {\eta_2^-}^2 + 2\rho\gamma_1^-\eta_2^-)} \int_{-\infty}^{\infty}\int_{-\sqrt{\tau}(\gamma_1^- + \rho\eta_2^-)}^{\infty} \exp\left\{-\frac{{y_1'}^2 + {y_2'}^2 - 2y_1' y_2' \rho}{2(1-\rho^2)\tau}\right\} d\hat{y}_1 d\hat{y}_2\right) \\
&= e^{x_1^* c_1 + x_2 c_2 - \alpha\tau - \int_t^T r(s)ds} \exp\left\{-\frac{y_1^2 + y_2^2 - 2y_1 y_2 \rho}{2(1-\rho^2)\tau}\right\} \\
&\left(e^{\frac{\tau}{2}({\gamma_1^+}^2 + {\eta_2^+}^2 + 2\rho\gamma_1^+\eta_2^+)} \mathcal{N}\left(\sqrt{\tau}(\gamma_1^+ + \rho\eta_2^+)\right)\right. \\
&\left. -e^{\frac{\tau}{2}({\gamma_1^-}^2 + {\eta_2^-}^2 + 2\rho\gamma_1^-\eta_2^-)} \mathcal{N}\left(\sqrt{\tau}(\gamma_1^- + \rho\eta_2^-)\right)\right), \qquad \text{(A.69)}
\end{aligned}
$$

with

$$
\begin{aligned}
\gamma_1^+ &= -c_1\sigma_1 + \frac{y_1 - \rho y_2}{(1-\rho^2)\tau}, \\
\eta_2^+ &= (1-c_2)\sigma_2 + \frac{y_2 - \rho y_1}{(1-\rho^2)\tau}, \\
\gamma_1^- &= -c_1\sigma_1 + \frac{-y_1(1-2\rho^2) - \rho y_2}{(1-\rho^2)\tau}, \\
\eta_2^- &= \eta_2^+.
\end{aligned}
$$

To simplify we calculate

$$\begin{aligned}
\mathbf{d}_1 := \sqrt{\tau}(\gamma_1^+ + \rho\eta_2^+) &= \sqrt{\tau}\left(-c_1\sigma_1 + \rho(1-c_2)\sigma_2 + \frac{x_1^*}{\sigma_1\tau}\right) \\
&= \sqrt{\tau}\left(-\frac{\sigma_1 - \rho\sigma_2}{2(1-\rho^2)} + \rho\frac{\sigma_2(1-2\rho^2)+\rho\sigma_1}{2(1-\rho^2)} + \frac{x_1^*}{\sigma_1\tau}\right) \\
&= \frac{x_1^*}{\sigma_1\sqrt{\tau}} - \frac{\sigma_1\sqrt{\tau}}{2} + \rho\sigma_2\sqrt{\tau}, \\
\tilde{\mathbf{d}}_1 := \sqrt{\tau}(\gamma_1^- + \rho\eta_2^-) &= \sqrt{\tau}\left(-\frac{\sigma_1 - \rho\sigma_2}{2(1-\rho^2)} + \rho\frac{\sigma_2(1-2\rho^2)+\rho\sigma_1}{2(1-\rho^2)} - \frac{x_1^*}{\sigma_1\tau}\right) \\
&= -\frac{x_1^*}{\sigma_1\sqrt{\tau}} - \frac{\sigma_1\sqrt{\tau}}{2} + \rho\sigma_2\sqrt{\tau},
\end{aligned}$$

$$\begin{aligned}
\frac{\tau}{2}({\gamma_1^+}^2 + {\eta_2^+}^2 + 2\rho\gamma_1^+\eta_2^+) &= \frac{\tau}{2}\Bigg(c_1^2\sigma_1^2 + (1-c_2)^2\sigma_2^2 - 2\rho\sigma_1\sigma_2 c_1(1-c_2) \\
&\quad + \left(\frac{y_1-\rho y_2}{(1-\rho^2)\tau}\right)^2 - 2c_1\sigma_1\frac{y_1-\rho y_2}{(1-\rho^2)\tau} + \left(\frac{y_2-\rho y_1}{(1-\rho^2)\tau}\right)^2 \\
&\quad + 2(1-c_2)\sigma_2\frac{y_2-\rho y_1}{(1-\rho^2)\tau} + 2\rho(1-c_2)\sigma_2\frac{y_1-\rho y_2}{(1-\rho^2)\tau} \\
&\quad - 2\rho c_1\sigma_1\frac{y_2-\rho y_1}{(1-\rho^2)\tau} + 2\rho\frac{(y_2-\rho y_1)(y_1-\rho y_2)}{(1-\rho^2)^2\tau^2}\Bigg) \\
&= \frac{\tau}{2}\Bigg(\sigma_1^2\left(\frac{\sigma_1-\rho\sigma_2}{\sigma_1(1-\rho^2)2}\right)^2 + \left(\frac{\sigma_2-\rho\sigma_1}{\sigma_2(1-\rho^2)2}\right)^2\sigma_2^2 \\
&\quad + 2\rho\sigma_1\sigma_2\frac{\sigma_1-\rho\sigma_2}{\sigma_1(1-\rho^2)2}\frac{\sigma_2-\rho\sigma_1}{\sigma_2(1-\rho^2)2} + \sigma_2^2 - 2c_2\sigma_2^2 - 2\rho c_1\sigma_1\sigma_2 \\
&\quad - 2c_1\sigma_1\frac{y_1}{\tau} + 2(1-c_2)\sigma_2\frac{y_2}{\tau} \\
&\quad + \frac{y_1^2(1+\rho^2-2\rho^2) + y_2^2(\rho^2+1-2\rho^2) - 2\rho y_1 y_2(1-\rho^2)}{(1-\rho^2)^2\tau^2}\Bigg) \\
&= \frac{\tau}{2}\Bigg(\frac{\sigma_1^2(1-\rho^2) + \sigma_2^2(1-\rho^2) - 2\rho\sigma_1\sigma_2(1-\rho^2)}{4(1-\rho^2)^2} \\
&\quad + \sigma_2^2 - \sigma_2\frac{\sigma_2-\rho\sigma_1}{1-\rho^2} - \sigma_2\frac{\rho\sigma_1-\rho^2\sigma_2}{1-\rho^2}\Bigg) \\
&\quad + \left(-c_1x_1^* + (1-c_2)x_2 + \frac{y_1^2+y_2^2-2\rho y_1y_2}{2(1-\rho^2)\tau}\right) \\
&= \alpha\tau - c_1x_1^* + (1-c_2)x_2 + \frac{y_1^2+y_2^2-2\rho y_1y_2}{2(1-\rho^2)\tau}, \qquad \text{(A.70)}
\end{aligned}$$

$$\begin{aligned}
\frac{\tau}{2}(\gamma_1^{-2}+\eta_2^{-2}+2\rho\gamma_1^-\eta_2^-) &= \alpha\tau+\frac{\tau}{2}\Bigg(\left(\frac{-y_1(1-2\rho^2)-\rho y_2}{(1-\rho^2)\tau}\right)^2 \\
&\quad -2c_1\sigma_1\frac{-y_1(1-2\rho^2)-\rho y_2}{(1-\rho^2)\tau} \\
&\quad +\left(\frac{y_2-\rho y_1}{(1-\rho^2)\tau}\right)^2+2(1-c_2)\sigma_2\frac{y_2-\rho y_1}{(1-\rho^2)\tau} \\
&\quad +2\rho(1-c_2)\sigma_2\frac{-y_1(1-2\rho^2)-\rho y_2}{(1-\rho^2)\tau}-2\rho c_1\sigma_1\frac{y_2-\rho y_1}{(1-\rho^2)\tau} \\
&\quad +2\rho\frac{(y_2-\rho y_1)(-y_1(1-2\rho^2)-\rho y_2)}{(1-\rho^2)^2\tau^2}\Bigg) \\
&= \alpha\tau+\frac{\tau}{2}\Bigg(-2c_1\sigma_1\frac{-y_1(1-2\rho^2+\rho^2)}{(1-\rho^2)\tau} \\
&\quad +2(1-c_2)\sigma_2\frac{y_2(1-\rho^2)-y_1\rho(1+1-2\rho^2)}{(1-\rho^2)\tau} \\
&\quad +\frac{y_1^2(1+4\rho^4-4\rho^2+\rho^2+2\rho^2(1-2\rho^2))+y_2^2(\rho^2+1-2\rho^2)}{(1-\rho^2)^2\tau^2} \\
&\quad -\frac{2\rho y_1y_2(-(1-2\rho^2)+1+(1-2\rho^2)-\rho^2)}{(1-\rho^2)^2\tau^2}\Bigg) \\
&= \alpha\tau+c_1x_1^*+(1-c_2)\left(x_2-2\rho\sigma_2\frac{x_1^*}{\sigma_1}\right) \\
&\quad +\frac{y_1^2+y_2^2-2\rho y_1y_2}{2(1-\rho^2)\tau}, \qquad \text{(A.71)}
\end{aligned}$$

where $\alpha=\frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}{8(1-\rho^2)}$. Hence,

$$\begin{aligned}
C_S^D(t,S_1,S_2) &= e^{x_1^*c_1+x_2c_2-\alpha\tau-\int_t^T r(s)ds} \\
&\quad \Big(\exp\{\alpha\tau-c_1x_1^*+(1-c_2)x_2\}\,\mathcal{N}_2(\mathbf{d}_1,y_2,\rho) \\
&\quad -\exp\left\{\alpha\tau+c_1x_1^*-2(1-c_2)\frac{\sigma_2\rho x_1^*}{\sigma_1}+(1-c_2)x_2\right\} \\
&\quad \mathcal{N}_2\left(\tilde{\mathbf{d}}_1,y_2,\rho\right)\Big) \\
&= S_2e^{-\int_t^T d_2(s)ds}\Big(\mathcal{N}_2(\mathbf{d}_1,y_2,\rho) \\
&\quad -\exp\left\{2x_1^*(c_1-\rho\frac{\sigma_2}{\sigma_1}(1-c_2))\right\}\mathcal{N}_2\left(\tilde{\mathbf{d}}_1,y_2,\rho\right)\Big),
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{d}_1 &= \frac{x_1^*}{\sigma_1\sqrt{\tau}}-\frac{\sigma_1\sqrt{\tau}}{2}+\rho\sigma_2\sqrt{\tau}, \\
\tilde{\mathbf{d}}_1 &= -\frac{x_1^*}{\sigma_1\sqrt{\tau}}-\frac{\sigma_1\sqrt{\tau}}{2}+\rho\sigma_2\sqrt{\tau}.
\end{aligned}$$

For the stochastic covariance framework we have to derive $\hat{\mathfrak{h}}_S^D(u_1,u_2)$ using the transformations in Section A.2.1 analogue to (3.43).

$$\begin{aligned}
\hat{\mathfrak{h}}_S^D(u_1,u_2) &= \int_0^\infty\int_{-\infty}^\infty e^{-c_1x_1^{*\prime}+(1-c_2)x_2'}e^{iu_1z_1'+iu_2z_2'}dx_2'dx_1^{*\prime}\\
&= \int_0^\infty\int_{-\infty}^\infty e^{-c_1x_1^{*\prime}+(1-c_2)x_2'}e^{\left(\frac{x_1^{*\prime}}{\sigma_1}\left(-\frac{\rho}{\sqrt{1-\rho^2}}iu_2+iu_1\right)+iu_2\frac{x_2'}{\sigma_2\sqrt{1-\rho^2}}\right)}dx_2'dx_1^{*\prime}\\
&= \int_0^\infty\int_0^\infty e^{-c_1x_1^{*\prime}}e^{(1-c_2)x_2'}e^{\left(\frac{x_1^{*\prime}}{\sigma_1}\left(-\frac{\rho}{\sqrt{1-\rho^2}}iu_2+iu_1\right)+iu_2\frac{x_2'}{\sigma_2\sqrt{1-\rho^2}}\right)}dx_2'dx_1^{*\prime}\\
&\quad+\int_0^\infty\int_{-\infty}^0 e^{-c_1x_1^{*\prime}}e^{(1-c_2)x_2'}e^{\left(\frac{x_1^{*\prime}}{\sigma_1}\left(-\frac{\rho}{\sqrt{1-\rho^2}}iu_2+iu_1\right)+iu_2\frac{x_2'}{\sigma_2\sqrt{1-\rho^2}}\right)}dx_2'dx_1^{*\prime}\\
&= \int_0^\infty e^{-c_1x_1^{*\prime}}e^{\frac{x_1^{*\prime}}{\sigma_1}\left(-\frac{\rho}{\sqrt{1-\rho^2}}iu_2+iu_1\right)}\Bigg(\left[-\frac{1}{\frac{iu_2}{\sigma_2\sqrt{1-\rho^2}}+1-c_2}\right]_{\varpi_2>(1-c_2)\sigma_2\sqrt{1-\rho^2}}\\
&\quad+\left[\frac{1}{\frac{iu_2}{\sigma_2\sqrt{1-\rho^2}}+1-c_2}\right]_{\varpi_2<(1-c_2)\sigma_2\sqrt{1-\rho^2}}\Bigg)dx_1^{*\prime}.
\end{aligned}$$

Note that we can set $\Im(u_2)=\varpi_2=0$ for the second (first) fraction if $c_2<1$ ($c_2>1$). Then we can move the path of integration of the first (second) bracket to the real axis with Corollary 3. We compute with (2.56) the residue of the integrand f of $\hat{\mathfrak{h}}_S^D$ at $\breve{u}_2 = i(1-c_2)\sigma_2\sqrt{1-\rho^2}$.

$$\begin{aligned}
Res_{\breve{u}_2}f &= \lim_{u_2\to\breve{u}_1}\left(u_2-i\sigma_2\sqrt{1-\rho^2}(1-c_2)\right)\left(-\frac{\sigma_2\sqrt{1-\rho^2}}{i\left(u_2-i\sigma_2\sqrt{1-\rho^2}(1-c_2)\right)}\right)\\
&= i\sigma_2\sqrt{1-\rho^2} \qquad \text{(A.72)}
\end{aligned}$$

When we set now the path with Corollary 3 to $\varpi_2=0$ the fractions cancel and we are left with a Dirac delta function. Thus,

$$\begin{aligned}
\hat{\mathfrak{h}}_S^D(u_1,u_2) &= \int_0^\infty e^{-c_1x_1^{*\prime}}e^{\frac{x_1^{*\prime}}{\sigma_1}\left(-\frac{\rho}{\sqrt{1-\rho^2}}iu_2+iu_1\right)}\\
&\quad\left(-2\pi i\sigma_2\sqrt{1-\rho^2}i\delta(u_2-i\sigma_2\sqrt{1-\rho^2}(1-c_2))\right)dx_1^{*\prime}\\
&= \left(-\frac{\sigma_2\sqrt{1-\rho^2}2\pi}{\frac{iu_1}{\sigma_1}-\frac{\rho iu_2}{\sqrt{1-\rho^2}\sigma_1}-c_1}\right)_{\varpi_1>(-\sigma_1c_1+(1-c_2)\sigma_2\rho),\varpi_2=0}\\
&\quad\delta(u_2-i\sigma_2\sqrt{1-\rho^2}(1-c_2)).
\end{aligned}$$

□

Proof. (Proof of proposition 6)

With (A.65), $g(S_2) = \max(S_2 - K_2, 0)$ and the transformations performed according to Section A.2.1 (compare to the transformation performed after (3.70)) in the GBM framework the value of the call option in t is given by

$$\begin{aligned} C_C^D(\tau, y_1, y_2) &= K_2 e^{x_1^* c_1 + x_2 c_2 - \alpha\tau - \int_t^T r(s)ds} \Bigg(\int_0^\infty \int_0^\infty p^F(\tau, y_1' - y_1, y_2' - y_2) \\ & \quad (e^{-c_1 x_1^{*\prime} + (1-c_2)x_2'} - e^{-c_1 x_1^{*\prime} - c_2 x_2'}) dy_1' dy_2' \\ &- \int_0^\infty \int_0^\infty p^F(\tau, y_1' + y_1, y_2' + 2\rho y_1 - y_2) \\ & \quad (e^{-c_1 x_1^{*\prime} + (1-c_2)x_2'} - e^{-c_1 x_1^{*\prime} - c_2 x_2'}) dy_1' dy_2' \Bigg), \end{aligned}$$

where $x_2 = \ln(\frac{S_2 e^{\int_t^T r(s)ds}}{K_2})$. With (A.66)-(A.68) follows

$$\begin{aligned} C_C^D(\tau, y_1, y_2) &= K_2 e^{x_1^* c_1 + x_2 c_2 - \alpha\tau - \int_t^T r(s)ds} \exp\left\{ -\frac{y_1^2 + y_2^2 - 2y_1 y_2 \rho}{2(1-\rho^2)\tau} \right\} \frac{1}{2\pi\sqrt{1-\rho^2}\tau} \\ & \int_0^\infty \int_0^\infty \exp\left\{ -\frac{{y_1'}^2 + {y_2'}^2 - 2y_1' y_2' \rho}{2(1-\rho^2)\tau} \right\} \\ & \Bigg(\Big(\exp\left\{ \left(-c_1\sigma_1 + \frac{y_1 - \rho y_2}{(1-\rho^2)\tau} \right) y_1' + \left((1-c_2)\sigma_2 + \frac{y_2 - \rho y_1}{(1-\rho^2)\tau} \right) y_2' \right\} \\ & - \exp\left\{ \left(-c_1\sigma_1 + \frac{y_1 - \rho y_2}{(1-\rho^2)\tau} \right) y_1' + \left(-c_2\sigma_2 + \frac{y_2 - \rho y_1}{(1-\rho^2)\tau} \right) y_2' \right\} \Big) \\ & - \Big(\exp\left\{ \left(-c_1\sigma_1 - \frac{y_1 - 2\rho^2 y_1 + \rho y_2}{(1-\rho^2)\tau} \right) y_1' \right\} \\ & \exp\left\{ \left((1-c_2)\sigma_2 + \frac{y_2 - \rho y_1}{(1-\rho^2)\tau} \right) y_2' \right\} \\ & - \exp\left\{ \left(-c_1\sigma_1 - \frac{y_1 - 2\rho^2 y_1 + \rho y_2}{(1-\rho^2)\tau} \right) y_1' \right\} \\ & \exp\left\{ \left(-c_2\sigma_2 + \frac{y_2 - \rho y_1}{(1-\rho^2)\tau} \right) y_2' \right\} \Big) \Bigg) dy_1' dy_2' \\ &= K_2 e^{x_1^* c_1 + x_2 c_2 - \alpha\tau - \int_t^T r(s)ds} \exp\left\{ -\frac{y_1^2 + y_2^2 - 2y_1 y_2 \rho}{2(1-\rho^2)\tau} \right\} \\ & \Bigg(\Big(e^{\frac{\tau}{2}({\gamma_1^+}^2 + {\eta_2^+}^2 + 2\rho\gamma_1^+\eta_2^+)} \mathcal{N}_2 \left(\sqrt{\tau}(\gamma_1^+ + \rho\eta_2^+), \sqrt{\tau}(\eta_2^+ + \rho\gamma_1^+), \rho \right) \\ & - e^{\frac{\tau}{2}({\gamma_1^+}^2 + {\gamma_2^+}^2 + 2\rho\gamma_1^+\gamma_2^+)} \mathcal{N}_2 \left(\sqrt{\tau}(\gamma_1^+ + \rho\gamma_2^+), \sqrt{\tau}(\gamma_2^+ + \rho\gamma_1^+), \rho \right) \Big) \\ & - \Big(e^{\frac{\tau}{2}({\gamma_1^-}^2 + {\eta_2^-}^2 + 2\rho\gamma_1^-\eta_2^-)} \mathcal{N}_2 \left(\sqrt{\tau}(\gamma_1^- + \rho\eta_2^-), \sqrt{\tau}(\eta_2^- + \rho\gamma_1^-), \rho \right) \\ & e^{\frac{\tau}{2}({\gamma_1^-}^2 + {\gamma_2^-}^2 + 2\rho\gamma_1^-\gamma_2^-)} \mathcal{N}_2 \left(\sqrt{\tau}(\gamma_1^- + \rho\gamma_2^-), \sqrt{\tau}(\gamma_2^- + \rho\gamma_1^-), \rho \right) \Big) \Bigg), \end{aligned} \quad (A.73)$$

with

$$\begin{aligned}
\gamma_1^+ &= -c_1\sigma_1 + \frac{y_1 - \rho y_2}{(1-\rho^2)\tau},\\
\eta_2^+ &= (1-c_2)\sigma_2 + \frac{y_2 - \rho y_1}{(1-\rho^2)\tau},\\
\gamma_2^+ &= -c_2\sigma_2 + \frac{y_2 - \rho y_1}{(1-\rho^2)\tau},\\
\gamma_1^- &= -c_1\sigma_1 + \frac{-y_1(1-2\rho^2) - \rho y_2}{(1-\rho^2)\tau},\\
\eta_2^- &= \eta_2^+,\\
\gamma_2^- &= \gamma_2^+.
\end{aligned}$$

