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Reconstruction and Processing of Bandlimited Signals Based on Their Discrete Values

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Zusammenfassung

Diese Arbeit befasst sich mit dem Wechselspiel zwischen analoger und diskreter Welt. Die Umwandlung von zeitdiskreten in zeitkontinuierliche Signale ist essenziell, da Informationen heutzutage fast ausschließlich digital verarbeitet werden, während reale Signale analog sind. Neben der bandbegrenzten Interpolation wird die Rekonstruktion von bandbegrenzen Signalen anhand ihrer Abtastwerte für verschiedene Signalräume untersucht. Es werden fundamentale Grenzen aufgezeigt und zahlreiche neue Resultate gewonnen, z.B. für nichtäquidistante Abtastung, Überabtastung und stochastische Prozesse. Für Anwendungen ist die Verarbeitung von Signalen durch lineare zeitinvariante Systeme von großer Bedeutung. Das klassische und distributionelle Konvergenzverhalten von verschiedenen Faltungs-Systemrepräsentationen wird analysiert. Im Fokus stehen Abtast-Systemrepräsentationen, die nur die Abtastwerte des Eingangssignals zur Berechnung des Systemausgangs verwenden. Abschließend wird der Einfluss von Quantisierung auf die Signalrekonstruktion und die Systemapproximation untersucht.

Abstract

This dissertation analyzes the interplay between the analog and the digital worlds. The conversion between discrete-time signals and continuous-time signals is important because today most information is processed digitally while real world signals are analog. Bandlimited interpolation is studied, as well as the reconstruction of bandlimited signals from their samples for different signal spaces. Fundamental limits are discovered and results are obtained in several directions, e.g., for non-equidistant sampling, oversampling, and stochastic processes. The processing of signals with linear time-invariant systems is important for applications. The classical and distributional convergence behavior of different convolution-type system representations is analyzed. Attention is paid to sampling-type representations that use only the samples of the input signal to compute the system output. Finally, the effects of quantization on the signal reconstruction and the system approximation are studied.

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Contents

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Introduction

1.1 Motivation

Claude Shannon's fundamental paper "Communication in the Presence of Noise" was published in 1949 [1,2]. A central part of this paper was the sampling theorem, which states that certain bandlimited signals are uniquely determined by their equidistant samples if the samples are taken at least at the Nyquist rate, and that these signals can be perfectly reconstructed from the samples using a series, which is nowadays called Shannon's sampling series.

More precisely, the Shannon sampling theorem states the following. Given a signal f with finite energy and bandlimited to σ, i.e., a signal f that has the representation

$$f(t) = \frac{1}{2\pi} \int_{-\sigma}^{\sigma} g(\omega)\, \mathrm{e}^{\mathrm{i}\omega t}\ \mathrm{d}\omega, \quad t \in \mathbb{R},$$

for some $g \in L^2[-\sigma, \sigma]$, then the samples of the signal $\{f(k\pi/\sigma)\}_{k\in\mathbb{Z}}$ uniquely determine the continuous-time signal f, and f can be reconstructed using the series

$$f(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k\pi}{\sigma}\right) \frac{\sin\left(\sigma\left(t - \frac{k\pi}{\sigma}\right)\right)}{\sigma\left(t - \frac{k\pi}{\sigma}\right)}, \quad t \in \mathbb{R}. \tag{1.1}$$

Although this sampling theorem is often called the Shannon sampling theorem, Shannon was not the first to discover this sampling theorem. It was already known in the mathematical literature [3]. However, its significance in many engineering applications and theoretical concepts made it famous.

Today, most information is processed digitally. In our daily life we are surrounded by devices that operate with digital signals: computers, mobile phones, portable media players, and many more. However, the real world signals are not digital but analog in nature, and therefore we need to convert the signals back and forth between the analog and digital domains. The sampling theorem established the basis for performing this conversion, and thus the work of Shannon can be seen as the starting point of the information and communication age in which we are living.

The current technological advance in digital devices is driven by Moore's law, which states that the complexity of electronic circuits, measured in the number of transistors on a chip, doubles approximately every two years. It was postulated in 1965 by Gordon Moore, who originally stated, based on the development between 1959 and 1965, a doubling every year, which would continue for at least ten years [4]. In 1975, he corrected his prediction to the current two year statement.

The prediction of Moore's law is astonishingly precise, and indeed, this remarkable progress in technology could be observed over the last 50 years. However, it is clear that the miniaturization cannot be continued forever due to physical limitations of atomic structure or power density. Regardless of whether this driving force will stop in 10 or 20 years, by then we need to find other methods and concepts to maintain the technological progress.

In this thesis we study the interplay between the analog and the digital worlds. We will find many results that go beyond Shannon's sampling theorem and discover fundamental limits. It is important to better understand the basic principles, because knowing the theoretically feasible can help improve future systems.

1.2 Contribution and Organization of this Thesis

There are many applications where the conversion between digital and analog signals is important. For example in modern wireless communications, the information to be transmitted to the receiver is present in the form of a digital signal at the sender. In order to be able to physically transmit this signal through the air we have to convert it into an analog signal, and later at the receiver we have to convert it back to a digital signal.

The digitization of an analog signal, which is visualized in Fig. 1.1, consists of sampling and quantization. Some authors already call the discrete-time signal a "digital signal". However, in this thesis we distinguish between these terms and call a discrete-time signal a "digital signal" only if it is additionally quantized. Further, we will use the terms analog signal and continuous-time signal synonymously.

It is well known that the sampling step is reversible if the signal and the sampling pattern satisfy certain properties. For example, from Shannon's sampling theorem we know that the sampling step is fully reversible if the analog signal is bandlimited, has finite energy, and is sampled equidistantly at least at the Nyquist rate. In contrast, the quantization step inevitably entails a loss of information and thus is not reversible in general.

While for finite-energy bandlimited signals the conversion from the analog signal to the discrete-time signal and the conversion back from the discrete-time signal to the analog signal can be performed without problems by sampling and interpolation

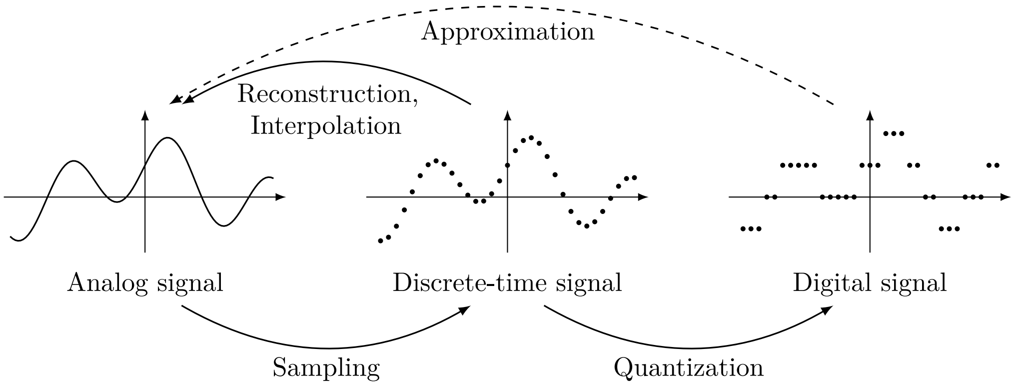

Figure 1.1: Analog to digital conversion and reconstruction.

with the Shannon sampling series, the latter transition can be problematic for other signal spaces. In the first part of this thesis we will analyze the interpolation problem, which is depicted in Fig. 1.1, for larger signal spaces.

Another essential question is, whether certain properties of the signal in one domain carry over to the other domain. It is a well-known fact that if the discrete signal has finite energy, then the bandlimited continuous-time signal, which is obtained by interpolation with Shannon's sampling series, also has finite energy. Conversely, the sampled version of a bandlimited signal with finite energy has finite energy. By sampling and interpolation one can switch between both representations and the finite-energy property of the signal is preserved. We will analyze whether a similar correspondence is also true for general bounded bandlimited signals.

Further, in applications it can occur that the sampling process is not ideal. Then the samples of the signal are disturbed in one or both of the following ways:

1. The sample positions are changed.
2. The sample values are changed.

Consideration of the first disturbance leads directly to the problem of non-equidistant sampling. Although the sampling points are often chosen equidistantly for ease of reconstruction, imperfections in the sampling procedure inevitably lead to a jitter in the sampling positions. The influence of non-equidistant sampling patterns will also be studied in this thesis.

A special case of the second disturbance is quantization, where the sample values are changed according to a specific quantization rule. Quantization is important because—as an integral part of digitization—it is present in any real system. We examine quantization and thresholding, which is closely related to quantization, and study their effects on signal reconstruction. In this thesis the quantization operator is treated deterministically. The deterministic analysis is difficult because of the non-linear nature of the quantization operator, but it reveals some properties of the quantization process, which cannot be analyzed with an additive noise description of the quantization error.

Another big part of this thesis is devoted to the analysis of representations of stable linear time-invariant (LTI) systems. There we are not interested in the reconstruction of a bandlimited signal from its samples but in the calculation of the output signal of a stable LTI system. This problem is interesting because the processing of signals by systems is at the heart of signal processing and, for applications, is probably even more important than the mere reconstruction of signals. We will analyze the convergence of certain time domain convolution-type system representations. Further, since bandlimited signals are determined by their samples, we also consider sampling-type system representations, which only use the samples of the bandlimited input signal to compute the continuous-time output signal of a stable LTI system. A priori it is not clear whether such a sampling based signal processing can be performed in a stable way.

Outline

The outline of this thesis is as follows.

In **Chapter 2** we introduce some notation and definitions. Further, we define different spaces of bandlimited signals and state their basic properties.

In **Chapter 3** we treat the interplay between discrete-time and continuous-time signals. First, in Section 3.1, the bandlimited interpolation is studied. Then, in Section 3.2, the reconstruction of bandlimited signals from their equidistant samples is analyzed. After reviewing the Shannon sampling theorem for different spaces of bandlimited signals, we analyze the convergence behavior of a whole class of axiomatically defined Nyquist-rate reconstruction processes for signals in $\mathcal{PW}_{\pi}^{1}$. It is shown for this very general class, which contains all common sampling series including the Shannon sampling series, that a reconstruction that is uniformly convergent on compact subsets of the real line and uniformly bounded on the entire real line is not possible. Further, we consider oversampling and give a sufficient condition for the uniform convergence of the Shannon sampling series without oversampling. Additionally, we analyze the convergence behavior of the non-symmetric sampling series and a variant of the Shannon sampling series which is truncated symmetrically around t. Finally, in Section 3.3, non-equidistant sampling series are studied for general bounded bandlimited signals. We consider sampling patterns that are given by the zeros of sine-type functions and analyze the local and global convergence behavior of the sampling series. It is shown that the series converge locally uniformly for bounded bandlimited signals that vanish at infinity. Moreover, we discuss the influence of oversampling on the global approximation behavior and the convergence speed of the sampling series. Parts of the results are published in [5–13] or will be published in [14].

In **Chapter 4** we analyze time domain convolution-type representations of stable linear time-invariant (LTI) systems operating on bandlimited signals. Although a frequency domain representation of such systems is always possible, a time domain representation can be problematic. Two convolution integrals as well as the discrete counterpart, the convolution sum, are treated. We identify the differences in the convergence behavior of the two convolution integrals, and show that there exist stable LTI systems operating on the Paley–Wiener space $\mathcal{PW}_\pi^1$ for which the convolution integral representation does not exist because the integral is divergent, even if the convergence is interpreted in a distributional sense. Furthermore, we compare the classical and the distributional convergence behavior and completely characterize all stable LTI systems for which a time domain convolution representation is possible by giving a necessary and sufficient condition for convergence. For the sampling-type system representation we consider, in addition to equidistant sampling at Nyquist rate, non-equidistant sampling patterns and oversampling. Parts of the results are published in [13, 15, 16] or will be published in [14].

In **Chapter 5** we analyze the convergence behavior of the symmetric and the non-symmetric Shannon sampling series for bandlimited continuous-time wide-sense stationary stochastic processes that have absolutely continuous spectral measure. We completely characterize the processes for which the approximation error variance of the symmetric sampling series is uniformly bounded on the whole real axis and the processes for which the symmetric as well as the non-symmetric sampling series converge in the mean square sense uniformly on compact subsets of the real axis. Moreover, it is shown that there are I-processes for which the mean square approximation error of the non-symmetric sampling series diverges pointwise. This shows that there is a significant difference between the convergence behavior of the symmetric and the non-symmetric sampling series. Parts of the results are published in [17].

In **Chapter 6** we analyze the approximation behavior of sampling series where the sample values are disturbed either by the nonlinear threshold operator or the nonlinear quantization operator. We perform the analysis for several spaces of bandlimited signals and completely characterize the spaces for which an approximation is possible. Additionally, we study the approximation of outputs of stable linear time-invariant systems operating on $\mathcal{PW}_\pi^1$ by sampling series that use only the samples of the input signal, for the case where the samples are disturbed by the threshold operator or the quantization operator. We show that there exist stable systems that become unstable under thresholding and quantization and that the approximation error is unbounded irrespective of how small the quantization step size is chosen. Further, we give a necessary and sufficient condition for the pointwise and the uniform convergence of the series. Surprisingly, the condition for the uniform convergence is the well-known condition for bounded-input bounded-output (BIBO) stability. Parts of the results are published in [9, 15, 18].

Finally, in **Chapter 7** we conclude the thesis and point out open problems and possible future research directions.

Further Results that are not Part of this Thesis

During my time as a research assistant at the Technische Universität Berlin and the Technische Universität München we obtained further interesting results, which are not included in this thesis:

- In [19] the convergence behavior of the Shannon sampling series was analyzed for Hardy spaces. It was shown that there exist signals in the Hardy space such that the peak value of the Shannon sampling series diverges unboundedly.

- In [20] the convergence behavior of a convolution representation of stable linear time-invariant (LTI) systems operating on the Zakai class of bandlimited signals was analyzed. It was shown that the convergence of the convolution integral is problematic if the system is the Hilbert transform or the ideal low-pass filter with bandwidth less than or equal to the signal bandwidth. Moreover, using a previously obtained result of Habib [21], it was proved that the class of stable LTI systems that map the Zakai class into itself does not include the Hilbert transform and the ideal low-pass filter with bandwidth less than or equal to the signal bandwidth.

- In [22] sampling series that are disturbed by the non-linear threshold operator were studied. The set of $\mathcal{PW}_\pi^1$-signals for which the sampling series diverges as the threshold goes to zero was characterized, and it was shown that this set is a residual set.

- In [23] the approximation of the outputs of linear time-invariant systems by sampling series that use only the samples of the input signal was analyzed for the case where the samples are disturbed by the threshold operator. It was shown for the Hilbert transform that the peak approximation error can grow arbitrarily large for some signals in $\mathcal{PW}_\pi^1$ when the threshold approaches zero. Furthermore, a game theoretic interpretation of the problem in the setting of a game against nature was given.

- In [24] the existence of efficient bandpass-type systems for the space of bounded bandlimited signals was analyzed. Here efficient means that the system fulfills the following properties: every output signal contains only frequencies within the passband; every input signal that has only frequencies within the passband is not disturbed by the system; and the system is stable. It was proved that a linear realization cannot exist; however, a nonlinear realization is possible. Further, it was shown that a splitting of bounded bandlimited signals according to their frequency content can be problematic.

A complete list of publications is given in Appendix B.

2

Notation

For $t_1, t_2 \in \mathbb{R}$, $C[t_1,t_2]$ denotes the space of all continuous functions on $[t_1,t_2]$, and $C_0^\infty[t_1,t_2]$ the space of all infinitely differentiable functions on $\mathbb{R}$ whose support is contained in $[t_1,t_2]$.

$L^p(X)$, $1 \le p < \infty$, is the space of complex-valued measurable functions, defined on $X \subset \mathbb{R}$ that are Lebesgue integrable to the pth power, and $\|\cdot\|_{L^p(X)}$ denotes the usual L^p-norm. If $X = \mathbb{R}$ we use the abbreviation $\|\cdot\|_p := \|\cdot\|_{L^p(\mathbb{R})}$. Moreover, $L^\infty(X)$ denotes the space of all complex-valued functions defined on $X \subset \mathbb{R}$ for which the essential supremum norm $\|\cdot\|_{L^\infty(X)}$ is finite. If $X = \mathbb{R}$ we use the abbreviation $\|\cdot\|_\infty := \|\cdot\|_{L^\infty(\mathbb{R})}$. λ denotes the Lebesgue measure. l^p, $1 \le p < \infty$, is the space of sequences $x = \{x_k\}_{k\in\mathbb{Z}} \subset \mathbb{C}$, for which $\|x\|_{l^p} := \left(\sum_{k=-\infty}^{\infty} |x_k|^p\right)^{1/p} < \infty$. Moreover, l^∞ is the space of all bounded sequences, i.e., sequences for which the supremum norm $\|\cdot\|_{l^\infty}$ is finite.

Further, let $\hat{f}$ denote the Fourier transform of a function f. For functions in $L^1(\mathbb{R})$ the Fourier transform is defined in the classical sense, according to $\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, \mathrm{e}^{-\mathrm{i}\omega t}\ \mathrm{d}t$, for functions in $L^p(\mathbb{R})$, $1 < p \le 2$ as the limit of $\int_{-N}^{N} f(t)\, \mathrm{e}^{-\mathrm{i}\omega t}\ \mathrm{d}t$ in $L^q(\mathbb{R})$, $1/p + 1/q = 1$, and for functions in $L^p(\mathbb{R})$, $p > 2$, in the distributional sense.

Throughout the thesis, we denote by C and C_1, C_2, … positive constants unless otherwise stated. The numbering of the constants is by chapter.

2.1 Bandlimited Signals

The two most important families of spaces of bandlimited signals that we will deal with are the Bernstein spaces $\mathcal{B}_\sigma^p$ and the Paley–Wiener spaces $\mathcal{PW}_\sigma^p$, $1 \le p \le \infty$, $0 < \sigma < \infty$.

The following definition shows that the concept of bandlimited signals is closely related to entire functions of exponential type [25, p. 4]. For $0 < \sigma < \infty$ the space

$\mathcal{B}_\sigma$ is defined to be the set of all entire functions f with the property that for all $\epsilon > 0$ there exists a constant $C(f, \epsilon)$ with

$$|f(z)| \leq C(f, \epsilon)\, \mathrm{e}^{(\sigma+\epsilon)|z|}$$

for all $z \in \mathbb{C}$. For $1 \leq p \leq \infty$, the Bernstein space $\mathcal{B}_\sigma^p$ consists of all functions in $\mathcal{B}_\sigma$ whose restriction to the real line is in $L^p(\mathbb{R})$. The norm for $\mathcal{B}_\sigma^p$, $1 \leq p \leq \infty$ is given by $\|f\|_{\mathcal{B}_\sigma^p} = \|f\|_p$. A function in $\mathcal{B}_\sigma^p$ is called bandlimited to σ. $\mathcal{B}_\sigma^\infty$ is the space of bandlimited signals that are bounded on the real axis. We call a signal in $\mathcal{B}_\pi^\infty$ a bounded bandlimited signal. By $\mathcal{B}_{\sigma,0}^\infty$ we denote the set of all signals $f \in \mathcal{B}_\sigma^\infty$ with the property $\lim_{|t|\to\infty} f(t) = 0$. It is well known that $\mathcal{B}_\sigma^p \subset \mathcal{B}_\sigma^s$ for $1 \leq p \leq s \leq \infty$ [26, p. 49]. Hence, every signal in $\mathcal{B}_\sigma^p$, $1 \leq p \leq \infty$, is bounded on the real axis. An important inequality for functions $f \in \mathcal{B}_\sigma^p$, $1 \leq p \leq \infty$, is Bernstein's inequality $\|f^{(r)}\|_p \leq \sigma^r \|f\|_p$, $r \in \mathbb{N}$ [26, p. 49], which gives an upper bound on the norm of the r-th derivative of f.

As we can see from the above definition, all signals in $\mathcal{B}_\sigma^p$, $1 \leq p \leq \infty$, $0 < \sigma < \infty$, are defined on the complex plane and are entire functions of exponential type at most σ. However, in practical applications the signals are usually considered to be a function of a real variable, which often represents time. Since all signals in $\mathcal{B}_\sigma^p$ are entire functions, they are uniquely determined by their values on the real line. Therefore, we will not distinguish between signals defined on the complex plane and signals defined on the real axis in the following. For example, if f is a function defined on the real axis and we write $f \in \mathcal{B}_\sigma^p$, we mean that f can be extended to an entire function, defined on the complex plane, which is in $\mathcal{B}_\sigma^p$. In the same way, if f is an entire function and we write $f \in L^p(\mathbb{R})$, we mean that the restriction of f to the real axis is in $L^p(\mathbb{R})$.

The special case $p = 2$ gives the commonly used space $\mathcal{B}_\sigma^2$ of bandlimited signals with finite energy. The Paley–Wiener theorem [27, p. 13], [26, p. 68] provides a connection between the exponential type σ of a function $f \in \mathcal{B}_\sigma^2$ and the support of the Fourier transform $\hat{f}$ of f.

Paley–Wiener Theorem. *Let $f \in L^2(\mathbb{R})$. Then f has an analytic extension to $\mathbb{C}$ which belongs to $\mathcal{B}_\sigma$ if and only if* $\operatorname{supp} \hat{f} \subset [-\sigma, \sigma]$.

Thus, the Paley–Wiener theorem shows that

$$\mathcal{B}_\sigma^2 = \left\{ f \in L^2(\mathbb{R}) : \operatorname{supp} \hat{f} \subset [-\sigma, \sigma] \right\}$$

can alternatively be used to define the space $\mathcal{B}_\sigma^2$.

For $0 < \sigma < \infty$ and $1 \leq p \leq \infty$ we denote by $\mathcal{PW}_\sigma^p$ the Paley–Wiener space of signals f with a representation

$$f(z) = \frac{1}{2\pi} \int_{-\sigma}^{\sigma} g(\omega)\, \mathrm{e}^{\mathrm{i}z\omega}\, \mathrm{d}\omega, \quad z \in \mathbb{C}, \tag{2.1}$$

for some $g \in L^p[-\sigma, \sigma]$. If $f \in \mathcal{PW}_\sigma^p$ then $g(\omega) = \hat{f}(\omega)$. The norm for $\mathcal{PW}_\sigma^p$, $1 \leq p < \infty$, is given by

$$\|f\|_{\mathcal{PW}_\sigma^p} = \left(\frac{1}{2\pi} \int_{-\sigma}^{\sigma} |\hat{f}(\omega)|^p \, \mathrm{d}\omega \right)^{1/p}.$$

Remark 2.1. The nomenclature we have introduced so far concerning the Bernstein and Paley–Wiener spaces is not consistent in the literature. Sometimes the space that we call the Bernstein space is called the Paley–Wiener space [28]. We adhere to the notation used in [26] by Higgins.

From (2.1) we see that

$$|f(z)| \leq \frac{1}{2\pi} \int_{-\sigma}^{\sigma} |g(\omega)| \, \mathrm{e}^{-\operatorname{Im}(z)\omega} \, \mathrm{d}\omega \leq \frac{\|g\|_{L^1[-\sigma,\sigma]}}{2\pi} \, \mathrm{e}^{\sigma|z|},$$

which shows that every signal $f \in \mathcal{PW}_\sigma^p$, $1 \leq p \leq \infty$, also belongs to $\mathcal{B}_\sigma$. Thus, the Hausdorff–Young inequality leads to $\mathcal{B}_\sigma^q \supset \mathcal{PW}_\sigma^p$ for $1 < p \leq 2$, $1/p + 1/q = 1$. For $p = q = 2$ we see from Parseval's equality and the Paley–Wiener theorem that every signal $f \in \mathcal{B}_\sigma^2$ has the representation (2.1) for some $g \in L^2[-\sigma, \sigma]$, and hence is in $\mathcal{PW}_\sigma^2$. Thus, we have $\mathcal{B}_\sigma^2 = \mathcal{PW}_\sigma^2$. Furthermore, Hölder's inequality leads to $\|f\|_{\mathcal{PW}_\pi^p} \leq \|f\|_{\mathcal{PW}_\pi^s}$ for $f \in \mathcal{PW}_\pi^s$, $1 \leq p < s \leq \infty$, and consequently to the inclusion $\mathcal{PW}_\sigma^p \supset \mathcal{PW}_\sigma^s$, $1 \leq p < s \leq \infty$. Moreover, for $f \in \mathcal{PW}_\sigma^1$ we have $\|f\|_\infty \leq \|f\|_{\mathcal{PW}_\pi^1}$, which implies that every signal in $\mathcal{PW}_\sigma^p$, $1 \leq p \leq \infty$, is bounded on the real axis.

The above facts show that

$$\mathcal{PW}_\sigma^2 = \mathcal{B}_\sigma^2 \subset \mathcal{PW}_\sigma^1 \subset \mathcal{B}_\sigma^\infty.$$

This inclusion relation will be intensely used in this thesis.

Note that without loss of generality we can restrict our further investigations to signals with bandwidth $\sigma = \pi$, because every bandlimited signal with bandwidth different to π can be scaled such that the resulting signal is bandlimited to π.

3

Discrete-Time and Continuous-Time Signals

In this chapter we treat the topics which are visualized in the left part of Fig. 1.1, that is, we analyze the conversion of analog signals into discrete-time signals and, vice versa, the conversion of discrete-time signals into analog signals.

The effects of quantization, which are important for practical applications, are neglected for the moment in order to focus on the main ideas and problems of sampling and interpolation. The impact of quantization on signal reconstruction will be analyzed in Chapter 6. Since quantization is not considered in this chapter, we refer to analog signals as continuous-time signals to emphasize the difference to discrete-time signals.

Modern signal processing is performed nearly exclusively with digital processors while the physical quantities of the real world are analog. Therefore, the conversion of continuous-time signals into discrete-time signals and the conversion of discrete-time signals back into continuous-time signals is essential. The first conversion is done by sampling and the second one by a reconstruction process. The Shannon sampling series (1.1) is probably the most prominent example of a reconstruction process. However, many other reconstruction processes are possible, for example processes designed for non-equidistant samples and non-bandlimited or non-linear reconstruction processes.

A very natural requirement for applications is the boundedness of the involved signals. Hence, both conversions should be stable in the sense that a bounded signal in one domain is converted into a bounded signal in the other domain. Thus, for signal processing applications it would be useful to have a correspondence between the space $\mathcal{B}_\pi^\infty$ of bounded and bandlimited continuous-time signals and the space l^∞ of bounded discrete-time signals, in the sense that

i) every continuous-time signal $f \in \mathcal{B}_\pi^\infty$ leads to a bounded discrete-time signal $x = \{f(t_k)\}_{k\in\mathbb{Z}} \in l^\infty$ if it is sampled at the sampling points $\{t_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$, and

ii) for every discrete-time signal $x = \{x_k\}_{k\in\mathbb{Z}} \in l^\infty$ there exists a continuous-time signal $f \in \mathcal{B}_\pi^\infty$ such that $f(t_k) = x_k$ for all $k \in \mathbb{Z}$.

Since signals are usually considered to be functions on the real axis—the time axis—we restrict ourselves to sampling point sequences of real numbers.

In order to further motivate the importance of ii), we consider an example from mobile communications. In orthogonal frequency-division multiplexing (OFDM) systems, where complex trigonometric polynomials are the baseband signals, high peak-to-average power ratios (PAPRs) are problematic because high peak values can overload the power amplifier, which in turn leads to undesired out-of-band radiation. Thus, the relation between the peak value of the discrete-time signal and the peak value of the bandlimited continuous-time signal is of interest.

For bandlimited signals that are, in particular, continuous, the sampling process does not create any problems. Obviously, if $f \in \mathcal{B}_\pi^\infty$, then we have $\sup_{t\in\mathbb{R}} |f(t)| < \infty$, by definition. As a consequence, given any sequence of sampling points $\{t_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$, the sampled signal $x = \{f(t_k)\}_{k\in\mathbb{Z}}$ satisfies $\|x\|_{l^\infty} \leq \|f\|_\infty < \infty$ and thus is in l^∞. This shows that the sampling operation is unproblematic for all signals in the large signal space $\mathcal{B}_\pi^\infty$. Sampling of any bounded bandlimited signal leads to a discrete-time signal that is also bounded.

However, the inverse problem of sampling—bandlimited interpolation—is more intricate. It is not clear a priori whether for every bounded discrete-time signal $x \in l^\infty$ it is possible to construct a bounded bandlimited continuous-time signal $f \in \mathcal{B}_\pi^\infty$ that interpolates x at the points $\{t_k\}_{k\in\mathbb{Z}}$. Whether this is possible depends strongly on the sequence $\{t_k\}_{k\in\mathbb{Z}}$. But even for equidistant interpolation points $t_k = k$, $k \in \mathbb{Z}$, it will turn out that for general bounded discrete-time signals the existence of the bounded bandlimited interpolation cannot be guaranteed. Only if the set of discrete-time signals is further restricted is bounded bandlimited interpolation always possible. We will analyze the existence of bandlimited interpolation under various aspects in Section 3.1.

One possible way to further restrict the set of discrete-time signals is to only consider those which are created by sampling certain continuous-time signals. This restriction immediately leads to the following question: which continuous-time signals can be reconstructed from their samples? In Sections 3.2 and 3.3 we analyze this question, which covers both sampling and interpolation and hence the whole circle in the left part of Fig. 1.1.

We distinguish between equidistant (or uniform) sampling, where the distance between every pair of consecutive sampling points is the same, and non-equidistant (or non-uniform) sampling, where we do not have this restriction on the distances between the sampling points. Equidistant sampling will be treated in Section 3.2 and non-equidistant sampling in Section 3.3.

For a reconstruction process like, for example, the Shannon sampling series, the type of convergence is important. For practical applications it is desirable to have a reconstruction process that converges uniformly on all of $\mathbb{R}$ to the sampled signal f. In this case the supremum of the approximation error can be bounded and goes to zero. However, global uniform convergence is a very demanding property for a reconstruction process. A weaker condition, which is also a necessary condition for global uniform convergence, is local uniform convergence and global uniform boundedness. The uniform convergence on bounded intervals is important for the reconstruction behavior in the time interval of interest, and the bounded peak value on $\mathbb{R}$ assures that the reconstructed signal is well behaved outside the interval.

3.1 Bandlimited Interpolation

In this section we study the interpolation of discrete-time signals, i.e., the procedure which is represented by the upper left arrow in Fig. 1.1.

We say a signal f interpolates a sequence $\{x_k\}_{k \in I} \subset \mathbb{C}$ at the points $\{t_k\}_{k \in I} \subset \mathbb{R}$, where I is an arbitrary index set, if $f(t_k) = x_k$ for all $k \in I$. There are numerous ways to interpolate a given sequence: piecewise constant interpolation, linear interpolation, or spline interpolation, for example. However, the interpolants obtained by these methods are generally not bandlimited. In this thesis we focus on bandlimited interpolants because bandlimited signals have several nice properties which are important in many applications. For example, the bandlimitedness guarantees that the interpolant is smooth.

3.1.1 Bandlimited Interpolation for Discrete-Time Signals

Next, we analyze the question whether for every bounded discrete-time signal $x = \{x_k\}_{k \in \mathbb{Z}} \in l^\infty$ it is possible to construct a bounded bandlimited continuous-time signal $f \in \mathcal{B}_\pi^\infty$ which interpolates x at the integers.

Definition 3.1. We call a signal $f \in \mathcal{B}_\pi^\infty$ bounded bandlimited interpolation of a discrete-time signal (sequence) $\{x_k\}_{k \in \mathbb{Z}} \in l^\infty$ if $f(k) = x_k$ for all $k \in \mathbb{Z}$.

For the subspaces l^p, $1 \leq p < \infty$, this question can be answered in the affirmative, and the Shannon sampling series

$$f(t) = \sum_{k=-\infty}^{\infty} x_k \frac{\sin(\pi(t-k))}{\pi(t-k)} \tag{3.1}$$

provides a way to obtain the bandlimited interpolation. This can be easily seen. For $t = l \in \mathbb{Z}$ we have $f(l) = x_l$ because

$$\frac{\sin(\pi(l-k))}{\pi(l-k)} = \begin{cases} 1, & k = l \\ 0, & k \in \mathbb{Z} \setminus \{l\}, \end{cases}$$

and for $t \in \mathbb{R} \setminus \mathbb{Z}$, $1 \le p < \infty$, and $1/p + 1/q = 1$ it follows by Hölder's inequality that

$$\left| \sum_{k=-\infty}^{\infty} x_k \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| \le \left(\sum_{k=-\infty}^{\infty} |x_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=-\infty}^{\infty} \left| \frac{\sin(\pi(t-k))}{\pi(t-k)} \right|^q \right)^{\frac{1}{q}} .$$

The first term is finite by assumption and the finiteness of the second term follows from the convergence of $\sum_{k=-\infty}^{\infty} 1/k^q$, $q > 1$. Thus, equation (3.1) can be used to obtain the bounded bandlimited interpolation of any discrete-time signal $\{x_k\}_{k\in\mathbb{Z}} \in l^p$, $1 \le p < \infty$.

As is well known, for arbitrary discrete-time signals $x = \{x_k\}_{k\in\mathbb{Z}} \in l^\infty$ the series (3.1) cannot be used to obtain the bandlimited interpolation, because the Shannon sampling series is divergent for certain bounded discrete-time signals [5,29]. A simple example of a discrete-time signal in l^∞ which creates divergence is given by

$$\tilde{x}_k = \begin{cases} 0, & k \le 0 \\ \frac{(-1)^k}{\log(1+k)}, & k \ge 1. \end{cases} \tag{3.2}$$

For this signal we have

$$\lim_{N\to\infty} \left| \sum_{k=-N}^{N} \tilde{x}_k \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| = \infty \tag{3.3}$$

for all $t \in \mathbb{R} \setminus \mathbb{Z}$. Equation (3.3) will be proved later in Theorem 3.27.

The result that (3.1) diverges for all points between two integers shows that the Shannon sampling series does not produce a signal $f \in \mathcal{B}_\pi^\infty$ out of $\tilde{x}$. In order to prove that for $\tilde{x}$ there exists no $f \in \mathcal{B}_\pi^\infty$ such that $\tilde{x}_k = f(k)$ for all $k \in \mathbb{Z}$, we need the Valiron interpolation series [30, p. 12], which is sometimes called Tschakaloff's series [26, p. 60].

The Valiron interpolation series

$$f(t) = f(0) + f'(0)\frac{\sin(\pi t)}{\pi} + t \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{f(k) - f(0)}{k} \frac{\sin(\pi(t-k))}{\pi(t-k)} \tag{3.4}$$

is a valid representation for all signals $f \in \mathcal{B}_\pi^\infty$, i.e., every signal $f \in \mathcal{B}_\pi^\infty$ can be represented in the form of the right-hand side of equation (3.4). Conversely, for every sequence $x = \{x_k\}_{k\in\mathbb{Z}} = \{f(k)\}_{k\in\mathbb{Z}}$ that is constructed out of the samples $\{f(k)\}_{k\in\mathbb{Z}}$ of a signal $f \in \mathcal{B}_\pi^\infty$, the series

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{x_k - x_0}{k} \frac{\sin(\pi(t-k))}{\pi(t-k)} \tag{3.5}$$

converges, and the function

$$f_x(t) := t \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{x_k - x_0}{k} \frac{\sin(\pi(t-k))}{\pi(t-k)} \tag{3.6}$$

is a function in $\mathcal{B}_\pi^\infty$.

The Valiron interpolation series enables us to prove the following theorem.

Theorem 3.2. *Given the discrete-time signal $\tilde{x} = \{\tilde{x}_k\}_{k \in \mathbb{Z}} \in l^\infty$ as defined in (3.2), there exists no signal $f \in \mathcal{B}_\pi^\infty$ with $\tilde{x}_k = f(k)$ for all $k \in \mathbb{Z}$.*

Proof. The idea for the proof is as follows. We show that the signal $f_{\tilde{x}}$, which is constructed according to (3.6), is not in $\mathcal{B}_\pi^\infty$. This implies that the sequence $\tilde{x}$ cannot be obtained by sampling any signal $f \in \mathcal{B}_\pi^\infty$, since otherwise $f_{\tilde{x}}$ would be in $\mathcal{B}_\pi^\infty$. In other words, there exists no signal $f \in \mathcal{B}_\pi^\infty$ such that $\tilde{x}_k = f(k)$ for all $k \in \mathbb{Z}$. Hence, we have proved Theorem 3.2.

By Hölder's inequality, the series $\sum_{k=1}^{\infty} 1/k^2 = \pi^2/6$, and Parseval's equality it follows that

$$\begin{aligned}
\sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \left| \frac{x(k)}{k} \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| &\leq \left(\sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{|x(k)|^2}{k^2} \right)^{\frac{1}{2}} \left(\sum_{k=-\infty}^{\infty} \left| \frac{\sin(\pi(t-k))}{\pi(t-k)} \right|^2 \right)^{\frac{1}{2}} \\
&\leq \frac{\pi}{\sqrt{3}} \|x\|_{l^\infty}.
\end{aligned}$$

Thus, the series (3.5) converges pointwise for every $x \in l^\infty$. It remains to show that $f_{\tilde{x}} \notin \mathcal{B}_\pi^\infty$. Although $f_{\tilde{x}}(k) = \tilde{x}_k$ for all $k \in \mathbb{Z}$ and consequently $\sup_{k \in \mathbb{Z}} |f_{\tilde{x}}(k)| = \|\tilde{x}\|_{l^\infty} < \infty$, we can show that for $N \in \mathbb{N}$ we have

$$\lim_{N \to \infty} f_{\tilde{x}}\left(2N + \tfrac{1}{2}\right) = +\infty$$

and

$$\lim_{N \to \infty} f_{\tilde{x}}\left(2N + \tfrac{3}{2}\right) = -\infty.$$

Let $N \in \mathbb{N}$ be arbitrarily but fixed. Then we have

$$\begin{aligned}
f_{\tilde{x}}(N+\tfrac{1}{2}) =& \left(N+\frac{1}{2}\right)\sum_{k=1}^{\infty}\frac{(-1)^k}{\log(1+k)k}\frac{\sin\left(\pi(N+\frac{1}{2}-k)\right)}{\pi\left(N+\frac{1}{2}-k\right)}\\
=&\Bigg[\underbrace{\left(N+\frac{1}{2}\right)\left(\sum_{k=1}^{N}\frac{1}{\log(1+k)k\left(N+\frac{1}{2}-k\right)}\right)}_{=R_N^1}\\
&+\underbrace{\left(N+\frac{1}{2}\right)\left(\sum_{k=N+1}^{\infty}\frac{1}{\log(1+k)k\left(N+\frac{1}{2}-k\right)}\right)}_{=R_N^2}\Bigg]\cdot\frac{\sin\left(\pi\left(N+\frac{1}{2}\right)\right)}{\pi}.
\end{aligned}\tag{3.7}$$

Since

$$R_N^1 > \sum_{k=1}^{N}\frac{1}{\log(1+k)(1+k)}$$

and

$$\frac{1}{\log(1+k)(1+k)} > \int_k^{k+1}\frac{1}{\log(1+\tau)(1+\tau)}\,\mathrm{d}\tau,$$

we obtain

$$R_N^1 > \int_1^{N+1}\frac{1}{\log(1+\tau)(1+\tau)}\,\mathrm{d}\tau > \log(\log(N+2)).\tag{3.8}$$

The modulus of the second term R_N^2 can be bounded from above by

$$\begin{aligned}
|R_N^2| =& \left(N+\frac{1}{2}\right)\sum_{k=N+1}^{\infty}\frac{1}{\log(1+k)k\left(k-\left(N+\frac{1}{2}\right)\right)}\\
\leq&\frac{N+\frac{1}{2}}{\log(2+N)}\sum_{k=N+1}^{\infty}\frac{1}{k\left(k-\left(N+\frac{1}{2}\right)\right)}\\
=&\frac{1}{\log(2+N)}\lim_{M\to\infty}\left(\sum_{k=N+1}^{M}\frac{1}{k-(N+\frac{1}{2})}-\sum_{k=N+1}^{M}\frac{1}{k}\right).
\end{aligned}\tag{3.9}$$

Since

$$\frac{1}{k-\left(N+\frac{1}{2}\right)} < \int_{k-1}^{k}\frac{1}{\tau-\left(N+\frac{1}{2}\right)}\,\mathrm{d}\tau$$

for $k \geq N+2$, it is possible to find an upper bound for the first sum in (3.9), namely

$$\begin{aligned}\sum_{k=N+1}^{M} \frac{1}{k-\left(N+\frac{1}{2}\right)} &= 2 + \sum_{k=N+2}^{M} \frac{1}{k-\left(N+\frac{1}{2}\right)} \\ &\leq 2 + \int_{N+1}^{M} \frac{1}{\tau-\left(N+\frac{1}{2}\right)} \,\mathrm{d}\tau \\ &= 2 + \log(2M-2N-1).\end{aligned}$$

The second sum in (3.9) can be bounded from below by

$$\sum_{k=N+1}^{M} \frac{1}{k} \geq \sum_{k=N+1}^{M} \int_{k}^{k+1} \frac{1}{\tau} \,\mathrm{d}\tau = \int_{N+1}^{M+1} \frac{1}{x} \,\mathrm{d}x = \log\left(\frac{M+1}{N+1}\right).$$

Combining both bounds yields

$$\begin{aligned}|R_N^2| &\leq \frac{1}{\log(N+2)} \lim_{M\to\infty} \left[2 + \log(2M-2N-1) - \log\left(\frac{M+1}{N+1}\right)\right] \\ &= \frac{1}{\log(N+2)} \lim_{M\to\infty} \left[2 + \log(N+1) + \log\left(\frac{2M-2N-1}{M+1}\right)\right] \\ &= \frac{2+\log(N+1)+\log(2)}{\log(N+2)}. \end{aligned} \tag{3.10}$$

Now we are in the position to evaluate $f_{\tilde{x}}(2N+1/2)$ and $f_{\tilde{x}}(2N+3/2)$. Combining (3.7), (3.8), and (3.10) we obtain

$$f_{\tilde{x}}\left(2N+\tfrac{1}{2}\right) \geq \frac{1}{\pi}\left[\log(\log(2N+2)) - \frac{2+\log(2N+1)+\log(2)}{\log(2N+2)}\right]$$

and consequently $\lim_{N\to\infty} f_{\tilde{x}}(2N+1/2) = \infty$. For $f_{\tilde{x}}(2N+3/2)$ we obtain

$$f_{\tilde{x}}\left(2N+\tfrac{3}{2}\right) \leq -\frac{1}{\pi}\left[\log(\log(2N+3)) - \frac{2+\log(2N+2)+\log(2)}{\log(2N+3)}\right]$$

and thus $\lim_{N\to\infty} f_{\tilde{x}}(2N+3/2) = -\infty$. Therefore, we have $f_{\tilde{x}} \notin \mathcal{B}_\pi^\infty$. □

In Theorem 3.2 we have seen that there exist discrete-time signals in l^∞ that have no bounded bandlimited interpolation in $\mathcal{B}_\pi^\infty$. Discrete-time signals that lead to divergence, like the signal $\tilde{x}$ above, can emerge from discrete-time signals that have a bounded bandlimited interpolation when simple signal processing operations are applied. For example, the discrete-time signal $\tilde{x}$ can be obtained by truncating the discrete-time signal $y = \{y_k\}_{k\in\mathbb{Z}} \in l^\infty$, given by

$$y_k = \begin{cases} 0, & k = 0 \\ \frac{(-1)^{|k|}}{\log(1+|k|)}, & |k| \geq 1. \end{cases}$$

It is easy to see that y has a bandlimited interpolation, because y can be obtained by sampling the bounded and bandlimited signal $f_2 \in \mathcal{PW}_\pi^1 \subset \mathcal{B}_\pi^\infty$, given by

$$f_2(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}_2(\omega) e^{\mathrm{i}\omega t} \, \mathrm{d}\omega, \tag{3.11}$$

where

$$\hat{f}_2(\omega) = 2 \sum_{k=1}^{\infty} \frac{(-1)^k}{\log(1+k)} \cos(k\omega), \quad |\omega| \le \pi.$$

A short calculation shows that $f_2(k) = y_k$, $k \in \mathbb{Z}$. The bandlimitedness of f_2 is obvious, and so it remains to show that $\hat{f}_2 \in L^1[-\pi, \pi]$. Since

$$\hat{g}(\omega) = \sum_{k=1}^{\infty} \frac{\cos(k\omega)}{\log(1+k)}$$

is in $L^1[-\pi, \pi]$ [31, p. 183], it follows that the function $\hat{G}$, $\hat{G}(e^{\mathrm{i}\omega}) = \hat{g}(\omega)$, is in $L^1(\partial D)$, $\partial D = \{z \in \mathbb{C} : |z| = 1\}$. Now, if we rotate $\hat{G}$ by π we obtain the function

$$\hat{G}(e^{j(\omega+\pi)}) = \sum_{k=1}^{\infty} \frac{\cos(k(\omega+\pi))}{\log(1+k)} = \sum_{k=1}^{\infty} \frac{(-1)^k \cos(k\omega)}{\log(1+k)},$$

which also is in $L^1(\partial D)$. Therefore, $\hat{f}_2$ is in $L^1(-\pi, \pi)$.

This shows that even simple signal processing operations like truncation can lead to discrete-time signals that have no bounded bandlimited interpolation. It would be interesting to know the operations that behave well with respect to the bounded bandlimited interpolation.

3.1.2 Bandlimited Interpolation for Continuous-Time Signals

A problem closely related to the one analyzed in Section 3.1.1 is the following. Given an arbitrary continuous and bounded, but not necessarily bandlimited signal g, we want to find a bandlimited signal $f \in \mathcal{B}_\pi^\infty$ which interpolates $\{g(k)\}_{k\in\mathbb{Z}}$ at the integers. In the literature, f is known as the bandlimited interpolation of g [32, p. 144].

The bandlimited interpolation is a frequently used concept in signal processing. Therefore it would be interesting to know the largest signal space which possess a bounded bandlimited interpolation. The Banach algebra $\mathcal{W}$ that consists of all continuous functions g with the property that $\hat{g} \in L^1(\mathbb{R})$ exists in the distributional sense and

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(\omega) \, e^{\mathrm{i}\omega t} \, \mathrm{d}\omega$$

is—to the best of our knowledge—the largest known space of signals, all of which have a bounded bandlimited interpolation. However, as we have seen in Section 3.1.1, simple signal processing operations like truncation of the discrete-time signal are not stable, i.e. the bandlimited interpolation of the new signal can be unbounded.

Next, we show that for all signals $g \in \mathcal{W}$ there exists a bandlimited interpolation $f \in \mathcal{PW}_\pi^1$. To this end, let $g \in \mathcal{W}$ be arbitrary but fixed and consider

$$\hat{f}(\omega) = \begin{cases} \sum_{k=-\infty}^{\infty} \hat{g}(\omega + 2k\pi), & |\omega| \le \pi \\ 0, & |\omega| > \pi, \end{cases} \tag{3.12}$$

where the series in (3.12) converges for almost all $\omega \in [-\pi, \pi]$. Since $\hat{g} \in L^1(\mathbb{R})$, we have

$$\int_{-\infty}^{\infty} |\hat{f}(\omega)| \, \mathrm{d}\omega \le \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} |\hat{g}(\omega + 2k\pi)| \, \mathrm{d}\omega = \int_{-\infty}^{\infty} |\hat{g}(\omega)| \, \mathrm{d}\omega < \infty,$$

which shows that $\hat{f} \in L^1(\mathbb{R})$ and therefore $f \in \mathcal{PW}_\pi^1 \subset \mathcal{B}_\pi^\infty$. Furthermore, a simple calculation reveals that $f(k) = g(k)$, $k \in \mathbb{Z}$. Thus $\hat{f}$ is indeed the Fourier transform of the bandlimited interpolation f. The result that all signals in $\mathcal{W}$ have a bandlimited interpolation, which is in $\mathcal{PW}_\pi^1$, was already obtained by Brown in [33].

Because the bandlimited interpolation is a frequently used concept in signal processing it would be important to know the largest space of signals which possess a bounded, bandlimited interpolation, even when common signal processing operations like truncation are performed.

Remark 3.3. In the definition of the bounded bandlimited interpolation (Definition 3.1 on page 15) we required the bandwidth of the interpolant to be π, that is, the rate of the interpolation points is the Nyquist rate of the interpolant. If we relax the bandwidth constraint on the interpolant, i.e., if we allow f to be in $\mathcal{B}_{a\pi}^\infty$ for some $1 < a < \infty$, then the bounded bandlimited interpolation exists for every bounded discrete-time signal.

3.2 Equidistant Sampling

The goal of sampling based signal reconstruction is to reconstruct a signal f from its samples $\{f(t_k)\}_{k\in\mathbb{Z}}$, where $\{t_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ is the sequence of sampling points. Clearly, a reconstruction of a signal from its samples can only be successful if we put further restrictions on the signal. For an arbitrary non-continuous signal there is no hope of successful reconstruction if we know only the samples of the signal, because the signal can take arbitrary values between the sampling points, and thus is not uniquely determined by the samples.

In this section we analyze the reconstruction of bandlimited signals from their equidistant samples, i.e., we assume that $t_k = k/a$, $k \in \mathbb{Z}$, where $a \ge 1$ denotes the oversampling factor. Without loss of generality the bandwidth of all signals is assumed to be π.

3.2.1 The Classical Shannon Sampling Theorem

Sampling theory has its roots in the mathematical literature. Several mathematicians dealt with that topic: Borel, Hadamard, La Vallée Poussin and E. T. Whittaker are the most famous. The Shannon sampling series as it is known today was probably first described in 1915 by Whittaker in [3], where he called it the cardinal function. However, Kotel'nikov [34], Raabe [35], and Shannon [1] where the first to introduce the theory in the realm of communications.

Kotel'nikov published it in 1933 [34], Raabe in 1939 [35, 36] and Shannon in 1949 [1, 2]. Since the work by Kotel'nikov was published in Russian and Shannon's paper was already written in 1940 [37], it is reasonable to assume that all three publications were created independently. The reader who is further interested in the historical development of the sampling theorem is referred to [37–40], [41, Chapter 1], and [26, Chapter 1], where several historical notes can be found.

Shannon introduced the sampling theorem for bandlimited signals in $L^2(\mathbb{R})$. In its original form the Shannon sampling theorem makes two assertions. First, it states that a finite-energy signal with bandwidth $\sigma > 0$, i.e. a signal $f \in \mathcal{PW}_\sigma^2$, is uniquely determined by its samples $\{f(k\pi/\sigma)\}_{k\in\mathbb{Z}}$. And second, it states that the Shannon sampling series (1.1) can be used to reconstruct the continuous-time signal f from the samples $\{f(k\pi/\sigma)\}_{k\in\mathbb{Z}}$. The Shannon sampling series converges to f in the $\mathcal{PW}_\sigma^2$-norm and uniformly on $\mathbb{R}$.

The Nyquist rate $r_{\text{Ny}} = \sigma/\pi$ is the lowest possible sampling rate r for which all signals $f \in \mathcal{PW}_\sigma^2$ can be reconstructed without error from the samples $\{f(k/r)\}_{k\in\mathbb{Z}}$. If the signal is sampled at a rate larger than the Nyquist rate then the convergence speed of the Shannon sampling series can be increased by choosing other kernels which decay faster than the sinc-kernel [42–45]. Oversampling will be treated in Section 3.2.4.

As we restrict the bandwidth to $\sigma = \pi$ in this section, the Shannon sampling series without oversampling takes the form

$$\sum_{k=-\infty}^{\infty} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}. \tag{3.13}$$

Since the initial publishing of Shannon's sampling theorem for bandlimited finite-energy signals, much effort has been put in extending these results to a broader class of signals [46–51], and in analyzing the effects of changing the positions of the samples [52–61], of changing the sample values [62], or of truncating the series [63–67]. For an overview of various developments in sampling theory, see for example [38, 68–70].

3.2.2 Sampling Theorems for Larger Signal Spaces

Next, we review the convergence of the Shannon sampling series for signal spaces larger than the commonly used space $\mathcal{PW}_\pi^2$ of bandlimited signals with finite energy.

A very useful tool in the convergence analysis of the Shannon sampling series is the Plancherel–Pólya theorem [71, p. 48], [72, p. 22], [73].

Theorem 3.4 (Plancherel–Pólya Theorem). *Let $1 < p < \infty$. There exist two constants $C_L(p) > 0$ and $C_R(p) > 0$, depending only on p, such that for all $f \in \mathcal{B}_\pi^p$*

$$C_L(p) \left(\sum_{k=-\infty}^{\infty} |f(k)|^p \right)^{\frac{1}{p}} \leq \left(\int_{-\infty}^{\infty} |f(t)|^p \, \mathrm{d}t \right)^{\frac{1}{p}} \leq C_R(p) \left(\sum_{k=-\infty}^{\infty} |f(k)|^p \right)^{\frac{1}{p}}. \quad (3.14)$$

Note that the first inequality in (3.14) also holds for $p = 1$ and $p = \infty$, whereas, as is known in the mathematical literature, the second inequality (3.14) cannot be valid for $p = 1$ and $p = \infty$ (see corresponding remarks in [72, p. 11 and p. 22] and [73, p. 130]).

Convergence Behavior of the Shanon Sampling Series for $\mathcal{B}_\pi^p$, $1 \leq p < \infty$

For signals $f \in \mathcal{B}_\pi^p$, $1 \leq p < \infty$, the uniform convergence on all of $\mathbb{R}$ of the Shannon sampling series can be easily shown by using the Plancherel–Pólya theorem (Theorem 3.4). Let $1 < p < \infty$ and $f \in \mathcal{B}_\pi^p$ be arbitrary but fixed. Since

$$(S_N f)(t) := \sum_{k=-N}^{N} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}, \quad (3.15)$$

is a finite linear combination of sinc functions, $f - S_N f$ is in $\mathcal{B}_\pi^p$ too. Therefore, we have

$$\|f - S_N f\|_p \leq C_{\mathrm{R}}(p) \left(\sum_{k=-\infty}^{\infty} |f(k) - (S_N f)(k)|^p \right)^{\frac{1}{p}} \leq C_{\mathrm{R}}(p) \left(\sum_{|k|>N} |f(k)|^p \right)^{\frac{1}{p}},$$

and consequently $\lim_{N\to\infty} \|f - S_N f\|_p = 0$. Note that $\|f - S_N f\|_\infty \leq C_1(p) \|f - S_N f\|_p$, for some constant $C_1(p)$. Thus, for $f \in \mathcal{B}_\pi^p$, $1 < p < \infty$, the peak value of the approximation error $\|f - S_N f\|_\infty$, made by the truncation of the Shannon sampling series to N summands, can be bounded above and goes to zero for $N \to \infty$. Since $\mathcal{B}_\pi^1 \subset \mathcal{B}_\pi^2$, this result is also valid for $p = 1$.

Thus, we have proved the following well-known theorem about the convergence behavior of the Shanon sampling series for $\mathcal{B}_\pi^p$, $1 \leq p < \infty$ [30, p. 9].

Theorem 3.5. *For all $f \in \mathcal{B}_\pi^p$, $1 \leq p < \infty$, we have*

$$f(t) = \sum_{k=-\infty}^{\infty} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}.$$

The series converges absolutely for $t \in \mathbb{R}$ and uniformly on $\mathbb{R}$.

Furthermore, we have the uniform convergence of the Shannon sampling series for the Paley–Wiener spaces $\mathcal{PW}_\pi^p$, $1 < p \leq \infty$, because $\mathcal{B}_\pi^q \supset \mathcal{PW}_\pi^p$ for $1 < p \leq 2$, $1/p + 1/q = 1$, and $\mathcal{PW}_\sigma^p \subset \mathcal{PW}_\sigma^2$ for $p > 2$.

Theorem 3.5 makes no statement about the convergence of the Shannon sampling series for the Bernstein space $\mathcal{B}_\pi^\infty$ and the Paley–Wiener space $\mathcal{PW}_\pi^1$. As for the Bernstein space $\mathcal{B}_\pi^\infty$, it is easy to see that the Shannon sampling series (3.13) cannot reconstruct all signals in this space by considering the signal $f_1(t) = \sin(\pi t)$. For this signal we have $f_1(k) = 0$ for all $k \in \mathbb{Z}$ and consequently $(S_N f_1)(t) \equiv 0$ for all $N \in \mathbb{N}$. The convergence behavior of the Shannon sampling series for the space $\mathcal{PW}_\pi^1 \subset \mathcal{B}_\pi^\infty$ will be discussed next.

Global Convergence Behavior of the Shannon Sampling Series for $\mathcal{PW}_\pi^1$

A well-known fact [30,33,37] about the convergence behavior of the Shannon sampling series is its uniform convergence on compact subsets of $\mathbb{R}$ for all $f \in \mathcal{PW}_\pi^1$.

Theorem 3.6 (Brown's Theorem). *For all $f \in \mathcal{PW}_\pi^1$ and $\tau > 0$ fixed we have*

$$\lim_{N\to\infty} \max_{t\in[-\tau,\tau]} \left| f(t) - \sum_{k=-N}^{N} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| = 0.$$

This theorem plays a fundamental role in applications because it establishes the uniform convergence on compact subsets of $\mathbb{R}$ for a large class of signals, namely $\mathcal{PW}_\pi^1$, which is the largest space within the scale of Paley–Wiener spaces. The space $\mathcal{PW}_\pi^1$ is interesting, because this space is larger than the commonly used $\mathcal{PW}_\pi^2$-space of signals with finite energy and because the convergence behavior of sampling series for signals in $\mathcal{PW}_\pi^1$ is closely related to the convergence behavior of sampling series for bandlimited wide-sense stationary stochastic processes [74].

Although the Shannon sampling series is locally uniformly convergent for all $f \in \mathcal{PW}_\pi^1$, the series is not globally uniformly convergent in general. This could be expected for the following reason: For signals $f \in \mathcal{PW}_\pi^1$ we have $\lim_{|t|\to\infty} f(t) = 0$ by the Riemann–Lebesgue lemma. Furthermore, the finite Shannon sampling series vanishes for infinite t, too. Thus, both the signal and the partial sums are zero for infinite t. Together with the good (uniform) local convergence behavior of the Shannon sampling series, one could conjecture the same good (uniform) convergence behavior of the Shannon sampling series on the whole real axis.

However, this is not the case. The next theorem shows that the peak value $\|S_N f\|_\infty$ can increase unboundedly for certain signals $f \in \mathcal{PW}_\pi^1$ as N tends to infinity. Thus, the Shannon sampling series as a reconstruction process is not globally uniformly convergent, not even globally uniformly bounded.

Theorem 3.7. *There exists a signal $f_1 \in \mathcal{PW}_\pi^1$, such that*

$$\limsup_{N\to\infty} \max_{t\in\mathbb{R}} \left| \sum_{k=-N}^{N} f_1(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| = \infty.$$

Theorem 3.7 shows that the suggested argumentation at the beginning of this section, namely that the Shannon sampling series could be uniformly convergent on all of $\mathbb{R}$ for all $f \in \mathcal{PW}_\pi^1$, is indeed wrong. We will not prove Theorem 3.7 directly, but a more general result in Theorem 3.11, which also contains Theorem 3.7 as a special case.

3.2.3 General Reconstruction Processes

As the Shannon sampling series is not uniformly convergent for $\mathcal{PW}_\pi^1$, the question arises whether there are other reconstruction processes which are uniformly convergent on all of $\mathbb{R}$, or, at least locally uniformly convergent and globally uniformly bounded for all signals in $\mathcal{PW}_\pi^1$. In the following we will analyze this question for a very general class of reconstruction processes and show that none of the reconstruction processes in this class is globally uniformly bounded.

Certainly, for $f \in \mathcal{PW}_\pi^1$ the set $\mathbb{Z}$ is a set of uniqueness, i.e., given any two signals $f, g \in \mathcal{PW}_\pi^1$, $f(k) = g(k)$ for all $k \in \mathbb{Z}$ and implies $f \equiv g$. Thus, every signal $f \in \mathcal{PW}_\pi^1$ is uniquely determined by its samples $\{f(k)\}_{k\in\mathbb{Z}}$. But here the question is whether a locally uniformly convergent and globally uniformly bounded reconstruction is possible for $\mathcal{PW}_\pi^1$ by using only the signal samples.

For the analysis we consider reconstruction processes of the general structure

$$(Tf)(t) + \sum_{k=-\infty}^{\infty} f(k)\phi_k(t), \tag{3.16}$$

where $T : \mathcal{PW}_\pi^1 \to \mathcal{B}_\pi^\infty$ is a linear and continuous operator and $\phi_k \in \mathcal{B}_\pi^\infty$, $k \in \mathbb{Z}$. Furthermore, we make three assumptions, which the reconstruction process must satisfy:

P1) The expression (3.16) shall converge for all $t \in \mathbb{R}$ and all $f \in \mathcal{PW}_\pi^2$ to $f(t)$.

P2) $\phi_k \in \mathcal{B}_\pi^\infty$ and $\|\phi_k\|_\infty \le C_2$ for all $k \in \mathbb{Z}$, where C_2 is a positive constant.

P3) The operator $T : \mathcal{PW}_\pi^1 \to \mathcal{B}_\pi^\infty$ is linear and has the property that there exists a number $R > 0$ and a constant $C_3 > 0$, such that for all $f \in \mathcal{PW}_\pi^1$

$$\sup_{t\in\mathbb{R}} |(Tf)(t)| \le C_3 \max_{|z|\le R} |f(z)|.$$

Definition 3.8. A reconstruction process is called to be of *type* $\mathcal{P}$ if it has the structure (3.16) and satisfies the assumptions P1–P3.

Remark 3.9. Assumptions P1 and P2 are no real restrictions for the practical use of the reconstruction processes. They are easily satisfied by all relevant reconstruction processes. In contrast, assumption P3 poses a real restriction. The goal is to control the global behavior of the reconstruction process by using only finitely many local samples, which is feasible for $f \in \mathcal{PW}_\pi^2$. Therefore it is reasonable to require a good local concentration of the operator T. It is possible that there are reconstruction processes that do not satisfy assumption P3. However, all reconstruction processes that are commonly analyzed in the literature satisfy P1–P3. A necessary and sufficient condition for assumption P1 to hold is

$$\sum_{k=-\infty}^{\infty} |\phi_k(t)|^2 < \infty. \tag{3.17}$$

For the examples i)–iii) in the next section we have (3.17). For the reconstruction processes i) and ii) we even have $\sum_{k=-\infty}^{\infty} |\phi_k(t)| < \infty$.

Possible Nyquist Set Reconstruction Processes

Next, we present and analyze some reconstruction processes that satisfy the assumptions P1–P3. Series of the type i) and ii) are called Valiron interpolation series [30, p. 12] or Tschakaloff's series [39, p. 53] [26, p. 60]. A special case of the reconstruction processes i) and ii) is discussed in [26, p. 60].

i) Let $t_0 \in \mathbb{R} \setminus \mathbb{Z}$ arbitrary but fixed and consider the series

$$f(t) = f(t_0)\frac{\sin(\pi t)}{\sin(\pi t_0)} + (t - t_0) \sum_{k=-\infty}^{\infty} \frac{f(k)}{k - t_0} \frac{\sin(\pi(t-k))}{\pi(t-k)}.$$

ii) The second series of concern is, for $m \in \mathbb{Z}$,

$$\begin{aligned} f(t) = & f(m)\frac{\sin(\pi(t-m))}{\pi(t-m)} + f'(m)\frac{\sin(\pi(t-m))}{\pi} \\ & + (t-m) \sum_{\substack{k=-\infty \\ k\neq m}}^{\infty} \frac{f(k)}{k-m} \frac{\sin(\pi(t-k))}{\pi(t-k)}. \end{aligned}$$

iii) And the third is the well-known Shannon sampling series

$$f(t) = \sum_{k=-\infty}^{\infty} f(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}.$$

All those reconstruction processes possess the structure (3.16), i.e.,

$$f(t) = (Tf)(t) + \sum_{k=-\infty}^{\infty} f(k)\phi_k(t),$$

where the functions ϕ_k, $k \in \mathbb{Z}$, and the linear operator T are given by

i)
$$\phi_k^{\mathrm{a}}(t) = \left(\frac{t-t_0}{k-t_0}\right)\frac{\sin\big(\pi(t-k)\big)}{\pi(t-k)},$$

$$(T^{\mathrm{a}}f)(t) = f(t_0)\frac{\sin(\pi t)}{\sin(\pi t_0)},$$

ii)
$$\phi_k^{\mathrm{b}}(t) = \left(\frac{t-m}{k-m}\right)\frac{\sin\big(\pi(t-k)\big)}{\pi(t-k)}, \quad k \neq m,$$

$$(T^{\mathrm{b}}f)(t) = f(m)\frac{\sin\big(\pi(t-m)\big)}{\pi(t-m)} + f'(m)\frac{\sin\big(\pi(t-m)\big)}{\pi},$$

and

iii)
$$\phi_k^{\mathrm{c}}(t) = \frac{\sin\big(\pi(t-k)\big)}{\pi(t-k)},$$

$$(T^{\mathrm{c}}f)(t) \equiv 0.$$

Note, all reconstruction processes i)–iii) have the properties that

i) they are concentrated in the Nyquist points,

ii) they are locally uniformly convergent for all $f \in \mathcal{PW}_\pi^1$, and

iii) they are of type $\mathcal{P}$.

Observation 3.10. *The operators T^a, T^b and T^c have the Property P3.*

For completeness the proof is given in Appendix A.1.

Characterization of the Approximation Behavior

The next theorem states that a uniformly convergent reconstruction of signals $f \in \mathcal{PW}_\pi^1$ with reconstruction processes of type $\mathcal{P}$ is not possible in general.

