Low-dimensional partial integro-differential equations for high-dimensional Asian options

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Asian options on a single asset under a jump-diffusion model can be priced by solving a partial integro-differential equation (PIDE). We consider the more challenging case of an option whose payoff depends on a large number (or even a continuum) of assets. Possible applications include options on a stock basket index and electricity contracts with a delivery period. Both of these can be modeled with an exponential, time-inhomogeneous, Hilbert space valued jump-diffusion process. We derive the corresponding high- or even infinite-dimensional PIDE for Asian option prices in this setting and show how to approximate it with a low-dimensional PIDE. To this end, we employ proper orthogonal decomposition (POD) to reduce the dimension. We generalize the convergence results known for European options to the case of Asian options and give an estimate for the approximation error.

1 Introduction

It is well known that the price of an Asian option on a single asset driven by a geometric Brownian motion is the solution of a partial differential equation [15]. This equation depends on two space variables, the value of the underlying and its average up to the current time. If we add jumps to the model, we obtain an additional integral term which yields a partial integro-differential equation (PIDE). In fact, there are several ways to derive such a PIDE. Using clever parametrizations, it is possible to obtain a PIDE with only one space variable [18].

The PIDEs corresponding to Asian options in general cannot be solved analytically. They are, however, the basis for numerical pricing methods. Using appropriate algorithms, the PIDEs can be solved in a numerically stable way, see [19, 6] and the

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references therein. For an overview of methods for pricing Asian options, we refer to [17].

In the present article, we consider arithmetic average Asian options depending on more than one underlying asset. More precisely, we will use the time-inhomogeneous, Hilbert space valued jump-diffusion model introduced in [10]. This is a quite general approach suitable for a wide range of applications. We may, e.g., price Asian options written on an index depending on a large basket of stocks. In this case, we would choose the Hilbert space to be finite-dimensional, the dimension equals to the number of stocks. There are, however, also markets in which the option depends on a continuum of assets. This happens, among others, in electricity markets. Electricity option payoffs depend on the forward curve of prices which can be modeled with a function-valued process [8]. We discuss our model and the driving stochastic process, which is applicable to both stock baskets and electricity contracts, in Section 2.

Introducing the arithmetic average as an additional space variable, the option price can be written as a function of time, the average value, and the Hilbert space valued variable describing the state of the underlying assets. This is a high-dimensional (possibly infinite-dimensional) object. The main objective of this article is to derive a low-dimensional PIDE which approximates the option price. To this end, we generalize the dimension reduction method for European options presented in [9] to Asian options. The reduction is based on proper orthogonal decomposition (POD) and uses a similar idea as principal component analysis. In Section 3, we first describe the POD method for Asian options in detail. Then, we derive the low-dimensional PIDE satisfied by the approximated price process. We show convergence of the PIDE solution to the true value of the Asian option in Theorem 3.9, which is the main result of this paper. The numerical solution of the PIDE is beyond the scope of this article. This will be a topic for future research. All the results presented here are also applicable to European options as a special case.

2 Hilbert Space Valued Jump-Diffusion

In this section, we state our market model. We first define the driving stochastic process, a time-inhomogeneous Hilbert space valued jump-diffusion. Then, we construct the exponential of this process, which we will use to model the underlying assets. Finally, we discuss the payoff of an Asian option.

2.1 Driving Stochastic Process

Since we consider Hilbert space valued processes, we will make use of infinite-dimensional stochastic analysis. For a definition of integrals with respect to Hilbert space valued Brownian motion see, e.g., [5, 11]. An overview of Poisson random measures in Hilbert spaces can be found in [7], the case of Lévy processes is treated in [13].

Let $(D, \mathcal{F}_D, \mu_D)$ be a finite measure space. We consider the separable Hilbert space

$$H := L^2(D; \mu_D).$$

For every $h \in H$, we denote the corresponding norm by

$$\|h\|_H := \sqrt{\int_D \left[h(u)\right]^2 \mu_D(u)}.$$

This is the state space for the underlying assets of the Asian option. To model, e.g., a basket of stocks, we could choose a discrete set D, with $\|\cdot\|_H$ denoting the Euclidean norm. For a continuum of assets, on the other hand, we may consider a compact interval $D \subset \mathbb{R}$ and the Lebesgue measure μ_D.

We assume that our model is stated under the risk neutral measure. The driving stochastic process for our model is the H-valued process

$$X_t := \int_0^t \gamma_s \, ds + \int_0^t \sigma_s \, dW(s) + \int_0^t \int_H \eta_s(\xi) \, \widetilde{M}(d\xi, ds), \quad t \geq 0. \tag{1}$$

The diffusion part is driven by an H-valued Wiener process W whose covariance is a symmetric nonnegative definite trace class operator Q. The jumps are characterized by $\widetilde{M}$, the compensated random measure of an H-valued compound Poisson process

$$J_t = \sum_{i=1}^{N_t} Y_i, \quad t \geq 0,$$

which is independent of W. Here, N denotes a Poisson process with intensity λ and $Y_i \sim P^Y$ ($i = 1, 2, \ldots$) are iid on H (and independent of N). The corresponding Lévy measure is denoted by $\nu = \lambda P^Y$. We denote by $\mathrm{L}(H, H)$ the space of all bounded linear operators on H. We assume the drift $\gamma : [0, T] \to H$, the volatility $\sigma : [0, T] \to \mathrm{L}(H, H)$, and the jump dampening factor $\eta : [0, T] \to \mathrm{L}(H, H)$ to be deterministic functions. Let further $(\Omega, (\mathcal{F}_t)_{t\in[0,T]})$ be the filtered measurable space on which the risk neutral measure is defined, with the natural filtration $(\mathcal{F}_t)_{t\in[0,T]}$ generated by X. We make the following assumption, which is similar to the finite-dimensional moment conditions in [16, sec. 25].