To simplify we calculate

$$\begin{aligned}
\mathbf{d}_1^* := \sqrt{\tau}(\gamma_1^+ + \rho\gamma_2^+) &= \sqrt{\tau}\left(-c_1\sigma_1 - \rho c_2\sigma_2 + \frac{x_1^*}{\sigma_1\tau}\right)\\
&= \sqrt{\tau}\left(-\frac{\sigma_1 - \rho\sigma_2}{2(1-\rho^2)} - \rho\frac{\sigma_2 - \rho\sigma_1}{2(1-\rho^2)} + \frac{x_1^*}{\sigma_1\tau}\right)\\
&= \frac{x_1^*}{\sigma_1\sqrt{\tau}} - \frac{\sigma_1\sqrt{\tau}}{2},\\
\tilde{\mathbf{d}}_1^* := \sqrt{\tau}(\gamma_1^- + \rho\gamma_2^-) &= \sqrt{\tau}\left(-\frac{\sigma_1 - \rho\sigma_2}{2(1-\rho^2)} - \rho\frac{\sigma_2 - \rho\sigma_1}{2(1-\rho^2)} - \frac{x_1^*}{\sigma_1\tau}\right)\\
&= -\frac{x_1^*}{\sigma_1\sqrt{\tau}} - \frac{\sigma_1\sqrt{\tau}}{2},\\
\mathbf{d}_2 := \sqrt{\tau}(\eta_2^+ + \rho\gamma_1^+) &= \sqrt{\tau}\left((1-c_2)\sigma_2 - \rho c_1\sigma_1 + \frac{x_2}{\sigma_2\tau}\right)\\
&= \frac{x_2}{\sigma_2\sqrt{\tau}} + \sqrt{\tau}\left(\frac{\sigma_2(1-2\rho^2) + \rho\sigma_1}{2(1-\rho^2)} - \rho\frac{\sigma_1 - \rho\sigma_2}{2(1-\rho^2)}\right)\\
&= \frac{x_2}{\sigma_2\sqrt{\tau}} + \frac{1}{2}\sqrt{\tau}\sigma_2,\\
\mathbf{d}_2^* := \sqrt{\tau}(\gamma_2^+ + \rho\gamma_1^+) &= \sqrt{\tau}\left(-c_2\sigma_2 - \rho c_1\sigma_1 + \frac{x_2}{\sigma_2\tau}\right)\\
&= \frac{x_2}{\sigma_2\sqrt{\tau}} + \sqrt{\tau}\left(-\frac{\sigma_2 - \rho\sigma_1}{2(1-\rho^2)} - \rho\frac{\sigma_1 - \rho\sigma_2}{2(1-\rho^2)}\right)\\
&= \frac{x_2}{\sigma_2\sqrt{\tau}} - \frac{1}{2}\sqrt{\tau}\sigma_2,
\end{aligned}$$

$$
\begin{aligned}
\tilde{\mathbf{d}}_2 := \sqrt{\tau}(\eta_2^- + \rho\gamma_1^-) &= \frac{1}{2}\sqrt{\tau}\sigma_2 + \sqrt{\tau}\left(\frac{y_2 - \rho y_1}{(1-\rho^2)\tau} - \rho\frac{y_1(1-2\rho^2) + \rho y_1}{(1-\rho^2)\tau}\right) \\
&= \frac{1}{2}\sqrt{\tau}\sigma_2 + \sqrt{\tau}\left(\frac{y_2(1-\rho^2)}{(1-\rho^2)\tau} - \frac{\rho y_1(2-2\rho^2)}{(1-\rho^2)\tau}\right) \\
&= \frac{1}{2}\sqrt{\tau}\sigma_2 + \frac{y_2}{\sqrt{\tau}} - 2\rho\frac{y_1}{\sqrt{\tau}}, \\
\tilde{\mathbf{d}}_2^* := \sqrt{\tau}(\gamma_2^- + \rho\gamma_1^-) &= -\frac{1}{2}\sqrt{\tau}\sigma_2 + \sqrt{\tau}\left(\frac{y_2 - \rho y_1}{(1-\rho^2)\tau} - \rho\frac{y_1(1-2\rho^2) + \rho y_1}{(1-\rho^2)\tau}\right) \\
&= -\frac{1}{2}\sqrt{\tau}\sigma_2 + \frac{y_2}{\sqrt{\tau}} - 2\rho\frac{y_1}{\sqrt{\tau}},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\tau}{2}({\gamma_1^+}^2 + {\gamma_2^+}^2 + 2\rho\gamma_1^+\gamma_2^+) &= \frac{\tau}{2}(c_1^2\sigma_1^2 - c_2^2\sigma_2^2 + 2\rho\sigma_1\sigma_2 c_1 c_2) \\
&\quad + \left(\frac{y_1 - \rho y_2}{(1-\rho^2)\tau}\right)^2 - 2c_1\sigma_1\frac{y_1 - \rho y_2}{(1-\rho^2)\tau} + \left(\frac{y_2 - \rho y_1}{(1-\rho^2)\tau}\right)^2 \\
&\quad - 2c_2\sigma_2\frac{y_2 - \rho y_1}{(1-\rho^2)\tau} - 2\rho c_2\sigma_2\frac{y_1 - \rho y_2}{(1-\rho^2)\tau} \\
&\quad - 2\rho c_1\sigma_1\frac{y_2 - \rho y_1}{(1-\rho^2)\tau} + 2\rho\frac{(y_2 - \rho y_1)(y_1 - \rho y_2)}{(1-\rho^2)^2\tau^2} \\
&= \frac{\tau}{2}\left(\frac{\sigma_1^2(1-\rho^2) + \sigma_2^2(1-\rho^2) - 2\rho\sigma_1\sigma_2(1-\rho^2)}{4(1-\rho^2)^2}\right) \\
&\quad + \left(-c_1 x_1^* - c_2 x_2 + \frac{y_1^2 + y_2^2 - 2\rho y_1 y_2}{2(1-\rho^2)\tau}\right) \\
&= \alpha\tau - c_1 x_1^* - c_2 x_2 + \frac{y_1^2 + y_2^2 - 2\rho y_1 y_2}{2(1-\rho^2)\tau},
\end{aligned}
$$

$$
\frac{\tau}{2}({\gamma_1^-}^2 + {\gamma_2^-}^2 + 2\rho\gamma_1^-\gamma_2^-) = \alpha\tau + c_1 x_1^* - c_2(x_2 - 2\rho\sigma_2\frac{x_1^*}{\sigma_1}) + \frac{y_1^2 + y_2^2 - 2\rho y_1 y_2}{2(1-\rho^2)\tau},
$$

$$\begin{aligned}
C_C^D(t,S_1,S_2,K_2) &= K_2 e^{x_1^* c_1 + x_2 c_2 - \alpha\tau - \int_t^T r(s)ds} \\
&\Bigg(\bigg(\exp\left\{-c_1 x_1^* + (1-c_2)x_2 + \alpha\tau\right\}\mathcal{N}_2\left(\mathbf{d}_1,\mathbf{d}_2,\rho\right) \\
&- \exp\left\{-c_1 x_1^* - c_2 x_2 + \alpha\tau\right\}\mathcal{N}_2\left(\mathbf{d}_1^*,\mathbf{d}_2^*,\rho\right)\bigg) \\
&-\bigg(\exp\left\{\alpha\tau + c_1 x_1^* - 2(1-c_2)\frac{\sigma_2\rho x_1^*}{\sigma_1} + (1-c_2)x_2\right\}\mathcal{N}_2\left(\tilde{\mathbf{d}}_1,\tilde{\mathbf{d}}_2,\rho\right) \\
&- \exp\left\{\alpha\tau + c_1 x_1^* + 2c_2\frac{\sigma_2\rho x_1^*}{\sigma_1} - c_2 x_2\right\}\mathcal{N}_2\left(\tilde{\mathbf{d}}_1^*,\tilde{\mathbf{d}}_2^*,\rho\right)\bigg)\Bigg) \\
&= \Bigg(\bigg(S_2 e^{-\int_t^T d_2(s)ds}\mathcal{N}_2\left(\mathbf{d}_1,\mathbf{d}_2,\rho\right) - K_2 e^{-\int_t^T r(s)ds}\mathcal{N}_2\left(\mathbf{d}_1^*,\mathbf{d}_2^*,\rho\right)\bigg) \\
&-\bigg(S_2 e^{-\int_t^T d_2(s)ds}\exp\left\{2x_1^*(c_1 - (1-c_2)\frac{\sigma_2\rho}{\sigma_1})\right\}\mathcal{N}_2\left(\tilde{\mathbf{d}}_1,\tilde{\mathbf{d}}_2,\rho\right) \\
&-K_2 e^{-\int_t^T r(s)ds}\exp\left\{2x_1^*(c_1 + c_2\frac{\sigma_2\rho}{\sigma_1})\right\}\mathcal{N}_2\left(\tilde{\mathbf{d}}_1^*,\tilde{\mathbf{d}}_2^*,\rho\right)\bigg)\Bigg),
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{d}_1 &= \tfrac{x_1^*}{\sigma_1\sqrt{\tau}} - \tfrac{1}{2}\sigma_1\sqrt{\tau} + \rho\sigma_2\sqrt{\tau}, & \mathbf{d}_2 &= \frac{x_2}{\sigma_2\sqrt{\tau}} + \frac{1}{2}\sigma_2\sqrt{\tau}, \\
\tilde{\mathbf{d}}_1 &= -\tfrac{x_1^*}{\sigma_1\sqrt{\tau}} - \tfrac{1}{2}\sigma_1\sqrt{\tau} + \rho\sigma_2\sqrt{\tau}, & \tilde{\mathbf{d}}_2 &= \frac{x_2}{\sigma_2\sqrt{\tau}} - 2\rho\frac{x_1^*}{\sigma_1\sqrt{\tau}} + \frac{1}{2}\sigma_2\sqrt{\tau}, \\
\mathbf{d}_1^* &= \mathbf{d}_1 - \rho\sigma_2\sqrt{\tau}, & \mathbf{d}_2^* &= \mathbf{d}_2 - \sigma_2\sqrt{\tau}, \\
\tilde{\mathbf{d}}_1^* &= \tilde{\mathbf{d}}_1 - \rho\sigma_2\sqrt{\tau}, & \tilde{\mathbf{d}}_2^* &= \tilde{\mathbf{d}}_2 - \sigma_2\sqrt{\tau}.
\end{aligned}$$

For the stochastic covariance framework we derive $\hat{\mathfrak{h}}_C^D$:

$$\begin{aligned}
\hat{\mathfrak{h}}_C^D(u_1,u_2) &= K_2\int_0^\infty\int_0^\infty e^{-c_1 x_1^{*\prime}}\left(e^{x_2'(1-c_2)} - e^{-x_2' c_2}\right) \\
&\quad e^{\left(\frac{x_1^{*\prime}}{\sigma_1}\left(-\frac{\rho}{\sqrt{1-\rho^2}}iu_2 + iu_1\right) + iu_2\frac{x_2'}{\sigma_2\sqrt{1-\rho^2}}\right)}dx_2' dx_1^{*\prime} \\
&= K_2\frac{1}{i\left(\frac{u_1}{\sigma_1} - \frac{u_2\rho}{\sigma_1\sqrt{1-\rho^2}}\right) - c_1}\frac{1}{\left(i\frac{u_2}{\sigma_2\sqrt{1-\rho^2}} + 1 - c_2\right)\left(i\frac{u_2}{\sigma_2\sqrt{1-\rho^2}} - c_2\right)}.
\end{aligned}$$

Note that we have to set $\Im(u_2) > \sigma_2\sqrt{1-\rho^2}(1-c_2)$ and $\Im(u_1) > \frac{\rho}{\sqrt{1-\rho^2}}\Im(u_2) - \sigma_1 c_1$, i.e

$$S_{g_C^D} = \left\{\mathbf{u} = \mathbf{w} + i\boldsymbol{\varpi} : \varpi_2 > \sigma_2\sqrt{1-\rho^2}(1-c_2) \wedge \varpi_1 > \frac{\rho}{\sqrt{1-\rho^2}}\varpi_2 - \sigma_1 c_1\right\}. \qquad \square$$

Proof. (Proof of proposition 7)

Analogue to the proof above for the call option the value of the digital call option in t in the GBM framework with payoff $g(S_2) = \mathbb{1}_{S_2(T)>K}$ (for the transformation compare with the explanations after (3.66)) follows

$$\begin{aligned} C_D^D(t, S_1, S_2, K_2) &= e^{-\int_t^T r(s)ds}\Bigg(\mathcal{N}_2\left(\mathbf{d}_1^*, \mathbf{d}_2^*, \rho\right) - \exp\left\{2x_1^*(c_1 + c_2\frac{\sigma_2\rho}{\sigma_1})\right\} \\ &\qquad \mathcal{N}_2\left(\tilde{\mathbf{d}}_1^*, \tilde{\mathbf{d}}_2^*, \rho\right)\Bigg), \end{aligned}$$

and,

$$\mathbf{d}_2^* = \frac{x_2}{\sigma_2\sqrt{\tau}} - \frac{1}{2}\sigma_2\sqrt{\tau},\ \tilde{\mathbf{d}}_2^* = \frac{x_2}{\sigma_2\sqrt{\tau}} - 2\rho\frac{x_1^*}{\sigma_1\sqrt{\tau}} - \frac{1}{2}\sigma_2\sqrt{\tau}.$$

The result for the stochastic covariance framework follows with Corollary 5 with $b_2 = 0$. □

Proof. (Proof of proposition 8)

(A.60) with additional boundary condition $p(\tau, y_1, b_2) = 0$ can be solved by applying Theorem 36 in connection with Remark 14.
(A.59) with the additional boundary condition is solved by the application of Theorem 36. The results follow. □

Appendix B

Appendix for Chapter 4

B.1 Appendix for Section 4.2

Proof. (Proof of Corollary 16)

Assume the affine form (4.7) and plug it in the PDE (4.6). We obtain then the following PDE:

$$\begin{aligned}
& -\frac{\partial A_H^*(\tau,\mathbf{u})}{\partial \tau} - \sum_{j=1}^{2}\left(\frac{\delta_j^2}{\sigma_{v_j}^2}\frac{\partial B_{H1j}^*(\tau,\mathbf{u})}{\partial \tau} v_j\right) \\
& + i\sum_{l=1}^{2}\left(r - \sum_{j=1}^{2}\frac{a_{lj}^2 f(v_j)^2}{2}\right)u_l - \frac{1}{2}\sum_{l,k=1}^{2} u_l u_k \sum_{j=1}^{2} a_{lj}a_{kj} f(v_j)^2 \\
& + \sum_{j=1}^{2}\left(\frac{\delta_j^2}{\sigma_{v_j}^2} B_{H1j}^*(\tau,\mathbf{u})\frac{\kappa_{v_j}}{\delta_j^2}\left(\zeta_{v_j} - v_j\right) + \frac{1}{2}\frac{\delta_j^2}{\sigma_{v_j}^2} v_j B_{H1j}^*(\tau,\mathbf{u})^2\right. \\
& \left. + \sum_{l=1}^{2}\sum_{j=1}^{2} i u_l a_{lj} f(v_j)\left(B_{H1j}^*(\tau,\mathbf{u})\rho_{lj}^v\sqrt{v_j}\frac{\delta_j}{\sigma_{v_j}}\right)\right) = 0. \qquad \text{(B.1)}
\end{aligned}$$

If $f(v_j) = \sqrt{v_j}$ and $\rho_j^v = 0$ the PDE is affine and breaks down into several ODEs:

$$
\begin{aligned}
\frac{\partial A_H^*(\tau,\mathbf{u})}{\partial \tau} - i\sum_{l=1}^{2} u_l r - \sum_{j=1}^{2}\left(\frac{1}{\sigma_{v_j}^2} B_{H1j}^*(\tau,\mathbf{u})\,\kappa_{v_j}\zeta_{v_j}\right) &= 0, \\
A_H^*(0,\mathbf{u}) &= 0,
\end{aligned}
\tag{B.2}
$$

$$
\begin{aligned}
\frac{\delta_j^2}{\sigma_{v_j}^2}\frac{\partial B_{H1j}^*(\tau,\mathbf{u})}{\partial \tau} + i\sum_{i=1}^{2} u_i \frac{a_{ij}^2}{2} + \frac{1}{2}\sum_{i,k=1}^{2} u_i u_k a_{ij} a_{kj} & \\
+\frac{1}{\sigma_{v_j}^2} B_{H1j}^*(\tau,\mathbf{u})\,\kappa_{v_j} - \frac{\delta_j^2}{2\sigma_{v_j}^2} B_{H1j}^*(\tau,\mathbf{u})^2 &= 0, \\
B_{H1j}^*(0,\mathbf{u}) &= 0.
\end{aligned}
\tag{B.3}
$$

These ODEs can be solved analogously to (A.28). Thus, we set $B_{H1j}^* = -2\frac{\frac{\partial E_{H1j}^*}{\partial \tau}}{E_{H1j}^*}$. E_{H1j}^* is then given by

$$
E_{H1j}^*(\tau,\mathbf{u}) = e^{\frac{1}{2}\left(-\frac{\kappa_{v_j}}{\delta_j^2}+\mathfrak{d}_{v_j}\right)\tau}\,\frac{\frac{\kappa_{v_j}}{\delta_j^2} + \mathfrak{d}_{v_j} + \left(-\frac{\kappa_{v_j}}{\delta_j^2} + \mathfrak{d}_{v_j}\right)e^{-\mathfrak{d}_{v_j}\tau}}{2\mathfrak{d}_{v_j}},
\tag{B.4}
$$

with

$$
\mathfrak{d}_{v_j} = \mathfrak{d}_{v_j}(\mathbf{u}) = \sqrt{\frac{\kappa_{v_j}^2}{\delta_j^4} + \frac{\sigma_{v_j}^2}{\delta_j^2}\left(i\left(u_1 a_{1j}^2 + u_2 a_{2j}^2\right) + 2u_1u_2a_{1j}a_{2j} + u_1^2 a_{1j}^2 + u_2^2 a_{2j}^2\right)}.
$$

Thus,

$$
\begin{aligned}
A_H^*(\tau,\mathbf{u}) &= \tau\left(ir(u_1+u_2)\right) - 2\sum_{j=1}^{2}\left(\frac{\kappa_{v_j}\zeta_{v_j}}{\sigma_{v_j}^2}\ln E_{H1j}^*\right) \\
&= \tau\left(ir(u_1+u_2) + \sum_{j=1}^{2}\frac{\kappa_{v_j}^2\zeta_{v_j}}{\delta_j^2\sigma_{v_j}^2}\right. \\
&\quad \left. -2\sum_{j=1}^{2}\left(\frac{\kappa_{v_j}}{\sigma_{v_j}^2}\zeta_{v_j}\ln\left(\frac{\kappa_{v_j}}{\delta_j^2\mathfrak{d}_{v_j}}\sinh\left(\frac{\mathfrak{d}_{v_j}}{2}\tau\right) + \cosh\left(\frac{\mathfrak{d}_{v_j}}{2}\tau\right)\right)\right)\right),
\end{aligned}
$$

and B_{1j} follows from (A.35) with $\kappa := \frac{\kappa_{v_j}}{\delta_j^2}$ for B_{1j}.

Define $\mathfrak{D}_{v_j}^*(\mathbf{u}) = \mathfrak{d}_{v_j}(\mathbf{u})^2$. We see that $\mathfrak{D}_{v_j}^*(\mathbf{0}) = \frac{\kappa_{v_j}^2}{\delta_j^4} > 0$. Hence, we see that φ is well-defined and regular in a neighbourhood of the origin according to Cauchy's integral theorem, i.e. there exists a complex analytic extension of $\bar{M}(\mathbf{w})$, the moment generating function, to an open set $\mathcal{D} \subset \mathbb{C}$ in the neighbourhood of the origin (see Theorem 32 and Section 3.4.2). □

B.2 Appendix for Section 4.5

Proof. (Pricing of two-asset option in GBM model)

Pursue the transformations of the PDE according to A.11-A.14. The Green function is then given by

$$p(\tau, x_1', x_2', x_1, x_2) = \frac{1}{2\sigma_1\sigma_2\pi\sqrt{1-\rho^2}\tau} e^{-\frac{\frac{(x_1'-x_1)^2}{\overline{\sigma}_1^2} + \frac{(x_2'-x_2)^2}{\overline{\sigma}_2^2} - 2\frac{\rho(x_1'-x_1)(x_2'-x_2)}{\overline{\sigma}_1\overline{\sigma}_2}}{2(1-\rho^2)\tau}}. \tag{B.5}$$

Thus,

$$\begin{aligned} C_{0,0}(t, S_1, S_2) &= K_2 e^{c_1x_1+c_2x_2+\alpha\tau-\int_t^T r(s)ds} \int_0^\infty \int_0^\infty p(\tau, x_1 - x_1', x_2 - x_2') \\ &\quad \left(e^{-c_1x_1'+(1-c_2)x_2'} - e^{-c_1x_1'-c_2x_2'}\right) dx_1' dx_2'. \end{aligned} \tag{B.6}$$

We transform this expression to a bivariate normal distribution.