Theorem 3.11. *There exists a universal signal $f_1 \in \mathcal{PW}_\pi^1$ such that for all reconstruction processes of type $\mathcal{P}$ we have*

$$\limsup_{N\to\infty} \sup_{t\in\mathbb{R}} \left|(Tf_1)(t) + \sum_{k=-N}^{N} f_1(k)\phi_k(t)\right| = \infty.$$

Theorem 3.11 applied to the Shannon sampling series gives Theorem 3.7, which shows that Brown's theorem (Theorem 3.6) cannot be extended to uniform convergence on all of $\mathbb{R}$. Thus, Theorem 3.7 is a special case of Theorem 3.11.

Proof of Theorem 3.11. Let $f \in \mathcal{PW}_\pi^1$ be fixed and consider the finite Shannon sampling series

$$(S_N f)(z) = \sum_{k=-N}^{N} f(k) \frac{\sin(\pi(z-k))}{\pi(z-k)}, \quad z \in \mathbb{C}.$$

Then, for all $f \in \mathcal{PW}_\pi^1$ and all $R > 0$, we have [33]

$$\lim_{N\to\infty} \max_{|z|\le R} |f(z) - (S_N f)(z)| = 0.$$

Since

$$(S_N f)(k) = \begin{cases} f(k), & |k| \le N \\ 0, & |k| > N, \end{cases}$$

for $k \in \mathbb{Z}$, we obtain the decomposition

$$\begin{aligned} \sum_{k=-N}^{N} & f(k)\phi_k(t) + (Tf)(t) \\ &= \sum_{k=-N}^{N} (S_N f)(k)\phi_k(t) + (Tf)(t) \\ &= \sum_{k=-N}^{N} (S_N f)(k)\phi_k(t) + (TS_N f)(t) + (Tf)(t) - (TS_N f)(t) \\ &= \sum_{k=-\infty}^{\infty} (S_N f)(k)\phi_k(t) + (TS_N f)(t) + (Tf)(t) - (TS_N f)(t) \\ &= (S_N f)(t) + (Tf)(t) - (TS_N f)(t). \end{aligned} \tag{3.18}$$

The last equality is due to Property P1 and the fact that for $f \in \mathcal{PW}_\pi^1$ the finite Shannon sampling series $S_N f$ is a signal in $\mathcal{PW}_\pi^2$. Since the sequence $(S_N f)_{N\in\mathbb{N}}$ is uniformly convergent on $\{z \in \mathbb{C} : |z| \le R\}$ for a given finite $R > 0$ and all $f \in \mathcal{PW}_\pi^1$, it follows from the Banach–Steinhaus theorem that there exists a constant $C_4 > 0$, depending only on R, such that $\max_{|z|\le R} |(S_N f)(z)| \le C_4 \|f\|_{\mathcal{PW}_\pi^1}$ for all $f \in \mathcal{PW}_\pi^1$ and all $N \in \mathbb{N}$. Together with Property P3 it follows that

$$|(TS_N f)(t)| \le C_3 \max_{|z|\le R} |(S_N f)(z)| \le C_5 \|f\|_{\mathcal{PW}_\pi^1} \tag{3.19}$$

for all $t \in \mathbb{R}$ and all $f \in \mathcal{PW}_\pi^1$. Hence, combining (3.18) and (3.19) and using Property P3 again leads to

$$\begin{aligned} \left| \sum_{k=-N}^{N} f(k)\phi_k(t) + (Tf)(t) \right| &\ge |(S_N f)(t)| - C_3 \max_{|z|\le R} |f(z)| - C_5 \|f\|_{\mathcal{PW}_\pi^1} \\ &\ge |(S_N f)(t)| - (C_3 \exp(R\pi) + C_5) \|f\|_{\mathcal{PW}_\pi^1}, \end{aligned}$$

where the last inequality follows from

$$|f(z)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{f}(\omega)||e^{\mathrm{i}\omega z}| \, \mathrm{d}\omega \leq \exp(R\pi) \, \|f\|_{\mathcal{PW}_\pi^1} \tag{3.20}$$

for $|z| \leq R$.

Next we will construct a signal $f_1 \in \mathcal{PW}_\pi^1$, such that $\limsup_{N\to\infty} \|S_N f_1\|_\infty = \infty$ and thus complete the proof. Let $M > 1$ be an arbitrary natural number and consider the function $g_M \in \mathcal{PW}_\pi^1$ with

$$\hat{g}_M(\omega) = \begin{cases} M, & \pi\left(1 - \frac{1}{M}\right) < |\omega| < \pi \\ 0, & \text{otherwise.} \end{cases}$$

Then we have $\|g_M\|_{\mathcal{PW}_\pi^1} = 1$,

$$g_M(t) = \begin{cases} 1, & t = 0 \\ \frac{M}{\pi t}\left[\sin(\pi t) - \sin\left(\pi\left(1 - \frac{1}{M}\right)t\right)\right], & t \neq 0, \end{cases}$$

and

$$g_M(k) = \begin{cases} 1, & k = 0 \\ \frac{1}{\pi}\frac{(-1)^k \sin\left(\frac{k}{M}\pi\right)}{\frac{k}{M}}, & k \neq 0, \end{cases}$$

after using the identity $\sin(\pi(t-k)) = \sin(\pi t)(-1)^k$ for $t \in \mathbb{R}$, $k \in \mathbb{Z}$. Next, the partial sum $S_N g_M$, given by

$$\begin{aligned}(S_N g_M)(t) &= \sum_{k=-N}^{N} g_M(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \\ &= \frac{\sin(\pi t)}{\pi}\left(\frac{1}{t} + \sum_{\substack{k=-N \\ k\neq 0}}^{N} \frac{\sin\left(\frac{k}{M}\pi\right)}{\frac{k}{M}\pi}\frac{1}{t-k}\right),\end{aligned}$$

is analyzed. For $t = N + 1/2$ we obtain

$$(S_N g_M)\left(N + \tfrac{1}{2}\right) = \frac{\sin\left(\left(N + \frac{1}{2}\right)\pi\right)}{\pi}\left(\frac{1}{\left(N + \frac{1}{2}\right)} + \sum_{\substack{k=-N \\ k\neq 0}}^{N} \frac{\sin\left(\frac{k}{M}\pi\right)}{\frac{k}{M}\pi\left(N + \frac{1}{2} - k\right)}\right). \tag{3.21}$$

First, the case $1 \leq N \leq M/2$ is considered. The term in parentheses on the right-hand side of (3.21) can be bounded from below by using the inequality

$$\frac{\sin\left(\frac{k}{M}\pi\right)}{\frac{k}{M}\pi} \geq \frac{\sin\left(\frac{N}{M}\pi\right)}{\frac{N}{M}\pi} \geq \frac{\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} = \frac{2}{\pi},$$

which holds for $|k| \leq N$. We have

$$\begin{aligned}
\frac{1}{\left(N+\frac{1}{2}\right)} + \sum_{\substack{k=-N \\ k\neq 0}}^{N} \frac{\sin\left(\frac{k}{M}\pi\right)}{\frac{k}{M}\pi\left(N+\frac{1}{2}-k\right)} &> \frac{2}{\pi\left(N+\frac{1}{2}\right)} + \frac{2}{\pi}\sum_{\substack{k=-N \\ k\neq 0}}^{N} \frac{1}{N+\frac{1}{2}-k} \\
&= \frac{2}{\pi}\sum_{k=0}^{2N} \frac{1}{k+\frac{1}{2}} \\
&> \frac{2}{\pi}\sum_{k=0}^{2N} \int_k^{k+1} \frac{1}{\tau+\frac{1}{2}}\,\mathrm{d}\tau \\
&= \frac{2}{\pi}\int_0^{2N+1} \frac{1}{\tau+\frac{1}{2}}\,\mathrm{d}\tau \\
&> \frac{2}{\pi}\log\left(4N+1\right).
\end{aligned}$$

If N is even, then $\sin((N+1/2)\pi) = 1$, and we obtain

$$(S_N g_M)\left(N+\tfrac{1}{2}\right) > \frac{1}{\pi^3}\log(4N+1), \tag{3.22}$$

for all $1 \leq N \leq M/2$.

Now, we examine the behavior of $S_N g_M$ for $t = N + 1/2$ and $M \leq N$. The sum in equation (3.21) can be bounded from above by

$$\sum_{\substack{k=-N \\ k\neq 0}}^{N} \frac{\sin\left(\frac{k}{M}\pi\right)}{\frac{k}{M}\pi\left(N+\frac{1}{2}-k\right)} \leq \frac{M}{\pi\left(N+\frac{1}{2}\right)} \sum_{\substack{k=-N \\ k\neq 0}}^{N} \left(\frac{1}{k} + \frac{1}{N+\frac{1}{2}-k}\right).$$

But since

$$\sum_{\substack{k=-N \\ k\neq 0}}^{N} \frac{1}{|k|} = 2\sum_{k=1}^{N}\frac{1}{k} < 2 + 2\sum_{k=2}^{N}\int_{k-1}^{k}\frac{1}{\tau}\,\mathrm{d}\tau < 2 + 2\log(N+1),$$

and

$$\begin{aligned}
\sum_{\substack{k=-N \\ k\neq 0}}^{N} \frac{1}{N+\frac{1}{2}-k} < \sum_{k=0}^{2N}\frac{1}{k+\frac{1}{2}} &\leq 2 + \sum_{k=1}^{2N}\int_{k-1}^{k}\frac{1}{\tau+\frac{1}{2}}\,\mathrm{d}\tau \\
&= 2 + \int_0^{2N}\frac{1}{\tau+\frac{1}{2}}\,\mathrm{d}\tau = 2 + \log(4N+1) \\
&< 2 + 4\log(N+1),
\end{aligned} \tag{3.23}$$

it follows that

$$\left|(S_N g_M)\left(N+\tfrac{1}{2}\right)\right| < \frac{1}{N+\frac{1}{2}} + \frac{M}{\pi\left(N+\frac{1}{2}\right)}(4+6\log(N+1)). \tag{3.24}$$

In order to continue the proof, we need to define the sequence $M_k = 2^{(k^2)}$, $k \in \mathbb{N}$, and the signal

$$f_1(t) = \sum_{k=1}^{\infty} \frac{1}{k^{\frac{3}{2}}} g_{M_k}(t).$$

First note that $f_1 \in \mathcal{PW}_\pi^1$ because

$$\|f_1\|_{\mathcal{PW}_\pi^1} \leq \sum_{k=1}^{\infty} \frac{1}{k^{\frac{3}{2}}} \|g_{M_k}\|_{\mathcal{PW}_\pi^1} \leq \sum_{k=1}^{\infty} \frac{1}{k^{\frac{3}{2}}} < \infty.$$

Now, let $l \in \mathbb{N}$, arbitrary. Then

$$\begin{aligned}
\left|(S_{M_l} f_1)\left(M_l + \tfrac{1}{2}\right)\right| &= \left|\sum_{k=l}^{\infty} \frac{1}{k^{\frac{3}{2}}} (S_{M_l} g_{M_k})\left(M_l + \tfrac{1}{2}\right) + \sum_{k=1}^{l-1} \frac{1}{k^{\frac{3}{2}}} (S_{M_l} g_{M_k})\left(M_l + \tfrac{1}{2}\right)\right| \\
&\geq \left|\sum_{k=l}^{\infty} \frac{1}{k^{\frac{3}{2}}} (S_{M_l} g_{M_k})\left(M_l + \tfrac{1}{2}\right)\right| - \sum_{k=1}^{l-1} \frac{1}{k^{\frac{3}{2}}} \left|(S_{M_l} g_{M_k})\left(M_l + \tfrac{1}{2}\right)\right| \\
&\geq \sum_{k=l+1}^{\infty} \frac{1}{k^{\frac{3}{2}}} (S_{M_l} g_{M_k})\left(M_l + \tfrac{1}{2}\right) - \sum_{k=1}^{l-1} \frac{1}{k^{\frac{3}{2}}} \left|(S_{M_l} g_{M_k})\left(M_l + \tfrac{1}{2}\right)\right|.
\end{aligned} \tag{3.25}$$

The last inequality is due to the fact, that $(S_{M_l} g_{M_k})\left(M_l + \frac{1}{2}\right)$ is non-negative for all $k \geq l$. For $k \leq l-1$ we get from equation (3.24) that

$$\begin{aligned}
\left|(S_{M_l} g_{M_k})\left(M_l + \tfrac{1}{2}\right)\right| &< \frac{1}{2^{(l^2)} + \frac{1}{2}} + \frac{2^{(k^2)}}{\pi\left(2^{(l^2)} + \frac{1}{2}\right)} \left(4 + 6 \log\left(2^{(l^2)} + 1\right)\right) \\
&< 2 + \frac{2^{k^2 - l^2}}{\pi} \left(4 + 6[\log(2) + l^2 \log(2)]\right) \\
&\leq 2 + l^2 \frac{2^{(l-1)^2 - l^2}}{\pi} (4 + 12 \log(2)) \\
&= 2 + \frac{l^2\ 2\ (4 + 12 \log(2))}{2^{2l}\ \pi} \\
&\leq 2 + \frac{2}{\pi} (4 + 12 \log(2)) =: C_6,
\end{aligned} \tag{3.26}$$

where the last inequality is due to $l^2/2^{2l} \leq 1$.

On the other hand, for $k \geq l+1$ we can use equation (3.22) because M_l is even to obtain

$$(S_{M_l} g_{M_k})\left(M_l + \tfrac{1}{2}\right) > \frac{\log\left(4 \cdot 2^{(l^2)} + 1\right)}{\pi^3} > l^2 \frac{\log(2)}{\pi^3}. \tag{3.27}$$

Inserting (3.26) and (3.27) into equation (3.25) gives

$$\left|(S_{M_l} f_1)\left(M_l + \tfrac{1}{2}\right)\right| > l^2 \frac{\log(2)}{\pi^3} \sum_{k=l}^{\infty} \frac{1}{k^{\frac{3}{2}}} - C_6 \sum_{k=1}^{l-1} \frac{1}{k^{\frac{3}{2}}}.$$

But since

$$\sum_{k=l}^{\infty} \frac{1}{k^{\frac{3}{2}}} > \sum_{k=l}^{\infty} \int_{k}^{k+1} \frac{1}{\tau^{\frac{3}{2}}} \,\mathrm{d}\tau = \int_{l}^{\infty} \frac{1}{\tau^{\frac{3}{2}}} \,\mathrm{d}\tau = \frac{2}{\sqrt{l}}$$

and

$$\sum_{k=1}^{l-1} \frac{1}{k^{\frac{3}{2}}} < 1 + \sum_{k=2}^{l-1} \int_{k-1}^{k} \frac{1}{\tau^{\frac{3}{2}}} \,\mathrm{d}\tau = 1 + 2\left(1 - \frac{1}{\sqrt{l-1}}\right) < 3$$

we arrive at

$$\left|(S_{M_l} f_1)\left(M_l + \tfrac{1}{2}\right)\right| \geq l^{\frac{3}{2}} \frac{2\log(2)}{\pi^3} - 3C_6. \tag{3.28}$$

Finally, because (3.28) is valid for all l, it follows $\limsup_{N\to\infty} \|S_N f_1\|_\infty = \infty$, which concludes the proof. □

Remark 3.12. Despite all ϕ_k in the examples i)–iii) on page 26 have the interpolation property

$$\phi_k(l) = \begin{cases} 1 & l = k \\ 0 & l \neq k, \end{cases}$$

it is important to note that throughout the proof we do not require it. Therefore the divergence behavior of the reconstruction processes is not a consequence of the interpolation property.

Upper and Lower Bounds

In this section the behavior of the reconstruction processes is further examined. We will analyze the influence of N on the peak value of the finite sampling series and derive a lower and an upper bound. In the proof of Theorem 3.13 we will employ the same decomposition (3.18) that was used in the proof of Theorem 3.11.

Theorem 3.13. *Given a reconstruction process of type $\mathcal{P}$, then there exist three constants C_3, C_5, $R > 0$, such that*

i) for all $N \in \mathbb{N}$ and $f \in \mathcal{PW}_\pi^1$

$$\sup_{t\in\mathbb{R}} \left|(Tf)(t) + \sum_{k=-N}^{N} f(k)\phi_k(t)\right| < \left[2 + \frac{2}{\pi} + \frac{2}{\pi}\log(2N) + C_3\exp(R\pi) + C_5\right] \|f\|_{\mathcal{PW}_\pi^1},$$

ii) for all $N \in \mathbb{N}$

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1}=1} \sup_{t\in\mathbb{R}} \left|(Tf)(t) + \sum_{k=-N}^{N} f(k)\phi_k(t)\right| > \frac{1}{\pi}\log(4N+1) - (C_3\exp(R\pi) + C_5).$$

Remark 3.14. In Theorem 3.11 we had one single signal f_1, such that the divergence was universal for all reconstruction processes of type $\mathcal{P}$. In contrast, the constants C_3, C_5 and R in Theorem 3.13 depend on the specific process used for the reconstruction, i.e., they are not universal.

Proof of Theorem 3.13. As a consequence of the equations (3.18), (3.19) and (3.20) and Property P3 it follows for $f \in \mathcal{PW}_\pi^1$ that

$$\left|(Tf)(t) + \sum_{k=-N}^{N} f(k)\phi_k(t)\right| \leq |(S_N f)(t)| + (C_3 \exp(R\pi) + C_5)\|f\|_{\mathcal{PW}_\pi^1} \tag{3.29}$$

and

$$\left|(Tf)(t) + \sum_{k=-N}^{N} f(k)\phi_k(t)\right| \geq |(S_N f)(t)| - (C_3 \exp(R\pi) + C_5)\|f\|_{\mathcal{PW}_\pi^1}. \tag{3.30}$$

Hence it is sufficient to analyze $S_N f$. We have

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1}=1} \|S_N f\|_\infty \geq \|S_N g_M\|_\infty \geq \left|(S_N g_M)\left(N + \tfrac{1}{2}\right)\right|,$$

where g_M, $M > 1$, are the functions that were defined in the proof of Theorem 3.11 on page 29. Due to equation (3.21) it follows that

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1}=1} \|S_N f\|_\infty \geq \frac{1}{\pi\left(N + \frac{1}{2}\right)} + \frac{1}{\pi} \sum_{\substack{k=-N \\ k\neq 0}}^{N} \frac{\sin\left(\frac{k}{M}\pi\right)}{\frac{k}{M}\pi\left(N + \frac{1}{2} - k\right)}$$

for all $M > 1$. In the limit $M \to \infty$ we obtain

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1}=1} \|S_N f\|_\infty \geq \frac{1}{\pi} \sum_{k=-N}^{N} \frac{1}{N + \frac{1}{2} - k} > \frac{1}{\pi} \log(4N + 1).$$

Thus, assertion ii) follows immediately from (3.30).

On the other hand we have

$$\begin{aligned}
|(S_N f)(t)| &= \left|\sum_{k=-N}^{N} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}\right| \\
&\leq \max_{-N\leq k\leq N} |f(k)| \sum_{k=-N}^{N} \left|\frac{\sin(\pi(t-k))}{\pi(t-k)}\right| \\
&\leq \|f\|_{\mathcal{PW}_\pi^1} \sum_{k=-N}^{N} \left|\frac{\sin(\pi(t-k))}{\pi(t-k)}\right|.
\end{aligned}$$

For $t \in \mathbb{Z}$, $|(S_N f)(t)| = |f(k)| \leq \|f\|_{\mathcal{PW}_\pi^1}$ and for $t \in \mathbb{R} \setminus \mathbb{Z}$, three cases have to be distinguished: 1) $t > N$, 2) $-N < t < N$ and 3) $t < -N$.

1) For $t > N$ we have

$$\begin{aligned}\sum_{k=-N}^{N} \left| \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| &< 1 + \sum_{k=-N}^{N-1} \left| \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| \\ &\leq 1 + \frac{1}{\pi} \sum_{k=-N}^{N-1} \frac{1}{t-k} \leq 1 + \frac{1}{\pi} \sum_{k=-N}^{N-1} \frac{1}{N-k} \\ &= 1 + \frac{1}{\pi} \sum_{k=1}^{2N} \frac{1}{k} < 1 + \frac{1}{\pi} + \frac{1}{\pi} \log(2N),\end{aligned}$$

where the last inequality is obtained by the same technique as in (3.23).

2) For $-N < t < N$ let $L^{(1)}(t)$ be the largest integer k with $k < t$ and $L^{(2)}(t)$ the smallest integer k with $k > t$. Then we have

$$\begin{aligned}\sum_{k=-N}^{N} \left| \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| &= \sum_{k=-N}^{L^{(1)}(t)-1} \left| \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| + \left| \frac{\sin(\pi(t-L^{(1)}(t)))}{\pi(t-L^{(1)}(t))} \right| \\ &\quad + \left| \frac{\sin(\pi(t-L^{(2)}(t)))}{\pi(t-L^{(2)}(t))} \right| + \sum_{k=L^{(2)}(t)+1}^{N} \left| \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| \\ &\leq 2 + \frac{1}{\pi} \left(\sum_{k=-N}^{L^{(1)}(t)-1} \frac{1}{t-k} + \sum_{k=L^{(2)}(t)+1}^{N} \frac{1}{k-t} \right) \\ &\leq 2 + \frac{1}{\pi} \left(\sum_{k=-N}^{L^{(1)}(t)-1} \frac{1}{L^{(1)}(t)-k} + \sum_{k=L^{(2)}(t)+1}^{N} \frac{1}{k-L^{(2)}(t)} \right) \\ &= 2 + \frac{1}{\pi} \left(\sum_{k=1}^{L^{(1)}(t)+N} \frac{1}{k} + \sum_{k=1}^{N-L^{(2)}(t)} \frac{1}{k} \right) \\ &\leq 2 + \frac{2}{\pi} \sum_{k=1}^{2N} \frac{1}{k} < 2 + \frac{2}{\pi} + \frac{2}{\pi} \log(2N),\end{aligned}$$

where the last inequality is obtained analogously to (3.23).

3) For $t < -N$ the same estimate holds as in 1).

It follows that

$$|(S_N f)(t)| \leq \left(2 + \frac{2}{\pi} + \frac{2}{\pi} \log(2N) \right) \|f\|_{\mathcal{PW}_\pi^1}$$

for all $t \in \mathbb{R}$ and $N \in \mathbb{N}$. This, together with (3.29), proves assertion i). □

Using the following definition we can characterize the asymptotic behavior of the reconstruction process.

Definition 3.15. We call two functions $f(t)$ and $g(t)$ asymptotically equivalent $f(t) \sim g(t)$ for $t \to \infty$ if there exists a t_0 and two positive constants A and B such that $Ag(t) \le f(t) \le Bg(t)$ for all $t > t_0$.

Corollary 3.16. *Given a reconstruction process of type $\mathcal{P}$, then*

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1}=1} \sup_{t\in\mathbb{R}} \left| (Tf)(t) + \sum_{k=-N}^{N} f(k)\phi_k(t) \right| \sim \log N.$$

Discussion

We have shown in Theorem 3.11 that there exists a universal signal f_1, such that the peak value of all reconstruction schemes of type $\mathcal{P}$, in particular, the processes i)–iii) on page 26, diverges unboundedly. Although the set $\mathbb{Z}$ is a set of uniqueness for $\mathcal{PW}_\pi^1$, it is not sufficient to have the samples on this set for a locally uniformly convergent and globally uniformly bounded signal reconstruction.

This result may be disappointing. However, there is a solution to this dilemma. If the requirement of sampling at the Nyquist rate is relaxed, i.e. if we allow for oversampling, then a uniformly convergent reconstruction is possible. Of course this nice convergence behavior does not come for free. Oversampling in real applications always comes with the price of storing and processing more samples. We will treat oversampling in Section 3.2.4.

Further, a modified series that is symmetric around t converges uniformly on the whole real axis even at Nyquist rate. This centered sampling series will be analyzed in Section 3.2.9.

3.2.4 Signal Reconstruction with Oversampling

In this section the stability of reconstruction processes with oversampling is analyzed. Oversampling creates a degree of freedom in the choice of the reconstruction kernel [75], and if the kernel is chosen appropriately then, in general, the convergence behavior of the reconstruction process is improved [42, 43, 45, 76].

There are other topics of signal theory where oversampling is essential. One example is the estimation of the peak value $\|f\|_\infty$ of a signal $f \in \mathcal{B}_\pi^\infty$ by its samples on the lattice k/a, $a > 1$, $k \in \mathbb{Z}$ [77]. The best possible estimate is given by

$$\|f\|_\infty \le \frac{1}{\cos\left(\frac{\pi}{2a}\right)} \sup_{k\in\mathbb{Z}} \left| f\left(\frac{k}{a}\right) \right|. \tag{3.31}$$

It is interesting to note that the expression $1/\cos(\pi/(2a))$ will also appear in the analysis of the Shannon sampling series with oversampling.

As we will see, application of oversampling leads to a uniformly convergent reconstruction processes for all signals in $\mathcal{PW}_\pi^1$. Further, it will turn out that an elaborate kernel design is not necessary as far as only convergence is important.

In Theorem 3.17 we will see that even the Shannon sampling series with slightly increased bandwidth is uniformly convergent on all of $\mathbb{R}$ for all signals in $\mathcal{PW}_\pi^1$ when oversampling is applied.

Theorem 3.17. *Let $a > 1$ be fixed. Then for all $f \in \mathcal{PW}_\pi^1$ we have*

$$\lim_{N\to\infty} \max_{t\in\mathbb{R}} \left| f(t) - \sum_{k=-N}^{N} f\left(\frac{k}{a}\right) \frac{\sin(a\pi(t-\frac{k}{a}))}{a\pi(t-\frac{k}{a})} \right| = 0.$$

In order to prove Theorem 3.17 we need the Lemmas 3.18 and 3.19.

Lemma 3.18. *Let $a > 1$ be fixed. Then for all $f \in \mathcal{PW}_\pi^1$ and all $N \in \mathbb{N}$ we have*

$$\max_{t\in\mathbb{R}} \left| \sum_{k=-N}^{N} f\left(\frac{k}{a}\right) \frac{\sin(a\pi(t-\frac{k}{a}))}{a\pi(t-\frac{k}{a})} \right| \le 2\left(1 + \frac{2}{\pi\cos(\frac{\pi}{2a})}\right) \|f\|_{\mathcal{PW}_\pi^1}.$$

Proof. We have

$$\begin{aligned} \left| \sum_{k=-N}^{N} f\left(\frac{k}{a}\right) \frac{\sin(a\pi(t-\frac{k}{a}))}{a\pi(t-\frac{k}{a})} \right| &\le \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{f}(\omega)| \left| \sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\omega k/a} \frac{\sin(a\pi(t-\frac{k}{a}))}{a\pi(t-\frac{k}{a})} \right| \mathrm{d}\omega \\ &\le \max_{\omega\in[-\pi,\pi]} |g_N(t,\omega,a)| \, \|f\|_{\mathcal{PW}_\pi^1} \\ &\le 2\left(1 + \frac{2}{\pi\cos(\frac{\pi}{2a})}\right) \|f\|_{\mathcal{PW}_\pi^1}, \end{aligned}$$

where we introduced the abbreviation

$$g_N(t,\omega,a) := \sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\omega k/a} \frac{\sin(a\pi(t-\frac{k}{a}))}{a\pi(t-\frac{k}{a})} \tag{3.32}$$

and used Lemma 3.19 for the last inequality. □

Lemma 3.19. *For $g_N(t,\omega,a)$ as defined in* (3.32) *and all $a > 1$, $t \in \mathbb{R}$, $\omega \in [-\pi,\pi]$, and $N \in \mathbb{N}$ we have*

$$|g_N(t,\omega,a)| \le 2\left(1 + \frac{2}{\pi\cos(\frac{\pi}{2a})}\right).$$

Proof. Let $\omega \in [-\pi,\pi]$ and $a > 1$ be arbitrary but fixed. Only the case $t > 0$ has to be analyzed. For $t = 0$ we have $g_N(0,\omega,a) = 1$ and the case $t < 0$ is treated analogously to the case $t > 0$. If $t \in \mathcal{A} := \{k/a : k \in \mathbb{N}\}$ then

$$g_N(t,\omega,a) = \begin{cases} \mathrm{e}^{\mathrm{i}\omega t}, & |t| \le N/a \\ 0, & |t| > N/a, \end{cases}$$

and therefore $|g_N(t,\omega,a)| \le 1$. Hence, $t > 0$, $t \notin \mathcal{A}$ can be assumed. Let $N \in \mathbb{N}$ be arbitrary but fixed. Obviously,

$$\begin{aligned} g_N(t,\omega,a) &= \frac{\sin(a\pi t)}{a\pi} \sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}k(\omega/a+\pi)} \ \frac{1}{t-\frac{k}{a}} \\ &= \frac{\sin(a\pi t)}{a\pi} \sum_{k=-N}^{N} c_k \ d_k, \end{aligned} \tag{3.33}$$

where $c_k = \mathrm{e}^{\mathrm{i}k(\omega/a+\pi)}$ and $d_k = 1/(t-k/a)$. Let $C_k = \sum_{l=-N}^{k} c_l$, $|k| \le N$. Then we have

$$\begin{aligned} |C_k| &= \left|\sum_{l=-N}^{k} \mathrm{e}^{\mathrm{i}l(\omega/a+\pi)}\right| = \left|\mathrm{e}^{-\mathrm{i}N(\omega/a+\pi)} \sum_{l=0}^{N+k} \mathrm{e}^{\mathrm{i}l(\omega/a+\pi)}\right| \\ &= \left|\frac{1-\mathrm{e}^{\mathrm{i}(N+k+1)(\omega/a+\pi)}}{1-\mathrm{e}^{\mathrm{i}(\omega/a+\pi)}}\right| \le \frac{2}{\left|1+\mathrm{e}^{\mathrm{i}\omega/a}\right|} \\ &\le \frac{1}{\cos\left(\frac{\pi}{2a}\right)}. \end{aligned}$$

We begin with the case $t > (N+1)/a$. Summation by parts gives

$$\begin{aligned} \left|\sum_{k=-N}^{N} c_k \ d_k\right| &\le |C_N d_N| + \sum_{k=-N}^{N-1} |C_k(d_k - d_{k+1})| \\ &\le \frac{1}{\cos\left(\frac{\pi}{2a}\right)} \left(\frac{1}{t-\frac{N}{a}} + \sum_{k=-N}^{N-1} \left|\frac{1}{t-\frac{k}{a}} - \frac{1}{t-\frac{k+1}{a}}\right|\right) \\ &= \frac{1}{\cos\left(\frac{\pi}{2a}\right)} \left(\frac{1}{t-\frac{N}{a}} + \sum_{k=-N}^{N-1} \left(\frac{1}{t-\frac{k+1}{a}} - \frac{1}{t-\frac{k}{a}}\right)\right) \\ &\le \frac{1}{\cos\left(\frac{\pi}{2a}\right)} \left(a + \sum_{k=-N}^{N-1} \left(\frac{1}{t-\frac{k+1}{a}} - \frac{1}{t-\frac{k}{a}}\right)\right). \end{aligned} \tag{3.34}$$

The right-hand side of (3.34) can be further simplified by evaluating the telescoping series

$$\sum_{k=-N}^{N-1} \left(\frac{1}{t-\frac{k+1}{a}} - \frac{1}{t-\frac{k}{a}}\right) = \frac{1}{t-\frac{N}{a}} - \frac{1}{t+\frac{N}{a}} \le a \tag{3.35}$$

for $t > (N+1)/a$. Combining equations (3.33), (3.34) and (3.35) leads to

$$|g_N(t,\omega,a)| \le \frac{2}{\pi\cos\left(\frac{\pi}{2a}\right)}.$$

Next, the case $t < (N+1)/a$ is treated. Let N_t be the largest natural number such that $N_t/a < t$. Then

$$g_N(t,\omega,a) = \frac{\sin(a\pi t)}{a\pi}\left(\sum_{k=-N}^{N_t-1} c_k d_k + \mathrm{e}^{\mathrm{i}N_t(\frac{\omega}{a}+\pi)}\frac{1}{t-\frac{N_t}{a}} \right.$$
$$\left. + \mathrm{e}^{\mathrm{i}(N_t+1)(\frac{\omega}{a}+\pi)}\frac{1}{t-\frac{N_t+1}{a}} + \sum_{k=N_t+2}^{N} c_k d_k\right)$$

and

$$|g_N(t,\omega,a)| \le \frac{1}{a\pi}\left(\left|\sum_{k=-N}^{N_t-1} c_k d_k\right| + \left|\sum_{k=N_t+2}^{N} c_k d_k\right|\right) + 2. \tag{3.36}$$

The fist sum on the right-hand side of equation (3.36) can be bounded from above by

$$\left|\sum_{k=-N}^{N_t-1} c_k d_k\right| \le \frac{2a}{\cos(\frac{\pi}{2a})}$$

exactly in the same way as before and the second sum by

$$\left|\sum_{k=N_t+2}^{N} c_k d_k\right| \le \frac{2a}{\cos(\frac{\pi}{2a})}.$$

This completes the proof. □

Now we are in the position to prove Theorem 3.17.

Proof of Theorem 3.17. Let $f \in \mathcal{PW}_\pi^1$ be arbitrary but fixed and $\epsilon > 0$. Then there exists a $f_\epsilon \in \mathcal{PW}_\pi^2$ such that $f_\epsilon(k)$ is different from zero for only finitely many $k \in \mathbb{Z}$ and $\|f - f_\epsilon\|_{\mathcal{PW}_\pi^1} < \epsilon$. Obviously, $f_\epsilon \in \mathcal{PW}_\pi^2 \subset \mathcal{PW}_{a\pi}^2$. Therefore, there exists a $N_0 = N_0(\epsilon)$ such that

$$\max_{t\in\mathbb{R}} \left| f_\epsilon(t) - \sum_{k=-N}^{N} f_\epsilon\left(\frac{k}{a}\right)\frac{\sin(a\pi(t-\frac{k}{a}))}{a\pi(t-\frac{k}{a})}\right| < \epsilon$$

for all $N \geq N_0$. Moreover, for all $N \geq N_0$ we have

$$\begin{aligned}
&\left| f(t) - \sum_{k=-N}^{N} f\left(\frac{k}{a}\right) \frac{\sin\left(a\pi\left(t-\frac{k}{a}\right)\right)}{a\pi\left(t-\frac{k}{a}\right)} \right| \\
&\qquad \leq \left| f(t) - f_\epsilon(t) + f_\epsilon(t) - \sum_{k=-N}^{N} f_\epsilon\left(\frac{k}{a}\right) \frac{\sin\left(a\pi\left(t-\frac{k}{a}\right)\right)}{a\pi\left(t-\frac{k}{a}\right)} \right. \\
&\qquad\qquad \left. + \sum_{k=-N}^{N} \left(f_\epsilon\left(\frac{k}{a}\right) - f\left(\frac{k}{a}\right) \right) \frac{\sin\left(a\pi\left(t-\frac{k}{a}\right)\right)}{a\pi\left(t-\frac{k}{a}\right)} \right| \\
&\qquad < \epsilon + \epsilon + 2\epsilon \left(1 + \frac{2}{\pi \cos\left(\frac{\pi}{2a}\right)}\right) \\
&\qquad = 4\epsilon \left(1 + \frac{1}{\pi \cos\left(\frac{\pi}{2a}\right)}\right),
\end{aligned} \tag{3.37}$$

where Lemma 3.18, applied to $f_\epsilon - f$, has been used for the last inequality. Since (3.37) holds for all $\epsilon > 0$ the proof is complete. □

A crucial part in the proof was Lemma 3.18, which can be used to analyze the influence oversampling on the peak value of the partial sum of the Shannon sampling series. It is interesting to note that the $1/\cos(\pi/(2a))$ term from equation (3.31) reappears as integral part of Lemma 3.18.

Convergence Speed

Theorem 3.17 shows that if oversampling is used we can have global uniform convergence for all $f \in \mathcal{PW}_\pi^1$ even with the Shannon sampling series. An interesting question concerns the rate of convergence of the sampling series: Given some $a > 1$ and $f \in \mathcal{PW}_\pi^1$, can we find two constants $\gamma = \gamma(f, a) > 0$ and $C_7 = C_7(f, a) < \infty$ such that

$$\max_{t \in \mathbb{R}} \left| f(t) - \sum_{k=-N}^{N} f\left(\frac{k}{a}\right) \frac{\sin\left(a\pi\left(t-\frac{k}{a}\right)\right)}{a\pi\left(t-\frac{k}{a}\right)} \right| \leq C_7 N^{-\gamma}?$$

Is it even possible to find a γ independently of f? The answer to both questions is given by the next theorem.

Theorem 3.20. *Let $a > 1$ be fixed. For each arbitrary sequence $\{\epsilon_N\}_{N\in\mathbb{N}}$ of positive numbers that converges to zero, there exists a $f \in \mathcal{PW}_\pi^1$ such that*

$$\limsup_{N\to\infty} \frac{1}{\epsilon_N} \left(\max_{t\in\mathbb{R}} \left| f(t) - \sum_{k=-N}^{N} f\left(\frac{k}{a}\right) \frac{\sin\left(a\pi\left(t-\frac{k}{a}\right)\right)}{a\pi\left(t-\frac{k}{a}\right)} \right| \right) = \infty.$$

Proof. For $N \in \mathbb{N}$ we introduce the operators $R_N : \mathcal{PW}_\pi^1 \to \mathcal{B}_\pi^\infty$,

$$(R_N f)(t) := f(t) - \sum_{k=-N}^{N} f\left(\frac{k}{a}\right) \frac{\sin\left(a\pi\left(t-\frac{k}{a}\right)\right)}{a\pi\left(t-\frac{k}{a}\right)}.$$

Obviously we have

$$\begin{aligned}
\|R_N\| &= \sup_{\|f\|_{\mathcal{PW}_\pi^1}\le 1} \|R_N f\|_\infty \\
&\ge \sup_{\|f\|_{\mathcal{PW}_\pi^1}\le 1} |(R_N f)(t)| \\
&= \sup_{\|f\|_{\mathcal{PW}_\pi^1}\le 1} \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}(\omega)\left(\mathrm{e}^{\mathrm{i}\omega t} - \sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\omega k/a} \frac{\sin\left(a\pi\left(t-\frac{k}{a}\right)\right)}{a\pi\left(t-\frac{k}{a}\right)}\right) \mathrm{d}\omega \right| \\
&= \max_{|\omega|\le\pi} \left| \mathrm{e}^{\mathrm{i}\omega t} - \sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\omega k/a} \frac{\sin\left(a\pi\left(t-\frac{k}{a}\right)\right)}{a\pi\left(t-\frac{k}{a}\right)} \right|. \qquad (3.38)
\end{aligned}$$

Since (3.38) is valid for all $t \in \mathbb{R}$, we can choose $t = (N+1)/a$. Then we have

$$\sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\omega k/a} \frac{\sin\left(a\pi\left(t-\frac{k}{a}\right)\right)}{a\pi\left(t-\frac{k}{a}\right)} = 0$$

and consequently

$$\|R_N\| = \sup_{\|f\|_{\mathcal{PW}_\pi^1}\le 1} \|R_N f\|_\infty \ge 1.$$

Moreover, the operators $\tilde{R}_N : \mathcal{PW}_\pi^1 \to \mathcal{B}_\pi^\infty$, $N \in \mathbb{N}$, defined by

$$(\tilde{R}_N f)(t) := \frac{1}{\epsilon_N}(R_N f)(t),$$

are linear and bounded, and we have

$$\lim_{N\to\infty} \|\tilde{R}_N\| \ge \lim_{N\to\infty} \frac{1}{\epsilon_N} = \infty.$$

Hence, by the Banach–Steinhaus theorem [78, p. 98] there exists a signal $f_1 \in \mathcal{PW}_\pi^1$ such that

$$\limsup_{N\to\infty} \|\tilde{R}_N f_1\|_\infty = \infty,$$

which completes the proof. □

Theorem 3.20 shows that the convergence speed of the Shannon sampling series with oversampling for the space $\mathcal{PW}_\pi^1$ can be arbitrarily slow and that no convergence rates can be given.

3.2.5 A Sufficient Condition for Uniform Convergence Without Oversampling

In Theorem 3.7 it has been shown that, for some $f_1 \in \mathcal{PW}_\pi^1$, the Shannon sampling series

$$\sum_{k=-\infty}^{\infty} f_1(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}$$

does not converge uniformly on all of $\mathbb{R}$ to f_1. Next, we will use the results from Section 3.2.4 to analyze the convergence behavior of the Shannon sampling series for signals in $\mathcal{PW}_\pi^1$ without oversampling and to find a sufficient condition for the uniform convergence of the reconstruction process. If $\hat{f}(\omega)$ satisfies certain integrability conditions in the vicinity of $\omega = \pm\pi$ then the Shannon sampling series without oversampling converges uniformly on all of $\mathbb{R}$.

Theorem 3.21. *If $f \in \mathcal{PW}_\pi^1$ has the property that there exists a $\delta > 0$ and a $p > 1$, such that*

$$\int_{\pi-\delta \le |\omega| \le \pi} |\hat{f}(\omega)|^p \, \mathrm{d}\omega < \infty,$$

then we have

$$\lim_{N\to\infty} \max_{t\in\mathbb{R}} \left| f(t) - \sum_{k=-N}^{N} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| = 0.$$

Proof. Two auxiliary bandlimited signals f_1 and f_2 defined by

$$\hat{f}_1(\omega) = \begin{cases} \hat{f}(\omega), & |\omega| < \pi - \delta, \\ 0, & |\omega| \ge \pi - \delta \end{cases}$$

and

$$\hat{f}_2(\omega) = \begin{cases} \hat{f}(\omega), & \pi - \delta \le |\omega| \le \pi, \\ 0, & |\omega| \in \mathbb{R}\backslash[\pi - \delta, \pi] \end{cases}$$

are needed for the proof. Obviously, $f_1 \in \mathcal{PW}_{\pi-\delta}^1$ and $f(t) = f_1(t) + f_2(t)$, $t \in \mathbb{R}$. Furthermore, by assumption we have that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{f}_2(\omega)|^p \, \mathrm{d}\omega < \infty.$$

Therefore, $f_2 \in \mathcal{PW}_\pi^p$. Let $\epsilon > 0$ be arbitrarily chosen. Then there exists a $N_0 = N_0(\epsilon)$, such that

$$\max_{t\in\mathbb{R}} \left| f_2(t) - \sum_{k=-N}^{N} f_2(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| < \epsilon$$

for all $N \geq N_0$. Since $f_1 \in \mathcal{PW}^1_{\pi-\delta}$, there exists, according to Theorem 3.17, a $N_1 = N_1(\epsilon)$, such that

$$\max_{t\in\mathbb{R}} \left| f_1(t) - \sum_{k=-N}^{N} f_1(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| < \epsilon$$

for all $N \geq N_1$. Consequently, for all $N \geq \max(N_0, N_1)$ we have

$$\begin{aligned} \left| f(t) - \sum_{k=-N}^{N} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| \leq& \left| f_1(t) - \sum_{k=-N}^{N} f_1(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| \\ &+ \left| f_2(t) - \sum_{k=-N}^{N} f_2(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| \\ \leq& 2\epsilon. \end{aligned} \tag{3.39}$$

Since $\epsilon > 0$ was arbitrary and inequality (3.39) is valid for all $t \in \mathbb{R}$, the proof is complete. □

Theorem 3.21 shows that the divergence of the peak value of the Shannon sampling series for signals $f \in \mathcal{PW}^1_\pi$ is only a consequence of the behavior of $\hat{f}(\omega)$ in the vicinity of $\omega = \pi$. If certain integrability conditions are satisfied in this region, then the divergence does not occur.

3.2.6 Oversampling and Reconstruction Bandwidth

In this section we want to examine whether oversampling is really a universal remedy for circumventing convergence problems. We start with the following observation: The finite Shannon sampling series

$$\frac{1}{a\pi} \sum_{k=-N}^{N} f\left(\frac{k}{a}\right) \frac{\sin\left(a\pi\left(t-\frac{k}{a}\right)\right)}{t-\frac{k}{a}} \tag{3.40}$$

with oversampling factor $a > 1$ is not bandlimited to π, but to $a\pi$. Can we use reconstruction functions in the approximation formula that are bandlimited to π itself?

For $f \in \mathcal{PW}^2_\pi$ this is obviously possible. By expanding $\hat{f}$ into a Fourier series in the interval $[-a\pi, a\pi]$, $a > 1$, we have

$$\lim_{N\to\infty} \int_{-a\pi}^{a\pi} \left| \hat{f}(\omega) - \frac{1}{a} \sum_{k=-N}^{N} f\left(\frac{k}{a}\right) \mathrm{e}^{-\mathrm{i}\omega k/a} \right|^2 \mathrm{d}\omega = 0.$$

Moreover, using the definition

$$\chi_\pi(\omega) := \begin{cases} 1, & |\omega| < \pi, \\ \frac{1}{2}, & |\omega| = \pi, \\ 0, & |\omega| > \pi \end{cases}$$

of the characteristic function χ_π, it holds that

$$\begin{aligned} &f(t) - \frac{1}{a\pi} \sum_{k=-N}^{N} f\left(\frac{k}{a}\right) \frac{\sin(\pi(t-\frac{k}{a}))}{t-\frac{k}{a}} \\ &= \frac{1}{2\pi} \int_{-a\pi}^{a\pi} \left(\hat{f}(\omega) - \frac{1}{a} \sum_{k=-N}^{N} f\left(\frac{k}{a}\right) \mathrm{e}^{-\mathrm{i}\omega k/a} \right) \chi_\pi(\omega)\, \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}\omega \end{aligned}$$

and, using the Cauchy–Schwarz inequality, it follows that

$$\begin{aligned} &\left| f(t) - \frac{1}{a\pi} \sum_{k=-N}^{N} f\left(\frac{k}{a}\right) \frac{\sin(\pi(t-\frac{k}{a}))}{t-\frac{k}{a}} \right| \\ &\leq \left(\frac{1}{2\pi} \int_{-a\pi}^{a\pi} \left| \hat{f}(\omega) - \frac{1}{a} \sum_{k=-N}^{N} f\left(\frac{k}{a}\right) \mathrm{e}^{-\mathrm{i}\omega k/a} \right|^2 \mathrm{d}\omega \right)^{\frac{1}{2}} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} 1\, \mathrm{d}\omega \right)^{\frac{1}{2}}. \end{aligned}$$

Therefore, we have

$$\lim_{N\to\infty} \max_{t\in\mathbb{R}} \left| f(t) - \frac{1}{a\pi} \sum_{k=-N}^{N} f\left(\frac{k}{a}\right) \frac{\sin(\pi(t-\frac{k}{a}))}{t-\frac{k}{a}} \right| = 0.$$

As a consequence, for $f \in \mathcal{PW}_\pi^2$ it is possible to use

$$\frac{1}{a\pi} \sum_{k=-N}^{N} f\left(\frac{k}{a}\right) \frac{\sin(\pi(t-\frac{k}{a}))}{t-\frac{k}{a}} \tag{3.41}$$

for signal reconstruction. This has the advantage that $f \in \mathcal{PW}_\pi^2$ can be approximated according to equation (3.41) by a signal which is bandlimited to π. The sampling series (3.41) can be obtained filtering the signal generated by the sampling series in equation (3.40) with a low-pass filter with bandwidth π.

Since for all signals $f \in \mathcal{PW}_\pi^1$ the series in (3.40) converges uniformly on all of $\mathbb{R}$ to the signal f, it is reasonable to ask whether a low-pass filtering of (3.40) preserves the uniform convergence, even for $f \in \mathcal{PW}_\pi^1$. Then, this would be the projection on the desired frequency interval. However, the following theorem gives a negative answer.

Theorem 3.22. *There is a* $f_1 \in \mathcal{PW}_\pi^1$ *such that*

$$\limsup_{N\to\infty} \max_{t\in\mathbb{R}} \left| \frac{1}{a\pi} \sum_{k=-N}^{N} f_1\left(\frac{k}{a}\right) \frac{\sin(\pi(t-\frac{k}{a}))}{t-\frac{k}{a}} \right| = \infty. \tag{3.42}$$

Proof. For $t \in \mathbb{R}$, $a > 1$, $N \in \mathbb{N}$ fixed, we have

$$\begin{aligned}
&\sup_{\|f\|_{\mathcal{PW}_\pi^1} \le 1} \left| \frac{1}{a\pi} \sum_{k=-N}^{N} f\left(\frac{k}{a}\right) \frac{\sin(\pi(t - \frac{k}{a}))}{t - \frac{k}{a}} \right| \\
&= \sup_{\|f\|_{\mathcal{PW}_\pi^1} \le 1} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) \sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\omega k/a} \frac{\sin(\pi(t - \frac{k}{a}))}{a\pi(t - \frac{k}{a})} \,\mathrm{d}\omega \right| \\
&= \max_{\omega \in [-\pi,\pi]} \left| \sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\omega k/a} \frac{\sin(\pi(t - \frac{k}{a}))}{a\pi(t - \frac{k}{a})} \right| .
\end{aligned}$$

Next,

$$q_N(t,\omega,a) := \sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\omega k/a} \frac{\sin(\pi(t - \frac{k}{a}))}{a\pi(t - \frac{k}{a})}$$

is analyzed. For $\omega = \pi$ and $t_N = (N + 1/2)/a$ we get

$$\begin{aligned}
q_N(t_N,\pi,a) &= \frac{1}{2\mathrm{i}\pi} \left(\sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\pi t_N} \frac{\mathrm{e}^{\mathrm{i}k(\pi/a - \pi/a)}}{N + \frac{1}{2} - k} - \sum_{k=-N}^{N} \mathrm{e}^{-\mathrm{i}\pi t_N} \frac{\mathrm{e}^{\mathrm{i}k(\pi/a + \pi/a)}}{N + \frac{1}{2} - k} \right) \\
&= \frac{1}{2\mathrm{i}\pi} \left(\mathrm{e}^{\mathrm{i}\pi t_N} \sum_{k=-N}^{N} \frac{1}{N + \frac{1}{2} - k} - \mathrm{e}^{-\mathrm{i}\pi t_N} \sum_{k=-N}^{N} \frac{\mathrm{e}^{\mathrm{i}k2\pi/a}}{N + \frac{1}{2} - k} \right)
\end{aligned}$$

and

$$\begin{aligned}
|q_N(t_N,\pi,a)| &\ge \frac{1}{2\pi} \left(\left| \sum_{k=-N}^{N} \frac{1}{N + \frac{1}{2} - k} \right| - \left| \sum_{k=-N}^{N} \frac{\mathrm{e}^{\mathrm{i}k2\pi/a}}{N + \frac{1}{2} - k} \right| \right) \qquad (3.43) \\
&> \frac{1}{2\pi} \log(N) - \frac{4}{\sin\left(\frac{\pi}{a}\right)} .
\end{aligned}$$

The second sum in equation (3.43) was evaluated in the same way as the sum in (3.33). Consequently, for $t_N = (N + 1/2)/a$ we have

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1} \le 1} \left| \frac{1}{a\pi} \sum_{k=-N}^{N} f\left(\frac{k}{a}\right) \frac{\sin(\pi(t_N - \frac{k}{a}))}{t_N - \frac{k}{a}} \right| > \frac{1}{2\pi} \log(N) - \frac{4}{\sin\left(\frac{\pi}{a}\right)} .$$

Hence, by the Banach–Steinhaus theorem [78, p. 98] there exists a signal $f_1 \in \mathcal{PW}_\pi^1$ that fulfills (3.42). □

The result of Theorem 3.22 is interesting. It shows that a low-pass filtering destroys the uniform convergence. Not even the uniform boundedness is preserved. This means that the sole redundancy in the set $\{f(k/a)\}_{k \in \mathbb{Z}}$, $a > 1$, is not sufficient for uniform convergence. It is necessary to use a proper kernel. Consequently, a simultaneous projection onto the space $\mathcal{PW}_\pi^1$ during reconstruction is not possible without the peak value of the reconstruction process becoming divergent.

It is easy to see that for $f \in \mathcal{PW}_\pi^1$ no reconstruction process of the form

$$\sum_{k=-N}^{N} f\left(\frac{k}{a}\right) \phi\left(t - \frac{k}{a}\right) \tag{3.44}$$

and with the property that (3.44) is bandlimited to π can be uniformly bounded. If (3.44) is bandlimited to π then the Fourier transform

$$\frac{1}{a} \sum_{k=-N}^{N} f\left(\frac{k}{a}\right) \mathrm{e}^{\mathrm{i}\omega k/a} (a\hat{\phi}(\omega))$$

is supported in $[\pi, \pi]$. Since $\sum_{k=-N}^{N} f\,(k/a)\,\mathrm{e}^{\mathrm{i}\omega k/a}$ has only isolated zeros, $\hat{\phi}(\omega) = 0$ for $|\omega| > \pi$ must hold. Moreover, the assumed convergence of (3.44) implies that $a\hat{\phi}(\omega) = 1$ for $|\omega| < \pi$. Hence, $\phi(t) = (\sin(\pi t))/(a\pi t)$. But Theorem 3.22 has shown that the reconstruction process for this kernel is neither uniformly convergent nor uniformly bounded on all of $\mathbb{R}$.

Thus, it is impossible to have a locally uniformly convergent and globally uniformly bounded reconstruction for all $f \in \mathcal{PW}_\pi^1$ on the basis of the samples $\{f(k/a)\}_{k\in\mathbb{Z}}$ if the reconstruction process is of the form (3.44) and ϕ is bandlimited to π. It is possible to approach the bandwidth π arbitrarily closely, but to have ϕ exactly bandlimited with π is impossible if uniform boundedness is desired.

3.2.7 Oversampling with General Kernels

Due to oversampling many different reconstruction kernels are possible, not only the sinc-kernel of the Shannon sampling series without oversampling [42, 43, 45, 76]. In particular, all kernels ϕ in $\mathcal{M}(a)$ can be used.

Definition 3.23. $\mathcal{M}(a)$, $a > 1$, is the set of functions $\phi \in \mathcal{B}_{a\pi}^1$ with $\hat{\phi}(\omega) = 1/a$ for $|\omega| \le \pi$.

The functions in $\phi \in \mathcal{M}(a)$, $a > 1$, are suitable kernels for the sampling series

$$(S_{N,\phi}^a f)(t) = \sum_{k=-N}^{N} f\left(\frac{k}{a}\right) \phi\left(t - \frac{k}{a}\right)$$

because for all $f \in \mathcal{PW}_\pi^2$ we have $\lim_{N\to\infty} \|f - S_{N,\phi}^a f\|_{\mathcal{PW}_{a\pi}^2} = 0$ [79], and consequently

$$\lim_{N\to\infty} \|f - S_{N,\phi}^a f\|_\infty = 0. \tag{3.45}$$

Example 3.24. Two well-known classes of kernels in $\mathcal{M}(a)$, $a > 1$, are the kernels with a trapezoidal shape in the frequency domain and the kernels with a cosine roll-off characteristic in the frequency domain.

One important property of the kernels $\phi \in \mathcal{M}(a)$, $a > 1$, is stated in the following lemma.

Lemma 3.25. *For all $a > 1$ and $\phi \in \mathcal{B}^1_{a\pi}$ there exists a constant C_8 such that*

$$\sum_{k=-\infty}^{\infty} \left| \phi\left(t - \frac{k}{a}\right) \right| \le C_8 \|\phi\|_{\mathcal{B}^1_{a\pi}}$$

for all $t \in \mathbb{R}$.

Lemma 3.25 is a direct consequence of Nikol'skiĭ's inequality [26, p. 49]. Nevertheless, for the sake of a self contained presentation, we have included the short proof in Appendix A.2.

We have seen in (3.45) that the sampling series $S^a_{N,\phi} f$ converges uniformly on all or $\mathbb{R}$ for all signals in $\mathcal{PW}^2_\pi$. The next theorem shows that we also have the uniform convergence for all signals in $\mathcal{PW}^1_\pi$.

Theorem 3.26. *Let $\phi \in \mathcal{M}(a)$, $a > 1$. Then we have*

$$\lim_{N\to\infty} \left\| f - \sum_{k=-N}^{N} f\left(\frac{k}{a}\right) \phi\left(\cdot - \frac{k}{a}\right) \right\|_\infty = 0$$

for all $f \in \mathcal{PW}^1_\pi$.

We could prove Theorem 3.26 directly, however, using Theorem 4.40, which is stated later in Section 4.6 on p. 122, the proof can be considerably shortened. Hence, we base the proof of Theorem 3.26 on Theorem 4.40.

Proof. Let $\phi \in \mathcal{M}(a)$, $a > 1$, be arbitrary but fixed. Then, according to Lemma 3.25, we have

$$\sup_{t\in\mathbb{R}} \max_{|\omega|\le\pi} \left| \sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\omega k/a} \phi\left(t - \frac{k}{a}\right) \right| \le \sup_{t\in\mathbb{R}} \sum_{k=-N}^{N} \left| \phi\left(t - \frac{k}{a}\right) \right| \le C_8 \|\phi\|_{\mathcal{B}^1_{a\pi}},$$

and the assertion follows from Theorem 4.40 for $T = Id$. □

3.2.8 Non-Symmetric Sampling Series

Many attempts have been made to prove convergence results that are true for the symmetric sampling series for the non-symmetric case. However, often the non-symmetric sampling series exhibits a significantly different convergence behavior compared to the symmetric sampling series. This is particularly true for the Shannon sampling series and the space $\mathcal{PW}^1_\pi$: In this section we will show that there exists a signal $f_2 \in \mathcal{PW}^1_\pi$ such that for all $t \in \mathbb{R} \setminus \mathbb{Z}$

$$\limsup_{M,N\to\infty} \left| \sum_{k=-M}^{N} f_2(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| = \infty,$$

which means that the non-symmetric Shannon sampling series diverges for all $t \in \mathbb{R} \setminus \mathbb{Z}$. Consequently, a theorem stating the uniform convergence of the non-symmetric Shannon sampling series on compact subsets of $\mathbb{R}$, similar to Theorem 3.6, which states exactly this for the symmetric Shannon sampling series, cannot exist.

The double limit $\lim_{M,N\to\infty}$ in the equation above is defined as usual [80]. For a double sequence $\{a_{MN}\}_{M,N\in\mathbb{N}} \subset \mathbb{C}$ we write $\lim_{M,N\to\infty} a_{MN} = c$ if for all $\epsilon > 0$ there exists a $n \in \mathbb{N}$ such that $|a_{MN} - c| < \epsilon$ for all $M > n$ and $N > n$. Moreover, we write $\limsup_{M,N\to\infty} a_{MN} = \infty$ if for all $K > 0$ there exist two natural numbers M and N such that $a_{MN} \geq K$.

For the proof we use the same signal $f_2 \in \mathcal{PW}_\pi^1$ that was used in the proof of Theorem 3.2 and defined in (3.11) on page 20. We need the fact that the samples of f_2 are given by

$$f_2(k) = \begin{cases} 0, & k = 0 \\ \frac{(-1)^{|k|}}{\log(1+|k|)}, & |k| \geq 1. \end{cases}$$

Theorem 3.27. *Let f_2 be defined as in equation (3.11). Then, for all $t \in \mathbb{R} \setminus \mathbb{Z}$, we have*

$$\lim_{N\to\infty} \left| \sum_{k=1}^{N} f_2(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| = \infty.$$

Proof. For $N \in \mathbb{N}$, $N > 1$, consider the finite sums

$$(A_N f_2)(t) := \sum_{k=1}^{N} f_2(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}.$$

First, we prove that $\lim_{N\to\infty}(A_N f_2)(-1/2) = \infty$. We have

$$\begin{aligned} (A_N f_2)(-\tfrac{1}{2}) &= \sin\left(-\frac{\pi}{2}\right) \sum_{k=1}^{N} f_2(k) \frac{(-1)^k}{\pi\left(-\frac{1}{2} - k\right)} \\ &> \frac{1}{\pi} \sum_{k=1}^{N} \frac{1}{\log(1+k)} \frac{1}{1+k}, \end{aligned}$$

where we used the identity $\sin(\pi(t-k)) = \sin(\pi t)(-1)^k$ for $t \in \mathbb{R}$, $k \in \mathbb{Z}$, in the first line. But, since

$$\frac{1}{\log(1+k)(1+k)} > \int_k^{k+1} \frac{1}{\log(1+x)(x+1)} \,\mathrm{d}x,$$

we obtain

$$\begin{aligned} (A_N f_2)(-\tfrac{1}{2}) &> \frac{1}{\pi} \sum_{k=1}^{N} \int_k^{k+1} \frac{1}{\log(1+x)(x+1)} \,\mathrm{d}x \\ &= \frac{1}{\pi} \int_1^{N+1} \frac{1}{\log(1+x)(x+1)} \,\mathrm{d}x > \frac{1}{\pi} \log\left(\frac{\log N}{2}\right), \end{aligned}$$

and hence $\lim_{N\to\infty}(A_N f_2)(-1/2) = \infty$.

Now, let $t_1, t_2 \in \mathbb{R} \setminus \mathbb{Z}$ be arbitrarily chosen and consider

$$\begin{aligned} D &:= \frac{(A_N f_2)(t_1)}{\sin(\pi t_1)} - \frac{(A_N f_2)(t_2)}{\sin(\pi t_2)} \\ &= \frac{1}{\pi} \sum_{k=1}^{N} f_2(k)(-1)^k \left(\frac{1}{t_1 - k} - \frac{1}{t_2 - k} \right) \\ &= \frac{t_2 - t_1}{\pi} \sum_{k=1}^{N} f_2(k) \frac{(-1)^k}{(t_1 - k)(t_2 - k)}. \end{aligned}$$

The modulus of D can be bounded above as follows:

$$\begin{aligned} |D| &\le \frac{|t_2 - t_1|}{\pi} \sum_{k=1}^{N} |f_2(k)| \frac{1}{|t_1 - k||t_2 - k|} \\ &\le \frac{|t_2 - t_1| \|f_2\|_{\mathcal{PW}_\pi^1}}{\pi} \sum_{k=1}^{\infty} \frac{1}{|t_1 - k||t_2 - k|} \\ &= \|f_2\|_{\mathcal{PW}_\pi^1} \, C_9(t_1, t_2), \end{aligned}$$

with a constant $C_9(t_1, t_2)$, which depends only on t_1 and t_2. Thus, it follows from $\lim_{N\to\infty}(A_N f_2)(-1/2) = \infty$ that $\lim_{N\to\infty} |(A_N f_2)(t)| = \infty$ for all $t \in \mathbb{R} \setminus \mathbb{Z}$. □

Theorem 3.27 also answers the question of how the non-symmetric Shannon sampling series behaves for signals $f \in \mathcal{PW}_\pi^1$.

Corollary 3.28. *Let f_2 be defined as in equation (3.11). Then, for all $t \in \mathbb{R} \setminus \mathbb{Z}$, we have*

$$\limsup_{M,N\to\infty} \left| \sum_{k=-M}^{N} f_2(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| = \infty.$$

The preceding considerations have been made for the case where the samples are taken at Nyquist rate, i.e., where no oversampling is applied. The convergence behavior of the non-symmetric Shannon sampling series changes completely if we use oversampling. A closer look on the proof of Lemma 3.19 shows that the inequality

$$\left| \sum_{k=-M}^{N} \mathrm{e}^{\mathrm{i}\omega k/a} \, \frac{\sin\bigl(a\pi(t - \frac{k}{a})\bigr)}{a\pi\bigl(t - \frac{k}{a}\bigr)} \right| \le 2 \left(1 + \frac{2}{\cos(\frac{\pi}{2a})} \right)$$

holds for all $M, N \in \mathbb{N}$, $t \in \mathbb{R}$ and $|\omega| \le \pi$. Therefore, it is possible to get an analogous result to Lemma 3.19 for the non-symmetric sampling series. Consequently, a theorem similar to Theorem 3.17 can be derived for the non-symmetric sampling series.

Theorem 3.29. *Let $a > 1$ be fixed. Then we have for all $f \in \mathcal{PW}_\pi^1$*

$$\lim_{M,N\to\infty} \max_{t\in\mathbb{R}} \left| f(t) - \sum_{k=-M}^{N} f\left(\frac{k}{a}\right) \frac{\sin\left(a\pi\left(t-\frac{k}{a}\right)\right)}{a\pi\left(t-\frac{k}{a}\right)} \right| = 0.$$

Proof. Analogously to Theorem 3.17. □

We have seen that for $f \in \mathcal{PW}_\pi^1$ there are differences in the convergence behavior between the symmetric Shannon sampling series and the non-symmetric Shanon sampling series when no oversampling is applied. However, with oversampling both have the same good convergence behavior, i.e., both are uniformly convergent on all of $\mathbb{R}$. Here we see that oversampling, which is applied in practice for several reasons, is also theoretically justified by the reconstruction behavior of the non-symmetric Shannon sampling series for signals in $\mathcal{PW}_\pi^1$.

3.2.9 Centered Sampling Series

In Theorem 3.11 we have seen that a whole class of reconstruction processes, including the Shannon sampling series, is not uniformly convergent on $\mathbb{R}$ and not even uniformly bounded on $\mathbb{R}$ for signals in $\mathcal{PW}_\pi^1$ in general. One further possible way of truncating the series (3.13), which was considered in [63, 64, 75, 81], is to truncate the series (3.13) symmetrically around $t \in \mathbb{R}$. Including oversampling with oversampling factor $a > 1$, the reconstruction process is then given by

$$(A_N^a f)(t) := \sum_{k=K(t)-N}^{K(t)+N} f\left(\frac{k}{a}\right) \frac{\sin\left(a\pi\left(t-\frac{k}{a}\right)\right)}{a\pi\left(t-\frac{k}{a}\right)}, \tag{3.46}$$

where $K(t)$ denotes the largest integer that is smaller than or equal to $t + 1/2$. For every fixed point in time only $2N + 1$ signal values are needed, but as t ranges from $-\infty$ to ∞, infinitely many samples are necessary to reconstruct the whole signal.