Assumption 2.1. *We assume that the second exponential moment of the jump distribution Y exists:*

$$E[e^{2\|Y\|_H}] = \int_H e^{2\|\xi\|_H} P^Y(d\xi) < \infty.$$

We assume further $\gamma \in \mathrm{L}^2(0, T; H)$, $\sigma \in \mathrm{L}^2(0, T; \mathrm{L}(H, H))$, and

$$\|\eta_t\|_{\mathrm{L}(H,H)} \leq 1 \quad \textit{for every } t \in [0, T].$$

In a finite-dimensional setting ($\dim H < \infty$), the value of each underlying asset at time $t \geq 0$ is modeled by the exponential of one component of the driving process X,

$$S_i(t) = S_i(0) \, e^{X_i(t)} \in \mathbb{R}, \quad i = 1, \ldots, \dim H, \tag{2}$$

where $S_i(0) \in \mathbb{R}$ denotes the initial value. For a generalization of the exponential to an infinite-dimensional Hilbert space, let $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of H. We then define

$$S_t := \sum_{k\in\mathbb{N}} \langle S_0, e_k\rangle_H \, e^{\langle X_t, e_k\rangle_H} \, e_k \in H, \tag{3}$$

for $t > 0$, with the initial value $S_0 \in H$. While it might not be obvious that S_t is an element of H again, this is indeed a consequence of Assumption 2.1, see [8, Thm. 2.2]. Note that this definition reproduces (2) in the finite-dimensional case, if we choose e_i to be standard unit vectors.

2.2 Value of an Asian Option

Before we can define the value of an arithmetic average Asian option, we need to clarify what exactly *average* is supposed to mean in our Hilbert space valued setting. Consider the application of our model to a basket of stocks. An index on such a basket is basically a weighted sum of the individual stock values. The Asian option is then written on the time-average of this sum. The weight factors are nothing more than a linear mapping working on the vector of asset prices. More generally, we consider an arbitrary bounded linear mapping $w : H \to \mathbb{R}$, which we identify with $w \in H$ by the representation theorem of Fréchet–Riesz. The arithmetic average up to time $t > 0$ is then given by

$$A_t := \frac{1}{t} \int_0^t \langle w, S_u\rangle_H \, du \in \mathbb{R}. \tag{4}$$

Using the Jensen inequality, the Cauchy–Schwarz inequality, and Fubini's theorem, we obtain

$$E\left[A_t^2\right] = \frac{1}{t^2} E\left[\left(\int_0^t \langle w, S_u\rangle_H \, du\right)^2\right] \le \frac{1}{t^2} \|w\|_H^2 \int_0^t E\,\|S_u\|_H^2 \, du.$$

This expression is finite by [8, Thm. 2.2]. Hence, the average is a well defined random variable in $L^2(\Omega)$ for $t > 0$. The defining equation (4) is, however, not valid for $t = 0$. Intuitively,

$$A_0 := \langle w, S_0\rangle_H \tag{5}$$

is the obvious continuation for A. The following theorem shows that this is indeed the correct choice.

Proposition 2.2. *The following convergence holds almost surely:*

$$\lim_{t\to 0} A_t = \langle w, S_0\rangle_H \, .$$

Proof. Using the definition of A, we find

$$|A_t - \langle w, S_0\rangle_H| \leq \frac{1}{t}\int_0^t |\langle w, S_u - S_0\rangle_H|\, du. \tag{6}$$

In order to find a bound for $\langle w, S_u - S_0\rangle_H$, we consider the driving process X. From the proof of [10, Thm. 2.2], we know that

$$E\,\|X_t\|_H^2 \leq \int_0^t \Big(\|\gamma_s\|_H^2 + (\operatorname{tr} Q)\ \|\sigma_s\|^2_{\mathrm{L}(H,H)} + C\int_H \|\eta_s(\xi)\|_H^2\ \nu(d\xi)\Big) ds.$$

Thus, $\lim_{t\to 0}\|X_t\|_H = 0$ in $\mathrm{L}^2(\Omega)$. Consequently, there is a sequence $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ satisfying $\lim_{n\to\infty} t_n = 0$ such that almost surely

$$\lim_{n\to\infty}\|X_{t_n}\|_H = 0.$$

Moreover, almost surely there exists $\delta > 0$ such that the path of X is continuous in $[0,\delta)$. Consequently, we have almost surely

$$\lim_{t\to 0}\|X_t\|_H = 0.$$

Due to the Cauchy–Schwarz inequality, this yields almost surely $\lim_{t\to 0}\langle X_t, e_k\rangle_H = 0$ and thus

$$\lim_{t\to 0} e^{\langle X_t, e_k\rangle_H} = 1$$

uniformly in k. Hence, we have almost surely

$$|\langle w, S_t - S_0\rangle_H| = \sum_{k\in\mathbb{N}} \langle S_0, e_k\rangle_H\ \langle w, e_k\rangle_H \left(e^{\langle X_t, e_k\rangle_H} - 1\right) \to 0 \quad \text{for } t\to 0.$$

We apply this limit to (6) and the proof is complete. □

Let $T > 0$ be the maturity of an Asian option. By definition, the value of the option depends on A_T. In addition, it may depend on the state S_T of the underlying at maturity, e.g., in the case of a floating strike. The state S_T in turn is a function of the driving process X_T, defined in (3). It turns out that in view of the dimension reduction methods which we will discuss in Section 3 it is useful to introduce the centered process

$$Z_t := X_t - E[X_t], \quad t \geq 0. \tag{7}$$

Hence, $S_t = S_t(Z_t)$ is completely determined by Z_t. We can write it as the function

$$S_t : \begin{cases} H & \to H, \\ z & \mapsto \sum_{k\in\mathbb{N}} \langle S_0, e_k\rangle_H\ e^{\left\langle \int_0^t \gamma(u)\,du + z, e_k\right\rangle_H} e_k. \end{cases} \tag{8}$$