With L given in (A.66) follows

$$\begin{aligned} C_{0,0}(t, S_1, S_2) &= K_2 e^{c_1x_1+c_2x_2+\alpha\tau-\int_t^T r(s)ds} \Bigg(e^{-\alpha\tau-c_1x_1+(1-c_2)x_2} \mathcal{N}_2(\mathbf{d}_2, \mathbf{d}_1, \overline{\rho}) \\ &\quad - e^{-\alpha\tau-c_1x_1-c_2x_2} \mathcal{N}_2(\mathbf{d}_2^*, \mathbf{d}_1^*, \overline{\rho}) \Bigg), \end{aligned} \tag{B.7}$$

where

$$\begin{aligned} \tau = T - t, &\qquad x_i = \ln \frac{S_i e^{-\int_t^T r(s)ds}}{K_i}, \\ \mathbf{d}_1^* = \mathbf{d}_1 - \overline{\rho}\overline{\sigma}_2\sqrt{\tau}, &\qquad \mathbf{d}_1 = \frac{x_1}{\overline{\sigma}_1\sqrt{\tau}} - \frac{1}{2}\overline{\sigma}_1\sqrt{\tau} + \overline{\rho}\overline{\sigma}_2\sqrt{\tau}, \\ \mathbf{d}_2^* = \mathbf{d}_2 - \overline{\sigma}_2\sqrt{\tau}, &\qquad \mathbf{d}_2 = \frac{x_2}{\overline{\sigma}_2\sqrt{\tau}} + \frac{1}{2}\overline{\sigma}_2\sqrt{\tau}. \end{aligned}$$

□

Proof. (Pricing of two-asset barrier options in GBM model)

Follow the transformations in Section A.1.1 for constant barrier options and apply (A.17) to (A.18). For $\overline{\rho} = -\cos\left(\frac{2\pi k}{n}\right)$ the Green function is then given by (see also Section 3.4

for the method of images in a wedge)

$$p(\tau, z_1', z_2', z_1, z_2) = \sum_{k=0}^{n-1} \frac{1}{2\pi\tau}\Big(e^{-\frac{1}{2\tau}\left(\left(z_1' - r_p \cos\left(\frac{2\pi k}{n}+\theta_p\right)\right)^2 + \left(z_2' - r_p \sin\left(\frac{2\pi k}{n}+\theta_p\right)\right)^2\right)} - e^{-\frac{1}{2}\left(\left(z_1' - r_p \cos\left(\frac{2\pi k}{n}-\theta_p\right)\right)^2 + \left(z_2' - r_p \sin\left(\frac{2\pi k}{n}-\theta_p\right)\right)^2\right)}\Big). \tag{B.8}$$

Hence,

$$C_{B,0,0}(t, S_1, S_2) = K_2 e^{c_1 x_1 + c_2 x_2 - \alpha\tau - \int_t^T r(s)ds} \int_0^\infty \int_{-z_1 \frac{\sqrt{1-\overline{\rho}^2}}{\overline{\rho}}}^\infty p(\tau, z_1', z_2', z_1, z_2) \left(e^{-c_1 x_1' + (1-c_2)x_2'} - e^{-c_1 x_1' - c_2 x_2'}\right) dz_2' dz_1'. \tag{B.9}$$

With (A.66)

$$\begin{aligned}
L_2 &= \int_0^\infty \int_{-z_1 \frac{\sqrt{1-\overline{\rho}^2}}{\overline{\rho}}}^\infty \frac{1}{2\pi\tau} \exp\left\{ -\frac{1}{2\tau}\left(\left(z_1' - r_p\cos\left(\frac{2\pi k}{n} \pm \theta_p\right)\right)^2 \right.\right. \\
&\quad \left.\left. + \left(z_2' - r_p \sin\left(\frac{2\pi k}{n} \pm \theta_p\right)\right)^2\right)\right\} \exp\left\{k_1 x_1' + k_2 x_2'\right\} dz_1' dz_2' \\
&= e^{-\frac{1}{2\tau}r_p^2} \int_{-\frac{b_1}{\overline{\sigma}_1}}^\infty \int_{-\frac{b_2}{\overline{\sigma}_2}}^\infty \frac{1}{2\pi\tau\sqrt{1-\overline{\rho}^2}} \\
&\quad \exp\left\{ -\frac{1}{2(1-\overline{\rho}^2)\tau}\left({y_1'}^2 - 2\overline{\rho} y_1' y_2' + {y_2'}^2\right)\right\} \\
&\quad \exp\left\{ \frac{y_1'}{\sqrt{1-\overline{\rho}^2}\tau} r_p \cos\left(\frac{2\pi k}{n} \pm \theta_p\right) \right. \\
&\quad \left. + \frac{y_2'}{\tau}\left(r_p \sin(\frac{2\pi k}{n} \pm \theta_p) - \frac{\overline{\rho}}{\sqrt{1-\overline{\rho}^2}} r_p \cos\left(\frac{2\pi k}{n} \pm \theta_p\right)\right)\right\} \\
&\quad \exp\left\{k_1 \overline{\sigma}_1 y_1' + k_2 \overline{\sigma}_2 y_2' + k_1 b_1 + k_2 b_2\right\} dy_2' dy_1' \\
&= e^{-\frac{1}{2\tau}r_p^2 + k_1 b_1 + k_2 b_2} \exp\left\{ \frac{\tau\left({\gamma_1^{\pm}}^2 + 2\overline{\rho}\gamma_1^{\pm}\gamma_2^{\pm} + {\gamma_2^{\pm}}^2\right)}{2}\right\} \\
&\quad \mathcal{N}_2\left(\sqrt{\tau}(\gamma_1^{\pm} + \overline{\rho}\gamma_2^{\pm}) + \frac{b_1}{\overline{\sigma}_1\sqrt{\tau}}, \sqrt{\tau}(\overline{\rho}\gamma_1^{\pm} + \gamma_2^{\pm}) + \frac{b_2}{\overline{\sigma}_2\sqrt{\tau}}\right),
\end{aligned} \tag{B.10}$$

where $y_i = \frac{x_i - b_i}{\sigma_i}$, $z_1^2 := \frac{1}{(1-\overline{\rho}^2)}(y_1 - \overline{\rho} y_2)^2 = \frac{1}{(1-\overline{\rho}^2)}(y_1^2 + y_2^2\overline{\rho}^2 - 2y_1y_2\overline{\rho})$, and $z_2^2 := y_2^2$,

$$
\begin{aligned}
\gamma_1^{\pm} &= \frac{1}{\tau\sqrt{1-\overline{\rho}^2}} r_p \cos\left(\frac{2\pi k}{n} \mp \theta_p\right) + k_1\overline{\sigma}_1 \\
&= \frac{1}{\tau\sqrt{1-\overline{\rho}^2}} \left(r_p \cos(\frac{2\pi k}{n}) \cos(\theta_p) \pm r_p \sin(\frac{2\pi k}{n}) \sin(\theta_p) \right) + k_1\overline{\sigma}_1 \\
&= \frac{1}{\tau\sqrt{1-\overline{\rho}^2}} \left(z_1 \cos(\frac{2\pi k}{n}) \pm z_2 \sin(\frac{2\pi k}{n}) \right) + k_1\overline{\sigma}_1 \\
&= \frac{1}{\tau\sqrt{1-\overline{\rho}^2}} \left(\frac{1}{\sqrt{1-\overline{\rho}^2}} (y_1 - \overline{\rho} y_2) \cos(\frac{2\pi k}{n}) \pm y_2 \sin(\frac{2\pi k}{n}) \right) + k_1\overline{\sigma}_1 \\
&= k_1\overline{\sigma}_1 + y_1 \frac{1}{\tau(1-\overline{\rho}^2)} \cos\left(\frac{2\pi k}{n}\right) \\
&\quad + y_2 \left(\pm \frac{1}{\tau\sqrt{1-\overline{\rho}^2}} \sin\left(\frac{2\pi k}{n}\right) - \frac{\overline{\rho}}{\tau\left(1-\overline{\rho}^2\right)} \cos\left(\frac{2\pi k}{n}\right) \right),
\end{aligned}
\tag{B.11}
$$

$$
\begin{aligned}
\gamma_2^{\pm} &= \frac{1}{\tau\sqrt{1-\overline{\rho}^2}} \left(\sqrt{1-\overline{\rho}^2} r_p \sin\left(\frac{2\pi k}{n} \mp \theta_p\right) - \overline{\rho} r_p \cos\left(\frac{2\pi k}{n} \mp \theta_p\right) \right) + k_2\overline{\sigma}_2 \\
&= \frac{1}{\tau\sqrt{1-\overline{\rho}^2}} \Bigg(\sqrt{1-\overline{\rho}^2} \left(\mp r_p \sin(\theta_p) \cos(\frac{2\pi k}{n}) + r_p \cos(\theta_p) \sin(\frac{2\pi k}{n}) \right) \\
&\quad - \overline{\rho} \left(r_p \cos(\frac{2\pi k}{n}) \cos(\theta_p) \pm r_p \sin(\frac{2\pi k}{n}) \sin(\theta_p) \right) \Bigg) + k_2\overline{\sigma}_2 \\
&= \frac{1}{\tau\sqrt{1-\overline{\rho}^2}} \Bigg(\sqrt{1-\overline{\rho}^2} \left(\mp z_2 \cos(\frac{2\pi k}{n}) + z_1 \sin(\frac{2\pi k}{n}) \right) \\
&\quad - \overline{\rho} \left(z_1 \cos(\frac{2\pi k}{n}) \pm z_2 \sin(\frac{2\pi k}{n}) \right) \Bigg) + k_2\overline{\sigma}_2 \\
&= \frac{1}{\tau\sqrt{1-\overline{\rho}^2}} \Bigg(\sqrt{1-\overline{\rho}^2} \left(\mp y_2 \cos(\frac{2\pi k}{n}) + \frac{1}{\sqrt{1-\overline{\rho}^2}} (y_1 - \overline{\rho} y_2) \sin(\frac{2\pi k}{n}) \right) \\
&\quad - \overline{\rho} \left(\frac{1}{\sqrt{1-\overline{\rho}^2}} (y_1 - \overline{\rho} y_2) \cos(\frac{2\pi k}{n}) \pm y_2 \sin(\frac{2\pi k}{n}) \right) \Bigg) + k_2\overline{\sigma}_2
\end{aligned}
$$

$$\begin{cases} = & \frac{1}{\tau\sqrt{1-\overline{\rho}^2}}\left(y_2\left(\sqrt{1-\overline{\rho}^2}\cos\left(\frac{2\pi k}{n}\right)+\frac{\overline{\rho}^2}{\sqrt{1-\overline{\rho}^2}}\cos\left(\frac{2\pi k}{n}\right)\right)\right. \\ & \left.+y_1\left(\sin\left(\frac{2\pi k}{n}\right)-\frac{\overline{\rho}}{\sqrt{1-\overline{\rho}^2}}\cos\left(\frac{2\pi k}{n}\right)\right)\right)+k_2\overline{\sigma}_2 \quad \text{for } \gamma_2^+ \\ = & \frac{1}{\tau\sqrt{1-\overline{\rho}^2}}\left(y_2\left(-\sqrt{1-\overline{\rho}^2}\cos\left(\frac{2\pi k}{n}\right)+\frac{\overline{\rho}^2}{\sqrt{1-\overline{\rho}^2}}\cos\left(\frac{2\pi k}{n}\right)-2\overline{\rho}\sin(\frac{2\pi k}{n})\right)\right. \\ & \left.+y_1\left(\sin\left(\frac{2\pi k}{n}\right)-\frac{\overline{\rho}}{\sqrt{1-\overline{\rho}^2}}\cos\left(\frac{2\pi k}{n}\right)\right)\right)+k_2\overline{\sigma}_2 \quad \text{for } \gamma_2^- \end{cases}$$

$$\begin{cases} = & \frac{1}{\tau\sqrt{1-\overline{\rho}^2}}\left(y_2\frac{1}{\sqrt{1-\overline{\rho}^2}}\cos\left(\frac{2\pi k}{n}\right)\right. \\ & \left.+y_1\left(\sin\left(\frac{2\pi k}{n}\right)-\frac{\overline{\rho}}{\sqrt{1-\overline{\rho}^2}}\cos\left(\frac{2\pi k}{n}\right)\right)\right)+k_2\overline{\sigma}_2 \quad \text{for } \gamma_2^+ \\ = & \frac{1}{\tau\sqrt{1-\overline{\rho}^2}}\left(y_2\left(-\frac{1-2\overline{\rho}^2}{\sqrt{1-\overline{\rho}^2}}\cos\left(\frac{2\pi k}{n}\right)-2\overline{\rho}\sin(\frac{2\pi k}{n})\right)\right. \\ & \left.+y_1\left(\sin\left(\frac{2\pi k}{n}\right)-\frac{\overline{\rho}}{\sqrt{1-\overline{\rho}^2}}\cos\left(\frac{2\pi k}{n}\right)\right)\right)+k_2\overline{\sigma}_2 \quad \text{for } \gamma_2^- \end{cases} \tag{B.12}$$

Hence, $C_{B,0,0}$ is given by

$$
\begin{aligned}
C_{B,0,0}(t,S_1,S_2) \;=\; & \sum_{k=0}^{n-1} e^{c_1x_1+c_2x_2-\alpha\tau} e^{-\frac{1}{2\tau}r_p^2-c_1b_1-c_2b_2} \\
& \Bigg(B_2 \Bigg(\exp\left\{ \frac{\tau\left(\gamma_1^{+2}+2\overline{\rho}\gamma_1^+\eta_2^+ + \eta_2^{+2}\right)}{2} \right\} \\
& \mathcal{N}_2\left(\sqrt{\tau}(\gamma_1^+ + \overline{\rho}\eta_2^+) + \frac{b_1}{\overline{\sigma}_1\sqrt{\tau}}, \sqrt{\tau}(\overline{\rho}\gamma_1^+ + \eta_2^+) + \frac{b_2}{\overline{\sigma}_2\sqrt{\tau}} \right) \\
& - \exp\left\{ \frac{\tau\left(\gamma_1^{-2}+2\overline{\rho}\gamma_1^-\eta_2^- + \eta_2^{-2}\right)}{2} \right\} \\
& \mathcal{N}_2\left(\sqrt{\tau}(\gamma_1^- + \overline{\rho}\eta_2^-) + \frac{b_1}{\overline{\sigma}_1\sqrt{\tau}}, \sqrt{\tau}(\overline{\rho}\gamma_1^- + \eta_2^-) + \frac{b_2}{\overline{\sigma}_2\sqrt{\tau}} \right) \Bigg) \\
& - K_2 e^{-\int_t^T r(s)ds} \Bigg(\exp\left\{ \frac{\tau\left(\gamma_1^{+2}+2\overline{\rho}\gamma_1^+\gamma_2^+ + \gamma_2^{+2}\right)}{2} \right\} \\
& \mathcal{N}_2\left(\sqrt{\tau}(\gamma_1^+ + \overline{\rho}\gamma_2^+) + \frac{b_1}{\overline{\sigma}_1\sqrt{\tau}}, \sqrt{\tau}(\overline{\rho}\gamma_1^+ + \gamma_2^+) + \frac{b_2}{\overline{\sigma}_2\sqrt{\tau}} \right) \\
& - \exp\left\{ \frac{\tau\left(\gamma_1^{-2}+2\overline{\rho}\gamma_1^-\gamma_2^- + \gamma_2^{-2}\right)}{2} \right\} \\
& \mathcal{N}_2\left(\sqrt{\tau}(\gamma_1^- + \overline{\rho}\gamma_2^-) + \frac{b_1}{\overline{\sigma}_1\sqrt{\tau}}, \sqrt{\tau}(\overline{\rho}\gamma_1^- + \gamma_2^-) + \frac{b_2}{\overline{\sigma}_2\sqrt{\tau}} \right) \Bigg) \Bigg),
\end{aligned} \tag{B.13}
$$

where $\gamma_1^\pm$, $\gamma_2^\pm$ are as indicated in (B.11)-(B.12) with $k_1 = -c_1$ and $k_2 = -c_2$, $\eta_2^\pm$ follows $\gamma_2^\pm$ with $k_2 = (1-c_2)$.

Now, we simplify the singular multiplying factors of the normal distributions by inserting (B.11)-(B.12) and we use our calculations in (A.70) for $\alpha\tau$, i.e. $\frac{\tau}{2}\left(c_1^2\overline{\sigma}_1^2 + (1-c_2)^2\overline{\sigma}_2^2 - 2\overline{\rho}\overline{\sigma}_1\overline{\sigma}_2 c_1(1-c_2)\right) = \frac{\tau}{2}\left(c_1^2\overline{\sigma}_1^2 + c_2^2\overline{\sigma}_2^2 + 2\overline{\rho}\overline{\sigma}_1\overline{\sigma}_2 c_1 c_2\right) = \alpha\tau$.

$$
\begin{aligned}
& \frac{\tau}{2}(\gamma_1^{\pm 2} + 2\overline{\rho}\gamma_1^{\pm}\eta_2^{\pm} + \eta_2^{\pm 2}) \\
= \; & \frac{\tau}{2}\left(c_1^2\overline{\sigma}_1^2 + (1-c_2)^2\overline{\sigma}_2^2 - 2\overline{\rho}c_1(1-c_2)\overline{\sigma}_1\overline{\sigma}_2\right) \\
& + \frac{\tau}{2}\left(\frac{1}{\tau^2(1-\overline{\rho}^2)} r_p^2 \cos\left(\frac{2\pi k}{n} \pm \theta_p\right) + \frac{1}{\tau^2(1-\overline{\rho}^2)}\right. \\
& \left((1-\overline{\rho}^2) r_p^2 \sin^2\left(\frac{2\pi k}{n} \pm \theta_p\right) + \overline{\rho}^2 r_p^2 \cos^2\left(\frac{2\pi k}{n} \pm \theta_p\right)\right) \\
& -2\overline{\rho}\sqrt{1-\overline{\rho}^2} r_p^2 \cos\left(\frac{2\pi k}{n} \pm \theta_p\right) \sin\left(\frac{2\pi k}{n} \pm \theta_p\right) \\
& +2\frac{\overline{\rho}}{\tau^2(1-\overline{\rho}^2)}\left(\sqrt{1-\overline{\rho}^2} r_p^2 \sin\left(\frac{2\pi k}{n} \pm \theta_p\right) \cos\left(\frac{2\pi k}{n} \pm \theta_p\right) - \overline{\rho} r_p^2 \cos^2\left(\frac{2\pi k}{n} \pm \theta_p\right)\right) \\
& -2\frac{c_1\overline{\sigma}_1}{\tau\sqrt{1-\overline{\rho}^2}} r_p \cos\left(\frac{2\pi k}{n} \pm \theta_p\right) + 2\frac{(1-c_2)\overline{\sigma}_2}{\tau\sqrt{1-\overline{\rho}^2}} \\
& \left(\sqrt{1-\overline{\rho}^2} r_p \sin\left(\frac{2\pi k}{n} \pm \theta_p\right) - \overline{\rho} r_p \cos\left(\frac{2\pi k}{n} \pm \theta_p\right)\right) \\
& -2\frac{\overline{\rho}c_1\overline{\sigma}_1}{\tau\sqrt{1-\overline{\rho}^2}}\left(\sqrt{1-\overline{\rho}^2} r_p \sin\left(\frac{2\pi k}{n} \pm \theta_p\right) - \overline{\rho} r_p \cos\left(\frac{2\pi k}{n} \pm \theta_p\right)\right) \\
& \left. +2\frac{\overline{\rho}(1-c_2)\overline{\sigma}_2}{\tau\sqrt{1-\overline{\rho}^2}} r_p \cos\left(\frac{2\pi k}{n} \pm \theta_p\right)\right) \\
= \; & \alpha\tau + \frac{1}{2\tau} r_p^2 + \frac{\tau}{2}\Big(-2\frac{c_1\overline{\sigma}_1(1-\overline{\rho}^2)}{\tau\sqrt{1-\overline{\rho}^2}} r_p \cos\left(\frac{2\pi k}{n} \pm \theta_p\right) \\
& +2\frac{1}{\tau}\left((1-c_2)\overline{\sigma}_2 - \overline{\rho}c_1\overline{\sigma}_1\right) r_p \left(\frac{2\pi k}{n} \pm \theta_p\right)\Big) \\
= \; & \alpha\tau + \frac{1}{2\tau} r_p^2 - c_1\overline{\sigma}_1\sqrt{1-\overline{\rho}^2} r_p \cos\left(\frac{2\pi k}{n} \pm \theta_p\right) + \left((1-c_2)\overline{\sigma}_2 - \overline{\rho}c_1\overline{\sigma}_1\right) r_p \sin\left(\frac{2\pi k}{n} \pm \theta_p\right) \\
= \; & \alpha\tau + \frac{1}{2\tau} r_p^2 - c_1\overline{\sigma}_1\sqrt{1-\overline{\rho}^2}\left(r_p \cos\left(\frac{2\pi k}{n}\right)\cos(\theta_p) \mp r_p \sin\left(\frac{2\pi k}{n}\right)\sin(\theta_p)\right) \\
& + \left((1-c_2)\overline{\sigma}_2 - \overline{\rho}c_1\overline{\sigma}_1\right)\left(\pm r_p \cos\left(\frac{2\pi k}{n}\right)\sin(\theta_p) + r_p \sin\left(\frac{2\pi k}{n}\right)\cos(\theta_p)\right) \\
= \; & \alpha\tau + \frac{1}{2\tau} r_p^2 - c_1\overline{\sigma}_1\sqrt{1-\overline{\rho}^2}\left(\frac{1}{\sqrt{1-\rho^2}}(y_1 - \overline{\rho}y_2)\cos\left(\frac{2\pi k}{n}\right) \mp y_2 \sin\left(\frac{2\pi k}{n}\right)\right) \\
& + \left((1-c_2)\,\overline{\sigma}_2 - \overline{\rho}c_1\overline{\sigma}_1\right)\left(\pm \cos\left(\frac{2\pi k}{n}\right) y_2 + \sin\left(\frac{2\pi k}{n}\right)\frac{1}{\sqrt{1-\overline{\rho}^2}}(y_1 - \rho y_2)\right)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \alpha\tau + \frac{1}{2\tau} r_p^2 + y_1\Big(-c_1\overline{\sigma}_1 \cos\left(\frac{2\pi k}{n}\right) + \frac{1}{\sqrt{1-\overline{\rho}^2}}\left((1-c_2)\overline{\sigma}_2 - \overline{\rho}c_1\overline{\sigma}_1\right) \sin\left(\frac{2\pi k}{n}\right) \\
& + y_2\Bigg(\overline{\rho}c_1\sigma_1 \cos\left(\frac{2\pi k}{n}\right) \pm c_1\overline{\sigma}_1\sqrt{1-\overline{\rho}^2} \sin\left(\frac{2\pi k}{n}\right) \\
& + \left((1-c_2)\overline{\sigma}_2 - \overline{\rho}c_1\overline{\sigma}_1\right)\left(\pm \cos\left(\frac{2\pi k}{n}\right) - \frac{\overline{\rho}}{\sqrt{1-\overline{\rho}^2}} \sin\left(\frac{2\pi k}{n}\right)\right)\Bigg)
\end{aligned}
$$

$$
\begin{cases}
= \alpha\tau + \frac{1}{2\tau} r_p^2 + y_1\left(-c_1\overline{\sigma}_1 \cos\left(\frac{2\pi k}{n}\right) + \frac{1}{\sqrt{1-\overline{\rho}^2}}\left((1-c_2)\overline{\sigma}_2 - \overline{\rho}c_1\overline{\sigma}_1\right) \sin\left(\frac{2\pi k}{n}\right)\right) \\
+ y_2\left((1-c_2)\overline{\sigma}_2 \cos\left(\frac{2\pi k}{n}\right) + \frac{1}{\sqrt{1-\overline{\rho}^2}}\left(-(1-c_2)\overline{\sigma}_2\overline{\rho} + \overline{\sigma}_1 c_1\right) \sin\left(\frac{2\pi k}{n}\right)\right) \\
= \alpha\tau + \frac{1}{2\tau} r_p^2 + y_1\left(-c_1\overline{\sigma}_1 \cos\left(\frac{2\pi k}{n}\right) + \frac{1}{\sqrt{1-\overline{\rho}^2}}\left((1-c_2)\overline{\sigma}_2 - \overline{\rho}c_1\overline{\sigma}_1\right) \sin\left(\frac{2\pi k}{n}\right)\right) \\
+ y_2\Big(\left(-(1-c_2)\overline{\sigma}_2 + 2\overline{\sigma}_1 c_1\overline{\rho}\right) \cos\left(\frac{2\pi k}{n}\right) \\
+ \frac{1}{\sqrt{1-\overline{\rho}^2}}\left(-(1-c_2)\overline{\sigma}_2\overline{\rho} - \overline{\sigma}_1 c_1(1-2\overline{\rho}^2)\right) \sin\left(\frac{2\pi k}{n}\right) \Big).
\end{cases}
$$