In the case where $a > 1$, one can use other kernels than the sinc-kernel $\sin(a\pi(t - k/a))/(a\pi(t - k/a))$ that is used in (3.46) [68]. In [75] kernels that are the product of $\sin(a\pi(t - k/a))/(a\pi(t - k/a))$ and a function h_0 with certain properties were considered in order to reduce the reconstruction error.

In [63] an upper bound was given for $\|f - A_N^a f\|_\infty$. However, as pointed out in [75], for the important case $a \to 1$ this bound tends to infinity. We will show that this bound does not reflect the true behavior of $\|f - A_N^a f\|_\infty$ for $a \to 1$ because the bound is not tight. In fact, it holds that $\sup_{N\in\mathbb{N}} \|f - A_N^1 f\|_\infty \le C_{10} < \infty$ for all $f \in \mathcal{PW}_\pi^1$ with $\|f\|_{\mathcal{PW}_\pi^1} \le 1$.

In this section we analyze the convergence behavior of

$$(A_N f)(t) := (A_N^1 f)(t) = \sum_{k=K(t)-N}^{K(t)+N} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}.$$

It is shown that $A_N f$ converges uniformly on all of $\mathbb{R}$ to f for all $f \in \mathcal{PW}_\pi^1$. This fact is expressed in Theorem 3.30, the proof of which is postponed until we have discussed Theorem 3.30 and compared it with our previous results.

Theorem 3.30. *For all $f \in \mathcal{PW}_\pi^1$ we have*

$$\lim_{N\to\infty} \max_{t\in\mathbb{R}} \left| f(t) - \sum_{k=K(t)-N}^{K(t)+N} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| = 0.$$

At first glance, Theorem 3.30 seems to be a contradiction to the result given in Theorem 3.11, but a closer examination of A_N reveals that the reconstructed signal $A_N f$ is not bandlimited. This fact permits the good convergence properties.

Theorem 3.31. *$A_N f$ is not bandlimited for $f \in \mathcal{PW}_\pi^1$ in general.*

Proof. Consider for example a function $f \in \mathcal{PW}_\pi^2 \subset \mathcal{PW}_\pi^1$ with $f(k) = 0$ for $|k| \geq M$ and $f(k) \neq 0$ for $|k| < M$ for some $M \in \mathbb{N}$. Then $(A_N f)(t) = 0$ for all $|t| > M + N$. If $A_N f$ was bandlimited then it would follow that $A_N f \equiv 0$, which would be a contradiction to $(A_N f)(k) = f(k)$, $k \in \mathbb{Z}$. □

Despite the good convergence behavior, A_N has two drawbacks for practical applications:

1. The resulting functions $A_N f$ are not bandlimited;
2. Infinitely many samples are needed for the calculation of $(A_N f)(t)$ as t ranges over the whole real axis.

The proof of Theorem 3.30 requires two lemmas, namely Lemma 3.32 and Lemma 3.33.

Lemma 3.32. *Let $D_N(x) = 1/2 + \sum_{k=1}^N \cos(kx)$ be the Dirichlet kernel. Then we have*

$$\left| \int_{-\pi}^{\tau} D_N(\omega - \omega_1)\, \mathrm{d}\omega_1 \right| \leq 3\pi$$

for all $N \in \mathbb{N}$, $\omega \in \mathbb{R}$ and $|\tau| \leq \pi$.

Proof. Using

$$\int_{-\pi}^{\tau} D_N(\omega - \omega_1)\, \mathrm{d}\omega_1 = \frac{1}{2}(\tau - \pi) + \sum_{k=1}^{N} \frac{\sin(k(\omega + \pi))}{k} - \sum_{k=1}^{N} \frac{\sin(k(\omega - \tau))}{k}$$

together with

$$\left| \sum_{k=1}^{N} \frac{\sin(kx)}{k} \right| \leq \pi \quad \text{for all } x \in \mathbb{R},$$

which is a consequence of Gibb's phenomenon [82, p. 61], gives

$$\left|\int_{-\pi}^{\tau} D_N(\omega-\omega_1)\,\mathrm{d}\omega_1\right| \le \pi+\pi+\pi = 3\pi$$

for all $N \in \mathbb{N}$, $\omega \in \mathbb{R}$ and $|\tau| \le \pi$. □

Lemma 3.33. *For all $f \in \mathcal{PW}_\pi^2$ we have*

$$\lim_{N\to\infty} \max_{t\in\mathbb{R}} \left| f(t) - \sum_{k=K(t)-N}^{K(t)+N} f(k)\frac{\sin(\pi(t-k))}{\pi(t-k)} \right| = 0.$$

Proof. Let $f \in \mathcal{PW}_\pi^2$ and $t \in \mathbb{R}$ be arbitrary but fixed and $N \ge 2$. Then we have

$$\begin{aligned}
\left| f(t) - \sum_{k=K(t)-N}^{K(t)+N} f(k)\frac{\sin(\pi(t-k))}{\pi(t-k)} \right| &= \left| \sum_{|k-K(t)|>N} f(k)\frac{\sin(\pi(t-k))}{\pi(t-k)} \right| \\
&\le \left(\sum_{|k-K(t)|>N} |f(k)|^2 \right)^{\frac{1}{2}} \left(\sum_{|k-K(t)|>N} \left| \frac{\sin(\pi(t-k))}{\pi(t-k)} \right|^2 \right)^{\frac{1}{2}} \\
&\le \left(\sum_{k=-\infty}^{\infty} |f(k)|^2 \right)^{\frac{1}{2}} \left(\sum_{|k-K(t)|>N} \frac{1}{\pi^2(t-k)^2} \right)^{\frac{1}{2}},
\end{aligned}$$

and the upper bound

$$\begin{aligned}
\sum_{|k-K(t)|>N} \frac{1}{(t-k)^2} &= \sum_{k=-\infty}^{K(t)-N-1} \frac{1}{(t-k)^2} + \sum_{k=K(t)+N+1}^{\infty} \frac{1}{(t-k)^2} \\
&\le \sum_{k=-\infty}^{K(t)-N-1} \frac{1}{(K(t)-k-1)^2} + \sum_{k=K(t)+N+1}^{\infty} \frac{1}{(k-K(t)-1)^2} \\
&= \sum_{k=-\infty}^{-N} \frac{1}{k^2} + \sum_{k=N}^{\infty} \frac{1}{k^2} \le \frac{2}{N-1}
\end{aligned}$$

leads to

$$|f(t) - (A_N f)(t)| \le \|f\|_{\mathcal{PW}_\pi^2} \left(\frac{2}{N-1} \right)^{\frac{1}{2}}.$$

Therefore, we have $\lim_{N\to\infty} \|f - A_N f\|_\infty = 0$ for all $f \in \mathcal{PW}_\pi^2$. □

Now we are in the position to proof Theorem 3.30.

Proof of Theorem 3.30. Let $f \in \mathcal{PW}_\pi^1$ be arbitrary but fixed. Then we have

$$
\begin{aligned}
(A_N f)(t) &= \sum_{k=K(t)-N}^{K(t)+N} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) \left(\sum_{k=K(t)-N}^{K(t)+N} \mathrm{e}^{\mathrm{i}\omega k} \frac{\sin(\pi(t-k))}{\pi(t-k)} \right) \mathrm{d}\omega \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega)\, q_N(t,\omega)\, \mathrm{d}\omega. \qquad (3.47)
\end{aligned}
$$

Next, we analyze

$$
q_N(t,\omega) := \sum_{k=K(t)-N}^{K(t)+N} \mathrm{e}^{\mathrm{i}\omega k} \frac{\sin(\pi(t-k))}{\pi(t-k)}.
$$

Since

$$
\frac{\sin(\pi(t-k))}{\pi(t-k)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}\omega_1(t-k)}\, \mathrm{d}\omega_1,
$$

we obtain

$$
\begin{aligned}
q_N(t,\omega) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}\omega_1 t} \sum_{k=K(t)-N}^{K(t)+N} \mathrm{e}^{\mathrm{i}k(\omega-\omega_1)}\, \mathrm{d}\omega_1 \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}\omega_1 t}\, \mathrm{e}^{\mathrm{i}K(t)(\omega-\omega_1)} \sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}k(\omega-\omega_1)}\, \mathrm{d}\omega_1 \\
&= \frac{\mathrm{e}^{\mathrm{i}K(t)\omega}}{\pi} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}\omega_1(t-K(t))} \left(\frac{1}{2} + \sum_{k=1}^{N} \cos(k(\omega-\omega_1)) \right) \mathrm{d}\omega_1 \\
&= \frac{\mathrm{e}^{\mathrm{i}K(t)\omega}}{\pi} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}\omega_1(t-K(t))}\, D_N(\omega-\omega_1)\, \mathrm{d}\omega_1, \qquad (3.48)
\end{aligned}
$$

and denote by $D_N(x)$ the Dirichlet kernel $1/2 + \sum_{k=1}^{N} \cos(kx)$. The integral on the right-hand side of (3.48) is further analyzed. Integration by parts and using $\int_{-\pi}^{\pi} D_N(\omega-\omega_1)\, \mathrm{d}\omega_1 = \pi$ gives

$$
\begin{aligned}
&\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}\omega_1(t-K(t))}\, D_N(\omega-\omega_1)\, \mathrm{d}\omega_1 \\
&\qquad = \pi\, \mathrm{e}^{\mathrm{i}\pi(t-K(t))} - \int_{-\pi}^{\pi} \int_{-\pi}^{\tau} D_N(\omega-\omega_1)\, \mathrm{d}\omega_1\, \mathrm{i}(t-K(t))\, \mathrm{e}^{\mathrm{i}\tau(t-K(t))}\, \mathrm{d}\tau
\end{aligned}
$$

and

$$
\left| \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}\omega_1(t-K(t))}\, D_N(\omega-\omega_1)\, \mathrm{d}\omega_1 \right| \le \pi + \int_{-\pi}^{\pi} \left| \int_{-\pi}^{\tau} D_N(\omega-\omega_1)\, \mathrm{d}\omega_1 \right| \cdot |t-K(t)|\, \mathrm{d}\tau.
$$

By Lemma 3.32, we have

$$\left| \int_{-\pi}^{\tau} D_N(\omega - \omega_1) \, \mathrm{d}\omega_1 \right| \leq 3\pi$$

for all $N \in \mathbb{N}$, $\omega \in \mathbb{R}$ and $|\tau| \leq \pi$. Since $|t - K(t)| \leq 1/2$ for all $t \in \mathbb{R}$, we obtain

$$\left| \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}\omega_1 (t - K(t))} \, D_N(\omega - \omega_1) \, \mathrm{d}\omega_1 \right| \leq \pi + 3\pi^2$$

and as a consequence

$$|q_N(t, \omega)| \leq 1 + 3\pi \tag{3.49}$$

for all $N \in \mathbb{N}$, $t \in \mathbb{R}$ and $|\omega| \leq \pi$. Finally, equations (3.47) and (3.49) can be used to derive the upper bound

$$\begin{aligned} |(A_N f)(t)| &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{f}(\omega)| \, |q_N(t, \omega)| \, \mathrm{d}\omega \\ &\leq (1 + 3\pi) \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{f}(\omega)| \, \mathrm{d}\omega \\ &= (1 + 3\pi) \|f\|_{\mathcal{PW}_\pi^1}, \end{aligned} \tag{3.50}$$

which is valid for all $N \in \mathbb{N}$, all $t \in \mathbb{R}$, and all $f \in \mathcal{PW}_\pi^1$.

In order to continue the proof we fix an arbitrary $\epsilon > 0$. Then, there exists a function $f_\epsilon \in \mathcal{PW}_\pi^2$ such that $\|f - f_\epsilon\|_{\mathcal{PW}_\pi^1} < \epsilon$. Furthermore,

$$\begin{aligned} |f(t) - (A_N f)(t)| &= |f(t) - f_\epsilon(t) + f_\epsilon(t) - (A_N f_\epsilon)(t) + (A_N (f - f_\epsilon))(t)| \\ &\leq |f(t) - f_\epsilon(t)| + |f_\epsilon(t) - (A_N f_\epsilon)(t)| + |(A_N (f - f_\epsilon))(t)| \\ &\leq \|f - f_\epsilon\|_{\mathcal{PW}_\pi^1} + |f_\epsilon(t) - (A_N f_\epsilon)(t)| + (1 + 3\pi) \|f - f_\epsilon\|_{\mathcal{PW}_\pi^1} \qquad (3.51) \\ &< \epsilon (2 + 3\pi) + |f_\epsilon(t) - (A_N f_\epsilon)(t)|, \end{aligned}$$

holds for all $N \in \mathbb{N}$ and $t \in \mathbb{R}$. To obtain (3.51) we applied (3.50) on $f - f_\epsilon$ and used the inequality $|f(t)| \leq \|f\|_{\mathcal{PW}_\pi^1}$, which holds for all $t \in \mathbb{R}$ and $f \in \mathcal{PW}_\pi^1$. Due to Lemma 3.33, there is a $N_0 = N_0(\epsilon)$ such that $|f_\epsilon(t) - (A_N f_\epsilon)(t)| < \epsilon$ for all $t \in \mathbb{R}$ and all $N \geq N_0$. Hence,

$$|f(t) - (A_N f)(t)| < \epsilon (3 + 3\pi)$$

for all $t \in \mathbb{R}$ and all $N \geq N_0$. Since $\epsilon > 0$ was arbitrary, the proof of Theorem 3.30 is complete. □

As we have seen in Theorem 3.30, the reconstruction process $A_N f$ converges uniformly on all of $\mathbb{R}$ to f for all $f \in \mathcal{PW}_\pi^1$.

Now we analyze the non-symmetric version

$$(A_{M,N} f)(t) := \sum_{k=K(t)-M}^{K(t)+N} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}. \tag{3.52}$$

We have seen in Corollary 3.28 that there are signals in $\mathcal{PW}_\pi^1$, such that the non-symmetric Shannon sampling series

$$(S_{M,N}f)(t) := \sum_{k=-M}^{N} f(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}$$

diverges unboundedly for all $t \in \mathbb{R} \setminus \mathbb{Z}$. Since for certain $f \in \mathcal{PW}_\pi^1$ the symmetric Shannon sampling series $S_N f$ is not uniformly convergent on all of $\mathbb{R}$ and the reconstruction process $A_N f$ is uniformly convergent on all of $\mathbb{R}$ for all $f \in \mathcal{PW}_\pi^1$, we might expect a better behavior of $A_{M,N}f$ compared to the non-symmetric Shannon sampling series $S_{M,N}f$.

However, this is not the case, as shown by the following result.

Theorem 3.34. *There exists a signal $f_2 \in \mathcal{PW}_\pi^1$, such that for all $t \in \mathbb{R} \setminus \mathbb{Z}$*

$$\limsup_{M,N\to\infty} \left| f_2(t) - \sum_{k=K(t)-M}^{K(t)+N} f_2(k)\frac{\sin(\pi(t-k))}{\pi(t-k)} \right| = \infty.$$

Proof. For the proof we use the same signal $f_2 \in \mathcal{PW}_\pi^1$ that was used in the proof of Theorem 3.2 and defined in (3.11) on page 20.

To simplify the notation, we use the abbreviation $(A_{M,N}f)(t)$ that was defined in (3.52). Let $t_1, t_2 \in \mathbb{R}\backslash\mathbb{Z}$ arbitrary but fixed. We will show that there is a constant $C_{11}(t_1,t_2) < \infty$ such that for all $f \in \mathcal{PW}_\pi^1$ and all $M, N \in \mathbb{N}$ we have

$$\left| \frac{(A_{M,N}f)(t_1)}{\sin(\pi t_1)} - \frac{(A_{M,N}f)(t_2)}{\sin(\pi t_2)} \right| \le C_{11}(t_1,t_2)\|f\|_{\mathcal{PW}_\pi^1}. \tag{3.53}$$

Suppose (3.53) has been proved, then it is enough to show that

$$\limsup_{M,N\to\infty} |(A_{M,N}f_2)(1/4)| = \infty, \tag{3.54}$$

in order to finish the proof. But equation (3.54) follows directly from Corollary 3.28, because $(A_{M,N}f_2)(1/4) = (S_{M,N}f_2)(1/4)$.

It remains to show that (3.53) is true. Without loss of generality, we assume $t_1 < t_2$. Next,

$$\begin{aligned} D &= \frac{(A_{M,N}f)(t_1)}{\sin(\pi t_1)} - \frac{(A_{M,N}f)(t_2)}{\sin(\pi t_2)} \\ &= \frac{1}{\pi}\left(\sum_{k=K(t_1)-M}^{K(t_1)+N} f(k)\frac{(-1)^k}{t_1-k} - \sum_{k=K(t_2)-M}^{K(t_2)+N} f(k)\frac{(-1)^k}{t_2-k} \right) \end{aligned} \tag{3.55}$$

is analyzed. We have to distinguish two cases.

The first is the case where $K(t_1) + N < K(t_2) - M$. In this case, both M and N must be smaller than $K(t_2) - K(t_1)$, which implies that both sums in (3.55) can be bounded from above by some constant independently of M and N. Thus $|D| \leq C_{12}(t_1, t_2)\|f\|_{\mathcal{PW}_\pi^1}$.

In the second case, i.e. the case where $K(t_1) + N \geq K(t_2) - M$, we rearrange the sums and obtain

$$D = \frac{1}{\pi}\left(\sum_{k=K(t_1)-M}^{K(t_2)-M-1} f(k)\frac{(-1)^k}{t_1 - k} + \sum_{k=K(t_2)-M}^{K(t_1)+N} f(k)(-1)^k\left(\frac{1}{t_1 - k} - \frac{1}{t_2 - k}\right)\right.$$
$$\left. - \sum_{k=K(t_1)+N+1}^{K(t_2)+N} f(k)\frac{(-1)^k}{t_2 - k}\right). \quad (3.56)$$

The modulus of the first sum in (3.56) can be bounded from above according to

$$\left|\sum_{k=K(t_1)-M}^{K(t_2)-M-1} f(k)\frac{(-1)^k}{t_1 - k}\right| \leq \sup_{t\in\mathbb{R}} |f(t)| \sum_{k=K(t_1)}^{K(t_2)-1} \frac{1}{|t_1 - k + M|} \leq C_{13}(t_1, t_2)\|f\|_{\mathcal{PW}_\pi^1},$$

because the number of summands does not depend on M. Applying the same arguments to the third sum in (3.56) gives

$$\left|\sum_{k=K(t_1)+N+1}^{K(t_2)+N} f(k)\frac{(-1)^k}{t_2 - k}\right| \leq C_{14}(t_1, t_2)\|f\|_{\mathcal{PW}_\pi^1}.$$

The modulus of the second sum in (3.56) can be upper bounded by

$$\left|\sum_{k=K(t_2)-M}^{K(t_1)+N} f(k)(-1)^k\left(\frac{1}{t_1 - k} - \frac{1}{t_2 - k}\right)\right| \leq \|f\|_{\mathcal{PW}_\pi^1} \sum_{k=K(t_2)-M}^{K(t_1)+N} \frac{t_2 - t_1}{|t_1 - k| \cdot |t_2 - k|}$$
$$= C_{15}(t_1, t_2)\|f\|_{\mathcal{PW}_\pi^1}.$$

Therefore $|D| \leq C_{11}(t_1, t_2)\|f\|_{\mathcal{PW}_\pi^1}$, which finishes the proof. □

3.3 Non-Equidistant Sampling

In many practical applications non-equidistant sampling patterns are of interest [58, 61, 83]. In this case, sampling series like

$$\sum_{k=-\infty}^{\infty} f(t_k)\phi_k(t), \tag{3.57}$$

where ϕ_k, $k \in \mathbb{Z}$, are certain reconstruction functions, and $\{t_k\}_{k\in\mathbb{Z}}$ is the sequence of sampling points, can be used for the reconstruction of the signal f from its samples $\{f(t_k)\}_{k\in\mathbb{Z}}$. Of course the convergence of the series cannot be taken for granted and has to be checked from case to case.

Throughout this thesis we assume that the sequence of sampling points $\{t_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ is real, and, without loss of generality, we further assume that $t_0 = 0$ and that the sequence of sampling points is ordered strictly increasingly, i.e.,

$$\ldots < t_{-N} < t_{-N+1} < \ldots < t_{-1} < t_0 = 0 < t_1 < \ldots < t_{N-1} < t_N < \ldots. \tag{3.58}$$

Sensor networks are one example where non-equidistant sampling is important. In a sensor network a large number of sensors is used to monitor some physical quantity, e.g., the temperature or the electric field intensity. This physical quantity varies continuously in space and, thus, can be viewed as a signal in space. In general, the sensors are placed non-equidistantly according to the given spacial settings. Thus, in sampling theoretic terminology, the sensors perform a non-equidistant sampling of the signal. At the fusion center, where the data from all sensors is gathered, the task is to reconstruct the signal from the samples.

In this section we discuss under what conditions on the sampling patterns and the signals it is possible to use (3.57) for the reconstruction.

The convergence behavior of (3.57) certainly depends strongly on the signal space under consideration. There is a vast amount of literature discussing the properties of sampling series with non-equidistant sampling points for the space of bandlimited signals with finite energy: The papers [84] and [85] analyze the stability of such sampling series, and [86] derives series representations. Sampling series involving derivatives are considered in [87], and the case where only a finite number of sampling points differs from the integer grid is considered in [88]. Aspects of numerical computation in the reconstruction of bandlimited signals with finite energy from irregular samples are treated in [89–91].

Only few papers [54, 92] discuss non-equidistant sampling for larger signal spaces than the space of bandlimited signals with finite energy. In [92] Seip proves the uniform convergence of (3.57) on all compact subsets of the complex plane for bounded bandlimited signals if oversampling is used and the sequence $\{t_k\}_{k\in\mathbb{Z}}$ of real sampling points satisfies

$$\sup_{k\in\mathbb{Z}} |t_k - k| \le D$$

for some $D < 1/4$. Hinsen [54] analyzes (3.57) without oversampling for bandlimited signals that are in L^p, $1 \le p < \infty$, when restricted to the real line. He gives a condition on D which is sufficient for (3.57) to be uniformly convergent on all compact subsets of the complex plane.

3.3.1 Complete Interpolating Sequences

An important class of sampling patterns are complete interpolating sequences.

Definition 3.35. We say that a sequence $\{t_k\}_{k\in\mathbb{Z}}$ is a complete interpolating sequence for $\mathcal{PW}_\pi^2$ if the interpolation problem $f(t_k) = c_k$, $k \in \mathbb{Z}$ has exactly one solution $f \in \mathcal{PW}_\pi^2$ for every sequence $\{c_k\}_{k\in\mathbb{Z}}$ satisfying $\sum_{k=-\infty}^{\infty}|c_k|^2 < \infty$.

Complete interpolating sequences are useful sampling patterns because every signal $f \in \mathcal{PW}_\pi^2$ is completely determined by its sample values $\{f(t_k)\}_{k\in\mathbb{Z}}$ if $\{t_k\}_{k\in\mathbb{Z}}$ is a complete interpolating sequence for $\mathcal{PW}_\pi^2$. Throughout this section we assume that the sequence of sampling points $\{t_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ is a complete interpolating sequence for $\mathcal{PW}_\pi^2$.

If the sequence of sampling points $\{t_k\}_{k\in\mathbb{Z}}$ is a complete interpolating sequence for $\mathcal{PW}_\pi^2$, it follows by definition that, for each $k \in \mathbb{Z}$, there is exactly one function $\phi_k \in \mathcal{PW}_\pi^2$ that solves the interpolation problem

$$\phi_k(t_l) = \begin{cases} 1, & l = k \\ 0, & l \neq k. \end{cases} \tag{3.59}$$

Moreover, the product

$$\phi(z) = z \lim_{R\to\infty} \prod_{\substack{|t_k|\le R \\ k\neq 0}} \left(1 - \frac{z}{t_k}\right) \tag{3.60}$$

converges uniformly on $|z| \le R$ for all $R < \infty$ and ϕ is an entire function of exponential type π [25, p. 134, Theorem 4]. It can bee seen from (3.60) that ϕ, which is often called a generating function, has the zeros $\{t_k\}_{k\in\mathbb{Z}}$. Since $\{t_k\}_{k\in\mathbb{Z}}$ is a complete interpolating sequence, it follows that

$$\phi_k(t) = \frac{\phi(t)}{\phi'(t_k)(t - t_k)} \tag{3.61}$$

is the unique function in $\mathcal{PW}_\pi^2$ that solves the interpolation problem (3.59). For further details we would like to refer the reader to [58, Chapter 3].

If $\{t_k\}_{k\in\mathbb{Z}}$ is a complete interpolating sequence for $\mathcal{PW}_\pi^2$ then ϕ_k is a Riesz basis for $\mathcal{PW}_\pi^2$.

Definition 3.36. A sequence of vectors $\{\phi_k\}_{k\in\mathbb{Z}}$ in a separable Hilbert space $\mathcal{H}$ is called a Riesz basis if $\{\phi_k\}_{k\in\mathbb{Z}}$ is complete in $\mathcal{H}$ and there exist positive constants A and B such that for all $M, N \in \mathbb{N}$ and arbitrary scalars c_k we have

$$A \sum_{k=-M}^{N} |c_k|^2 \le \left\| \sum_{k=-M}^{N} c_k\,\phi_k \right\|^2 \le B \sum_{k=-M}^{N} |c_k|^2.$$

If $\{t_k\}_{k\in\mathbb{Z}}$ is a complete interpolating sequence for $\mathcal{PW}_\pi^2$ then it follows from Definition 3.36 that there exist two constants $0 < C_{\mathrm{L}}^{\mathrm{Riesz}}, C_{\mathrm{R}}^{\mathrm{Riesz}} < \infty$ such that

$$C_{\mathrm{L}}^{\mathrm{Riesz}} \|f\|_{\mathcal{PW}_\pi^2} \le \left(\sum_{k=-\infty}^{\infty} |f(t_k)|^2 \right)^{1/2} \le C_{\mathrm{R}}^{\mathrm{Riesz}} \|f\|_{\mathcal{PW}_\pi^2} \tag{3.62}$$

for all $f \in \mathcal{PW}_\pi^2$.

The norm equivalence (3.62) is very useful for the convergence analysis of the sampling series (3.57). For $f \in \mathcal{PW}_\pi^2$ we have

$$\begin{aligned}\left\| f - \sum_{k=-N}^{N} f(t_k)\phi_k \right\|_{\mathcal{PW}_\pi^2} &\le \frac{1}{C_{\mathrm{L}}^{\mathrm{Riesz}}} \left(\sum_{l=-\infty}^{\infty} \left| f(t_l) - \sum_{k=-N}^{N} f(t_k)\phi_k(t_l) \right|^2 \right)^{1/2} \\ &= \frac{1}{C_{\mathrm{L}}^{\mathrm{Riesz}}} \left(\sum_{|k|>N} |f(t_k)|^2 \right)^{1/2},\end{aligned}$$

where we used (3.62) in the first inequality and (3.59) in the last equality. Since

$$\left(\sum_{k=-\infty}^{\infty} |f(t_k)|^2 \right)^{1/2} \le C_{\mathrm{R}}^{\mathrm{Riesz}} \|f\|_{\mathcal{PW}_\pi^2}^2 < \infty,$$

according to (3.62), it follows that

$$\lim_{N\to\infty} \left(\sum_{|k|>N} |f(t_k)|^2 \right)^{1/2} = 0$$

and consequently

$$\lim_{N\to\infty} \left\| f - \sum_{k=-N}^{N} f(t_k)\phi_k \right\|_{\mathcal{PW}_\pi^2} = 0 \tag{3.63}$$

for all $f \in \mathcal{PW}_\pi^2$. Moreover, since $\|f\|_\infty \le \|f\|_{\mathcal{PW}_\pi^2}$, this implies that

$$\lim_{N\to\infty} \left\| f - \sum_{k=-N}^{N} f(t_k)\phi_k \right\|_{\infty} = 0 \tag{3.64}$$

for all signals $f \in \mathcal{PW}_\pi^2$.

Example 3.37. The Shannon sampling series is a special case of the general sampling series that are considered in this section. Let $\phi(t) = \sin(\pi t)$ with zeros $t_k = k$, $k \in \mathbb{Z}$. Then $\phi'(t_k) = \pi \cos(\pi t_k) = \pi(-1)^k$ and

$$\phi_k(t) = \frac{\phi(t)}{\phi'(t_k)(t - t_k)} = \frac{(-1)^k \sin(\pi t)}{\pi(t - t_k)} = \frac{\sin(\pi(t-k))}{\pi(t-k)}$$

is the well-known sinc-kernel of the Shannon sampling series.

Remark 3.38. A well-known fact concerning the relationship between Riesz bases and frames is the following [93, p. 157]. A sequence of vectors $\{\phi_k\}_{k\in\mathbb{Z}}$ in a separable Hilbert space $\mathcal{H}$ is a Riesz basis if and only if it is an exact frame. For further information about frames see for example [94].

3.3.2 Sine-Type Sampling Patterns

For arbitrary complete interpolating sequences, the functions ϕ can have a complicated behavior, which makes an analysis of (3.57) difficult. Therefore we restrict our analysis to sampling point sequences $\{t_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ that are given by the zeros of functions of sine type in this section. In Lemma 3.41 we will see that all these sequences are also complete interpolating sequences, which means that we restrict our analysis to a subclass of complete interpolating sequences. The use of sine-type functions makes the analysis easier because they have several helpful properties. In order to illustrate them we discuss equivalent definitions and characterizations, and state some of their key properties. For further information about sine-type functions see [93] and [25].

Definition 3.39. An entire function f of exponential type π is said to be of sine type if

i) the zeros of f are separated and simple, and

ii) there exist positive constants A, B, and H such that $A\,\mathrm{e}^{\pi|y|} \leq |f(x + \mathrm{i}y)| \leq B\,\mathrm{e}^{\pi|y|}$ whenever x and y are real and $|y| \geq H$.

We use Definition 3.39 to define sine-type functions in this thesis. In addition to Definition 3.39 there are other possible equivalent definitions that have advantages as well. However, Definition 3.39 explicitly states the structure of sine-type functions that we need to obtain our main results.

An equivalent definition [95] of sine-type functions is obtained if i) and ii) are replaced by the conditions that

i') the zeros of f are separated and lie in $\{z \in \mathbb{C} : |\mathrm{Im}(z)| \leq h\}$ for some $h > 0$, and

ii') there is a $y_0 \in \mathbb{R}$ and $A', B' > 0$ such that $A' \leq |f(x + \mathrm{i}y_0)| \leq B'$ for all $x \in \mathbb{R}$.

The definitions presented so far are valid for arbitrary functions of sine type with complex zeros. However, in this thesis we assume that all zeros are real. With this restriction it is easier to characterize functions of sine type, because we can determine whether a function ϕ is a function of sine type solely based on its behavior on the real axis. We assume that $\{t_k\}_{k\in\mathbb{Z}}$ is a real complete interpolating sequence. Then the generating function ϕ is not necessarily bounded on the real axis. However, according to part ii) of Definition 3.39 and the Phragmén–Lindelöf Theorem [93, pp. 68], ϕ has to be bounded on the real axis in order to be a sine-type function. This is the first condition on ϕ, which refers only to its behavior on the real axis. Additionally, we require that for every $\epsilon > 0$ there is a constant $C_{16}(\epsilon) > 0$ such that

$$|\phi(t)| \geq C_{16}(\epsilon) \tag{3.65}$$

for all $t \in \mathbb{R} \setminus \bigcup_{k\in\mathbb{Z}}(t_k - \epsilon, t_k + \epsilon)$. Of course, every sine-type function fulfills (3.65) [25, p. 163]. However, the converse is nontrivial and contained in the following interesting result [96].

Let ϕ be the generating function of a real complete interpolating sequence $\{t_k\}_{k\in\mathbb{Z}}$. Then all zeros are simple, and we can assume without loss of generality that $\phi(t) > 0$ for $t \in (t_0, t_1)$. Next, consider the sequence $\{c_k\}_{k\in\mathbb{Z}}$ defined by

$$c_k = \begin{cases} \max_{t\in(t_k,t_{k+1})} \phi(t), & \text{for } k \text{ even,} \\ \min_{t\in(t_k,t_{k+1})} \phi(t), & \text{for } k \text{ odd.} \end{cases}$$

We have $c_k(-1)^k > 0$ for all $k \in \mathbb{Z}$. ϕ is a function of sine type if and only if there exist two constants A, B such that

$$0 < A \leq |c_k| \leq B < \infty \tag{3.66}$$

for all $k \in \mathbb{Z}$ [96]. Condition (3.66) implies that ϕ is bounded on the real axis and that the maximum of $|\phi(t)|$ on $[t_k, t_{k+1}]$ is bounded from below by a positive constant, which is independent of $k \in \mathbb{Z}$. Of course, the requirement (3.66) is weaker than requirement (3.65) plus boundedness. Nevertheless, both conditions are sufficient to characterize sine-type functions with real zeros that are a complete interpolating sequence.

Example 3.40. $\sin(\pi z)$ is a function of sine type and its zeros are $t_k = k$, $k \in \mathbb{Z}$.

There is an important connection between the set of zeros $\{t_k\}_{k\in\mathbb{Z}}$ of a function of sine type, the basis properties of the system of exponentials $\{\mathrm{e}^{\mathrm{i}\omega t_k}\}_{k\in\mathbb{Z}}$, and complete interpolating sequences [93, pp. 143–144].

Lemma 3.41. *If $\{t_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ is the set of zeros of a function of sine type, then the system $\{\mathrm{e}^{\mathrm{i}\omega t_k}\}_{k\in\mathbb{Z}}$ is a Riesz basis for $L^2[-\pi,\pi]$, and $\{t_k\}_{k\in\mathbb{Z}}$ is a complete interpolating sequence for $\mathcal{PW}^2_\pi$.*

Proof. This lemma is a simple consequence of Theorems 9 and 10 on pages 143 and 144, respectively, in [93]. □

Lemma 3.41 implies that if ϕ is a function of sine type with zeros $\{t_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ then $\{\phi_k\}_{k\in\mathbb{Z}}$, where ϕ_k is given by (3.61), is a Riesz basis for $\mathcal{PW}_\pi^2$ [25, p. 169, Theorem 1].

In this section we analyze the convergence behavior of (3.57) for sampling points $\{t_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ that are the zeros of some sine-type function. Since $\{t_k\}_{k\in\mathbb{Z}}$ is a also complete interpolating sequence for $\mathcal{PW}_\pi^2$ according to Lemma 3.41, it follows that the product (3.60) converges uniformly on $|z| \le R$ for all $R < \infty$, and that ϕ is an entire function of exponential type π. Moreover, it follows that (3.61) is the unique function in $\mathcal{PW}_\pi^2$ that solves the interpolation problem (3.59).

Although we restrict the sampling patterns to the zeros of functions of sine type, there are many possible sampling patterns since the class of sine-type functions is very large. In Section 3.3.6 we will present a possibility to construct such functions.

Two important properties of sine-type functions, which will be used in the proofs, are stated in Lemmas 3.42 and 3.43.

Lemma 3.42. *Let f be a function of sine type, whose zeros $\{\lambda_k\}_{k\in\mathbb{Z}}$ are ordered increasingly according to their real parts. Then we have*

$$\inf_{k\in\mathbb{Z}} |\lambda_{k+1} - \lambda_k| \ge \underline{\delta} > 0 \tag{3.67}$$

and

$$\sup_{k\in\mathbb{Z}} |\lambda_{k+1} - \lambda_k| \le \overline{\delta} < \infty \tag{3.68}$$

for some constants $\underline{\delta}$ and $\overline{\delta}$.

Proof. Equation (3.67) follows directly from Definition 3.39 and the proof of (3.68) can be found in [25, p. 164]. □

Lemma 3.43. *Let f be a function of sine type. For each $\epsilon > 0$ there exists a number $C_{17} > 0$ such that*

$$|f(x + \mathrm{i}y)| \ge C_{17}\,\mathrm{e}^{\pi|y|}$$

outside the circles of radius ϵ centered at the zeros of f.

Proof. A proof of Lemma 3.43 can be found in [93, p. 144]. □

For further information about sine-type functions see for example [25] and [93].

3.3.3 Convergence Behavior without Oversampling

In this and the next section we analyze the sampling series (3.57) for signals from the Bernstein spaces $\mathcal{B}_{\beta\pi}^{\infty}$ and $\mathcal{B}_{\beta\pi,0}^{\infty}$. $\beta = 1$ corresponds to the case where no oversampling is used and $0 < \beta < 1$ to the case where oversampling is used. The case $\beta = 1$ is analyzed in this section, whereas the case $0 < \beta < 1$ is treated in Section 3.3.4.

The following lemma, which gives an upper bound on the increase of a function $f \in \mathcal{B}_{\beta\pi}^{\infty}$ parallel to the imaginary axis, will be important to obtain the results.

Lemma 3.44. *Let $f \in \mathcal{B}_{\beta\pi}^{\infty}$, $0 < \beta \leq 1$. Then we have*

$$|f(x + \mathrm{i}y)| \leq \mathrm{e}^{\beta\pi|y|}\|f\|_{\infty} \tag{3.69}$$

for all $x, y \in \mathbb{R}$.

Proof. Lemma 3.44 is a consequence of the Phragmén–Lindelöf principle. For a proof see [25, Lecture 6]. □

Lemma 3.44 enables us to derive some interesting convergence results for signals from the Bernstein spaces $\mathcal{B}_{\beta\pi}^{\infty}$ and $\mathcal{B}_{\beta\pi,0}^{\infty}$, $0 < \beta \leq 1$.

Local Convergence Behavior

Our first theorem shows that the approximation error is locally uniformly bounded for all signals in $\mathcal{B}_{\pi}^{\infty}$.

Theorem 3.45. *Let ϕ be a function of sine type, whose zeros $\{t_k\}_{k\in\mathbb{Z}}$ are all real and ordered increasingly according to (3.58). Furthermore, let ϕ_k be defined as in (3.61). Then, for all $\tau > 0$ there exists a constant $C_{18} = C_{18}(\tau)$ such that*

$$\sup_{N\in\mathbb{N}} \max_{t\in[-\tau,\tau]} \left| f(t) - \sum_{k=-N}^{N} f(t_k)\phi_k(t) \right| \leq C_{18}\|f\|_{\infty}$$

for all $f \in \mathcal{B}_{\pi}^{\infty}$.

For the proof of Theorem 3.45 we need the following lemma.

Lemma 3.46. *Let ϕ be a function of sine type, whose zeros are all real, and $Y_0 > 0$. Then there exists a constant C_{19} such that, for all $0 < \beta \leq 1$, $|Y| \geq Y_0$, $A, B \in \mathbb{R}$, $A \leq B$, $t \in \mathbb{R}$, and $f \in \mathcal{B}_{\beta\pi}^{\infty}$, we have*

$$\int_A^B \left| \frac{f(x + \mathrm{i}Y)}{\phi(x + \mathrm{i}Y)} \frac{\phi(t)}{x + \mathrm{i}Y - t} \right| \, \mathrm{d}x \leq \frac{(B - A)\|f\|_{\infty}\|\phi\|_{\infty}}{C_{19}|Y|}.$$

Proof. For all $x, Y \in \mathbb{R}$, we have $|f(x+\mathrm{i}Y)| \le \mathrm{e}^{\beta\pi|Y|}\|f\|_\infty$ according to Lemma 3.44. Furthermore, since ϕ is a function of sine type with all zeros being real, it follows from Lemma 3.43 that there exists a positive constant C_{19} such that $|\phi(x+\mathrm{i}Y)| \ge C_{19}\,\mathrm{e}^{\pi|Y|}$ for all $x \in \mathbb{R}$ and all $|Y| \ge Y_0$. Therefore, we obtain

$$\begin{aligned} \int_A^B \left| \frac{f(x+\mathrm{i}Y)}{\phi(x+\mathrm{i}Y)} \frac{\phi(t)}{x+\mathrm{i}Y-t} \right| \mathrm{d}x &\le \frac{\|f\|_\infty \|\phi\|_\infty}{C_{19}} \int_A^B \frac{\mathrm{e}^{-\pi(1-\beta)|Y|}}{|x+\mathrm{i}Y-t|} \mathrm{d}x \\ &\le \frac{(B-A)\|f\|_\infty\|\phi\|_\infty}{C_{19}|Y|}, \end{aligned}$$

which completes the proof. □

Proof of Theorem 3.45. Let $\tau > 0$ and $f \in \mathcal{B}_\pi^\infty$ be arbitrary but fixed and let

$$\tilde{t}_n = \begin{cases} (t_{n+1}+t_n)/2, & \text{for } n \ge 1 \\ (t_{n-1}+t_n)/2, & \text{for } n \le -1. \end{cases} \tag{3.70}$$

Furthermore, consider, for $N \in \mathbb{N}$ and $Y > 0$, the path $P_N(Y)$ in the complex plane that is depicted in Fig. 3.1. For all $N \in \mathbb{N}$ and $t \in \mathbb{R}$ we have the equality

$$\sum_{k=-N}^{N} f(t_k)\phi_k(t) = \frac{1}{2\pi \mathrm{i}} \oint_{P_N(Y)} \frac{\phi(\zeta)-\phi(t)}{\zeta - t} \frac{f(\zeta)}{\phi(\zeta)} \,\mathrm{d}\zeta. \tag{3.71}$$

Equation (3.71) can be easily seen by using the method of residues. Note that by the choice of $P_N(Y)$ we have $\phi(\zeta) \ne 0$ for all $\zeta \in P_N(Y)$. Furthermore, for all $N \in \mathbb{N}$ and $t \in \mathbb{R}$ with $\tilde{t}_{-N} < t < \tilde{t}_N$, we have

$$\begin{aligned} \frac{1}{2\pi \mathrm{i}} \oint_{P_N(Y)} \frac{\phi(\zeta)-\phi(t)}{\zeta - t} \frac{f(\zeta)}{\phi(\zeta)} \,\mathrm{d}\zeta &= \frac{1}{2\pi \mathrm{i}} \oint_{P_N(Y)} \frac{f(\zeta)}{\zeta - t} \,\mathrm{d}\zeta - \frac{1}{2\pi \mathrm{i}} \oint_{P_N(Y)} \frac{\phi(t)}{\zeta - t} \frac{f(\zeta)}{\phi(\zeta)} \,\mathrm{d}\zeta \\ &= f(t) - \frac{1}{2\pi \mathrm{i}} \oint_{P_N(Y)} \frac{\phi(t)}{\zeta - t} \frac{f(\zeta)}{\phi(\zeta)} \,\mathrm{d}\zeta. \end{aligned} \tag{3.72}$$

Combining (3.71) and (3.72), it follows that

$$f(t) - \sum_{k=-N}^{N} f(t_k)\phi_k(t) = \frac{1}{2\pi \mathrm{i}} \oint_{P_N(Y)} \frac{\phi(t)}{\zeta - t} \frac{f(\zeta)}{\phi(\zeta)} \,\mathrm{d}\zeta \tag{3.73}$$

for all $N \in \mathbb{N}$ and $t \in \mathbb{R}$ with $\tilde{t}_{-N} < t < \tilde{t}_N$.

According to Lemma 3.42 there exist two positive constants $\overline{\delta}$ and $\underline{\delta}$ such that (3.67) and (3.68) are fulfilled. Next choose $Y_N = N\overline{\delta}$, $N \in \mathbb{N}$. Since $|K|\underline{\delta} < |\tilde{t}_K| < (|K|+1)\overline{\delta}$, $K \in \mathbb{Z} \setminus \{0\}$, it follows that there are two positive constants C_{20} and C_{21} such that

$$\frac{Y_{|K|}}{C_{20}} < |\tilde{t}_K| < \frac{Y_{|K|}}{C_{21}} \tag{3.74}$$

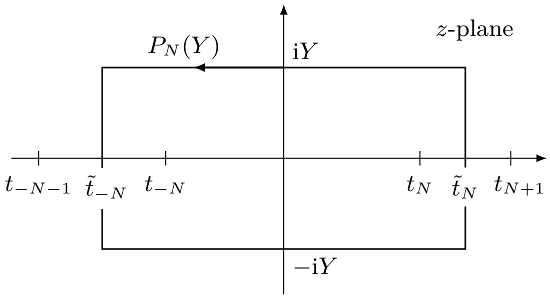

Figure 3.1: Path $P_N(Y)$ in the complex plane.

for all $K \in \mathbb{Z} \setminus \{0\}$.

Let N_0 be the smallest natural number for which $\min(t_{N_0}, |t_{-N_0}|) > \tau$. By using the identity (3.73), we obtain

$$\left| f(t) - \sum_{k=-N}^{N} f(t_k)\phi_k(t) \right| \leq \frac{1}{2\pi} \int_{-Y_N}^{Y_N} \left| \frac{f(\tilde{t}_N + \mathrm{i}y)}{\phi(\tilde{t}_N + \mathrm{i}y)} \right| \frac{|\phi(t)|}{|\tilde{t}_N + \mathrm{i}y - t|} \,\mathrm{d}y + \frac{1}{2\pi} \int_{-Y_N}^{Y_N} \left| \frac{f(\tilde{t}_{-N} + \mathrm{i}y)}{\phi(\tilde{t}_{-N} + \mathrm{i}y)} \right| \frac{|\phi(t)|}{|\tilde{t}_{-N} + \mathrm{i}y - t|} \,\mathrm{d}y + \frac{1}{2\pi} \int_{\tilde{t}_{-N}}^{\tilde{t}_N} \left| \frac{f(x + \mathrm{i}Y_N)}{\phi(x + \mathrm{i}Y_N)} \right| \frac{|\phi(t)|}{|x + \mathrm{i}Y_N - t|} \,\mathrm{d}x + \frac{1}{2\pi} \int_{\tilde{t}_{-N}}^{\tilde{t}_N} \left| \frac{f(x - \mathrm{i}Y_N)}{\phi(x - \mathrm{i}Y_N)} \right| \frac{|\phi(t)|}{|x - \mathrm{i}Y_N - t|} \,\mathrm{d}x \tag{3.75}$$

for all $N \geq N_0$ and $t \in [-\tau, \tau]$. Next, we will bound the right-hand side of (3.75) from above by analyzing each integral separately. It is important that this bound is independent of N.

For all $x, y \in \mathbb{R}$, we have $|f(x + \mathrm{i}y)| \leq \mathrm{e}^{\pi|y|}\|f\|_\infty$ according to Lemma 3.44. Furthermore, since ϕ is a function of sine type it follows from (3.67) and Lemma 3.43 that there exists a constant C_{22} such that $|\phi(\tilde{t}_K + \mathrm{i}y)| \geq C_{22}\,\mathrm{e}^{\pi|y|}$ for all $K \in \mathbb{Z} \setminus \{0\}$ and all $y \in \mathbb{R}$. Consequently, for the first term on the right-hand side of (3.75) we have

$$\begin{aligned} \frac{1}{2\pi} \int_{-Y_N}^{Y_N} \left| \frac{f(\tilde{t}_N + \mathrm{i}y)}{\phi(\tilde{t}_N + \mathrm{i}y)} \right| \frac{|\phi(t)|}{|\tilde{t}_N + \mathrm{i}y - t|} \,\mathrm{d}y &\leq \frac{\|f\|_\infty \|\phi\|_\infty}{C_{22}} \frac{1}{2\pi} \int_{-Y_N}^{Y_N} \frac{1}{|\tilde{t}_N + \mathrm{i}y - t|} \,\mathrm{d}y \\ &\leq \frac{\|f\|_\infty \|\phi\|_\infty}{C_{22}} \frac{Y_N}{\pi(\tilde{t}_N - \tau)} \\ &\leq \frac{\|f\|_\infty \|\phi\|_\infty}{C_{22}} \frac{\tilde{t}_N C_{20}}{(\tilde{t}_N - \tau)} \end{aligned}$$

for all $N \geq N_0$ and $t \in [-\tau, \tau]$, where we used (3.74) in the last inequality. Similarly, for the second term we obtain

$$\begin{aligned}\frac{1}{2\pi}\int_{-Y_N}^{Y_N}\left|\frac{f(\tilde{t}_{-N}+\mathrm{i}y)}{\phi(\tilde{t}_{-N}+\mathrm{i}y)}\right|\frac{|\phi(t)|}{|\tilde{t}_{-N}+\mathrm{i}y-t|}\,\mathrm{d}y &\leq \frac{\|f\|_\infty\|\phi\|_\infty}{C_{22}}\frac{Y_N}{\pi(|\tilde{t}_{-N}|-\tau)}\\ &\leq \frac{\|f\|_\infty\|\phi\|_\infty}{C_{22}}\frac{|\tilde{t}_{-N}|C_{20}}{(|\tilde{t}_{-N}|-\tau)}\end{aligned}$$

for all $N \geq N_0$ and $t \in [-\tau, \tau]$. The third term on the right-hand side of (3.75) can be bounded from above by using Lemma 3.46. For all $N \geq N_0$ we have

$$\begin{aligned}\frac{1}{2\pi}\int_{\tilde{t}_{-N}}^{\tilde{t}_N}\left|\frac{f(x+\mathrm{i}Y_N)}{\phi(x+\mathrm{i}Y_N)}\right|\frac{|\phi(t)|}{|x+\mathrm{i}Y_N-t|}\,\mathrm{d}x &\leq \frac{\|f\|_\infty\|\phi\|_\infty}{\pi C_{22}}\frac{(\tilde{t}_N-\tilde{t}_{-N})}{2Y_N}\\ &\leq \frac{\|f\|_\infty\|\phi\|_\infty}{\pi C_{22}C_{21}},\end{aligned}$$

where we used (3.74) again in the last inequality. Similarly, we obtain for the fourth term that

$$\frac{1}{2\pi}\int_{\tilde{t}_{-N}}^{\tilde{t}_N}\left|\frac{f(x-\mathrm{i}Y_N)}{\phi(x-\mathrm{i}Y_N)}\right|\frac{|\phi(t)|}{|x-\mathrm{i}Y_N-t|}\,\mathrm{d}x \leq \frac{\|f\|_\infty\|\phi\|_\infty}{\pi C_{22}C_{21}}$$

for all $N \geq N_0$. Next, we choose $N_1 \geq N_0$ such that

$$\max\left(\frac{\tilde{t}_{N_1}}{\tilde{t}_{N_1}-\tau}, \frac{|\tilde{t}_{-N_1}|}{|\tilde{t}_{-N_1}|-\tau}\right) \leq 2.$$

Note that N_1 depends only on τ and not on f. Consequently, for all $N \geq N_1$ and $t \in [-\tau, \tau]$, we have

$$\left|f(t) - \sum_{k=-N}^{N} f(t_k)\phi_k(t)\right| \leq C_{23}\|f\|_\infty,$$

where C_{23} is some constant. □

Theorem 3.45 shows that the approximation error

$$\left|f(t) - \sum_{k=-N}^{N} f(t_k)\phi_k(t)\right| \tag{3.76}$$

is uniformly bounded on all compact subsets of $\mathbb{R}$ as N tends to infinity. If we further assume that $\lim_{|t|\to\infty} f(t) = 0$, i.e., that $f \in \mathcal{B}_{\pi,0}^\infty$, we can prove that the approximation error (3.76) converges to zero uniformly on all compact subsets of $\mathbb{R}$.

Theorem 3.47. *Let ϕ be a function of sine type, whose zeros $\{t_k\}_{k\in\mathbb{Z}}$ are all real and ordered increasingly according to* (3.58)*. Furthermore, let ϕ_k be defined as in* (3.61)*. Then, for all $\tau > 0$ and all $f \in \mathcal{B}^{\infty}_{\pi,0}$ we have*

$$\lim_{N\to\infty} \max_{t\in[-\tau,\tau]} \left| f(t) - \sum_{k=-N}^{N} f(t_k)\phi_k(t) \right| = 0.$$

Theorem 3.47 generalizes Brown's Theorem (Theorem 3.6), where the uniform convergence on compact subsets of $\mathbb{R}$ was stated for equidistant sampling and signals in $\mathcal{PW}^1_\pi$, towards non-equidistant sampling and the larger signal space $\mathcal{B}^{\infty}_{\pi,0}$.

Proof of Theorem 3.47. Let $\tau > 0$ and $f \in \mathcal{B}^{\infty}_{\pi,0}$ be arbitrary but fixed. Since $\mathcal{B}^{\infty}_{\pi,0} \subset \mathcal{B}^{\infty}_{\pi}$, it follows by Theorem 3.45 that there exists a constant C_{18} such that

$$\left| f(t) - \sum_{k=-N}^{N} f(t_k)\phi_k(t) \right| \le C_{18}\|f\|_\infty \tag{3.77}$$

for all $N \in \mathbb{N}$ and $t \in [-\tau,\tau]$. In order to complete the proof, we use (3.77), the fact that $\mathcal{PW}^2_\pi$ is dense in $\mathcal{B}^{\infty}_{\pi,0}$, and the uniform convergence of the series for $\mathcal{PW}^2_\pi$. Let $\epsilon > 0$ be arbitrary but fixed. There exists a function $f_\epsilon \in \mathcal{PW}^2_\pi$ such that $\|f - f_\epsilon\|_\infty < \epsilon$. As a consequence, we have

$$\begin{aligned} &\left| f(t) - \sum_{k=-N}^{N} f(t_k)\phi_k(t) \right| \\ &\quad\le \left| f(t) - f_\epsilon(t) - \sum_{k=-N}^{N} (f(t_k) - f_\epsilon(t_k))\phi_k(t) \right| + \left| f_\epsilon(t) - \sum_{k=-N}^{N} f_\epsilon(t_k)\phi_k(t) \right| \\ &\quad\le C_{18}\|f - f_\epsilon\|_\infty + \left| f_\epsilon(t) - \sum_{k=-N}^{N} f_\epsilon(t_k)\phi_k(t) \right| \\ &\quad\le C_{18}\epsilon + \left| f_\epsilon(t) - \sum_{k=-N}^{N} f_\epsilon(t_k)\phi_k(t) \right| \end{aligned} \tag{3.78}$$

for all $N \in \mathbb{N}$ and all $t \in [-\tau,\tau]$. Furthermore, since $f_\epsilon \in \mathcal{PW}^2_\pi$, we can use (3.64), i.e., the uniform convergence of the series. It follows that there exists a natural number $N_2(\epsilon)$ such that

$$\left| f_\epsilon(t) - \sum_{k=-N}^{N} f_\epsilon(t_k)\phi_k(t) \right| < \epsilon \tag{3.79}$$

for all $N \geq N_2(\epsilon)$ and $t \in \mathbb{R}$. Combining (3.78) and (3.79) gives

$$\max_{t\in[-\tau,\tau]} \left| f(t) - \sum_{k=-N}^{N} f(t_k)\phi_k(t) \right| \le (C_{18} + 1)\,\epsilon$$

for all $N \geq N_2(\epsilon)$, which completes the proof, because ϵ was arbitrary. □

Although Theorem 3.47 states the local uniform convergence for all $f \in \mathcal{B}_{\pi,0}^{\infty}$, it makes no assertion about the convergence speed. We have the following conjecture.

Conjecture 3.48. *The convergence speed in Theorem 3.47 can be arbitrarily slow. That is, for every $\tau > 0$ and every positive sequence ϵ_N converging to zero, there exists a signal $f_1 \in \mathcal{B}_{\pi,0}^{\infty}$ such that*

$$\limsup_{N\to\infty} \frac{1}{\epsilon_N} \max_{t\in[-\tau,\tau]} \left| f_1(t) - \sum_{k=-N}^{N} f_1(t_k)\phi_k(t) \right| = \infty.$$

In Section 3.3.4 we will see that the situation is different if oversampling is used. With oversampling it is possible to specify the convergence speed.

The next conjecture, Conjecture 3.49, makes a statement about the local convergence behavior of the sampling series (3.57) for sampling patterns that are complete interpolating sequences but where the sampling points are not the zeros of a sine-type function.

Conjecture 3.49. *For all $\tau > 0$ there exist a complete interpolating sequence $\{t_k\}_{k\in\mathbb{Z}}$ for $\mathcal{PW}_{\pi}^2$ and a signal $f_1 \in \mathcal{B}_{\pi,0}^{\infty}$ such that*

$$\lim_{N\to\infty} \max_{t\in[-\tau,\tau]} \left| f_1(t) - \sum_{k=-N}^{N} f_1(t_k)\phi_k(t) \right| = \infty.$$

If Conjecture 3.49 is true, it shows the importance of the assumption in Theorem 3.47 that ϕ is a sine-type function.

Global Convergence Behavior

In Theorem 3.11 we have shown for the space $\mathcal{PW}_{\pi}^1$ and a large class of reconstruction processes that neither a globally uniformly convergent nor a locally uniformly convergent and globally bounded signal reconstruction is possible if the samples are taken equidistantly at Nyquist rate. In contrast, for non-equidistant sampling the global convergence behavior of sampling based reconstruction processes is unknown in general. By using non-equidistant sampling, an additional degree of freedom is created, which may help to improve the convergence behavior. However, we suspect that non-equidistant sampling is not capable to improve the global convergence behavior.

In Theorem 3.47 we have seen that the sampling series (3.57) is locally uniformly convergent for all signals in $\mathcal{B}_{\pi,0}^{\infty}$. Next, we show that the global behavior is different. There are signals in $\mathcal{B}_{\pi,0}^{\infty}$ for which the peak value of the approximation error diverges.

Theorem 3.50. *Let ϕ be a function of sine type, whose zeros $\{t_k\}_{k\in\mathbb{Z}}$ are all real and ordered increasingly according to* (3.58)*. Furthermore, let ϕ_k be defined as in* (3.61)*. Then there exists a signal $f_1 \in \mathcal{B}^\infty_{\pi,0}$ such that*

$$\limsup_{N\to\infty} \max_{t\in\mathbb{R}} \left| f_1(t) - \sum_{k=-N}^{N} f_1(t_k)\phi_k(t) \right| = \infty.$$

Proof. Let $N \in \mathbb{N}$ be arbitrary but fixed. For the operator $A_N^\phi : \mathcal{B}^\infty_{\pi,0} \to \mathcal{B}^\infty_{\pi,0}$, defined by

$$(A_N^\phi f)(t) := \sum_{k=-N}^{N} f(t_k)\phi_k(t), \tag{3.80}$$

we analyze the operator norm

$$\|A_N^\phi\| = \sup_{\substack{f\in\mathcal{B}^\infty_{\pi,0}\\ \|f\|_\infty\le 1}} \|A_N^\phi f\|_\infty.$$

For $0 < \epsilon < 1$, let

$$f_\epsilon(t) = \phi((1-\epsilon)t)\frac{\sin(\epsilon\pi t)}{\epsilon\pi t}.$$

Note that $f_\epsilon \in \mathcal{B}^\infty_{\pi,0}$. A simple calculation shows that $\lim_{\epsilon\to 0} f_\epsilon'(t) = \phi'(t)$ for all $t \in \mathbb{R}$. Next, choose a constant $C_{24} > 0$, so large that $\|f_\epsilon'\|_\infty / C_{24} \le 1$ for all $0 < \epsilon < 1$. It follows that

$$\begin{aligned}
\|A_N^\phi\| &\ge \lim_{\epsilon\to 0} \frac{1}{C_{24}} \left| \sum_{k=-N}^{N} f_\epsilon'(t_k)\phi_k(\tilde{t}_N) \right| \\
&= \lim_{\epsilon\to 0} \frac{1}{C_{24}} \left| \sum_{k=-N}^{N} \frac{f_\epsilon'(t_k)\phi(\tilde{t}_N)}{\phi'(t_k)(\tilde{t}_N - t_k)} \right| \\
&= \frac{1}{C_{24}} \left| \phi(\tilde{t}_N) \sum_{k=-N}^{N} \frac{1}{\tilde{t}_N - t_k} \right|,
\end{aligned} \tag{3.81}$$

where $\tilde{t}_N = (t_{N+1} + t_N)/2$. Because of (3.67) and Lemma 3.43, there exists a constant $C_{25} > 0$ such that $|\phi(\tilde{t}_N)| \ge C_{25}$ for all $N \in \mathbb{N}$. Furthermore, according to (3.68), we have $\tilde{t}_N - t_k \le (N+1-k)\overline{\delta}$ for all $|k| \le N$, and it follows that

$$\sum_{k=-N}^{N} \frac{1}{\tilde{t}_N - t_k} \ge \sum_{k=-N}^{N} \frac{1}{(N+1-k)\overline{\delta}} = \frac{1}{\overline{\delta}} \sum_{k=1}^{2N+1} \frac{1}{k} \ge \frac{1}{\overline{\delta}} \log(2N+2). \tag{3.82}$$

Combining (3.81) and (3.82) gives

$$\|A_N^\phi\| \ge \frac{C_{25}}{C_{24}\overline{\delta}} \log(2N+2).$$

Thus, the Banach–Steinhaus theorem [78, p. 98] implies that there exists a signal $f_1 \in \mathcal{B}_{\pi,0}^{\infty}$ such that

$$\limsup_{N\to\infty} \max_{t\in\mathbb{R}} \left| \sum_{k=-N}^{N} f_1(t_k)\phi_k(t) \right| = \infty. \tag{3.83}$$

Since

$$\left| f_1(t) - \sum_{k=-N}^{N} f_1(t_k)\phi_k(t) \right| \geq \left| \sum_{k=-N}^{N} f_1(t_k)\phi_k(t) \right| - \|f_1\|_\infty,$$

the proof is complete. □

3.3.4 Convergence Behavior with Oversampling

Local Convergence Behavior

In Theorem 3.45, where we analyzed the sampling series (3.57) for signals in $\mathcal{B}_{\pi}^{\infty}$, we could only prove the boundedness of the approximation error on compact subsets of $\mathbb{R}$, but not the convergence of the series. However, if we consider oversampling, i.e., signals in $\mathcal{B}_{\beta\pi}^{\infty}$, $0 < \beta < 1$, then the sampling series (3.57) is uniformly convergent on all compact subsets of $\mathbb{R}$.

Theorem 3.51. *Let ϕ be a function of sine type, whose zeros $\{t_k\}_{k\in\mathbb{Z}}$ are all real and ordered increasingly according to* (3.58)*. Furthermore, let ϕ_k be defined as in* (3.61) *and $0 < \beta < 1$. Then, for all $\tau > 0$ and all $f \in \mathcal{B}_{\beta\pi}^{\infty}$ we have*

$$\lim_{N\to\infty} \max_{t\in[-\tau,\tau]} \left| f(t) - \sum_{k=-N}^{N} f(t_k)\phi_k(t) \right| = 0.$$

Proof. Let $\tau > 0$ be arbitrary but fixed. We start with the same identity as in the proof of Theorem 3.45, namely

$$f(t) - \sum_{k=-N}^{N} f(t_k)\phi_k(t) = \frac{1}{2\pi \mathrm{i}} \oint_{P_N(Y)} \frac{\phi(t)}{(\zeta - t)} \frac{f(\zeta)}{\phi(\zeta)} \,\mathrm{d}\zeta, \tag{3.84}$$

which is valid for all $N \in \mathbb{N}$, $Y > 0$, and $t \in \mathbb{R}$ with $\tilde{t}_{-N} < t < \tilde{t}_N$. $\tilde{t}_N$ is defined as in (3.70), and the integration path $P_N(Y)$ is depicted in Fig. 3.1. Let N_0 be the smallest natural number for which $N_0\underline{\delta} > \tau$. Since $\tilde{t}_N \geq N\underline{\delta}$ for all $N \in \mathbb{N}$, it follows that $\tilde{t}_{N_0} > \tau$. Furthermore, let $Y_N = N\overline{\delta}$. By using the identity (3.84), we obtain

$$\begin{aligned}\left| f(t) - \sum_{k=-N}^{N} f(t_k)\phi_k(t) \right| \leq & \frac{1}{2\pi} \int_{-Y_N}^{Y_N} \left| \frac{f(\tilde{t}_N + \mathrm{i}y)}{\phi(\tilde{t}_N + \mathrm{i}y)} \right| \frac{|\phi(t)|}{|\tilde{t}_N + \mathrm{i}y - t|} \, \mathrm{d}y \\ & + \frac{1}{2\pi} \int_{-Y_N}^{Y_N} \left| \frac{f(\tilde{t}_{-N} + \mathrm{i}y)}{\phi(\tilde{t}_{-N} + \mathrm{i}y)} \right| \frac{|\phi(t)|}{|\tilde{t}_{-N} + \mathrm{i}y - t|} \, \mathrm{d}y \\ & + \frac{1}{2\pi} \int_{\tilde{t}_{-N}}^{\tilde{t}_N} \left| \frac{f(x + \mathrm{i}Y_N)}{\phi(x + \mathrm{i}Y_N)} \right| \frac{|\phi(t)|}{|x + \mathrm{i}Y_N - t|} \, \mathrm{d}x \\ & + \frac{1}{2\pi} \int_{\tilde{t}_{-N}}^{\tilde{t}_N} \left| \frac{f(x - \mathrm{i}Y_N)}{\phi(x - \mathrm{i}Y_N)} \right| \frac{|\phi(t)|}{|x - \mathrm{i}Y_N - t|} \, \mathrm{d}x \end{aligned} \tag{3.85}$$

for all $N \geq N_0$ and all $t \in [-\tau, \tau]$. Next, we will analyze each integral on the right-hand side of (3.85) separately for $N \geq N_0$ and $t \in [-\tau, \tau]$.