We denote the value of the option at time $t \in [0,T]$, discounted to time 0, by

$$\widehat{V}(t,z,a) := e^{-rT} E[G(Z_T, A_T) | Z_t = z, A_t = a] \quad \text{for every } z \in H, a \in \mathbb{R}. \tag{9}$$

This is the conditional expectation of the payoff $G : H \times \mathbb{R} \to \mathbb{R}$ at maturity T given the current state $z \in H$ of the underlying assets and the average $a \in \mathbb{R}$. We make the following assumption concerning the payoff.

Assumption 2.3. *We assume that there are constants L_z^G and L_a^G such that the payoff function G satisfies the Lipschitz conditions*

$$\begin{aligned} |G(z_1,a) - G(z_2,a)| &\le L_z^G \, \|z_1 - z_2\|_H \quad \text{for every } z_1, z_2 \in H, \ a \in \mathbb{R}, \\ |G(z,a_1) - G(z,a_2)| &\le L_a^G \, |a_1 - a_2| \qquad \text{for every } z \in H, \ a_1, a_2 \in \mathbb{R}. \end{aligned}$$

Note that this assumption is satisfied, e.g., for Asian call and put options on A_T with fixed or floating strike.

Similar to the finite-dimensional case, the option value $\widehat{V}$ satisfies a PIDE. In order to derive this PIDE in the Hilbert space valued setting, we need H-valued generalizations of two concepts: covariances and derivatives. Covariance matrices are replaced by covariance operators which can be interpreted as possibly infinite dimensional matrices. By [10, Thm. 2.4],

$$\mathcal{C}_{X_T} : \begin{cases} H & \to H', \\ h & \mapsto E\left[\langle X_T - E[X_T], h \rangle_H \, \langle X_T - E[X_T], \cdot \rangle_H \right] \end{cases} \tag{10}$$

is a well defined, symmetric, nonnegative definite trace class operator (and thus compact). We are particularly interested in the subspace of H where $\mathcal{C}_{X_T}$ is strictly positive definite, i.e., the orthogonal complement of its kernel. We denote this space by $E_0(\mathcal{C}_{X_T})^\perp$ (E_0 denoting the eigenspace corresponding to eigenvalue 0).

Furthermore, we denote by $D_z \widehat{V}(t,z,a) \in \mathrm{L}(H,\mathbb{R})$ the Fréchet derivative of $\widehat{V}$ at $(t,z,a) \in [0,T] \times H \times \mathbb{R}$ with respect to z. The second derivative is $D_z^2 \widehat{V}(t,z,a) \in \mathrm{L}(H,H)$. The derivatives are continuous linear operators such that for every $t \in [0,T]$, $z \in H$, and $a \in \mathbb{R}$ we have

$$\widehat{V}(t, z + \zeta, a) = \widehat{V}(t,z,a) + [D_z \widehat{V}(t,z,a)](\zeta) + \frac{1}{2} \left\langle [D_z^2 \widehat{V}(t,z,a)](\zeta), \zeta \right\rangle_H + o(\|\zeta\|_H^2)$$

for every $\zeta \in H$. It is often convenient to identify $D_z^2 \widehat{V}(t,z,a)$ with a bilinear form on $H \times H$, setting

$$[D_z^2 \widehat{V}(t,z,a)](\zeta_1, \zeta_2) := \left\langle [D_z^2 \widehat{V}(t,z,a)](\zeta_1), \zeta_2 \right\rangle_H \quad \text{for every } \zeta_1, \zeta_2 \in H.$$

The one-dimensional partial derivatives of $\widehat{V}$ with respect to time and average are denoted by $\partial_t \widehat{V}$ and $\partial_a \widehat{V}$, respectively. We can now state the Hilbert space valued PIDE for $\widehat{V}$. We denote the trace operator by $\mathrm{tr}(\cdot)$, and the adjoint operator of $\sigma_t \in \mathrm{L}(H,H)$ by σ_t^*.

Theorem 2.4. *Suppose that the discounted price $\widehat{V}$ defined in (9) is continuously differentiable with respect to t and twice continuously differentiable with respect to z and a. Moreover, assume that the second derivative with respect to z restricted to an arbitrary bounded subset of H is a uniformly continuous mapping to the Hilbert–Schmidt space $\mathrm{L}_{\mathrm{HS}}(H,H)$. Then $\widehat{V}$ is a classical solution of the PIDE*

$$
\begin{aligned}
-\partial_t \widehat{V}(t,z,a) &= \frac{1}{2}\operatorname{tr}\left(D_z^2 \widehat{V}(t,z,a)\sigma_t Q \sigma_t^*\right) + \frac{1}{t}\left(\langle w, S_t(z)\rangle_H - a\right)\partial_a \widehat{V}(t,z,a) \\
&\quad + \int_H \left[\widehat{V}(t,z+\eta_t(\zeta),a) - \widehat{V}(t,z,a) - \left[D_z \widehat{V}(t,z,a)\right]\eta_t(\zeta)\right]\nu(d\zeta)
\end{aligned}
\tag{11}
$$

with terminal condition

$$\widehat{V}(T,z,a) = e^{-rT} G(z,a)$$

for every $t \in (0,T)$, $z \in E_0(\mathcal{C}_{X_T})^{\perp}$, and $a \in \mathbb{R}$.