Analogously, we obtain

$$
\begin{aligned}
& \frac{\tau}{2}(\gamma_1^{\pm 2} + 2\overline{\rho}\gamma_1^{\pm}\eta_2^{\pm} + \eta_2^{\pm 2}) \\
= \quad & \alpha\tau + \frac{1}{2\tau} r_p^2 - c_1\overline{\sigma}_1\sqrt{1-\overline{\rho}^2}\, r_p \cos\left(\frac{2\pi k}{n} \pm \theta_p\right) + \left(-c_2\overline{\sigma}_2 - \overline{\rho}c_1\overline{\sigma}_1\right) r_p \sin\left(\frac{2\pi k}{n} \pm \theta_p\right).
\end{aligned}
$$

$$
\begin{cases}
= \alpha\tau + \frac{1}{2\tau} r_p^2 + y_1\left(-c_1\overline{\sigma}_1 \cos\left(\frac{2\pi k}{n}\right) + \frac{1}{\sqrt{1-\overline{\rho}^2}}\left(-c_2\overline{\sigma}_2 - \overline{\rho}c_1\overline{\sigma}_1\right) \sin\left(\frac{2\pi k}{n}\right)\right) \\
+ y_2\left(-c_2\overline{\sigma}_2 \cos\left(\frac{2\pi k}{n}\right) + \frac{1}{\sqrt{1-\overline{\rho}^2}}\left(c_2\overline{\sigma}_2\overline{\rho} + \overline{\sigma}_1 c_1\right) \sin\left(\frac{2\pi k}{n}\right)\right) \\
= \alpha\tau + \frac{1}{2\tau} r_p^2 + y_1\left(-c_1\overline{\sigma}_1 \cos\left(\frac{2\pi k}{n}\right) + \frac{1}{\sqrt{1-\overline{\rho}^2}}\left(-c_2\overline{\sigma}_2 - \overline{\rho}c_1\overline{\sigma}_1\right) \sin\left(\frac{2\pi k}{n}\right)\right) \\
+ y_2\Big(\left(c_2\overline{\sigma}_2 + 2\overline{\sigma}_1 c_1\overline{\rho}\right) \cos\left(\frac{2\pi k}{n}\right) \\
+ \frac{1}{\sqrt{1-\overline{\rho}^2}}\left(c_2\overline{\sigma}_2\overline{\rho} - \overline{\sigma}_1 c_1(1-2\overline{\rho}^2)\right) \sin\left(\frac{2\pi k}{n}\right) \Big)
\end{cases}
$$

Thus,

$$\begin{aligned}
C_{B,0,0}(t,S_1,S_2) &= \sum_{k=0}^{n-1} e^{y_1\left(c_1\overline{\sigma}_1 - c_1\overline{\sigma}_1\cos\left(\frac{2\pi k}{n}\right) + \frac{1}{\sqrt{1-\overline{\rho}^2}}(-c_2\overline{\sigma}_2 - \overline{\rho}c_1\overline{\sigma}_1)\sin\left(\frac{2\pi k}{n}\right)\right)} e^{y_2 c_2 \overline{\sigma}_2} \qquad \text{(B.14)}\\
&\left(B_2 e^{y_1 \frac{\overline{\sigma}_2 \sin\left(\frac{2\pi k}{n}\right)}{\sqrt{1-\overline{\rho}^2}}}\left(e^{y_2\left((1-c_2)\overline{\sigma}_2\cos\left(\frac{2\pi k}{n}\right) + \frac{1}{\sqrt{1-\overline{\rho}^2}}(-(1-c_2)\overline{\sigma}_2\overline{\rho} + \overline{\sigma}_1 c_1)\sin\left(\frac{2\pi k}{n}\right)\right)}\right.\right.\\
&\mathcal{N}_2\left(\sqrt{\tau}(\gamma_1^+ + \overline{\rho}\eta_2^+) + \frac{b_1}{\overline{\sigma}_1\sqrt{\tau}}, \sqrt{\tau}(\overline{\rho}\gamma_1^+ + \eta_2^+) + \frac{b_2}{\overline{\sigma}_2\sqrt{\tau}}\right)\\
&-e^{y_2\left((-(1-c_2)\overline{\sigma}_2 + 2\overline{\sigma}_1 c_1\overline{\rho})\cos\left(\frac{2\pi k}{n}\right) + \frac{1}{\sqrt{1-\overline{\rho}^2}}\left(-(1-c_2)\overline{\sigma}_2\overline{\rho} - \overline{\sigma}_1 c_1(1-2\overline{\rho}^2)\right)\sin\left(\frac{2\pi k}{n}\right)\right)}\\
&\left.\mathcal{N}_2\left(\sqrt{\tau}(\gamma_1^- + \overline{\rho}\eta_2^-) + \frac{b_1}{\overline{\sigma}_1\sqrt{\tau}}, \sqrt{\tau}(\overline{\rho}\gamma_1^- + \eta_2^-) + \frac{b_2}{\overline{\sigma}_2\sqrt{\tau}}\right)\right)\\
&-e^{-\int_t^T r(s)ds} K_2\left(e^{y_2\left(-c_2\overline{\sigma}_2\cos\left(\frac{2\pi k}{n}\right) + \frac{1}{\sqrt{1-\overline{\rho}^2}}(c_2\overline{\sigma}_2\overline{\rho} + \overline{\sigma}_1 c_1)\sin\left(\frac{2\pi k}{n}\right)\right)}\right.\\
&\mathcal{N}_2\left(\sqrt{\tau}(\gamma_1^+ + \overline{\rho}\gamma_2^+) + \frac{b_1}{\overline{\sigma}_1\sqrt{\tau}}, \sqrt{\tau}(\overline{\rho}\gamma_1^+ + \gamma_2^+) + \frac{b_2}{\overline{\sigma}_2\sqrt{\tau}}\right)\\
&-e^{y_2\left((c_2\overline{\sigma}_2 + 2\overline{\sigma}_1 c_1\overline{\rho})\cos\left(\frac{2\pi k}{n}\right) + \frac{1}{\sqrt{1-\overline{\rho}^2}}\left(c_2\overline{\sigma}_2\overline{\rho} - \overline{\sigma}_1 c_1(1-2\overline{\rho}^2)\right)\sin\left(\frac{2\pi k}{n}\right)\right)}\\
&\left.\left.\mathcal{N}_2\left(\sqrt{\tau}(\gamma_1^- + \overline{\rho}\gamma_2^-) + \frac{b_1}{\overline{\sigma}_1\sqrt{\tau}}, \sqrt{\tau}(\overline{\rho}\gamma_1^- + \gamma_2^-) + \frac{b_2}{\overline{\sigma}_2\sqrt{\tau}}\right)\right)\right),
\end{aligned}$$

□

Proof. (Explicit expression of differences for Lemma 2)

$$
\begin{aligned}
&\left|C_{BS}\left(t, S_1e^{\Lambda_1+r\epsilon}, S_2e^{\Lambda_2+r\epsilon}, \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\rho}\right) - C_{BS}^{\tilde{\epsilon},\delta}\left(t, S_1e^{\Lambda_1}, S_2e^{\Lambda_2}, \hat{\sigma}_1, \hat{\sigma}_2, \hat{\rho}\right)\right| \\
&= \Bigg|S_2e^{\Lambda_2}\Bigg(e^{-r\tilde{\epsilon}}\mathcal{N}_2\left(\frac{x_1+\Lambda_1+r\epsilon-\frac{1}{2}\tilde{\sigma}_1^2}{\tilde{\sigma}_1}+\tilde{\rho}\tilde{\sigma}_2, \frac{x_2+\Lambda_2+r\epsilon+\frac{1}{2}\tilde{\sigma}_2^2}{\tilde{\sigma}_2}, \tilde{\rho}\right) \\
&\qquad -\mathcal{N}_2\left(\frac{x_1+\Lambda_1-\frac{1}{2}\hat{\sigma}_1^2}{\hat{\sigma}_1}+\hat{\rho}\hat{\sigma}_2, \frac{x_2+\Lambda_2+\frac{1}{2}\hat{\sigma}_2^2}{\hat{\sigma}_2}, \hat{\rho}\right)\Bigg) \\
&\qquad -K_2e^{-r(T-t)}\Bigg(e^{-r\tilde{\epsilon}}\mathcal{N}_2\left(\frac{x_1+\Lambda_1+r\epsilon-\frac{1}{2}\tilde{\sigma}_1^2}{\tilde{\sigma}_1}, \frac{x_2+\Lambda_2+r\epsilon-\frac{1}{2}\tilde{\sigma}_2^2}{\tilde{\sigma}_2}, \tilde{\rho}\right) \\
&\qquad -\mathcal{N}_2\left(\frac{x_1+\Lambda_1-\frac{1}{2}\hat{\sigma}_1^2}{\hat{\sigma}_1}, \frac{x_2+\Lambda_2-\frac{1}{2}\hat{\sigma}_2^2}{\hat{\sigma}_2}, \hat{\rho}\right)\Bigg)\Bigg| \\
&\le \Bigg|S_2e^{\Lambda_2}\Bigg(e^{-r\tilde{\epsilon}}\mathcal{N}_2\left(\frac{x_1+\Lambda_1+r\epsilon-\frac{1}{2}\tilde{\sigma}_1^2}{\tilde{\sigma}_1}+\tilde{\rho}\tilde{\sigma}_2, \frac{x_2+\Lambda_2+r\epsilon+\frac{1}{2}\tilde{\sigma}_2^2}{\tilde{\sigma}_2}, \tilde{\rho}\right) \\
&\qquad -\mathcal{N}_2\left(\frac{x_1+\Lambda_1-\frac{1}{2}\hat{\sigma}_1^2}{\hat{\sigma}_1}+\hat{\rho}\hat{\sigma}_2, \frac{x_2+\Lambda_2+\frac{1}{2}\hat{\sigma}_2^2}{\hat{\sigma}_2}, \hat{\rho}\right)\Bigg)\Bigg| \\
&+\Bigg|-K_2e^{-r(T-t)}\Bigg(e^{-r\tilde{\epsilon}}\mathcal{N}_2\left(\frac{x_1+\Lambda_1+r\epsilon-\frac{1}{2}\tilde{\sigma}_1^2}{\tilde{\sigma}_1}, \frac{x_2+\Lambda_2+r\epsilon-\frac{1}{2}\tilde{\sigma}_2^2}{\tilde{\sigma}_2}, \tilde{\rho}\right) \\
&\qquad -\mathcal{N}_2\left(\frac{x_1+\Lambda_1-\frac{1}{2}\hat{\sigma}_1^2}{\hat{\sigma}_1}, \frac{x_2+\Lambda_2-\frac{1}{2}\hat{\sigma}_2^2}{\hat{\sigma}_2}, \hat{\rho}\right)\Bigg)\Bigg|.
\end{aligned}
$$

Note that

$$\frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{\mathbf{d}_1}\int_{-\infty}^{\mathbf{d}_2} e^{-\frac{x_1^2+x_2^2-2\rho x_1x_2}{2(1-\rho^2)}}dx_2dx_1 = \frac{1}{2\pi}\int_{-\infty}^{\mathbf{d}_1}\int_{-\infty}^{\frac{\mathbf{d}_2-\rho z_1}{\sqrt{1-\rho^2}}} e^{-\frac{z_1^2+z_2^2}{2}}dz_2dz_1, \quad \text{(B.15)}$$

as we set $z_1 = x_1$ and $z_2 = \frac{1}{\sqrt{1-\rho^2}}(x_2-\rho x_1)$. Hence, $x_2 = \sqrt{1-\rho^2}z_2+\rho z_1$, $x_2^2 = (1-\rho^2)z_2^2+\rho^2z_1^2+2\rho\sqrt{1-\rho^2}z_1z_2$, and $-2\rho x_1x_2 = -2\rho z_1(\sqrt{1-\rho^2}z_2+\rho z_1)$. In the following

$$
\begin{aligned}
\mathbf{d}_1 &= \frac{x_1+\Lambda_1}{\hat{\sigma}_1}-\frac{1}{2}\hat{\sigma}_1+\hat{\rho}\hat{\sigma}_2, \\
\mathbf{d}_2 &= \frac{x_2+\Lambda_2}{\hat{\sigma}_2}+\frac{1}{2}\hat{\sigma}_2, \\
\mathbf{d}_1^{\tilde{\epsilon}} &= \frac{x_1+\Lambda_1+r\tilde{\epsilon}}{\tilde{\sigma}_1}-\frac{1}{2}\tilde{\sigma}_1+\tilde{\rho}\tilde{\sigma}_2, \\
\mathbf{d}_2 &= \frac{x_2+\Lambda_2}{\tilde{\sigma}_2}+\frac{1}{2}\tilde{\sigma}_2.
\end{aligned}
$$

To simplify the following calculation we compute the two expressions with (4.144) and (4.145)

$$
\begin{aligned}
&\left(\frac{\tilde{\rho}}{\sqrt{1-\tilde{\rho}^2}} - \frac{\hat{\rho}}{\sqrt{1-\hat{\rho}^2}}\right)\\
= \;& \frac{\tilde{\rho}\tilde{\sigma}_1\tilde{\sigma}_2}{\sqrt{1-\tilde{\rho}^2}\tilde{\sigma}_1\tilde{\sigma}_2} - \frac{\hat{\rho}\hat{\sigma}_1\hat{\sigma}_2}{\sqrt{1-\hat{\rho}^2}\hat{\sigma}_1\hat{\sigma}_2}\\
= \;& \frac{\hat{\rho}\hat{\sigma}_1\hat{\sigma}_2 + \epsilon\overline{\rho}\;\overline{\sigma}_1\overline{\sigma}_2}{\sqrt{1-\tilde{\rho}^2}\tilde{\sigma}_1\tilde{\sigma}_2} - \frac{\hat{\rho}\hat{\sigma}_1\hat{\sigma}_2}{\sqrt{1-\hat{\rho}^2}\hat{\sigma}_1\hat{\sigma}_2}\\
\leq \;& \epsilon\tilde{\tilde{c}}_1 + \hat{\rho}\hat{\sigma}_1\hat{\sigma}_2\left(\frac{\sqrt{1-\hat{\rho}^2}\hat{\sigma}_1\hat{\sigma}_2 - \sqrt{1-\tilde{\rho}^2}\tilde{\sigma}_1\tilde{\sigma}_2}{\hat{\sigma}_1\hat{\sigma}_2\tilde{\sigma}_1\tilde{\sigma}_2\sqrt{1-\hat{\rho}^2}\sqrt{1-\tilde{\rho}^2}}\right)\\
= \;& \tilde{\epsilon}\tilde{\tilde{c}}_1 + \hat{\rho}\hat{\sigma}_1\hat{\sigma}_2\frac{\hat{\sigma}_1^2\hat{\sigma}_2^2 - \hat{\rho}^2\hat{\sigma}_1^2\hat{\sigma}_2^2 - \tilde{\sigma}_1^2\tilde{\sigma}_2^2 + \tilde{\sigma}_1^2\tilde{\sigma}_2^2\tilde{\rho}^2}{\hat{\sigma}_1\hat{\sigma}_2\tilde{\sigma}_1\tilde{\sigma}_2\sqrt{1-\hat{\rho}^2}\sqrt{1-\tilde{\rho}^2}}\frac{1}{\hat{\sigma}_1\hat{\sigma}_2\sqrt{1-\hat{\rho}^2} + \sqrt{1-\tilde{\rho}^2}\tilde{\sigma}_1\tilde{\sigma}_2}\\
= \;& \tilde{\epsilon}\tilde{\tilde{c}}_1 + \hat{\rho}\hat{\sigma}_1\hat{\sigma}_2\underbrace{\frac{\hat{\sigma}_1^2\hat{\sigma}_2^2 - \hat{\rho}^2\hat{\sigma}_1^2\hat{\sigma}_2^2 - (\hat{\sigma}_1^2 + \tilde{\epsilon}\overline{\sigma}_1^2)(\hat{\sigma}_2^2 + \tilde{\epsilon}\overline{\sigma}_2^2) + (\hat{\rho}\hat{\sigma}_1\hat{\sigma}_2 + \overline{\sigma}_1\overline{\sigma}_2\overline{\rho}\tilde{\epsilon})^2}{\hat{\sigma}_1\hat{\sigma}_2\tilde{\sigma}_1\tilde{\sigma}_2\sqrt{1-\hat{\rho}^2}\sqrt{1-\tilde{\rho}^2}}}_{\text{bounded in } \tilde{\epsilon}}\\
&\frac{1}{\hat{\sigma}_1\hat{\sigma}_2\sqrt{1-\hat{\rho}^2} + \sqrt{1-\tilde{\rho}^2}\tilde{\sigma}_1\tilde{\sigma}_2}\\
\leq \;& \tilde{\epsilon}\tilde{\tilde{c}}_2,
\end{aligned}
\tag{B.16}
$$

and

$$
\begin{aligned}
&\left|\frac{d_1^{\tilde{\epsilon}}\tilde{\sigma}_1}{\sqrt{1-\tilde{\rho}^2}\tilde{\sigma}_1} - \frac{d_1\hat{\sigma}_1}{\sqrt{1-\hat{\rho}^2}\hat{\sigma}_1}\right|\\
= \;& \left|\frac{d_1\hat{\sigma}_1 + \epsilon(r - \frac{1}{2}\overline{\sigma}_1^2 + \overline{\rho\sigma}_1\overline{\sigma}_2)}{\sqrt{1-\tilde{\rho}^2}\tilde{\sigma}_1} - \frac{d_1\hat{\sigma}_1}{\sqrt{1-\hat{\rho}^2}\hat{\sigma}_1}\right|\\
\leq \;& \tilde{\epsilon}\tilde{\tilde{c}}_3 + \left|\frac{d_1\hat{\sigma}_1^2\sqrt{1-\hat{\rho}^2} - d_1\hat{\sigma}_1\tilde{\sigma}_1\sqrt{1-\tilde{\rho}^2}}{\sqrt{1-\tilde{\rho}^2}\sqrt{1-\hat{\rho}^2}\hat{\sigma}_1\tilde{\sigma}_1}\right|\\
= \;& \tilde{\epsilon}\tilde{\tilde{c}}_3 + \left|d_1\hat{\sigma}_1\underbrace{\frac{\hat{\sigma}_1\sqrt{1-\hat{\rho}^2} - \tilde{\sigma}_1\sqrt{1-\tilde{\rho}^2}}{\hat{\sigma}_1\tilde{\sigma}_1\sqrt{1-\tilde{\rho}^2}\sqrt{1-\hat{\rho}^2}}}_{\text{see above: bounded in } \tilde{\epsilon}}\right|\\
\leq \;& \tilde{\epsilon}\tilde{\tilde{c}}_4(1 + |\Lambda_1|).
\end{aligned}
\tag{B.17}
$$

The assessment for the respective expression with $d_2^{\tilde{\epsilon}}$ follows analogously.

Thus,

$$
\begin{aligned}
&\left|\left(e^{-r\tilde{\epsilon}}\mathcal{N}_2\left(\frac{x_1+\Lambda_1+r\epsilon-\frac{1}{2}\tilde{\sigma}_1^2}{\tilde{\sigma}_1},\frac{x_2+\Lambda_2+r\epsilon-\frac{1}{2}\tilde{\sigma}_2^2}{\tilde{\sigma}_2},\tilde{\rho}\right)\right.\right.\\
&\left.\left.-\mathcal{N}_2\left(\frac{x_1+\Lambda_1-\frac{1}{2}\hat{\sigma}_1^2}{\hat{\sigma}_1},\frac{x_2+\Lambda_2-\frac{1}{2}\hat{\sigma}_2^2}{\hat{\sigma}_2},\hat{\rho}\right)\right)\right|\\
\le\ &\left|\int_{-\infty}^{\mathbf{d}_1}\int_{-\infty}^{\frac{\mathbf{d}_2^{\tilde{\epsilon}}-\tilde{\rho}z_1}{\sqrt{1-\tilde{\rho}^2}}} e^{-\frac{z_1^2+z_2^2}{2}}dz_2dz_1+\int_{\mathbf{d}_1}^{\mathbf{d}_1^{\tilde{\epsilon}}}\int_{-\infty}^{\frac{\mathbf{d}_2^{\tilde{\epsilon}}-\tilde{\rho}z_1}{\sqrt{1-\tilde{\rho}^2}}} e^{-\frac{z_1^2+z_2^2}{2}}dz_2dz_1\right.\\
&\left.-\int_{-\infty}^{\mathbf{d}_1}\int_{-\infty}^{\frac{\mathbf{d}_2-\hat{\rho}z_1}{\sqrt{1-\hat{\rho}^2}}} e^{-\frac{z_1^2+z_2^2}{2}}dz_2dz_1\right|\\
=\ &\left|\int_{-\infty}^{\mathbf{d}_1}\int_{-\infty}^{\frac{\mathbf{d}_2-\hat{\rho}z_1}{\sqrt{1-\hat{\rho}^2}}} e^{-\frac{z_1^2+z_2^2}{2}}dz_2dz_1+\int_{-\infty}^{\mathbf{d}_1}\int_{\frac{\mathbf{d}_2-\hat{\rho}}{\sqrt{1-\hat{\rho}^2}}}^{\frac{\mathbf{d}_2^{\tilde{\epsilon}}-\tilde{\rho}z_1}{\sqrt{1-\tilde{\rho}^2}}} e^{-\frac{z_1^2+z_2^2}{2}}dz_2dz_1\right.\\
&\left.+\int_{\mathbf{d}_1}^{\mathbf{d}_1^{\tilde{\epsilon}}}\int_{-\infty}^{\frac{\mathbf{d}_2^{\tilde{\epsilon}}-\tilde{\rho}}{\sqrt{1-\tilde{\rho}^2}}} e^{-\frac{z_1^2+z_2^2}{2}}dz_2dz_1-\int_{-\infty}^{\mathbf{d}_1}\int_{-\infty}^{\frac{\mathbf{d}_2-\hat{\rho}z_1}{\sqrt{1-\hat{\rho}^2}}} e^{-\frac{z_1^2+z_2^2}{2}}dz_2dz_1\right|\\
\le\ &\left|\int_{-\infty}^{\mathbf{d}_1}\int_{\frac{\mathbf{d}_2-\hat{\rho}z_1}{\sqrt{1-\hat{\rho}^2}}}^{\frac{\mathbf{d}_2^{\tilde{\epsilon}}-\tilde{\rho}z_1}{\sqrt{1-\tilde{\rho}^2}}} \underbrace{e^{-\frac{z_1^2+z_2^2}{2}}}_{\le 1}dz_2dz_1\right|+\left|\int_{\mathbf{d}_1}^{\mathbf{d}_1^{\tilde{\epsilon}}}\int_{-\infty}^{\frac{\mathbf{d}_2^{\tilde{\epsilon}}-\tilde{\rho}z_1}{\sqrt{1-\tilde{\rho}^2}}} e^{-\frac{z_1^2+z_2^2}{2}}dz_2dz_1\right|\\
\le\ &\left|\int_{-\infty}^{\mathbf{d}_1}\left(\frac{\mathbf{d}_2^{\tilde{\epsilon}}-\tilde{\rho}z_1}{\sqrt{1-\tilde{\rho}^2}}-\frac{\mathbf{d}_2+\hat{\rho}z_1}{\sqrt{1-\hat{\rho}^2}}\right)e^{-\frac{1}{2}z_1^2}dz_1\right|+\left|\int_{-\infty}^{\mathbf{d}_2}\left(\frac{\mathbf{d}_1^{\tilde{\epsilon}}-\tilde{\rho}z_2}{\sqrt{1-\tilde{\rho}^2}}-\frac{\mathbf{d}_1+\hat{\rho}z_2}{\sqrt{1-\hat{\rho}^2}}\right)e^{-\frac{1}{2}z_2^2}dz_2\right|\\
\le\ &\tilde{c}_1\left|(1+\mathbf{d}_1)\left(\frac{\mathbf{d}_2^{\tilde{\epsilon}}}{\sqrt{1-\tilde{\rho}^2}}-\frac{\mathbf{d}_2}{\sqrt{1-\hat{\rho}^2}}\right)\right|+\left|-\int_{-\infty}^{\mathbf{d}_1}e^{-\frac{1}{2}z_1^2}\left(\frac{\tilde{\rho}}{\sqrt{1-\tilde{\rho}^2}}-\frac{\hat{\rho}}{\sqrt{1-\hat{\rho}^2}}\right)z_1dz_1\right|\\
&+\tilde{c}_2\left|(1+\mathbf{d}_2)\left(\frac{\mathbf{d}_1^{\tilde{\epsilon}}}{\sqrt{1-\tilde{\rho}^2}}-\frac{\mathbf{d}_1}{\sqrt{1-\hat{\rho}^2}}\right)\right|+\left|-\int_{-\infty}^{\mathbf{d}_2}e^{-\frac{1}{2}z_2^2}\left(\frac{\tilde{\rho}}{\sqrt{1-\tilde{\rho}^2}}-\frac{\hat{\rho}}{\sqrt{1-\hat{\rho}^2}}\right)z_2dz_2\right|\\
\le\ &\tilde{c}_3\left|(1+\mathbf{d}_1)\left(\frac{\mathbf{d}_2^{\tilde{\epsilon}}}{\sqrt{1-\tilde{\rho}^2}}-\frac{\mathbf{d}_2}{\sqrt{1-\hat{\rho}^2}}\right)\right|+\left|e^{-\frac{\mathbf{d}_1^2}{2}}\left(\frac{\tilde{\rho}}{\sqrt{1-\tilde{\rho}^2}}-\frac{\hat{\rho}}{\sqrt{1-\hat{\rho}^2}}\right)\right|\\
&+\tilde{c}_4\left|(1+\mathbf{d}_2)\left(\frac{\mathbf{d}_1^{\tilde{\epsilon}}}{\sqrt{1-\tilde{\rho}^2}}-\frac{\mathbf{d}_1}{\sqrt{1-\hat{\rho}^2}}\right)\right|+\left|e^{-\frac{\mathbf{d}_2^2}{2}}\left(\frac{\tilde{\rho}}{\sqrt{1-\tilde{\rho}^2}}-\frac{\hat{\rho}}{\sqrt{1-\hat{\rho}^2}}\right)\right|\\
\le\ &\tilde{c}_5\tilde{\epsilon}(1+|\Lambda_1|+|\Lambda_2|+|\Lambda_1|\,|\Lambda_2|). \qquad \text{(B.18)}
\end{aligned}
$$