Because of (3.67) and the definition of $\tilde{t}_N$, it follows that the distance between $\tilde{t}_N$ and the nearest zero of ϕ is at least $\underline{\delta}/2$. Hence, according to Lemma 3.43, there exists a constant $C_{26} > 0$ such that $|\phi(\tilde{t}_N + \mathrm{i}y)| \geq C_{26}\, \mathrm{e}^{\pi|y|}$. Therefore, for the first integral we obtain

$$\frac{1}{2\pi} \int_{-Y_N}^{Y_N} \left| \frac{f(\tilde{t}_N + \mathrm{i}y)}{\phi(\tilde{t}_N + \mathrm{i}y)} \right| \frac{|\phi(t)|}{|\tilde{t}_N + \mathrm{i}y - t|} \, \mathrm{d}y \leq \frac{\|f\|_\infty \|\phi\|_\infty}{2\pi C_{26}(\underline{\delta}/2)} \int_{-Y_N}^{Y_N} \frac{\mathrm{e}^{-\pi(1-\beta)|y|}}{|\tilde{t}_N + \mathrm{i}y - t|} \, \mathrm{d}y, \tag{3.86}$$

by using (3.69). Furthermore, we have

$$\begin{aligned}\int_{-Y_N}^{Y_N} \frac{\mathrm{e}^{-\pi(1-\beta)|y|}}{|\tilde{t}_N + \mathrm{i}y - t|} \, \mathrm{d}y &\leq \frac{2}{\tilde{t}_N - \tau} \int_0^{Y_N} \mathrm{e}^{-\pi(1-\beta)|y|} \, \mathrm{d}y \\ &\leq \frac{2}{\tilde{t}_N - \tau} \frac{1 - \mathrm{e}^{-\pi(1-\beta)Y_N}}{\pi(1-\beta)} \\ &\leq \frac{2}{(N\underline{\delta} - \tau)\pi(1-\beta)}. \end{aligned} \tag{3.87}$$

Combining (3.86) and (3.87) gives

$$\frac{1}{2\pi} \int_{-Y_N}^{Y_N} \left| \frac{f(\tilde{t}_N + \mathrm{i}y)}{\phi(\tilde{t}_N + \mathrm{i}y)} \right| \frac{|\phi(t)|}{|\tilde{t}_N + \mathrm{i}y - t|} \, \mathrm{d}y \leq \frac{\|f\|_\infty \|\phi\|_\infty}{\pi^2 C_{26}(N\underline{\delta} - \tau)(1-\beta)}. \tag{3.88}$$

Similarly, for the second integral, we obtain

$$\frac{1}{2\pi} \int_{-Y_N}^{Y_N} \left| \frac{f(\tilde{t}_{-N} + \mathrm{i}y)}{\phi(\tilde{t}_{-N} + \mathrm{i}y)} \right| \frac{|\phi(t)|}{|\tilde{t}_{-N} + \mathrm{i}y - t|} \, \mathrm{d}y \leq \frac{\|f\|_\infty \|\phi\|_\infty}{\pi^2 C_{26}(N\underline{\delta} - \tau)(1-\beta)}.$$

Next, we treat the third integral on the right-hand side of (3.85). Since all zeros of ϕ are real and $Y_N = N\overline{\delta} \geq \overline{\delta}$, it follows from Lemma 3.43 that there exists a constant

$C_{27} > 0$ such that $|\phi(x + \mathrm{i}Y_N)| \geq C_{27}\,\mathrm{e}^{\pi Y_N}$ for all $x \in \mathbb{R}$. Thus, we obtain

$$\begin{aligned}\frac{1}{2\pi}\int_{\tilde{t}_{-N}}^{\tilde{t}_N}\left|\frac{f(x+\mathrm{i}Y_N)}{\phi(x+\mathrm{i}Y_N)}\right|\frac{|\phi(t)|}{|x+\mathrm{i}Y_N-t|}\,\mathrm{d}x &\leq \frac{\|f\|_\infty\|\phi\|_\infty}{2\pi C_{27}}\int_{\tilde{t}_{-N}}^{\tilde{t}_N}\frac{\mathrm{e}^{-\pi(1-\beta)Y_N}}{|x+\mathrm{i}Y_N-t|}\\ &\leq \frac{\|f\|_\infty\|\phi\|_\infty(\tilde{t}_N-\tilde{t}_{-N})}{2\pi C_{27}Y_N}\,\mathrm{e}^{-\pi(1-\beta)Y_N}\\ &\leq \frac{2\|f\|_\infty\|\phi\|_\infty}{\pi C_{27}}\,\mathrm{e}^{-\pi(1-\beta)N\overline{\delta}}. \qquad (3.89)\end{aligned}$$

by using Lemma 3.44 and the fact that $\tilde{t}_N - \tilde{t}_{-N} \leq (2N+1)\overline{\delta}$ and $Y_N = N\overline{\delta}$. For the fourth integral we obtain the same upper bound

$$\frac{1}{2\pi}\int_{\tilde{t}_{-N}}^{\tilde{t}_N}\left|\frac{f(x-\mathrm{i}Y_N)}{\phi(x-\mathrm{i}Y_N)}\right|\frac{|\phi(t)|}{|x-\mathrm{i}Y_N-t|}\,\mathrm{d}x \leq \frac{2\|f\|_\infty\|\phi\|_\infty}{\pi C_{27}}\,\mathrm{e}^{-\pi(1-\beta)N\overline{\delta}}. \qquad (3.90)$$

Finally, combining the partial results (3.88)–(3.90) with (3.85) yields

$$\left|f(t) - \sum_{k=-N}^{N} f(t_k)\phi_k(t)\right| \leq \|f\|_\infty\|\phi\|_\infty\left(\frac{2}{\pi^2 C_{26}(N\underline{\delta}-\tau)(1-\beta)} + \frac{4\,\mathrm{e}^{-\pi(1-\beta)N\overline{\delta}}}{\pi C_{27}}\right) \qquad (3.91)$$

for $N \geq N_0$ and $t \in [-\tau, \tau]$. Taking the maximum $\max_{t\in[-\tau,\tau]}$ and the limit $N \to \infty$ on both sides of (3.91) completes the proof. □

The proof of Theorem 3.51 shows the significance of oversampling. In (3.91) we have an upper bound on the approximation error, which decreases asymptotically like $1/N$. In contrast, in Theorem 3.45, where we treated the situation without oversampling, we only have the boundedness of the approximation error and no statement whether the approximation error converges to zero as N goes to infinity. Even in Theorem 3.47, where we proved the local uniform convergence for the subspace $\mathcal{B}_{\pi,0}^\infty \subset \mathcal{B}_\pi^\infty$, we have no such convergence speed. As formulated in Conjecture 3.48, we conjecture that the convergence in Theorem 3.47 can be arbitrarily slow.

Global Convergence Behavior

The next two theorems treat the global convergence behavior of the sampling series (3.57) if oversampling is used. For signals in $\mathcal{B}_{\beta\pi}^\infty$, $0 < \beta < 1$, the approximation error is globally uniformly bounded.

Theorem 3.52. *Let ϕ be a function of sine type, whose zeros $\{t_k\}_{k\in\mathbb{Z}}$ are all real and ordered increasingly according to* (3.58). *Furthermore, let ϕ_k be defined as in* (3.61) *and $0 < \beta < 1$. Then, there exists a constant C_{28} such that*

$$\sup_{N\in\mathbb{N}}\max_{t\in\mathbb{R}}\left|f(t) - \sum_{k=-N}^{N} f(t_k)\phi_k(t)\right| \leq C_{28}\|f\|_\infty$$

for all $f \in \mathcal{B}_{\beta\pi}^\infty$.

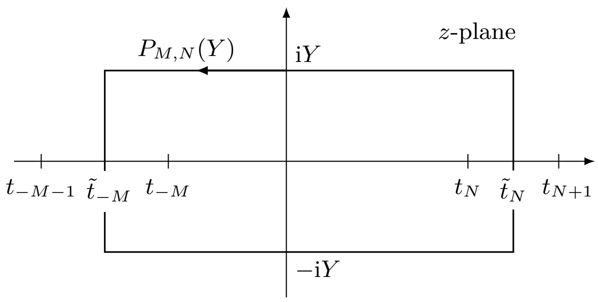

Figure 3.2: Path $P_{M,N}(Y)$ in the complex plane.

For the proof of Theorem 3.52 we need the following lemma.

Lemma 3.53. *Let ϕ be a function of sine type, whose zeros $\{t_k\}_{k\in\mathbb{Z}}$ are all real and ordered increasingly according to* (3.58)*, $0 < \beta < 1$, and $\tilde{t}_n$ as defined in* (3.70)*. Then there exist a constant C_{29} such that for all $K \in \mathbb{Z} \setminus \{0\}$, $Y > 0$, $t \in \mathbb{R}$ with $|t - \tilde{t}_K| \geq \underline{\delta}$, and $f \in \mathcal{B}^\infty_{\beta\pi}$ we have*

$$\int_{-Y}^{Y} \left| \frac{f(\tilde{t}_K + \mathrm{i}y)}{\phi(\tilde{t}_K + \mathrm{i}y)} \frac{\phi(t)}{\tilde{t}_K + \mathrm{i}y - t} \right| \, \mathrm{d}y \leq \frac{2\|f\|_\infty \|\phi\|_\infty}{\pi C_{29} \underline{\delta}(1-\beta)}.$$

Proof. For all $x, y \in \mathbb{R}$, we have $|f(x + \mathrm{i}y)| \leq \mathrm{e}^{\beta\pi|y|}\|f\|_\infty$ according to Lemma 3.44. Furthermore, since ϕ is a function of sine type it follows from Lemma 3.43 and (3.67) that there exists a constant C_{29} such that $|\phi(\tilde{t}_K + \mathrm{i}y)| \geq C_{29}\, \mathrm{e}^{\pi|y|}$ for all $K \in \mathbb{Z} \setminus \{0\}$ and $y \in \mathbb{R}$. Therefore, we obtain

$$\begin{aligned} \int_{-Y}^{Y} \left| \frac{f(\tilde{t}_K + \mathrm{i}y)}{\phi(\tilde{t}_K + \mathrm{i}y)} \frac{\phi(t)}{\tilde{t}_K + \mathrm{i}y - t} \right| \, \mathrm{d}y &\leq \frac{\|f\|_\infty \|\phi\|_\infty}{C_{29}} \int_{-Y}^{Y} \left| \frac{\mathrm{e}^{-\pi(1-\beta)|y|}}{\tilde{t}_K + \mathrm{i}y - t} \right| \, \mathrm{d}y \\ &\leq \frac{2\|f\|_\infty \|\phi\|_\infty}{C_{29}\underline{\delta}} \int_0^Y \mathrm{e}^{-\pi(1-\beta)y} \, \mathrm{d}y \\ &\leq \frac{2\|f\|_\infty \|\phi\|_\infty}{\pi C_{29} \underline{\delta}(1-\beta)}, \end{aligned}$$

which completes the proof. □

Proof of Theorem 3.52. Let $0 < \beta < 1$ and $f \in \mathcal{B}^\infty_{\beta\pi}$ be arbitrary but fixed. Furthermore, let $\tilde{t}_n$ be defined as in (3.70) and consider, for $M, N \in \mathbb{N}$ and $Y > 0$, the path $P_{M,N}(Y)$ in the complex plane that is depicted in Fig. 3.2.

For all $M, N \in \mathbb{N}$ and $t \in \mathbb{R}$ we have

$$\sum_{k=-M}^{N} f(t_k)\phi_k(t) = \frac{1}{2\pi\mathrm{i}} \oint_{P_{M,N}(Y)} \frac{\phi(\zeta) - \phi(t)}{\zeta - t} \frac{f(\zeta)}{\phi(\zeta)} \, \mathrm{d}\zeta. \tag{3.92}$$

Note that by the choice of $P_{M,N}(Y)$ we have $\phi(\zeta) \neq 0$ for all $\zeta \in P_{M,N}(Y)$. If $t \in (\tilde{t}_{-M}, \tilde{t}_N)$, we have

$$\frac{1}{2\pi \mathrm{i}} \oint_{P_{M,N}(Y)} \frac{\phi(\zeta) - \phi(t)}{\zeta - t} \frac{f(\zeta)}{\phi(\zeta)} \, \mathrm{d}\zeta = f(t) - \frac{1}{2\pi \mathrm{i}} \oint_{P_{M,N}(Y)} \frac{\phi(t)}{\zeta - t} \frac{f(\zeta)}{\phi(\zeta)} \, \mathrm{d}\zeta, \tag{3.93}$$

and, by combining (3.92) and (3.93), it follows that

$$f(t) - \sum_{k=-M}^{N} f(t_k)\phi_k(t) = \frac{1}{2\pi \mathrm{i}} \oint_{P_{M,N}(Y)} \frac{\phi(t)}{\zeta - t} \frac{f(\zeta)}{\phi(\zeta)} \, \mathrm{d}\zeta \tag{3.94}$$

for all $M, N \in \mathbb{N}$ and $t \in (\tilde{t}_{-M}, \tilde{t}_N)$. If $t \in \mathbb{R} \setminus [\tilde{t}_{-M}, \tilde{t}_N]$ we have

$$\oint_{P_{M,N}(Y)} \frac{f(\zeta)}{\zeta - t} \, \mathrm{d}\zeta = 0,$$

and it follows that

$$\sum_{k=-M}^{N} f(t_k)\phi_k(t) = \frac{-1}{2\pi \mathrm{i}} \oint_{P_{M,N}(Y)} \frac{\phi(t)}{\zeta - t} \frac{f(\zeta)}{\phi(\zeta)} \, \mathrm{d}\zeta \tag{3.95}$$

for all $M, N \in \mathbb{N}$ and $t \in \mathbb{R} \setminus [\tilde{t}_{-M}, \tilde{t}_N]$.

Let $N \in \mathbb{N}$, $N > 2$ be arbitrary, but fixed. According to Lemma 3.42 there exist two positive constants $\overline{\delta}$ and $\underline{\delta}$ such that (3.67) and (3.68) are fulfilled. Next choose $Y_N = N\overline{\delta}$, $N \in \mathbb{N}$. Without loss of generality, we can assume that $t > 0$, because negative t are treated analogously to positive. We have to distinguish two cases, first $0 \leq t \leq \tilde{t}_N$, and second $t > \tilde{t}_N$.

For $0 \leq t \leq \tilde{t}_N$ we have

$$\begin{aligned} \left| f(t) - \sum_{k=-N}^{N} f(t_k)\phi_k(t) \right| &\leq |f(t_{N+1})\phi_{N+1}(t)| + \left| f(t) - \sum_{k=-N}^{N+1} f(t_k)\phi_k(t) \right| \\ &\leq \|f\|_\infty \|\phi_{N+1}\|_\infty + \left| f(t) - \sum_{k=-N}^{N+1} f(t_k)\phi_k(t) \right|. \end{aligned} \tag{3.96}$$

Furthermore, by using the identity (3.94), we obtain

$$\begin{aligned} \left| f(t) - \sum_{k=-N}^{N+1} f(t_k)\phi_k(t) \right| \leq & \frac{1}{2\pi} \int_{-Y_N}^{Y_N} \left| \frac{f(\tilde{t}_{N+1} + \mathrm{i}y)}{\phi(\tilde{t}_{N+1} + \mathrm{i}y)} \right| \frac{|\phi(t)|}{|\tilde{t}_{N+1} + \mathrm{i}y - t|} \, \mathrm{d}y \\ & + \frac{1}{2\pi} \int_{-Y_N}^{Y_N} \left| \frac{f(\tilde{t}_{-N} + \mathrm{i}y)}{\phi(\tilde{t}_{-N} + \mathrm{i}y)} \right| \frac{|\phi(t)|}{|\tilde{t}_{-N} + \mathrm{i}y - t|} \, \mathrm{d}y \\ & + \frac{1}{2\pi} \int_{\tilde{t}_{-N}}^{\tilde{t}_{N+1}} \left| \frac{f(x + \mathrm{i}Y_N)}{\phi(x + \mathrm{i}Y_N)} \right| \frac{|\phi(t)|}{|x + \mathrm{i}Y_N - t|} \, \mathrm{d}x \\ & + \frac{1}{2\pi} \int_{\tilde{t}_{-N}}^{\tilde{t}_{N+1}} \left| \frac{f(x - \mathrm{i}Y_N)}{\phi(x - \mathrm{i}Y_N)} \right| \frac{|\phi(t)|}{|x - \mathrm{i}Y_N - t|} \, \mathrm{d}x. \end{aligned} \tag{3.97}$$

Since $\tilde{t}_{N+1} - t \geq \underline{\delta}$ and $t - \tilde{t}_{-N} > \underline{\delta}$, we can use Lemma 3.53 to bound the first and second term on the right-hand side of (3.97) from above. In particular we get

$$\frac{1}{2\pi} \int_{-Y_N}^{Y_N} \left| \frac{f(\tilde{t}_{N+1} + \mathrm{i}y)}{\phi(\tilde{t}_{N+1} + \mathrm{i}y)} \right| \frac{|\phi(t)|}{|\tilde{t}_{N+1} + \mathrm{i}y - t|} \,\mathrm{d}y \leq \frac{\|f\|_\infty \|\phi\|_\infty}{\pi^2 C_{29} \underline{\delta}(1-\beta)} \tag{3.98}$$

and

$$\frac{1}{2\pi} \int_{-Y_N}^{Y_N} \left| \frac{f(\tilde{t}_{-N} + \mathrm{i}y)}{\phi(\tilde{t}_{-N} + \mathrm{i}y)} \right| \frac{|\phi(t)|}{|\tilde{t}_{-N} + \mathrm{i}y - t|} \,\mathrm{d}y \leq \frac{\|f\|_\infty \|\phi\|_\infty}{\pi^2 C_{29} \underline{\delta}(1-\beta)}, \tag{3.99}$$

where C_{29} is some constant, which is independent of N. The third term on the right-hand side of (3.97) is bounded from above by

$$\begin{aligned} \frac{1}{2\pi} \int_{\tilde{t}_{-N}}^{\tilde{t}_{N+1}} \left| \frac{f(x + \mathrm{i}Y_N)}{\phi(x + \mathrm{i}Y_N)} \right| \frac{|\phi(t)|}{|x + \mathrm{i}Y_N - t|} \,\mathrm{d}x &\leq \frac{(\tilde{t}_{N+1} - \tilde{t}_{-N}) \|f\|_\infty \|\phi\|_\infty}{2\pi C_{19} Y_N} \\ &\leq \frac{(2N+3)\overline{\delta} \|f\|_\infty \|\phi\|_\infty}{2\pi C_{19} N \overline{\delta}} \\ &\leq C_{30} \|f\|_\infty \|\phi\|_\infty, \end{aligned} \tag{3.100}$$

because of Lemma 3.46 and (3.68). C_{30} is a constant, which is independent of N. Analogously, the fourth term on the right-hand side of (3.97) is bounded above by

$$\frac{1}{2\pi} \int_{\tilde{t}_{-N}}^{\tilde{t}_{N+1}} \left| \frac{f(x - \mathrm{i}Y_N)}{\phi(x - \mathrm{i}Y_N)} \right| \frac{|\phi(t)|}{|x + \mathrm{i}Y_N - t|} \,\mathrm{d}x \leq C_{30} \|f\|_\infty \|\phi\|_\infty. \tag{3.101}$$

Combining (3.98), (3.99), (3.100), and (3.101) gives

$$\left| f(t) - \sum_{k=-N}^{N+1} f(t_k) \phi_k(t) \right| \leq C_{31} \|f\|_\infty \|\phi\|_\infty,$$

which inserted in (3.96) leads to

$$\left| f(t) - \sum_{k=-N}^{N} f(t_k) \phi_k(t) \right| \leq \|f\|_\infty (\|\phi_{N+1}\|_\infty + C_{31} \|\phi\|_\infty) \tag{3.102}$$

if $0 < t \leq \tilde{t}_N$. The constant C_{31} depends on β. Since we assumed β to be fixed throughout the proof we suppress this dependence.

Next, we treat the case $t > \tilde{t}_N$. We have

$$\left| f(t) - \sum_{k=-N}^{N} f(t_k) \phi_k(t) \right| \leq \|f\|_\infty (1 + \|\phi_N\|_\infty) + \left| \sum_{k=-N}^{N-1} f(t_k) \phi_k(t) \right|. \tag{3.103}$$

Using (3.95) and Lemma 3.53, the sum on the right-hand side of (3.103) can be bounded from above, similarly to the previous case, by

$$\left| \sum_{k=-N}^{N-1} f(t_k) \phi_k(t) \right| \leq C_{32} \|f\|_\infty \|\phi\|_\infty,$$

where the constant C_{32} is independent of N. The dependence of C_{32} on β is suppressed, because we assumed β to be fixed throughout the proof. Moreover, by (3.103), we finally obtain

$$\left| f(t) - \sum_{k=-N}^{N} f(t_k)\phi_k(t) \right| \leq \|f\|_\infty (1 + \|\phi_N\|_\infty + C_{32}\|\phi\|_\infty). \tag{3.104}$$

if $t > \tilde{t}_N$.

From (3.102) and (3.104) we see that

$$\left| f(t) - \sum_{k=-N}^{N} f(t_k)\phi_k(t) \right| \leq C_{33}\|f\|_\infty$$

for all $t > 0$, with some constant C_{33} that is independent of t and N. Since $N > 2$ was arbitrary and the same upper bound holds for negative t, it follows that

$$\sup_{N\in\mathbb{N}} \max_{t\in\mathbb{R}} \left| f(t) - \sum_{k=-N}^{N} f(t_k)\phi_k(t) \right| \leq C_{28}\|f\|_\infty. \tag{3.105}$$

□

Next, we can use Theorem 3.52 to derive the uniform convergence of the sampling series (3.57) for signals in $\mathcal{B}^\infty_{\beta\pi,0}$, $0 < \beta < 1$.

Theorem 3.54. *Let ϕ be a function of sine type, whose zeros $\{t_k\}_{k\in\mathbb{Z}}$ are all real and ordered increasingly according to* (3.58). *Furthermore, let ϕ_k be defined as in* (3.61) *and $0 < \beta < 1$. Then, for all $f \in \mathcal{B}^\infty_{\beta\pi,0}$, we have*

$$\lim_{N\to\infty} \max_{t\in\mathbb{R}} \left| f(t) - \sum_{k=-N}^{N} f(t_k)\phi_k(t) \right| = 0.$$

Proof. From Theorem 3.52 we know that there exists a constant C_{28} such that

$$\left| f(t) - \sum_{k=-N}^{N} f(t_k)\phi_k(t) \right| \leq C_{28}\|f\|_\infty$$

for all $N \in \mathbb{N}$ and all $t \in \mathbb{R}$. Finally, the uniform convergence of the sampling series (3.57) for $\mathcal{PW}^2_{\beta\pi}$ and the fact that $\mathcal{PW}^2_{\beta\pi}$ is dense in $\mathcal{B}^\infty_{\beta\pi,0}$ together imply the assertion. □

Since $\mathcal{PW}^1_{\beta\pi} \subset \mathcal{B}^\infty_{\beta\pi,0}$, $0 < \beta < 1$, the previous theorem also implies the global uniform convergence of the sampling series (3.57) for signals in $\mathcal{PW}^1_{\beta\pi}$, $1 < \beta < 1$.

Remark 3.55. For the proof it was important to find the upper bound (3.105). A closer look at the proof of Theorem 3.52 reveals that oversampling, i.e., $\beta < 1$, is essential for the approach that was used to obtain (3.105). The same approach cannot be used for $\beta \to 1$ because the right-hand sides of (3.98) and (3.99) diverge.

3.3.5 Discrete-Time Signals, Continuous-Time Signals, and Two Interesting Open Questions

At the beginning of Chapter 3 we have discussed the correspondence between the space $\mathcal{B}_\pi^\infty$ of bounded bandlimited continuous-time signals and the space l^∞ of bounded discrete-time signals. We argued that a correspondence in the following sense would be desirable:

i) Every continuous-time signal $f \in \mathcal{B}_\pi^\infty$ leads to a bounded discrete-time signal $x = \{f(t_k)\}_{k\in\mathbb{Z}} \in l^\infty$ if it is sampled at the sampling points $\{t_k\}_{k\in\mathbb{Z}}$, and

ii) for every discrete-time signal $x = \{x_k\}_{k\in\mathbb{Z}} \in l^\infty$ there exists a continuous-time signal $f \in \mathcal{B}_\pi^\infty$ such that $f(t_k) = x_k$ for all $k \in \mathbb{Z}$.

However, ii) is not true in general. For equidistant sampling at Nyquist rate, a simple bounded discrete-time signal $\tilde{x} = \{\tilde{x}_k\}_{k\in\mathbb{Z}} \in l^\infty$ was given in Theorem 3.2, for which there exists no signal $f \in \mathcal{B}_\pi^\infty$ such that $f(k) = \tilde{x}_k$ for all $k \in \mathbb{Z}$. A valid question is whether the situation improves if non-equidistant sampling patterns are used.

For convenience, we denote by $\mathcal{S}$ the set of all sampling patterns $\{t_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ that are made of the zeros of sine-type functions.

Using Theorem 3.45, it follows immediately that the situation does not improve if we consider non-equidistant sampling patterns from $\mathcal{S}$. To see this, let $t \in (t_0, t_1)$ and choose the sequence $\{x_k^*\}_{k\in\mathbb{Z}} = \{\operatorname{sgn}(\phi_k(t))\}_{k\in\mathbb{Z}}$, where sgn denotes the signum function. We suppose there exists a signal $f^* \in \mathcal{B}_\pi^\infty$ such that $f^*(t_k) = x_k^*$ for all $k \in \mathbb{Z}$ and construct a contradiction. According to Theorem 3.45 there exists a constant C_{18} such that

$$\left| \sum_{k=-N}^{N} f^*(t_k)\phi_k(t) \right| \le C_{18} \|f^*\|_\infty \tag{3.106}$$

for all $N \in \mathbb{N}$. On the other hand, for $N \in \mathbb{N}$, we have

$$\begin{aligned}
\sum_{k=-N}^{N} f^*(t_k)\phi_k(t) &= \sum_{k=-N}^{N} x_k \phi_k(t) = \sum_{k=-N}^{N} |\phi_k(t)| \\
&\ge \frac{|\phi(t)|}{\pi\|\phi\|_\infty} \sum_{k=-N}^{N} \frac{1}{|t-t_k|} \\
&\ge \frac{|\phi(t)|}{\pi\|\phi\|_\infty} \sum_{k=1}^{N} \frac{1}{t_k - t_0} \ge \frac{|\phi(t)|}{\bar{\delta}\pi\|\phi\|_\infty} \sum_{k=1}^{N} \frac{1}{k} \\
&\ge \frac{|\phi(t)|}{\bar{\delta}\pi\|\phi\|_\infty} \log(N+1),
\end{aligned}$$

which is a contradiction to (3.106). This shows that the correspondence, which was discussed above, does not exist, regardless what sampling pattern is chosen from $\mathcal{S}$.

First Open Question We have seen that ii) is not true in general. Next, we ask whether at least a less stringent version of ii) is true. Does there exist a constant C_{34} such that

$$\|f\|_\infty \le C_{34} \sup_{k\in\mathbb{Z}} |f(t_k)| \tag{3.107}$$

for all $f \in \mathcal{B}^\infty_{\pi,0}$? A relation like inequality (3.107) would be extremely helpful in all applications where the peak value of the continuous-time signal $\|f\|_\infty$ has to be estimated from the peak value of the the samples $\sup_{k\in\mathbb{Z}}|f(t_k)|$.

A potential application where the estimation of the peak value is relevant are sensor networks that monitor the temperature in some area. It has to be ensured that the peak temperature in the whole area is below some critical temperature ϑ_c. If (3.107) was true we could conclude that the peak value of the temperature is below ϑ_c if $\sup_{k\in\mathbb{Z}}|f(t_k)| < \vartheta_\mathrm{c}/C_{34}$, i.e., if the peak value of the samples is less than $\vartheta_\mathrm{c}/C_{34}$.

As explained before, OFDM transmission systems are another example where we are interested in the peak value of the continuous-time signal.

Note the simplicity of the peak value estimation via (3.107). We do not have to reconstruct the continuous-time signal according to some complicated reconstruction process involving reconstruction functions that are difficult to compute in order to obtain the peak value of the continuous-time signal.

Unfortunately, not even the weakened version of ii) as formulated above is true. Using the results from Theorem 3.50, we can easily construct a contradiction, which shows that (3.107) cannot be true. Let A_N^ϕ be defined as in (3.80) and assume that (3.107) is true for all $f \in \mathcal{B}^\infty_{\pi,0}$. From (3.83) in the proof of Theorem 3.50 we know that there exists a signal $f_1 \in \mathcal{B}^\infty_{\pi,0}$ such that

$$\limsup_{N\to\infty} \|A_N^\phi f_1\|_\infty = \infty. \tag{3.108}$$

Since $A_N^\phi f_1 \in \mathcal{B}^\infty_{\pi,0}$, we have

$$\|A_N^\phi f_1\|_\infty \le C_{34} \sup_{k\in\mathbb{Z}} |(A_N^\phi f_1)(t_k)| \le C_{34} \sup_{k\in\mathbb{Z}} |f_1(t_k)| \le C_{34} \|f_1\|_\infty$$

for all $N \in \mathbb{N}$, according to the assumption and the fact that $|(A_N^\phi f_1)(t_k)| \le |f_1(t_k)|$, $N \in \mathbb{N}$. However, this is a contradiction to (3.108).

As we have seen in Theorems 3.51 and 3.54, oversampling can help improve the convergence behavior of sampling series. So it is natural to ask whether the inequality (3.107) is true if oversampling is used.

Question 3.56. *Let $0 < \beta < 1$ and $\{t_k\}_{k\in\mathbb{Z}} \in \mathcal{S}$. Does there exist a constant $C_{35} = C_{35}(\beta)$ such that*

$$\|f\|_\infty \le C_{35} \sup_{k\in\mathbb{Z}} |f(t_k)| \tag{3.109}$$

for all $f \in \mathcal{B}^\infty_{\beta\pi}$?

Of course, the peak value of f can be determined with arbitrarily high precision from the samples $\{f(t_k)\}_{k\in\mathbb{Z}}$ because for $0 < \beta < 1$ we know from Theorem 3.54 that $\lim_{N\to\infty}\|f - A_N^{\phi} f\|_\infty = 0$ for all $f \in \mathcal{B}_{\beta\pi,0}^{\infty}$. Nevertheless, since this procedure requires the computation of the reconstruction $A_N^{\phi} f$, it would be favorable to have the relation (3.109).

For the special case of equidistant sampling it has been shown in [77] that (3.109) is true. Moreover, the implication of this result on the PAPR problem of OFDM systems has been analyzed in [97].

Example 3.57. Let $0 < \beta < 1$ and $t_k = k$, $k \in \mathbb{Z}$. Then we have

$$\|f\|_\infty \leq \frac{1}{\cos\left(\frac{\beta\pi}{2}\right)} \sup_{k\in\mathbb{Z}} |f(k)|$$

for all $f \in \mathcal{B}_{\beta\pi,0}^{\infty}$.

However, we do not know whether the relation (3.109) does also hold for arbitrary sampling patterns $\{t_k\}_{k\in\mathbb{Z}} \in \mathcal{S}$.

Second Open Question A further interesting open question concerns the stability of the sampling patterns with respect to jitter in the sampling points. Assume that $\{t_k\}_{k\in\mathbb{Z}}$ is a given sampling pattern that is made of the zeros of some sine-type function and consider the new sampling pattern $\{t_k^*\}_{k\in\mathbb{Z}}$ which is generated by changing the location of each sampling point t_k by some small amount ϵ_k, the absolute value of which is bounded above by some constant δ. That is, consider $t_k^* = t_k + \epsilon_k$, where $|\epsilon_k| < \delta$ for all $k \in \mathbb{Z}$. The question is whether, for small enough δ, the assertion of Theorem 3.47 is still true for this disturbed sampling pattern. More precisely:

Question 3.58. *Let $\{t_k\}_{k\in\mathbb{Z}} \in \mathcal{S}$. Does there exist a $\delta > 0$ such that, given any sampling point sequence $\{t_k^*\}_{k\in\mathbb{Z}}$ with $|t_k^* - t_k| < \delta$ for all $k \in \mathbb{Z}$, we have*

$$\lim_{N\to\infty} \max_{t\in[-\tau,\tau]} \left| f(t) - \sum_{k=-N}^{N} f(t_k^*)\phi_k^*(t) \right| = 0$$

for all $\tau > 0$ and all $f \in \mathcal{B}_{\pi,0}^{\infty}$, where the ϕ_k^* are constructed according to (3.61) *and* (3.60)*, using the disturbed sapling points $\{t_k^*\}_{k\in\mathbb{Z}}$.*

Remark 3.59. For the sampling pattern $t_k = k$, $k \in \mathbb{Z}$, and $\delta < 1/4$ we have $|t_k^* - k| < \delta$, and, according to Kadec's 1/4-theorem [93, p. 36], $\{t_k^*\}_{k\in\mathbb{Z}}$ is a complete interpolating sequence for $\mathcal{PW}_\pi^2$. Thus, for this specific sampling pattern the property of $\{t_k\}_{k\in\mathbb{Z}}$ being a complete interpolating sequence is preserved under slight jitter of the sampling points. Certainly, this does not answer the question whether $\{t_k\}_{k\in\mathbb{Z}} \in \mathcal{S}$ implies $\{t_k^*\}_{k\in\mathbb{Z}} \in \mathcal{S}$.

3.3.6 Construction of Sine-Type Functions and Sampling Patterns

In this section we present a method to construct sine-type functions and hence possible sampling patterns.

Consider for an arbitrary function $g \in \mathcal{PW}_\pi^1$ that is real on the real axis, with $\|g\|_{\mathcal{PW}_\pi^1} < 1$, the function

$$\phi_g(t) = g(t) - \cos(\pi t). \tag{3.110}$$

Functions of this kind were analyzed for example in [98], [99] and [100]. The next theorem shows that the function ϕ_g is a function of sine type.

Theorem 3.60. *Let $g \in \mathcal{PW}_\pi^1$ be a function that is real on the real axis. Then $\phi_g(t) = g(t) - \cos(\pi t)$ is a function of sine type.*

Proof. The zeros $\{t_k\}_{k\in\mathbb{Z}}$ of ϕ_g are all real and simple, because we assumed that g is real-valued and $\|g\|_\infty \le \|g\|_{\mathcal{PW}_\pi^1} < 1$ [101]. Moreover, we have $\phi_g'(t) = \pi\sin(\pi t) + g'(t)$ and consequently, for $0 < \epsilon < 1/2$ and $k \in \mathbb{Z}$,

$$|\phi_g'(k+\epsilon)| \ge \pi\cos(\pi\epsilon) - |g'(k+\epsilon)| \ge \pi\cos(\pi\epsilon) - \pi\|g\|_\infty.$$

Now, we choose $\epsilon_1 > 0$ such that $\pi\cos(\pi\epsilon_1) - \pi\|f\|_\infty > 0$. Hence we have

$$\max_{t\in[k-\epsilon_1,k+\epsilon_1]} |\phi'(t)| > 0$$

for all $k \in \mathbb{Z}$. This shows that ϕ is either strictly increasing or strictly decreasing in $[k-\epsilon_1, k+\epsilon_1]$, and therefore has at most one zero in this interval. Furthermore, by the Riemann–Lebesgue lemma, we have $\lim_{t\to\infty} f(t) = 0$. It follows that there exists a T_0 such that for all k with $|t_k| \ge T_0$ there exists a $n_k \in \mathbb{Z}$ such that $|t_k - n_k| < \epsilon_1$. But, we know already that there is at most one zero in every interval $[n_k-\epsilon_1, n_k+\epsilon_1]$, $k \in \mathbb{Z}$. This implies that $n_{k+1} - n_k \ge 1$ for all k with $|t_k| \ge T_0$. It follows that $t_{k+1} - t_k \ge n_{k+1} - \epsilon_1 - (n_k + \epsilon_1) \ge 1 - 2\epsilon_1$ for all k with $|t_k| \ge T_0$. Since ϕ_g has only finitely many zeros in $[-T_0, T_0]$, we have $\inf_{k\in\mathbb{Z}}|t_{k+1} - t_k| > 0$, i.e., the zeros $\{t_k\}_{k\in\mathbb{Z}}$ are separated. Thus, item i) of Definition 3.39 is fulfilled.

Further, ϕ_g has the representation

$$\begin{aligned}\phi_g(t) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{g}(\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\;\mathrm{d}\omega - \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}\omega t}\;\mathrm{d}\mu_1(\omega)\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}\omega t}\;\mathrm{d}\mu_2(\omega)\end{aligned}$$

as a Lebesgue–Stieltjes integral with

$$\mu_1(\omega) = \begin{cases} 0, & \omega < -\pi \\ -\frac{1}{2\mathrm{i}}, & -\pi \le \omega < \pi \\ 0, & \omega \ge \pi \end{cases}$$

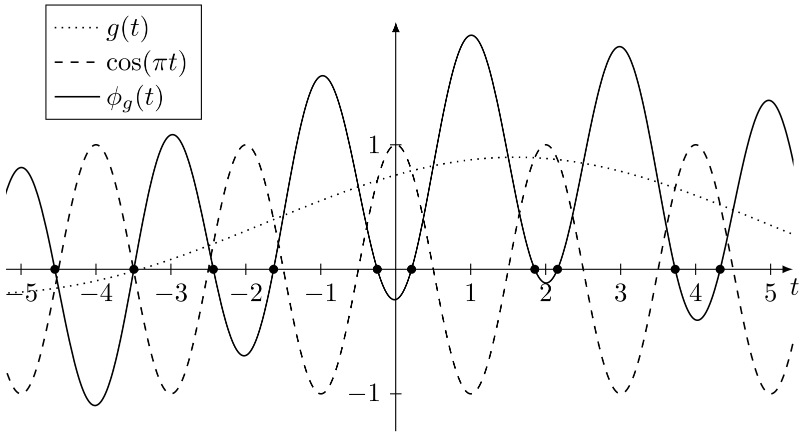

Figure 3.3: Construction of possible sine-type sampling patterns. ϕ_g is a sine-type function and the black dots mark the positions of a possible sampling pattern.

and $\mu_2(\omega) = -\mu_1(\omega) + (1/(2\pi)) \int_{-\pi}^{\omega} \hat{g}(\omega_1)\, \mathrm{d}\omega_1$. The total variation $V_{\mu_2}[-\pi, \pi]$ of μ_2 satisfies

$$
\begin{aligned}
V_{\mu_2}[-\pi, \pi] &\le V_{\mu_1}[-\pi, \pi] + \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{g}(\omega_1)|\, \mathrm{d}\omega_1 \\
&\le 1 + \|g\|_{\mathcal{PW}_\pi^1} \\
&< \infty,
\end{aligned}
$$

which shows that μ_2 is of bounded variation on $[-\pi, \pi]$. Moreover, μ_2 has jump discontinuities at each of the endpoints of the interval. Thus, it follows from [93, p. 143] that ϕ_g also satisfies item ii) of Definition 3.39, and hence is a sine-type function. □

Thus, by equation (3.110) we have a method to construct arbitrarily many functions of sine type ϕ_g and hence arbitrarily many sampling patterns $\{t_k\}_{k\in\mathbb{Z}}$ for which the theorems in Sections 3.3.3 and 3.3.4 are valid. The sampling points $\{t_k\}_{k\in\mathbb{Z}}$ are nothing else than the crossings of some bandlimited function $g \in \mathcal{PW}_\pi^1$ that is real on the real axis and satisfies $\|g\|_{\mathcal{PW}_\pi^1} < 1$, with the cosine function.

Example 3.61. In Fig. 3.3 the construction of a sine-type function and a sampling pattern is illustrated. For the example we have chosen the function

$$
g(t) = \frac{9}{10} \frac{\sin\left(\frac{2}{10}\pi t - 1\right)}{\frac{2}{10}\pi t - 1}.
$$

4

System Representations

In signal processing applications a main goal is to process signals. A widely used method to perform such a processing is to use filters, i.e., linear time-invariant (LTI) systems.

In general, a system T takes a signal f from some input space and produces an output signal Tf in some output space. If the input and the output space consist of analog signals—as it is the case in analog signal processing—we have to use an "analog" system T^{A} that maps analog signals in analog signals. This analog signal processing is depicted in the upper half of Fig. 4.1.

The first question that we treat concerns the representation of stable LTI systems. A common representation of stable LTI systems on $\mathcal{PW}_\pi^2$ is the time domain convolution representation where the system output signal is given by the convolution of the system input signal with the impulse response of the system. However, this is not the only possible representation. The problem of finding representations of stable LTI systems has been studied for a long time, and several results for spaces of bandlimited signals, which are larger than the space of bandlimited, finite energy signals, have been presented [21, 47, 102–104]. In [21] Habib derived a convolution integral and a series representation for systems operating on bandlimited signals in the Zakai space [47, 105]. In this chapter we analyze different system representations for the spaces $\mathcal{PW}_\pi^1$ and $\mathcal{B}_\pi^p$, $1 < p < \infty$.

The second question that we treat is whether "analog" systems can be implemented digitally. That is, given a stable "analog" system T^{A}, can we implement this system by a "digital" system T^{D} that uses only the quantized samples of the signal. The operation of the "digital" system is illustrated in the lower half of Fig. 4.1. In this chapter we treat the simplified version of this question where the quantization is not present. Quantization effects in the context of systems, i.e., the full problem, will be analyzed in Section 6.3.

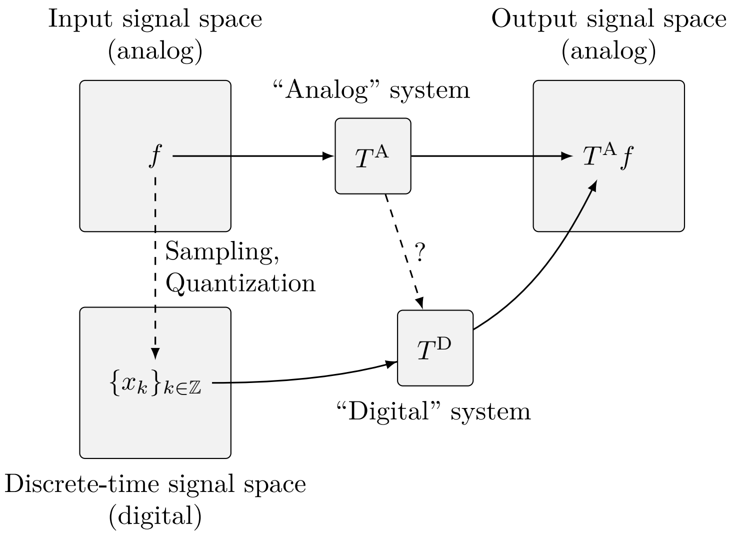

Figure 4.1: Analog versus digital signal processing.

4.1 Definitions and Notation

Before we continue the discussion, we briefly review some definitions and facts about stable LTI systems. A linear system $T : \mathcal{A} \to \mathcal{B}$, mapping signals from the input signal space $\mathcal{A}$ to the output signal space $\mathcal{B}$ is called stable if the operator T is bounded, i.e., if $\|T\| = \sup_{\|f\|_{\mathcal{A}}\le 1}\|Tf\|_{\mathcal{B}} < \infty$. Furthermore, it is called time-invariant if $(Tf(\,\cdot\, - a))(t) = (Tf)(t-a)$ for all $f \in \mathcal{A}$ and $t, a \in \mathbb{R}$.

Remark 4.1. Note that our definition of stability is with respect to the norms of the spaces $\mathcal{A}$ and $\mathcal{B}$, and thus is different from the concept of bounded-input bounded-output (BIBO) stability in general.

One important class of systems is the set of all stable LTI systems $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ that map $\mathcal{PW}_\pi^1$ into $\mathcal{PW}_\pi^1$. Important LTI systems like the Hilbert transform and the ideal low-pass filter belong to this class. For every stable LTI system $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ there exists exactly one function $\hat{h}_T \in L^\infty[-\pi,\pi]$ such that

$$(Tf)(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}(\omega)\hat{h}_T(\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega \tag{4.1}$$

for all $f \in \mathcal{PW}_\pi^1$. The operator norm of T is given by $\|T\| = \|\hat{h}_T\|_\infty$ and the impulse response h_T by $h_T = T\,\mathrm{sinc}$, where $\mathrm{sinc}(t) = \sin(\pi t)/(\pi t)$ for $t \neq 0$ and $\mathrm{sinc}(t) = 1$ for $t = 0$. Conversely, every function $\hat{h}_T \in L^\infty[-\pi,\pi]$ defines a stable LTI system $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$. Thus, the space of all stable LTI systems defined on $\mathcal{PW}_\pi^1$ is isometrically isomorphic to $L^\infty[-\pi,\pi]$.

Furthermore, it can be shown that the representation (4.1) with a unique function $\hat{h}_T \in L^\infty[-\pi,\pi]$ is also valid for all stable LTI systems $T : \mathcal{PW}_\pi^2 \to \mathcal{PW}_\pi^2$ and that all $\hat{h}_T \in L^\infty[-\pi,\pi]$ define a stable LTI system $T : \mathcal{PW}_\pi^2 \to \mathcal{PW}_\pi^2$. Consequently, we do not have to distinguish between stable LTI systems that map $\mathcal{PW}_\pi^1$ into $\mathcal{PW}_\pi^1$ and stable LTI systems that map $\mathcal{PW}_\pi^2$ into $\mathcal{PW}_\pi^2$, because both can be identified with $\hat{h} \in L^\infty[-\pi,\pi]$. Therefore, every stable LTI system that maps $\mathcal{PW}_\pi^1$ in $\mathcal{PW}_\pi^1$ maps $\mathcal{PW}_\pi^2$ in $\mathcal{PW}_\pi^2$ and vice versa.

Note that $\hat{h}_T \in L^\infty[-\pi,\pi] \subset L^2[-\pi,\pi]$ and consequently $h_T \in \mathcal{PW}_\pi^2$. We always assume that T is non-trivial, i.e., we assume that T is not the operator that maps all input signals to the zero function.

Before we continue, we introduce the Hilbert transform and the ideal low-pass filter, which will be needed subsequently and which serve as illustrative examples of two stable LTI systems. Both systems are important in theoretical analyses [106,107]. Although it is not possible to realize them exactly in practice, they can be seen as the limit case of realizable systems. The Hilbert transform Hf of a signal f is defined by

$$(Hf)(t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} (-\mathrm{i}\,\mathrm{sgn}(\omega))\,\hat{f}(\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\;\mathrm{d}\omega,$$

where sgn denotes the signum function. It is well known that the Hilbert transform H is a translation invariant, linear, and bounded operator that maps $\mathcal{PW}_\pi^p$ into $\mathcal{PW}_\pi^p$, $1 \le p \le \infty$, and that $\|H\| = \sup_{\|f\|_{\mathcal{PW}_\pi^p} \le 1} \|Hf\|_{\mathcal{PW}_\pi^p} = 1$ for $1 \le p \le \infty$. This implies that H is a stable LTI system.

The Hilbert transform has many applications [106–108]. For example, in communication theory the Hilbert transform is used to define the analytical signal

$$f_+ = f + \mathrm{i}Hf.$$

One of the key properties of the analytical signal is the fact that its Fourier transform $\hat{f}_+$ is zero for negative frequencies. For $f \in \mathcal{PW}_\pi^2$ the analytical signal f_+ is well defined, and we have

$$\|f_+\|_{\mathcal{PW}_\pi^2} \le \|f\|_{\mathcal{PW}_\pi^2} + \|Hf\|_{\mathcal{PW}_\pi^2} = 2\|f\|_{\mathcal{PW}_\pi^2}.$$

The ideal low-pass filter $L_{\omega_g} : \mathcal{PW}_\pi^2 \to \mathcal{PW}_{\omega_g}^2$ with bandwidth $0 < \omega_g \le \pi$ is defined by

$$(L_{\omega_g} f)(t) := \frac{1}{2\pi}\int_{-\omega_g}^{\omega_g} \hat{f}(\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\;\mathrm{d}\omega.$$

Obviously, $L_{\omega_g} : \mathcal{PW}_\pi^2 \to \mathcal{PW}_{\omega_g}^2$ is, like the Hilbert transform, a stable LTI system with $\|L_{\omega_g}\| = \sup_{\|f\|_{\mathcal{PW}_\pi^2} \le 1} \|L_{\omega_g} f\|_{\mathcal{PW}_\pi^2} = 1$.

Remark 4.2. The mathematical theory of multipliers is closely related to the problems analyzed in this chapter. The theory of multipliers studies integrals of the form (4.1) or, more generally, linear operators that are defined in the frequency domain by a multiplication of the Fourier transform of the signal with some other function. For more information about multipliers we would like to refer the reader to [109].

4.2 Possible Representations

Mathematically, a system is an operator, i.e., a rule by which an input signal is transformed into an output signal. This operator can have different representations. For example, one possible representation for stable LTI systems operating on signals in $\mathcal{PW}_\pi^2$ is the frequency domain representation (4.1).

In addition to the frequency domain representation, there are other representations for stable LTI systems operating on signals in $\mathcal{PW}_\pi^2$. One is the following time domain representation in the form of a convolution integral. For every stable LTI system $T : \mathcal{PW}_\pi^2 \to \mathcal{PW}_\pi^2$ we have

$$(Tf)(t) = \int_{-\infty}^{\infty} f(t-\tau) h_T(\tau)\, \mathrm{d}\tau = \int_{-\infty}^{\infty} f(\tau) h_T(t-\tau)\, \mathrm{d}\tau \tag{4.2}$$

for all $f \in \mathcal{PW}_\pi^2$, where $h_T = T \operatorname{sinc} \in \mathcal{PW}_\pi^2$. Another representation is the discrete counterpart of (4.2), the convolution sum

$$(Tf)(t) = \sum_{k=-\infty}^{\infty} f(k) h_T(t-k). \tag{4.3}$$

The representation (4.3) has the advantage that it uses only the samples $\{f(k)\}_{k\in\mathbb{Z}}$ of the input signal f to compute the system output Tf. We therefore call it sampling-type representation.

However, for stable LTI systems operating on other signal spaces, a convolution integral representation like (4.2) and a convolution sum representation like (4.3) do not necessarily exist, because of convergence problems of the integrals and the sum.

In the next sections we will analyze the convergence behavior of (4.2) and (4.3) for larger signal spaces than $\mathcal{PW}_\pi^2$. In Section 4.3 we analyze (4.2) and (4.3) for signals in $\mathcal{PW}_\pi^1$. It will turn out that a time domain representation in the form of (4.2) or (4.3) is not always possible for $\mathcal{PW}_\pi^1$, even in a distributional setting. Further, in Section 4.4 we will show that the convolution sum (4.3) is a valid representation for all stable LTI systems on $\mathcal{B}_\pi^p$, $1 < p < \infty$.

4.3 Convolution-Type System Representations for $\mathcal{PW}_\pi^1$

Many engineering books [110, 111] give the impression that any LTI system T can be represented as a convolution integral in the form

$$(Tf)(t) = \int_{-\infty}^{\infty} f(\tau) h_T(t-\tau)\, \mathrm{d}\tau, \tag{4.4}$$

where h_T is the impulse response of the system and f is the input signal. Of course this is true for example for stable LTI systems operating on bandlimited signals with finite energy. However, it is not necessarily true for stable LTI systems acting on other signal spaces. In [104] it has been shown that the integral in (4.4) is generally not convergent for signals from the Paley–Wiener space $\mathcal{PW}_\pi^1$.

Although the integral in (4.4) does not necessarily exist in the classical sense for $\mathcal{PW}_\pi^1$, it might be possible that it can still be meaningfully interpreted in a distributional sense. Indeed, distributions can provide a way out of many convergence problems that are present in the classical non-distributional setting. One example is given by the convergence of Fourier series: It is well known that there are signals in $L^1[-\pi, \pi]$ whose Fourier series diverge almost everywhere. In a distributional sense however, the Fourier series converges for all signals in $L^1[-\pi, \pi]$. This example shows that there are situation where a distributional interpretation can resolve convergence problems. Unfortunately, many engineering textbooks about LTI systems do not treat distributions in a rigorous mathematical manner. Often heuristic arguments prevail.

Another problem which has gained a lot of attention concerns the existence of the impulse response for stable LTI systems operating on general, not necessarily bandlimited, spaces, and the question whether the impulse response gives a complete description of the system [112–115]. In [112] it was shown that the class of stable (with respect to the L^∞-norm) LTI systems that map bounded uniformly continuous signals into bounded uniformly continuous signals contains systems whose impulse response is the zero function, but which take certain inputs into nonzero outputs. Consequently, there exist two different stable LTI systems that have the same impulse response. [113] treats systems operating on bounded signals and finds a necessary and sufficient under which a systems has the representation (4.4).

The fact that the impulse response h_T may not exist is one reason why a representation of the form (4.4) can be problematic. In [114, 115] LTI systems were studied in a distributional way. The authors proved that in a distributional setting and under certain assumptions, it is possible to define in a certain sense an impulse response for every stable LTI system. One assumption that was made in order to obtain their results was that the space of input signals contains the space of test functions $\mathcal{D}$. Since functions in $\mathcal{D}$ are compactly supported, they cannot be bandlimited. Therefore, the results are not applicable for systems operating on spaces of bandlimited signals.

Fortunately, we do not have to face these problems here: Since we consider bandlimited input signals, the impulse response is always a well-defined bandlimited function, which uniquely determines the system. However, although the impulse response exists, it will turn out that stable LTI systems can generally not be represented in the form (4.4) because the integral diverges. In contrast to the common perception, this divergence cannot be circumvented by considering a distributional setting.

In this section we analyze the distributional convergence behavior of the two convolution integrals

$$(A_N^T f)(t) := \int_{-N}^{N} f(\tau) h_T(t-\tau)\,\mathrm{d}\tau \tag{4.5}$$

and

$$(B_N^T f)(t) := \int_{-N}^{N} f(t-\tau) h_T(\tau)\,\mathrm{d}\tau, \tag{4.6}$$

and the convolution sum

$$(S_N^T f)(t) := \sum_{k=-N}^{N} f(k) h_T(t-k) \tag{4.7}$$

for signals f in the Paley–Wiener space $\mathcal{PW}_\pi^1$ and stable LTI systems $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$. We show that the perception that any stable LTI system acting on bandlimited signals can—at least in a distributional setting—be represented as a convolution integral is problematic and not justified in general. Moreover, we completely characterize all stable LTI systems for which the approximation processes (4.5), (4.6), and (4.7) converge to Tf for all $f \in \mathcal{PW}_\pi^1$ as N tends to infinity, and compare the distributional convergence behavior and the classical convergence behavior.

For practical applications we need the convergence of an approximation process for all signals from the signal space because generally it is not known in advance which signal from the signal space occurs in the application at hand. This is the reason why we want to characterize the stable LTI systems T for which the approximation processes (4.5), (4.6), and (4.7) converge for all $f \in \mathcal{PW}_\pi^1$.

4.3.1 Distributions and Convergence

In order to be able to state our key results, we additionally need the concept of distributions. Distributions are continuous linear functionals on some space of test functions. In this thesis we deal with two different test functions spaces. The first one is the space $\mathcal{D}$ of all functions $\varphi : \mathbb{R} \to \mathbb{C}$ that have continuous derivatives of all orders and are zero outside some finite interval. $\mathcal{D}'$ denotes the dual space of $\mathcal{D}$, i.e., the space of all distributions that can be defined on $\mathcal{D}$. The other space of test functions that we use in this thesis is the Schwartz space $\mathcal{S}$ of all continuous functions $\varphi : \mathbb{R} \to \mathbb{C}$ that have continuous derivatives of all orders and fulfill

$$\sup_{t\in\mathbb{R}} |t^a \varphi^{(b)}(t)| < \infty$$

for all $a, b \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\mathcal{S}'$ denotes the dual space of $\mathcal{S}$. From the definition of the spaces $\mathcal{D}$ and $\mathcal{S}$, it follows immediately that $\mathcal{D}$ is a proper subspace of $\mathcal{S}$, and that $\mathcal{S}'$ is a proper subspace of $\mathcal{D}'$. Furthermore, we have $\|\varphi\|_\infty < \infty$ and $\|\varphi\|_1 < \infty$ for all $\varphi \in \mathcal{S}$, and consequently for all $\varphi \in \mathcal{D}$. The Fourier transform maps the space $\mathcal{S}$ onto itself. These properties of φ will be used extensively in the proofs.

For a locally integrable functions g we can define the linear functional

$$\varphi \mapsto \int_{-\infty}^{\infty} g(t)\varphi(t)\,\mathrm{d}t \tag{4.8}$$

on the space $\mathcal{D}$. It can be proved that this functional is continuous and thus defines a distribution [116]. If g further fulfills

$$\int_{-\infty}^{\infty} |g(t)|(1+|t|)^{-m}\,\mathrm{d}t < \infty$$

for some $m \geq 0$ then (4.8) also defines a continuous linear functional on $\mathcal{S}$. Distributions of the type (4.8) are called regular distributions.

A sequence of distributions $\{f_k\}_{k\in\mathbb{N}}$ in $\mathcal{D}'$ is said to converge in $\mathcal{D}'$ if for every $\varphi \in \mathcal{D}$ the sequence of numbers $\{f_k\varphi\}_{k\in\mathbb{N}}$ converges. Similarly, a sequence of distributions $\{f_k\}_{k\in\mathbb{N}}$ in $\mathcal{S}'$ is said to converge in $\mathcal{S}'$ if for every $\varphi \in \mathcal{S}$ the sequence of numbers $\{f_k\varphi\}_{k\in\mathbb{N}}$ converges. Thus, a sequence of regular distributions, which is induced by a sequence of functions $\{g_k\}_{k\in\mathbb{N}}$ according to (4.8), converges in $\mathcal{S}'$ if for every $\varphi \in \mathcal{S}$ the sequence of numbers $\{\int_{-\infty}^{\infty} g_k(t)\varphi(t)\,\mathrm{d}t\}_{k\in\mathbb{N}}$ converges.

Convergence in $\mathcal{S}'$ and convergence in $\mathcal{D}'$ are connected in the following way.

Observation 4.3. *If $\{f_k\}_{k\in\mathbb{Z}}$ is a sequence in $\mathcal{S}'$ it is also a sequence in $\mathcal{D}'$, and, since $\mathcal{D} \subset \mathcal{S}$, convergence in $\mathcal{S}'$ implies convergence in $\mathcal{D}'$.*

For further details about distributions, and for a definition of convergence in the test spaces, we would like to refer the reader to [116].

4.3.2 Convolution Integral

In this section we analyze the convergence behavior of the two convolution integrals (4.5) and (4.6) for stable LTI systems T. Note that, for all $N \in \mathbb{N}$, $A_N^T f$ and $B_N^T f$ are bounded and continuous functions and therefore can be identified with a regular distribution according to (4.8).

The theory for stable LTI systems operating on bandlimited signals with finite energy is simple. It is well known that every stable LTI system $T : \mathcal{PW}_\pi^2 \to \mathcal{PW}_\pi^2$ has the representation

$$(Tf)(t) = \int_{-\infty}^{\infty} f(t-\tau)h_T(\tau)\,\mathrm{d}\tau = \int_{-\infty}^{\infty} f(\tau)h_T(t-\tau)\,\mathrm{d}\tau$$

with $h_T = T\,\mathrm{sinc} \in \mathcal{PW}_\pi^2$. That is, the system output is the convolution of the system input with the impulse response h_T of the system T.

Example 4.4. For the ideal low-pass filter L_{ω_g} with bandwidth $0 < \omega_g \leq \pi$ and the Hilbert transform H we have the impulse responses

$$h_{L_{\omega_g}}(t) = (L_{\omega_g} \operatorname{sinc})(t) = \frac{\sin(\omega_g t)}{\pi t}$$

and

$$h_H(t) = (H \operatorname{sinc})(t) = \frac{\sin^2\left(\frac{\pi}{2}t\right)}{\frac{\pi}{2}t}.$$

However, the situation for signals $f \in \mathcal{PW}_\pi^1$ is more difficult. In [104] it has been shown that the convolution integrals (4.5) and (4.6) have a significantly different convergence behavior. For example, it has been shown for the Hilbert transform that (4.6) is globally uniformly convergent for all $f \in \mathcal{PW}_\pi^1$, but that there are signals in $\mathcal{PW}_\pi^1$ for which the peak value of (4.5) diverges. Further, the class of systems for which (4.5) and (4.6) converge pointwise has been completely characterized. It turned out that there are stable LTI systems for which the integrals (4.5) and (4.6) diverge pointwise. More precisely, for every $t \in \mathbb{R}$ there is a stable LTI systems T such that (4.5) diverges for some signal $f \in \mathcal{PW}_\pi^1$ as N tends to infinity. The same is true for the convolution integral (4.6).

Although the convolution integrals are not necessarily convergent in the classical (pointwise) sense, it may be possible that (4.5) and (4.6), interpreted as a sequence of regular distributions, converge in the distributional sense for all stable LTI systems T and all $f \in \mathcal{PW}_\pi^1$. If this were true the common conception that every stable LTI system has a time domain representation in the form of a convolution integral would have a rigorous theoretical foundation for the space $\mathcal{PW}_\pi^1$, at least in a distributional sense.

In this section we analyze this question and show that there are stable LTI systems and signals in $\mathcal{PW}_\pi^1$ for which (4.5) and (4.6) diverge even in the distributional sense. Furthermore, we completely characterize all stable LTI systems for which we have convergence in the distributional sense by giving a necessary and sufficient condition for convergence. By characterizing the distributional convergence behavior we extend the results from [104].

Convergence Behavior of the Convolution Integral I

We start our analysis with the convergence behavior of the convolution integral (4.5). For notational convenience, we introduce the abbreviation

$$A_{N,\varphi}^T f := \int_{-\infty}^{\infty} (A_N^T f)(t)\varphi(t)\, \mathrm{d}t.$$

In the following theorem we completely characterize the stable LTI systems for which $(A_N^T f)(t)$ converges in the classical (pointwise) sense to $(Tf)(t)$ for all $f \in \mathcal{PW}_\pi^1$. Moreover, we characterize the stable LTI systems for which $A_N^T f$ converges in the distributional sense to Tf for all $f \in \mathcal{PW}_\pi^1$.

Theorem 4.5. *Let $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ be a stable LTI system.*

i) For all $t \in \mathbb{R}$ and all $f \in \mathcal{PW}_\pi^1$ we have

$$\lim_{N\to\infty} |(Tf)(t) - (A_N^T f)(t)| = 0$$

if and only if there exists a constant $C_1 < \infty$ such that

$$\max_{\omega\in[-\pi,\pi]} \left| \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{h}_T(\omega_1) \frac{\sin(N(\omega - \omega_1))}{\omega - \omega_1} \, \mathrm{d}\omega_1 \right| \le C_1 \tag{4.9}$$

for all $N \in \mathbb{N}$. In addition, if (4.9) is not fulfilled, then for every $t \in \mathbb{R}$ there exists a signal $f_1 \in \mathcal{PW}_\pi^1$ such that

$$\limsup_{N\to\infty} |(A_N^T f_1)(t)| = \infty. \tag{4.10}$$

ii) Moreover, we have

$$\lim_{N\to\infty} \left| A_{N,\varphi}^T f - \int_{-\infty}^{\infty} (Tf)(t)\varphi(t) \, \mathrm{d}t \right| = 0 \tag{4.11}$$

for all $f \in \mathcal{PW}_\pi^1$ and all $\varphi \in \mathcal{S}$ if and only if for all $\varphi \in \mathcal{S}$ there exists a constant $C_2 = C_2(\varphi) < \infty$ such that

$$\max_{\omega\in[-\pi,\pi]} \left| \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{h}_T(\omega_1)\hat{\varphi}(-\omega_1) \frac{\sin(N(\omega - \omega_1))}{\omega - \omega_1} \, \mathrm{d}\omega_1 \right| \le C_2(\varphi) \tag{4.12}$$

for all $N \in \mathbb{N}$. In addition, if (4.12) is not fulfilled for some $\varphi \in \mathcal{S}$, then there exists a signal $f_1 \in \mathcal{PW}_\pi^1$ such that

$$\limsup_{N\to\infty} |A_{N,\varphi}^T f_1| = \infty. \tag{4.13}$$

Remark 4.6. Since (4.9) does not depend on t, we have the special situation that the convergence of $(A_N^T f)(t)$ for some $t \in \mathbb{R}$ and all $f \in \mathcal{PW}_\pi^1$ implies the convergence of $(A_N^T f)(t)$ for all $t \in \mathbb{R}$ and all $f \in \mathcal{PW}_\pi^1$. Due to this special behavior we are able to derive the interesting result in Theorem 4.16 that pointwise convergence for some $t \in \mathbb{R}$ and all $f \in \mathcal{PW}_\pi^1$ is equivalent to distributional convergence for all $f \in \mathcal{PW}_\pi^1$. Moreover, we will see in Section 4.3.3 that the convolution sum does not possess this behavior.

In addition to the pointwise convergence behavior, Theorem 4.5 characterizes the convergence of $A_N^T f$ in $\mathcal{S}'$. $A_N^T f$ converges to Tf in $\mathcal{S}'$ for all $f \in \mathcal{PW}_\pi^1$ if and only if for all $\varphi \in \mathcal{S}$ there exists a constant $C_2(\varphi)$ such that (4.12) is fulfilled for all $N \in \mathbb{N}$. Moreover, if (4.12) is not fulfilled for some $\varphi \in \mathcal{S}$ then we have distributional divergence of $A_N^T f_1$ for some $f_1 \in \mathcal{PW}_\pi^1$ in the sense of (4.13).