Proof. The proof is very similar to the one of [8, Thm. 4.5]. Applying Itô's formula for Hilbert space valued processes [13, Thm. D.2] to $\widehat{V}(t, Z_t, A_t)$, $t > 0$, yields

$$
\begin{aligned}
&\widehat{V}(t,Z_t,A_t) = \\
&\quad \widehat{V}(0,Z_0,A_0) + \int_0^t \partial_t \widehat{V}(u-,Z_{u-},A_{u-})\,du + \int_0^t D_z \widehat{V}(u-,Z_{u-},A_{u-})\,dZ_u \\
&\quad + \int_0^t \partial_a \widehat{V}(u-,Z_{u-},A_{u-})\,dA_u + \frac{1}{2}\int_0^t D_z^2 \widehat{V}(u-,Z_{u-},A_{u-})\,d[Z,Z]_u^c \\
&\quad + \sum_{0\le u\le t}\left[\widehat{V}(u,Z_u,A_u) - \widehat{V}(u-,Z_{u-},A_{u-}) - \left[D_z \widehat{V}(u-,Z_{u-},A_{u-})\right](Z_u - Z_{u-})\right],
\end{aligned}
\tag{12}
$$

where $[Z,Z]^c$ denotes the continuous part of the square bracket process as defined in [13]. Note that the average process A is continuous and of finite variation. Hence, the jump part of the equation does not contain the partial derivative $\partial_a \widehat{V}$. For the same reason, the square bracket processes $[A,A]$ and $[A,Z]$ do not occur in the equation.

We first simplify the covariation term. By the properties of quadratic variations for real-valued processes and [5, Cor. 4.14], we obtain

$$
\begin{aligned}
[Z,Z]_t^c &= \sum_{i,j\in\mathbb{N}} e_i \otimes e_j \left[X_i^c, X_j^c\right]_t \\
&= \sum_{i,j\in\mathbb{N}} e_i \otimes e_j \left[\left\langle \int_0^\cdot \sigma_u\,dW_u, e_i\right\rangle_H, \left\langle \int_0^\cdot \sigma_u\,dW_u, e_j\right\rangle_H\right]_t \\
&= \sum_{i,j\in\mathbb{N}} e_i \otimes e_j \left(\int_0^t \left\langle [\sigma_u Q \sigma_u^*] e_j, e_i\right\rangle_H du\right),
\end{aligned}
$$

where $e_i \otimes e_j$ denotes the tensor product of the two basis elements (compare also the proof of [8, Lemma 4.4]). Thus, we get

$$\begin{aligned}
&\int_0^t D_z^2 \widehat{V}(u-, Z_{u-}, A_{u-})\, d[Z,Z]_u^c \\
&\quad = \int_0^t \sum_{i,j\in\mathbb{N}} \left[D_z^2 \widehat{V}(u-, Z_{u-}, A_{u-})\right](e_i, e_j) \left\langle [\sigma_u Q \sigma_u^*] e_j, e_i \right\rangle_H du \\
&\quad = \int_0^t \sum_{j\in\mathbb{N}} \left[D_z^2 \widehat{V}(u-, Z_{u-}, A_{u-})\right] \Big([\sigma_u Q \sigma_u^*] e_j, e_j \Big)\, du \\
&\quad = \int_0^t \operatorname{tr} \Big(D_z^2 \widehat{V}(u-, Z_{u-}, A_{u-}) \sigma_u Q \sigma_u^* \Big)\, du.
\end{aligned}$$

Next we calculate dA_u. By definition (4) of A we have

$$\langle w, S_u \rangle_H\, du = d(uA_u) = A_u du + u dA_u.$$

Hence, we obtain

$$dA_u = \frac{1}{u} \left(\langle w, S_u(Z_u) \rangle_H - A_u \right) du.$$

Finally, we reorganize the jump terms in (12) exactly in the same way as in the proof of [8, Lemma 4.4]. The result is

$$\begin{aligned}
&d\widehat{V}(t, Z_t, A_t) = \\
&\quad \partial_t \widehat{V}(t-, Z_{t-}, A_{t-}) dt + \frac{1}{2} \operatorname{tr} \Big(D_z^2 \widehat{V}(t-, Z_{t-}, A_{t-}) \sigma_t Q \sigma_t^* \Big)\, dt \\
&\quad + \frac{1}{t} \left(\langle w, S_t(Z_{t-}) \rangle_H - A_{t-} \right) \partial_a \widehat{V}(t-, Z_{t-}, A_{t-})\, dt \\
&\quad + \int_H \Big[\widehat{V}(t, Z_{t-} + \eta_t(\zeta), A_{t-}) - \widehat{V}(t-, Z_{t-}, A_{t-}) - D_z \widehat{V}(t-, Z_{t-}, A_{t-}) \eta_t(\zeta) \Big]\, \nu(d\zeta)\, dt \\
&\quad + D_z \widehat{V}(t-, Z_{t-}, A_{t-}) \sigma_t\, dW_t \\
&\quad + \int_H \Big[\widehat{V}(t, Z_{t-} + \eta_t(\zeta), A_{t-}) - \widehat{V}(t-, Z_{t-}, A_{t-}) \Big]\, \widetilde{M}(d\zeta, dt).
\end{aligned}$$

The last two summands in this equation are local martingales by definition of the stochastic integral [13, Thms. 8.7,8.23]. Due to the fact that continuous local martingales of finite variation are almost surely constant [14, Ch. II,Thm. 27], the sum of the remaining integral terms must equal 0. This yields the PIDE. □

3 Approximate Pricing with POD

The PIDE derived in the previous section depends on H-valued objects. In order to obtain a lower-dimensional equation which allows for a numerical solution, we reduce the dimension using POD. The basic idea is to find a small set of orthonormal vectors in H which allow for an accurate approximation of the state S_t of the underlying assets for every $t \in [0, T]$. The POD method has been discussed in [9] in the context of European options. We generalize the approach to Asian options. In particular, we state an error estimate for the solution of the approximating equation.