The inequality in the third last line is derived by separating $\int_{-\infty}^{0} e^{-\frac{1}{2}z_1^2} = \frac{1}{2}$ and $\left|\int_0^{\mathbf{d}_1} \underbrace{e^{-\frac{1}{2}z_1^2}}_{\le 1} dz_1\right| \le |d_1|$. The inequality in the second last line is derived from integrating $\int_{-\infty}^{\mathbf{d}_2} e^{-\frac{1}{2}z_2^2}(-z_2)dz_2 = \left| e^{-\frac{1}{2}z_2^2} \right|_{-\infty}^{\mathbf{d}_2}$. Now, we easily see that

$$\begin{aligned} &\left| C_{BS}(t, S_1 e^{\Lambda_1 + r\epsilon}, S_2 e^{\Lambda_2 + r\epsilon}, \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\rho}) - C_{BS}^{\tilde{\epsilon},\delta}(t, S_1 e^{\Lambda_1}, S_2 e^{\Lambda_2}, \hat{\sigma}_1, \hat{\sigma}_2, \hat{\rho}) \right| \\ &\qquad \leq \tilde{c}_4 \tilde{\epsilon} \left(e^{\Lambda_2} + 1 \right) \left(1 + |\Lambda_1| + |\Lambda_2| + |\Lambda_1| \, |\Lambda_2| \right). \end{aligned} \tag{B.19}$$

□

($C_{3,0}$, $C_{0,3}$ and explicit representation of $F_i^{\tilde{\epsilon}}(t, S_1, S_2)$)

(4.60) is a Poisson equation in v_2. Hence, we can write

$$\begin{aligned} \mathcal{L}_0^1 C_{3,0} &= -\left\langle \mathcal{L}_1^1 C_{2,0} + \mathcal{L}_2 C_{1,0} \right\rangle_{v_2} \\ &= -\Bigg[-\Bigg\langle \mathcal{L}_1^1 \Bigg(\frac{1}{2}\phi_1 \left(a_{11}^2 S_1^2 \frac{\partial^2}{\partial S_1^2} + a_{21}^2 S_2^2 \frac{\partial^2}{\partial S_2^2} + 2a_{11}a_{21}S_1S_2 \frac{\partial^2}{\partial S_1 \partial S_2} \right) \\ &\quad + f_{2,0}^e(t, S_1, S_2) \Bigg) \Bigg\rangle_{v_2} + \langle \mathcal{L}_2 \rangle_{v_2} C_{1,0} \underbrace{- \langle\langle \mathcal{L}_2 \rangle\rangle C_{1,0} + \mathcal{A}_1 C_{0,0}}_{=0} \Bigg] \\ &= -\langle \mathcal{L}_2 \rangle_{v_2} C_{1,0} + \langle\langle \mathcal{L}_2 \rangle\rangle C_{1,0} \\ &\quad + \mathcal{L}_1^1 \Bigg(\frac{1}{2}\phi_1 \left(a_{11}^2 S_1^2 \frac{\partial^2}{\partial S_1^2} + a_{21}^2 S_2^2 \frac{\partial^2}{\partial S_2^2} + 2a_{11}a_{21}S_1S_2 \frac{\partial^2}{\partial S_1 \partial S_2} \right) \\ &\quad + f_{2,0}^e(t, S_1, S_2) \Bigg) - \mathcal{A}_1 C_{0,0} \\ &= \frac{1}{2}(T-t)\left(v_1 - \overline{\sigma}_1^2 \right) \Bigg(a_{11}^2 S_1^2 \frac{\partial^2}{\partial S_1^2} + a_{21}^2 S_2^2 \frac{\partial^2}{\partial S_2^2} \\ &\quad + 2a_{11}a_{21}S_1S_2 \frac{\partial^2}{\partial S_1 \partial S_2} \Bigg) \mathcal{A}_1 C_{0,0} \\ &\quad + \rho_1^v \sigma_{v_1} \left(v_1 \frac{\partial \phi_1}{\partial v_1} - \left\langle v_1 \frac{\partial \phi_1}{\partial v_1} \right\rangle \right) (S_1 \frac{a_{11}}{2} \frac{\partial}{\partial S_1} + S_2 \frac{a_{21}}{2} \frac{\partial}{\partial S_2}) \\ &\quad \left(a_{11}^2 S_1^2 \frac{\partial^2}{\partial S_1^2} + a_{21}^2 S_2^2 \frac{\partial^2}{\partial S_2^2} + 2a_{11}a_{21}S_1S_2 \frac{\partial^2}{\partial S_1 \partial S_2} \right) C_{0,0}. \end{aligned}$$

Thus,

$$\begin{aligned} C_{3,0} &= \frac{1}{2}(T-t)\phi_1 \left(a_{11}^2 S_1^2 \frac{\partial^2}{\partial S_1^2} + a_{21}^2 S_2^2 \frac{\partial^2}{\partial S_2^2} + 2a_{11}a_{21}S_1S_2 \frac{\partial^2}{\partial S_1 \partial S_2} \right) \mathcal{A}_1 C_{0,0} \\ &\quad + \rho_1^v \sigma_{v_1} \xi_1 \left(S_1 \frac{a_{11}}{2} \frac{\partial}{\partial S_1} + S_2 \frac{a_{21}}{2} \frac{\partial}{\partial S_2} \right) \\ &\quad \left(a_{11}^2 S_1^2 \frac{\partial^2}{\partial S_1^2} + a_{21}^2 S_2^2 \frac{\partial^2}{\partial S_2^2} + 2a_{11}a_{21}S_1S_2 \frac{\partial^2}{\partial S_1 \partial S_2} \right) C_{0,0} \\ &\quad + f_{3,0}^e(t, S_1, S_2), \end{aligned} \tag{B.20}$$

where $\mathcal{L}_0^1\xi_1 = \left(v_1\frac{\partial\phi_1}{\partial v_1} - \left\langle v_1\frac{\partial\phi_1}{\partial v_1}\right\rangle\right)$. $C_{0,3}$ can be solved accordingly.

Inserting (4.156) in (4.155) we obtain

$$\begin{aligned}
F_1^{\tilde{\epsilon}}(t,S_1,S_2) \;=\; & (T-t)\rho_2^v\sigma_{v_2}v_2\frac{\partial\phi_2}{\partial v_2}\left(\frac{a_{12}}{2}S_1\frac{\partial}{\partial S_1}+S_2\frac{a_{22}}{2}\frac{\partial}{\partial S_2}\right)\\
&\left(a_{12}^2S_1^2\frac{\partial^2}{\partial S_1^2}+S_2^2a_{22}^2\frac{\partial^2}{\partial S_2^2}+2a_{12}a_{22}S_1S_2\frac{\partial^2}{\partial S_1\partial S_2}\right)\mathcal{A}_1C_{0,0}\\
&-2(T-t)\mathcal{A}_1\mathcal{A}_2C_{0,0}+(T-t)^2\mathcal{A}_1\mathcal{A}_2\mathcal{L}_2C_{0,0}\\
&(T-t)\rho_1^v\sigma_{v_1}\frac{\partial\phi_1}{\partial v_1}\left(\frac{a_{11}}{2}S_1\frac{\partial}{\partial S_1}+S_2\frac{a_{21}}{2}\frac{\partial}{\partial S_2}\right)\\
&\left(a_{11}^2S_1^2\frac{\partial^2}{\partial S_1^2}+S_2^2a_{21}^2\frac{\partial^2}{\partial S_2^2}+2a_{11}a_{21}S_1S_2\frac{\partial^2}{\partial S_1\partial S_2}\right)\mathcal{A}_2C_{0,0}\\
&+\delta_1\Bigg(-\frac{1}{2}\phi_1\Bigg(a_{11}^2S_1^2\frac{\partial^2}{\partial S_1^2}+S_2^2a_{21}^2\frac{\partial^2}{\partial S_2^2}\\
&+2a_{11}a_{21}S_1S_2\frac{\partial^2}{\partial S_1\partial S_2}\Bigg)\mathcal{A}_2C_{0,0}+\frac{1}{2}(T-t)\phi_1\\
&\left(a_{11}^2S_1^2\frac{\partial^2}{\partial S_1^2}+S_2^2a_{21}^2\frac{\partial^2}{\partial S_2^2}+2a_{11}a_{21}S_1S_2\frac{\partial^2}{\partial S_1\partial S_2}\right)\mathcal{A}_2\mathcal{L}_2C_{0,0}\Bigg)\\
&+\delta_2\Bigg(-\frac{1}{2}\phi_1\Bigg(a_{12}^2S_1^2\frac{\partial^2}{\partial S_1^2}+S_2^2a_{22}^2\frac{\partial^2}{\partial S_2^2}\\
&+2a_{12}a_{22}S_1S_2\frac{\partial^2}{\partial S_1\partial S_2}\Bigg)\mathcal{A}_2C_{0,0}+\frac{1}{2}(T-t)\phi\\
&\left(a_{12}^2S_1^2\frac{\partial^2}{\partial S_1^2}+S_2^2a_{22}^2\frac{\partial^2}{\partial S_2^2}+2a_{12}a_{22}S_1S_2\frac{\partial^2}{\partial S_1\partial S_2}\right)\mathcal{A}_2\mathcal{L}_2C_{0,0}\Bigg)\\
&+\tilde{q}_4(t,S_1,S_2),
\end{aligned}$$

$$
\begin{aligned}
F_2^{\tilde{\epsilon}}(t,S_1,S_2) \;=\; & -\frac{1}{2}\phi_1\left(a_{11}^2S_1^2\frac{\partial^2}{\partial S_1^2}+a_{21}^2S_2^2\frac{\partial^2}{\partial S_2^2}+2a_{11}a_{21}S_1S_2\frac{\partial^2}{\partial S_1\partial S_2}\right)\mathcal{L}_2C_{0,0} \\
& +\rho_1^v\sigma_{v_1}(T-t)v_1\frac{\partial\phi_1}{\partial v_1}\left(S_1\frac{a_{11}}{2}\frac{\partial}{\partial S_1}+S_2\frac{a_{21}}{2}\frac{\partial}{\partial S_2}\right) \\
& \left(a_{11}^2S_1^2\frac{\partial^2}{\partial S_1^2}+a_{21}^2S_2^2\frac{\partial^2}{\partial S_2^2}+2a_{11}a_{21}S_1S_2\frac{\partial^2}{\partial S_1\partial S_2}\right)\mathcal{A}_1C_{0,0} \\
& +\rho_1^{v2}\sigma_{v_1}^2\frac{\partial\xi_1}{\partial v_1}\left(S_1\frac{a_{11}}{2}\frac{\partial}{\partial S_1}+S_2\frac{a_{21}}{2}\frac{\partial}{\partial S_2}\right) \\
& \left(a_{11}^2S_1^2\frac{\partial^2}{\partial S_1^2}+a_{21}^2S_2^2\frac{\partial^2}{\partial S_2^2}+2a_{11}a_{21}S_1S_2\frac{\partial^2}{\partial S_1\partial S_2}\right)C_{0,0} \\
& +\delta_1\Bigg(-\frac{1}{2}\phi_1\Bigg(a_{11}^2S_1^2\frac{\partial^2}{\partial S_1^2}+S_2^2a_{21}^2\frac{\partial^2}{\partial S_2^2} \\
& +2a_{11}a_{21}S_1S_2\frac{\partial^2}{\partial S_1\partial S_2}\Bigg)\mathcal{A}_1C_{0,0}+\frac{1}{2}(T-t)\phi_1 \\
& \left(a_{11}^2S_1^2\frac{\partial^2}{\partial S_1^2}+S_2^2a_{21}^2\frac{\partial^2}{\partial S_2^2}+2a_{11}a_{21}S_1S_2\frac{\partial^2}{\partial S_1\partial S_2}\right)\mathcal{A}_1\mathcal{L}_2C_{0,0} \\
& +\rho_1^v\sigma_{v_1}\xi_1\left(S_1\frac{a_{11}}{2}\frac{\partial}{\partial S_1}+S_2\frac{a_{21}}{2}\frac{\partial}{\partial S_2}\right) \\
& \left(a_{11}^2S_1^2\frac{\partial^2}{\partial S_1^2}+a_{21}^2S_2^2\frac{\partial^2}{\partial S_2^2}+2a_{11}a_{21}S_1S_2\frac{\partial^2}{\partial S_1\partial S_2}\right)\mathcal{L}_2C_{0,0}\Bigg) \\
& +\tilde{q}_5(t,S_1,S_2).
\end{aligned}
$$

$F_3^{\tilde{\epsilon}}(t,S_1,S_2)$ follows accordingly.

In the following we derive the form of the derivatives of $C_{0,0}$:

$$
\begin{aligned}
\frac{\partial C_{0,0}}{\partial x_2} &= e^{x_2} K_2 e^{-r\tau^{\tilde{\epsilon}}} \mathcal{N}_2(\mathbf{d}_2^{\tilde{\epsilon}}, \mathbf{d}_1^{\tilde{\epsilon}}, \overline{\rho}) + \frac{S_1}{2\pi\overline{\sigma}_2\sqrt{1-\overline{\rho}^2}} \int_{-\infty}^{d_1^{\tilde{\epsilon}}} e^{-\frac{(y_1^2 + d_2^{\tilde{\epsilon}\,2} - 2\overline{\rho} y_1 d_2^{\tilde{\epsilon}})}{2(1-\overline{\rho}^2)}} dy_1 \\
&\quad - \frac{K_2 e^{-r\tau^{\tilde{\epsilon}} r}}{2\pi\overline{\sigma}_2\sqrt{1-\overline{\rho}^2}\sqrt{\tau^{\tilde{\epsilon}}}} \int_{-\infty}^{d_1^{\tilde{\epsilon}} - \overline{\rho}\overline{\sigma}_2\sqrt{\tau^{\tilde{\epsilon}}}} e^{-\frac{(y_1^2 + (d_2^{\tilde{\epsilon}} + \overline{\sigma}_2\sqrt{\tau^{\tilde{\epsilon}}})^2 - 2\overline{\rho} y_1 (d_2^{\tilde{\epsilon}} + \overline{\sigma}_2\sqrt{\tau^{\tilde{\epsilon}}}))}{2(1-\overline{\rho}^2)}} dy_1 \\
&= e^{x_2} K_2 e^{-\tau^{\tilde{\epsilon}} r} \mathcal{N}_2(\mathbf{d}_2^{\tilde{\epsilon}}, \mathbf{d}_1^{\tilde{\epsilon}}, \overline{\rho}) + \frac{S_1}{2\pi\overline{\sigma}_2\sqrt{1-\overline{\rho}^2}} \int_{-\infty}^{d_1^{\tilde{\epsilon}}} e^{-\frac{(y_1^2 + d_2^{\tilde{\epsilon}\,2} - 2\overline{\rho} y_1 d_2^{\tilde{\epsilon}})}{2(1-\overline{\rho}^2)}} dy_1 \\
&\quad - \frac{K_2 e^{-r\tau^{\tilde{\epsilon}}}}{2\pi\overline{\sigma}_2\sqrt{1-\overline{\rho}^2}\sqrt{\tau^{\tilde{\epsilon}}}} \int_{-\infty}^{d_1^{\tilde{\epsilon}}} e^{-\frac{(y_1 + \overline{\rho}\overline{\sigma}_2\sqrt{\tau^{\tilde{\epsilon}}})^2 + (d_2^{\tilde{\epsilon}} + \overline{\sigma}_2\sqrt{\tau^{\tilde{\epsilon}}})^2 - 2\overline{\rho}(y_1 + \overline{\rho}\tilde{\sigma}_2\sqrt{\tau^{\tilde{\epsilon}}})(d_2^{\tilde{\epsilon}} + \overline{\sigma}_2\sqrt{\tau^{\tilde{\epsilon}}})}{2(1-\overline{\rho}^2)}} dy_1 \\
&= e^{x_2} K_2 e^{-\tau^{\tilde{\epsilon}} r} \mathcal{N}_2(\mathbf{d}_2^{\tilde{\epsilon}}, \mathbf{d}_1^{\tilde{\epsilon}}, \overline{\rho}) + \frac{S_1}{2\pi\overline{\sigma}_2\sqrt{1-\overline{\rho}^2}} \int_{-\infty}^{d_1^{\tilde{\epsilon}}} e^{-\frac{(y_1^2 + d_2^{\tilde{\epsilon}\,2} - 2\overline{\rho} y_1 d_2^{\tilde{\epsilon}})}{2(1-\overline{\rho}^2)}} dy_1 \\
&\quad - \frac{K_2 e^{-r\tau^{\tilde{\epsilon}}}}{2\pi\overline{\sigma}_2\sqrt{1-\overline{\rho}^2}\sqrt{\tau^{\tilde{\epsilon}}}} \int_{-\infty}^{d_1^{\tilde{\epsilon}}} e^{-\frac{y_1^2 + d_2^{\tilde{\epsilon}\,2} - 2\overline{\rho} y_1 d_2^{\tilde{\epsilon}}}{2(1-\overline{\rho}^2)}} \\
&\quad e^{-\frac{(2y_1\overline{\rho}\overline{\sigma}_2\sqrt{\tau^{\tilde{\epsilon}}} + \overline{\rho}^2\overline{\sigma}_2^2\tau^{\tilde{\epsilon}} + 2d_2^{\tilde{\epsilon}}\overline{\sigma}_2\sqrt{\tau^{\tilde{\epsilon}}} + \overline{\sigma}_2^2\tau^{\tilde{\epsilon}} - 2\overline{\rho} y_1\overline{\sigma}_2\sqrt{\tau^{\tilde{\epsilon}}} - 2\overline{\rho}^2\overline{\sigma}_2\sqrt{\tau^{\tilde{\epsilon}}} d_2^{\tilde{\epsilon}} - 2\overline{\rho}^2\overline{\sigma}_2^2\tau^{\tilde{\epsilon}})}{2(1-\overline{\rho}^2)}} dy_1 \\
&= e^{x_2} K_2 e^{-r\tau^{\tilde{\epsilon}}} \mathcal{N}_2(\mathbf{d}_2^{\tilde{\epsilon}}, \mathbf{d}_1^{\tilde{\epsilon}}, \overline{\rho}) + \frac{S_1}{2\pi\overline{\sigma}_2\sqrt{1-\overline{\rho}^2}} \int_{-\infty}^{d_1^{\tilde{\epsilon}}} e^{-\frac{(y_1^2 + d_2^{\tilde{\epsilon}\,2} - 2\overline{\rho} y_1 d_2^{\tilde{\epsilon}})}{2(1-\overline{\rho}^2)}} dy_1 \\
&\quad - \frac{K_2 e^{-r\tau^{\tilde{\epsilon}}}}{2\pi\overline{\sigma}_2\sqrt{1-\overline{\rho}^2}\sqrt{\tau^{\tilde{\epsilon}}}} \int_{-\infty}^{d_1^{\tilde{\epsilon}}} e^{-\frac{y_1^2 + d_2^{\tilde{\epsilon}\,2} - 2\overline{\rho} y_1 d_2^{\tilde{\epsilon}}}{2(1-\overline{\rho}^2)}} e^{-\frac{(\overline{\sigma}_2^2(1-\overline{\rho}^2)\tau^{\tilde{\epsilon}} + 2d_2^{\tilde{\epsilon}}\overline{\sigma}_2\sqrt{\tau^{\tilde{\epsilon}}}(1-\overline{\rho}^2))}{2(1-\overline{\rho}^2)}} dy_1 \\
&= e^{x_2} K_2 e^{-r\tau^{\tilde{\epsilon}}} \mathcal{N}_2(\mathbf{d}_2^{\tilde{\epsilon}}, \mathbf{d}_1^{\tilde{\epsilon}}, \overline{\rho}),
\end{aligned}
$$

because $e^{x_2} K_2 e^{-\tau^{\tilde{\epsilon}} r} - K_2 e^{-\tau^{\tilde{\epsilon}} r} e^{-\frac{1}{2}\overline{\sigma}_2^2\tau^{\tilde{\epsilon}} - d_2^{\tilde{\epsilon}}\overline{\sigma}_2\sqrt{\tau^{\tilde{\epsilon}}}} = 0$.