Remark 4.7. Note that $\int_{-\infty}^{\infty}(Tf)(t)\varphi(t)\,\mathrm{d}t$ is always some finite number because $Tf \in \mathcal{PW}_\pi^1$ is bounded. For this reason (4.13) implies that

$$\limsup_{N\to\infty}\left|A_{N,\varphi}^T f_1 - \int_{-\infty}^{\infty}(Tf_1)(t)\varphi(t)\,\mathrm{d}t\right| = \infty.$$

For the proof of Theorem 4.5 we need Lemma 4.8, Lemma 4.9, and Lemma 4.11, which in turn is based on Lemma 4.10. The proofs of Lemmas 4.8 and 4.9 are given in Appendices A.3 and A.4.

Lemma 4.8. *Let $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ be a stable LTI system, $t \in \mathbb{R}$, and $N \in \mathbb{N}$. Then we have*

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1}\le 1}\left|(A_N^T f)(t)\right| = \max_{\omega\in[-\pi,\pi]}\left|\int_{t-N}^{t+N} h_T(\tau)\,\mathrm{e}^{-\mathrm{i}\omega\tau}\,\mathrm{d}\tau\right|.$$

Lemma 4.9. *Let T be a stable LTI system, $f \in \mathcal{PW}_\pi^1$ with $\|f\|_{\mathcal{PW}_\pi^1} \le 1$, $t \in \mathbb{R}$, and $N \in \mathbb{N}$. Then we have*

$$|(A_N^T f)(t)| \le \|T\|\frac{2}{\pi}\left(\pi + \frac{2}{\pi} + \frac{2}{\pi}\log(2N-1)\right).$$

Lemma 4.10. *For the operator $U : \mathcal{PW}_\pi^1 \to \mathbb{C}$ defined by*

$$Uf = \frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{f}(\omega)K(\omega)\,\mathrm{d}\omega,$$

where $K \in L^\infty[-\pi,\pi]$, we have

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1}\le 1}|Uf| = \|K\|_{L^\infty[-\pi,\pi]}.$$

Proof. Lemma 4.10 is a direct consequence of Lemma 17 in [9]. □

Lemma 4.11. *Let $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ be a stable LTI system, $\varphi \in \mathcal{S}$, and $N \in \mathbb{N}$. Then we have*

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1}\le 1}|A_{N,\varphi}^T f| = \max_{\omega\in[-\pi,\pi]}\left|\frac{1}{\pi}\int_{-\pi}^{\pi}\hat{h}_T(\omega_1)\hat{\varphi}(-\omega_1)\frac{\sin(N(\omega-\omega_1))}{\omega-\omega_1}\,\mathrm{d}\omega_1\right|.$$

Proof. Let $\varphi \in \mathcal{S}$, $N \in \mathbb{N}$, and the stable LTI systems T be arbitrary but fixed. For $f \in \mathcal{PW}_\pi^1$ we have

$$\begin{aligned}
A_{N,\varphi}^T f &= \int_{-\infty}^{\infty}\int_{-N}^{N} f(\tau)h_T(t-\tau)\varphi(t)\,\mathrm{d}\tau\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty}\int_{-N}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{f}(\omega)\,\mathrm{e}^{\mathrm{i}\omega\tau}\,h_T(t-\tau)\varphi(t)\,\mathrm{d}\omega\,\mathrm{d}\tau\,\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{f}(\omega)\int_{-N}^{N}\mathrm{e}^{\mathrm{i}\omega\tau}\int_{-\infty}^{\infty}h_T(t-\tau)\varphi(t)\,\mathrm{d}t\,\mathrm{d}\tau\,\mathrm{d}\omega,
\end{aligned}$$

where the order of the integrals was interchanged according to Fubini's theorem, which is applicable because

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-N}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi}|\hat{f}(\omega)\,\mathrm{e}^{\mathrm{i}\omega\tau}\,h_T(t-\tau)\varphi(t)|\;\mathrm{d}\omega\;\mathrm{d}\tau\;\mathrm{d}t&\\ \le 2N\|h_T\|_\infty\int_{-\infty}^{\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}|\hat{f}(\omega)\varphi(t)|\;\mathrm{d}\omega\;\mathrm{d}t&\\ \le 2N\|T\|\,\|f\|_{\mathcal{PW}_\pi^1}\|\varphi\|_1<\infty.&\end{aligned}\tag{4.14}$$

Moreover, since $h_T \in L^2(\mathbb{R})$ and $\varphi \in L^2(\mathbb{R})$, we obtain

$$\int_{-\infty}^{\infty} h_T(t-\tau)\varphi(t)\;\mathrm{d}t = \frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{h}_T(\omega_1)\hat{\varphi}(-\omega_1)\,\mathrm{e}^{-\mathrm{i}\omega_1\tau}\;\mathrm{d}\omega_1$$

by applying the generalized Parseval equality. Thus, it follows that

$$A_{N,\varphi}^T f = \frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{f}(\omega)\frac{1}{\pi}\int_{-\pi}^{\pi}\hat{h}_T(\omega_1)\hat{\varphi}(-\omega_1)\frac{\sin(N(\omega-\omega_1))}{\omega-\omega_1}\;\mathrm{d}\omega_1\;\mathrm{d}\omega,$$

and the assertion is a direct consequence of Lemma 4.10 □

Now we are in the position to prove Theorem 4.5.

Remark 4.12. Part i) of Theorem 4.5 was proved in [104] by abstractly showing the existence of the signal f_1 using the Banach–Steinhaus theorem. In this thesis we give an alternative proof, in which the divergence-creating signal f_1 is explicitly constructed.

Proof of Theorem 4.5. The proof is divided into two parts. In the first part we prove the "⇐" direction of the first "if and only if" statement, and (4.10), which implies the "⇒" direction of the first "if and only if" statement. In the second part we prove the "⇐" direction of the "if and only if" statement, and second, the "⇒" direction as well as (4.13).

First part, "⇐" direction: Let T be a stable LTI system and $t \in \mathbb{R}$ be arbitrary but fixed. Now suppose (4.9) is true. Since

$$\begin{aligned}&\left|\int_{t-N}^{t+N}\mathrm{e}^{-\mathrm{i}\omega\tau}\,h_T(\tau)\;\mathrm{d}\tau\right|\\ &=\left|\int_{-N}^{N}\mathrm{e}^{-\mathrm{i}\omega\tau}\,h_T(\tau)\;\mathrm{d}\tau+\int_{N}^{t+N}\mathrm{e}^{-\mathrm{i}\omega\tau}\,h_T(\tau)\;\mathrm{d}\tau-\int_{-N}^{t-N}\mathrm{e}^{-\mathrm{i}\omega\tau}\,h_T(\tau)\;\mathrm{d}\tau\right|\\ &\le\left|\int_{-N}^{N}\mathrm{e}^{-\mathrm{i}\omega\tau}\,h_T(\tau)\;\mathrm{d}\tau\right|+2|t|\cdot\|T\|\\ &=\left|\int_{-N}^{N}\mathrm{e}^{-\mathrm{i}\omega\tau}\frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{h}_T(\omega_1)\,\mathrm{e}^{\mathrm{i}\omega_1\tau}\;\mathrm{d}\omega_1\;\mathrm{d}\tau\right|+2|t|\cdot\|T\|\\ &=\left|\frac{1}{\pi}\int_{-\pi}^{\pi}\hat{h}_T(\omega_1)\frac{\sin(N(\omega-\omega_1))}{\omega-\omega_1}\;\mathrm{d}\omega_1\right|+2|t|\cdot\|T\|,\end{aligned}\tag{4.15}$$

equation (4.9) implies that

$$\max_{\omega\in[-\pi,\pi]} \left| \int_{t-N}^{t+N} \mathrm{e}^{-\mathrm{i}\omega\tau}\, h_T(\tau)\, \mathrm{d}\tau \right| \leq C_1 + 2|t| \cdot \|T\| =: C_3(t). \tag{4.16}$$

Furthermore, let $f \in \mathcal{PW}_\pi^1$ and $\epsilon > 0$ be arbitrary. Since $\mathcal{PW}_\pi^2$ is dense in $\mathcal{PW}_\pi^1$, there exists a signal $f_\epsilon \in \mathcal{PW}_\pi^2$ with $\|f - f_\epsilon\|_{\mathcal{PW}_\pi^1} < \epsilon$. Thus, according to (4.16) we obtain

$$\begin{aligned}
|(A_N^T(f - f_e))(t)| &= \left| \int_{-N}^{N} (f(\tau) - f_\epsilon(\tau)) h_T(t-\tau)\, \mathrm{d}\tau \right| \\
&= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} (\hat{f}(\omega) - \hat{f}_\epsilon(\omega))\, \mathrm{e}^{\mathrm{i}\omega t} \int_{t-N}^{t+N} h_T(\tau)\, \mathrm{e}^{-\mathrm{i}\omega\tau}\, \mathrm{d}\tau\, \mathrm{d}\omega \right| \\
&\leq \|f - f_\epsilon\|_{\mathcal{PW}_\pi^1} \max_{\omega\in[-\pi,\pi]} \left| \int_{t-N}^{t+N} \mathrm{e}^{-\mathrm{i}\omega\tau}\, h_T(\tau)\, \mathrm{d}\tau \right| \\
&< \epsilon C_3(t)
\end{aligned}$$

for all $N \in \mathbb{N}$. Moreover, we have

$$\begin{aligned}
&\left| (Tf)(t) - (A_N^T f)(t) \right| \\
&\quad = \left| (Tf)(t) - (Tf_\epsilon)(t) + (Tf_\epsilon)(t) - (A_N^T f_\epsilon)(t) + (A_N^T(f_\epsilon - f))(t) \right| \\
&\quad \leq \|T\| \|f - f_\epsilon\|_{\mathcal{PW}_\pi^1} + \left| (Tf_\epsilon)(t) - (A_N^T f_\epsilon)(t) \right| + \epsilon C_3(t).
\end{aligned} \tag{4.17}$$

Since $f_\epsilon, h_T \in \mathcal{PW}_\pi^2$ we obtain for fixed $t \in \mathbb{R}$ and all $N \in \mathbb{N}$

$$\int_{-N}^{N} |f_\epsilon(\tau)||h_T(t-\tau)|\, \mathrm{d}\tau \leq \|f_\epsilon\|_2 \|h_T\|_2 < \infty,$$

i.e. the function $f_\epsilon(\tau) h_T(t-\tau)$ is absolutely integrable with respect to τ. Furthermore, application of Parseval's equation gives

$$\int_{-\infty}^{\infty} f_\epsilon(\tau) h_T(t-\tau)\, \mathrm{d}\tau = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}_\epsilon(\omega) \hat{h}_T(\omega)\, \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}\omega = (Tf_\epsilon)(t),$$

where the last equality is due to (4.1). Thus, there exists a $N_0 = N_0(\epsilon, t)$ such that

$$\left| (Tf_\epsilon)(t) - (A_N^T f_\epsilon)(t) \right| = \left| (Tf_\epsilon)(t) - \int_{-N}^{N} f_\epsilon(\tau) h_T(t-\tau)\, \mathrm{d}\tau \right| < \epsilon$$

for all $N \geq N_0(\epsilon, t)$. Consequently, using (4.17), we obtain

$$\left| (Tf)(t) - (A_N^T f)(t) \right| < (\|T\| + C_3(t) + 1)\epsilon$$

for all $N \geq N_0(\epsilon, t)$, which completes the proof of the "$\Leftarrow$" direction of the first part.

First part, "$\Rightarrow$" direction: Let T be a stable LTI system and $t \in \mathbb{R}$ be arbitrary but fixed. We prove this direction by proving (4.10), i.e., by showing that $\limsup_{N\to\infty} \left|(A_N^T f_1)(t)\right| = \infty$ for some signal $f_1 \in \mathcal{PW}_\pi^1$ follows from

$$\limsup_{N\to\infty} \max_{\omega\in[-\pi,\pi]} \left| \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{h}_T(\omega_1) \frac{\sin(N(\omega-\omega_1))}{\omega-\omega_1} \, \mathrm{d}\omega_1 \right| = \infty. \tag{4.18}$$

Since

$$\left| \int_{t-N}^{t+N} \mathrm{e}^{-\mathrm{i}\omega\tau} \, h_T(\tau) \, \mathrm{d}\tau \right| \geq \left| \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{h}_T(\omega_1) \frac{\sin(N(\omega-\omega_1))}{\omega-\omega_1} \, \mathrm{d}\omega_1 \right| - 2|t| \cdot \|T\|,$$

by the same considerations as in (4.15), equation (4.18) implies that

$$\limsup_{N\to\infty} \max_{\omega\in[-\pi,\pi]} \left| \int_{t-N}^{t+N} \mathrm{e}^{-\mathrm{i}\omega\tau} \, h_T(\tau) \, \mathrm{d}\tau \right| = \infty.$$

Thus, there exists a strictly increasing sequence $\{M_N\}_{N\in\mathbb{N}}$ of natural numbers such that

$$C(N) := \max_{\omega\in[-\pi,\pi]} \left| \int_{t-M_N}^{t+M_N} \mathrm{e}^{-\mathrm{i}\omega\tau} \, h_T(\tau) \, \mathrm{d}\tau \right|$$

tends monotonically to infinity, i.e., we have $\lim_{N\to\infty} C(N) = \infty$. Furthermore, since $\mathcal{PW}_\pi^2$ is dense in $\mathcal{PW}_\pi^1$, there is, according to Lemma 4.8, for every $N \in \mathbb{N}$ a function $f_N \in \mathcal{PW}_\pi^2$ with $\|f_N\|_{\mathcal{PW}_\pi^1} = 1$ such that

$$\left|(A_{M_N}^T f_N)(t)\right| \geq C(N) - 1. \tag{4.19}$$

Moreover, since $f_N \in \mathcal{PW}_\pi^2$, there exists for every $\epsilon > 0$ a natural number $K_0 = K_0(\epsilon, N)$ such that for all $K \geq K_0$ we have

$$\left|(A_K^T f_N)(t) - (T f_N)(t)\right| < \epsilon.$$

Next, we consider a sequence $\{N_k\}_{k\in\mathbb{N}}$ of natural numbers that satisfies

$$C(N_{k+1}) \geq 2C(N_k), \tag{4.20}$$

$$\sqrt{C(N_{k+1})} \geq \frac{\sqrt{2}}{\sqrt{2}-1} 5 \log(2M_{N_k}), \tag{4.21}$$

and for $1 \leq l \leq k-1$

$$\left|(A_{M_{N_k}}^T f_{N_l})(t) - (T f_{N_l})(t)\right| \leq 1. \tag{4.22}$$

A direct consequence of (4.20) is the inequality

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{C(N_k)}} \leq \frac{1}{\sqrt{C(N_1)}} \sum_{k=0}^{\infty} \frac{1}{2^{k/2}} = \frac{1}{\sqrt{C(N_1)}} \frac{\sqrt{2}}{\sqrt{2}-1}. \tag{4.23}$$

Furthermore, we define the signal f_1 by

$$f_1(t) := \sum_{k=1}^{\infty} \frac{1}{\sqrt{C(N_k)}} f_{N_k}(t),$$

and since

$$\|f_1\|_{\mathcal{PW}_\pi^1} \leq \sum_{k=1}^{\infty} \frac{1}{\sqrt{C(N_k)}} \|f_{N_k}\|_{\mathcal{PW}_\pi^1} \leq \frac{1}{\sqrt{C(N_1)}} \frac{\sqrt{2}}{\sqrt{2}-1},$$

where we used (4.23) in the last inequality, we have $f_1 \in \mathcal{PW}_\pi^1$. For $r \in \mathbb{N}$ arbitrary, we obtain

$$\begin{aligned} \left|(A_{M_{N_r}}^T f_l)(t)\right| &= \left|\sum_{k=1}^{r-1} \frac{(A_{M_{N_r}}^T f_{N_k})(t)}{\sqrt{C(N_k)}} + \frac{(A_{M_{N_r}}^T f_{N_r})(t)}{\sqrt{C(N_r)}} + \sum_{k=r+1}^{\infty} \frac{(A_{M_{N_r}}^T f_{N_k})(t)}{\sqrt{C(N_k)}}\right| \\ &\geq \frac{\left|(A_{M_{N_r}}^T f_{N_r})(t)\right|}{\sqrt{C(N_r)}} - \left|\sum_{k=1}^{r-1} \frac{(A_{M_{N_r}}^T f_{N_k})(t)}{\sqrt{C(N_k)}}\right| - \left|\sum_{k=r+1}^{\infty} \frac{(A_{M_{N_r}}^T f_{N_k})(t)}{\sqrt{C(N_k)}}\right|. \end{aligned} \tag{4.24}$$

Next, the individual terms on right-hand side of (4.24) are analyzed. For the first term we have

$$\frac{\left|(A_{M_{N_r}}^T f_{N_r})(t)\right|}{\sqrt{C(N_r)}} \geq \frac{C(N_r)-1}{\sqrt{C(N_r)}},$$

because of (4.19). The second term is bounded above by

$$\begin{aligned} \left|\sum_{k=1}^{r-1} \frac{(A_{M_{N_r}}^T f_{N_k})(t)}{\sqrt{C(N_k)}}\right| &\leq (1+\|T\|) \sum_{k=1}^{r-1} \frac{1}{\sqrt{C(N_k)}} \\ &\leq (1+\|T\|) \frac{1}{\sqrt{C(N_1)}} \frac{\sqrt{2}}{\sqrt{2}-1} \end{aligned}$$

where we used inequality (4.23) and the fact that

$$\begin{aligned} \left|(A_{M_{N_r}}^T f_{N_k})(t)\right| &\leq \left|(A_{M_{N_r}}^T f_{N_k})(t) - (T f_{N_k})(t)\right| + |(T f_{N_k})(t)| \\ &\leq 1 + \|T f_{N_k}\|_{\mathcal{PW}_\pi^1} \leq 1 + \|T\|, \end{aligned}$$

which is due to (4.22). Finally, using Lemma 4.9, we obtain

$$\begin{aligned}\left|\sum_{k=r+1}^{\infty} \frac{(A_{M_{N_r}}^T f_{N_k})(t)}{\sqrt{C(N_k)}}\right| &\le \sum_{k=r+1}^{\infty} \frac{\left|(A_{M_{N_r}}^T f_{N_k})(t)\right|}{\sqrt{C(N_k)}} \\ &\le \sum_{k=r+1}^{\infty} \frac{1}{\sqrt{C(N_k)}} \|T\| \frac{2}{\pi}\left(\pi + \frac{2}{\pi} + \frac{2}{\pi}\log(2M_{N_r}-1)\right) \\ &\le \sum_{k=r+1}^{\infty} \frac{\|T\| 5\log(2M_{N_r})}{\sqrt{C(N_k)}}\end{aligned}$$

for the third term. This can be further simplified according to

$$\begin{aligned}\sum_{k=r+1}^{\infty} \frac{\|T\| 5\log(2M_{N_r})}{\sqrt{C(N_k)}} &\le \|T\| 5\log(2M_{N_r}) \sum_{k=r+1}^{\infty} \frac{1}{\sqrt{C(N_{r+1})}} \frac{1}{\sqrt{2^{k-(r+1)}}} \\ &\le \|T\| \frac{5\log(2M_{N_r})}{\sqrt{C(N_{r+1})}} \sum_{l=0}^{\infty} \frac{1}{2^{l/2}} \\ &= \|T\| \frac{5\log(2M_{N_r})}{\sqrt{C(N_{r+1})}} \frac{\sqrt{2}}{\sqrt{2}-1} \\ &\le \|T\|,\end{aligned}$$

by using (4.20) and (4.21). Summarizing we have

$$\left|(A_{M_{N_r}}^T f_1)(t)\right| \ge \frac{C(N_r)-1}{\sqrt{C(N_r)}} - (1+\|T\|)\frac{1}{\sqrt{C(N_1)}}\frac{\sqrt{2}}{\sqrt{2}-1} - \|T\|. \tag{4.25}$$

Since (4.25) is valid for arbitrary $r \in \mathbb{N}$ it follows that $\lim_{r\to\infty}\left|(A_{M_{N_r}}^T f_1)(t)\right| = \infty$, which completes the proof of the "$\Rightarrow$" direction of the first part.

Second part, "$\Leftarrow$" direction: Let $f \in \mathcal{PW}_\pi^1$, $\varphi \in \mathcal{S}$, and $\epsilon > 0$ be arbitrary but fixed. Since $\mathcal{PW}_\pi^2$ is dense in $\mathcal{PW}_\pi^1$, there exists a function $f_\epsilon \in \mathcal{PW}_\pi^2$ with $\|f - f_\epsilon\|_{\mathcal{PW}_\pi^1} < \epsilon$. According to Lemma 4.11 and the assumption (4.12) we have

$$\begin{aligned}A_{N,\varphi}^T(f-f_\epsilon) &\le \|f-f_\epsilon\|_{\mathcal{PW}_\pi^1} \max_{\omega\in[-\pi,\pi]}\left|\frac{1}{\pi}\int_{-\pi}^{\pi} \hat{h}_T(\omega_1)\hat{\varphi}(-\omega_1)\frac{\sin(N(\omega-\omega_1))}{\omega-\omega_1}\,\mathrm{d}\omega_1\right| \\ &\le \epsilon C_2(\varphi)\end{aligned}$$

for all $N \in \mathbb{N}$. Therefore, we obtain

$$\begin{aligned}&\left|\int_{-\infty}^{\infty}(Tf)(t)\varphi(t)\,\mathrm{d}t - A_{N,\varphi}^T f\right| \\ &= \left|\int_{-\infty}^{\infty}(T(f-f_\epsilon))(t)\varphi(t)\,\mathrm{d}t + \int_{-\infty}^{\infty}(Tf_\epsilon)(t)\varphi(t)\,\mathrm{d}t - A_{N,\varphi}^T f_\epsilon - A_{N,\varphi}^T(f-f_\epsilon)\right| \\ &< \|T\|\,\|f-f_\epsilon\|_{\mathcal{PW}_\pi^1}\int_{-\infty}^{\infty}|\varphi(t)|\,\mathrm{d}t + \left|\int_{-\infty}^{\infty}(Tf_\epsilon)(t)\varphi(t)\,\mathrm{d}t - A_{N,\varphi}^T f_\epsilon\right| + \epsilon C_2(\varphi) \\ &< \epsilon\|T\|\,\|\varphi\|_1 + \left|\int_{-\infty}^{\infty}(Tf_\epsilon)(t)\varphi(t)\,\mathrm{d}t - A_{N,\varphi}^T f_\epsilon\right| + \epsilon C_2(\varphi).\end{aligned} \tag{4.26}$$

Further, we have

$$A_{N,\varphi}^T f_\epsilon = \int_{-\infty}^{\infty} (A_N^T f_\epsilon)(t)\varphi(t)\,\mathrm{d}t$$
$$= \int_{-\infty}^{\infty} \int_{-N}^{N} f_\epsilon(t) h_T(t-\tau)\,\mathrm{d}\tau\varphi(t)\,\mathrm{d}t.$$

Since $f_\epsilon, h_T \in \mathcal{PW}_\pi^2$,

$$\left| \int_{-N}^{N} f_\epsilon(t) h_T(t-\tau)\,\mathrm{d}\tau\varphi(t) \right| \leq \int_{-\infty}^{\infty} |f_\epsilon(t) h_T(t-\tau)|\,\mathrm{d}\tau|\varphi(t)|$$
$$\leq \|f_\epsilon\|_2 \|h_T\|_2 |\varphi(t)|,$$

and $\varphi \in L^1(\mathbb{R})$, we can apply Lebesgue's dominated convergence theorem, which leads to

$$\lim_{N\to\infty} A_{N,\varphi}^T f_\epsilon = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_\epsilon(t) h_T(t-\tau)\,\mathrm{d}\tau\varphi(t)\,\mathrm{d}t$$
$$= \int_{-\infty}^{\infty} (Tf_\epsilon)(t)\varphi(t)\,\mathrm{d}t,$$

where the last equality follows from

$$\int_{-\infty}^{\infty} f_\epsilon(\tau) h_T(t-\tau)\,\mathrm{d}\tau = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}_\epsilon(\omega) \hat{h}_T(\omega)\,\mathrm{e}^{\mathrm{i}\omega t}$$
$$= (Tf_\epsilon)(t),$$

according to the generalized Parseval equality. Thus, there is a $N_0 = N_0(\epsilon)$ such that

$$\left| \int_{-\infty}^{\infty} (Tf_\epsilon)(t)\varphi(t)\,\mathrm{d}t - A_{N,\varphi}^T f_\epsilon \right| < \epsilon \tag{4.27}$$

for all $N \geq N_0(\epsilon)$. Combining (4.26) and (4.27), we obtain

$$\left| \int_{-\infty}^{\infty} (Tf)(t)\varphi(t)\,\mathrm{d}t - A_{N,\varphi}^T f \right| < \epsilon(\|T\|\,\|\varphi\|_1 + C_2(\varphi) + 1)$$

for all $N \geq N_0(\epsilon)$. This completes this part of the proof, because $\epsilon > 0$ was arbitrary.

Second part, "⇒" direction: Let $\varphi \in \mathcal{S}$ be arbitrary but fixed. Since

$$|A_{N,\varphi}^T f| \leq \left| A_{N,\varphi}^T f - \int_{-\infty}^{\infty} (Tf)(t)\varphi(t)\,\mathrm{d}t \right| + \left| \int_{-\infty}^{\infty} (Tf)(t)\varphi(t)\,\mathrm{d}t \right|$$

for all $N \in \mathbb{N}$ and all $f \in \mathcal{PW}_\pi^1$, equation (4.11) and the fact that

$$\left| \int_{-\infty}^{\infty} (Tf)(t)\varphi(t)\,\mathrm{d}t \right| \leq \|T\|\,\|f\|_{\mathcal{PW}_\pi^1} \|\varphi\|_1 < \infty$$

imply that $\sup_{N\in\mathbb{N}}|A_{N,\varphi}^T f| < \infty$ for all $f \in \mathcal{PW}_\pi^1$. From (4.14) we see that $A_{N,\varphi}^T : \mathcal{PW}_\pi^1 \to \mathbb{C}$ is a bounded linear operator for all $N \in \mathbb{N}$. It follows from the Banach–Steinhaus theorem [78, p. 98] that

$$\sup_{N\in\mathbb{N}} \sup_{\|f\|_{\mathcal{PW}_\pi^1}\le 1} |A_{N,\varphi}^T f| < \infty.$$

Consequently, we have

$$\sup_{N\in\mathbb{N}} \max_{\omega\in[-\pi,\pi]} \left| \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{h}_T(\omega_1)\hat{\varphi}(-\omega_1) \frac{\sin(N(\omega-\omega_1))}{\omega-\omega_1} \, \mathrm{d}\omega_1 \right| < \infty,$$

by Lemma 4.11, which completes this part of the "if and only if" statement.

On the other hand if (4.12) is not fulfilled, i.e., if

$$\limsup_{N\to\infty} \max_{\omega\in[-\pi,\pi]} \left| \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{h}_T(\omega_1)\hat{\varphi}(-\omega_1) \frac{\sin(N(\omega-\omega_1))}{\omega-\omega_1} \, \mathrm{d}\omega_1 \right| = \infty,$$

we have

$$\sup_{N\in\mathbb{N}} \sup_{\|f\|_{\mathcal{PW}_\pi^1}\le 1} |A_{N,\varphi}^T f| = \infty$$

according to Lemma 4.11. Thus, the Banach–Steinhaus theorem [78, p. 98] implies that there exists a signal $f_1 \in \mathcal{PW}_\pi^1$ such that (4.13) is true. Using the same steps as in the proof of the first part, it would be even possible to construct such a divergence creating signal f_1. □

Remark 4.13. In the previous theorem we have seen that if (4.12) is not fulfilled then there exists a signal $f_1 \in \mathcal{PW}_\pi^1$ such that $A_N^T f_1$ diverges in $\mathcal{S}'$. In this case, there is not only one single divergence creating signal. In fact, the set of signals for which we have divergence is large: It is a residual set, i.e., the complement of a set of the first category, and therefore it is dense in $\mathcal{PW}_\pi^1$ [117, p. 12].

From the proof of part ii) of Theorem 4.5 we see that the same arguments hold if we replace $\varphi \in \mathcal{S}$ with $\varphi \in \mathcal{D}$. Thus, part ii) of Theorem 4.5 is also true if we replace $\varphi \in \mathcal{S}$ with $\varphi \in \mathcal{D}$. With that we also have a characterization of the convergence of $A_N^T f$ in $\mathcal{D}'$.

Corollary 4.14. *Part ii) of Theorem 4.5 remains true if $\varphi \in \mathcal{S}$ is replaced with $\varphi \in \mathcal{D}$.*

In Theorem 4.17 we will show that there really exists a stable LTI system such that (4.12) is not fulfilled for some $\varphi \in \mathcal{D}$, i.e., that there exists a stable LTI system T_1 such that $A_N^{T_1} f_1$ diverges in $\mathcal{D}'$ for some $f_1 \in \mathcal{PW}_\pi^1$.

It would be interesting to have a connection between the pointwise convergence of $A_N^T f$, the convergence of $A_N^T f$ in $\mathcal{D}'$, and the convergence of $A_N^T f$ in $\mathcal{S}'$. The following lemma, the proof of which is given in Appendix A.5, is the main step towards Theorem 4.16, where we identify this connection.

Lemma 4.15. *Let $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ be a stable LTI system.*

i) If for all $\varphi \in \mathcal{D}$ there exists a constant $C_2 = C_2(\varphi) < \infty$ such that (4.12) *in Theorem 4.5 is fulfilled for all $N \in \mathbb{N}$ then there exists a constant $C_1 < \infty$ such that* (4.9) *in Theorem 4.5 is fulfilled for all $N \in \mathbb{N}$.*

ii) Further, if there exists a constant $C_1 < \infty$ such that (4.9) *in Theorem 4.5 is fulfilled for all $N \in \mathbb{N}$ then, for all $\varphi \in \mathcal{S}$ there exists a constant $C_2 = C_2(\varphi) < \infty$ such that* (4.12) *in Theorem 4.5 is fulfilled for all $N \in \mathbb{N}$.*

Theorem 4.16 establishes the connection between the classical (pointwise) convergence and the distributional convergence of $A_N^T f$.

Theorem 4.16. *Let $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ be a stable LTI system. The following statements are equivalent.*

i) $A_N^T f$ converges in $\mathcal{D}'$ for all $f \in \mathcal{PW}_\pi^1$.

ii) $A_N^T f$ converges in $\mathcal{S}'$ for all $f \in \mathcal{PW}_\pi^1$.

iii) $(A_N^T f)(t)$ converges pointwise for some $t \in \mathbb{R}$ and all $f \in \mathcal{PW}_\pi^1$.

iv) $(A_N^T f)(t)$ converges pointwise for all $t \in \mathbb{R}$ and all $f \in \mathcal{PW}_\pi^1$.

Proof. "*iii) ⇒ ii)*": This follows from Theorem 4.5 i), Lemma 4.15 ii), and Theorem 4.5 ii). "*ii) ⇒ i)*": Observation 4.3. "*i) ⇒ iv)*": This follows from Corollary 4.14, Lemma 4.15 i), and Theorem 4.5 i). "*iv) ⇒ iii)*": Obvious. □

In general, convergence in $\mathcal{S}'$ is a stronger statement than convergence in $\mathcal{D}'$, because the former implies the latter. However, in Theorem 4.16 we have the situation that $A_N^T f$ converges in $\mathcal{S}'$ if and only if it converges in $\mathcal{D}'$.

Moreover, Theorem 4.16 shows that we do not gain anything regarding the convergence behavior of $A_N^T f$ for stable LTI systems T and signals f in $\mathcal{PW}_\pi^1$ if we consider the more relaxed concept of distributional convergence. If $(A_N^T f)(t)$ diverges in the classical (pointwise) sense for some signal $f \in \mathcal{PW}_\pi^1$ and some $t \in \mathbb{R}$ then $A_N^T f$ diverges also in $\mathcal{D}'$ and consequently in $\mathcal{S}'$ for some signal $f \in \mathcal{PW}_\pi^1$.

The following theorem states that there exists a stable LTI system T_1 and a signal $f_1 \in \mathcal{PW}_\pi^1$ such that $A_N^{T_1} f_1$ diverges in $\mathcal{D}'$ as N tends to infinity.

Theorem 4.17. *There exists a stable LTI system $T_1 : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ and a signal $f_1 \in \mathcal{PW}_\pi^1$ such that*

$$\limsup_{N\to\infty} |A_{N,\varphi_1}^{T_1} f_1| = \infty \tag{4.28}$$

for some $\varphi_1 \in \mathcal{D}$.

Proof. We can prove this theorem by finding an explicit system T_1 such that

$$\limsup_{N\to\infty} \max_{\omega\in[-\pi,\pi]} \left| \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{h}_{T_1}(\omega_1) \frac{\sin(N(\omega-\omega_1))}{\omega-\omega_1} \, \mathrm{d}\omega_1 \right| = \infty. \tag{4.29}$$

Then it follows by Lemma 4.15 i) that there exists a $\varphi_1 \in \mathcal{D}$ such that

$$\limsup_{N\to\infty} \max_{\omega\in[-\pi,\pi]} \left| \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{h}_{T_1}(\omega_1)\hat{\varphi}_1(-\omega_1) \frac{\sin(N(\omega-\omega_1))}{\omega-\omega_1} \, \mathrm{d}\omega_1 \right| = \infty,$$

which in turn implies, by Corollary 4.14, that there exists a signal $f_1 \in \mathcal{PW}_\pi^1$ such that (4.28) is true.

Next, we construct the system T_1. To this end, consider

$$h_H(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (-\mathrm{i}\,\mathrm{sgn}(\omega))\, \mathrm{e}^{\mathrm{i}\omega t} \, \mathrm{d}\omega = \frac{\sin^2\left(\frac{\pi}{2}t\right)}{\frac{\pi}{2}t}.$$

and the function h_N, defined by

$$h_N(\tau) = \frac{h_H(\tau+N) - h_H(\tau-N)}{2}.$$

It follows that

$$\hat{h}_N(\omega) = \sin(N\omega)\,\mathrm{sgn}(\omega), \quad |\omega| \le \pi,$$

and

$$\begin{aligned}
\int_{-N}^{N} h_N(\tau) \, \mathrm{d}\tau &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{h}_N(\omega) \int_{-N}^{N} \mathrm{e}^{\mathrm{i}\omega\tau} \, \mathrm{d}\tau \, \mathrm{d}\omega \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(N\omega)\,\mathrm{sgn}(\omega) \frac{\sin(N\omega)}{\omega} \, \mathrm{d}\omega \\
&= \frac{2}{\pi} \int_{0}^{\pi} \frac{\sin^2(N\omega)}{\omega} \, \mathrm{d}\omega \\
&> \frac{1}{\pi} \log(2N),
\end{aligned}$$

because

$$\begin{aligned}
\frac{2}{\pi} \int_{0}^{\pi} \frac{\sin^2(N\omega)}{\omega} \, \mathrm{d}\omega &= \int_{0}^{2N} \frac{\sin^2\left(\frac{\pi}{2}\tau\right)}{\frac{\pi}{2}\tau} \, \mathrm{d}\tau \\
&> \frac{2}{\pi} \sum_{k=0}^{2N-1} \frac{1}{k+1} \int_{k}^{k+1} \sin^2\left(\frac{\pi}{2}\tau\right) \mathrm{d}\tau \\
&= \frac{1}{\pi} \sum_{k=0}^{2N-1} \frac{1}{k+1} \\
&> \frac{1}{\pi} \log(2N).
\end{aligned} \tag{4.30}$$

Now let $N_k = 2^{n_k}$, $k \in \mathbb{N}$, be a sequence of dyadic numbers, where $\{n_k\}_{k=1}^{\infty}$ is a sequence of natural numbers satisfying $n_{k+1} > n_k$, $k \in \mathbb{N}$, and let $\{\epsilon_k\}_{k=1}^{\infty}$ be a sequence of positive numbers with $\sum_{k=1}^{\infty} \epsilon_k < \infty$. Consider the function

$$\hat{h}_{T_1}(\omega) = \sum_{k=1}^{\infty} \epsilon_k \hat{h}_{N_k}(\omega). \tag{4.31}$$

Since

$$\|\hat{h}_{T_1}\|_{L^\infty[-\pi,\pi]} \leq \sum_{k=1}^{\infty} \epsilon_k \|\hat{h}_{N_k}\|_{L^\infty[-\pi,\pi]} \leq \sum_{k=1}^{\infty} \epsilon_k < \infty,$$

the operator T_1 defined by

$$(T_1 f)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) \hat{h}_{T_1}(\omega) \, \mathrm{e}^{\mathrm{i}\omega t} \, \mathrm{d}\omega$$

is a stable LTI system, and the impulse response h_{T_1} is a well defined continuous function. Next, we analyze the integral $\int_{-M}^{M} h_{T_1}(\tau) \, \mathrm{d}\tau$ for arbitrary dyadic numbers $M \geq 2$. We have

$$\begin{aligned} \int_{-M}^{M} h_{T_1}(\tau) \, \mathrm{d}\tau &= \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{h}_{T_1}(\omega) \frac{\sin(M\omega)}{\omega} \, \mathrm{d}\omega \\ &= \sum_{k=1}^{\infty} \epsilon_k \frac{1}{\pi} \int_{-\pi}^{\pi} \operatorname{sgn}(\omega) \sin(N_k \omega) \frac{\sin(M\omega)}{\omega} \, \mathrm{d}\omega \\ &= \sum_{k=1}^{\infty} \epsilon_k \frac{2}{\pi} \int_{0}^{\pi} \sin(N_k \omega) \frac{\sin(M\omega)}{\omega} \, \mathrm{d}\omega. \end{aligned} \tag{4.32}$$

The right-hand side of (4.32) is further analyzed. For all $k \in \mathbb{N}$ and $M = N_k$ we have

$$\frac{2}{\pi} \int_{0}^{\pi} \sin(N_k \omega) \frac{\sin(M\omega)}{\omega} \, \mathrm{d}\omega \geq \frac{1}{\pi} \log(2N_k).$$

Furthermore, for every $k \in \mathbb{N}$ and $M \neq N_k$ we can decompose the integral into three parts by splitting the integration interval and using the identity for the product of sine functions

$$\begin{aligned} \frac{2}{\pi} \int_{0}^{\pi} \sin(N_k \omega) \frac{\sin(M\omega)}{\omega} \, \mathrm{d}\omega = {} & \frac{2}{\pi} \int_{0}^{\delta} \sin(N_k \omega) \frac{\sin(M\omega)}{\omega} \, \mathrm{d}\omega \\ & + \frac{1}{\pi} \int_{\delta}^{\pi} \frac{\cos((M - N_k)\omega)}{\omega} \, \mathrm{d}\omega \\ & - \frac{1}{\pi} \int_{\delta}^{\pi} \frac{\cos((M + N_k)\omega)}{\omega} \, \mathrm{d}\omega. \end{aligned}$$

By choosing $\bar{\delta}_{k,M} = \pi/(2|N_k - M|)$ it can be shown that, for all $k \in \mathbb{N}$ and N_k, M dyadic with $N_k \neq M$,

$$\left| \frac{2}{\pi} \int_{0}^{\bar{\delta}_{k,M}} \sin(N_k \omega) \frac{\sin(M\omega)}{\omega} \, \mathrm{d}\omega \right| \leq 1,$$

$$\left| \frac{1}{\pi} \int_{\bar{\delta}_{k,M}}^{\pi} \frac{\cos((M - N_k)\omega)}{\omega} \,\mathrm{d}\omega \right| \le C_4$$

and

$$\left| \frac{1}{\pi} \int_{\bar{\delta}_{k,M}}^{\pi} \frac{\cos((M + N_k)\omega)}{\omega} \,\mathrm{d}\omega \right| \le C_5.$$

Hence, for N_k, M dyadic and $N_k \ne M$ we have

$$\left| \frac{2}{\pi} \int_0^{\pi} \sin(N_k\omega) \frac{\sin(M\omega)}{\omega} \,\mathrm{d}\omega \right| \le C_6$$

for all $k \in \mathbb{N}$. By setting $M = N_r$, we obtain

$$\left| \int_{-N_r}^{N_r} h_{T_1}(\tau) \,\mathrm{d}\tau \right| \ge \epsilon_r \frac{1}{\pi} \log(2N_r) - C_6 \sum_{\substack{k=1 \\ k \ne r}}^{\infty} \epsilon_k.$$

The function h_{T_1}, which was defined in (4.31), certainly depends on the concrete choice of the sequences $\{\epsilon_k\}_{k=1}^{\infty}$ and $\{N_k\}_{k=1}^{\infty}$. We can choose $\epsilon_k = 1/k^2$ and $N_k = 2^{(k^3)}$. Then the function h_{T_1} satisfies

$$\left| \int_{-N_r}^{N_r} h_{T_1}(\tau) \,\mathrm{d}\tau \right| \ge \frac{1}{\pi} \frac{1}{r^2} \log\left(2^{(r^3)}\right) - C_6 \sum_{\substack{k=1 \\ k \ne r}}^{\infty} \frac{1}{k^2} \ge \frac{\log(2)}{\pi} r - C_6 \frac{\pi^2}{6}$$

and consequently

$$\limsup_{N\to\infty} \left| \int_{-N}^{N} h_{T_1}(\tau) \,\mathrm{d}\tau \right| = \infty.$$

Moreover, since

$$\begin{aligned} \limsup_{N\to\infty} \left| \int_{-N}^{N} h_{T_1}(\tau) \,\mathrm{d}\tau \right| &\le \limsup_{N\to\infty} \max_{\omega\in[-\pi,\pi]} \left| \int_{-N}^{N} \mathrm{e}^{-\mathrm{i}\omega\tau}\, h_{T_1}(\tau) \,\mathrm{d}\tau \right| \\ &= \limsup_{N\to\infty} \max_{\omega\in[-\pi,\pi]} \left| \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{h}_{T_1}(\omega_1) \frac{\sin(N(\omega-\omega_1))}{\omega-\omega_1} \,\mathrm{d}\omega_1 \right|, \end{aligned}$$

we have (4.29). □

Theorem 4.17 shows that a convolution-type representation of stable LTI systems in the form (4.5) is not possible in general for the space $\mathcal{PW}_\pi^1$, even if the convergence is treated in the distributional sense. In Theorem 4.23 we will see that the same is true for the second convolution integral (4.6).

Test Signals

Before we treat the second convolution integral, we give an interesting interpretation of condition (4.9) in terms of test signals. Since

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi} \hat{h}_T(\omega_1)\frac{\sin(N(\omega-\omega_1))}{\omega-\omega_1}\,\mathrm{d}\omega_1 &= \int_{-N}^{N} \mathrm{e}^{\mathrm{i}\omega\tau}\, h_T(-\tau)\,\mathrm{d}\tau \\ &= (A_N^T\,\mathrm{e}^{\mathrm{i}\omega\,\cdot})(0) \qquad (4.33)\end{aligned}$$

we see that (4.9) is equivalent to

$$\max_{\omega\in[-\pi,\pi]} |(A_N^T f_\omega^{\mathrm{test}})(0)| \le C_1,$$

where $f_\omega^{\mathrm{test}}(t) = \mathrm{e}^{\mathrm{i}\omega t}$. Thus, we can regard the exponential function $\mathrm{e}^{\mathrm{i}\omega t}$ as a test signal. If $|(A_N^T f_\omega^{\mathrm{test}})(0)|$ is uniformly bounded for all test signals f_ω^{test}, where the parameter ω ranges from $-\pi$ to π, and all $N \in \mathbb{N}$, then $A_N^T f$ converges pointwise, and due to Theorem 4.16 also in $\mathcal{S}'$, for all $f \in \mathcal{PW}_\pi^1$. That is, we have

$$\lim_{N\to\infty} \left| A_{N,\varphi}^T f - \int_{-\infty}^{\infty} (Tf)(t)\varphi(t)\,\mathrm{d}t \right| = 0$$

for all $f \in \mathcal{PW}_\pi^1$ and all $\varphi \in \mathcal{S}$.

However, the converse statement might be more useful in practice. If we find one test signal $f_{\omega_1}^{\mathrm{test}}$, $\omega_1 \in [-\pi,\pi]$, such that

$$\lim_{N\to\infty} |(A_N^T f_{\omega_1}^{\mathrm{test}})(0)| = \infty$$

then we have both pointwise divergence and divergence in $\mathcal{D}'$. That is, there exists a signal $f_1 \in \mathcal{PW}_\pi^1$ such that

$$\limsup_{N\to\infty} |A_{N,\varphi_1}^T f_1| = \infty$$

for some $\varphi_1 \in \mathcal{D}$.

Although the test signals do not belong to the signal space $\mathcal{PW}_\pi^1$, they have an appealingly simple structure. They are just scaled versions of one basic function $\mathrm{e}^{\mathrm{i}t}$.

Convergence Behavior of the Convolution Integral II

Now we treat (4.6), i.e., the second convolution integral.

The next theorem analyzes the global convergence behavior of B_N^T and the distributional convergence behavior of B_N^T in $\mathcal{S}'$. For each type of convergence, we completely characterize the stable LTI systems T for which $B_N^T f$ converges to Tf for all $f \in \mathcal{PW}_\pi^1$, by giving a necessary and sufficient condition for convergence.

Theorem 4.18. *Let $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ be a stable LTI system.*

i) For all $f \in \mathcal{PW}_\pi^1$ we have

$$\lim_{N\to\infty} \|Tf - B_N^T f\|_\infty = 0 \tag{4.34}$$

if and only if there exists a constant $C_7 < \infty$ such that

$$\max_{\omega\in[-\pi,\pi]} \left| \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{h}_T(\omega_1) \frac{\sin(N(\omega-\omega_1))}{\omega-\omega_1} \, \mathrm{d}\omega_1 \right| \le C_7 \tag{4.35}$$

for all $N \in \mathbb{N}$. In addition, if (4.35) is not fulfilled, then there exists a signal $f_1 \in \mathcal{PW}_\pi^1$ such that

$$\limsup_{N\to\infty} \|B_N^T f_1\|_\infty = \infty.$$

ii) Moreover, we have

$$\lim_{N\to\infty} \left| B_{N,\varphi}^T f - \int_{-\infty}^{\infty} (Tf)(t)\varphi(t) \, \mathrm{d}t \right| = 0$$

for all $f \in \mathcal{PW}_\pi^1$ and all $\varphi \in \mathcal{S}$ if and only if for all $\varphi \in \mathcal{S}$ there exists a constant $C_8 = C_8(\varphi) < \infty$ such that

$$\max_{\omega\in[-\pi,\pi]} \left| \hat{\varphi}(-\omega) \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{h}_T(\omega_1) \frac{\sin(N(\omega-\omega_1))}{\omega-\omega_1} \, \mathrm{d}\omega_1 \right| \le C_8(\varphi) \tag{4.36}$$

for all $N \in \mathbb{N}$. In addition, if (4.36) is not fulfilled for some $\varphi \in \mathcal{S}$, then there exists a signal $f_1 \in \mathcal{PW}_\pi^1$ such that

$$\limsup_{N\to\infty} |B_{N,\varphi}^T f_1| = \infty.$$

The proof of Theorem 4.18 is done analogously to the proof of Theorem 4.5.

We see that the conditions (4.9) and (4.35) are the same. Therefore, $A_N^T f$ converges pointwise for all $f \in \mathcal{PW}_\pi^1$ if and only if $B_N^T f$ converges uniformly for all $f \in \mathcal{PW}_\pi^1$.

Example 4.19. For the Hilbert transform H and the ideal low-pass filter L_{ω_g} with bandwidth $0 < \omega_g \le \pi$, the condition (4.35) is fulfilled. Thus, for all $f \in \mathcal{PW}_\pi^1$, we have (4.34), i.e., uniform convergence of $B_N^H f$ and $B_N^{L_{\omega_g}} f$.

Moreover, Theorem 4.18 gives a necessary and sufficient condition for the convergence of $B_N^T f$ in $\mathcal{S}'$. $B_N^T f$ converges to Tf in $\mathcal{S}'$ for all $f \in \mathcal{PW}_\pi^1$ if and only if for all $\varphi \in \mathcal{S}$ there exists a constant $C_8(\varphi)$ such that (4.36) is fulfilled for all $N \in \mathbb{N}$.

Since the proof of part ii) of Theorem 4.18 is analogous to the proof of Theorem 4.5, we have the same situation here and can replace $\varphi \in \mathcal{S}$ with $\varphi \in \mathcal{D}$. This observation leads to the next corollary about the convergence of $B_N^T f$ in $\mathcal{D}'$.

Corollary 4.20. *Part ii) of Theorem 4.18 remains true if $\varphi \in \mathcal{S}$ is replaced with $\varphi \in \mathcal{D}$.*

Corollary 4.20 provides a necessary and sufficient condition for the convergence of $B_N^T f$ in $\mathcal{D}'$. $B_N^T f$ converges to Tf in $\mathcal{D}'$ for all $f \in \mathcal{PW}_\pi^1$ if and only if for all $\varphi \in \mathcal{D}$ there exists a constant $C_8(\varphi)$ such that (4.36) is fulfilled for all $N \in \mathbb{N}$.

Of course we are again interested in a connection between the uniform convergence of $B_N^T f$, the convergence of $B_N^T f$ in $\mathcal{D}'$, and the convergence of $B_N^T f$ in $\mathcal{S}'$. The following lemma, the proof of which is given in Appendix A.6, is the main step towards Theorem 4.22, where we identify this connection.

Lemma 4.21. *Let $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ be a stable LTI system.*

i) If for all $\varphi \in \mathcal{D}$ there exists a constant $C_8 = C_8(\varphi) < \infty$ such that (4.36) in Theorem 4.18 is fulfilled for all $N \in \mathbb{N}$ then there exists a constant $C_7 < \infty$ such that (4.35) in Theorem 4.18 is fulfilled for all $N \in \mathbb{N}$.

ii) Further, if there exists a constant $C_7 < \infty$ such that (4.35) in Theorem 4.18 is fulfilled for all $N \in \mathbb{N}$ then, for all $\varphi \in \mathcal{S}$, there exists a constant $C_8 = C_8(\varphi) < \infty$ such that (4.36) in Theorem 4.18 is fulfilled for all $N \in \mathbb{N}$.

Theorem 4.22 shows that again we do not have to distinguish between convergence in $\mathcal{D}'$ and convergence in $\mathcal{S}'$ because they are equivalent.

Theorem 4.22. *Let $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ be a stable LTI system. The following statements are equivalent.*

i) $B_N^T f$ converges in $\mathcal{D}'$ for all $f \in \mathcal{PW}_\pi^1$.

ii) $B_N^T f$ converges in $\mathcal{S}'$ for all $f \in \mathcal{PW}_\pi^1$.

iii) $B_N^T f$ converges uniformly on all of $\mathbb{R}$ for all $f \in \mathcal{PW}_\pi^1$.

Proof. "*iii*) ⇒ *ii*)": This follows from Theorem 4.18 i), Lemma 4.21 ii), and Theorem 4.18 ii). "*ii*) ⇒ *i*)": Observation 4.3. "*i*) ⇒ *iii*)": This follows from Corollary 4.20, Lemma 4.21 i), and Theorem 4.18 i). □

With Corollary 4.20 we have completely characterized all stable LTI systems T for which $B_N^T f$ converges in $\mathcal{D}'$ for all $f \in \mathcal{PW}_\pi^1$. Next we show that there actually exists a stable LTI system T_1 such that $B_N^{T_1} f_1$ diverges in $\mathcal{D}'$ for some $f_1 \in \mathcal{PW}_\pi^1$.

Theorem 4.23. *There exists a stable LTI system $T_1 : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ and a signal $f_1 \in \mathcal{PW}_\pi^1$ such that*

$$\limsup_{N\to\infty} |B_{N,\varphi_1}^{T_1} f_1| = \infty \tag{4.37}$$

for some $\varphi_1 \in \mathcal{D}$.

Proof. In the proof of Theorem 4.17 we have constructed a stable LTI system T_1 such that

$$\limsup_{N\to\infty} \max_{\omega\in[-\pi,\pi]} \left| \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{h}_{T_1}(\omega_1) \frac{\sin(N(\omega-\omega_1))}{\omega-\omega_1} \, \mathrm{d}\omega_1 \right| = \infty.$$

It follows from Lemma 4.21 i) that there exists a $\varphi_1 \in \mathcal{D}$ such that

$$\limsup_{N\to\infty} \max_{\omega\in[-\pi,\pi]} \left| \hat{\varphi_1}(-\omega) \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{h}_{T_1}(\omega_1) \frac{\sin(N(\omega-\omega_1))}{\omega-\omega_1} \, \mathrm{d}\omega_1 \right| = \infty,$$

which in turn implies, by Corollary 4.20, that there exists a signal $f_1 \in \mathcal{PW}_\pi^1$ for which (4.37) is true. □

Comparison of the Convergence of the Convolution Integrals I and II

In general, the convolution integrals (4.5) and (4.6) have a different convergence behavior. This can be seen for example, if the stable LTI system is the Hilbert transform H. For the Hilbert transform, $B_N^H f$ is uniformly convergent for all $f \in \mathcal{PW}_\pi^1$. In contrast, $A_N^H f$ is not uniformly convergent for all $f \in \mathcal{PW}_\pi^1$ because the peak value of $A_N^H f_1$ diverges for some signal $f_1 \in \mathcal{PW}_\pi^1$.

Theorem 4.24. *We have*

$$\lim_{N\to\infty} \|Hf - B_N^H f\|_\infty = 0$$

for all $f \in \mathcal{PW}_\pi^1$, but there exists a signal $f_1 \in \mathcal{PW}_\pi^1$ such that

$$\limsup_{N\to\infty} \|Hf_1 - A_N^H f_1\|_\infty = \infty.$$

For completeness, the proof of Theorem 4.24 is given in Appendix A.7.

Next, we compare the distributional convergence behavior of the convolution integrals. Since the conditions (4.9) and (4.35) are the same, i.e., since $A_N^T f$ converges pointwise for all $f \in \mathcal{PW}_\pi^1$ if and only if $B_N^T f$ converges uniformly on all of $\mathbb{R}$ for all $f \in \mathcal{PW}_\pi^1$, we can combine Theorem 4.16 and Theorem 4.22 to obtain the following interesting result about the distributional convergence behavior of the convolution integrals $A_N^T f$ and $B_N^T f$.

Corollary 4.25. *Let $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ be a stable LTI system. The following statements are equivalent.*

i) $A_N^T f$ converges in $\mathcal{S}'$ for all $f \in \mathcal{PW}_\pi^1$.

ii) $A_N^T f$ converges in $\mathcal{D}'$ for all $f \in \mathcal{PW}_\pi^1$.

iii) $B_N^T f$ converges in $\mathcal{S}'$ for all $f \in \mathcal{PW}_\pi^1$.

iv) $B_N^T f$ converges in $\mathcal{D}'$ for all $f \in \mathcal{PW}_\pi^1$.

Corollary 4.25 shows that both convolution integrals $A_N^T f$ and $B_N^T f$ have the same distributional convergence behavior. Thus, there is a difference between the classical convergence behavior of the convolution integrals and the distributional convergence behavior. In the classical setting, the integrals (4.5) and (4.6) exhibit a different convergence behavior, whereas in the distributional setting we do not have to distinguish between the integrals because both have the same convergence behavior.

4.3.3 Convolution Sum

The discrete counterpart of the convolution integral (4.5), which is given by the convolution sum (4.7), naturally emerges from the finite Shannon sampling series

$$(S_N f)(t) := \sum_{k=-N}^{N} f(k) \operatorname{sinc}(t-k)$$

when some LTI operator T is applied because

$$\begin{aligned}(T S_N f)(t) &= \sum_{k=-N}^{N} f(k)(T \operatorname{sinc}(\,\cdot\, - k))(t) \\ &= \sum_{k=-N}^{N} f(k) h_T(t-k) \\ &= (S_N^T f)(t).\end{aligned}$$

The sum in (4.7) is important for practical applications because it uses only the samples $\{f(k)\}_{k\in\mathbb{Z}}$ of the signal f. If $(S_N^T f)(t)$ converges to $(Tf)(t)$ for all $t \in \mathbb{R}$ as N tends to infinity, then $(S_N^T f)(t)$ can be used to approximate $(Tf)(t)$. Of course the convergence of $(S_N^T f)(t)$ is not guaranteed and depends on the signal f and the stable LTI system T.

For signals in $\mathcal{PW}_\pi^2$ the situation is simple, because we have

$$(Tf)(t) = \sum_{k=-\infty}^{\infty} f(k) h_T(t-k)$$

for all stable LTI systems $T : \mathcal{PW}_\pi^2 \to \mathcal{PW}_\pi^2$ and all signals $f \in \mathcal{PW}_\pi^2$. This is due to the convergence of the Shannon sampling series in the $\mathcal{PW}_\pi^2$-norm and the continuity and linearity of T.

Unfortunately, for signals $f \in \mathcal{PW}_\pi^1$ and stable LTI systems T operating on $\mathcal{PW}_\pi^1$, $(S_N^T f)(t)$ does not always converge to $(Tf)(t)$. There are stable LTI systems T for which $(S_N^T f)(t)$ diverges for some signal $f \in \mathcal{PW}_\pi^1$ [13,118]. In part i) of Theorem 4.26 we characterize the stable LTI systems T for which $(S_N^T f)(t)$ converges pointwise to $(Tf)(t)$ for all $f \in \mathcal{PW}_\pi^1$.

Further, we analyze the distributional convergence behavior of $S_N^T f$. For this purpose we introduce the abbreviation

$$S_{N,\varphi}^T f := \int_{-\infty}^{\infty} (S_N^T f)(t)\varphi(t)\ \mathrm{d}t.$$

In part ii) of Theorem 4.26 we characterize the stable LTI systems T for which the convolution sum $S_N^T f$ converges in $\mathcal{S}'$ for all $f \in \mathcal{PW}_\pi^1$.

Theorem 4.26. *Let $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ be a stable LTI system.*

i) For all $t \in \mathbb{R}$ and all $f \in \mathcal{PW}_\pi^1$ we have

$$\lim_{N\to\infty} |(Tf)(t) - (S_N^T f)(t)| = 0$$

if and only if there exists a constant $C_9 = C_9(t) < \infty$ such that

$$\max_{\omega\in[-\pi,\pi]} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{h}_T(\omega_1)\, \mathrm{e}^{\mathrm{i}\omega_1 t}\, \frac{\sin\left[\left(N+\frac{1}{2}\right)(\omega-\omega_1)\right]}{\sin\left(\frac{\omega-\omega_1}{2}\right)}\ \mathrm{d}\omega_1 \right| \le C_9(t) \tag{4.38}$$

for all $N \in \mathbb{N}$. In addition, if (4.38) is not fulfilled, then there exists a signal $f_1 \in \mathcal{PW}_\pi^1$ such that

$$\limsup_{N\to\infty} |(S_N^T f_1)(t)| = \infty.$$

ii) Moreover, we have

$$\lim_{N\to\infty} \left| S_{N,\varphi}^T f - \int_{-\infty}^{\infty} (Tf)(t)\varphi(t)\ \mathrm{d}t \right| = 0$$

for all $f \in \mathcal{PW}_\pi^1$ and all $\varphi \in \mathcal{S}$ if and only if for all $\varphi \in \mathcal{S}$ there exists a constant $C_{10} = C_{10}(\varphi) < \infty$ such that

$$\max_{\omega\in[-\pi,\pi]} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{h}_T(\omega_1)\hat{\varphi}(-\omega_1) \frac{\sin\left[\left(N+\frac{1}{2}\right)(\omega-\omega_1)\right]}{\sin\left(\frac{\omega-\omega_1}{2}\right)}\ \mathrm{d}\omega_1 \right| \le C_{10}(\varphi) \tag{4.39}$$

for all $N \in \mathbb{N}$. In addition, if (4.39) is not fulfilled for some $\varphi \in \mathcal{S}$, then there exists a signal $f_1 \in \mathcal{PW}_\pi^1$ such that

$$\limsup_{N\to\infty} |S_{N,\varphi}^T f_1| = \infty.$$

Part i) of Theorem 4.26 was proved in [118], and the proof of part ii) is done analogously to the proof of part ii) of Theorem 4.5.

Like the proofs of the Theorems 4.5 and 4.18, the proof of Theorem 4.26 does not rely on the fact that $\varphi \in \mathcal{S}$. All these arguments also hold if $\varphi \in \mathcal{D}$. This observation leads to the following corollary about the convergence of $S_N^T f$ in $\mathcal{D}'$.

Corollary 4.27. *Part ii) of Theorem 4.26 remains true if $\varphi \in \mathcal{S}$ is replaced with $\varphi \in \mathcal{D}$.*

In Theorem 4.30 we will use the characterization that is provided by Corollary 4.27 to show that there exists a stable LTI system T_1 for which $S_N^{T_1} f_1$ diverges in $\mathcal{D}'$ for some $f_1 \in \mathcal{PW}_\pi^1$.

4.3.4 Differences Between the Convolution Integral and the Convolution Sum

In Theorem 4.5 we have the special situation that the convergence of $(A_N^T f)(t)$ for one $t \in \mathbb{R}$ and all $f \in \mathcal{PW}_\pi^1$ implies the convergence of $(A_N^T f)(t)$ for all $t \in \mathbb{R}$ and all $f \in \mathcal{PW}_\pi^1$. In this section we investigate the question whether the convolution sum exhibits the same behavior, i.e., whether the convergence of $(S_N^T f)(t)$ for one $t \in \mathbb{R}$ and all $f \in \mathcal{PW}_\pi^1$ implies the convergence of $(S_N^T f)(t)$ for all $t \in \mathbb{R}$ and all $f \in \mathcal{PW}_\pi^1$. Despite the obvious similarities between the convolution integral and the convolution sum, the surprising answer to this question is no.

Theorem 4.28. *For every $t \in \mathbb{R}$ and every $t^* \in \mathbb{R} \setminus (t + \mathbb{Z})$ there exists a stable LTI system $T_1 : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ such that*

$$\lim_{N\to\infty} |(T_1 f)(t) - (S_N^{T_1} f)(t)| = 0 \tag{4.40}$$

for all $f \in \mathcal{PW}_\pi^1$ and

$$\limsup_{N\to\infty} |(S_N^{T_1} f_1)(t^*)| = \infty \tag{4.41}$$

for some $f_1 \in \mathcal{PW}_\pi^1$.

Proof. Let $t \in \mathbb{R}$ and $t^* \in \mathbb{R}\setminus(t+\mathbb{Z})$ be arbitrary but fixed. According to Theorem 4.26 i) and the equality

$$\begin{aligned}
&\frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{h}_T(\omega_1)\, \mathrm{e}^{\mathrm{i}\omega_1 t} \, \frac{\sin\left[\left(N+\frac{1}{2}\right)(\omega-\omega_1)\right]}{\sin\left(\frac{\omega-\omega_1}{2}\right)} \,\mathrm{d}\omega_1 \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{h}_T(\omega_1)\, \mathrm{e}^{\mathrm{i}\omega_1 t} \sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}k(\omega-\omega_1)} \,\mathrm{d}\omega_1 \\
&= \sum_{k=-N}^{N} h_T(t-k)\, \mathrm{e}^{\mathrm{i}\omega k},
\end{aligned}$$

we have (4.40) for all $f \in \mathcal{PW}_\pi^1$ if and only if

$$\sup_{N\in\mathbb{N}} \max_{\omega\in[-\pi,\pi]} \left| \sum_{k=-N}^{N} h_{T_1}(t-k)\, \mathrm{e}^{\mathrm{i}\omega k} \right| < \infty. \tag{4.42}$$

Furthermore, we have (4.41) for some $f_1 \in \mathcal{PW}_\pi^1$ if and only if

$$\limsup_{N\to\infty} \max_{\omega\in[-\pi,\pi]} \left| \sum_{k=-N}^{N} h_{T_1}(t^*-k)\, \mathrm{e}^{\mathrm{i}\omega k} \right| = \infty. \tag{4.43}$$

Thus, we have to show that there exists a stable LTI system T_1 such that (4.42) and (4.43) is true.