3.1 POD for the Driving Process

We start with an approximation of the centered driving process Z at maturity $T > 0$.

Definition 3.1. *A sequence of orthonormal elements $\{p_l\}_{l \in \mathbb{N}} \subset H$ is called a* POD-basis *for Z_T, if it solves the minimization problem*

$$\min_{\langle p_i, p_j \rangle_H = \delta_{ij}} E \left\| Z_T - \sum_{l=1}^{d} p_l \langle Z_T, p_l \rangle_H \right\|_H^2$$

for every $d \in \mathbb{N}$.

In other words, a POD basis is a set of deterministic orthonormal functions such that we expect the projection of the random vector $Z_T = X_T - E[X_T] \in H$ onto the first d elements of this basis to be a good approximation. Projecting to a POD basis is equivalent to using the partial sum of the first d elements of a Karhunen–Loève expansion, which itself is closely connected to the eigenvector problem of the covariance operator $\mathcal{C}_{X_T}$ defined in (10). The following proposition is quoted from [10, Thm. 3.3]. It shows that the eigenvectors of $\mathcal{C}_{X_T}$ are indeed a POD basis.

Proposition 3.2. *Every sequence of orthonormal eigenvectors $(p_l)_{l \in \mathbb{N}}$ of the operator $\mathcal{C}_{X_T}$, ordered by descending size of the corresponding eigenvalues $\mu_1 \geq \mu_2 \geq ... \geq 0$, solves the maximization problem*

$$\max_{\langle p_i, p_j \rangle_H = \delta_{ij}} \sum_{l=1}^{d} \langle \mathcal{C}_{X_T} p_l, p_l \rangle_H$$

for every $d \in \{1, 2, \ldots, \dim H\}$. The maximum value is

$$\sum_{l=1}^{d} \langle \mathcal{C}_{X_T} p_l, p_l \rangle_H = \sum_{l=1}^{d} \mu_l.$$

Moreover, the eigenvectors are a POD basis in the sense of Definition 3.1, and the expectation of the projection error is

$$E \left\| Z_T - \sum_{l=1}^{d} p_l \langle Z_T, p_l \rangle_H \right\|_H^2 = \sum_{l=d+1}^{\dim H} \mu_l. \tag{13}$$

Subsequently, let $(p_l)_{l \in \mathbb{N}}$ and $(\mu_l)_{l \in \mathbb{N}}$ denote the orthonormal basis and eigenvalues from Proposition 3.2. Further, let

$$U_d := \operatorname{span}\{p_1, p_2, \ldots, p_d\} \subset H$$

be the d-dimensional subspace spanned by the eigenvectors corresponding to the largest eigenvalues. We will assume that $\mu_1 \geq \ldots \geq \mu_d > 0$, as there is no need to include eigenvectors of the covariance operator corresponding to eigenvalue 0. We define the projection operator

$$\mathcal{P}_d : \begin{cases} H & \to U_d \cong \mathbb{R}^d, \\ z & \mapsto \sum_{l=1}^{d} \langle z, p_l \rangle_H \, p_l. \end{cases}$$

Hence, we can rewrite (13) as

$$E \big\| Z_T - \mathcal{P}_d Z_T \big\|_H^2 = \sum_{l=d+1}^{\dim H} \mu_l.$$

Whenever necessary, we will identify U_d with $\mathbb{R}^d$ via the isometry

$$\iota : \begin{cases} \big(U_d, \|\cdot\|_H \big) & \to \big(\mathbb{R}^d, \|\cdot\| \big), \\ x & \mapsto \big(\langle x, p_l \rangle_H \big)_{l=1}^{d}. \end{cases}$$

So far, we have approximated the value of Z only at time T. It turns out, however, that this is indeed sufficient to obtain small projection errors for arbitrary $t \in [0, T]$.

Proposition 3.3. *Let Z be the centered jump-diffusion defined in (7). For every $t \in [0, T]$, we have*

$$E \left[\| Z_t - \mathcal{P}_d Z_t \|_H^2 \right] \leq \sum_{l=d+1}^{\dim H} \mu_l. \tag{14}$$

Proof. This is a direct consequence of the independent increments of Z. Using the Pythagorean theorem, we obtain

$$\begin{aligned} E \, \| Z_T - \mathcal{P}_d Z_T \|_H^2 &= E \, \| Z_t - \mathcal{P}_d Z_t + (Z_T - Z_t) - \mathcal{P}_d (Z_T - Z_t) \|_H^2 \\ &= E \, \| Z_t - \mathcal{P}_d Z_t \|_H^2 + E \, \| (Z_T - Z_t) - \mathcal{P}_d (Z_T - Z_t) \|_H^2 \\ &\geq E \, \| Z_t - \mathcal{P}_d Z_t \|_H^2 . \end{aligned}$$

Applying Proposition 3.2 yields (14). □

Consequently, it is not necessary to change Definition 3.1 in order to approximate the whole path Z_t, $t \in [0, T]$. This is due to the fact that by approximating Z_T, we obviously capture also the events up to time T. In the time-homogeneous case, we even obtain the following t-dependent equality.