Thus,

$$
\begin{aligned}
\frac{\partial^k C_{0,0}}{\partial x_2^k} &= S_2 \mathcal{N}_2\left(\mathbf{d}_2^{\tilde{\epsilon}}, \mathbf{d}_1^{\tilde{\epsilon}}, \overline{\rho}\right) + \sum_{\mathfrak{k}=0}^{k-2} e^{x_2} \int_{-\infty}^{\mathbf{d}_1^{\tilde{\epsilon}}} \frac{\tilde{b}_{\mathfrak{k}}^1}{\sqrt{\tau^{\tilde{\epsilon}}}} \\
&\quad \frac{\partial^{\mathfrak{k}}}{x_2^{\mathfrak{k}}} \left(\exp\left\{ -\frac{1}{2(1-\overline{\rho}^2)} \left(y_1^2 + \mathbf{d}_2^{\tilde{\epsilon}\,2} - 2\overline{\rho}\mathbf{d}_2^{\tilde{\epsilon}} y_2 \right) \right\} \right) dy_1, \qquad \text{(B.21)}
\end{aligned}
$$

where $\tilde{b}_{\mathfrak{k}}^{1}$ is some constant term. For derivatives in x_1 we find:

$$\begin{aligned}\frac{\partial C_{0,0}}{\partial x_1} &= \frac{K_2 e^{-\tau^{\tilde{\epsilon}} r}}{2\pi\overline{\sigma}_2\sqrt{1-\overline{\rho}^2}\sqrt{\tau^{\tilde{\epsilon}}}}\Bigg[e^{x_2} e^{-\frac{1}{2}\mathbf{d}_1^{\tilde{\epsilon}\,2}} \\ &\int_{-\infty}^{\mathbf{d}_2^{\tilde{\epsilon}}} \exp\left\{-\frac{1}{2(1-\overline{\rho}^2)}(y_2-\overline{\rho}\mathbf{d}_1^{\tilde{\epsilon}})^2\right\} dy_2 \\ &-K_2 e^{-r(T-t)} e^{-\frac{1}{2}\mathbf{d}_1^{\tilde{\epsilon}*2}} \\ &\int_{-\infty}^{\mathbf{d}_2^{\tilde{\epsilon}*}} \exp\left\{-\frac{1}{2(1-\overline{\rho}^2)}(y_2-\overline{\rho}\mathbf{d}_1^{\tilde{\epsilon}*})^2\right\} dy_2\Bigg],\end{aligned}$$

and, thus, we deduce

$$\begin{aligned}\frac{\partial^i C_{0,0}}{\partial x_1^i} &= e^{x_2}\int_{-\infty}^{\mathbf{d}_2^{\tilde{\epsilon}}} \frac{\tilde{b}_i^2}{\sqrt{\tau^{\tilde{\epsilon}}}}\frac{\partial^{i-1}}{\partial x_1^{i-1}} \exp\left\{-\frac{\mathbf{d}_1^{\tilde{\epsilon}\,2}+y_2^2-2\overline{\rho}\mathbf{d}_1^{\tilde{\epsilon}}y_2}{2\left(1-\overline{\rho}^2\right)}\right\} dy_2 \\ &-K_2 e^{-r(T-t)}\int_{-\infty}^{\mathbf{d}_2^{\tilde{\epsilon}}-\sigma_2\sqrt{\tau^{\tilde{\epsilon}}}} \frac{\tilde{b}_i^3}{\sqrt{\tau^{\tilde{\epsilon}}}}\frac{\partial^{i-1}}{\partial x_1^{i-1}} \\ &\exp\left\{-\frac{1}{2(1-\overline{\rho}^2)}\left(y_2^2+\mathbf{d}_1^{\tilde{\epsilon}*2}-2\overline{\rho}y_2\mathbf{d}_1^{\tilde{\epsilon}*}\right)\right\} dy_2.\end{aligned}$$

We further see that

$$\begin{aligned}\frac{C_{0,0}}{\partial x_2\partial x_1} &= \frac{e^{x_2}K_2 e^{-\tau^{\tilde{\epsilon}} r}}{2\pi\sigma_1\sqrt{\left(1-\overline{\rho}^2\right)(T-t)}} \\ &\int_{-\infty}^{\mathbf{d}_2^{\tilde{\epsilon}}} \exp\left\{-\frac{\mathbf{d}_1^{\tilde{\epsilon}\,2}+y_2^2-2\overline{\rho}\mathbf{d}_1^{\tilde{\epsilon}}y_2}{2\left(1-\overline{\rho}^2\right)}\right\} dy_2,\end{aligned} \tag{B.22}$$

and

$$\frac{\partial^{i+1} C_{0,0}}{\partial x_2\partial x_1^i} = e^{x_2}\int_{-\infty}^{\mathbf{d}_2^{\tilde{\epsilon}}} \frac{\tilde{b}_i^4}{\sqrt{\tau^{\tilde{\epsilon}}}}\frac{\partial^{i-1}}{\partial x_1^{i-1}} \exp\left\{-\frac{\mathbf{d}_1^{\tilde{\epsilon}\,2}+y_2^2-2\overline{\rho}\mathbf{d}_1^{\tilde{\epsilon}}y_2}{2\left(1-\overline{\rho}^2\right)}\right\} dy_2, \tag{B.23}$$

as well as

$$\begin{aligned}\frac{\partial^{i+k} C_{0,0}}{\partial x_2^k\partial x_1^i} &= e^{x_2}\int_{-\infty}^{\mathbf{d}_2^{\tilde{\epsilon}}} \frac{\tilde{b}_i^4}{\sqrt{\tau^{\tilde{\epsilon}}}}\frac{\partial^{i-1}}{\partial x_1^{i-1}} \exp\left\{-\frac{\mathbf{d}_1^{\tilde{\epsilon}\,2}+y_2^2-2\overline{\rho}\mathbf{d}_1^{\tilde{\epsilon}}y_2}{2\left(1-\overline{\rho}^2\right)}\right\} dy_2 \\ &+\sum_{k=0}^{n-2} e^{x_2}\frac{\tilde{b}_k^5}{T-t} \\ &\frac{\partial^{i+k-1}}{\partial x_1^{i-1}\partial x_2^k} \exp\left\{-\frac{\mathbf{d}_1^{\tilde{\epsilon}\,2}+\mathbf{d}_2^{\tilde{\epsilon}\,2}-2\overline{\rho}\mathbf{d}_1^{\tilde{\epsilon}}\mathbf{d}_2^{\tilde{\epsilon}}}{2\left(1-\overline{\rho}^2\right)}\right\}.\end{aligned} \tag{B.24}$$

B.3 Poisson equation with CIR operator

Let χ solve

$$\mathcal{L}_0^j \chi(v_j) + h_j = 0, \quad \text{for } j = 1, 2 \tag{B.25}$$

with $\mathcal{L}_0^i$ as defined in Equation (4.43) or (4.44) and with h_j satisfying the centering condition

$$\langle h_1 \rangle_{v_1} = 0, \tag{B.26}$$

or

$$\langle h_2 \rangle_{v_2} = 0, \tag{B.27}$$

respectively. The averaging is done as explained before with respect to the invariant density $p^{inv}(v_j)$ of the Cox-Ingersoll-Ross (CIR) process [36]

$$p^{inv}(v_j) = \frac{\mu_j^{CIR a_j^{CIR}}}{\Gamma(a_j^{CIR})} v_j^{a_j^{CIR}-1} e^{-\mu_j^{CIR} v_j} \mathbb{1}_{v_j > 0}, \tag{B.28}$$

where

$$\begin{aligned} a_j^{CIR} &= \frac{2\kappa_j \zeta_j}{\sigma_{v_j}^2}, \\ \mu_j^{CIR} &= \frac{2\kappa_j}{\sigma_{v_j}^2}. \end{aligned}$$

Equation (B.25) has to satisfy Equation (B.26) (B.27 respectively) to be solvable (see Theorem 30). We see that by taking into account the following relationship

$$\begin{aligned} \langle h_j \rangle_{v_j} = -\left\langle \mathcal{L}_0^j \xi(v_j) \right\rangle_{v_j} &= -\int_0^\infty \left(\mathcal{L}_0^j \xi(v_j) \right) p^{inv}(v_j) dv_j \\ &\overset{Def.45}{=} \int_0^\infty \xi(v_j) \left(\mathcal{L}_0^{j*} p^{inv}(v_j) \right) dv_j \\ &= 0, \end{aligned} \tag{B.29}$$

where $\mathcal{L}_0^{j*}$ is the adjoint operator of $\mathcal{L}_0^j$ (see Definition (45)) and the inequality in line two is due to the definition of the adjoint operator. In the third line we exploit that the invariant distribution solves the adjoint equation $\mathcal{L}_0^{j*} p^{inv}(v_j) = 0$ (see Remark 6).

In the following we derive an upper boundary on the absolute value of $\frac{\partial \chi}{\partial v_j}$ and $\chi(v_j)$.

Hence, we derive an equivalent expression for $\mathcal{L}_0^j\chi$:

$$\begin{aligned}
\mathcal{L}_0^j\chi &= \frac{1}{2}\left(\frac{\partial\chi}{\partial v_j}\left(2\kappa_{v_j}\zeta_{v_j}\right) - 2v_j\kappa_{v_j}\frac{\partial\chi}{\partial v_j} + \sigma_{v_j}^2 v_j\frac{\partial^2\chi}{\partial v_j^2}\right)\\
&= \frac{\sigma_{v_j}^2}{2}\left(\frac{\partial\chi}{\partial v_j} + v_j\frac{\partial\chi}{\partial v_j}\frac{1}{v_j}\left(a_j-1\right) + v_j\frac{\partial\chi}{\partial v_j}\left(-\mu_j\right) + v\frac{\partial^2\chi}{\partial v_j^2}\right)\\
&= \frac{\sigma_{v_j}^2}{2p^{inv}(v_j)}\left(p^{inv}(v_j)\frac{\partial\chi}{\partial v_j} + v_j\frac{\partial p^{inv}(v_j)}{\partial v_j}\frac{\partial\chi}{\partial v_j} + v_jp^{inv}(v_j)\frac{\partial^2\chi}{\partial v_j^2}\right)\\
&= \frac{\sigma_{v_j}^2}{2p^{inv}(v_j)}\frac{\partial}{\partial v_j}\left(v_jp^{inv}(v_j)\frac{\partial\chi}{\partial v_j}\right)\\
&= -h_j, \qquad\qquad (B.30)
\end{aligned}$$

where the last line follows due to (B.25). Thus,

$$\frac{\sigma_{v_j}^2}{2p^{inv}(v_j)}\frac{\partial}{\partial v_j}\left(v_jp^{inv}(v_j)\frac{\partial\chi}{\partial v_j}\right) = -h_j, \qquad (B.31)$$

and we can solve (B.31) for $\frac{\partial\chi}{\partial v_j}$.

$$\frac{\partial\chi}{\partial v_j} = -\frac{2}{\sigma_{v_j}^2 v_jp^{inv}(v_j)}\int_0^{v_j} h_j(w)p(w)dw, \qquad (B.32)$$

When $|h_j(v_j)| \leq \bar{c}_1\left(1+|v_j|^l\right)$ we obtain for $v\to\infty, 0$

$$\begin{aligned}
\left|\frac{\partial\chi}{\partial v_j}\right| &= \left|-\frac{2}{\sigma_{v_j}^2 v_jp^{inv}(v_j)}\int_0^{v_j} h(w)p(w)dw\right|\\
&\leq \left|-\frac{\bar{c}_2}{p^{inv}(v_j)v_j\sigma_{v_j}^2}\int_0^{v_j} w^lp(w)dw\right|\\
&= \left|\frac{\bar{c}_2}{p^{inv}(v_j)v_j\sigma_{v_j}^2}\int_0^{v_j}(-\mu_j^{CIR})w^{a_j^{CIR}+l-1}e^{-\mu^{CIR}w}\frac{\mu^{CIR a_j^{CIR}-1}}{\Gamma(a_j^{CIR})}dw\right|\\
&= \left|\frac{\bar{c}_2}{p^{inv}(v_j)v_j\sigma_{v_j}^2}\left(\left[e^{-\mu^{CIR}w}w^{a_j^{CIR}+l-1}\right]_0^{v_j} - \int_0^{v_j} w^{a_j^{CIR}+l-2}e^{-\mu^{CIR}w}\frac{\mu^{CIR a_j^{CIR}}}{\Gamma(a_j^{CIR})}dz\right)\right|\\
&\sim \bar{c}_3v_j^{l-1}, \qquad\qquad (B.33)
\end{aligned}$$

where we use the inequality for the absolute value of $h_j(v_j)$ in the second line, the third line follows when we insert (B.28). We partially integrate the expression in the third line to obtain the forth line. Note that the first part of the partial integration increases like v_j^{l-1} because evaluating the expression at the upper limit we get $\frac{\bar{c}_3p^{inv}(v_j)v^l}{p^{inv}(v_j)}$ while the expression at the lower limit is zero. The second part, the integral, could be again evaluated using partial integration. Note that the result would increase like v_j^{l-2} and so on. Thus, the absolute value of the above expression increases at infinity or zero at most like v_j^{l-1}.

For $l = 0$ we further analyse the behaviour of $|\chi'|$ at 0 (the upper limit is $\bar{c}_3 \frac{1}{v_j}$):

$$\lim_{v_j \to 0} \left| \frac{-2\bar{c}_4}{\sigma_{v_j}^2 v_j p^{inv}(v_j)} F^{inv}(v_j) \right| \overset{\text{(L'Hôpital)}}{=} \lim_{v_j \to 0} \left| \frac{-2\bar{c}_4 \frac{\partial F^{inv}(v_j)}{\partial v_j}}{\sigma_{v_j}^2 \left(p^{inv}(v_j) + v_j \frac{\partial p^{inv}(v_j)}{\partial v_j} \right)} \right| = \bar{c}_5. \quad \text{(B.34)}$$

The last equality follows as $\frac{\partial F^{inv}(v_j)}{\partial v_j}$ and $p^{inv}(v_j)$ cancel and $v_j \frac{\partial p^{inv}(v_j)}{\partial v_j} = 0$. From (B.33) and (B.34) we can show for $l = 0$ that

$$|\chi(v_j)| \quad \leq \quad \bar{c}_6 \left(1 + \ln\left(1 + |v_j|\right)\right). \quad \text{(B.35)}$$

B.4 Autocorrelation function for CIR processes

In the following we omit the index j, which indicates the order of the eigenvalue. As mentioned above we assume no correlation between the processes for the stock prices and the processes driving the volatility (i.e. $\rho_j^v = 0$).

Due to the stationarity of $\{\bar{L}^d(n)\}$ the variogram defined by $V^{d,N}(\mathfrak{k}) = \frac{1}{N-\mathfrak{k}} \sum_{n=1}^{N-\mathfrak{k}} (\bar{L}^d(n+\mathfrak{k}) - \bar{L}^d(n))^2$ is an estimator of the value $\mathbb{E}\left[\left(\bar{L}^d(j) - \bar{L}^d(0)\right)^2\right]$. For the discrete equilibrium processes $\{\bar{v}(n)\}$ it is true that (see 2.114 for $t \to \infty$):

$$\text{Cov}\left(\bar{v}(\mathfrak{k}), \bar{v}(0)\right) = \vartheta_v^2 e^{-\kappa_v^* \mathfrak{k} \sqrt{\Delta t}}, \quad \text{(B.36)}$$

where

$$\vartheta_v^2 \quad = \quad \frac{\sigma_v^2 \zeta_v}{2\kappa_v}.$$

By applying (B.36) we compute for the function $f(v) = \exp\{v\}$:

$$\begin{aligned}
&\mathbb{E}\left[\left(\bar{L}^d(n+\mathfrak{k}) - \bar{L}^d(n)\right)^2\right] = \mathbb{E}\left[\left(\bar{L}^d(\mathfrak{k}) - \bar{L}^d(0)\right)^2\right] \\
= \ & \mathbb{E}\left[\left(\ln f\left(\bar{v}(\mathfrak{k})\right) - \ln f\left(\bar{v}(0)\right)\right)^2\right] + \mathbb{E}\left[\left(\ln\left|\frac{\Delta W(\mathfrak{k})}{\sqrt{\Delta t}}\right| - \ln\left|\frac{\Delta W(0)}{\sqrt{\Delta t}}\right|\right)^2\right] \\
= \ & 2\mathbb{E}\left[(\bar{v})^2\right] - 2\mathbb{E}\left[\bar{v}(\mathfrak{k})\bar{v}(0)\right] + 2\mathbf{Var}\left(\ln\left(\left|\frac{\Delta W}{\sqrt{\Delta t}}\right|\right)\right) \\
= \ & 2\vartheta_v^2\left(1 - e^{-\kappa_v^* \mathfrak{k} \sqrt{\Delta t}}\right) + 2\mathbf{Var}\left(\ln\left(\left|\frac{\Delta W}{\sqrt{\Delta t}}\right|\right)\right).
\end{aligned}$$

B.5 Appendix to Section 4.5.2

In the following we fist show the expansion of the problem in ε_1. We only provide those expansions which are necessary to solve for the explicitly indicated terms in (4.195). The analysis of the number of necessary terms for a good convergence will be left for future research.

$$C^{\delta,\varepsilon} = \sum_{n=0}^{\infty} C_n^{\delta,\varepsilon_2} \varepsilon_1^n.$$

This series is substituted in (4.191) and the first two leading order terms are set to zero.

$$\begin{aligned}
\varepsilon_1^0 : &\left[\frac{1}{\delta_1^2}\mathcal{L}_0^1 + \frac{1}{\delta_2^2}\mathcal{L}_0^2 + \frac{1}{\delta_1}\mathcal{L}_1^1 + \frac{1}{\delta_2}\mathcal{L}_1^2 + \mathcal{L}_2 \right. \\
&\left. + \varepsilon_2^2\mathcal{M}_0^2 + \varepsilon_2\mathcal{M}_1^2 + \frac{\varepsilon_2}{\delta_2}\mathcal{M}_3^2 \right] C_0^{\delta,\varepsilon_2} = 0, \qquad \text{(B.37)} \\
\varepsilon_1^1 : &\left[\frac{1}{\delta_1^2}\mathcal{L}_0^1 + \frac{1}{\delta_2^2}\mathcal{L}_0^2 + \frac{1}{\delta_1}\mathcal{L}_1^1 + \frac{1}{\delta_2}\mathcal{L}_1^2 + \mathcal{L}_2 + \varepsilon_2^2\mathcal{M}_0^2 + \varepsilon_2\mathcal{M}_1^2 + \frac{\varepsilon_2}{\delta_2}\mathcal{M}_3^2 \right] C_1^{\delta,\varepsilon_2} \\
&+ \left[\mathcal{M}_1^1 + \frac{1}{\delta_1}\mathcal{M}_3^1 \right] C_0^{\delta,\varepsilon_2} = 0. \qquad \text{(B.38)}
\end{aligned}$$

Next we expand in ε_2

$$\begin{aligned}
C_0^{\delta,\varepsilon_2} &= \sum_{n=0}^{\infty} C_{0,n}^{\delta} \varepsilon_2^n, \\
C_1^{\delta,\varepsilon_2} &= \sum_{n=0}^{\infty} C_{1,n}^{\delta} \varepsilon_2^n.
\end{aligned}$$

Inserting these expansions in (B.37) as well as (B.38) and forming terms of equal order in ε_2 we get for the first two leading terms

$$\begin{aligned}
\varepsilon_1^0,\ \varepsilon_2^0 : &\left[\frac{1}{\delta_1^2}\mathcal{L}_0^1 + \frac{1}{\delta_2^2}\mathcal{L}_0^2 + \frac{1}{\delta_1}\mathcal{L}_1^1 + \frac{1}{\delta_2}\mathcal{L}_1^2 + \mathcal{L}_2 \right] C_{0,0}^{\delta} = 0, \qquad \text{(B.39)} \\
\varepsilon_1^0,\ \varepsilon_2^1 : &\left[\frac{1}{\delta_1^2}\mathcal{L}_0^1 + \frac{1}{\delta_2^2}\mathcal{L}_0^2 + \frac{1}{\delta_1}\mathcal{L}_1^1 + \frac{1}{\delta_2}\mathcal{L}_1^2 + \mathcal{L}_2 \right] C_{0,1}^{\delta} \\
&+ \left[\mathcal{M}_1^2 + \frac{1}{\delta_2}\mathcal{M}_3^2 \right] C_{0,0}^{\delta} = 0, \qquad \text{(B.40)}
\end{aligned}$$

$$\varepsilon_1^1,\ \varepsilon_2^0 : \left[\frac{1}{\delta_1^2}\mathcal{L}_0^1 + \frac{1}{\delta_2^2}\mathcal{L}_0^2 + \frac{1}{\delta_1}\mathcal{L}_1^1 + \frac{1}{\delta_2}\mathcal{L}_1^2 + \mathcal{L}_2\right] C_{1,0}^\delta + \left[\mathcal{M}_1^1 + \frac{1}{\delta_1}\mathcal{M}_3^1\right] C_{0,0}^\delta = 0, \tag{B.41}$$

$$\varepsilon_1^1,\ \varepsilon_2^1 : \left[\frac{1}{\delta_1^2}\mathcal{L}_0^1 + \frac{1}{\delta_2^2}\mathcal{L}_0^2 + \frac{1}{\delta_1}\mathcal{L}_1^1 + \frac{1}{\delta_2}\mathcal{L}_1^2 + \mathcal{L}_2\right] C_{1,1}^\delta + \left[\mathcal{M}_1^1 + \frac{1}{\delta_1}\mathcal{M}_3^1\right] C_{0,1}^\delta + \left[\mathcal{M}_1^2 + \frac{1}{\delta_2}\mathcal{M}_3^2\right] C_{1,0}^\delta = 0. \tag{B.42}$$

We expand in δ_1

$$C_{0,0}^\delta = \sum_{n=0}^{\infty} C_{0,0,n}^{\delta_2}\delta_1^n, C_{0,1}^\delta = \sum_{n=0}^{\infty} C_{0,1,n}^{\delta_2}\delta_1^n, C_{1,0}^\delta = \sum_{n=0}^{\infty} C_{1,0,n}^{\delta_2}\delta_1^n.$$