To this end, we consider the space $\mathcal{K}$ that consists of all functions h with a representation $h(\tau) = 1/(2\pi) \int_{-\pi}^{\pi} \hat{h}(w)\, \mathrm{e}^{\mathrm{i}\omega\tau} \,\mathrm{d}\omega$, $t \in \mathbb{R}$, for some $\hat{h} \in C[-\pi,\pi]$ and with finite norm

$$\|h\|_{\mathcal{K},t} := \|\hat{h}\|_\infty + \sup_{N\in\mathbb{N}} \max_{\omega\in[-\pi,\pi]} \left| \sum_{k=-N}^{N} h(t-k)\, \mathrm{e}^{\mathrm{i}\omega k} \right|.$$

The space $\mathcal{K}$, equipped with the norm $\|\cdot\|_{\mathcal{K},t}$ is a Banach space. We prove this fact, which will become important at the end of this proof, in Appendix A.8. Next, we consider the sequence of bounded linear operators $\{U_N\}_{N\in\mathbb{N}}$ that map $\mathcal{K}$ into $(C[-\pi,\pi], \|\cdot\|_{L^\infty[-\pi,\pi]})$, defined by

$$(U_N h)(\omega) = \sum_{k=-N}^{N} h\left(t^*-k\right) \mathrm{e}^{\mathrm{i}\omega k}.$$

Further, we need the functions h_n, $n \in \mathbb{N}$, given by

$$h_n(\tau) = \frac{\sin(\pi(\tau-n-t))}{2\pi(\tau-n-t)}$$

and the fact that

$$h_n\left(t^*-k\right) = \frac{(-1)^{k+n}\sin(\pi(t^*-t))}{2\pi(t^*-t-k-n)}$$

for all $k \in \mathbb{Z}$. Since

$$\|h_n\|_{\mathcal{K},t} = \frac{1}{2} + \sup_{N\in\mathbb{N}} \max_{\omega\in[-\pi,\pi]} \left| \sum_{k=-N}^{N} \frac{\sin(\pi(k+n))}{2\pi(k+n)} \mathrm{e}^{\mathrm{i}\omega k} \right| = 1,$$

we have

$$\begin{aligned}
\|U_N\| &= \sup_{\|h\|_{\mathcal{K},t}\le 1} \max_{\omega\in[-\pi,\pi]} |(U_N h)(\omega)| \\
&\ge \max_{\omega\in[-\pi,\pi]} |(U_N h_N)(\omega)| \\
&= \max_{\omega\in[-\pi,\pi]} \left| \sum_{k=-N}^{N} \frac{(-1)^{k+N}\sin(\pi(t^*-t))}{2\pi(t^*-t-k-N)} \mathrm{e}^{\mathrm{i}\omega k} \right| \\
&\ge \left| \frac{\sin(\pi(t^*-t))}{2\pi} \sum_{k=-N}^{N} \frac{(-1)^k}{t^*-t-k-N} \mathrm{e}^{\mathrm{i}\pi k} \right| \\
&= \left| \frac{\sin(\pi(t^*-t))}{2\pi} \sum_{k=-N}^{N} \frac{1}{t^*-t-k-N} \right|
\end{aligned}$$

for all $N \in \mathbb{N}$. Moreover, for $K_0 = \max(\lceil t^* - t \rceil, 1)$ and $N \geq K_0$ we obtain

$$\begin{aligned}
\left| \sum_{k=-N}^{N} \frac{1}{t^* - t - k - N} \right| &= \left| \sum_{k=0}^{2N} \frac{1}{k - (t^* - t)} \right| \\
&\geq \sum_{k=K_0}^{2N} \frac{1}{k - (t^* - t)} - \left| \sum_{k=0}^{K_0 - 1} \frac{1}{k - (t^* - t)} \right| \\
&\geq \sum_{k=K_0}^{2N} \int_k^{k+1} \frac{1}{\tau - (t^* - t)} \, \mathrm{d}\tau - C_{11} \\
&= \int_{K_0}^{2N+1} \frac{1}{\tau - (t^* - t)} \, \mathrm{d}\tau - C_{11} \\
&= \log\left(\frac{2N + 1 - (t^* - t)}{K_0 - (t^* - t)} \right) - C_{11}.
\end{aligned}$$

Since C_{11} is some constant, which is independent of N, we see that

$$\lim_{N \to \infty} \left| \sum_{k=-N}^{N} \frac{1}{t^* - t - k - N} \right| = \infty,$$

which implies that $\sup_{N \in \mathbb{N}} \|U_N\| = \infty$. Thus, according to the Banach–Steinhaus theorem [78, p. 98] there exists a function $h_{T_1} \in \mathcal{K}$ such that

$$\limsup_{N \to \infty} \|U_N h_{T_1}\|_{L^\infty[-\pi,\pi]} = \limsup_{N \to \infty} \max_{\omega \in [-\pi,\pi]} \left| \sum_{k=-N}^{N} h_{T_1}(t^* - k) \, \mathrm{e}^{\mathrm{i}\omega k} \right| = \infty.$$

By the definition of the space $\mathcal{K}$, h_{T_1} fulfills

$$\sup_{N \in \mathbb{N}} \max_{\omega \in [-\pi,\pi]} \left| \sum_{k=-N}^{N} h_{T_1}(t - k) \, \mathrm{e}^{\mathrm{i}\omega k} \right| < \infty.$$

So T_1 is the desired stable LTI system. □

According to Theorem 4.28 we cannot conclude the convergence of $(S_N^T f)(t)$ for all $t \in \mathbb{R}$ and all $f \in \mathcal{PW}_\pi^1$ from the convergence of $(S_N^T f)(t)$ for some fixed $t \in \mathbb{R}$ and all $f \in \mathcal{PW}_\pi^1$. This is in contrast to the situation in Theorem 4.5 (compare Remark 4.6) where exactly this was possible. Consequently, for $(S_N^T f)(t)$ we cannot obtain an equivalence like the equivalence between item iii) and item iv) in Theorem 4.16.

Nevertheless, it would be satisfying if the convergence types

S1) $S_N^T f$ converges in $\mathcal{D}'$ for all $f \in \mathcal{PW}_\pi^1$,

S2) $S_N^T f$ converges in $\mathcal{S}'$ for all $f \in \mathcal{PW}_\pi^1$, S3) $(S_N^T f)(t)$ converges pointwise for all $t \in \mathbb{R}$ and all $f \in \mathcal{PW}_\pi^1$, and,

S4) $(S_N^T f)(t)$ converges uniformly on all compact subsets of $\mathbb{R}$ for all $f \in \mathcal{PW}_\pi^1$

could be related to each other. In general, the analysis of the convolution sum $(S_N^T f)(t)$ is more intricate than the analysis of the convolution integral, because of the periodicity of the Dirichlet kernel

$$\frac{\sin\left[\left(N+\frac{1}{2}\right)(\omega)\right]}{\sin\left(\frac{\omega}{2}\right)}.$$

We do not fully know the relation between S1), S2), S3), and S4). However, we have the following connections.

Theorem 4.29. *Let $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ be a stable LTI system.*

i) S2) implies S1).

ii) S2) implies S3).

iii) S4) implies S1).

Proof. *i)*: Observation 4.3.

ii): Let $t \in \mathbb{R}$ be arbitrary but fixed. Since $S_N^T f$ converges in $\mathcal{S}'$ for all $f \in \mathcal{PW}_\pi^1$ there exists, according to part ii) of Theorem 4.26, for every $\varphi \in \mathcal{S}$ a constant $C_{10}(\varphi)$ such that (4.39) is true for all $N \in \mathbb{N}$. For the specific $\varphi_1 \in \mathcal{S}$ with $\hat{\varphi}_1(\omega) = \mathrm{e}^{-\mathrm{i}\omega t}$ for $\omega \in [-\pi, \pi]$ we obtain

$$\max_{\omega\in[-\pi,\pi]} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{h}_T(\omega_1)\, \mathrm{e}^{\mathrm{i}\omega_1 t} \frac{\sin\left[\left(N+\frac{1}{2}\right)(\omega-\omega_1)\right]}{\sin\left(\frac{\omega-\omega_1}{2}\right)} \,\mathrm{d}\omega_1 \right| \le C_{10}(\varphi_1)$$

for all $N \in \mathbb{N}$. Thus, the assertion follows from part i) of Theorem 4.26.

iii): Let $\varphi \in \mathcal{D}$ be arbitrary but fixed. Since φ is concentrated on some compact set $I \subset \mathbb{R}$, we have

$$S_{N,\varphi}^T f = \int_{-\infty}^{\infty} (S_N^T f)(t)\varphi(t)\,\mathrm{d}t = \int_I (S_N^T f)(t)\varphi(t)\,\mathrm{d}t.$$

It follows that

$$\begin{aligned}
\lim_{N\to\infty} S_{N,\varphi}^T f &= \int_I \lim_{N\to\infty} (S_N^T f)(t)\varphi(t)\,\mathrm{d}t \\
&= \int_I f(t)\varphi(t)\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} f(t)\varphi(t)\,\mathrm{d}t,
\end{aligned}$$

because $(S_N^T f)(t)$ converges uniformly on I by assumption. Since $\varphi \in \mathcal{D}$ was arbitrary, the proof is complete. □

From the results in [13, 118] it can be seen that there exists a stable LTI system T_1 such that $(S_N^{T_1} f_1)(t)$ diverges for some $t \in \mathbb{R}$ and $f_1 \in \mathcal{PW}_\pi^1$. Hence, item ii) of Theorem 4.29 implies that $S_N^{T_1} f_1$ diverges in $\mathcal{S}'$ for some $f_1 \in \mathcal{PW}_\pi^1$. This shows that there are stable LTI systems for which the convolution sum diverges even in $\mathcal{S}'$. However, since we do not know whether S1) implies S3), we cannot immediately conclude the divergence of $S_N^{T_1} f_1$ in $\mathcal{D}'$. Regardless, the following theorem shows that we have this divergence.

Theorem 4.30. *There exists a stable LTI system $T_1 : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ and a signal $f_1 \in \mathcal{PW}_\pi^1$ such that*

$$\limsup_{N\to\infty} |S_{N,\varphi_1}^{T_1} f_1| = \infty$$

for some $\varphi_1 \in \mathcal{D}$.

Proof. Using the characterization that was provided by Corollary 4.27, we have to show that there exists a stable LTI system T_1 and a function $\varphi_1 \in \mathcal{D}$ such that

$$\sup_{N\in\mathbb{N}} \max_{\omega\in[-\pi,\pi]} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{h}_{T_1}(\omega_1)\hat{\varphi}_1(-\omega_1) \frac{\sin\left[\left(N+\frac{1}{2}\right)(\omega-\omega_1)\right]}{\sin\left(\frac{\omega-\omega_1}{2}\right)} \, \mathrm{d}\omega_1 \right| = \infty. \tag{4.44}$$

Let

$$\hat{h}_{T,N}(\omega_1) = \operatorname{sgn}\left(\frac{\sin\left[\left(N+\frac{1}{2}\right)(\omega_1)\right]}{\sin\left(\frac{\omega_1}{2}\right)} \right),$$

and choose some $\varphi_1 \in \mathcal{D}$ such that $\hat{\varphi}_1$ is real valued and $\hat{\varphi}_1(\omega) \geq 1$ for all $\omega \in [-\pi, \pi]$. Next, we analyze the sequence of bounded linear functionals $K_N : L^\infty[-\pi,\pi] \to \mathbb{C}$, $N \in \mathbb{N}$, given by

$$K_N f = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\omega_1)\hat{\varphi}_1(-\omega_1) \frac{\sin\left[\left(N+\frac{1}{2}\right)(\omega_1)\right]}{\sin\left(\frac{\omega_1}{2}\right)} \, \mathrm{d}\omega_1.$$

For $N \in \mathbb{N}$, we have

$$\begin{aligned}
|K_N \hat{h}_{T,N}| &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{\varphi}_1(-\omega_1) \left| \frac{\sin\left[\left(N+\frac{1}{2}\right)(\omega_1)\right]}{\sin\left(\frac{\omega_1}{2}\right)} \right| \, \mathrm{d}\omega_1 \\
&\geq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\sin\left[\left(N+\frac{1}{2}\right)(\omega_1)\right]}{\sin\left(\frac{\omega_1}{2}\right)} \right| \, \mathrm{d}\omega_1 \\
&\geq \frac{2}{\pi^2} \log(1+N),
\end{aligned}$$

where the last inequality follows from the well-known divergence of the L^1-norm of the Dirichlet kernel [78, p. 102]. It follows that

$$\begin{aligned}
\|K_N\| &= \sup_{\|\hat{h}_T\|_{L^\infty[-\pi,\pi]}\leq 1} |K_N \hat{h}_T| \\
&\geq |K_N \hat{h}_{T,N}| \\
&\geq \frac{2}{\pi^2} \log(1+N)
\end{aligned}$$

for all $N \in \mathbb{N}$, and consequently that $\sup_{N\in\mathbb{N}}\|K_N\| = \infty$. Thus, the Banach–Steinhaus theorem [78, p. 98] implies that there exists a $\hat{h}_{T_1} \in L^\infty[-\pi,\pi]$ such that

$$\sup_{N\in\mathbb{N}}\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{h}_{T_1}(\omega_1)\hat{\varphi}_1(-\omega_1)\frac{\sin\left[\left(N+\frac{1}{2}\right)(\omega_1)\right]}{\sin\left(\frac{\omega_1}{2}\right)}\,\mathrm{d}\omega_1\right| = \infty. \tag{4.45}$$

The proof is complete because (4.45) implies (4.44). □

Although we cannot say whether S3) is a necessary condition for S1), we see from Theorem 4.30 that there are stable LTI systems and signals in $\mathcal{PW}_\pi^1$ such that the convolution sum (4.7) diverges in $\mathcal{D}'$. In this regard, we have the same situation as in Section 4.3.2, where we analyzed the convolution integral: The divergence of the convolution sum in the classical, non-distributional setting cannot be circumvented by considering the more relaxed concept of distributional convergence. Therefore, a convolution-type representation of stable LTI systems in the form (4.7) is not possible in general for the space $\mathcal{PW}_\pi^1$, even if the convergence is treated in the distributional sense.

4.3.5 Discussion

In this section we analyzed the convergence behavior of three commonly used time domain convolution-type system representations—two convolution integrals (4.5), (4.6) and one convolution sum (4.7)—for the Paley–Wiener space $\mathcal{PW}_\pi^1$. Although the convolution integrals have a different classical convergence behavior, it turned out that they have the same distributional convergence behavior. Unfortunately, there exist stable LTI systems and signals for which the convolution integrals diverge even in a distributional sense. The same holds for the convolution sum. Hence, the more relaxed concept of distributional convergence cannot circumvent the convergence problems of the convolution integrals and the convolution sum that are encountered in the classical, non-distributional setting. This result is interesting because it shows that a convolution-type time domain representation of stable LTI systems operating on $\mathcal{PW}_\pi^1$ is not always possible, even though such systems always have a frequency domain representation. Further, we completely characterized all stable LTI systems for which a convolution-type system representation is possible.

Although the convergence of the analyzed convolution-type system representations (4.5)–(4.7) is problematic, it is not obvious what other—more complicated—representations exist, which are convergent for all stable LTI systems and all signals in $\mathcal{PW}_\pi^1$. To find such representations, especially for important systems like the Hilbert transform, would be a challenging task for further research.

4.4 Sampling-Type System Representations for $\mathcal{B}_\pi^p$, $1 < p < \infty$

Although for $\mathcal{PW}_\pi^1$, a sampling-type representation does not necessarily exist for all stable LTI systems, because $S_N^T f$ diverges even in a distributional sense for some signal $f \in \mathcal{PW}_\pi^1$ and some stable LTI system $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$, we can find a positive result for signals from the spaces $\mathcal{B}_\pi^p$, $1 < p < \infty$.

The next theorem shows that $S_N^T f$ can be used to approximate Tf for all signals $f \in \mathcal{B}_\pi^p$, $1 < p < \infty$ and all stable LTI systems $T : \mathcal{B}_\pi^p \to \mathcal{B}_\pi^p$.

Theorem 4.31. *Let $1 < p < \infty$. For all $f \in \mathcal{B}_\pi^p$ and all stable LTI systems $T : \mathcal{B}_\pi^p \to \mathcal{B}_\pi^p$ we have*

$$\lim_{N\to\infty} \|S_N^T f - Tf\|_{\mathcal{B}_\pi^p} = 0.$$

Proof. According to (3.14) we have

$$\|S_N f - f\|_{\mathcal{B}_\pi^p} \le C_{\mathrm{R}}(p) \left(\sum_{|k|>N} |f(k)|^p \right)^{1/p}$$

and consequently

$$\lim_{N\to\infty} \|S_N f - f\|_{\mathcal{B}_\pi^p} = 0 \tag{4.46}$$

because

$$\left(\sum_{k=-\infty}^{\infty} |f(k)|^p \right)^{1/p} \le \frac{\|f\|_{\mathcal{B}_\pi^p}}{C_{\mathrm{L}}(p)}.$$

Since T is linear and continuous (4.46) implies that

$$\lim_{N\to\infty} \|S_N^T f - Tf\|_{\mathcal{B}_\pi^p} = 0.$$ □

Remark 4.32. For signals $f \in \mathcal{B}_\pi^p$, $1 < p < \infty$, the convergence of $TS_N f$ is in the $\mathcal{B}_\pi^p$-norm, and, due to $\|f\|_\infty \le C_{12}(p)\|f\|_{\mathcal{B}_\pi^p}$, also globally uniform.

4.5 Non-Equidistant Sampling

In Theorem 4.28 we have seen that for every $t \in \mathbb{R}$ there exists a stable LTI system $T_1 : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ and a signal $f_1 \in \mathcal{PW}_\pi^1$ such that

$$\limsup_{N\to\infty} \left| \sum_{k=-N}^{N} f_1(k) h_{T_1}(t-k) \right| = \infty.$$

Next, we consider more flexible sampling patterns. By using non-equidistant sampling, an additional degree of freedom is created, which may help to improve the convergence behavior. We will analyze whether this additional degree of freedom can be exploited to construct approximation processes that are convergent for all signals in $\mathcal{PW}_\pi^1$ and all stable LTI systems. More precisely, we analyze the convergence behavior of the sampling series

$$\sum_{k=-N}^{N} f(t_k)(T\phi_k)(t), \tag{4.47}$$

for stable LTI systems $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ and signals $f \in \mathcal{PW}_\pi^1$. We consider sampling patterns $\{t_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ that are a complete interpolating sequence for $\mathcal{PW}_\pi^2$. The reconstruction functions ϕ_k, $k \in \mathbb{Z}$, are defined as in (3.61).

Before we treat (4.47) for $\mathcal{PW}_\pi^1$ we review the situation for signals in $\mathcal{PW}_\pi^2$. If the sequence of sampling points $\{t_k\}_{k\in\mathbb{Z}}$ is a complete interpolating sequence for $\mathcal{PW}_\pi^2$, sampling based signal processing is possible for $f \in \mathcal{PW}_\pi^2$, because for all stable LTI systems $T : \mathcal{PW}_\pi^2 \to \mathcal{PW}_\pi^2$ and $f \in \mathcal{PW}_\pi^2$ we have

$$\begin{aligned} \left\| Tf - \sum_{k=-N}^{N} f(t_k) T\phi_k \right\|_{\mathcal{PW}_\pi^2} &= \left\| T\left(f - \left(\sum_{k=-N}^{N} f(t_k)\phi_k \right)\right)\right\|_{\mathcal{PW}_\pi^2} \\ &\leq \|T\| \left\| f - \sum_{k=-N}^{N} f(t_k)\phi_k \right\|_{\mathcal{PW}_\pi^2}, \end{aligned} \tag{4.48}$$

and the right-hand side of (4.48) converges to zero as $N \to \infty$ according to (3.63). Thus, we have

$$\lim_{N\to\infty} \left\| Tf - \sum_{k=-N}^{N} f(t_k) T\phi_k \right\|_{\mathcal{PW}_\pi^2} = 0 \tag{4.49}$$

and, due to $\|f\|_\infty \leq \|f\|_{\mathcal{PW}_\pi^2}$, that

$$\lim_{N\to\infty} \left\| Tf - \sum_{k=-N}^{N} f(t_k) T\phi_k \right\|_{\infty} = 0 \tag{4.50}$$

for all signals $f \in \mathcal{PW}_\pi^2$.

Equations (4.49) and (4.50) show that, for $f \in \mathcal{PW}_\pi^2$, the sampling series with transformed kernel (4.47) converges to the transformed signal Tf in the $\mathcal{PW}_\pi^2$-norm and in the maximum-norm. This means, the transformed signal Tf can be arbitrarily well approximated by the finite sampling series (4.47).

For $\mathcal{PW}_\pi^1$ the situation is different. Theorem 4.33 gives a necessary and sufficient condition for the convergence of (4.47) to the transformed signal Tf for all $f \in \mathcal{PW}_\pi^1$. Later, in Theorem 4.36, we will see that for every sampling pattern $\{t_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ that is a complete interpolating sequence for $\mathcal{PW}_\pi^2$ there exists a stable LTI system $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ and a signal $f \in \mathcal{PW}_\pi^1$ such that (4.47) diverges for a fixed $t \in \mathbb{R}$. Thus, it will turn out that the additional degree of freedom in the choice of the sampling points, compared to equidistant sampling, cannot prevent the approximation process (4.47) to diverge for some stable LTI system $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ and some signal $f \in \mathcal{PW}_\pi^1$.

Theorem 4.33. *Let $\{t_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ be a complete interpolating sequence for $\mathcal{PW}_\pi^2$, ϕ_k, $k \in \mathbb{Z}$, the corresponding reconstruction functions as defined in* (3.61)*, T a stable LTI system, and $\mathcal{I}$ a closed subset of $\mathbb{R}$. For all $f \in \mathcal{PW}_\pi^1$ we have*

$$\lim_{N\to\infty} \max_{t\in\mathcal{I}} \left| (Tf)(t) - \sum_{k=-N}^{N} f(t_k)(T\phi_k)(t) \right| = 0$$

if and only if there exists a constant $C_{13} < \infty$ such that

$$\max_{t\in\mathcal{I}} \max_{|\omega|\le\pi} \left| \sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\omega t_k} (T\phi_k)(t) \right| \le C_{13} \tag{4.51}$$

for all $N \in \mathbb{N}$. If (4.51) *is not fulfilled, then there exists a signal $f_1 \in \mathcal{PW}_\pi^1$ such that*

$$\limsup_{N\to\infty} \max_{t\in\mathcal{I}} \left| (Tf_1)(t) - \sum_{k=-N}^{N} f_1(t_k)(T\phi_k)(t) \right| = \infty. \tag{4.52}$$

Remark 4.34. Note that due to the generality of the set $\mathcal{I}$, Theorem 4.33 comprises several results for different types of convergence. If the set $\mathcal{I}$ contains only a single point then the theorem makes a statement about the pointwise convergence, and if $\mathcal{I} = \mathbb{R}$ then Theorem 4.33 deals with uniform convergence. Uniform convergence on all of $\mathbb{R}$ is important whenever the peak value of the reconstruction has to be controlled over the whole real axis.

For the proof of Theorem 4.33 we need Lemma 4.35, the proof of which is similar to the proof of Lemma 4.8 and therefore omitted.

Lemma 4.35. *Let $\{t_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ be a complete interpolating sequence for $\mathcal{PW}_\pi^2$, ϕ_k, $k \in \mathbb{Z}$, the corresponding reconstruction functions as defined in* (3.61)*, T a stable LTI system, $t \in \mathbb{R}$, and $N \in \mathbb{N}$. Then we have*

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1}\le 1} \left| \sum_{k=-N}^{N} f(t_k)(T\phi_k)(t) \right| = \max_{|\omega|\le\pi} \left| \sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\omega t_k} (T\phi_k)(t) \right|,$$

Proof of Theorem 4.33. The proof consists of two parts. The first part proves the "⇐" direction of the "if and only if" assertion and the second part the second assertion of the theorem. Since the second assertion implies the "⇒" direction of the "if and only if" assertion, the whole theorem is proved.

For the proof we introduce the abbreviation

$$(T_N f)(t) := \sum_{k=-N}^{N} f(t_k)(T\phi_k)(t).$$

First part, "$\Leftarrow$": Let (4.51) be fulfilled, and $f \in \mathcal{PW}_\pi^1$ be arbitrary but fixed. For each $\epsilon > 0$ there exists a $g \in \mathcal{PW}_\pi^2$ such that $\|f - g\|_{\mathcal{PW}_\pi^1} < \epsilon$ and consequently $\max_{t\in\mathcal{I}} |(Tf)(t) - (Tg)(t)| < \epsilon\|T\|$. Furthermore,

$$\begin{aligned}
\max_{t\in\mathcal{I}} |(T_N(g-f))(t)| &= \max_{t\in\mathcal{I}} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\hat{g}(\omega) - \hat{f}(\omega)\right) \sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\omega t_k} (T\phi_k)(t) \,\mathrm{d}\omega \right| \\
&\le \max_{t\in\mathcal{I}} \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{g}(\omega) - \hat{f}(\omega)| \left| \sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\omega t_k} (T\phi_k)(t) \right| \mathrm{d}\omega \\
&\le \max_{t\in\mathcal{I}} \max_{|\omega|\le\pi} \left| \sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\omega t_k} (T\phi_k)(t) \right| \|g - f\|_{\mathcal{PW}_\pi^1} \le C_{13}\epsilon,
\end{aligned}$$

where we used the assumption (4.51) in the last inequality. Moreover, because of (4.50), there exists a $N_0 = N_0(\epsilon)$ such that $\max_{t\in\mathcal{I}} |(Tg)(t) - (T_N g)(t)| < \epsilon$ for all $N \ge N_0$. Since

$$\begin{aligned}
\max_{t\in\mathcal{I}} |(Tf)(t) - (T_N f)(t)| &= \max_{t\in\mathcal{I}} |(Tf)(t) - (Tg)(t) + (Tg)(t) \\
&\quad - (T_N g)(t) + (T_N(g-f))(t)| \\
&\le \max_{t\in\mathcal{I}} |(Tf)(t) - (Tg)(t)| + \max_{t\in\mathcal{I}} |(Tg)(t) - (T_N g)(t)| \\
&\quad + \max_{t\in\mathcal{I}} |(T_N(g-f))(t)|,
\end{aligned}$$

we obtain $\max_{t\in\mathcal{I}} |(Tf)(t) - (T_N f)(t)| < (1 + \|T\| + C_{13})\epsilon$ for all $N \ge N_0$, and the proof of the first part is complete because ϵ was arbitrary.

Second part: If (4.51) is not fulfilled we have

$$\limsup_{N\to\infty} \max_{t\in\mathcal{I}} \max_{|\omega|\le\pi} \left| \sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\omega t_k} (T\phi_k)(t) \right| = \limsup_{N\to\infty} \max_{t\in\mathcal{I}} \sup_{\|f\|_{\mathcal{PW}_\pi^1}\le 1} |(T_N f)(t)| = \infty,$$

because

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1}\le 1} |(T_N f)(t)| = \max_{|\omega|\le\pi} \left| \sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\omega t_k} (T\phi_k)(t) \right|,$$

by Lemma 4.35. Thus, the Banach–Steinhaus theorem [78, p. 98] implies that there exists a signal $f_1 \in \mathcal{PW}_\pi^1$ such that

$$\limsup_{N\to\infty} \max_{t\in\mathcal{I}} |(T_N f_1)(t)| = \infty$$

Since $\max_{t\in\mathcal{I}} |(Tf_1)(t)| \le \|Tf_1\|_\infty < \infty$, we have (4.52) and the proof is complete. □

In Theorem 4.33 we have completely characterized the cases where (4.47) converges to the signal Tf for all $f \in \mathcal{PW}_\pi^1$.

In [14] it was shown that for every sampling pattern $\{t_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ that is a complete interpolating sequence for $\mathcal{PW}_\pi^2$ there really exists a stable LTI system $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ such that (4.47) diverges for some $f \in \mathcal{PW}_\pi^1$. For the proof, which uses a deep result by Szarek [119], we would like to refer to [14].

Theorem 4.36. *Let $\{t_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ be a complete interpolating sequence for $\mathcal{PW}_\pi^2$ and ϕ_k, $k \in \mathbb{Z}$, the corresponding reconstruction functions as defined in* (3.61). *Then, for all $t \in \mathbb{R}$ there exists a stable LTI system T_1 with continuous $\hat{h}_{T_1}$ and a signal $f_1 \in \mathcal{PW}_\pi^1$ such that*

$$\limsup_{N\to\infty} \left| (T_1 f_1)(t) - \sum_{k=-N}^{N} f_1(t_k)(T_1\phi_k)(t) \right| = \infty.$$

Theorem 4.36 shows that, in general, it is not possible to approximate the output of a stable LTI system Tf by the sampling series (4.47), because for some arbitrary given $t \in \mathbb{R}$, we can find a signal $f \in \mathcal{PW}_\pi^1$ and a stable LTI system T such that (4.47) diverges.

4.6 Oversampling

In this section we discuss the system approximation problem for $\mathcal{PW}_\pi^1$ if oversampling is applied. In Section 4.3.3 we have seen that $S_N^T f$ diverges pointwise for some signal $f \in \mathcal{PW}_\pi^1$ and some stable LTI system $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$. Although oversampling can improve the convergence behavior of sampling series, as shown in the case of the Shannon sampling series for $\mathcal{PW}_\pi^1$ in Section 3.2.4, it will turn out that oversampling cannot correct the divergence of $S_N^T f$.

Using oversampling, the approximation process takes the form

$$(S_{N,\phi}^{T,a} f)(t) := \sum_{k=-N}^{N} f\left(\frac{k}{a}\right)(T\phi)\left(t - \frac{k}{a}\right). \tag{4.53}$$

As in Section 3.2.7, $a > 1$ denotes the oversampling factor and $\phi \in \mathcal{M}(a)$ are suitable reconstruction kernels.

The following theorem shows that the convergence behavior of $S_{N,\phi}^{T,a} f$, i.e., the approximation process with oversampling, is not improved compared to the approximation process $S_N^T f$ without oversampling.

Theorem 4.37. *Let $t \in \mathbb{R}$ and $\phi \in \mathcal{M}(a)$, $a > 1$. Then there exists a stable LTI system $T_1 : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ and a signal $f_1 \in \mathcal{PW}_\pi^1$ such that*

$$\limsup_{N\to\infty} |(T_1 f_1)(t) - (S_{N,\phi}^{T_1,a} f_1)(t)| = \infty.$$

For the proof of Theorem 4.37 we need Lemma 4.38.

Lemma 4.38. *For $a > 1$ and $N \in \mathbb{N}$, $N \geq (2a-1)/2$, we have*

$$\int_0^{\pi/a} \frac{\sin^2\left(\left(N+\frac{1}{2}\right)\omega\right)}{\sin\left(\frac{\omega}{2}\right)} \,\mathrm{d}\omega > \log\left(\frac{2N+1}{2a} - 1\right).$$

Proof. Let $L_{N,a}$ be the largest natural number such that $2L_{N,a}\pi/(2N+1) \leq \pi/a$, which implies $L_{N,a} \leq (2N+1)/(2a)$. Then we obtain

$$\begin{aligned}
\int_0^{\pi/a} \frac{\sin^2\left(\left(N+\frac{1}{2}\right)\omega\right)}{\sin\left(\frac{\omega}{2}\right)} \,\mathrm{d}\omega &\geq \sum_{k=0}^{L_{N,a}-1} \int_{\frac{2k\pi}{2N+1}}^{\frac{2(k+1)\pi}{2N+1}} \frac{\sin^2\left(\left(N+\frac{1}{2}\right)\omega\right)}{\sin\left(\frac{\omega}{2}\right)} \,\mathrm{d}\omega \\
&> \sum_{k=0}^{L_{N,a}-1} \frac{1}{\sin\left(\frac{(k+1)\pi}{2N+1}\right)} \int_{\frac{2k\pi}{2N+1}}^{\frac{2(k+1)\pi}{2N+1}} \sin^2\left(\left(N+\frac{1}{2}\right)\omega\right) \mathrm{d}\omega \\
&= \frac{2}{2N+1} \int_0^{\pi} \sin^2\omega \,\mathrm{d}\omega \sum_{k=0}^{L_{N,a}-1} \frac{1}{\sin\left(\frac{(k+1)\pi}{2N+1}\right)} \\
&> \sum_{k=0}^{L_{N,a}-1} \frac{1}{k+1},
\end{aligned}$$

where we used the fact that $\sin x < x$ for all $x > 0$ in the last inequality. But since

$$\sum_{k=0}^{L_{N,a}-1} \frac{1}{k+1} \geq \log(L_{N,a}) > \log\left(\frac{2N+1}{2a} - 1\right),$$

the proof is complete. □

Proof of Theorem 4.37. Due to the translation invariance of T we can assume $t_1 = 0$ without loss of generality. Clearly,

$$(S_{N,\phi}^{T,a} f)(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) \sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\omega k/a}\, (T\phi)\left(-\frac{k}{a}\right) \,\mathrm{d}\omega.$$

Since $T\phi$ has the representation

$$(T\phi)(t) = \frac{1}{2\pi} \int_{-a\pi}^{a\pi} \hat{\phi}(\omega)\hat{h}_T(\omega)\, \mathrm{e}^{\mathrm{i}\omega t} \,\mathrm{d}\omega$$

with some $\hat{h}_T(\omega) \in L^\infty[-a\pi, a\pi]$, we obtain

$$\begin{aligned}\sum_{k=-N}^{N} (T\phi)\left(-\frac{k}{a}\right) =& \frac{1}{2\pi a}\int_{-\pi}^{\pi} \hat{h}_T(\omega) \sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\omega k/a}\,\mathrm{d}\omega \\ &+ \underbrace{\frac{1}{2\pi}\int_{\pi\le|\omega|\le a\pi} \hat{h}_T(\omega)\hat{\phi}(\omega) \sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\omega k/a}\,\mathrm{d}\omega}_{=:R_N} \\ =& \frac{1}{2\pi a}\int_{-\pi}^{\pi} \hat{h}_T(\omega)\frac{\sin\left(\left(N+\frac{1}{2}\right)\frac{\omega}{a}\right)}{\sin\left(\frac{\omega}{2a}\right)}\,\mathrm{d}\omega + R_N,\end{aligned}$$

where we identified $\sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\omega k/a}$ as the Dirichlet kernel. The modulus of R_N can be bounded above independently of N by

$$|R_N| \le C_{14}\frac{\|\hat{\phi}\|_\infty \|\hat{h}_T\|_\infty}{|1-\mathrm{e}^{\mathrm{i}\pi/a}|}.$$

We use the test function

$$\hat{g}_N(\omega) = \sin\left(\left(N+\frac{1}{2}\right)\frac{\omega}{a}\right)\hat{g}(\omega),$$

where $\hat{g}$ is an even, continuous function with $\hat{g}(\omega) = 1$, $0 \le |\omega| \le \pi$ and $\hat{g}(\omega) = 0$, $|\omega| \ge a\pi$. Then, using Lemma 4.38, we obtain

$$\begin{aligned}\frac{1}{2\pi a}\int_{-\pi}^{\pi} \hat{g}_N(\omega)\frac{\sin\left(\left(N+\frac{1}{2}\right)\frac{\omega}{a}\right)}{\sin\left(\frac{\omega}{2a}\right)}\,\mathrm{d}\omega &= \frac{1}{\pi a}\int_0^{\pi} \frac{\sin^2\left(\left(N+\frac{1}{2}\right)\frac{\omega}{a}\right)}{\sin\left(\frac{\omega}{2a}\right)}\,\mathrm{d}\omega \\ &= \frac{1}{\pi}\int_0^{\pi/a} \frac{\sin^2\left(\left(N+\frac{1}{2}\right)\omega\right)}{\sin\left(\frac{\omega}{2}\right)}\,\mathrm{d}\omega \\ &> \frac{1}{\pi}\log\left(\frac{2N+1}{2a}-1\right)\end{aligned}$$

for $N \ge (2a-1)/2$. By the Banach–Steinhaus theorem there exists a function $\hat{h}_{T_1}$ such that

$$\limsup_{N\to\infty}\left|\frac{1}{2\pi a}\int_{-\pi}^{\pi} \hat{h}_{T_1}(\omega)\frac{\sin\left(\left(N+\frac{1}{2}\right)\frac{\omega}{a}\right)}{\sin\left(\frac{\omega}{2a}\right)}\,\mathrm{d}\omega\right| = \infty$$

Since

$$\begin{aligned}\limsup_{N\to\infty}\sup_{\|f\|_{\mathcal{PW}_\pi^1}\le 1}|(S_{N,\phi}^{T_1,a}f)(0)| &\ge \limsup_{N\to\infty}\left|\sum_{k=-N}^{N}(T_1\phi)\left(-\frac{k}{a}\right)\right| \\ &= \infty,\end{aligned}$$

we can again apply the Banach–Steinhaus theorem, which states the existence of a signal $f_1 \in \mathcal{PW}_\pi^1$ such that $\limsup_{N\to\infty}|(S_{N,\phi}^{T_1,a}f_1)(0)| = \infty$. □

Theorem 4.37 reveals the very intricate convergence behavior of the approximation processes. We have shown that there is no universal approximation process of the shape (4.53) that is convergent for all stable LTI systems $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ and all signals $f \in \mathcal{PW}_\pi^1$. It would be useful to have a simple criterion for checking whether an approximation process is convergent for a given operator or not. Next, we will provide a simple test for convergence.

Theorem 4.39. *Let $\phi \in \mathcal{M}(a)$, $a > 1$, $t \in \mathbb{R}$, and $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ be a stable LTI system. Then we have:*

$$\lim_{N\to\infty} (S_{N,\phi}^{T,a} f)(t) = (Tf)(t)$$

for all $f \in \mathcal{PW}_\pi^1$ if and only if there exists a constant $C_{15} = C_{15}(t)$ such that

$$\max_{|\omega|\le\pi} \left| \sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\omega k/a} (T\phi)\left(t - \frac{k}{a}\right) \right| \le C_{15}(t)$$

for all $N \in \mathbb{N}$.

Proof. First part, "⇒": Let $t \in \mathbb{R}$ be arbitrary but fixed. Then, by assumption, $\lim_{N\to\infty} |(Tf)(t) - (S_{N,\phi}^{T,a} f)(t)| = 0$ for all $f \in \mathcal{PW}_\pi^1$, which implies that $\sup_{N\in\mathbb{N}} \sup_{\|f\|_{\mathcal{PW}_\pi^1}\le 1} |(S_{N,\phi}^{T,a} f)(t)| < \infty$. This together with

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1}\le 1} |(S_{N,\phi}^{T,a} f)(t)| = \max_{|\omega|\le\pi} \left| \sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\omega k/a} (T\phi)\left(t - \frac{k}{a}\right) \right|$$

completes the first part.

Second part, "⇐": Let $t \in \mathbb{R}$ be arbitrary but fixed. For each $\epsilon > 0$ there exists a $g \in \mathcal{PW}_\pi^2$ such that $\|f - g\|_{\mathcal{PW}_\pi^1} < \epsilon$ and consequently $|(Tf)(t) - (Tg)(t)| < \epsilon\|T\|$. Obviously, we have

$$\begin{aligned} &|(Tf)(t) - (S_{N,\phi}^{T,a} f)(t)| \\ &\qquad = |(Tf)(t) - (Tg)(t) + (Tg)(t) - (S_{N,\phi}^{T,a} g)(t) + (S_{N,\phi}^{T,a}(g-f))(t)| \\ &\qquad \le |(Tf)(t) - (Tg)(t)| + |(Tg)(t) - (S_{N,\phi}^{T,a} g)(t)| + |(S_{N,\phi}^{T,a}(g-f))(t)|. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} |(S_{N,\phi}^{T,a}(g-f))(t)| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\hat{g}(\omega) - \hat{f}(\omega)\right) \sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\omega k/a} (T\phi)\left(t - \frac{k}{a}\right) \,\mathrm{d}\omega \right| \\ &\le \max_{|\omega|\le\pi} \left| \sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\omega k/a} (T\phi)\left(t - \frac{k}{a}\right) \right| \|g - f\|_{\mathcal{PW}_\pi^1} \\ &\le C_{15}(t)\epsilon, \end{aligned}$$

where we used the assumption in the last inequality. Moreover, there exists a $N_0 = N_0(\epsilon)$ such that $|(Tg)(t) - (S_{N,\phi}^{T,a} g)(t)| < \epsilon$ for all $N \geq N_0$. Therefore, we have

$$|(Tf)(t) - (S_{N,\phi}^{T,a} f)(t)| < (1 + \|T\| + C_{15}(t))\epsilon$$

for all $N \geq N_0$ and the proof is complete because ϵ was arbitrary. □

Similar to Theorem 4.39, where the pointwise convergence was treated, we can give a necessary and sufficient condition for the uniform convergence of the approximation process (4.53).

Theorem 4.40. *Let $\phi \in \mathcal{M}(a)$, $a > 1$, and $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ be a stable LTI system. Then we have:*

$$\lim_{N\to\infty} \|Tf - S_{N,\phi}^{T,a} f\|_\infty = 0$$

for all $f \in \mathcal{PW}_\pi^1$ if and only if there exists a constant C_{16}, independently of t, such that

$$\sup_{t\in\mathbb{R}} \max_{|\omega|\leq\pi} \left| \sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\omega k/a} (T\phi)\left(t - \frac{k}{a}\right) \right| \leq C_{16}$$

for all $N \in \mathbb{N}$ and $t \in \mathbb{R}$.

Proof. Analogously to the proof of Theorem 4.39. □

Example 4.41. As an example, we investigate the convergence behavior of the approximation process $S_{N,\phi}^{T,a} f$ for $T = D$, where D is the differential operator, and $\phi \in \mathcal{M}(a)$, $a > 1$. For $\phi \in \mathcal{M}(a)$, $a > 1$, we have $D\phi \in \mathcal{B}_{a\pi}^1$ and

$$\begin{aligned} \sup_{t\in\mathbb{R}} \max_{|\omega|\leq\pi} \left| \sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\omega k/a} \left(\frac{k}{a}\right) (D\phi)\left(t - \frac{k}{a}\right) \right| &\leq \sup_{t\in\mathbb{R}} \sum_{k=-N}^{N} \left| (D\phi)\left(t - \frac{k}{a}\right) \right| \\ &\leq C_8 \|\phi\|_{\mathcal{B}_{a\pi}^1}, \end{aligned}$$

where we used Lemma 3.25 in the last inequality. Thus, Theorem 4.40 implies that $S_{N,\phi}^{D,a} f$ converges uniformly on $\mathbb{R}$ to Df for all $f \in \mathcal{PW}_\pi^1$ and all $\phi \in \mathcal{M}(a)$, $a > 1$.

5

Stochastic Processes

In addition to the reconstruction of deterministic signals from their samples, the reconstruction of stochastic processes is important because they often appear in the modeling of physical processes. By now many results for the reconstruction of stochastic processes have been presented. In [120] Balakrishnan gave a rigorous proof that the Shannon sampling series converges in the mean square sense for bandlimited wide-sense stationary stochastic processes that have either a spectral density or a spectral distribution which is continuous at the endpoints of the spectrum. The almost sure convergence of the Shannon sampling series with oversampling was proved in [121] for wide-sense stationary processes. Zakai [47] generalized the notion of bandlimited processes and proved almost sure convergence for this new class. Later, [122] and [123] extended the results to hold for a broader classes of non-stationary second order processes. In [124] Brown analyzed the truncation error for bandlimited wide-sense stationary stochastic processes with continuous power spectral density, and in [51] upper bounds for the truncation error were derived for stochastic processes which are bandlimited in the sense of Zakai under the assumption of a guard band. The problem of reconstructing a bandlimited stochastic processes from non-equidistant samples was investigated in [125]. In [21] Habib analyzed sampling representations of bounded linear operators acting on stochastic processes that are bandlimited in the sense of Zakai [47] and Lee [126]. For a general overview of sampling theorems for stochastic processes see for example [38], [68, Chapter 9], and [30].

5.1 Notation

We restrict our analyses to wide-sense stationary processes, i.e., the class of continuous-time, complex valued stochastic processes $X = X(t)$ with zero mean $\mathbb{E}(X(t)) = 0$ and finite second moment $\mathbb{E}(|X(t)|^2) < \infty$ for all $t \in \mathbb{R}$, and whose correlation function $\Gamma(t,t') = \mathbb{E}(X(t)X^*(t'))$, where * denotes the complex conjugate, is only a function of the difference $t - t'$. This enables us to define the correlation function as a function of one variable

$$R(\tau) := \mathbb{E}(X(\tau)X^*(0)),$$

and it can be easily seen that R is nonnegative definite. Furthermore, we assume that X is mean square continuous, which implies that R is continuous. Then the correlation function R has the representation

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega\tau}\ \mathrm{d}\mu(\omega),$$

for a positive and finite measure μ. For details and further facts about wide-sense stationary processes we refer to the standard literature, for example [127] or [128]. We additionally assume that the measure μ is absolutely continuous with respect to the Lebesgue measure λ, which implies that there exists a function $S \in L^1(\mathbb{R})$ such that $\mathrm{d}\mu = S\ \mathrm{d}\lambda$. Furthermore, since μ is positive, it follows that $S(\omega) \geq 0$ almost everywhere (a.e.). S is called the power spectral density. We say the wide-sense stationary process X is bandlimited with bandwidth $\sigma > 0$, if R can be extended to an entire function, and for all $\epsilon > 0$ there exists a constant $C(\epsilon)$ with $|R(z)| \leq C(\epsilon)\exp((\sigma+\epsilon)|z|)$ for all $z \in \mathbb{C}$. It follows that almost all sample functions are entire functions of exponential type at most σ [121].

Definition 5.1. We call a bandlimited wide-sense stationary mean square continuous process an *I-process* if its correlation function R has the representation

$$R(\tau) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S(\omega)\,\mathrm{e}^{\mathrm{i}\omega\tau}\ \mathrm{d}\omega, \tag{5.1}$$

for some $S \in L^1[-\pi,\pi]$. Further, if R has the representation (5.1) then the function S is unique. Note that the fact $S(\omega) \geq 0$ a.e. will be essential for the proofs. “I” stands for integrability.

5.2 Behavior of the Mean Square Error

In this section we analyze the reconstruction of I-processes X from their samples $\{X(k)\}_{k\in\mathbb{Z}}$ using the symmetric Shannon sampling series

$$\sum_{k=-N}^{N} X(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}, \tag{5.2}$$

$N \in \mathbb{N}$, and the non-symmetric Shannon sampling series

$$\sum_{k=-M}^{N} X(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}, \tag{5.3}$$

$M, N \in \mathbb{N}$.

One basic question, which has been studied from the beginning, is how well the I-process can be approximated in the mean square sense by using (5.2) or (5.3), and whether the approximation error converges to zero if more and more samples are used for the approximation, i.e., if N, or M and N, tend to infinity in equations (5.2) and (5.3), respectively. Of course, such a behavior would be desirable and is intuitively expected for the symmetric as well as the non-symmetric sampling series. The early researchers who studied the convergence behavior of the Shannon sampling series for stochastic processes were probably also led by this intuition, and therefore thought that there is no difference in the convergence behavior of the symmetric and the non-symmetric sampling series [65, 124, 129].

As for the symmetric sampling series, it is well known [124] that for all I-processes X and $0 < \tau < \infty$ fixed, we have

$$\lim_{N\to\infty} \max_{t\in[-\tau,\tau]} \mathbb{E} \left| X(t) - \sum_{k=-N}^{N} X(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right|^2 = 0, \tag{5.4}$$

i.e., the variance of the reconstruction error of the symmetric Shannon sampling series is bounded on all compact subsets of $\mathbb{R}$ and converges to zero as $N \to \infty$. It was believed that this result is equally true for the non-symmetric Shannon sampling series.

Disregarding the differences between the symmetric and the non-symmetric Shannon sampling series, Brown claims in the introduction to [124] that

$$\mathbb{E} \left| X(t) - \sum_{k=-M}^{N} X(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right|^2 \tag{5.5}$$

converges to zero for all I-processes X. In Theorem 5.5 we will see that this is not true. However, the theorems in [124] concerning the upper bounds on the mean square approximation error are correct, because in the theorems additional assumptions on the I-processes are made. One assumption is that S is continuous, and the other is that a guard band is present. So, for a restricted class of I-processes we have the mean square convergence of the non-symmetric Shannon sampling series. In Theorem 5.5 we will completely characterize this subclass, i.e., the I-processes for which the "claim" in the introduction to [124] is true, by giving a necessary and sufficient condition for the convergence of (5.5).

5.2.1 Symmetric Shannon Sampling Series

In this section we analyze the convergence behavior of the mean square approximation error of the symmetric sampling series.

Theorem 5.2. *Let X be an I-process. We have*

$$\sup_{N\in\mathbb{N}} \max_{t\in\mathbb{R}} \mathbb{E}\left|X(t) - \sum_{k=-N}^{N} X(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}\right|^2 < \infty$$

if and only if the power spectral density S satisfies

$$\int_{-\pi}^{\pi} S(\omega)\left|\log\left(2\cos\left(\frac{\omega}{2}\right)\right)\right|^2 \,\mathrm{d}\omega < \infty. \tag{5.6}$$

If (5.6) *is not satisfied, then we have*

$$\lim_{N\to\infty} \max_{t\in\mathbb{R}} \mathbb{E}\left|X(t) - \sum_{k=-N}^{N} X(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}\right|^2 = \infty.$$

We have seen in (5.4) that the variance of the reconstruction error converges to zero on all compact subsets of $\mathbb{R}$. However, this is not enough in order to characterize the global approximation behavior of the Shannon sampling series. Although $\mathbb{E}|X(t)|^2$ is constant, there are I-processes such that

$$\max_{t\in\mathbb{R}} \mathbb{E}\left|\sum_{k=-N}^{N} X(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}\right|^2$$

is unbounded as N goes to infinity.

Theorem 5.2 shows that the variance of the reconstruction error is uniformly bounded on all of $\mathbb{R}$ if and only if the power spectral density satisfies (5.6). Consequently, if (5.6) is satisfied, we do not only have perfect local convergence, but also good global convergence behavior.

Remark 5.3. Of course we can neither expect the symmetric nor the non-symmetric sampling series to be globally uniformly convergent in the mean square sense. This is because X is an I-process and hence

$$\mathbb{E}|X(t)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} S(\omega)\,\mathrm{d}\omega = C_1,$$

where C_1 is a constant that is independent of $t \in \mathbb{R}$. Thus, for all $M, N \in \mathbb{N}$ we have

$$\lim_{t\to\infty} \mathbb{E}\left|X(t) - \sum_{k=-M}^{N} X(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}\right|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} S(\omega)\,\mathrm{d}\omega,$$

and consequently

$$\max_{t\in\mathbb{R}} \mathbb{E}\left|X(t)-\sum_{k=-M}^{N} X(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}\right|^2 \geq \frac{1}{2\pi}\int_{-\pi}^{\pi} S(\omega)\,\mathrm{d}\omega.$$

In order to prove the theorems, we need Lemma 5.4, the proof of which is given in Appendix A.9.

Lemma 5.4. *There exists a constant C_2 such that*

$$\left|\sum_{k=M}^{N} \mathrm{e}^{\mathrm{i}\omega k}\,\frac{\sin(\pi(t-k))}{\pi(t-k)}\right| \leq \frac{2}{\pi}\left|\log\left(2\cos\left(\frac{\omega}{2}\right)\right)\right| + C_2$$

for all $t\in\mathbb{R}$, $M,N\in\mathbb{Z}$, $M\leq N$, and $\omega\in[-\pi,\pi]$.

Now we are in the position to prove Theorem 5.2.

Proof of Theorem 5.2. First, we prove the "⇐" part of the "if and only if" assertion. Then we prove the second assertion of the theorem. But the second assertion implies the "⇒" part of the first assertion. Then the entire theorem will be proved. Note that the fact that $S(\omega)\geq 0$ a.e. is essential for the proof.

"⇐": Suppose (5.6) is true. Then we have

$$\begin{aligned}
&\left(\mathbb{E}\left|X(t)-\sum_{k=-N}^{N} X(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}\right|^2\right)^{1/2}\\
&=\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} S(\omega)\left|\mathrm{e}^{\mathrm{i}\omega t}-\sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\omega k}\,\frac{\sin(\pi(t-k)}{\pi(t-k)}\right|^2\,\mathrm{d}\omega\right)^{1/2}\\
&\leq\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} S(\omega)\,\mathrm{d}\omega\right)^{1/2}+\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} S(\omega)\left|\sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\omega k}\,\frac{\sin(\pi(t-k)}{\pi(t-k)}\right|^2\,\mathrm{d}\omega\right)^{1/2}\\
&\leq C_3.
\end{aligned}$$

The last inequality follows by Lemma 5.4, the assumption (5.6), and the fact that $S\in L^1[-\pi,\pi]$. This completes the proof of the "⇐" part of the first assertion.

"⇒": Now suppose (5.6) is not satisfied, i.e.,

$$\int_{-\pi}^{\pi} S(\omega)\left|\log\left(2\cos\left(\frac{\omega}{2}\right)\right)\right|^2\,\mathrm{d}\omega=\infty.$$

Let $N \in \mathbb{N}$ be arbitrary. Since

$$\begin{aligned}
&\left(\mathbb{E}\left|X(t) - \sum_{k=-N}^{N} X(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}\right|^2\right)^{1/2} \\
&= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} S(\omega)\left|\mathrm{e}^{\mathrm{i}\omega t} - \sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\omega k}\,\frac{\sin(\pi(t-k)}{\pi(t-k)}\right|^2\,\mathrm{d}\omega\right)^{1/2} \\
&\geq \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} S(\omega)\left|\sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\omega k}\,\frac{\sin(\pi(t-k)}{\pi(t-k)}\right|^2\,\mathrm{d}\omega\right)^{1/2} - \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} S(\omega)\,\mathrm{d}\omega\right)^{1/2}, \quad (5.7)
\end{aligned}$$

it is sufficient to analyze

$$\left(\int_{-\pi}^{\pi} S(\omega)\left|\sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\omega k}\,\frac{\sin(\pi(t-k)}{\pi(t-k)}\right|^2\,\mathrm{d}\omega\right)^{1/2}.$$

For $t_N = N + \frac{1}{2}$ and $0 < \delta < \pi$ we have

$$\begin{aligned}
\int_{-\pi}^{\pi} S(\omega)\left|\sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\omega k}\,\frac{\sin(\pi(t_N-k))}{\pi(t_N-k)}\right|^2\,\mathrm{d}\omega &= \int_{-\pi}^{\pi} S(\omega)\left|\sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\omega k}\,\frac{(-1)^{N-k}}{\pi(N+\frac{1}{2}-k)}\right|^2\,\mathrm{d}\omega \\
&= \int_{-\pi}^{\pi} S(\omega)\left|\sum_{k=0}^{2N} \mathrm{e}^{\mathrm{i}\omega k}\,\frac{(-1)^k}{\pi(k+\frac{1}{2})}\right|^2\,\mathrm{d}\omega \\
&\geq \int_{-\pi+\delta}^{\pi-\delta} S(\omega)\left|\sum_{k=0}^{2N} \mathrm{e}^{\mathrm{i}\omega k}\,\frac{(-1)^k}{\pi(k+\frac{1}{2})}\right|^2\,\mathrm{d}\omega
\end{aligned}$$

and consequently

$$\begin{aligned}
&\max_{t\in\mathbb{R}} \left(\int_{-\pi}^{\pi} S(\omega)\left|\sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\omega k}\,\frac{\sin(\pi(t-k)}{\pi(t-k)}\right|^2\,\mathrm{d}\omega\right)^{1/2} \\
&\qquad\geq \left(\int_{-\pi}^{\pi} S(\omega)\left|\sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\omega k}\,\frac{\sin(\pi(t_N-k)}{\pi(t_N-k)}\right|^2\,\mathrm{d}\omega\right)^{1/2} \\
&\qquad\geq \left(\int_{-\pi+\delta}^{\pi-\delta} S(\omega)\left|\sum_{k=0}^{2N} \mathrm{e}^{\mathrm{i}\omega k}\,\frac{(-1)^k}{\pi(k+\frac{1}{2})}\right|^2\,\mathrm{d}\omega\right)^{1/2} \\
&\qquad\geq \left(\int_{-\pi+\delta}^{\pi-\delta} S(\omega)\left|\sum_{k=1}^{2N} \mathrm{e}^{\mathrm{i}\omega k}\,\frac{(-1)^k}{\pi k}\right|^2\,\mathrm{d}\omega\right)^{1/2} - C_4,
\end{aligned}$$

because

$$\begin{aligned}\left|\sum_{k=1}^{2N} \mathrm{e}^{\mathrm{i}\omega k}\,\frac{(-1)^k}{\pi k}\right| &\leq \left|\sum_{k=1}^{2N} \frac{\mathrm{e}^{\mathrm{i}\omega k}(-1)^k}{\pi}\left(\frac{1}{k}-\frac{1}{k+\frac{1}{2}}\right)\right| + \left|\sum_{k=1}^{2N} \frac{\mathrm{e}^{\mathrm{i}\omega k}(-1)^k}{\pi(k+\frac{1}{2})}\right| \\ &= \left|\sum_{k=1}^{2N} \frac{\mathrm{e}^{\mathrm{i}\omega k}(-1)^k}{2\pi}\,\frac{1}{k(k+\frac{1}{2})}\right| + \left|\sum_{k=1}^{2N} \frac{\mathrm{e}^{\mathrm{i}\omega k}(-1)^k}{\pi(k+\frac{1}{2})}\right| \\ &\leq C_5 + \left|\sum_{k=0}^{2N} \mathrm{e}^{\mathrm{i}\omega k}\,\frac{(-1)^k}{\pi(k+\frac{1}{2})}\right|.\end{aligned}$$

It follows that

$$\begin{aligned}&\liminf_{N\to\infty}\max_{t\in\mathbb{R}}\left(\int_{-\pi}^{\pi} S(\omega)\left|\sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}\omega k}\,\frac{\sin(\pi(t-k))}{\pi(t-k)}\right|^2 \mathrm{d}\omega\right)^{1/2} \\ &\quad\geq \liminf_{N\to\infty}\left(\int_{-\pi+\delta}^{\pi-\delta} S(\omega)\left|\sum_{k=1}^{2N} \mathrm{e}^{\mathrm{i}\omega k}\,\frac{(-1)^k}{\pi k}\right|^2 \mathrm{d}\omega\right)^{1/2} - C_4 \\ &\quad\geq \frac{1}{\pi}\left(\int_{-\pi+\delta}^{\pi-\delta} S(\omega)\left(\log\left(2\cos\left(\frac{\omega}{2}\right)\right)\right)^2 \mathrm{d}\omega\right)^{1/2} - C_6, \qquad (5.8)\end{aligned}$$

where we used Fatou's Lemma [78, p. 23] and

$$\liminf_{N\to\infty}\left|\sum_{k=1}^{N} \mathrm{e}^{\mathrm{i}\omega k}\,\frac{(-1)^k}{\pi k}\right|^2 \geq \frac{1}{\pi^2}\left(\left|\log\left(2\cos\left(\frac{\omega}{2}\right)\right)\right| - \frac{\pi}{2}\right)^2 \qquad (5.9)$$

for $|\omega| < \pi$. Equation (5.9) follows from

$$\begin{aligned}-\sum_{k=1}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\omega k}}{k} &= \frac{1}{2}\log(2-2\cos(\omega)) + \frac{\mathrm{i}}{2}\,\mathrm{sgn}(\omega)(|\omega|-\pi) \\ &= \log\left(1-\mathrm{e}^{\mathrm{i}\omega}\right)\end{aligned}$$

for $\omega \in [-\pi,\pi]$, $\omega \neq 0$, which is a consequence of 1.441 in [130], the identity

$$\sin\left(\frac{\omega+\pi}{2}\right) = \cos\left(\frac{\omega}{2}\right), \qquad (5.10)$$

and the fact that

$$|\log(1-\mathrm{e}^{\mathrm{i}\omega})| \geq \left|\log\left(2\left|\sin\left(\frac{\omega}{2}\right)\right|\right)\right| - \frac{\pi}{2}$$

for $\omega \in [-\pi, \pi]$, $\omega \neq 0$. Combining (5.7) and (5.8) leads to

$$\liminf_{N\to\infty} \max_{t\in\mathbb{R}} \left(\mathbb{E} \left| X(t) - \sum_{k=-N}^{N} X(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right|^2 \right)^{1/2}$$
$$\geq \left(\frac{1}{2\pi^3} \int_{-\pi+\delta}^{\pi-\delta} S(\omega) \left(\log \left(2\cos\left(\frac{\omega}{2}\right)\right)\right)^2 \mathrm{d}\omega \right)^{1/2} - C_7. \tag{5.11}$$

Since (5.11) is valid for all $0 < \delta < \pi$, the proof is complete. □

5.2.2 Non-Symmetric Shannon Sampling Series

The convergence of the non-symmetric sampling series is treated in the next theorem.

Theorem 5.5. *Let X be an I-process. Then for all $0 < \tau < \infty$ we have*

$$\lim_{N,M\to\infty} \max_{t\in[-\tau,\tau]} \mathbb{E} \left| X(t) - \sum_{k=-M}^{N} X(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right|^2 = 0$$

if and only if the power spectral density S satisfies (5.6)*. If* (5.6) *is not satisfied, then we have*

$$\limsup_{N,M\to\infty} \mathbb{E} \left| X(t) - \sum_{k=-M}^{N} X(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right|^2 = \infty \tag{5.12}$$

for all $t \in \mathbb{R} \setminus \mathbb{Z}$.

Theorem 5.5 gives a necessary and sufficient condition for the local uniform convergence in the mean square sense of the non-symmetric sampling series. That is, the non-symmetric sampling series is locally uniformly convergent in the mean square sense if and only if the condition (5.6) on the power spectral density is satisfied. This highlights the difference between the non-symmetric and the symmetric Shannon sampling series, where we always have—according to (5.4)—local uniform convergence, regardless of the power spectral density S.

The symmetric sampling series is a special case of the non-symmetric sampling series with $M = N$. Thus, some properties of the symmetric sampling series can be inferred from the properties of the non-symmetric sampling series. For example, according to the definition of the convergence of the non-symmetric sampling series, the convergence of the symmetric sampling series follows directly from the convergence of the non-symmetric sampling series. However, the divergence of the non-symmetric sampling series does not imply the divergence of the symmetric sampling series. This is because M and N can tend independently to infinity in the non-symmetric sampling series. In Section 5.3 we will give an example of a power spectral density for which the mean square approximation error of the non-symmetric sampling series diverges for all $t \in \mathbb{R} \setminus \mathbb{Z}$, whereas the mean square approximation error of the symmetric sampling series converges to zero uniformly on all compact subsets of $\mathbb{R}$.

For the proof of Theorem 5.5 we need Lemma 5.6, the proof of which is given in Appendix A.10.

Lemma 5.6. *Given any $\epsilon > 0$, $0 < \delta < \pi$, and $\tau > 0$, there exist two natural numbers $N_0 = N_0(\epsilon, \delta, \tau)$ and $M_0 = M_0(\epsilon, \delta, \tau)$ such that for all $N \geq N_0$, $M \geq M_0$,*

$$\max_{t \in [-\tau, \tau]} \max_{|\omega| \leq \pi - \delta} \left| \mathrm{e}^{\mathrm{i}\omega t} - \sum_{k=-M}^{N} \mathrm{e}^{\mathrm{i}\omega k} \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| < \epsilon.$$

Proof of Theorem 5.5. "$\Leftarrow$": Let $\tau \in \mathbb{R}$ with $0 < \tau < \infty$ be arbitrary but fixed. Since (5.6) is satisfied, for any $\epsilon > 0$ there exists a $0 < \delta_0 < \pi$ such that

$$\frac{1}{2\pi} \int_{\pi - \delta_0 \leq |\omega| \leq \pi} S(\omega) \left| \log \left(2 \cos \left(\frac{\omega}{2} \right) \right) \right|^2 \mathrm{d}\omega < \epsilon^2 \tag{5.13}$$

and

$$\left| \log \left(2 \cos \left(\frac{\omega}{2} \right) \right) \right| > 1$$

for all $|\omega| \in [\pi - \delta_0, \pi]$. Now let δ_0 be fixed. Then for $t \in [-\tau, \tau]$ we have

$$\begin{aligned} &\left(\mathbb{E} \left| X(t) - \sum_{k=-M}^{N} X(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right|^2 \right)^{\frac{1}{2}} \\ &\quad = \left(\frac{1}{2\pi} \int_{|\omega| \leq \pi - \delta_0} S(\omega) \left| \mathrm{e}^{\mathrm{i}\omega t} - \sum_{k=-M}^{N} \mathrm{e}^{\mathrm{i}\omega k} \frac{\sin(\pi(t-k))}{\pi(t-k)} \right|^2 \mathrm{d}\omega \right)^{\frac{1}{2}} \\ &\qquad + \left(\frac{1}{2\pi} \int_{\pi - \delta_0 \leq |\omega| \leq \pi} S(\omega) \left| \mathrm{e}^{\mathrm{i}\omega t} - \sum_{k=-M}^{N} \mathrm{e}^{\mathrm{i}\omega k} \frac{\sin(\pi(t-k))}{\pi(t-k)} \right|^2 \mathrm{d}\omega \right)^{\frac{1}{2}} . \end{aligned} \tag{5.14}$$

The first term on the right-hand side of (5.14) can be upper bounded as follows: By Lemma 5.6 we know that, given any $\epsilon > 0$, there are two constants $N_0 = N_0(\epsilon, \delta, \tau)$ and $M_0 = M_0(\epsilon, \delta, \tau)$ such that for all $N \geq N_0$ and $M \geq M_0$

$$\max_{t \in [-\tau, \tau]} \max_{|\omega| \leq \pi - \delta_0} \left| \mathrm{e}^{\mathrm{i}\omega t} - \sum_{k=-M}^{N} \mathrm{e}^{\mathrm{i}\omega k} \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| < \epsilon.$$

It follows that

$$\left(\frac{1}{2\pi} \int_{|\omega| \leq \pi - \delta_0} S(\omega) \left| \mathrm{e}^{\mathrm{i}\omega t} - \sum_{k=-M}^{N} \mathrm{e}^{\mathrm{i}\omega k} \frac{\sin(\pi(t-k))}{\pi(t-k)} \right|^2 \mathrm{d}\omega \right)^{\frac{1}{2}} \leq C_8 \epsilon.$$

The second term on the right-hand side of (5.14) gives

$$\begin{aligned}
&\left(\frac{1}{2\pi} \int_{\pi-\delta_0 \le |\omega| \le \pi} S(\omega) \left| \mathrm{e}^{\mathrm{i}\omega t} - \sum_{k=-M}^{N} \mathrm{e}^{\mathrm{i}\omega k} \frac{\sin(\pi(t-k))}{\pi(t-k)} \right|^2 \mathrm{d}\omega \right)^{\frac{1}{2}} \\
&\quad \le \left(\frac{1}{2\pi} \int_{\pi-\delta_0 \le |\omega| \le \pi} S(\omega) \, \mathrm{d}\omega \right)^{\frac{1}{2}} \\
&\quad\quad + \left(\frac{1}{2\pi} \int_{\pi-\delta_0 \le |\omega| \le \pi} S(\omega) \left| \sum_{k=-M}^{N} \mathrm{e}^{\mathrm{i}\omega k} \frac{\sin(\pi(t-k))}{\pi(t-k)} \right|^2 \mathrm{d}\omega \right)^{\frac{1}{2}} .
\end{aligned}$$

The first term is smaller than ϵ, and for the second term we obtain

$$\begin{aligned}
&\left(\frac{1}{2\pi} \int_{\pi-\delta_0 \le |\omega| \le \pi} S(\omega) \left| \sum_{k=-M}^{N} \mathrm{e}^{\mathrm{i}\omega k} \frac{\sin(\pi(t-k))}{\pi(t-k)} \right|^2 \mathrm{d}\omega \right)^{\frac{1}{2}} \\
&\quad \le \left(\frac{1}{2\pi} \int_{\pi-\delta_0 \le |\omega| \le \pi} S(\omega) \left(\frac{2}{\pi} \left| \log \left(2 \cos \left(\frac{\omega}{2} \right) \right) \right| + C_2 \right)^2 \mathrm{d}\omega \right)^{\frac{1}{2}} \\
&\quad \le C_9 \epsilon,
\end{aligned}$$

were we used Lemma 5.4 in the second to last inequality and (5.13) in the last inequality. Combining all partial results, we get

$$\mathbb{E} \left| X(t) - \sum_{k=-M}^{N} X(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right|^2 \le (C_8 + 1 + C_9)^2 \epsilon^2$$

for all $N \ge N_0$, $M \ge M_0$, and $t \in [-\tau, \tau]$. Since ϵ was arbitrary, this part of the proof is complete.