Proposition 3.4. *Let Z be the centered jump-diffusion defined in (7). Suppose Z is a time-homogeneous jump-diffusion process, i.e., σ and η in (1) do not depend on t. For every $t \in [0, T]$, we then have*

$$E \, \| Z_t - \mathcal{P}_d Z_t \|_H^2 = \frac{t}{T} \sum_{l=d+1}^{\dim H} \mu_l. \tag{15}$$

Proof. Due to i.i.d. increments, the covariance operator of $Z(t)$ is given by

$$\mathcal{C}_{X_t} = \frac{t}{T}\mathcal{C}_{X_T}.$$

Hence, the eigenpairs of $\mathcal{C}_{X_t}$ are given by $(\frac{t}{T}\mu_l, p_l)$, $l \in \mathbb{N}$. Applying Proposition 3.2 (setting $T = t$) yields (15). □

3.2 POD for the Average

Besides the centered driving process Z, the payoff G of the Asian option also depends on the average process A which is a function of the exponential S. Thus, to approximate (11) with a low-dimensional PIDE, we need to show that A and S can be accurately represented with the POD basis as well. To this end, recall that S is defined as a deterministic function of Z by (8). If we apply this function to $\mathcal{P}_d Z_t$ for arbitrary $t \in [0, T]$, we obtain

$$S_t(\mathcal{P}_d Z_t) = \sum_{k\in\mathbb{N}} \langle S_0, e_k\rangle_H \, e^{\left\langle \int_0^t \gamma(u)\,du + \mathcal{P}_d Z_t, e_k\right\rangle_H} e_k \in H.$$

The following theorem is the central part of generalizing the POD method to Asian options.

Theorem 3.5. *There is a constant $C > 0$ (depending on T) such that*

$$E\Big|\langle w, S_t(Z_t)\rangle_H - \langle w, S_t(\mathcal{P}_d Z_t)\rangle_H\Big| \le C\,\|w\|_H \left(\sum_{l=d+1}^{\dim H} \mu_l\right)^{\frac{1}{2}}$$

for every $t \in [0, T]$.

Proof. From the definition of S_t, we get

$$\begin{aligned} &E\Big|\langle w, S_t(Z_t)\rangle_H - \langle w, S_t(\mathcal{P}_d Z_t)\rangle_H\Big| \\ &= E\left|\sum_{k\in\mathbb{N}} \langle w, e_k\rangle_H \,\langle S_0, e_k\rangle_H \left(e^{\left\langle \int_0^t \gamma(u)\,du + Z_t, e_k\right\rangle_H} - e^{\left\langle \int_0^t \gamma(u)\,du + \mathcal{P}_d Z_t, e_k\right\rangle_H}\right)\right| \\ &\le E\sum_{k\in\mathbb{N}} \left|\langle w, e_k\rangle_H \,\langle S_0, e_k\rangle_H \, e^{\int_0^t \langle \gamma(u), e_k\rangle_H du} \left(e^{\langle Z_t, e_k\rangle_H} - e^{\langle \mathcal{P}_d Z_t, e_k\rangle_H}\right)\right|. \end{aligned} \tag{16}$$

For the term depending on γ, we use Assumption 2.1 and obtain

$$\left|\int_0^t \langle \gamma(u), e_k\rangle_H\,du\right| \le \int_0^t \|\gamma(u)\|_H\,du \le C_1 \left(\int_0^t \|\gamma(u)\|_H^2\,du\right)^{\frac{1}{2}} \le C_2,$$

with positive constants C_1, C_2 depending on T but not on t. Next, we apply the mean-value theorem to the exponential function and make use of the self-adjointness of the projection operator $\mathcal{P}_d$ for the estimate

$$\begin{aligned}\left|e^{\langle Z_t, e_k\rangle_H} - e^{\langle \mathcal{P}_d Z_t, e_k\rangle_H}\right| &\leq e^{\max\{\langle Z_t, e_k\rangle_H, \langle \mathcal{P}_d Z_t, e_k\rangle_H\}} \left|\langle Z_t - \mathcal{P}_d Z_t, e_k\rangle_H\right| \\ &\leq e^{\max\{\langle Z_t, e_k\rangle_H, \langle Z_t, \mathcal{P}_d e_k\rangle_H\}} \left\|Z_t - \mathcal{P}_d Z_t\right\|_H\end{aligned}$$

for every $k \in \mathbb{N}$. Inserting these results into (16) and using the monotone convergence theorem yields

$$\begin{aligned}&E\left|\langle w, S_t(Z_t)\rangle_H - \langle w, S_t(\mathcal{P}_d Z_t)\rangle_H\right| \\ &\quad\leq C \sum_{k\in\mathbb{N}} \left|\langle w, e_k\rangle_H \, \langle S_0, e_k\rangle_H\right| \, E\left[e^{\max\{\langle Z_t, e_k\rangle_H, \langle Z_t, \mathcal{P}_d e_k\rangle_H\}} \left\|Z_t - \mathcal{P}_d Z_t\right\|_H\right].\end{aligned}$$

With the Cauchy–Schwarz inequality, we find

$$\begin{aligned}&E\left|\langle w, S_t(Z_t)\rangle_H - \langle w, S_t(\mathcal{P}_d Z_t)\rangle_H\right| \\ &\quad\leq C \sum_{k\in\mathbb{N}} \left|\langle w, e_k\rangle_H \, \langle S_0, e_k\rangle_H\right| \left(E\left[e^{2\max\{\langle Z_t, e_k\rangle_H, \langle Z_t, \mathcal{P}_d e_k\rangle_H\}}\right]\right)^{\frac{1}{2}} \left(E\left\|Z_t - \mathcal{P}_d Z_t\right\|_H^2\right)^{\frac{1}{2}}.\end{aligned}$$