Again these expansions are substituted in (B.39), (B.40), and (B.41). We set the first four order terms for (B.39), the first three ones for (B.40) and (B.41) in δ_1 to zero and obtain

for (B.39)

$$\varepsilon_1^0,\ \varepsilon_2^0,\ \delta_1^{-2} : \mathcal{L}_0^1 C_{0,0,0}^{\delta_2} = 0 \Rightarrow C_{0,0,0}^{\delta_2} \text{ does not depend on } v_1, \tag{B.43}$$

$$\varepsilon_1^0,\ \varepsilon_2^0,\ \delta_1^{-1} : \mathcal{L}_0^1 C_{0,0,1}^{\delta_2} + \underbrace{\mathcal{L}_1^1 C_{0,0,0}^{\delta_2}}_{=0} = 0 \Rightarrow C_{0,0,1}^{\delta_2} \text{ does not depend on } v_1, \tag{B.44}$$

$$\varepsilon_1^0,\ \varepsilon_2^0,\ \delta_1^0 : \mathcal{L}_0^1 C_{0,0,2}^{\delta_2} + \underbrace{\mathcal{L}_1^1 C_{0,0,1}^{\delta_2}}_{=0} + \left[\mathcal{L}_2 + \frac{1}{\delta_2^2}\mathcal{L}_0^2 + \frac{1}{\delta_2}\mathcal{L}_1^2\right] C_{0,0,0}^{\delta_2} = 0, \tag{B.45}$$

$$\varepsilon_1^0,\ \varepsilon_2^0,\ \delta_1^1 : \mathcal{L}_0^1 C_{0,0,3}^{\delta_2} + \mathcal{L}_1^1 C_{0,0,2}^{\delta_2} + \left[\mathcal{L}_2 + \frac{1}{\delta_2^2}\mathcal{L}_0^2 + \frac{1}{\delta_2}\mathcal{L}_1^2\right] C_{0,0,1}^{\delta_2} = 0. \tag{B.46}$$

Note that we use here the same argument as in (4.51) and choose a v_1-independent solution for $C^{\delta_2}_{0,0,0}$ and $C^{\delta_2}_{0,0,1}$. Furthermore, insert the expansion

$$\begin{aligned}
&\text{in (B.40)}\\
\varepsilon_1^0,\ \varepsilon_2^1,\ \delta_1^{-2} &: \mathcal{L}_0^1 C^{\delta_2}_{0,1,0} = 0 \Rightarrow C^{\delta_2}_{0,1,0} \text{ does not depend on } v_1, \qquad \text{(B.47)}\\
\varepsilon_1^0,\ \varepsilon_2^1,\ \delta_1^{-1} &: \mathcal{L}_0^1 C^{\delta_2}_{0,1,1} + \underbrace{\mathcal{L}_1^1 C^{\delta_2}_{0,1,0}}_{=0} = 0 \Rightarrow C^{\delta_2}_{0,1,1} \text{ does not depend on } v_1, \qquad \text{(B.48)}\\
\varepsilon_1^0,\ \varepsilon_2^1,\ \delta_1^{0} &: \mathcal{L}_0^1 C^{\delta_2}_{0,1,2} + \underbrace{\mathcal{L}_1^1 C^{\delta_2}_{0,1,1}}_{=0} + \left[\mathcal{L}_2 + \frac{1}{\delta_2^2}\mathcal{L}_0^2 + \frac{1}{\delta_2}\mathcal{L}_1^2\right] C^{\delta_2}_{0,1,0}\\
&+ \left[\mathcal{M}_1^2 + \frac{1}{\delta_2}\mathcal{M}_3^2\right] C^{\delta_2}_{0,0,0} = 0, \qquad \text{(B.49)}
\end{aligned}$$

$$\begin{aligned}
&\text{in (B.41)}\\
\varepsilon_1^1,\ \varepsilon_2^0,\ \delta_1^{-2} &: \mathcal{L}_0^1 C^{\delta_2}_{1,0,0} = 0 \Rightarrow C^{\delta_2}_{1,0,0} \text{ does not depend on } v_1, \qquad \text{(B.50)}\\
\varepsilon_1^1,\ \varepsilon_2^0,\ \delta_1^{-1} &: \mathcal{L}_0^1 C^{\delta_2}_{1,0,1} + \underbrace{\mathcal{L}_1^1 C^{\delta_2}_{1,0,0} + \mathcal{M}_3^1 C^{\delta_2}_{0,0,0}}_{=0} = 0\\
&\Rightarrow C^{\delta_2}_{1,0,1} \text{ does not depend on } v_1, \qquad \text{(B.51)}\\
\varepsilon_1^1,\ \varepsilon_2^0,\ \delta_1^{0} &: \mathcal{L}_0^1 C^{\delta_2}_{1,0,2} + \underbrace{\mathcal{L}_1^1 C^{\delta_2}_{1,0,1}}_{=0} + \left[\mathcal{L}_2 + \frac{1}{\delta_2^2}\mathcal{L}_0^2 + \frac{1}{\delta_2}\mathcal{L}_1^2\right] C^{\delta_2}_{1,0,0}\\
&+ \mathcal{M}_1^1 C^{\delta_2}_{0,0,0} + \mathcal{M}_3^1 C^{\delta_2}_{0,0,1} = 0. \qquad \text{(B.52)}
\end{aligned}$$

Finally, we expand in δ_2

$$\begin{aligned}
C^{\delta_2}_{0,0,0} &= \sum_{n=0}^{\infty} C_{0,0,0,n}\delta_2^n, C^{\delta_2}_{0,0,1} = \sum_{n=0}^{\infty} C_{0,0,1,n}\delta_2^n,\\
C^{\delta_2}_{0,1,0} &= \sum_{n=0}^{\infty} C_{0,1,0,n}\delta_2^n, C^{\delta_2}_{1,0,0} = \sum_{n=0}^{\infty} C_{1,0,0,n}\delta_2^n.
\end{aligned}$$

Inserting these expressions in (B.45), (B.46), (B.49) as well as (B.52), forming terms of equal power in δ_2 and setting the first four leading terms in (B.45) and the first three leading terms in (B.46), (B.49) and (B.52) to zero we receive

for (B.45)

$$\varepsilon_1^0,\ \varepsilon_2^0,\ \delta_1^0,\ \delta_2^{-2} : \mathcal{L}_0^2 C_{0,0,0,0} = 0 \Rightarrow C_{0,0,0,0} \text{ does not depend on } v_2, \tag{B.53}$$

$$\varepsilon_1^0,\ \varepsilon_2^0,\ \delta_1^0,\ \delta_2^{-1} : \mathcal{L}_0^2 C_{0,0,0,1} + \underbrace{\mathcal{L}_1^2 C_{0,0,0,0}}_{=0} = 0 \Rightarrow C_{0,0,0,1} \text{ does not depend on } v_2, \tag{B.54}$$

$$\varepsilon_1^0,\ \varepsilon_2^0,\ \delta_1^0,\ \delta_2^0 : \mathcal{L}_0^2 C_{0,0,0,2} + \underbrace{\mathcal{L}_1^2 C_{0,0,0,1}}_{=0} + \mathcal{L}_2 C_{0,0,0,0} + \mathcal{L}_0^1 C_{0,0,2,0} = 0, \tag{B.55}$$

$$\varepsilon_1^0,\ \varepsilon_2^0,\ \delta_1^0,\ \delta_2^1 : \mathcal{L}_0^2 C_{0,0,0,3} + \mathcal{L}_1^2 C_{0,0,0,2} + \mathcal{L}_2 C_{0,0,0,1} + \mathcal{L}_0^1 C_{0,0,2,1} = 0, \tag{B.56}$$

for (B.46)

$$\varepsilon_1^0,\ \varepsilon_2^0,\ \delta_1^1,\ \delta_2^{-2} : \mathcal{L}_0^2 C_{0,0,1,0} = 0 \Rightarrow C_{0,0,1,0} \text{ does not depend on } v_2, \tag{B.57}$$

$$\varepsilon_1^0,\ \varepsilon_2^0,\ \delta_1^1,\ \delta_2^{-1} : \mathcal{L}_0^2 C_{0,0,1,1} + \underbrace{\mathcal{L}_1^2 C_{0,0,1,0}}_{=0} = 0 \Rightarrow C_{0,0,1,1} \text{ does not depend on } v_2, \tag{B.58}$$

$$\varepsilon_1^0,\ \varepsilon_2^0,\ \delta_1^1,\ \delta_2^0 : \mathcal{L}_0^2 C_{0,0,1,2} + \underbrace{\mathcal{L}_1^2 C_{0,0,1,1}}_{=0} + \mathcal{L}_2 C_{0,0,1,0} + \mathcal{L}_0^1 C_{0,0,3,0} + \mathcal{L}_1^1 C_{0,0,2,0} = 0, \tag{B.59}$$

for (B.49)

$$\varepsilon_1^0,\ \varepsilon_2^1,\ \delta_1^0,\ \delta_2^{-2} : \mathcal{L}_0^2 C_{0,1,0,0} = 0 \Rightarrow C_{0,1,0,0} \text{ does not depend on } v_2, \tag{B.60}$$

$$\varepsilon_1^0,\ \varepsilon_2^1,\ \delta_1^0,\ \delta_2^{-1} : \mathcal{L}_0^2 C_{0,1,0,1} + \underbrace{\mathcal{L}_1^2 C_{0,1,0,0} + \mathcal{M}_3^2 C_{0,0,0,0}}_{=0} = 0$$

$$\Rightarrow C_{1,0,0,1} \text{ does not depend on } v_2, \tag{B.61}$$

$$\varepsilon_1^0,\ \varepsilon_2^1,\ \delta_1^0,\ \delta_2^0 : \mathcal{L}_0^2 C_{0,1,0,2} + \underbrace{\mathcal{L}_1^2 C_{0,1,0,1}}_{=0} + \mathcal{L}_2 C_{0,1,0,0} + \mathcal{L}_0^1 C_{0,1,2,0}$$

$$+ \mathcal{M}_1^2 C_{0,0,0,0} + \underbrace{\mathcal{M}_3^2 C_{0,0,0,1}}_{=0} = 0, \tag{B.62}$$

for (B.52)

$$\varepsilon_1^1,\ \varepsilon_2^0,\ \delta_1^0,\ \delta_2^{-2} : \mathcal{L}_0^2 C_{1,0,0,0} = 0 \Rightarrow C_{1,0,0,0} \text{ does not depend on } v_2, \tag{B.63}$$

$$\varepsilon_1^1,\ \varepsilon_2^0,\ \delta_1^0,\ \delta_2^{-1} : \mathcal{L}_0^2 C_{1,0,0,1} + \underbrace{\mathcal{L}_1^2 C_{1,0,0,0}}_{=0} = 0$$

$$\Rightarrow C_{1,0,0,1} \text{ does not depend on } v_2, \tag{B.64}$$

$$\varepsilon_1^1,\ \varepsilon_2^0,\ \delta_1^0,\ \delta_2^0 : \mathcal{L}_0^1 C_{1,0,2,0} + \mathcal{L}_0^2 C_{1,0,0,2} + \underbrace{\mathcal{L}_1^2 C_{1,0,0,1}}_{=0} + \mathcal{L}_2 C_{1,0,0,0}$$

$$+ \mathcal{M}_1^1 C_{0,0,0,0} + \underbrace{\mathcal{M}_3^1 C_{0,0,1,0}}_{=0} = 0. \tag{B.65}$$

Solving the PDE for the leading terms $C_{0,0,0,0}$ and $C_{B,0,0,0,0}$

We see that $C_{0,0,0,0}$ is independent from v_1 and v_2 (see (B.43) and (B.53)) and that Equation (B.55) is a Poisson equation in $C_{0,0,0,2}$ and $C_{0,0,2,0}$ with respect to v_1 and v_2. All other steps are analogue to Section 4.5.1 (compare (4.66) and the explanations before) and thus,

$$C_{0,0,0,0}\,(t, S_1, S_2) = S_2 \mathcal{N}_2(\mathbf{d}_2, \mathbf{d}_1, \overline{\rho}) - K_2 e^{-r\tau} \mathcal{N}_2(\mathbf{d}_2^*; \mathbf{d}_1^*, \overline{\rho}), \tag{B.66}$$

where

$$\tau = T - t, \qquad x_i = \ln \frac{S_i e^{-\int_t^T r(s)ds}}{K_i},$$

$$\mathbf{d}_1^* = \mathbf{d}_1 - \overline{\rho}\overline{\sigma}_2\sqrt{\tau}, \qquad \mathbf{d}_1 = \frac{x_1}{\overline{\sigma}_1\sqrt{\tau}} - \frac{1}{2}\overline{\sigma}_1\sqrt{\tau} + \overline{\rho}\overline{\sigma}_2\sqrt{\tau},$$

$$\mathbf{d}_2^* = \mathbf{d}_2 - \overline{\sigma}_2\sqrt{\tau}, \qquad \mathbf{d}_2 = \frac{x_2}{\overline{\sigma}_2\sqrt{\tau}} + \frac{1}{2}\overline{\sigma}_2\sqrt{\tau},$$

with

$$\begin{aligned} \bar{\sigma}_1^2(y_1, y_2) &= a_{11}^2 \left\langle f_1^2 \right\rangle_{v_1} + a_{12}^2 \left\langle f_2^2 \right\rangle_{v_2}, \\ \bar{\sigma}_2^2(y_1, y_2) &= a_{21}^2 \left\langle f_1^2 \right\rangle_{v_1} + a_{22}^2 \left\langle f_2^2 \right\rangle_{v_2}, \\ \bar{\rho}(y_1, y_2) &= \frac{a_{11}a_{21} \left\langle f_1^2 \right\rangle_{v_1} + a_{12}a_{22} \left\langle f_2^2 \right\rangle_{v_2}}{\bar{\sigma}_1 \bar{\sigma}_2}. \end{aligned} \tag{B.67}$$

$C_{B,0,0,0,0}$, the respective expansion term for the two-asset option with two barriers, is hence for $\overline{\rho} = -\cos\left(\frac{2\pi k}{n}\right)$ given by (see also (4.67) and (4.68))

$$\begin{aligned} C_{B,0,0,0,0}\,(t, S_1, S_2) \;=\; & \sum_{k=0}^{n-1} e^{y_1 (c_1\overline{\sigma}_1 - c_1\overline{\sigma}_1 \cos\left(\frac{2\pi k}{n}\right) + \frac{1}{\sqrt{1-\overline{\rho}^2}}(-c_2\overline{\sigma}_2 - \overline{\rho} c_1 \overline{\sigma}_1)\sin(\frac{2\pi k}{n}))} \\ & \left(B_2(H_1^+ - H_1^-) - e^{-r\tau} K_2 (H_2^+ - H_2^-)\right), \end{aligned} \tag{B.68}$$

with the parameter given in (4.72).

Solving the PDE for the first-order corrections $C_{0,0,0,1}$, $C_{B,0,0,0,1}$ and $C_{0,0,1,0}$ as well as $C_{B,0,0,1,0}$

Equation (B.55) can be also seen as a Poisson equation in v_1 only (in v_2 respectively) for $C_{0,0,0,2}$ ($C_{0,0,2,0}$). We perform the steps analogously to the Solution (4.84) under the fast-mean reversion model in Section 4.5.1 and we can give the solutions:

$$C_{0,0,0,1} = -(T-t)\mathcal{A}_2 C_{0,0,0,0}, \tag{B.69}$$

where

$$\begin{aligned}\mathcal{A}_2 C_{0,0,0,0} &= \sigma_{v_2}\rho_2^v \left\langle f_2\sqrt{v_2}\frac{\partial\phi_2}{\partial v_2}\right\rangle_{v_2} \left(a_{12}S_1\frac{\partial}{\partial S_1} + a_{22}S_2\frac{\partial}{\partial S_2}\right)\\ &\quad \left(a_{12}a_{22}S_1S_2\frac{\partial^2}{\partial S_1\partial S_2} + \frac{1}{2}a_{12}^2S_1^2\frac{\partial^2}{\partial S_1^2} + \frac{1}{2}a_{22}^2S_2^2\frac{\partial^2}{\partial S_2^2}\right) C_{0,0,0,0}\end{aligned}$$

with $\frac{\partial\phi_2}{\partial v_2} = -\frac{2}{\sigma_{v_2}^2 v_2 p(v_2)}\int_0^{v_2}\left(f_2^2(z,y_2) - \langle f_2^2\rangle_{v_2}^2\right)p(z)dz$ (see (B.3)).
We obtain with Equation (B.59) an analogous expression for $C_{0,0,1,0}$.

$$C_{0,0,1,0} = -(T-t)\mathcal{A}_1 C_{0,0,0,0}, \tag{B.70}$$

with

$$\begin{aligned}\mathcal{A}_1 &= \sigma_{v_1}\rho_1^v \left\langle f_1\sqrt{v_1}\frac{\partial\phi_1}{\partial v_1}\right\rangle_{v_1} \left(a_{11}S_1\frac{\partial}{\partial S_1} + a_{21}S_2\frac{\partial}{\partial S_2}\right)\\ &\quad \left(a_{11}a_{21}S_1S_2\frac{\partial^2}{\partial S_1\partial S_2} + \frac{1}{2}a_{11}^2S_1^2\frac{\partial^2}{\partial S_1^2} + \frac{1}{2}a_{21}^2S_2^2\frac{\partial^2}{\partial S_2^2}\right).\end{aligned}$$

Analogously to the Solution (4.96) we can write down the solution of the barrier option term:

$$C_{B,0,0,0,1} = \hat{C}_{B,0,0,0,1} - \tilde{V}_{12}S_1\frac{\partial^2 C_{B,0,0,0,0}}{\partial S_1\partial\langle f_2^2\rangle_{v_2}} - \tilde{V}_{22}S_2\frac{\partial^2 C_{B,0,0,0,0}}{\partial S_2\partial\langle f_2^2\rangle_{v_2}}. \tag{B.71}$$

and

$$C_{B,0,0,1,0} = \hat{C}_{B,0,0,1,0} - \tilde{V}_{11}S_1\frac{\partial^2 C_{B,0,0,0,0}}{\partial S_1\partial\langle f_2^2\rangle_{v_2}} - \tilde{V}_{21}S_2\frac{\partial^2 C_{B,0,0,0,0}}{\partial S_2\partial\langle f_2^2\rangle_{v_2}}. \tag{B.72}$$

Solving the PDE for the first-order corrections $C_{1,0,0,0}$, $C_{B,1,0,0,0}$ and $C_{0,1,0,0}$ as well as $C_{B,0,1,0,0}$

We interpret Equation (B.65) as a Poisson equation in $C_{1,0,2,0}$ and $C_{1,0,0,2}$ with respect to v_1 and v_2. Hence,

$$\langle\langle\mathcal{L}_2\rangle\rangle_{v_1,v_2} C_{1,0,0,0} = -\left\langle\left\langle\mathcal{M}_1^1\right\rangle\right\rangle_{v_1,v_2} C_{0,0,0,0},$$

where

$$\left\langle\left\langle\mathcal{M}_1^1\right\rangle\right\rangle_{v_1,v_2} C_{0,0,0,0} = \rho_1^y \sigma_{y_1} \sqrt{y_1} \left\langle f_1\right\rangle_{v_1} \left(a_{11} S_1 \frac{\partial}{\partial S_1} + a_{21} S_2 \frac{\partial}{\partial S_2}\right) \frac{\partial}{\partial y_1} C_{0,0,0,0}. \tag{B.73}$$

We know that (compare (4.87))

$$\begin{aligned}
\mathcal{L}_{BS}(\bar{\sigma}_1,\bar{\sigma}_2,\rho)\frac{\partial C_{0,0,0,0}}{\partial\left\langle f_1^2\right\rangle_{v_1}} &= -\left(a_{11}a_{21}S_1S_2\frac{\partial^2}{\partial S_1\partial S_2} + \frac{1}{2}a_{11}^2S_1^2\frac{\partial^2}{\partial S_1^2} + \frac{1}{2}a_{21}^2S_2^2\frac{\partial^2}{\partial S_2^2}\right) \\
&\quad C_{0,0,0,0}, \\
\frac{\partial}{\partial\left\langle f_1^2\right\rangle_{v_1}} C_{0,0,0,0}\left(T,S_1,S_2\right) &= 0.
\end{aligned}$$

From there we can conclude

$$\frac{\partial}{\partial\left\langle f_1^2\right\rangle_{v_1}} C_{0,0,0,0} = (T-t)\left(a_{11}a_{21}S_1S_2\frac{\partial^2}{\partial S_1\partial S_2} + \frac{1}{2}a_{11}^2S_1^2\frac{\partial^2}{\partial S_1^2} + \frac{1}{2}a_{21}^2S_2^2\frac{\partial^2}{\partial S_2^2}\right) C_{0,0,0,0}.$$

We can then reformulate (B.73) to

$$\begin{aligned}
\left\langle\left\langle\mathcal{M}_1^1\right\rangle\right\rangle_{v_1,v_2} C_{0,0,0,0} &= (T-t)\rho_1^y\sigma_{y_1}\sqrt{y_1}\left\langle f_1\right\rangle_{v_1}\frac{\partial\left\langle f_1^2\right\rangle_{v_1}}{\partial y_1}\left(a_{11}S_1\frac{\partial}{\partial S_1} + a_{21}S_2\frac{\partial}{\partial S_2}\right) \\
&\quad \left(a_{11}a_{21}S_1S_2\frac{\partial^2}{\partial S_1\partial S_2} + \frac{1}{2}a_{11}^2S_1^2\frac{\partial^2}{\partial S_1^2} + \frac{1}{2}a_{21}^2S_2^2\frac{\partial^2}{\partial S_2^2}\right) C_{0,0,0,0}.
\end{aligned}$$

The following result easily follows as $\mathcal{L}_{BS}$ commutes with $S_i\frac{\partial^k}{\partial S_i^k}$ for $k = 1, 2$.

$$C_{1,0,0,0} = \frac{1}{2}(T-t)\left\langle\left\langle\mathcal{M}_1^1\right\rangle\right\rangle_{v_1,v_2} C_{0,0,0,0}. \tag{B.74}$$

Analogously, the respective corollary for $C_{0,1,0,0}$ follows from Equation (B.62).

$$C_{0,1,0,0} = \frac{1}{2}(T-t)\left\langle\left\langle\mathcal{M}_1^2\right\rangle\right\rangle_{v_1,v_2} C_{0,0,0,0}. \tag{B.75}$$

$$\begin{aligned}\left\langle\left\langle\mathcal{M}_1^2\right\rangle\right\rangle_{v_1,v_2} C_{0,0,0,0} &= (T-t)\rho_2^y\sigma_{y_2}\sqrt{y_2}\left\langle f_2\right\rangle_{v_2}\frac{\partial\left\langle f_2^2\right\rangle_{v_2}}{\partial y_2}\left(a_{12}S_1\frac{\partial}{\partial S_1}+a_{22}S_2\frac{\partial}{\partial S_2}\right)\\ &\quad\left(a_{12}a_{22}S_1S_2\frac{\partial^2}{\partial S_1\partial S_2}+\frac{1}{2}a_{12}^2S_1^2\frac{\partial^2}{\partial S_1^2}+\frac{1}{2}a_{22}^2S_2^2\frac{\partial^2}{\partial S_2^2}\right)C_{0,0,0,0}.\end{aligned}$$