"$\Rightarrow$": Analogously to the proof of Theorem 5.2 it is shown that if (5.6) is not satisfied then

$$\lim_{N\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} S(\omega) \left| \mathrm{e}^{\mathrm{i}\omega t} - \sum_{k=1}^{N} \mathrm{e}^{\mathrm{i}\omega k} \frac{\sin(\pi(t-k))}{\pi(t-k)} \right|^2 \mathrm{d}\omega = \infty$$

for all $t \in \mathbb{R} \setminus \mathbb{Z}$. This is equivalent to

$$\lim_{N\to\infty} \mathbb{E} \left| X(t) - \sum_{k=1}^{N} X(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right|^2 = \infty \tag{5.15}$$

for all $t \in \mathbb{R} \setminus \mathbb{Z}$, and (5.15) implies the assertion (5.12). $\square$

5.3 Discussion

As we have seen, is important to know when (5.6) is satisfied and when it is not. There are several special cases where (5.6) is true. One important case is when S is continuous and another is the case when $S \in L^p[-\pi,\pi]$, $1 < p \le \infty$. According to Hölder's inequality we have for $1 < p \le \infty$, $1/p + 1/q = 1$,

$$\begin{aligned} &\int_{-\pi}^{\pi} S(\omega) \left| \log\left(2\cos\left(\frac{\omega}{2}\right)\right)\right|^2 \, \mathrm{d}\omega \\ &\quad \le \left(\int_{-\pi}^{\pi} (S(\omega))^p \, \mathrm{d}\omega\right)^{1/p} \left(\int_{-\pi}^{\pi} \left| \log\left(2\cos\left(\frac{\omega}{2}\right)\right)\right|^{2q} \, \mathrm{d}\omega\right)^{1/q}. \end{aligned} \tag{5.16}$$

In both of the cases—S is continuous or $S \in L^p[-\pi,\pi]$—the first term on the right-hand side of (5.16) is finite. It remains to show that the second term on the right-hand side of (5.16) is finite. For $r \ge 1$ we have

$$\begin{aligned} &\int_{-\pi}^{\pi} \left| \log\left(2\cos\left(\frac{\omega}{2}\right)\right)\right|^r \, \mathrm{d}\omega \\ &\quad = \int_{-\frac{3\pi}{4}}^{\frac{3\pi}{4}} \left| \log\left(2\cos\left(\frac{\omega}{2}\right)\right)\right|^r \, \mathrm{d}\omega + 2\int_{\frac{3\pi}{4}}^{\pi} \left[-\log\left(2\cos\left(\frac{\omega}{2}\right)\right)\right]^r \, \mathrm{d}\omega. \end{aligned} \tag{5.17}$$

Since $\left|\log\left(2\cos\left(\omega/2\right)\right)\right|^r$ is continuous on $[-3\pi/4, 3\pi/4]$, the first summand on the right-hand side of (5.17) is finite. Furthermore, since $\cos(\omega/2) \ge 1 - \omega/\pi$ for all $\omega \in [3\pi/4, \pi]$, it follows that

$$\begin{aligned} 2\int_{\frac{3\pi}{4}}^{\pi} \left[-\log\left(2\cos\left(\frac{\omega}{2}\right)\right)\right]^r \, \mathrm{d}\omega &\le 2\int_{\frac{3\pi}{4}}^{\pi} \left[-\log\left(2 - \frac{2\omega}{\pi}\right)\right]^r \, \mathrm{d}\omega \\ &= \int_{\log(2)}^{\infty} u^r \, \mathrm{e}^{-u} \, \mathrm{d}u < \infty, \end{aligned} \tag{5.18}$$

where we used the substitution $u = -\log(2 - 2\omega/\pi)$ in the second to last line. So we have (5.6) if S is continuous or if $S \in L^p[-\pi,\pi]$, $1 < p \le \infty$.

On the other hand there are power spectral densities $S \in L^1[-\pi,\pi]$ for which (5.6) is not satisfied. One example is given by

$$S_1(\omega) = \frac{1}{(\omega+\pi)\left(\log\left(\frac{4\pi}{\omega+\pi}\right)\right)^2}.$$

The short calculation

$$\begin{aligned} \int_{-\pi}^{\pi} |S_1(\omega)| \, \mathrm{d}\omega &= \int_0^{2\pi} \frac{1}{\omega\left(\log\left(\frac{4\pi}{\omega}\right)\right)^2} \, \mathrm{d}\omega \\ &= \int_{\log(2)}^{\infty} \frac{1}{u^2} \, \mathrm{d}u \\ &= \frac{1}{\log(2)} < \infty \end{aligned}$$

shows that $S_1 \in L^1[-\pi,\pi]$. Next, we show that (5.6) is not satisfied for S_1. We have

$$\begin{aligned}&\int_{-\pi}^{\pi} S_1(\omega) \left|\log\left(2\cos\left(\frac{\omega}{2}\right)\right)\right|^2 \,\mathrm{d}\omega \\ &\quad = \int_{-\pi}^{-\frac{3\pi}{4}} \frac{\left|\log\left(2\cos\left(\frac{\omega}{2}\right)\right)\right|^2}{(\omega+\pi)\left(\log\left(\frac{4\pi}{\omega+\pi}\right)\right)^2} \,\mathrm{d}\omega + \int_{-\frac{3\pi}{4}}^{\pi} \frac{\left|\log\left(2\cos\left(\frac{\omega}{2}\right)\right)\right|^2}{(\omega+\pi)\left(\log\left(\frac{4\pi}{\omega+\pi}\right)\right)^2} \,\mathrm{d}\omega.\end{aligned}$$

The second integral is finite because

$$\int_{-\frac{3\pi}{4}}^{\pi} \frac{\left|\log\left(2\cos\left(\frac{\omega}{2}\right)\right)\right|^2}{(\omega+\pi)\left(\log\left(\frac{4\pi}{\omega+\pi}\right)\right)^2} \,\mathrm{d}\omega \le \int_{-\frac{3\pi}{4}}^{\pi} \frac{\left|\log\left(2\cos\left(\frac{\omega}{2}\right)\right)\right|^2}{\frac{\pi}{4}\left(\log(2)\right)^2} \,\mathrm{d}\omega < \infty,$$

according to (5.18). Furthermore, since

$$\begin{aligned}\int_{-\pi}^{-\frac{3\pi}{4}} \frac{\left|\log\left(2\cos\left(\frac{\omega}{2}\right)\right)\right|^2}{(\omega+\pi)\left(\log\left(\frac{4\pi}{\omega+\pi}\right)\right)^2} \,\mathrm{d}\omega &\ge \int_{-\pi}^{-\frac{3\pi}{4}} \frac{\left|\log\left(\omega+\pi\right)\right|^2}{(\omega+\pi)\left(\log\left(\frac{4\pi}{\omega+\pi}\right)\right)^2} \,\mathrm{d}\omega \\ &= \int_{\log(16)}^{\infty} \frac{(u-\log(4\pi))^2}{u^2} \,\mathrm{d}\omega \\ &\ge \int_{\log(16)}^{\infty} \frac{u-2\log(4\pi)}{u} \,\mathrm{d}\omega \\ &\ge [\log(16)-2\log(4\pi)] \int_{\log(16)}^{\infty} \frac{1}{u} \,\mathrm{d}\omega \\ &= \infty,\end{aligned}$$

where we used the inequality $\cos(\omega/2) \le (\omega+\pi)/2$, $\omega \in [-\pi, -3\pi/4]$, and the substitution $u = \log(4\pi/(\omega+\pi))$, we obtain that

$$\int_{-\pi}^{\pi} S_1(\omega) \left|\log\left(2\cos\left(\frac{\omega}{2}\right)\right)\right|^2 \,\mathrm{d}\omega = \infty.$$

Further discussion about the differences of the symmetric and the non-symmetric sampling series, together with a review of the historical development of this problem in the literature can be found in [17].

6

Impact of Thresholding and Quantization

The principle of digital signal processing relies on the fact that certain bandlimited signals can be perfectly reconstructed from their samples. This is only true if the sample values are known exactly. However, in real applications this can never be realized. One reason is that the quantization process in analog to digital conversion only has limited resolution and thus the samples of the digital signal are always disturbed by a quantization error in practice [131, 132]. In addition to this quantization error, there are often other reasons, which lead to disturbed samples, and thus render a perfect signal reconstruction impossible.

In this chapter, we consider two non-linear distortions of the samples and analyze their effect on the reconstruction behavior of the sampling series. The first is the threshold operator and the second is the quantization operator.

The threshold operator that we consider here differs from the often analyzed clipping operator [100, 133, 134]. We do our analysis for the threshold operator that sets all sample values whose absolute value is smaller than some threshold to zero, because few results are available for it, and because it has several interesting applications.

One application where the threshold operator is important is sensor networks. It is most natural to employ sampling theory in the performance analysis of sensor networks [135, 136], because the measurements of the sensors are nothing but a spatio-temporal sampling of the signal. The sensors sample some bandlimited signal in space and time and transmit the samples to the receiving signal processing unit. Then, using these samples, the receiver tries to reconstruct the signal, perfectly or at least approximately if a perfect reconstruction is not possible. In order to save energy, the sensors transmit only if the absolute value of the signal exceeds some threshold $\delta > 0$. Thus, the receiver has to reconstruct the signal by using only the samples whose absolute value is larger than or equal to the threshold δ.

The second distortion that we consider is the quantization operator. Due to its high practical importance, the analysis of the quantization error has gained a lot of attention in research. Often, the quantization operation is modeled as additive white noise [137, 138]. However, it turned out that this noise model is not always satisfactory because it can lead to false predictions [139, 140].

Since quantization is a deterministic process it is interesting to have a deterministic analysis in addition to the statistical approaches. The deterministic analysis is difficult because of the non-linear nature of the quantization operator, but it reveals some properties of the quantization process, which cannot be analyzed with the additive noise description of the quantization error.

There are numerous publications discussing the approximation error of sampling series in the presence of additive noise in the samples [70, 138]. However these publications do not consider the deterministic nature of the quantization. Only few publications treat the quantization error deterministically. One is [141], where the quantization error is analyzed for absolutely integrable bandlimited signals and certain non-bandlimited signals. Another paper is [142]. There the accuracy of analog to digital converters with oversampling is analyzed for bandlimited signals with finite energy. In [62, 79] the interpolation problem is analyzed for non-uniform quantized samples and a subset of the bandlimited signals with finite energy. Moreover, oversampled analog to digital conversion in shift-invariant spaces is treated in [143]. [132] discusses the effect of quantization threshold uncertainties in pulse code modulation and $\Sigma\Delta$ modulation analog to digital converters. An extensive account of the history of quantization and the discussion of several developments can be found in [131].

Those of the above publications which treat the quantization deterministically all concentrate on bandlimited signals with finite energy, or subspaces thereof. We do the analysis for the spaces $\mathcal{B}_\pi^p$, $1 < p < \infty$, and $\mathcal{PW}_\pi^p$, $1 \leq p \leq \infty$, i.e., in particular we also treat the signal spaces $\mathcal{PW}_\pi^1$ and $\mathcal{B}_\pi^p$, $2 < p < \infty$, which are larger than $\mathcal{PW}_\pi^2$.

6.1 Threshold and Quantization Operator

The first distortion is the threshold operator. For complex numbers $z \in \mathbb{C}$, the threshold operator κ_δ, $\delta > 0$, is defined by

$$\kappa_\delta z = \begin{cases} z, & |z| \geq \delta, \\ 0, & |z| < \delta. \end{cases}$$

The thresholding characteristic for real numbers is depicted in Fig. 6.1. Furthermore, for continuous signals $f : \mathbb{R} \to \mathbb{C}$, we define the threshold operator Θ_δ pointwise, i.e., $(\Theta_\delta f)(t) = \kappa_\delta f(t)$, $t \in \mathbb{R}$. The threshold operator κ_δ is applied on the samples $\{f(k)\}_{k\in\mathbb{Z}}$ of bandlimited signals f, which gives the disturbed samples $\{\kappa_\delta f(k)\}_{k\in\mathbb{Z}}$.

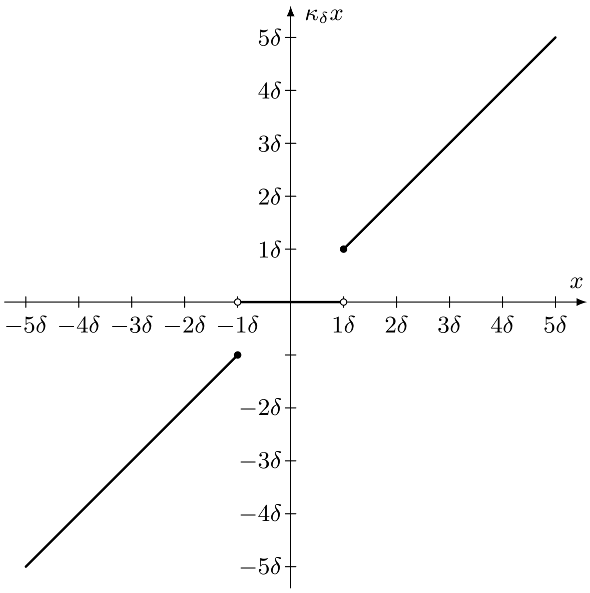

Figure 6.1: Thresholding characteristic.

This is, of course, equivalent to applying the threshold operator Θ_δ on the signal f itself and then taking the samples, i.e., $\{(\Theta_\delta f)(k)\}_{k\in\mathbb{Z}}$.

The resulting samples $\{(\Theta_\delta f)(k)\}_{k\in\mathbb{Z}}$ are used to build an approximation

$$(A_\delta f)(t) := \sum_{k=-\infty}^{\infty} (\Theta_\delta f)(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} = \sum_{\substack{k=-\infty \\ |f(k)|\geq\delta}}^{\infty} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \tag{6.1}$$

of the original signal f. By A_δ we denote the operator that maps f to $A_\delta f$ according to (6.1).

In general, $A_\delta f$ is only an approximation of f, and we want the reconstructed signal $A_\delta f$ to be close to f if δ is sufficiently small. Since the series in (6.1) uses all "important" samples of the signal, i.e., all samples that are larger than or equal than δ, one could expect $A_\delta f$ to be a good approximation for f, at least if δ is small. However, we will see that this is true only for certain signal spaces.

The second non-linear operator that we consider in this thesis is the simple but frequently used uniform mid-tread quantization, where each complex number $z \in \mathbb{C}$ is quantized to $q_\delta z$, depending on the quantization step size $2\delta > 0$, according to the rule

$$q_\delta z = \left\lfloor \frac{\operatorname{Re} z}{2\delta} + \frac{1}{2} \right\rfloor 2\delta + \left\lfloor \frac{\operatorname{Im} z}{2\delta} + \frac{1}{2} \right\rfloor 2\delta \mathrm{i}, \tag{6.2}$$

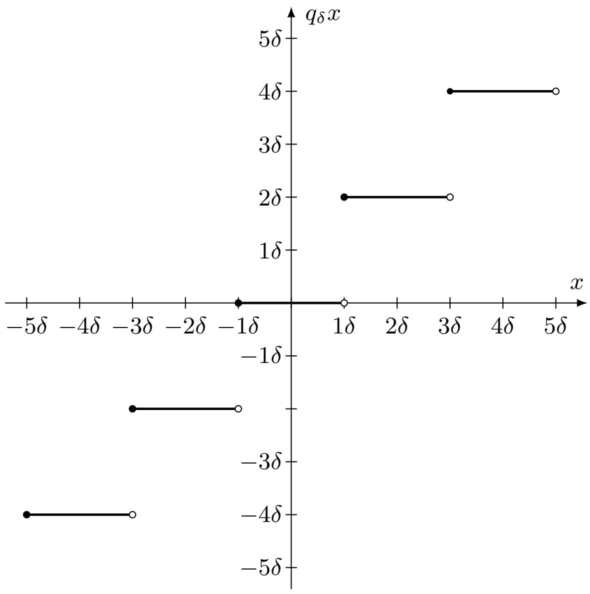

Figure 6.2: Quantization characteristic.

where $\lfloor x \rfloor$ denotes the largest integer smaller than or equal than x. As can be seen in (6.2), the quantization is done separately for the real and the imaginary part of z. The quantization characteristic for real numbers is depicted in Fig. 6.2. Furthermore, for continuous signals $f : \mathbb{R} \to \mathbb{C}$, we define the quantization operator Υ_δ pointwise, i.e., $(\Upsilon_\delta f)(t) = q_\delta f(t)$, $t \in \mathbb{R}$. For example, if the sample $f(k)$ is a real number and $f(k) \in [(2l-1)\delta, (2l+1)\delta)$ for some $l \in \mathbb{Z}$ then $(\Upsilon_\delta f)(k) = 2l\delta$.

As in the case of the threshold operator, the resulting samples $\{(\Upsilon_\delta f)(k)\}_{k\in\mathbb{Z}}$ are used to build an approximation

$$(B_\delta f)(t) := \sum_{k=-\infty}^{\infty} (\Upsilon_\delta f)(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \tag{6.3}$$

of the original signal f.

The convergence of the series in (6.1) and (6.3) for signals $f \in \mathcal{B}_\pi^p$, $1 \le p < \infty$, and signals $f \in \mathcal{PW}_\pi^p$, $1 \le p \le \infty$, is unproblematic. All these signals have the property that $\lim_{|t|\to\infty} f(t) = 0$, i.e., for every signal in these spaces and every $\delta > 0$ there exists a $t_0 = t_0(\delta)$ such that $|f(t)| < \delta$ for all $|t| \ge t_0$. As a consequence, we have $(\Theta_\delta f)(k) = 0$ and $(\Upsilon_\delta f)(k) = 0$ for all $|k| \ge t_0$. Hence the series in (6.1) and (6.3) have only finitely many summands, which implies that $A_\delta f \in \mathcal{PW}_\pi^2$ and $B_\delta f \in \mathcal{PW}_\pi^2$. In general, $A_\delta f$ and $B_\delta f$ are only approximations of f, and we want the approximation to be close to f if δ is sufficiently small.

The effect that $A_\delta f$ and $B_\delta f$ have only finitely many samples can be interpreted as a truncation of the Shannon sampling series

$$\sum_{k=-\infty}^{\infty} f(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}.$$

This truncation is controlled in the codomain of the signal because only the samples $f(k)$, $k \in \mathbb{Z}$, whose absolute value is larger than or equal to some threshold $\delta > 0$ are taken into account. As δ tends to zero, more and more samples are used for the approximation. Normally, the Shannon sampling series is truncated in the domain of the signal by considering only the samples $f(k)$, $k = -N, \ldots, N$. For this kind of truncation and signals $f \in \mathcal{PW}_\pi^1$ we have according to Brown's Theorem (Theorem 3.6) that

$$(S_N f)(t) = \sum_{k=-N}^{N} f(k)\frac{\sin(\pi(t-k))}{\pi(t-k)} \tag{6.4}$$

converges uniformly on compact subsets of $\mathbb{R}$ as N goes to infinity. It follows that for all $\tau > 0$ there exists a constant C_1 such that

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1} \le 1} \sup_{N \in \mathbb{N}} |(S_N f)(t)| \le C_1$$

for all $t \in [-\tau, \tau]$.

In contrast, we will see in Corollary 6.23 that, for $f \in \mathcal{PW}_\pi^1$, $A_\delta f$ and $B_\delta f$ behave completely different compared to (6.4). This shows that for signals in $\mathcal{PW}_\pi^1$ there is a significant difference between the truncation of the Shannon sampling series controlled in the codomain and the truncation controlled in the domain.

The approximation processes (6.1) and (6.3) are difficult to analyze because the threshold operator Θ_δ and the quantization operator Υ_δ are both non-linear. As a consequence, A_δ and B_δ are non-linear operators. Further properties of A_δ and B_δ are as follows. We only state and prove them for A_δ, nevertheless, they are equally true for B_δ.

1. For every $\delta > 0$, A_δ is a non-linear operator.
2. For every $\delta > 0$, the operator $A_\delta : (\mathcal{PW}_\pi^1, \|\cdot\|_{\mathcal{PW}_\pi^1}) \to (\mathcal{PW}_\pi^1, \|\cdot\|_\infty)$ is discontinuous, i.e., there exist a signal $f \in \mathcal{PW}_\pi^1$ and a constant $C_2 > 0$ such that for every $\epsilon > 0$ there exists a signal $g_\epsilon \in \mathcal{PW}_\pi^1$ satisfying $\|f - g_\epsilon\|_{\mathcal{PW}_\pi^1} < \epsilon$ and $\|A_\delta f - A_\delta g_\epsilon\|_\infty \ge C_2$.
3. Property 2 implies that $A_\delta : (\mathcal{PW}_\pi^1, \|\cdot\|_{\mathcal{PW}_\pi^1}) \to (\mathcal{PW}_\pi^1, \|\cdot\|_{\mathcal{PW}_\pi^1})$ is discontinuous for every $\delta > 0$.
4. For some $f \in \mathcal{PW}_\pi^1$, the operator A_δ is also discontinuous with respect to δ, i.e., there exist a signal $f \in \mathcal{PW}_\pi^1$ and a $t \in \mathbb{R}$ such that

$$\lim_{h \to 0} (A_{\delta+h} f)(t) \ne (A_\delta f)(t).$$

Proof. The proof is straightforward, but included for completeness. Proof of Property 1: For $f(t) = \delta \sin(\pi t)/(\pi t) \in \mathcal{PW}_\pi^1$ we have

$$0 = \left(A_\delta \frac{f}{2}\right)(t) \neq \frac{1}{2}(A_\delta f)(t) = \frac{1}{2}\frac{\delta \sin(\pi t)}{\pi t}.$$

Proof of Property 2: Let $f(t) = \delta \sin(\pi t)/(\pi t) \in \mathcal{PW}_\pi^1$ and $\epsilon > 0$ arbitrary but fixed. Then, for

$$g_\epsilon(t) = \begin{cases} (1 - \frac{\epsilon}{2\delta})f(t), & \epsilon < 1, \\ (1 - \frac{1}{2\delta})f(t), & \epsilon \geq 1, \end{cases}$$

we have $\|f - g_\epsilon\|_{\mathcal{PW}_\pi^1} \leq \frac{\epsilon}{2\delta}\|f\|_{\mathcal{PW}_\pi^1} < \epsilon$ and $\|A_\delta f - A_\delta g_\epsilon\|_\infty = \|A_\delta f\|_\infty = \|f\|_\infty = \delta$. Proof of Property 4: Take $f(t) = \delta \sin(\pi t)/(\pi t) \in \mathcal{PW}_\pi^1$. Then, for all $h > 0$, we have $(A_{\delta+h} f)(t) \equiv 0$, but $(A_\delta f)(t) = \delta \sin(\pi t)/(\pi t)$. □

6.2 Signal Approximation under Thresholding and Quantization

According to the previous discussion, for fixed $\delta > 0$ and $f \in \mathcal{B}_\pi^p$, $1 \leq p < \infty$, or $f \in \mathcal{PW}_\pi^p$, $1 \leq p \leq \infty$, we always have $A_\delta f \in \mathcal{B}_\pi^\infty$ and $B_\delta f \in \mathcal{B}_\pi^\infty$ and consequently $\|A_\delta f\|_\infty < \infty$ as well as $\|B_\delta f\|_\infty < \infty$. However, it is not clear whether, for fixed $\delta > 0$, $\|A_\delta f\|_\infty$ and $\|B_\delta f\|_\infty$ are bounded on all bounded sets of signals. This would be a necessary precondition for the practical application of the approximation processes A_δ and B_δ, because otherwise the peak value of $A_\delta f$ and $B_\delta f$ could increase arbitrarily, even though the norm of f is bounded by some fixed constant.

The following theorem shows that the norms $\|A_\delta f\|_{\mathcal{B}_\pi^p}$ and $\|B_\delta f\|_{\mathcal{B}_\pi^p}$ are uniformly bounded on all bounded sets of signals $f \in \mathcal{B}_\pi^p$, $1 < p < \infty$.

Theorem 6.1. *Let $1 < p < \infty$. For all $f \in \mathcal{B}_\pi^p$ and all $\delta > 0$ we have*

$$\|A_\delta f\|_{\mathcal{B}_\pi^p} \leq C_3(p)\|f\|_{\mathcal{B}_\pi^p} \tag{6.5}$$

and

$$\|B_\delta f\|_{\mathcal{B}_\pi^p} \leq C_4(p)\|f\|_{\mathcal{B}_\pi^p}, \tag{6.6}$$

where $C_3(p)$ and $C_4(p)$ are constants that depend only on p.

Proof. Let $f \in \mathcal{B}_\pi^p$, $1 < p < \infty$, and $\delta > 0$ be arbitrary but fixed.

First, we prove (6.5). According to (3.14) we have

$$\begin{aligned}
\|A_\delta f\|_{\mathcal{B}_\pi^p} &\le C_{\mathrm{R}}(p) \left(\sum_{k=-\infty}^{\infty} |(\Theta_\delta f)(k)|^p \right)^{1/p} \\
&= C_{\mathrm{R}}(p) \left(\sum_{|f(k)| \ge \delta} |f(k)|^p \right)^{1/p} \\
&\le C_{\mathrm{R}}(p) \left(\sum_{k=-\infty}^{\infty} |f(k)|^p \right)^{1/p} \\
&\le \frac{C_{\mathrm{R}}(p)}{C_{\mathrm{L}}(p)} \|f\|_{\mathcal{B}_\pi^p},
\end{aligned}$$

which completes the proof of (6.5).

Next, we prove (6.6). By the definition of the quantization operation, we have

$$|(\Upsilon_\delta f)(k)| \le 2|f(k)| \tag{6.7}$$

for all $k \in \mathbb{Z}$, and it follows that

$$\begin{aligned}
\|B_\delta f\|_{\mathcal{B}_\pi^p} &\le C_{\mathrm{R}}(p) \left(\sum_{k=-\infty}^{\infty} |(\Upsilon_\delta f)(k)|^p \right)^{1/p} \\
&\le 2C_{\mathrm{R}}(p) \left(\sum_{k=-\infty}^{\infty} |f(k)|^p \right)^{1/p} \\
&\le \frac{2C_{\mathrm{R}}(p)}{C_{\mathrm{L}}(p)} \|f\|_{\mathcal{B}_\pi^p},
\end{aligned}$$

where we used (3.14) again. □

Remark 6.2. The Plancherel–Pólya theorem, which we used twice in the proof, is only true for $1 < p < \infty$. Therefore, the spaces $\mathcal{B}_\pi^p$ with $p = 1$ and $p = \infty$ cannot be treated with the above proof technique. In fact, it is true that Theorem 6.1 does not hold for $\mathcal{B}_\pi^1$ and $\mathcal{B}_\pi^\infty$.

An immediate consequence of Theorem 6.1 is the following corollary, which shows that, for $1 < p < \infty$, the peak value of $A_\delta f$ and $B_\delta f$ is bounded on the set $\{f \in \mathcal{B}_\pi^p : \|f\|_{\mathcal{B}_\pi^p} \le 1\}$. This behavior cannot be taken for granted. We will see in Theorem 6.9 that we do not have this nice behavior for the signal space $\mathcal{PW}_\pi^1$.

Corollary 6.3. *Let* $1 < p < \infty$*. For all* $\delta > 0$ *we have*

$$\sup_{\|f\|_{\mathcal{B}_\pi^p} \le 1} \|A_\delta f\|_\infty < \infty$$

and

$$\sup_{\|f\|_{\mathcal{B}_\pi^p} \le 1} \|B_\delta f\|_\infty < \infty.$$

Due to the boundedness of the operators $A_\delta : \mathcal{B}_\pi^p \to \mathcal{B}_\pi^p$ and $B_\delta : \mathcal{B}_\pi^p \to \mathcal{B}_\pi^p$ for $1 < p < \infty$ and all $\delta > 0$, it is—in principle—possible to approximate $f \in \mathcal{B}_\pi^p$, $1 < p < \infty$, by $A_\delta f$ and $B_\delta f$.

Further, the following theorem shows that for fixed $f \in \mathcal{B}_\pi^p$, $1 < p < \infty$, $A_\delta f$ and $B_\delta f$ have a good behavior with respect to δ. The approximation error tends to zero as δ goes to zero. This is in accordance with the common intuition that a decreased quantization step size and a decreased threshold improves the approximation accuracy. Theorem 6.4 is important because we will use it in Section 6.3.1 to prove Theorem 6.16.

Theorem 6.4. *Let* $1 < p < \infty$*. For all* $f \in \mathcal{B}_\pi^p$ *we have*

$$\lim_{\delta \to 0} \|f - A_\delta f\|_{\mathcal{B}_\pi^p} = 0 \tag{6.8}$$

and

$$\lim_{\delta \to 0} \|f - B_\delta f\|_{\mathcal{B}_\pi^p} = 0. \tag{6.9}$$

Proof. Let $f \in \mathcal{B}_\pi^p$, $1 < p < \infty$ be arbitrary but fixed.

First, we prove (6.8). For $\delta > 0$ we have

$$\|f - A_\delta f\|_{\mathcal{B}_\pi^p} \le C_{\mathrm{R}}(p) \left(\sum_{k=-\infty}^{\infty} \underbrace{|f(k) - (\Theta_\delta f)(k)|}_{=u_\delta(k)}{}^p \right)^{1/p}, \tag{6.10}$$

according to (3.14). Moreover, since $|u_\delta(k)| \le |f(k)|$, for all $k \in \mathbb{Z}$ and all $\delta > 0$, we see that $u_\delta(k)$ is dominated by $|f(k)|$, and since

$$\sum_{k=-\infty}^{\infty} |f(k)|^p \le \left(\frac{\|f\|_{\mathcal{B}_\pi^p}}{C_{\mathrm{L}}(p)} \right)^p < \infty, \tag{6.11}$$

where we used (3.14) again, it follows that $\{|f(k)|\}_{k \in \mathbb{Z}}$ is in l^p. Using Lebesgue's Dominated Convergence Theorem [144, p. 463] and the fact that $\lim_{\delta \to 0} u_\delta(k) = 0$ for all $k \in \mathbb{Z}$ gives

$$\lim_{\delta \to 0} \left(\sum_{k=-\infty}^{\infty} |u_\delta(k)|^p \right)^{1/p} = \left(\sum_{k=-\infty}^{\infty} |\lim_{\delta \to 0} u_\delta(k)|^p \right)^{1/p} = 0.$$

Therefore, we obtain

$$\lim_{\delta\to 0}\|f - A_\delta f\|_{\mathcal{B}_\pi^p} \le C_{\mathrm{R}}(p) \lim_{\delta\to 0}\left(\sum_{k=-\infty}^{\infty} |u_\delta(k)|^p\right)^{1/p} = 0,$$

which completes the proof of (6.8).

The proof of (6.9) is analogously. For $\delta > 0$ we have

$$\|f - B_\delta f\|_{\mathcal{B}_\pi^p} \le C_{\mathrm{R}}(p)\left(\sum_{k=-\infty}^{\infty} |\underbrace{f(k) - (\Upsilon_\delta f)(k)}_{=v_\delta(k)}|^p\right)^{1/p}. \tag{6.12}$$

Since

$$\begin{aligned}|v_\delta(k)| &= |f(k) - (\Upsilon_\delta f)(k)|\\ &\le |f(k)| + |(\Upsilon_\delta f)(k)|\\ &\le 3|f(k)|,\end{aligned}$$

for all $k \in \mathbb{Z}$ and all $\delta > 0$, we see that $v_\delta(k)$ is dominated by $3|f(k)|$. Clearly, $\{|f(k)|\}_{k\in\mathbb{Z}}$ is in l^p by (6.11), and we have $\lim_{\delta\to 0} v_\delta(k) = 0$ for all $k \in \mathbb{Z}$. Thus, application of Lebesgue's Dominated Convergence Theorem gives

$$\lim_{\delta\to 0}\left(\sum_{k=-\infty}^{\infty} |v_\delta(k)|^p\right)^{1/p} = \left(\sum_{k=-\infty}^{\infty} |\lim_{\delta\to 0} v_\delta(k)|^p\right)^{1/p} = 0,$$

which, together with (6.12), leads to (6.9). □

Remark 6.5. For the proof it was important that we used (3.14) twice. Thanks to the inequality on the right-hand side of (3.14) we were able to derive (6.10) and (6.12), which transfer the difficult continuous-time approximation problem into an easier to solve discrete-time problem that involves only the samples of the signal. The left-hand side of (3.14) was used in (6.11) to show that $\{|f(k)|\}_{k\in\mathbb{Z}}$ is in l^p.

Up to now we have discussed the approximation behavior of $A_\delta f$ and $B_\delta f$ for the spaces $\mathcal{B}_\pi^p$, $1 < p < \infty$. Next, we will analyze their behavior for the Paley–Wiener spaces $\mathcal{PW}_\pi^p$, $1 < p \le \infty$. It will turn out that similar results to the above are possible for these spaces.

Theorem 6.6. *Let $1 < p \le \infty$. For all $f \in \mathcal{PW}_\pi^p$ and all $\delta > 0$ we have*

$$\|A_\delta f\|_\infty \le C_5(p)\|f\|_{\mathcal{PW}_\pi^p}$$

and

$$\|B_\delta f\|_\infty \le C_6(p)\|f\|_{\mathcal{PW}_\pi^p},$$

where $C_5(p)$ and $C_6(p)$ are constants that depend only on p.

Proof. We only need to prove the case $1 < p \le 2$, because the statement for $2 < p \le \infty$ follows from the result for $p = 2$ and the facts that $\mathcal{PW}_\pi^p \subset \mathcal{PW}_\pi^2$ for all $2 < p \le \infty$ and $\|f\|_{\mathcal{PW}_\pi^2} \le \|f\|_{\mathcal{PW}_\pi^p}$ for all $f \in \mathcal{PW}_\pi^p$ and all $2 < p \le \infty$.

Let $\delta > 0$ and $1 < p \le 2$ be arbitrary but fixed. Since $f \in \mathcal{PW}_\pi^p$, it follows from the Hausdorff–Young inequality [26, p. 19] that $f \in \mathcal{B}_\pi^q$ and

$$\|f\|_{\mathcal{B}_\pi^q} \le C_7(q)\|f\|_{\mathcal{PW}_\pi^p}, \tag{6.13}$$

where $1/p + 1/q = 1$. According to Theorem 6.1, which is applicable because $f \in \mathcal{B}_\pi^q$, we have

$$\begin{aligned} \|A_\delta f\|_{\mathcal{B}_\pi^q} &\le C_3(p)\|f\|_{\mathcal{B}_\pi^q} \\ &\le C_3(p)C_7(q)\|f\|_{\mathcal{PW}_\pi^p} \end{aligned} \tag{6.14}$$

and

$$\begin{aligned} \|B_\delta f\|_{\mathcal{B}_\pi^q} &\le C_4(p)\|f\|_{\mathcal{B}_\pi^q} \\ &\le C_4(p)C_7(q)\|f\|_{\mathcal{PW}_\pi^p}, \end{aligned} \tag{6.15}$$

where we used (6.13) in the last inequality of (6.14) and (6.15). Since

$$\|f\|_\infty \le C_8(p)\|f\|_{\mathcal{B}_\pi^q} \tag{6.16}$$

for all $f \in \mathcal{B}_\pi^q$, the proof is complete. □

An immediate consequence of Theorem 6.6 is the following corollary.

Corollary 6.7. *Let* $1 < p \le \infty$. *For all* $\delta > 0$ *we have*

$$\sup_{\|f\|_{\mathcal{PW}_\pi^p} \le 1} \|A_\delta f\|_\infty < \infty$$

and

$$\sup_{\|f\|_{\mathcal{PW}_\pi^p} \le 1} \|B_\delta f\|_\infty < \infty.$$

Exactly as for the spaces $\mathcal{B}_\pi^p$, $1 < p < \infty$, we have the pleasant result that, for all signals in $\mathcal{PW}_\pi^p$, $1 < p \le \infty$, the approximation error tends to zero as δ goes to zero.

Theorem 6.8. *Let* $1 < p \le \infty$. *For all* $f \in \mathcal{PW}_\pi^p$ *we have*

$$\lim_{\delta \to 0} \|f - A_\delta f\|_\infty = 0$$

and

$$\lim_{\delta \to 0} \|f - B_\delta f\|_\infty = 0.$$

Proof. We only need to prove the case $1 < p \leq 2$, because the statement for $2 < p \leq \infty$ follows from the result for $p = 2$ and the fact that $\mathcal{PW}_\pi^p \subset \mathcal{PW}_\pi^2$ for all $2 < p \leq \infty$.

Let $1 < p \leq 2$ be arbitrary but fixed. Since $f \in \mathcal{PW}_\pi^p$, it follows from the Hausdorff–Young inequality [26, p. 19] that $f \in \mathcal{B}_\pi^q$, $1/p + 1/q = 1$, and Theorem 6.4 implies that

$$\lim_{\delta \to 0} \|f - A_\delta f\|_{\mathcal{B}_\pi^q} = 0$$

as well as

$$\lim_{\delta \to 0} \|f - B_\delta f\|_{\mathcal{B}_\pi^q} = 0.$$

Due to (6.16), the proof is complete. □

6.2.1 Signal Approximation for $\mathcal{PW}_\pi^1$

We have seen that both $A_\delta f$ and $B_\delta f$ can be used to approximate f if f belongs to one of the Bernstein spaces $\mathcal{B}_\pi^p$, $1 < p < \infty$, or one of the Paley–Wiener spaces $\mathcal{PW}_\pi^p$, $1 < p \leq \infty$. In contrast, for $\mathcal{PW}_\pi^1$, $A_\delta : \mathcal{PW}_\pi^1 \to \mathcal{B}_\pi^\infty$ and $B_\delta : \mathcal{PW}_\pi^1 \to \mathcal{B}_\pi^\infty$ are unbounded operators for all $0 < \delta < 1/3$ as the next theorem shows, and thus cannot be used to approximate signals $f \in \mathcal{PW}_\pi^1$.

Theorem 6.9. *For all* $0 < \delta < 1/3$ *we have*

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1} \leq 1} \|A_\delta f\|_\infty = \infty \tag{6.17}$$

and

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1} \leq 1} \|B_\delta f\|_\infty = \infty. \tag{6.18}$$

We do not prove Theorem 6.9 here because it is a simple consequence of a more general result, stated in Theorem 6.17 and Corollary 6.23 in Section 6.3.2.

Remark 6.10. Although $A_\delta f \in \mathcal{PW}_\pi^2$ and $B_\delta f \in \mathcal{PW}_\pi^2$, for all $f \in \mathcal{PW}_\pi^1$, we have (6.17) and (6.18), which is no contradiction, because the maps $f \mapsto A_\delta f$ and $f \mapsto B_\delta f$ are non-linear. Moreover, since $\|f\|_\infty \leq \|f\|_{\mathcal{PW}_\pi^1} \leq 1$, the peak value of f in Theorem 6.9 is bounded. Nevertheless, the peak value of $A_\delta f$ and $B_\delta f$ can grow arbitrarily large.

Remark 6.11. Theorem 6.9 gives the nice additional result that Theorem 6.1 cannot be true for $f \in \mathcal{B}_\pi^\infty$. Since $\mathcal{PW}_\pi^1 \subset \mathcal{B}_\pi^\infty$, it follows that

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1} \leq 1} \|A_\delta f\|_\infty \leq \sup_{\|f\|_{\mathcal{B}_\pi^\infty} \leq 1} \|A_\delta f\|_\infty,$$

and Theorem 6.9 implies that $\sup_{\|f\|_{\mathcal{B}_\pi^\infty} \leq 1} \|A_\delta f\|_\infty = \infty$ for all $0 < \delta < 1/3$.

A direct consequence of Theorem 6.9 is the following corollary, which shows that the peak approximation error can grow arbitrarily large on the set of signals $f \in \mathcal{PW}_\pi^1$ with $\|f\|_{\mathcal{PW}_\pi^1} \leq 1$, regardless of how small the threshold δ is.

Corollary 6.12. *For all* $0 < \delta < 1/\pi$ *we have*

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1} \leq 1} \|f - A_\delta f\|_\infty = \infty$$

and

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1} \leq 1} \|f - B_\delta f\|_\infty = \infty.$$

Theorem 6.9 and Corollary 6.12 demonstrate that A_δ and B_δ cannot be used to approximate signals $f \in \mathcal{PW}_\pi^1$. For any given error level, we can find a signal with norm $\|f\|_{\mathcal{PW}_\pi^1} \leq 1$ such that the peak approximation error exceeds this level.

For signals in $\mathcal{PW}_\pi^1$, we have analyzed, so far, the behavior of A_δ and B_δ for fixed δ. The next theorem describes the behavior of $A_\delta f$ and $B_\delta f$ as the threshold δ tends to zero.

Theorem 6.13. *There exists a signal* $f_1 \in \mathcal{PW}_\pi^1$ *such that*

$$\limsup_{\delta \to 0} \|A_\delta f_1\|_\infty = \infty.$$

and

$$\limsup_{\delta \to 0} \|B_\delta f_1\|_\infty = \infty.$$

Hence a reduction of the threshold δ leads to an unbounded increase of the peak reconstruction error for some signals in $\mathcal{PW}_\pi^1$. This behavior is counterintuitive because one would suspect that the reconstruction behavior of $A_\delta f$ and $B_\delta f$ gets better as the threshold δ is reduced.

Remark 6.14. The divergence in Theorem 6.13 is with respect to the supremum norm. However, for $A_\delta f$ it is also possible to strengthen the result to pointwise divergence for every $t \in \mathbb{R} \setminus \mathbb{Z}$ as the threshold δ tends to zero. It can be shown that $(A_\delta f)(t)$, $t \in \mathbb{R} \setminus \mathbb{Z}$, diverges as $\delta \to 0$ for some signal in $f \in \mathcal{PW}_\pi^1$ [22]. The divergence of $(A_\delta f)(t)$ between the integers is remarkable because the approximation behavior on the integer grid is best possible. For all $t \in \mathbb{Z}$, $f \in \mathcal{PW}_\pi^1$, and $\delta > 0$ we have $|f(t) - (A_\delta f)(t)| < \delta$.

Proof of Theorem 6.13. We give the proof for the quantization operator, i.e., for B_δ. The proof for A_δ can be done analogously.

In order to construct the signal f_1 we use the functions

$$h_N(t) = \sum_{k=-\infty}^{\infty} h_N(k) \frac{\sin(\pi(t-k))}{\pi(t-k)},$$

where $h_N(k) = (-1)^k g_N(k)$ and

$$g_N(k) = \begin{cases} 1, & |k| \le N, \\ 2\left(1 - \frac{|k|}{2N}\right), & N < |k| < 2N, \\ 0, & |k| \ge 2N. \end{cases}$$

as basic building blocks. Next, a sequence $\{n_l\}_{l\in\mathbb{N}}$ of natural numbers is inductively constructed. Let $n_1 = 1$ and $N(k) = 2^{(k^3)}$, $k \in \mathbb{N}$. Furthermore, let n_{l+1} be the smallest natural number that is larger than n_l and fulfills

$$\frac{1}{n_l^2}\frac{N(n_l)-1}{N(n_l)} + \frac{1}{n_{l+1}-1} < \frac{1}{n_l^2}.$$

We define the signal

$$f_1(t) = \sum_{l=1}^{\infty} \frac{1}{n_l^2} h_{N(n_l)}(t).$$

First, note that $f \in \mathcal{PW}_\pi^1$, because

$$\|f_1\|_{\mathcal{PW}_\pi^1} \le \sum_{l=1}^{\infty} \frac{1}{n_l^2}\|h_{N(l)}\|_{\mathcal{PW}_\pi^1} \le 3\sum_{l=1}^{\infty}\frac{1}{n_l^2} < \infty.$$

Next, $f_1(k)$, $k \in \mathbb{Z}$, is analyzed. Let $r \in \mathbb{N}$ be arbitrary and $|k| \ge N(n_r)+1$. For $m < r$ we have $n_r \ge n_m + 1$ and consequently $n_r^3 \ge n_m^3 + 1$. It follows that $N(n_r) = 2^{(n_r^3)} \ge 2^{(n_m^3+1)} = 2 \cdot 2^{(n_m^3)} = 2N(n_m)$, which implies that $|k| \ge 2N(n_m)$. Thus, for $m < r$, $h_{N(n_m)}(k) = 0$ and

$$\begin{aligned} |f_1(k)| &= \left| \sum_{l=r}^{\infty} \frac{1}{n_l^2} h_{N(n_l)}(k) \right| \\ &= \left| (-1)^k \sum_{l=r}^{\infty} \frac{1}{n_l^2} g_{N(n_l)}(k) \right| \\ &= \frac{1}{n_r^2} g_{N(n_r)}(k) + \sum_{l=r+1}^{\infty} \frac{1}{n_l^2} g_{N(n_l)}(k) \\ &\le \frac{2}{n_r^2}\left(1 - \frac{N(n_r)+1}{2N(n_r)}\right) + \sum_{l=r+1}^{\infty}\frac{1}{n_l^2} \\ &\le \frac{1}{n_r^2}\frac{N(n_r)-1}{N(n_r)} + \sum_{l=n_{r+1}}^{\infty}\frac{1}{l^2} \\ &< \frac{1}{n_r^2}\frac{N(n_r)-1}{N(n_r)} + \frac{1}{n_{r+1}-1}. \end{aligned}$$

For δ_r with

$$\frac{1}{n_r^2}\frac{N(n_r)-1}{N(n_r)} + \frac{1}{n_{r+1}-1} < \delta_r < \frac{1}{n_r^2}$$

we have $(\Upsilon_{\delta_r} f_1)(k) = 0$ for $|k| \geq N(n_r) + 1$. For $|k| \leq N(n_r)$ and k even, we have

$$f_1(k) = \sum_{l=1}^{\infty} \frac{1}{n_l^2} g_{N(n_l)}(k) \geq \frac{1}{n_r^2} g_{N(n_r)}(k) = \frac{1}{n_r^2}$$

and

$$(\Upsilon_{\delta_r} f_1)(k) \geq 2\delta_r \geq \frac{1}{n_r^2},$$

because $\delta_r \geq 1/(2n_r^2)$. Similarly, it can be shown that for $|k| \leq N(n_r)$ and k odd

$$(\Upsilon_{\delta_r} f_1)(k) \leq -\frac{1}{n_r^2}$$

holds. This implies that

$$(B_{\delta_r} f_1)(t) = \sum_{k=-N(n_r)}^{N(n_r)} (\Upsilon_{\delta_r} f_1)(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}$$

and

$$\begin{aligned}
\left|(B_{\delta_r} f_1)(N(n_r) + \tfrac{1}{2})\right| &= \left| \sum_{k=-N(n_r)}^{N(n_r)} (\Upsilon_{\delta_r} f_1)(k) \frac{(-1)^k}{\pi(N(n_r) + \frac{1}{2} - k)} \right| \\
&\geq \frac{1}{\pi n_r^2} \sum_{k=-N(n_r)}^{N(n_r)} \frac{1}{N(n_r) + \frac{1}{2} - k} \\
&\geq \frac{1}{\pi n_r^2} \log(N(n_r)) \\
&= \frac{n_r}{\pi} \log(2).
\end{aligned}$$

Therefore, we have

$$\|B_{\delta_r} f_1\|_\infty \geq n_r \log(2)/\pi$$

and

$$\limsup_{\delta \to 0} \|B_\delta f_1\|_\infty \geq \limsup_{r \to \infty} \|B_{\delta_r} f_1\|_\infty = \infty. \qquad \square$$

6.2.2 Oversampling

We have seen that the threshold operator leads to a bad reconstruction behavior of the Shannon sampling series for $\mathcal{PW}_\pi^1$ if the samples are taken at Nyquist rate. It is well known that oversampling with oversampling factor $a > 1$ can help to resolve convergence problems, because better reconstruction kernels than the sinc-kernel can be used [42, 43, 61, 76]. Indeed, as shown in Theorem 3.17, for $f \in \mathcal{PW}_\pi^1$, the sampling series with oversampling and suitable kernel converges uniformly on the whole real axis to the signal f. As we will see in Theorem 6.15, oversampling can also improve the approximation behavior in our problem, where the samples are disturbed by the non-linear threshold operator.

However, often interest is not restricted to just reconstruction of signals, but includes the approximation of outputs of stable linear time-invariant systems from the samples of the input signals. This problem in connection with the threshold operator is studied in Section 6.3. In Theorem 6.29 we will prove that oversampling does not resolve the convergence problems in this case.

Next, we will analyze what happens in the case of oversampling, i.e., we consider $f \in \mathcal{PW}_\pi^1$ and $a > 1$. Using kernels $\phi \in \mathcal{M}(a)$, the reconstruction process with thresholding has the form

$$(A_{\delta,\phi}^a f)(t) := \sum_{k=-\infty}^{\infty} (\Theta_\delta f)\left(\frac{k}{a}\right) \phi\left(t - \frac{k}{a}\right).$$

Again, as in the case without oversampling, the series has only finitely many samples, which implies $A_{\delta,\phi}^a f \in \mathcal{PW}_{a\pi}^2 \subset \mathcal{PW}_{a\pi}^1$.

Now, we can show that $A_{\delta,\phi}^a f$ exhibits a good approximation behavior for $f \in \mathcal{PW}_\pi^1$.

Theorem 6.15. *For all $a > 1$ and $\phi \in \mathcal{M}(a)$ we have*

$$\lim_{\delta \to 0} \sup_{f \in \mathcal{PW}_\pi^1} \|f - A_{\delta,\phi}^a f\|_\infty = 0.$$

Proof. For all $t \in \mathbb{R}$ and all $f \in \mathcal{PW}_\pi^1$ we have

$$\begin{aligned}
|f(t) - (A_{\delta,\phi}^a f)(t)| &= \left| f(t) - \sum_{k=-\infty}^{\infty} (\Theta_\delta f)\left(\frac{k}{a}\right) \phi\left(t - \frac{k}{a}\right) \right| \\
&= \left| \sum_{\substack{k=-\infty \\ |f(\frac{k}{a})| < \delta}}^{\infty} f\left(\frac{k}{a}\right) \phi\left(t - \frac{k}{a}\right) \right| \\
&\leq \delta \sum_{k=-\infty}^{\infty} \left| \phi\left(t - \frac{k}{a}\right) \right| \\
&\leq \delta C_8 \|\phi\|_{\mathcal{B}_{a\pi}^1}.
\end{aligned} \tag{6.19}$$

Since the right-hand side of (6.19) neither depends on t nor on f, we obtain

$$\sup_{f \in \mathcal{PW}_\pi^1} \|f - A_{\delta,\phi}^a f\|_\infty \leq \delta C_8 \|\phi\|_{\mathcal{B}_{a\pi}^1}, \tag{6.20}$$

and the assertion follows immediately after taking the limit on both sides of (6.20). □

Theorem 6.15 shows that oversampling leads to a reconstruction process that is uniformly convergent on all of $\mathbb{R}$ for all signals in $\mathcal{PW}_\pi^1$, even if the samples are disturbed by the threshold operator. This is in contrast to the situation without oversampling where we have $\sup_{f\in\mathcal{PW}_\pi^1}\|f - A_\delta f\|_\infty = \infty$ for all $0 < \delta < 1/3$, according to Theorem 6.9.

Thus, with oversampling we can use $A^a_{\delta,\phi} f$ to approximate signals $f \in \mathcal{PW}_\pi^1$. Equation (6.20) gives an upper bound on the peak approximation error. This bound can be made arbitrarily small by decreasing the threshold δ. The constant C_8 does not depend on f, it only depends on the kernel ϕ and the oversampling factor a. For practical applications where a certain ϕ is given, it is possible to compute this constant and consequently the upper bound in (6.20).

6.3 System Approximation under Thresholding and Quantization

In Chapter 4 we already motivated the system approximation problem. If the samples $\{f(k)\}_{k\in\mathbb{Z}}$ are known perfectly we can use

$$\sum_{k=-N}^{N} f(k)\, T(\operatorname{sinc}(\,\cdot\, - k))(t) = \sum_{k=-N}^{N} f(k) h_T(t-k) \tag{6.21}$$

to obtain an approximation of Tf. The conditions under which (6.21) converges to Tf for $f \in \mathcal{PW}_\pi^1$ as N goes to infinity were analyzed in Theorem 4.26. In this section we analyze the signal approximation problem, where the samples are disturbed either by the non-linear threshold operator or by the non-linear quantization operator. More concretely, we want to approximate Tf either by

$$(A_\delta^T f)(t) := (TA_\delta f)(t) = \sum_{k=-\infty}^{\infty} (\Theta_\delta f)(k) h_T(t-k) \tag{6.22}$$

or by

$$(B_\delta^T f)(t) := (TB_\delta f)(t) = \sum_{k=-\infty}^{\infty} (\Upsilon_\delta f)(k) h_T(t-k). \tag{6.23}$$

In the following we will use the abbreviations $A_\delta^T := TA_\delta$ and $B_\delta^T := TB_\delta$.

The analysis of the approximation processes (6.22) and (6.23) is difficult, because the operators A_δ^T and B_δ^T have several properties, which complicate its treatment. Even though we list these properties only for A_δ^T, they are equally true for B_δ^T.

1. For every $\delta > 0$, $A_\delta^T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^2$ is a non-linear operator.
2. For every $\delta > 0$, the operator $A_\delta^T : \mathcal{PW}_\pi^2 \to \mathcal{PW}_\pi^2$ is discontinuous, i.e., there exist a signal $f \in \mathcal{PW}_\pi^2$ and a constant C_9 such that for every $\epsilon > 0$ there exists a signal $g_\epsilon \in \mathcal{PW}_\pi^2$ satisfying $\|f - g_\epsilon\|_{\mathcal{PW}_\pi^2} < \epsilon$ and $\|A_\delta^T f - A_\delta^T g_\epsilon\|_{\mathcal{PW}_\pi^2} \geq C_9$.

3. Property 2 implies that $A_\delta^T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^2$ is discontinuous for every $\delta > 0$.

4. For some $f \in \mathcal{PW}_\pi^1$, the operator A_δ^T is also discontinuous with respect to δ, i.e., there exist a signal $f \in \mathcal{PW}_\pi^1$ and a $t \in \mathbb{R}$ such that

$$\lim_{h \to 0} (A_{\delta+h}^T f)(t) \neq (A_\delta^T f)(t).$$

The proofs are very similar to the proofs of the analogous statements for the operators A_δ and B_δ in Section 6.2 on page 139, and thus are omitted here.

6.3.1 System Approximation for $\mathcal{B}_\pi^p$, $1 < p < \infty$

In this section we show that it is possible to approximate Tf by $A_\delta^T f$ and $B_\delta^T f$ for all $f \in \mathcal{B}_\pi^p$, $1 < p < \infty$.

Theorem 6.16. *Let $1 < p < \infty$. For all $f \in \mathcal{B}_\pi^p$ and all stable LTI systems $T : \mathcal{B}_\pi^p \to \mathcal{B}_\pi^p$ we have*

$$\lim_{\delta \to 0} \|Tf - A_\delta^T f\|_{\mathcal{B}_\pi^p} = 0$$

and

$$\lim_{\delta \to 0} \|Tf - B_\delta^T f\|_{\mathcal{B}_\pi^p} = 0.$$

Proof. From Theorem 6.4 we know that $\lim_{\delta \to 0} \|f - A_\delta f\|_{\mathcal{B}_\pi^p} = 0$ and $\lim_{\delta \to 0} \|f - B_\delta f\|_{\mathcal{B}_\pi^p} = 0$. Since $T : \mathcal{B}_\pi^p \to \mathcal{B}_\pi^p$ is a continuous operator, it follows that $\lim_{\delta \to 0} \|Tf - A_\delta^T f\|_{\mathcal{B}_\pi^p} = 0$ and $\lim_{\delta \to 0} \|Tf - B_\delta^T f\|_{\mathcal{B}_\pi^p} = 0$. □

According to Theorem 6.16 it is possible to use the series (6.22) and (6.23) to approximate Tf for all $f \in \mathcal{B}_\pi^p$, $1 < p < \infty$, and all stable LTI systems $T : \mathcal{B}_\pi^p \to \mathcal{B}_\pi^p$. Here, the intuition that a decreased quantization step size and a decreased threshold improves the approximation accuracy is true for $f \in \mathcal{B}_\pi^p$, $1 < p < \infty$. However, in Section 6.3.2 we will see that, for $f \in \mathcal{PW}_\pi^1$, the behavior is completely different.

6.3.2 System Approximation for $\mathcal{PW}_\pi^1$

In this section we analyze A_δ^T and B_δ^T for signals in $\mathcal{PW}_\pi^1$. Since it would be cumbersome to write every sentence for A_δ^T and B_δ^T we subsequently write A_δ^T in the text. Nevertheless, every information in this section that is true for A_δ^T is equally true for B_δ^T.

Again, we are interested in knowing whether $\sup_{\|f\|_{\mathcal{PW}_\pi^1} \le 1} |(A_\delta^T f)(t)| < \infty$. The following theorem gives a necessary and sufficient condition for this expression to be finite. In Theorem 6.20 we will see that the same condition is sufficient for a good approximation behavior of $(A_\delta^T f)(t)$.

Theorem 6.17. *Let $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ be a stable LTI system, $0 < \delta < 1/3$, and $t \in \mathbb{R}$. Then we have*

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1} \le 1} |(A_\delta^T f)(t)| < \infty$$

and

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1} \le 1} |(B_\delta^T f)(t)| < \infty$$

if and only if

$$\sum_{k=-\infty}^{\infty} |h_T(t-k)| < \infty. \tag{6.24}$$

We defer the proof of Theorem 6.17 and present a corollary and a theorem, which further discuss the approximation behavior of A_δ^T and B_δ^T.

Remark 6.18. Note that (6.24) is nothing else than the BIBO stability condition for discrete-time systems.

Theorem 6.17 implies that the pointwise approximation error $|(Tf)(t) - (A_\delta^T f)(t)|$ cannot be bounded on $\{f \in \mathcal{PW}_\pi^1 : \|f\|_{\mathcal{PW}_\pi^1} \le 1\}$ if the stable LTI system T does not fulfill (6.24).

Corollary 6.19. *Let $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ be a stable LTI system, $0 < \delta < 1/3$, and $t \in \mathbb{R}$. If* (6.24) *is not fulfilled then we have*

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1} \le 1} |(Tf)(t) - (A_\delta^T f)(t)| = \infty$$

and

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1} \le 1} |(Tf)(t) - (B_\delta^T f)(t)| = \infty.$$

Proof. We have

$$\begin{aligned}\sup_{\|f\|_{\mathcal{PW}_\pi^1} \le 1} |(Tf)(t) - (A_\delta^T f)(t)| &\ge \sup_{\|f\|_{\mathcal{PW}_\pi^1} \le 1} (|(A_\delta^T f)(t)| - \|T\| \, \|f\|_{\mathcal{PW}_\pi^1}) \\ &\ge \sup_{\|f\|_{\mathcal{PW}_\pi^1} \le 1} |(A_\delta^T f)(t)| - \|T\|\end{aligned}$$

and Theorem 6.17 implies that $\sup_{\|f\|_{\mathcal{PW}_\pi^1} \le 1} |(A_\delta^T f)(t)| = \infty$. □

Thus, if (6.24) is not fulfilled, the pointwise approximation error cannot be controlled, regardless of how small the threshold δ is. Clearly, for fixed $f \in \mathcal{PW}_\pi^1$, $|(A_\delta^T f)(t)|$ is bounded. However, according to Corollary 6.19, for any level $L > 0$ we can find a signal $f_1 \in \mathcal{PW}_\pi^1$ with $\|f_1\|_{\mathcal{PW}_\pi^1} \le 1$ such that $|(Tf_1)(t) - (A_\delta^T f_1)(t)| > L$.

On the other hand, if (6.24) is fulfilled, then we have a good pointwise approximation behavior because the approximation error converges to zero as the threshold δ decreases.

Theorem 6.20. *Let* $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ *be a stable LTI system and* $t \in \mathbb{R}$*. If* (6.24) *is fulfilled then we have*

$$\lim_{\delta \to 0} \sup_{f \in \mathcal{PW}_\pi^1} |(Tf)(t) - (A_\delta^T f)(t)| = 0$$

and

$$\lim_{\delta \to 0} \sup_{f \in \mathcal{PW}_\pi^1} |(Tf)(t) - (B_\delta^T f)(t)| = 0.$$

Proof. Let $t \in \mathbb{R}$ be arbitrary but fixed. From Theorem 4.31 we know that $(TS_N f)(t)$ converges to $(Tf)(t)$ for all $f \in \mathcal{PW}_\pi^2$. Let $U : \mathcal{PW}_\pi^2 \to \mathbb{C}$ denote the continuous linear operator

$$Uf = (Tf)(t) = \sum_{k=-\infty}^{\infty} f(k) h_T(t-k).$$

Moreover, we have

$$\begin{aligned}\left| \sum_{k=-\infty}^{\infty} f(k) h_T(t-k) \right| &\leq \sum_{k=-\infty}^{\infty} |f(k)|\, |h_T(t-k)| \\ &\leq \|f\|_{\mathcal{PW}_\pi^1} \sum_{k=-\infty}^{\infty} |h_T(t-k)| \\ &< \infty\end{aligned}$$

for all $f \in \mathcal{PW}_\pi^1$, because (6.24) is fulfilled by assumption. Thus,

$$\hat{U} f = \sum_{k=-\infty}^{\infty} f(k) h_T(t-k)$$

defines a continuous linear operator $\hat{U} : \mathcal{PW}_\pi^1 \to \mathbb{C}$. Since $\hat{U} f = Uf = (Tf)(t)$ for all f in $\mathcal{PW}_\pi^2$, which is dense in $\mathcal{PW}_\pi^1$, we can conclude that $\hat{U}$ is the unique linear extension of U, and consequently $\hat{U} f = (Tf)(t)$, i.e.,

$$(Tf)(t) = \sum_{k=-\infty}^{\infty} f(k) h_T(t-k)$$

for all $f \in \mathcal{PW}_\pi^1$. Taking the supremum on both sides of

$$\begin{aligned}|(Tf)(t) - (A_\delta^T f)(t)| &= \left| \sum_{k=-\infty}^{\infty} f(k) h_T(t-k) - \sum_{k=-\infty}^{\infty} (\Theta_\delta f)(k) h_T(t-k) \right| \\ &= \left| \sum_{k=-\infty}^{\infty} (f(k) - (\Theta_\delta f)(k)) h_T(t-k) \right| \\ &\leq \delta \sum_{k=-\infty}^{\infty} |h_T(t-k)| \qquad (6.25)\end{aligned}$$

proves the statement for A_δ^T. The proof for B_δ^T is the same. □

Remark 6.21. With (6.25) we have a universal bound for the approximation error which is independent of f.

In order to prove Theorem 6.17 we need Lemma 6.22.

Lemma 6.22. *For all stable LTI systems* $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$, $0 < \delta < 1/3$, *and* $t \in \mathbb{R}$ *we have*

$$\frac{\delta}{2} \sum_{k=-\infty}^{\infty} |h_T(t-k)| \leq \sup_{\|f\|_{\mathcal{PW}_\pi^1} \leq 1} |(A_\delta^T f)(t)| \leq \sum_{k=-\infty}^{\infty} |h_T(t-k)| \tag{6.26}$$

and

$$\frac{\delta}{2} \sum_{k=-\infty}^{\infty} |h_T(t-k)| \leq \sup_{\|f\|_{\mathcal{PW}_\pi^1} \leq 1} |(B_\delta^T f)(t)| \leq 2 \sum_{k=-\infty}^{\infty} |h_T(t-k)|. \tag{6.27}$$

Proof. The right inequality in (6.26) follows directly from

$$\begin{aligned} |(A_\delta^T f)(t)| &= \left| \sum_{k=-\infty}^{\infty} (\Theta f)(k) h_T(t-k) \right| \\ &\leq \sum_{|f(k)| \geq \delta} |f(k)||h_T(t-k)| \\ &\leq \|f\|_{\mathcal{PW}_\pi^1} \sum_{k=-\infty}^{\infty} |h_T(t-k)|, \end{aligned}$$

and the right inequality in (6.27) from

$$\begin{aligned} |(B_\delta^T f)(t)| &= \left| \sum_{k=-\infty}^{\infty} (\Upsilon_\delta f)(k) h_T(t-k) \right| \\ &\leq \sum_{k=-\infty}^{\infty} |2f(k)||h_T(t-k)| \\ &\leq 2\|f\|_{\mathcal{PW}_\pi^1} \sum_{k=-\infty}^{\infty} |h_T(t-k)|, \end{aligned}$$

where we used (6.7) in the first inequality.