For the first expectation, we use [8, Proposition. 2.3]:

$$\begin{aligned}E\left[e^{2\max\{\langle Z_t, e_k\rangle_H, \langle Z_t, \mathcal{P}_d e_k\rangle_H\}}\right] &= E\left[\max\{e^{\langle Z_t, 2e_k\rangle_H}, e^{\langle Z_t, 2\mathcal{P}_d e_k\rangle_H}\}\right] \\ &\leq E\left[e^{\langle Z_t, 2e_k\rangle_H} + e^{\langle Z_t, 2\mathcal{P}_d e_k\rangle_H}\right] \\ &\leq C_3 e^{C_4 T}\end{aligned}$$

with constants C_3, C_4. The Cauchy–Schwarz inequality in $l^2(\mathbb{N})$ yields the following bound for the remaining sum in k:

$$\sum_{k\in\mathbb{N}} \left|\langle w, e_k\rangle_H \, \langle S_0, e_k\rangle_H\right| \leq \|w\|_H \, \|S_0\|_H.$$

By Proposition 3.3, we thus get

$$\begin{aligned}E\left|\langle w, S_t(Z_t)\rangle_H - \langle w, S_t(\mathcal{P}_d Z_t)\rangle_H\right| &\leq C \|w\|_H \|S_0\|_H \left(E\left\|Z_t - \mathcal{P}_d Z_t\right\|_H^2\right)^{\frac{1}{2}} \\ &\leq C \|w\|_H \|S_0\|_H \left(\sum_{l=d+1}^{\dim H} \mu_l\right)^{\frac{1}{2}}.\end{aligned} \tag{17}$$

□

Although $S_t(\mathcal{P}_d Z_t)$ is still an element of the possibly infinite-dimensional Hilbert space H, it can be computed from the d-dimensional object $\mathcal{P}_d Z_t$. This makes the approximation suitable for numerical computations. Similar to (4), we define the arithmetic average corresponding to $S_t(\mathcal{P}_d Z_t)$ by

$$A_t^d := \frac{1}{t}\int_0^t \langle w, S_u(\mathcal{P}_d Z_u)\rangle_H \, du \in \mathbb{R}$$

for $t > 0$. Similar to (5), we set

$$A_0^d := \langle w, S_0(\mathcal{P}_d Z_0)\rangle_H = \langle w, S_0\rangle_H \, .$$

We find the following estimate for the approximation error.

Corollary 3.6. *There is a constant $C > 0$ (depending on T) such that*

$$E\left|A_t - A_t^d\right| \le C \, \|w\|_H \left(\sum_{l=d+1}^{\dim H} \mu_l\right)^{\frac{1}{2}}$$

for every $t \in [0, T]$.

Proof. By definition, $A_0^d = A_0$. For $t > 0$, we have

$$\begin{aligned} E\left|A_t - A_t^d\right| &= \frac{1}{t} E\left|\int_0^t \langle w, S_u(Z_u) - S_u(\mathcal{P}_d Z_u)\rangle_H \, du\right| \\ &\le \frac{1}{t} E\left[\int_0^t |\langle w, S_u(Z_u) - S_u(\mathcal{P}_d Z_u)\rangle_H| \, du\right]. \end{aligned}$$

Using Fubini's theorem and applying Theorem 3.5 yields

$$\begin{aligned} E\left|A_t - A_t^d\right| &\le \frac{1}{t}\int_0^t E\,|\langle w, S_u(Z_u) - S_u(\mathcal{P}_d Z_u)\rangle_H| \, du \\ &\le \frac{1}{t}\int_0^t C \, \|w\|_H \left(\sum_{l=d+1}^{\dim H} \mu_l\right)^{\frac{1}{2}} du. \end{aligned}$$

Since the integrand does no longer depend on the integration variable u, the proof is complete. □

As before, we obtain an t-dependent estimate for the approximation error in the time-homogeneous case.

Corollary 3.7. *Suppose that Z is a time-homogeneous jump-diffusion process. Then there is a constant $C > 0$ (depending on T) such that*

$$E\left|A_t - A_t^d\right| \le C \, \|w\|_H \sqrt{\frac{t}{T}} \left(\sum_{l=d+1}^{\dim H} \mu_l\right)^{\frac{1}{2}}$$

for every $t \in [0, T]$.

Proof. We apply Propostion 3.4 to equation (17) in the proof of 3.5 to obtain

$$E\Big|\langle w, S_t(Z_t)\rangle_H - \langle w, S_t(\mathcal{P}_d Z_t)\rangle_H\Big| \le C \, \|w\|_H \, \|S_0\|_H \sqrt{\frac{t}{T}} \left(\sum_{l=d+1}^{\dim H} \mu_l\right)^{\frac{1}{2}}.$$

We proceed as in the proof of Corollary 3.6 and find

$$E\left|A_t - A_t^d\right| \le \frac{1}{t}\int_0^t \sqrt{\frac{u}{T}} C \|w\|_H \left(\sum_{l=d+1}^{\dim H} \mu_l\right)^{\frac{1}{2}} du.$$

Since

$$\frac{1}{t}\int_0^t \sqrt{\frac{u}{T}}\, du = \frac{2}{3}\sqrt{\frac{t}{T}},$$

the proof is complete. □

3.3 Approximate Pricing

In the previous sections, we have seen how to approximate the processes on which the payoff G of the Asian option depends, the centered process Z, and the average A. Now, we use these results to find a finite-dimensional approximation of the discounted option value $\widehat{V}$. For $t \in [0, T]$, we define

$$\widehat{V}_d(t, z, a) := e^{-rT} E[G(\mathcal{P}_d Z_T, A_T^d) | \mathcal{P}_d Z_t = z, A_t^d = a] \quad \text{for every } z \in U_d, a \in \mathbb{R}. \tag{18}$$

In contrast to the definition of $\widehat{V}$ in (9), the payoff is applied to the projected random variables $\mathcal{P}_d Z_T$ and A_T^d here instead of Z_T and A_T. Thus, $\widehat{V}_d$ is defined on the finite dimensional domain $[0, T] \times U_d \times \mathbb{R}$ which allows for numerical discretization. Similar to Theorem 11, we find that $\widehat{V}_d$ satisfies a PIDE. The PIDE is finite-dimensional.