Introducing barriers leads to the following system for $C_{1,0,0,0}$

$$\begin{aligned}\mathcal{L}_{BS}C_{B,1,0,0,0} &= -\left\langle\left\langle\mathcal{M}_1^1\right\rangle\right\rangle_{v_1,v_2} C_{B,0,0,0,0},\\ C_{B,1,0,0,0}\left(T,S_1,S_2\right) &= 0,\\ C_{B,1,0,0,0}(t,B_1(t),S_2) &= 0,\\ C_{B,1,0,0,0}(t,S_1,B_2(t)) &= 0.\end{aligned}\tag{B.76}$$

Again, we transform the PDE in such a way that we solve a homogeneous PDE with inhomogeneous boundary conditions. Hence,

$$\begin{aligned}\mathcal{L}_{BS}\left(-(T-t)\left\langle\left\langle\mathcal{M}_1^1\right\rangle\right\rangle_{v_1,v_2} C_{0,0,0,0}\right) &= \langle\langle\mathcal{M}_1^1\rangle\rangle_{v_1,v_2}C_{0,0,0,0}\\ &\quad -(T-t)\rho_1^y\sigma_{y_1}\sqrt{y_1}\left\langle f_1\right\rangle_{v_1}\frac{1}{2}(a_{11}S_1\frac{\partial}{\partial S_1}+a_{21}S_2\frac{\partial}{\partial S_2})\\ &\quad\left(2a_{11}a_{21}S_1S_2\frac{\partial^2}{\partial S_1\partial S_2}+a_{11}^2S_1^2\frac{\partial^2}{\partial S_1^2}+a_{21}^2S_2^2\frac{\partial^2}{\partial S_2^2}\right)\\ &\quad C_{0,0,0,0},\end{aligned}$$

and

$$\begin{aligned}&\mathcal{L}_{BS}\Bigg(-\frac{(T-t)^2}{2}\left(\tilde{\tilde{V}}_{11}S_1\frac{\partial}{\partial S_1}+\tilde{\tilde{V}}_{21}S_2\frac{\partial}{\partial S_2}\right)\\ &\left(a_{11}a_{21}S_1S_2\frac{\partial^2}{\partial S_1\partial S_2}+\frac{1}{2}a_{11}^2S_1^2\frac{\partial^2}{\partial S_1^2}+\frac{1}{2}a_{21}^2S_2^2\frac{\partial^2}{\partial S_2^2}\right)C_{0,0,0,0}\Bigg)\\ =\;&(T-t)\left(\tilde{\tilde{V}}_{11}S_1\frac{\partial}{\partial S_1}+\tilde{\tilde{V}}_{21}S_2\frac{\partial}{\partial S_2}\right)\\ &\left(a_{11}a_{21}S_1S_2\frac{\partial^2}{\partial S_1\partial S_2}+\frac{1}{2}a_{11}^2S_1^2\frac{\partial^2}{\partial S_1^2}+\frac{1}{2}a_{21}^2S_2^2\frac{\partial^2}{\partial S_2^2}\right)C_{0,0,0,0},\end{aligned}$$

as $\mathcal{L}_{BS}$ commutes with $S_i \frac{\partial^k}{\partial S_i^k}, k = 1, 2$ and $\mathcal{L}_{BS} C_{0,0,0,0} = 0$. Thus, we define

$$\begin{aligned}
\hat{C}_{B,1,0,0,0} &= C_{B,1,0,0,0} - (T-t)\left\langle\left\langle \mathcal{M}_1^1 \right\rangle\right\rangle_{v_1,v_2} C_{0,0,0,0} - \frac{(T-t)^2}{2}\left(\tilde{\tilde{V}}_{11} S_1 \frac{\partial}{\partial S_1} + \tilde{\tilde{V}}_{21} S_2 \frac{\partial}{\partial S_2}\right) \\
&\quad \left(a_{11} a_{21} S_1 S_2 \frac{\partial^2}{\partial S_1 \partial S_2} + \frac{1}{2} a_{11}^2 S_1^2 \frac{\partial^2}{\partial S_1^2} + \frac{1}{2} a_{21}^2 S_2^2 \frac{\partial^2}{\partial S_2^2}\right) C_{0,0,0,0}, \\
\tilde{\tilde{V}}_{11} &= a_{11} \rho_1^y \sigma_{y_1} \sqrt{y_1} \left\langle f_1 \right\rangle_{v_1} \frac{\partial \left\langle f_1 \right\rangle_{v_1}}{\partial y_1}, \\
\tilde{\tilde{V}}_{21} &= a_{11} \rho_1^y \sigma_{y_1} \sqrt{y_1} \left\langle f_1 \right\rangle_{v_1} \frac{\partial \left\langle f_1 \right\rangle_{v_1}}{\partial y_1}.
\end{aligned} \tag{B.77}$$

Hence, we obtain

$$\begin{aligned}
\mathcal{L}_{\mathcal{BS}}(\bar{\rho}, \bar{\sigma}_1, \bar{\sigma}_2) \hat{C}_{B,1,0,0,0} &= 0, \\
\hat{C}_{B,1,0,0,0}\left(T, S_1, S_2\right) &= 0, \\
\hat{C}_{B,1,0,0,0}(t, B_1(t), S_2) &= \tilde{\tilde{\mathfrak{g}}}_{11}(t, B_1, S_2, y_1, y_2), \\
\hat{C}_{B,1,0,0,0}(t, S_1, B_2(t)) &= \tilde{\tilde{\mathfrak{g}}}_{12}(t, S_1, B_2, y_1, y_2),
\end{aligned} \tag{B.78}$$

with

$$\begin{aligned}
\tilde{\tilde{\mathfrak{g}}}_{11}(t, B_1, S_2, y_1, y_2) &= -(T-t)\left\langle\left\langle \mathcal{M}_1^1 \right\rangle\right\rangle_{v_1,v_2} C_{0,0,0,0} \\
&\quad -\frac{(T-t)^2}{2}\left(\tilde{\tilde{V}}_{11} S_1 \frac{\partial}{\partial S_1} + \tilde{\tilde{V}}_{21} S_2 \frac{\partial}{\partial S_2}\right) \\
&\quad \left(a_{11} a_{21} S_1 S_2 \frac{\partial^2}{\partial S_1 \partial S_2} + \frac{1}{2} a_{11}^2 S_1^2 \frac{\partial^2}{\partial S_1^2} + \frac{1}{2} a_{21}^2 S_2^2 \frac{\partial^2}{\partial S_2^2}\right) C_{0,0,0,0} \,\Big|_{S_1 = B_1(\iota_1)}, \\
\tilde{\tilde{\mathfrak{g}}}_{12}(t, S_1, B_2, y_1, y_2) &= -(T-t)\left\langle\left\langle \mathcal{M}_1^1 \right\rangle\right\rangle_{v_1,v_2} C_{0,0,0,0} \\
&\quad -\frac{(T-t)^2}{2}\left(\tilde{\tilde{V}}_{11} S_1 \frac{\partial}{\partial S_1} + \tilde{\tilde{V}}_{21} S_2 \frac{\partial}{\partial S_2}\right) \\
&\quad \left(a_{11} a_{21} S_1 S_2 \frac{\partial^2}{\partial S_1 \partial S_2} + \frac{1}{2} a_{11}^2 S_1^2 \frac{\partial^2}{\partial S_1^2} + \frac{1}{2} a_{21}^2 S_2^2 \frac{\partial^2}{\partial S_2^2}\right) C_{0,0,0,0} \,\Big|_{S_2 = B_2(\iota_2)}.
\end{aligned}$$

For the solution it holds

$$\begin{aligned}
\hat{C}_{B,1,0,0,0}(t, S_1, S_2, B_1, B_2) &= \int_0^T \int_0^\infty e^{-c_1 b_1 - c_2 b_2} e^{-c_2(\sigma_2 a_p' \sin\beta_p)} \tilde{\tilde{\mathfrak{g}}}_{11}(t', B_1, B_2 e^{\sigma_2 a_p' \sin\beta_p}) \\
&\quad p_{GBM}^x\left(\iota' \in dt', \theta_p' = \beta_p\right) da_p' dt' \\
&+ \int_0^T \int_0^\infty e^{-c_1 b_1 - c_2 b_2} e^{-c_1(\sigma_1 a_p' \sqrt{1-\rho^2})} \tilde{\tilde{\mathfrak{g}}}_{21}(t', B_1 e^{\sigma_1 \sqrt{1-\rho^2} a_p'}, B_2) \\
&\quad p_{GBM}^x\left(\iota' \in dt', \theta_p' = 0\right) da_p' dt'.
\end{aligned} \tag{B.79}$$

$C_{B,1,0,0,0}$ is given by

$$\begin{aligned} C_{B,1,0,0,0} &= \hat{C}^B_{1,0,0,0} + (T-t)\left\langle\left\langle\mathcal{M}_1^1\right\rangle\right\rangle_{v_1,v_2} C_{0,0,0,0} + \frac{(T-t)^2}{2}\left(\tilde{\tilde{V}}_{11}S_1\frac{\partial}{\partial S_1} + \tilde{\tilde{V}}_{21}S_2\frac{\partial}{\partial S_2}\right) \\ &\left(a_{11}a_{21}S_1S_2\frac{\partial^2}{\partial S_1\partial S_2} + \frac{1}{2}a_{11}^2S_1^2\frac{\partial^2}{\partial S_1^2} + \frac{1}{2}a_{21}^2S_2^2\frac{\partial^2}{\partial S_2^2}\right)C_{0,0,0,0}. \end{aligned} \tag{B.80}$$

Analogously, we can conclude

$$\begin{aligned} \hat{C}_{B,0,1,0,0}(t,S_1,S_2,B_1,B_2) &= \int_0^T\int_0^\infty e^{-c_1b_1-c_2b_2}e^{-c_2(\sigma_2a_p'\sin\beta_p)}\tilde{\tilde{\mathfrak{g}}}_{21}(t',B_1,B_2e^{\sigma_2a_p'\sin\beta_p}) \\ &\quad p^x_{GBM}\left(\iota'\in dt',\theta_p'=\beta_p\right)da_p'dt' \\ &+ \int_0^T\int_0^\infty e^{-c_1b_1-c_2b_2}e^{-c_1(\sigma_1a_p'\sqrt{1-\rho^2})}\tilde{\tilde{\mathfrak{g}}}_{22}(t',B_1e^{\sigma_1\sqrt{1-\rho^2}a_p'},B_2) \\ &\quad p^x_{GBM}\left(\iota'\in dt',\theta_p'=0\right)da_p'dt', \end{aligned} \tag{B.81}$$

with

$$\begin{aligned} \tilde{\tilde{\mathfrak{g}}}_{21}(t,B_1,S_2,y_1,y_2) &= (T-t)\left\langle\left\langle\mathcal{M}_1^2\right\rangle\right\rangle_{v_1,v_2} C_{0,0,0,0} \\ &\quad -\frac{(T-t)^2}{2}\left(\tilde{\tilde{V}}_{12}S_1\frac{\partial}{\partial S_1} + \tilde{\tilde{V}}_{22}S_2\frac{\partial}{\partial S_2}\right) \\ &\quad \left(a_{12}a_{22}S_1S_2\frac{\partial^2}{\partial S_1\partial S_2} + \frac{1}{2}a_{12}^2S_1^2\frac{\partial^2}{\partial S_1^2} + \frac{1}{2}a_{22}^2S_2^2\frac{\partial^2}{\partial S_2^2}\right)C_{0,0,0,0}\left|_{S_1=B_1(\iota_1)}\right., \\ \tilde{\tilde{\mathfrak{g}}}_{22}(t,S_1,B_2,y_1,y_2) &= -(T-t)\left\langle\left\langle\mathcal{M}_1^2\right\rangle\right\rangle_{v_1,v_2} C_{0,0,0,0} - \frac{(T-t)^2}{2}\left(\tilde{\tilde{V}}_{12}S_1\frac{\partial}{\partial S_1} + \tilde{\tilde{V}}_{22}S_2\frac{\partial}{\partial S_2}\right) \\ &\quad \left(a_{12}a_{22}S_1S_2\frac{\partial^2}{\partial S_1\partial S_2} + \frac{1}{2}a_{12}^2S_1^2\frac{\partial^2}{\partial S_1^2} + \frac{1}{2}a_{22}^2S_2^2\frac{\partial^2}{\partial S_2^2}\right)C_{0,0,0,0}\left|_{S_2=B_2(\iota_2)}\right., \end{aligned}$$

$$\begin{aligned} \tilde{\tilde{V}}_{22} &= a_{22}\rho_2^y\sigma_{y_2y_2}\left\langle f_1\right\rangle_{v_2}\frac{\partial\tilde{\sigma}_2}{\partial y_2}, \\ \tilde{\tilde{V}}_{12} &= a_{12}\rho_2^y\sigma_{y_2y_2}\left\langle f_1\right\rangle_{v_2}\frac{\partial\tilde{\sigma}_2}{\partial y_2}. \end{aligned}$$

$C_{B,0,1,0,0}$ is given by

$$\begin{aligned} C_{B,0,1,0,0} &= \hat{C}^B_{0,1,0,0} + \frac{(T-t)^2}{2}\left(\tilde{\tilde{V}}_{12}S_1\frac{\partial}{\partial S_1} + \tilde{V}_{22}S_2\frac{\partial}{\partial S_2}\right) \\ &\left(a_{12}a_{22}S_1S_2\frac{\partial^2}{\partial S_1\partial S_2} + \frac{1}{2}a_{12}^2S_1^2\frac{\partial^2}{\partial S_1^2} + \frac{1}{2}a_{22}^2S_2^2\frac{\partial^2}{\partial S_2^2}\right)C_{B,0,0,0,0} \\ &+(T-t)\langle\langle\mathcal{M}_1^2\rangle\rangle_{v_1,v_2}C_{0,0,0,0}. \end{aligned} \tag{B.82}$$

List of Symbols

$\mathbb{1}$ .... Indicator function
$\mathcal{A}^* f$ .... Operator of Kolmogorov forward equation
$\mathcal{A} f$ .... Infinitesimal generator
$\mathfrak{A}$ .... Set
A_H .... Parameter of Heston-type characteristic function
A_{S2} .... Parameter of Stein and Stein-type characteristic function
(a_{ij}) .... Matrix of eigenvectors
α_i^d .... Damping parameter
α_k .... kth algebraic moment
$\mathfrak{a}$ .... Real number
$\mathcal{B}(\mathbb{R})$ .... Borel sigma-algebra
B_H .... Parameter of Heston-type characteristic function
$B_i(t)$ .... Time-dependant barrier on stock i
B_{S2} .... Parameter of Stein and Stein-type characteristic function
β_k .... kth geometric moment
β_p .... $\tan^{-1}\left(-\frac{\sqrt{1-\bar{\rho}^2}}{\bar{\rho}}\right)$
$\mathfrak{b}$ .... Real number
$\mathbb{C}$ .... Complex numbers
$\mathbf{C}^k$ .... Collection of functions with continuous derivatives up to order k
$\mathbf{C}_0^k$ .... Subset of $\mathbf{C}^k$ of functions having compact support
C^D .... Defaultable derivative
$C_B(t, S_1, S_2, B_1, B_2)$ ...... Barrier option
$C_S^D(t, S_1, S_2)$ .... Defaultable investment in a stock
$C_Z(t)$ .... Zero coupon bond
$C_{1D}(t, S_1, S_2, B_2)$ .... Double-digital single-barrier option
$C_{1P}(t, S_2, K_2, B_2)$ .... Knock-out put option
$C_{2C}(t, S_1, S_2, B_1, B_2)$ ..... Correlation barrier option
$C_{2C}(t, S_1, S_2, B_2)$ .... Correlation single-barrier option

$C_{2D}(t, S_1, S_2, B_1, B_2)$ ..... Double-digital barrier option
$C_{Call}\,(t, S_2, K_2)$ .... Call option
$C_D\,(t, S_2, K_2)$ .... Digital option
C_{S2} .... Parameter of Stein and Stein-type characteristic function
CIR .... Cox-Ingersoll-Ross model
χ .... Probability measure
$\hat{c}$ .... Constant
$\mathfrak{c}_i$ .... Constant
$\tilde{c}$ .... Constant
c_1 .... $\frac{\sigma_1-\sigma_2\rho}{2\sigma_1(1-\rho^2)}$
c_2 .... $\frac{\sigma_2-\sigma_1\rho}{2\sigma_2(1-\rho^2)}$
$D_i(t)$ .... total debt per share of firm i
$\mathcal{D}$ .... Open set
$\mathrm{Def}(z)$ .... Space in which the moment generating function exists
∂D .... Boundary region of D
D .... Bounded region
δ_j .... Fast mean-reversion speed
$\mathfrak{d}(\mathbf{u})$ .... $\sqrt{\kappa^2 + \sigma_v^2\left(\mathbf{u}^2 + \frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}{4(1-\rho^2)}\right)}$
d_i .... Dividend yield of company i
$\mathbb{E}\,[X]$ .... Expected value of X
$\mathcal{E}\,(x)$ .... Exponential martingale of x
ϵ_j .... Slow mean-reversion speed
$\mathcal{F}$ .... σ-Algebra
$(F_1 * F_2)(y)$ .... Convolution of F_1 and F_2
$\mathbb{F}$ .... Filtration
Φ .... Set
$F(x) = \mathcal{Q}(X \leq x)$ .... Distribution function of X
$\hat{f}(u)$ .... Fourier transform of $f(x)$
$\overline{f}$ .... Complex conjugate of f
$\varphi(u)$ .... Characteristic function
φ_H .... Heston-type characteristic function
φ_{GBM} .... Characteristic function of geometric Brownian motion model
φ_{S2} .... Stein-and Stein-type characteristic function
f^d .... Function of the day effect
f^e .... Eigenfunction
$\bar{G}\,(t, \mathbf{x})$ .... Kernel, e.g. fundamental solution, Green function
$\bar{G}^F(t, x)$ .... Free space Green function

GBM .... Geometric Brownian motion model
$\tilde{\gamma}$ .... Stochastic process
$g(S_1, S_2)$ .... Payoff function dependent on S_1 and S_2 in T
$H(\mathbf{x})$ .... Function only depending on $\mathbf{x}$
$\hat{\mathfrak{h}}(\mathbf{x})$ .... Fourier transformed payoff function
$\Im(u)$ .... Imaginary part of u
$I_\varsigma(\frac{ab}{2c^2})$ .... Modified Bessel function of the first kind
ι .... Stopping time
j .... Counter with $j \in \mathbb{N}$
$\mathfrak{K}(t, \mathbf{x})$ .... Real-valued function
K_i .... Strike for asset i
κ .... Constant mean-reversion speed
$\mathfrak{k}$ .... Counter with $\mathfrak{k} \in \mathbb{N}$
k .... Counter with $k \in \mathbb{N}$
k_i .... $\ln K_i$
$\mathbf{L}^1$ .... Banach space of Lebesgue integrable functions in $\mathbb{R}^d$
$\mathbf{L}^2$ .... Hilbert space of square-integrable functions in $\mathbb{R}^d$
$\mathcal{L}f$ .... Operator for differential equation
$\bar{M}(u)$ .... Moment generating function
$\mathfrak{M} = (\mathfrak{M}(t))_{t\in[0,T]}$ .... Local ($\mathcal{Q}-$) martingale
$\mu(s)$ .... Drift parameter
${z_{k1}}^{\pm}$ .... $r_p \cos\left(\frac{2k\pi}{n} \pm \theta_p\right)$
${z_{k2}}^{\pm}$ .... $r_p \sin\left(\frac{2k\pi}{n} \pm \theta_p\right)$
$\mathbb{N}$ .... Natural number
$\mathbf{n}$ .... Exterior unit normal
$\mathfrak{n}(\mathbf{x})$ .... Non-negative function
ν .... Real number
$(\Omega, \mathcal{F}, \mathcal{Q}, \mathbb{F})$ .... Filtered probability space
$(\Omega, \mathcal{F}, \mathcal{Q})$ .... Probability space
$(\Omega, \mathcal{F})$ .... Measurable space
$\mathfrak{o}(\mathbf{x})$ .... Non-negative function
$\tilde{\omega}$ .... Random variable
ϖ .... Real variable
$\mathbf{P}$ .... Real vector
Ψ .... Bonus payment rate
$\mathfrak{p}$ .... Participation rate
$\vartheta(\mathbf{x}, t)$ .... Function
$p(x)$ .... Density function of x

$\bar{p}$	Number of considered eigenvectors
$\mathcal{Q}$	Probability measure
$q(\tau, \mathbf{x}', \mathbf{x}, v)$	Probability density function, integrated over v'
$\mathbb{R}$	Real numbers
$\mathfrak{R}$	Constant recovery rate
$\Re(x)$	Real part of x
$R^{\tilde{\epsilon},\delta}$	Error in the approximation for the regularised problem
ρ	Correlation
$r(s)$	Deterministic interest rate
r_p	Polar coordinate, radius
S_φ	Space in which φ is regular
S_g	Space, in which the transformed payoff function is Fourier integrable
S_i	Stock i
$S_i^d(n)$	Discrete normalised price returns of stock i at time $n\sqrt{\Delta t}$
σ_i	Diffusion parameter of stock i
σ_v	Diffusion parameter of v
ς_i	Real number
s	Time
θ_p	Polar coordinate,angle
$\mathfrak{T}$	Transformation
T	Maturity time
$T(t)$	Function only depending on t
t	Time
$\langle \mathbf{u}, \mathbf{x} \rangle$	Scalar product
$\langle \mathbf{u}, \mathbf{x} \rangle$	Scalar product
$u = w + i\varpi$	Complex variable
$V_j^{d,N}(\mathfrak{k})$	Variogram function with lag $\mathfrak{k}$ for eigenvalue j
v	Process driving the covariance
$\mathfrak{W}_i$	Value of firm i
$W = (W(t))_{t \geq 0}$	Brownian motion
w	Real variable
$(X(t))_{t \geq 0}$	Process
$\langle X \rangle := \langle X, X \rangle$	Quadratic variation process of X
Ξ	Constant
$\xi(t, \mathbf{x})$	Continuous function
x_i	$\ln\left(\frac{S_i(t)e^{\int_t^T r(s)ds}}{K_i}\right)$

$x_1^*(t)$ .... $\ln\left(\frac{S_1+D_1(t)}{D_1(t)}\right)$
y_i .... $\frac{x_i^*}{\sigma_i}$
y_j .... Driver of the covariance/correlation
$Z = (Z(t))_{t\geq 0}$ .... Brownian motion
ζ .... Constant mean-reversion level
z_1 .... $\frac{1}{\sqrt{1-\rho^2}}\left[\frac{x_1-b_1}{\sigma_1} - \rho\left(\frac{x_2-b_2}{\sigma_2}\right)\right]$
z_2 .... $\frac{x_2-b_2}{\sigma_2}$

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