The left inequality in (6.26) needs some more reasoning. Let $0 < \delta < 1/3$ and $t \in \mathbb{R}$ be arbitrary but fixed. Furthermore, let $\mathcal{Z}^+ = \{k \in \mathbb{Z} : h_T(t-k) \geq 0\}$ and $\mathcal{Z}^- = \{k \in \mathbb{Z} : h_T(t-k) < 0\}$. For $0 < \eta < 1$ and $N \in \mathbb{N}$, consider the function

$$h^+(t, \eta, N) := \sum_{k=-2N+1}^{2N-1} h^+(k, \eta, N) \frac{\sin(\pi(t-k))}{\pi(t-k)},$$

where

$$h^+(k,\eta,N) = \begin{cases} 1+\eta, & k \in \mathcal{Z}^+ \cap [-N,N], \\ 1-\eta, & k \in \mathcal{Z}^- \cap [-N,N], \\ 2-\frac{|k|}{N}, & N < |k| < 2N. \end{cases}$$

We have

$$h^+(t,\eta,N) = h^+(t,0,N) + \eta \underbrace{\sum_{\substack{k=-N \\ k\in\mathcal{Z}^+}}^{N} \frac{\sin(\pi(t-k))}{\pi(t-k)}}_{=:u_N^+(t)} - \eta \underbrace{\sum_{\substack{k=-N \\ k\in\mathcal{Z}^-}}^{N} \frac{\sin(\pi(t-k))}{\pi(t-k)}}_{=:u_N^-(t)},$$

and it follows that

$$\|h^+(\,\cdot\,,\eta,N)\|_{\mathcal{PW}_\pi^1} \le \|h^+(\,\cdot\,,0,N)\|_{\mathcal{PW}_\pi^1} + \eta\|u_N^+\|_{\mathcal{PW}_\pi^1} + \eta\|u_N^-\|_{\mathcal{PW}_\pi^1}.$$

Since $\|h^+(\,\cdot\,,0,N)\|_{\mathcal{PW}_\pi^1} < 3$, which is proved in the Appendix A.11, and $\|u_N^+\|_{\mathcal{PW}_\pi^1} < \infty$ as well as $\|u_N^-\|_{\mathcal{PW}_\pi^1} < \infty$ for all $N \in \mathbb{N}$, there exists an $\eta_0 = \eta_0(N)$ with $0 < \eta_0 < 1$ such that $\|h^+(\,\cdot\,,\eta_0,N)\|_{\mathcal{PW}_\pi^1} < 3$. Now, let $g^+(t) := \delta h^+(t,\eta_0,N)$. Note that $\|g^+\|_{\mathcal{PW}_\pi^1} < 1$. We have

$$\begin{aligned}(A_\delta^T g^+)(t) &= \sum_{\substack{k=-\infty \\ |g^+(k)|\ge\delta}}^{\infty} g^+(k) h_T(t-k) \\ &= (1+\eta_0)\delta \sum_{\substack{k=-N \\ k\in\mathcal{Z}^+}}^{N} h_T(t-k) \\ &> \delta \sum_{\substack{k=-N \\ k\in\mathcal{Z}^+}}^{N} h_T(t-k)\end{aligned}$$

and consequently

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1}\le 1} (A_\delta^T f)(t) \ge \delta \sum_{\substack{k=-\infty \\ k\in\mathcal{Z}^+}}^{\infty} h_T(t-k). \tag{6.28}$$

Analogously to $h^+(t,\eta,N)$ we define

$$h^-(t,\eta,N) := \sum_{k=-2N+1}^{2N-1} h^-(k,\eta,N) \frac{\sin(\pi(t-k))}{\pi(t-k)},$$

where

$$h^-(k,\eta,N) = \begin{cases} -(1+\eta), & k \in \mathcal{Z}^- \cap [-N,N], \\ -(1-\eta), & k \in \mathcal{Z}^+ \cap [-N,N], \\ -(2-\frac{|k|}{N}), & N < |k| < 2N, \end{cases}$$

and the function $g^-(t) := \delta h^-(t, \eta_1, N)$, where $\eta_1 = \eta_1(N)$, $0 < \eta_1 < 1$, is chosen such that $\|h^-(\,\cdot\,, \eta_1, N)\|_{\mathcal{PW}_\pi^1} < 3$, which implies that $\|g^-\|_{\mathcal{PW}_\pi^1} < 1$. Moreover, we have

$$
\begin{aligned}
(A_\delta^T g^-)(t) &= \sum_{\substack{k=-\infty \\ |g^-(k)|\geq\delta}}^{\infty} g^-(k) h_T(t-k) \\
&= -(1+\eta_1)\delta \sum_{\substack{k=-N \\ k\in\mathcal{Z}^-}}^{N} h_T(t-k) \\
&= (1+\eta_1)\delta \sum_{\substack{k=-N \\ k\in\mathcal{Z}^-}}^{N} |h_T(t-k)| \\
&> \delta \sum_{\substack{k=-N \\ k\in\mathcal{Z}^-}}^{N} |h_T(t-k)|
\end{aligned}
$$

and consequently

$$
\sup_{\|f\|_{\mathcal{PW}_\pi^1}\leq 1} (A_\delta^T f)(t) \geq \delta \sum_{\substack{k=-\infty \\ k\in\mathcal{Z}^-}}^{\infty} |h_T(t-k)|. \tag{6.29}
$$

Combining (6.28) and (6.29) finally gives

$$
\begin{aligned}
2 \sup_{\|f\|_{\mathcal{PW}_\pi^1}\leq 1} (A_\delta^T f)(t) &\geq \delta \sum_{\substack{k=-N \\ k\in\mathcal{Z}^+}}^{N} h_T(t-k) + \delta \sum_{\substack{k=-N \\ k\in\mathcal{Z}^-}}^{N} |h_T(t-k)| \\
&= \delta \sum_{k=-\infty}^{\infty} |h_T(t-k)|,
\end{aligned}
$$

which completes the proof of the left inequality in (6.26).

The proof of the left inequality in (6.27) is essentially the same as the proof of the left inequality in (6.26). □

Proof of Theorem 6.17. Theorem 6.17 follows directly from Lemma 6.22. □

The following corollary illustrates Theorem 6.17 and shows that even for common stable LTI systems like the ideal low-pass filter there are problems because (6.24) is not fulfilled.

Corollary 6.23. *Let L_π denote the ideal low-pass filter with $h_{L_\pi}(t) = \sin(\pi t)/(\pi t)$. Then we have for all $t \in \mathbb{R} \setminus \mathbb{Z}$ and $0 < \delta < 1/3$ that*

$$
\sup_{\|f\|_{\mathcal{PW}_\pi^1}\leq 1} |(A_\delta^{L_\pi} f)(t)| = \sup_{\|f\|_{\mathcal{PW}_\pi^1}\leq 1} |(A_\delta f)(t)| = \infty
$$

and

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1}\le 1} |(B_\delta^{L_\pi} f)(t)| = \sup_{\|f\|_{\mathcal{PW}_\pi^1}\le 1} |(B_\delta f)(t)| = \infty.$$

Proof. We have $\sum_{k=-\infty}^{\infty} |h_{L_\pi}(t-k)| = \infty$ for all $t \in \mathbb{R} \setminus \mathbb{Z}$, and the statement follows from Theorem 6.17. □

Here we see that the behavior of $A_\delta f$ for $f \in \mathcal{PW}_\pi^1$ is completely different to the behavior for $f \in \mathcal{PW}_\pi^p$, $1 < p \le \infty$, where we have $\sup_{\|f\|_{\mathcal{PW}_\pi^p}\le 1} \|A_\delta f\|_\infty < \infty$, according to Corollary 6.7.

Corollary 6.23 shows that, for $t \in \mathbb{R} \setminus \mathbb{Z}$ and any δ with $0 < \delta < 1/3$, the approximation error $|(L_\pi f)(t) - (A_\delta^{L_\pi} f)(t)| = |f(t) - (A_\delta f)(t)|$ can be arbitrarily large depending on the signal $f \in \mathcal{PW}_\pi^1$, $\|f\|_{\mathcal{PW}_\pi^1} \le 1$. This result is interesting because on the integer lattice $t = n \in \mathbb{Z}$ we have a good approximation behavior of $A_\delta f$. More precisely, for $n \in \mathbb{Z}$ we have $(A_\delta f)(n) = (\Theta f)(n)$, which implies that $\sup_{n\in\mathbb{Z}} |f(n) - (A_\delta f)(n)| \le 2\delta$ and $\lim_{\delta\to 0} \sup_{n\in\mathbb{Z}} |f(n) - (A_\delta f)(n)| = 0$. Thus, $(A_\delta f)(n)$ converges to $f(n)$ for all $n \in \mathbb{Z}$ and all $f \in \mathcal{PW}_\pi^1$. However, for $t \in \mathbb{R} \setminus \mathbb{Z}$, Corollary 6.23 shows that $|f(t) - (A_\delta f)(t)|$ can grow arbitrary large.

Remark 6.24. With Corollary 6.23 we have also proved Theorem 6.9, the proof of which was open until now.

Similar to Theorem 6.17, which characterizes the pointwise boundedness of $\sup_{\|f\|_{\mathcal{PW}_\pi^1}\le 1} |(A_\delta^T f)(t)|$, we can also give a necessary and sufficient condition for the uniform boundedness on the whole real axis.

Theorem 6.25. *Let $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ be a stable LTI system and $0 < \delta < 1/3$. We have*

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1}\le 1} \|A_\delta^T f\|_\infty < \infty \tag{6.30}$$

and

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1}\le 1} \|B_\delta^T f\|_\infty < \infty \tag{6.31}$$

if and only if

$$\sup_{0\le t\le 1} \sum_{k=-\infty}^{\infty} |h_T(t-k)| < \infty. \tag{6.32}$$

Proof. Theorem 6.25 follows directly from Lemma 6.22 by taking the supremum $\sup_{t\in\mathbb{R}}$ of all parts of (6.26) and (6.27) and the fact that $\sum_{k=-\infty}^{\infty} |h_T(t-k)|$ is periodic with period 1. □

Corollary 6.26. *Let $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ be a stable LTI system and $0 < \delta < 1/3$. We have (6.30) and (6.31) if and only if $h_T \in \mathcal{B}_\pi^1$, i.e., if and only if*

$$\int_{-\infty}^{\infty} |h_T(\tau)| \, \mathrm{d}\tau < \infty. \tag{6.33}$$

Proof. According to Nikol'skiĭ's inequality [26, p. 49], (6.32) is true if and only if $\int_{-\infty}^{\infty}|h_T(\tau)|\,\mathrm{d}\tau < \infty$. □

Remark 6.27. Note that (6.33) is nothing but the BIBO stability condition for continuous-time systems.

The next corollary shows the good global approximation behavior of $A_\delta^T f$ if (6.33) is fulfilled.

Corollary 6.28. *Let* $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ *be a stable LTI system. If* (6.33) *is fulfilled then we have*

$$\lim_{\delta\to\infty} \sup_{f\in\mathcal{PW}_\pi^1} \|Tf - A_\delta^T f\|_\infty = 0$$

and

$$\lim_{\delta\to\infty} \sup_{f\in\mathcal{PW}_\pi^1} \|Tf - B_\delta^T f\|_\infty = 0.$$

Proof. Analogously to the proof of Theorem 6.20. □

We have seen that, for $f \in \mathcal{PW}_\pi^1$, the class of stable LTI systems T that can be uniformly approximated by A_δ^T and B_δ^T is given by the set of LTI systems with $h_T \in \mathcal{B}_\pi^1$. This means that the class of stable LTI systems that are robust under thresholding and quantization is exactly the class of bounded-input bounded-output (BIBO) stable LTI systems.

6.3.3 Oversampling

In Section 6.2.2 we have seen that oversampling can be used to improve the convergence behavior of the sampling series and to reduce the reconstruction error for the case where the samples are disturbed by the quantization operator. However, oversampling cannot remove the instability encountered in Theorem 6.17 and Corollary 6.19. In [9] the following theorem was proved for the approximation process with oversampling

$$(A_{\delta,\phi}^{T,a} f)(t) = \sum_{k=-\infty}^{\infty} (\Theta_\delta f)\left(\frac{k}{a}\right)(T\phi)\left(t - \frac{k}{a}\right) \tag{6.34}$$

and the Hilbert transform $T = H$.

Theorem 6.29. *For all* $a > 1$, $\phi \in \mathcal{M}(a)$, *and* $0 < \delta < 1/\pi$ *we have*

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1}\le 1} \|A_{\delta,\phi}^{H,a} f\|_\infty = \infty$$

and consequently

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1}\le 1} \|Hf - A_{\delta,\phi}^{H,a} f\|_\infty = \infty.$$

Theorem 6.29 shows that the Hilbert transform cannot be approximated using (6.34) and the samples of $f \in \mathcal{PW}_\pi^1$, because the peak approximation error is not bounded on the set of signals $f \in \mathcal{PW}_\pi^1$ with $\|f\|_{\mathcal{PW}_\pi^1} \leq 1$. For any given error level, we can find a signal with norm $\|f\|_{\mathcal{PW}_\pi^1} \leq 1$ such that the peak approximation error exceeds this level.

In Theorem 6.29 we used the Hilbert transform as one specific stable LTI system, however, the result is not restricted to the Hilbert transform but is also valid for other stable LTI systems. Of course there are stable LTI systems for which the error is bounded and goes to zero as the threshold δ goes to zero. For example, if T is the identity operator Id, we have according to Theorem 6.15 that

$$\lim_{\delta\to 0} \sup_{f\in\mathcal{PW}_\pi^1} \|Idf - A_{\delta,\phi}^{Id,a} f\|_\infty = \lim_{\delta\to 0} \sup_{f\in\mathcal{PW}_\pi^1} \|f - A_{\delta,\phi}^{a} f\|_\infty = 0.$$

Hence the class of stable LTI systems for which Theorem 6.29 is valid, is a subset of all stable LTI systems.

6.4 Discussion

All the results in this chapter were obtained for equidistant sampling. It would be interesting to extend the results to non-equidistant sampling. Ordinary non-equidistant sampling series without quantization or thresholding were analyzed in Section 3.3. However, the more general problem which treats the convergence behavior of non-equidistant sampling series with sample values that are disturbed by the threshold operator or the quantization operator is difficult to analyze and still open. A first partial result towards the solution of this problem was obtained in [9].

7

Conclusion and Outlook

In this work we have studied the interplay between the analog and the digital worlds. The goal was to better understand the fundamentals of signal reconstruction and system approximation. We showed that many of the classical results can be extended, e.g., to larger signal spaces or to non-equidistant sampling patterns, but also that new problems occur.

The signal reconstruction and the system approximation problem in combination with quantization, which is essential for practical applications, turned out to be difficult to analyze because of the non-linearity of the quantization operator.

We have seen that the presence of stable linear time-invariant systems in the approximation process in general negatively affects the convergence behavior when compared to the signal reconstruction process without such a system. The convergence of system representations cannot be guaranteed and has to be checked from case to case. We provided sufficient and necessary conditions for the convergence.

Oversampling is generally known to resolve convergence problems and to improve the convergence behavior of sampling series. Although it turned out that this is also true for the reconstruction of signals in $\mathcal{PW}_{\pi}^{1}$, we have seen that oversampling cannot correct the divergence that occurs in the sampling based system approximation.

The main contributions of this thesis are as follows.

- We identified limits.

  We showed that the property of boundedness does not carry over from the discrete-time signals to the continuous-time signals because there exist bounded discrete-time signals for which a bounded bandlimited interpolation does not exist. Even simple signal processing operations like truncation can lead to such discrete-time signals.

  We analyzed a large class of Nyquist set reconstruction processes, containing the Shannon sampling series and the Valiron sampling series as spacial cases, and showed that a reconstruction that is uniformly convergent on compact subsets of the real line and uniformly bounded on the entire real line is not possible for $\mathcal{PW}_\pi^1$.

We studied representations of stable LTI systems on $\mathcal{PW}_\pi^1$. It turned out that there exist stable LTI systems and signals for which the convolution integral system representation diverges even in a distributional sense.

We showed that quantization and thresholding can lead to unexpected effects. In particular, there exist signals in $\mathcal{PW}_\pi^1$ such that the peak value of the reconstruction error increases unboundedly as the quantization step size is reduced. Moreover, it was shown that oversampling can improve this behavior.

- We extended classical results.

  We showed that the sampling series with oversampling is uniformly convergent on the entire real line for all signals in $\mathcal{PW}_\pi^1$, and that an elaborate kernel design is not necessary as far as only convergence is concerned—even the Shannon sampling series with slightly increased bandwidth has this good convergence behavior.

  We proved sampling theorems for non-equidistant sampling patterns and the signal spaces $\mathcal{B}_{\pi,0}^\infty$ and $\mathcal{B}_\pi^\infty$. These results extended the classical sampling theorems towards more flexible sampling points and larger signal spaces.

  We incorporated quantization and thresholding in the signal reconstruction and the system approximation process and analyzed their behavior with respect to the quantization step size. For the spaces $\mathcal{B}_\pi^p$, $1 < p < \infty$, and $\mathcal{PW}_\pi^p$, $1 < p \leq \infty$, we could prove that the reconstruction error goes to zero as the quantization step size is decreased.

- We characterized border cases.

  For $\mathcal{PW}_\pi^1$ we found a sufficient condition for the uniform convergence of the Shannon sampling series without oversampling.

  We completely characterized all the stable LTI systems on $\mathcal{PW}_\pi^1$ for which a convolution-type system representation is possible.

  We identified the differences in the convergence behavior of the symmetric and the non-symmetric Shannon sampling series for signals in $\mathcal{PW}_\pi^1$ as well as bandlimited stochastic processes (I-processes).

  We characterized all I-processes for which the non-symmetric Shannon sampling series converges locally uniformly in the mean square sense and for which the variance of the reconstruction error of the symmetric Shannon sampling series is globally uniformly bounded.

  We completely characterized the stable LTI systems on $\mathcal{PW}_\pi^1$ for which the system approximation process with quantization or thresholding behaves well with respect to the quantization step size.

7.1 Outlook and Open Problems

There are also open problems that require further research.

- In this thesis we derived several results for sine-type sampling patterns. However, the convergence behavior of sampling series for general non-equidistant sampling patterns is open in general, and the characterization of suitable sampling patterns for different signal spaces is certainly a challenging task.

  In particular the following open problems dealing with non-equidistant sampling patterns were addressed in this thesis: In Section 3.3.5 we raised two interesting unsolved questions. The first concerns the estimation of the peak value of a general bounded bandlimited signal from its non-equidistant samples and the second the stability of non-equidistant sine-type sampling patterns. Another open problem was implicitly formulated in Conjecture 3.49, where we conjectured convergence problems of the signal reconstruction process for bounded bandlimited signals that vanish at infinity if the restriction to sine-type sampling patterns is loosened and general complete interpolating sequences are allowed. A confirmation of this conjecture would show the importance of the restriction to sine-type sampling patterns for obtaining our results.

- Our analysis of the signal reconstruction and the system approximation with quantization and thresholding in Chapter 6 was done only for equidistant sampling points. Considering non-equidistant sampling patterns in this setting seems to be even more intricate. It would be interesting to know whether there exist complete interpolating sequences $\{t_k\}_{k\in\mathbb{Z}}$ such that

$$\lim_{\delta\to 0}\left\| Tf - \sum_{k=-\infty}^{\infty} (\Theta_\delta f)(t_k) T\phi_k \right\|_\infty = 0$$

  for all $f \in \mathcal{PW}_\pi^1$ and all stable LTI systems $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$. If this is true, it would imply that the problems that we encountered in Theorem 6.13 for equidistant sampling and the identity operator could be avoided by cleverly choosing the positions of the sampling points.

- The behavior of the system approximation process with thresholding and oversampling (6.34) was studied in Section 6.3.3 for fixed threshold δ and varying signals in $\mathcal{PW}_\pi^1$. Its behavior for a fixed signal and a threshold δ tending to zero is unknown.

- In Chapter 4 we have analyzed convolution-type system representations for stable LTI systems operating on $\mathcal{PW}_\pi^1$. We showed that there are systems and signals for which we have divergence even in a distributional sense. It would be important to find other representations that are convergent for all stable LTI systems and all signals in $\mathcal{PW}_\pi^1$.

A

Supplementary Proofs

A.1 Proof of Observation 3.10

Proof. i) Obviously,

$$|(T^a f)(t)| \leq \frac{1}{|\sin(\pi t_0)|}|f(t_0)| \leq \frac{1}{|\sin(\pi t_0)|} \max_{|z|\leq t_0} |f(z)|.$$

Choosing $R = t_0$ and $C_3 = |\sin(\pi t_0)|^{-1}$ and taking the supremum completes the proof for i).
ii) Since $f \in \mathcal{PW}_\pi^1$ is analytic on the whole complex plane we can use Cauchy's integral formula to obtain

$$f'(m) = \frac{1}{2\pi \mathrm{i}} \oint\limits_{|\xi|=R} \frac{f(\xi)}{(\xi - m)^2} \, \mathrm{d}\xi, \quad |m| + 1 < R$$

and as a consequence, the inequality

$$|f'(m)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(Re^{\mathrm{i}\theta})| \frac{R}{|Re^{\mathrm{i}\theta} - m|^2} \, \mathrm{d}\theta \leq R \max_{|\xi|=R} |f(\xi)|.$$

Using this inequality, we obtain

$$|(T^b f)(t)| \leq |f(m)| + R \max_{|\xi|=R} |f(\xi)| \leq (1 + R) \max_{|\xi|=R} |f(\xi)|,$$

which finishes the proof for ii) after taking the supremum and applying the maximum modulus principle. iii) Obvious. □

A.2 Proof of Lemma 3.25

Proof. For g with $\hat{g} \in C_0^\infty[-(a+1)\pi, (a+1)\pi]$ and $\hat{g}(\omega) = 1$, $|\omega| \le a\pi$, we have

$$\phi(t) = \int_{-\infty}^{\infty} \phi(\tau) g(t-\tau) \, \mathrm{d}\tau$$

and

$$\sum_{k=-N}^{N} \left| \phi\left(t - \frac{k}{a}\right) \right| \le \int_{-\infty}^{\infty} |\phi(\tau)| \sum_{k=-N}^{N} \left| g\left(t - \frac{k}{a} - \tau\right) \right| \, \mathrm{d}\tau. \tag{A.1}$$

Furthermore, since $\hat{g} \in C_0^\infty[-(a+1)\pi, (a+1)\pi]$, it follows that $|g(t)| \le C_1/(1+t^2)$ and consequently

$$\sum_{k=-N}^{N} \left| g\left(t - \frac{k}{a} - \tau\right) \right| \le C_8, \tag{A.2}$$

where C_1 and C_8 are some constants. Inserting (A.2) in (A.1) gives

$$\sum_{k=-\infty}^{\infty} \left| \phi\left(t - \frac{k}{a}\right) \right| \le C_8 \|\phi\|_{\mathcal{B}^1_{a\pi}}. \qquad \square$$

A.3 Proof of Lemma 4.8

Proof. Let $t \in \mathbb{R}$ and $N \in \mathbb{N}$ arbitrary but fixed. For convenience we introduce the function

$$g(\omega) := \int_{t-N}^{t+N} h_T(\tau) \, \mathrm{e}^{-\mathrm{i}\omega\tau} \, \mathrm{d}\tau.$$

We have

$$\begin{aligned} \left| \int_{-N}^{N} f(\tau) h_T(t-\tau) \, \mathrm{d}\tau \right| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) \, \mathrm{e}^{\mathrm{i}\omega t} \, g(\omega) \, \mathrm{d}\omega \right| \\ &\le \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{f}(\omega)| |g(\omega)| \, \mathrm{d}\omega \\ &\le \max_{|\omega| \le \pi} |g(\omega)| \, \|f\|_{\mathcal{PW}^1_\pi}, \end{aligned} \tag{A.3}$$

because g is continuous. Taking the supremum on both sides of (A.3) gives

$$\sup_{\|f\|_{\mathcal{PW}^1_\pi} \le 1} \left| \int_{-N}^{N} f(\tau) h_T(t-\tau) \, \mathrm{d}\tau \right| \le \max_{|\omega| \le \pi} |g(\omega)|. \tag{A.4}$$

Furthermore, since g is continuous on the compact interval $[-\pi, \pi]$, $|g|$ attains its maximum in some point $\omega^* \in [-\pi, \pi]$. For $n \in \mathbb{N}$ let

$$E_n = \left\{ \omega : |g(\omega)| \ge |g(\omega^*)| - \frac{1}{n} \right\}$$

and choose

$$f_n(\tau,t) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{2\pi}{\lambda(E_n)}\mathbf{1}_{E_n}(\omega)\,\mathrm{e}^{\mathrm{i}\phi(\omega)}\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{e}^{\mathrm{i}\omega\tau}\,\mathrm{d}\omega,$$

where $\phi(\omega) = -\arg(g(\omega))$, and $\mathbf{1}_E$ denotes the indicator function of the set E. Obviously, $\|f_n\|_{\mathcal{PW}_\pi^1} = 1$ for all $n \in \mathbb{N}$. It follows that

$$\begin{aligned}\left|\int_{-N}^{N} f_n(\tau,t)h_T(t-\tau)\,\mathrm{d}\tau\right| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{f}_n(\omega,t)\,\mathrm{e}^{\mathrm{i}\omega t}\,g(\omega)\,\mathrm{d}\omega\right|\\ &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{2\pi}{\lambda(E_n)}\mathbf{1}_{E_n}(\omega)\,\mathrm{e}^{\mathrm{i}\phi(\omega)}\,g(\omega)\,\mathrm{d}\omega\right|\\ &= \frac{1}{\lambda(E_n)}\int_{E_n}|g(\omega)|\,\mathrm{d}\omega \geq |g(\omega^*)| - \frac{1}{n},\end{aligned}$$

and

$$\begin{aligned}\sup_{\|f\|_{\mathcal{PW}_\pi^1}\leq 1}\left|\int_{-N}^{N} f(\tau)h_T(t-\tau)\,\mathrm{d}\tau\right| &\geq \lim_{n\to\infty}\left|\int_{-N}^{N} f_n(\tau,t)h_T(t-\tau)\,\mathrm{d}\tau\right|\\ &\geq |g(\omega^*)| = \max_{|\omega|\leq\pi}|g(\omega)|\,. \qquad \text{(A.5)}\end{aligned}$$

Combining (A.4) and (A.5) completes the proof. □

A.4 Proof of Lemma 4.9

Proof. We have

$$|(A_N^T f)(t)| = \left|\int_{-N}^{N} f(\tau)h_T(t-\tau)\,\mathrm{d}\tau\right| \leq \max_{\omega\in[-\pi,\pi]}\left|\int_{t-N}^{t+N} h_T(\tau)\,\mathrm{e}^{-\mathrm{i}\omega\tau}\,\mathrm{d}\tau\right|,$$

according to (A.3) because $\|f\|_{\mathcal{PW}_\pi^1} \leq 1$. Furthermore,

$$\begin{aligned}\int_{t-N}^{t+N} h_T(\tau)\,\mathrm{e}^{-\mathrm{i}\omega\tau}\,\mathrm{d}\tau &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{h}_T(\omega_1)\int_{t-N}^{t+N}\mathrm{e}^{\mathrm{i}\tau(\omega_1-\omega)}\,\mathrm{d}\tau\,\mathrm{d}\omega_1\\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}\hat{h}_T(\omega_1)\,\mathrm{e}^{\mathrm{i}t(\omega_1-\omega)}\,\frac{\sin\big(N(\omega_1-\omega)\big)}{\omega_1-\omega}\,\mathrm{d}\omega_1\end{aligned}$$

and consequently

$$\begin{aligned}\left|\int_{t-N}^{t+N} h_T(\tau)\,\mathrm{e}^{-\mathrm{i}\omega\tau}\right|\,\mathrm{d}\tau &\leq \frac{1}{\pi}\int_{-\pi}^{\pi}|\hat{h}_T(\omega_1)|\left|\frac{\sin\big(N(\omega_1-\omega)\big)}{\omega_1-\omega}\right|\,\mathrm{d}\omega_1\\ &\leq \|\hat{h}_T\|_{L^\infty[-\pi,\pi]}\frac{1}{\pi}\int_{-\pi}^{\pi}\left|\frac{\sin\big(N(\omega_1-\omega)\big)}{\omega_1-\omega}\right|\,\mathrm{d}\omega_1\\ &\leq \|T\|\frac{2}{\pi}\left(\pi+\frac{2}{\pi}+\frac{2}{\pi}\log(2N-1)\right)\end{aligned}$$

because for $\omega \in [-\pi, \pi]$,

$$\begin{aligned}\int_{-\pi}^{\pi} \left| \frac{\sin(N(\omega_1 - \omega))}{\omega_1 - \omega} \right| \, \mathrm{d}\omega_1 &= \int_{N(\omega-\pi)}^{N(\omega+\pi)} \left| \frac{\sin(\tau)}{\tau} \right| \, \mathrm{d}\tau \\ &\leq \int_{-2\pi N}^{2\pi N} \left| \frac{\sin(\tau)}{\tau} \right| \, \mathrm{d}\tau \\ &= 2 \int_{0}^{2N} \left| \frac{\sin(\pi\tau)}{\tau} \right| \, \mathrm{d}\tau \\ &\leq 2 \left(\pi + \sum_{k=1}^{2N-1} \int_{k}^{k+1} \left| \frac{\sin(\pi\tau)}{\tau} \right| \, \mathrm{d}\tau \right) \\ &\leq 2 \left(\pi + \sum_{k=1}^{2N-1} \frac{1}{k} \int_{k}^{k+1} |\sin(\pi\tau)| \, \mathrm{d}\tau \right) \\ &= 2 \left(\pi + \frac{2}{\pi} \sum_{k=1}^{2N-1} \frac{1}{k} \right) \\ &\leq 2 \left(\pi + \frac{2}{\pi} + \frac{2}{\pi} \log(2N-1) \right). \qquad \square\end{aligned}$$

A.5 Proof of Lemma 4.15

Proof. First, we derive a preliminary statement, which will be used in the proof of i) and ii). For $\omega \in [-\pi, \pi]$ and $\phi \in \mathcal{S}$ consider the difference

$$\begin{aligned}D_N(\hat{h}_T, \phi, \omega) := \Bigg| &\frac{1}{\pi} \int_{-\pi}^{\pi} \hat{h}_T(\omega_1) \hat{\phi}(-\omega_1) \frac{\sin(N(\omega - \omega_1))}{\omega - \omega_1} \, \mathrm{d}\omega_1 \\ &- \hat{\phi}(-\omega) \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{h}_T(\omega_1) \frac{\sin(N(\omega - \omega_1))}{\omega - \omega_1} \, \mathrm{d}\omega_1 \Bigg| \\ = &\left| \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{h}_T(\omega_1) \frac{\hat{\phi}(-\omega_1) - \hat{\phi}(-\omega)}{\omega - \omega_1} \sin(N(\omega - \omega_1)) \, \mathrm{d}\omega_1 \right|. \qquad \text{(A.6)}\end{aligned}$$

Since $\phi \in \mathcal{S}$ it follows that $\hat{\phi} \in \mathcal{S}$ and $\hat{\phi}' \in \mathcal{S}$. In particular, ϕ is Lipschitz continuous because $\|\hat{\phi}'\|_\infty < \infty$, and we have

$$\left| \frac{\hat{\phi}(-\omega_1) - \hat{\phi}(-\omega)}{\omega - \omega_1} \right| \leq \|\hat{\phi}'\|_\infty$$

for all $\omega \in [-\pi, \pi]$ and $\omega_1 \in [-\pi, \pi]$. Therefore, we obtain

$$D_N(\hat{h}_T, \phi, \omega) \leq 2\|\hat{\phi}'\|_\infty \|\hat{h}_T\|_{L^\infty[-\pi,\pi]} = C_2(\hat{h}_T, \phi) \qquad \text{(A.7)}$$

for all $\omega \in [-\pi, \pi]$ and all $\phi \in \mathcal{S}$, where $C_2(\hat{h}_T, \phi)$ is some positive constant that does not depend N. Since $\mathcal{D} \subset \mathcal{S}$, (A.7) is also true for all $\phi \in \mathcal{D}$.

“i)”: Let $N \in \mathbb{N}$ be arbitrary but fixed. Since (4.12) is true for all $\phi \in \mathcal{D}$ it is in particular true for the specific function $\phi_1 \in \mathcal{D}$ with real valued $\hat{\phi}_1$ and $\hat{\phi}_1(\omega) > 0$ for $\omega \in [-\pi, \pi]$. Therefore, it follows from (A.6) and (A.7) that

$$\begin{aligned}\hat{\phi}_1(-\omega)&\left|\frac{1}{\pi}\int_{-\pi}^{\pi}\hat{h}_T(\omega_1)\frac{\sin(N(\omega-\omega_1))}{\omega-\omega_1}\,\mathrm{d}\omega_1\right|\\&\leq C_2(\hat{h}_T,\phi_1)+\left|\frac{1}{\pi}\int_{-\pi}^{\pi}\hat{h}_T(\omega_1)\hat{\phi}_1(-\omega_1)\frac{\sin(N(\omega-\omega_1))}{\omega-\omega_1}\,\mathrm{d}\omega_1\right|\end{aligned}$$

for all $\omega \in [-\pi, \pi]$. Dividing by $\hat{\phi}_1(-\omega)$ and taking the maximum on both sides yields

$$\begin{aligned}&\max_{\omega\in[-\pi,\pi]}\left|\frac{1}{\pi}\int_{-\pi}^{\pi}\hat{h}_T(\omega_1)\frac{\sin(N(\omega-\omega_1))}{\omega-\omega_1}\,\mathrm{d}\omega_1\right|\\&\leq\max_{\omega\in[-\pi,\pi]}\frac{1}{\hat{\phi}_1(-\omega)}C_2(\hat{h}_T,\phi_1)\\&\quad+\max_{\omega\in[-\pi,\pi]}\frac{1}{\hat{\phi}_1(-\omega)}\left|\frac{1}{\pi}\int_{-\pi}^{\pi}\hat{h}_T(\omega_1)\hat{\phi}_1(-\omega_1)\frac{\sin(N(\omega-\omega_1))}{\omega-\omega_1}\,\mathrm{d}\omega_1\right|\\&\leq M(\phi_1)C_2(\hat{h}_T,\phi_1)+M(\phi_1)C_2(\phi_1)\\&<\infty,\end{aligned}$$

where $M(\phi_1) = \max_{\omega\in[-\pi,\pi]} 1/\hat{\phi}_1(-\omega)$.

“ii)”: Let $N \in \mathbb{N}$ be arbitrary but fixed. Suppose (4.9) is true, and let $\phi \in \mathcal{S}$ be arbitrary but fixed. From (A.6) and (A.7) it follows that

$$\begin{aligned}&\left|\frac{1}{\pi}\int_{-\pi}^{\pi}\hat{h}_T(\omega_1)\hat{\phi}(-\omega_1)\frac{\sin(N(\omega-\omega_1))}{\omega-\omega_1}\,\mathrm{d}\omega_1\right|\\&\qquad\leq C_2(\hat{h}_T,\phi)+|\hat{\phi}(-\omega)|\left|\frac{1}{\pi}\int_{-\pi}^{\pi}\hat{h}_T(\omega_1)\frac{\sin(N(\omega-\omega_1))}{\omega-\omega_1}\,\mathrm{d}\omega_1\right|\\&\qquad\leq C_2(\hat{h}_T,\phi)+\|\hat{\phi}\|_\infty C_1\end{aligned}$$

for all $\omega \in [-\pi, \pi]$. Taking the maximum on both sides yields

$$\begin{aligned}\max_{\omega\in[-\pi,\pi]}\left|\frac{1}{\pi}\int_{-\pi}^{\pi}\hat{h}_T(\omega_1)\hat{\phi}(-\omega_1)\frac{\sin(N(\omega-\omega_1))}{\omega-\omega_1}\,\mathrm{d}\omega_1\right|&\leq C_2(\hat{h}_T,\phi)+\|\hat{\phi}\|_\infty C_1\\&<\infty,\end{aligned}$$

which completes the proof. □

A.6 Proof of Lemma 4.21

Proof. "i)": Let $N \in \mathbb{N}$ be arbitrary but fixed. Since (4.36) is true for all $\phi \in \mathcal{D}$ it is in particular true for the specific function $\phi_1 \in \mathcal{D}$ with $\hat{\phi}_1(\omega) > 0$ for $\omega \in [-\pi, \pi]$. Therefore, we have

$$\begin{aligned}
&\left| \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{h}_T(\omega_1) \frac{\sin(N(\omega - \omega_1))}{\omega - \omega_1} \, \mathrm{d}\omega_1 \right| \\
&\qquad \leq M(\phi_1) \left| \hat{\phi}_1(-\omega) \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{h}_T(\omega_1) \frac{\sin(N(\omega - \omega_1))}{\omega - \omega_1} \, \mathrm{d}\omega_1 \right|,
\end{aligned}$$

where $M(\phi_1) = \max_{\omega \in [-\pi,\pi]} 1/\hat{\phi}_1(-\omega)$. Taking the maximum on both sides yields

$$\begin{aligned}
&\max_{\omega \in [-\pi,\pi]} \left| \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{h}_T(\omega_1) \frac{\sin(N(\omega - \omega_1))}{\omega - \omega_1} \, \mathrm{d}\omega_1 \right| \\
&\qquad \leq M(\phi_1) \max_{\omega \in [-\pi,\pi]} \left| \hat{\phi}_1(-\omega) \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{h}_T(\omega_1) \frac{\sin(N(\omega - \omega_1))}{\omega - \omega_1} \, \mathrm{d}\omega_1 \right| \\
&\qquad \leq M(\phi_1) C_8(\phi_1) < \infty.
\end{aligned}$$

"ii)": Let $N \in \mathbb{N}$ be arbitrary but fixed. Suppose (4.35) is true, and let $\phi \in \mathcal{S}$ be arbitrary but fixed. Then we have

$$\begin{aligned}
&\max_{\omega \in [-\pi,\pi]} \left| \hat{\phi}(-\omega) \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{h}_T(\omega_1) \frac{\sin(N(\omega - \omega_1))}{\omega - \omega_1} \, \mathrm{d}\omega_1 \right| \\
&\qquad \leq \|\hat{\phi}\|_\infty \max_{\omega \in [-\pi,\pi]} \left| \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{h}_T(\omega_1) \frac{\sin(N(\omega - \omega_1))}{\omega - \omega_1} \, \mathrm{d}\omega_1 \right| \\
&\qquad \leq \|\hat{\phi}\|_\infty C_7 < \infty,
\end{aligned}$$

which completes the proof. □

A.7 Proof of Theorem 4.24

Proof. We have

$$\begin{aligned}
\left| \int_{-N}^{N} h_H(\tau) \, \mathrm{e}^{\mathrm{i}\omega\tau} \, \mathrm{d}\tau \right| &= \left| 2\mathrm{i} \int_0^N \frac{\sin^2\left(\frac{\pi}{2}\tau\right)}{\frac{\pi}{2}\tau} \sin(\omega\tau) \, \mathrm{d}\tau \right| \\
&= \left| \frac{2\mathrm{i}}{\pi} \int_0^N \frac{1 - \cos(\pi\tau)}{\tau} \sin(\omega\tau) \, \mathrm{d}\tau \right| \\
&\leq \left| \frac{2}{\pi} \int_0^N \frac{\sin(\omega\tau)}{\tau} \, \mathrm{d}\tau \right| + \left| \frac{1}{\pi} \int_0^N \frac{\sin((\pi - \omega)\tau)}{\tau} \, \mathrm{d}\tau \right| + \left| \frac{1}{\pi} \int_0^N \frac{\sin((\pi + \omega)\tau)}{\tau} \, \mathrm{d}\tau \right| \\
&\leq \left| \frac{2}{\pi} \int_0^{\omega N} \frac{\sin(\tau)}{\tau} \, \mathrm{d}\tau \right| + \left| \frac{1}{\pi} \int_0^{(\pi - \omega)N} \frac{\sin(\tau)}{\tau} \, \mathrm{d}\tau \right| + \left| \frac{1}{\pi} \int_0^{(\pi + \omega)N} \frac{\sin(\tau)}{\tau} \, \mathrm{d}\tau \right| \\
&< C_3
\end{aligned}$$

for all $N \in \mathbb{N}$ and all $\omega \in [-\pi, \pi]$, because $\int_0^x \sin(\tau)/\tau \, \mathrm{d}\tau < \pi$, independently of $x \in \mathbb{R}$. Using (4.33) and Theorem 4.18 gives $\lim_{N\to\infty} \|Hf - B_N^H f\|_\infty = 0$.

Similar to Theorem 4.5 it can be shown that $\limsup_{N\to\infty} \|Hf - A_N^H f\|_\infty = \infty$ if and only if

$$\sup_{t\in\mathbb{R}} \sup_{N\in\mathbb{N}} \max_{\omega\in[-\pi,\pi]} \left| \int_{t-N}^{t+N} h_H(\tau)\, \mathrm{e}^{-\mathrm{i}\omega\tau} \, \mathrm{d}\tau \right| = \infty.$$

But choosing $t = N$ and $\omega = 0$, and using

$$\int_0^{2N} h_H(\tau)\, \mathrm{d}\tau = \int_0^{2N} \frac{\sin^2\left(\frac{\pi}{2}\tau\right)}{\frac{\pi}{2}\tau} \, \mathrm{d}\tau > \frac{1}{\pi} \log(2N),$$

where the last inequality follows from (4.30), shows exactly this. □

A.8 Supplement to the Proof of Theorem 4.28

In the proof of Theorem 4.28 we used the fact that $\mathcal{K}$ equipped with the norm $\|\cdot\|_{\mathcal{K},t}$ is a Banach space for all $t \in \mathbb{R}$. The proof thereof is given here.

Proof. We have to show that $(\mathcal{K}, \|\cdot\|_{\mathcal{K},t})$ is complete. Let $t \in \mathbb{R}$ be arbitrary but fixed, and let $\mathcal{U}$ denote the set of all functions h with a representation $h(\tau) = 1/(2\pi) \int_{-\pi}^{\pi} \hat{h}(w)\, \mathrm{e}^{\mathrm{i}\omega\tau} \, \mathrm{d}\omega$, $\tau \in \mathbb{R}$, for some $\hat{h} \in C[-\pi, \pi]$. Now, suppose $\{h_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{K}, \|\cdot\|_{\mathcal{K},t})$. Then, for every $\epsilon > 0$ there exists a $N_0 = N_0(\epsilon) \in \mathbb{N}$ such that $\|h_m - h_n\|_{\mathcal{K},t} < \epsilon$ for all $m, n \geq N_0$. It follows that

$$\|\hat{h}_m - \hat{h}_n\|_\infty < \epsilon \text{ for all } m, n \geq N_0 \tag{A.8}$$

and

$$\sup_{N\in\mathbb{N}} \max_{\omega\in[-\pi,\pi]} \left| \sum_{k=-N}^{N} (h_m(t-k) - h_n(t-k))\, \mathrm{e}^{\mathrm{i}\omega k} \right| < \epsilon \text{ for all } m, n \geq N_0. \tag{A.9}$$

From (A.8) we see that $\{\hat{h}_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(C[-\pi, \pi], \|\cdot\|_\infty)$. Since $(C[-\pi, \pi], \|\cdot\|_\infty)$ is complete, the sequence $\{\hat{h}_n\}_{n\in\mathbb{N}}$ has a limit $\hat{h} \in C[-\pi, \pi]$. Thus, there exists a $N_1 \in \mathbb{N}$ such that

$$\|\hat{h}_n - \hat{h}\|_\infty < \epsilon \text{ for all } n \geq N_1. \tag{A.10}$$

Moreover, since $\|f\|_\infty \leq \|\hat{f}\|_\infty$ for all $f \in \mathcal{U}$, it follows that $|h_n(t-k) - h(t-k)| \leq \|h_n - h\|_\infty \leq \|\hat{h}_n - \hat{h}\|_\infty$ for all $n \in \mathbb{N}$, $k \in \mathbb{Z}$, and consequently $\lim_{n\to\infty} h_n(t-k) = h(t-k)$ for all $k \in \mathbb{Z}$.

It remains to show that $h \in \mathcal{K}$ and that $\lim_{m\to\infty} \|h_m - h\|_{\mathcal{K}} = 0$. From (A.9) it follows that $|\sum_{k=-N}^{N} (h_m(t-k) - h_n(t-k))\, \mathrm{e}^{\mathrm{i}\omega k}| < \epsilon$ for all $m, n \geq N_0$, $N \in \mathbb{N}$, and $\omega \in [-\pi, \pi]$. Taking the limit $n \to \infty$, we obtain

$$\sup_{N\in\mathbb{N}} \max_{\omega\in[-\pi,\pi]} \left| \sum_{k=-N}^{N} (h_m(t-k) - h(t-k))\, \mathrm{e}^{\mathrm{i}\omega k} \right| \leq \epsilon \text{ for all } m \geq N_0. \tag{A.11}$$

From (A.10) and (A.11) we see that $h_m - h \in \mathcal{K}$ for all $m \geq M_0$, where $M_0 = \max(N_0, N_1)$. Since $h = h - h_{M_0} + h_{M_0}$, it follows that $h \in \mathcal{K}$. Moreover, using (A.10) and (A.11) again, we obtain $\|h_m - h\|_{\mathcal{K}} \leq 2\epsilon$ for all $m \geq M_0$, which implies that $\lim_{m\to\infty} \|h_m - h\|_{\mathcal{K}} = 0$. □

A.9 Proof of Lemma 5.4

The proof of Lemma 5.4 requires two more lemmas, namely Lemma A.1 and Lemma A.2.

Lemma A.1. *There exists a positive constant C_4, such that*

$$\left| \sum_{k=1}^{N} \frac{\mathrm{e}^{\mathrm{i}k\omega}}{k} \right| \leq \left| \log\left(2\left|\sin\left(\frac{\omega}{2}\right)\right|\right)\right| + C_4$$

for all $N \in \mathbb{N}$ and $\omega \in [-\pi, \pi]$.

Proof of Lemma A.1. We analyze $\sum_{k=N}^{M} \frac{\mathrm{e}^{\mathrm{i}\omega k}}{k}$ for $\omega \in [-\pi, \pi]$, $\omega \neq 0$, and $N, M \in \mathbb{N}$, $N < M$, using summation by parts. For $k \geq N$, let

$$D_{k,N}(\omega) := \sum_{l=N}^{k} \mathrm{e}^{\mathrm{i}\omega l}\,.$$

Then, using summation by parts, we obtain

$$\sum_{k=N+1}^{M} \frac{\mathrm{e}^{\mathrm{i}\omega k}}{k} = \frac{D_{M,N+1}(\omega)}{M} + \sum_{k=N+1}^{M-1} \frac{D_{k,N+1}(\omega)}{k(k+1)}. \tag{A.12}$$

Since

$$|D_{k,N+1}(\omega)| = \left| \sum_{l=N+1}^{k} \mathrm{e}^{\mathrm{i}\omega l} \right| = \left| \frac{1 - \mathrm{e}^{\mathrm{i}\omega(k-N)}}{1 - \mathrm{e}^{\mathrm{i}\omega}} \right| \leq \frac{1}{\left|\sin\left(\frac{\omega}{2}\right)\right|},$$

the first term in (A.12), i.e., $D_{M,N+1}(\omega)/M$, converges to zero for $M \to \infty$. Additionally, for $M > N+1$, we have

$$\sum_{k=N+1}^{M-1} \frac{|D_{k,N+1}(\omega)|}{k(k+1)} \leq \frac{1}{\left|\sin\left(\frac{\omega}{2}\right)\right|} \sum_{k=N+1}^{M-1} \frac{1}{k(k+1)} < \frac{1}{\left|\sin\left(\frac{\omega}{2}\right)\right| N}.$$

Thus, the sum in (A.12) is convergent for $M \to \infty$, and we obtain

$$\left| \sum_{k=N+1}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\omega k}}{k} \right| \leq \frac{1}{\left|\sin\left(\frac{\omega}{2}\right)\right| N} \tag{A.13}$$

for $\omega \in [-\pi, \pi]$, $\omega \neq 0$, and $N \in \mathbb{N}$. Further, since

$$\sum_{k=1}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\omega k}}{k} = \sum_{k=1}^{\infty} \frac{\cos(k\omega)}{k} + \mathrm{i} \sum_{k=1}^{\infty} \frac{\sin(k\omega)}{k},$$

it follows from 1.441 in [130] that

$$\begin{aligned} -\sum_{k=1}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\omega k}}{k} &= \frac{1}{2}\log(2 - 2\cos(\omega)) + \frac{\mathrm{i}}{2}\operatorname{sgn}(\omega)(|\omega| - \pi) \\ &= \log\left(1 - \mathrm{e}^{\mathrm{i}\omega}\right) \end{aligned}$$

for $\omega \in [-\pi, \pi]$, $\omega \neq 0$. Thus, we have

$$\begin{aligned} \left|\log\left(1 - \mathrm{e}^{\mathrm{i}\omega}\right) + \sum_{k=1}^{N} \frac{\mathrm{e}^{\mathrm{i}\omega k}}{k}\right| &= \left|\sum_{k=N+1}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\omega k}}{k}\right| \\ &< \frac{1}{\left|\sin\left(\frac{\omega}{2}\right)\right| N}, \end{aligned} \tag{A.14}$$

where we used (A.13) in the last inequality.

Next, we have to distinguish two cases. First, we analyze the case $2\sin(1/(2N)) \leq |\omega| \leq \pi$. We have

$$\frac{1}{\left|\sin\left(\frac{\omega}{2}\right)\right| N} \leq \frac{\pi}{|\omega| N} \leq \frac{\pi^2}{2}.$$

Thus, using (A.14), we obtain

$$\begin{aligned} \left|\sum_{k=1}^{N} \frac{\mathrm{e}^{\mathrm{i}\omega k}}{k}\right| &\leq \frac{\pi^2}{2} + \left|\log\left(1 - \mathrm{e}^{\mathrm{i}\omega}\right)\right| \\ &\leq \frac{\pi^2}{2} + \frac{\pi}{2} + \left|\log\left(2\left|\sin\left(\frac{\omega}{2}\right)\right|\right)\right| \end{aligned} \tag{A.15}$$

for $2\sin(1/(2N)) \leq |\omega| \leq \pi$, because

$$\left|\log\left(1 - \mathrm{e}^{\mathrm{i}\omega}\right)\right| \leq \left|\log\left(2\left|\sin\left(\frac{\omega}{2}\right)\right|\right)\right| + \frac{\pi}{2}.$$

The second case is $0 < |\omega| \leq 2\sin(1/(2N))$. We have

$$\left|\sum_{k=1}^{N} \frac{\mathrm{e}^{\mathrm{i}\omega k}}{k}\right| \leq \sum_{k=1}^{N} \frac{1}{k} < 1 + \log(2N).$$

Furthermore, a simple calculation shows that

$$\log(2N) \leq \log(2) + \left|\log\left(2\left|\sin\left(\frac{\omega}{2}\right)\right|\right)\right|,$$

and thus, it follows that

$$\left|\sum_{k=1}^{N}\frac{\mathrm{e}^{\mathrm{i}\omega k}}{k}\right| < 1+\log(2)+\left|\log\left(2\left|\sin\left(\frac{\omega}{2}\right)\right|\right)\right| \tag{A.16}$$

for $0<|\omega|\leq 2\sin(1/(2N))$. Combining (A.15) and (A.16), we have

$$\left|\sum_{k=1}^{N}\frac{\mathrm{e}^{\mathrm{i}\omega k}}{k}\right| \leq \frac{\pi^2}{2}+\frac{\pi}{2}+\left|\log\left(2\left|\sin\left(\frac{\omega}{2}\right)\right|\right)\right|$$

for $\omega\in[-\pi,\pi]$, $\omega\neq 0$, and $N\in\mathbb{N}$. For $\omega=0$ the assertion of the lemma is trivially fulfilled. □

Lemma A.2. *There exists a positive constants C_5, such that*

$$\left|\sum_{k=M}^{N}\frac{\mathrm{e}^{\mathrm{i}\omega k}}{k+\frac{1}{2}}\right| \leq 2\left|\log\left(2\left|\sin\left(\frac{\omega}{2}\right)\right|\right)\right|+C_5.$$

for all $M,N\in\mathbb{Z}$, $M\leq N$, and $\omega\in[-\pi,\pi]$.

Proof of Lemma A.2. Let $M,N\in\mathbb{Z}$, $M\leq N$ arbitrary but fixed. We have

$$\sum_{k=M}^{N}\frac{\mathrm{e}^{\mathrm{i}\omega k}}{k+\frac{1}{2}}-\sum_{\substack{k=M\\k\neq 0}}^{N}\frac{\mathrm{e}^{\mathrm{i}\omega k}}{k}=c_{M,N}-\frac{1}{2}\sum_{\substack{k=M\\k\neq 0}}^{N}\frac{\mathrm{e}^{\mathrm{i}\omega k}}{k\left(k+\frac{1}{2}\right)},$$

where

$$c_{M,N}=\begin{cases}2, & M\leq 0\leq N,\\ 0, & \text{otherwise},\end{cases}$$

and consequently

$$\left|\sum_{k=M}^{N}\frac{\mathrm{e}^{\mathrm{i}\omega k}}{k+\frac{1}{2}}-\sum_{\substack{k=M\\k\neq 0}}^{N}\frac{\mathrm{e}^{\mathrm{i}\omega k}}{k}\right| \leq 2+\frac{1}{2}\sum_{\substack{k=M\\k\neq 0}}^{N}\frac{1}{k\left(k+\frac{1}{2}\right)}\leq 4. \tag{A.17}$$

We proceed with the convention that an empty sum, i.e., a sum where the upper summation index is less than the lower summation index, is zero. Obviously,

$$\begin{aligned}\left|\sum_{\substack{k=M\\k\neq 0}}^{N}\frac{\mathrm{e}^{\mathrm{i}\omega k}}{k}\right| &\leq \left|\sum_{k=1}^{-M}\frac{\mathrm{e}^{\mathrm{i}\omega k}}{k}\right|+\left|\sum_{k=1}^{N}\frac{\mathrm{e}^{\mathrm{i}\omega k}}{k}\right|\\ &\leq 2\left(\left|\log\left(2\left|\sin\left(\frac{\omega}{2}\right)\right|\right)\right|+C_4\right),\end{aligned} \tag{A.18}$$

where we used Lemma A.1 in the last inequality. Combining (A.17) and (A.18) we obtain

$$\left|\sum_{k=M}^{N} \frac{\mathrm{e}^{\mathrm{i}\omega k}}{k+\frac{1}{2}}\right| \leq 2\left(\left|\log\left(2\left|\sin\left(\frac{\omega}{2}\right)\right|\right)\right| + C_4\right) + 4,$$

which completes the proof. □

Now we are in the position to prove Lemma 5.4

Proof of Lemma 5.4. Since the assertion of the lemma is obviously true for $t \in \mathbb{Z}$, we can restrict our analysis to the case $t \in \mathbb{R}\setminus\mathbb{Z}$. Let $\lfloor t \rfloor$ denote the largest integer that is smaller or equal to t. This implies that $|\lfloor t \rfloor + 1/2 - t| < 1/2$. Using the abbreviation

$$q_{M,N}(t,\omega) := \sum_{k=M}^{N} \mathrm{e}^{\mathrm{i}\omega k} \frac{\sin(\pi(t-k))}{\pi(t-k)},$$

we have

$$\begin{aligned}\left|\frac{q_{M,N}(t,\omega)}{\sin(\pi t)} - \frac{q_{M,N}(\lfloor t \rfloor + \frac{1}{2},\omega)}{\sin(\pi(\lfloor t \rfloor + \frac{1}{2}))}\right| &= \left|\frac{1}{\pi}\sum_{k=M}^{N} \mathrm{e}^{\mathrm{i}\omega k}(-1)^k \left(\frac{1}{t-k} - \frac{1}{\lfloor t \rfloor + \frac{1}{2} - k}\right)\right| \\ &\leq C_6 + \frac{1}{|t - \lfloor t \rfloor|} + \frac{1}{|t - \lfloor t+1 \rfloor|}.\end{aligned}$$

It follows that

$$\begin{aligned}|q_{M,N}(t,\omega)| &\leq C_6|\sin(\pi t)| + \frac{|q_{M,N}(\lfloor t \rfloor + \frac{1}{2},\omega)||\sin(\pi t)|}{|\sin(\pi(\lfloor t \rfloor + \frac{1}{2}))|} + \frac{|\sin(\pi t)|}{|t - \lfloor t \rfloor|} + \frac{|\sin(\pi t)|}{|t - \lfloor t+1 \rfloor|} \\ &\leq C_6 + |q_{M,N}(\lfloor t \rfloor + \tfrac{1}{2},\omega)| + 2. \qquad \text{(A.19)}\end{aligned}$$

Furthermore, we have

$$|q_{M,N}(\lfloor t \rfloor + \tfrac{1}{2},\omega - \pi)| = \left|\frac{1}{\pi}\sum_{k=\lfloor t \rfloor - N}^{\lfloor t \rfloor - M} \frac{\mathrm{e}^{\mathrm{i}\omega k}}{k+\frac{1}{2}}\right| \leq \frac{2}{\pi}\left|\log\left(2\left|\sin\left(\frac{\omega}{2}\right)\right|\right)\right| + C_7,$$

where the last inequality follows by Lemma A.2. Using (5.10) it follows that

$$|q_{M,N}(\lfloor t \rfloor + \tfrac{1}{2},\omega)| \leq \frac{2}{\pi}\left|\log\left(2\left|\cos\left(\frac{\omega}{2}\right)\right|\right)\right| + C_7.$$

and, using (A.19), that

$$|q_{M,N}(t,\omega)| \leq \frac{2}{\pi}\left|\log\left(2\left|\cos\left(\frac{\omega}{2}\right)\right|\right)\right| + C_2,$$

which completes the proof. □

A.10 Proof of Lemma 5.6

Proof. Let $0 < \delta < \pi$ and $\tau > 0$ be arbitrary but fixed. From the identity

$$\frac{\sin(\pi(t-k))}{\pi(t-k)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{e}^{-\mathrm{i}\omega k}\;\mathrm{d}\omega$$

we see that

$$\frac{\sin(\pi(t-k))}{\pi(t-k)}, \qquad k \in \mathbb{Z}$$

are the Fourier coefficients of the function $g(\omega, t)$ that is 2π-periodic in ω and identical to $\mathrm{e}^{\mathrm{i}\omega t}$ for $-\pi \le \omega < \pi$, where $|t| \le \tau$ is a parameter. Now, if

$$A_{M,N}(\omega,t) := \sum_{k=-M}^{N} \mathrm{e}^{\mathrm{i}\omega k}\,\frac{\sin(\pi(t-k))}{\pi(t-k)}, \qquad M, N \in \mathbb{N}$$

converges for $M, N \to \infty$ to a function $\tilde{g}(\omega, t)$ for $|t| \le \tau$ and $|\omega| \le \pi - \delta$, then $\tilde{g}(\omega, t) = g(\omega, t) = \mathrm{e}^{\mathrm{i}\omega t}$ for $|t| \le \tau$ and $|\omega| \le \pi - \delta$ by the representation theorem for Fourier series and the fact that $\mathrm{e}^{\mathrm{i}\omega t}$ is continuous differentiable in $[-\pi + \delta, \pi - \delta]$. It remains to show that $A_{M,N}(\omega, t)$ is uniformly convergent with respect to t and ω for $|t| \le \tau$ and $|\omega| \le \pi - \delta$ as $M, N \to \infty$.

In the following analysis we always assume that $|t| \le \tau$ and $|\omega| \le \pi - \delta$. For $N_1, N, M_1, M \in \mathbb{N}$ with $N_1 > N > \tau$ and $M_1 > M > \tau$ we have

$$\begin{aligned} &|A_{M_1,N_1}(\omega,t) - A_{M,N}(\omega,t)| \\ &\quad\le \left|\sum_{k=-M_1}^{-M-1} \mathrm{e}^{\mathrm{i}\omega k}\,\frac{\sin(\pi(t-k))}{\pi(t-k)}\right| + \left|\sum_{k=N+1}^{N_1} \mathrm{e}^{\mathrm{i}\omega k}\,\frac{\sin(\pi(t-k))}{\pi(t-k)}\right|. \end{aligned} \tag{A.20}$$

It is sufficient to analyze the case $t \notin \mathbb{Z}$, because $A_{M_1,N_1}(\omega, t) - A_{M,N}(\omega, t) = 0$ for $t \in \mathbb{Z}$, $|t| \le \tau$. The second term on the right-hand side of (A.20) can be bounded from above by

$$\left|\sum_{k=N+1}^{N_1} \mathrm{e}^{\mathrm{i}\omega k}\,\frac{\sin(\pi(t-k))}{\pi(t-k)}\right| \le \left|\sum_{k=N+1}^{N_1} \frac{\mathrm{e}^{\mathrm{i}(\omega+\pi)k}}{\pi(t-k)}\right|. \tag{A.21}$$

Defining $D_{k,N+1}(\omega) = \sum_{l=N+1}^{k} \mathrm{e}^{\mathrm{i}(\omega+\pi)l}$ and using summation by parts in the same way as in (A.12), we have

$$\sum_{k=N+1}^{N_1} \frac{\mathrm{e}^{\mathrm{i}(\omega+\pi)k}}{k-t} = \frac{D_{N+1,N+1}}{N_1 - t} + \sum_{k=N+1}^{N_1-1} \frac{D_{k,N+1}(\omega)}{(k-t)(k+1-t)},$$

and since $|D_{k,N+1}(\omega)| \le 1/\left|\sin\left(\frac{\omega+\pi}{2}\right)\right|$, we obtain

$$\left|\sum_{k=N+1}^{N_1} \frac{\mathrm{e}^{\mathrm{i}(\omega+\pi)k}}{k-t}\right| \le \frac{1}{\left|\sin\left(\frac{\omega+\pi}{2}\right)\right|}\left(\frac{1}{N_1-t}+\sum_{k=N+1}^{N_1-1}\frac{1}{(k-t)(k+1-t)}\right)$$
$$\le \frac{1}{\sin\left(\frac{\delta}{2}\right)}\left(\frac{1}{N-\tau}+\sum_{k=N+1}^{\infty}\frac{1}{(k-\tau)^2}\right). \tag{A.22}$$

The right-hand side of (A.22) converges to zero for $N \to \infty$. Thus, combining (A.21) and (A.22) we see that for all $\epsilon > 0$ there exists a natural number $N_0 = N_0(\epsilon, \delta, \tau)$ such that

$$\left|\sum_{k=N+1}^{N_1} \mathrm{e}^{\mathrm{i}\omega k}\frac{\sin(\pi(t-k))}{\pi(t-k)}\right| < \frac{\epsilon}{2}$$

for all $N \ge N_0$, $N_1 > N$, $|\omega| \le \pi - \delta$, and $|t| \le \tau$. The first term on the right-hand side of (A.20) can be treated in the same way.

Consequently, for all $\epsilon > 0$, $\tau > 0$, and $0 < \delta < \pi$ there exist two natural numbers $N_0 = N_0(\epsilon, \delta, \tau)$ and $M_0 = M_0(\epsilon, \delta, \tau)$ such that for all $|\omega| \le \pi - \delta$ and $|t| \le \tau$ we have

$$|A_{M_1,N_1}(\omega,t) - A_{M,N}(\omega,t)| < \epsilon$$

for all $N \ge N_0$, $N_1 > N$, and $M \ge M_0$, $M_1 > M$. It follows that

$$\lim_{N,M\to\infty} \max_{|t|\le\tau} \max_{|\omega|\le\pi-\delta} \left|\mathrm{e}^{\mathrm{i}\omega t} - \sum_{k=-M}^{N} \mathrm{e}^{\mathrm{i}\omega k}\frac{\sin(\pi(t-k))}{\pi(t-k)}\right| = 0,$$

and, therefore, the assertion of the lemma is proved. □

A.11 Proof of $\|h^{+}(\,\cdot\,,0,N)\|_{\mathcal{PW}_\pi^1} < 3$

Proof. The Fourier coefficients $F_N(k)$, $k \in \mathbb{Z}$, of the Fejér kernel

$$\hat{F}_N(\omega) = \frac{1}{N+1}\frac{\sin^2((N+1)\frac{\omega}{2})}{\sin^2(\frac{\omega}{2})}$$

are given by

$$F_N(k) = \begin{cases} 1 - \frac{|k|}{N}, & |k| < N, \\ 0, & |k| \ge N. \end{cases}$$

Thus, $h^{+}(k,0,N) = 2F_{2N}(k) - F_N(k)$, $k \in \mathbb{Z}$ and the Fourier transform of

$$h^{+}(t,0,N) = \sum_{k=-2N+1}^{2N-1} h^{+}(k,0,N)\frac{\sin(\pi(t-k))}{\pi(t-k)}$$

is

$$\hat{h}^+(\omega, 0, N) = 2\hat{F}_{2N}(\omega) - \hat{F}_N(\omega), \quad |\omega| \le \pi.$$

As a consequence we obtain

$$\begin{aligned}\|h^+(\,\cdot\,, 0, N)\|_{\mathcal{PW}_\pi^1} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |2\hat{F}_{2N}(\omega) - \hat{F}_N(\omega)| \,\mathrm{d}\omega \\ &< \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\hat{F}_{2N}(\omega) + \hat{F}_N(\omega) \,\mathrm{d}\omega = 3.\end{aligned}$$

In the last line we can write "$<$" instead of "$\le$" because $\hat{F}_{2N}$ and $\hat{F}_N$ are both non-negative. □

B

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