Theorem 3.8. *Suppose that the approximated option value $\widehat{V}_d$ defined in (18) is continuously differentiable with respect to t and twice continuously differentiable with respect to z and a. Then $\widehat{V}_d$ is a classical solution of the PIDE*

$$\begin{aligned} -\partial_t \widehat{V}_d(t, z, a) &= \frac{1}{2}\sum_{i,j=1}^{d} c_{ij}(t)\, \partial_i \partial_j \widehat{V}_d(t, z, a) + \frac{1}{t}\left(\langle w, S_t(z)\rangle_H - a\right) \partial_a \widehat{V}(t, z, a) \\ &\quad + \int_H \Big[\widehat{V}_d(t, z + \mathcal{P}_d \eta_t(\zeta), a) - \widehat{V}_d(t, z, a) \\ &\qquad - \sum_{i=1}^{d} \langle \eta_t(\zeta), p_i\rangle_H\, \partial_i \widehat{V}_d(t, z, a)\Big] \nu(d\zeta), \end{aligned} \tag{19}$$

with time-dependent coefficients

$$c_{ij}(t) := \left\langle \sigma_t Q \sigma_t^* p_i, p_j \right\rangle_H, \quad i, j = 1, \ldots, d,$$

and terminal condition

$$\widehat{V}_d(T, z, a) = e^{-rT} G(z, a)$$

for a.e. $t \in (0, T)$, $z \in U_d$, and $a \in \mathbb{R}$.

Proof. This can be shown along the very same lines as in the proof of Theorem 2.4. The main difference is that we make use of a finite-dimensional version of Itô's formula (see, e.g., [4, Prop. 8.19]). This yields finite sums of second derivatives instead of the trace operator. □

The value of the Asian option at time $t = 0$ is given by $\widehat{V}(0, 0, \langle w, S_0\rangle_H)$, since $Z_0 = 0 \in H$ and $A_0 = \langle w, S_0\rangle_H \in \mathbb{R}$ by definition. The solution of the finite-dimensional PIDE yields $\widehat{V}_d(0, 0, \langle w, S_0\rangle_H)$. The following theorem states an upper bound of the approximation error for the option value.

Theorem 3.9. *There is a constant $C > 0$ (depending on T) such that the difference of the true Asian option price and its finite dimensional approximation satisfies*

$$\left|\widehat{V}(0, 0, \langle w, S_0\rangle_H) - \widehat{V}_d(0, 0, \langle w, S_0\rangle_H)\right| \le C \left(\sum_{l=d+1}^{\dim H} \mu_l\right)^{\frac{1}{2}}. \tag{20}$$

Proof. We start with the definition of $\widehat{V}$ and $\widehat{V}_d$ and make use of Assumption 2.3 to find

$$\begin{aligned}\left|\widehat{V}(0, 0, A_0) - \widehat{V}_d(0, 0, A_0^d)\right| &= e^{-rT}\left|E[G(Z_T, A_T) - E[G(\mathcal{P}_d Z_T, A_T^d)]\right| \\ &\le e^{-rT} E\left[L_z^G \left\|Z_T - \mathcal{P}_d Z_T\right\|_H + L_a^G \left|A_T - A_T^d\right|\right] \\ &\le e^{-rT} \max\{L_z^G, L_a^G\}\left(E\left\|Z_T - \mathcal{P}_d Z_T\right\|_H + E\left|A_T - A_T^d\right|\right).\end{aligned}$$

With the Cauchy–Schwarz inequality, we get

$$\left|\widehat{V}(0, 0, A_0) - \widehat{V}_d(0, 0, A_0^d)\right| \le C\left(\left(E\left\|Z_T - \mathcal{P}_d Z_T\right\|_H^2\right)^{\frac{1}{2}} + E\left|A_T - A_T^d\right|\right).$$

Applying Proposition 3.2 to $E\left\|Z_T - \mathcal{P}_d Z_T\right\|_H^2$ and Corollary 3.6 to $E\left|A_T - A_T^d\right|$ completes the proof. □

The theorem shows that we can achieve a good approximation, if the right-hand side of (20) is small. In practice, we can first compute the eigenvalues μ_l, $l = 1, 2, \dots$, and then decide how many POD components we have to include in the projection in order to satisfy a given absolute tolerance.

For the discretization of the PIDE (19), sparse grid methods similar to those presented in [10] may be suitable. The nonlocal integral terms, which are due to the jumps in the model, can be discretized using a Galerkin approach with a wavelet basis [12]. The POD method in combination with sparse grids was already shown to be a promising approach to break the "curse of dimension" in the case of European options.

There is, however, an additional numerical difficulty when dealing with Asian options. The fact that there is no diffusion in the variable a representing the average requires special attention. Equations of this kind are often termed "degenerate parabolic" PIDEs. A large number of authors has dealt with such problems, see, e.g., [1, 2, 3, 19] and the references therein. Since the dimension reduced equation is finite-dimensional, the numerical schemes and convergence result presented there can be applied directly. These include, e.g., flux limiting methods, operator splitting, and difference-quadrature methods. Numerical experiments concerning the presented PIDE for Asian options will be a topic for future research.

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