Estimation for Non-negative Lévy-driven Ornstein-Uhlenbeck Processes

Peter J. Brockwell, Richard A. Davis* and Yu Yang,
Department of Statistics, Colorado State University,
Fort Collins, CO 80523-1877.
* Department of Statistics, Columbia University,
New York, NY 10027

Abstract

Continuous-time autoregressive moving average (CARMA) processes with a non-negative kernel and driven by a non-decreasing Lévy process constitute a very general class of stationary, non-negative continuous-time processes. In financial econometrics a stationary Ornstein-Uhlenbeck (or CAR(1)) process, driven by a non-decreasing Lévy process, was introduced by Barndorff-Nielsen and Shephard (2001) as a model for stochastic volatility to allow for a wide variety of possible marginal distributions and the possibility of jumps. For such processes we take advantage of the non-negativity of the increments of the driving Lévy process to study the properties of a highly efficient estimation procedure for the parameters when observations are available of the CAR(1) process at uniformly spaced times $0, h, \ldots, Nh$. We also show how to reconstruct the background driving Lévy process from a continuously observed realization of the process and use this result to estimate the increments of the Lévy process itself when h is small. Asymptotic properties of the coefficient estimator are derived and the results illustrated using a simulated gamma-driven Ornstein-Uhlenbeck process.

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1. Introduction.

This paper is concerned with estimation of the parameters of a non-negative Lévy-driven Ornstein-Uhlenbeck process and of the parameters of the background driving Lévy process, based on observations made at uniformly and closely-spaced times. We investigate the asymptotic properties of the estimator of the CAR(1) coefficient obtained by applying the method of Davis and McCormick (1989) to the estimation of the corresponding coefficient of the sampled AR(1) process. (The weak consistency of this estimator was shown by Jongbloed et al.(2005).) The estimator is then used to estimate the corresponding realization of the driving Lévy process. The exact recovery of the driving Lévy process requires continuous observation of the Ornstein-Uhlenbeck process. The integral expressions determining the driving Lévy process are therefore replaced by approximating sums using the available discrete-time observations.

In Section 2, we define the stationary Lévy-driven Ornstein-Uhlenbeck (or CAR(1)) process, $\{Y(t), t \geq 0\}$. In Section 3, we characterize the sampled AR(1) process, $\{Y_n^{(h)} = Y(nh), n = 0, 1, 2, \ldots\}$, and the distribution of its driving white noise sequence in terms of the parameters of the underlying CAR(1) process and its driving Lévy process. The autoregressive coefficient of the sampled process is then estimated using the method of Davis and McCormick (1989). From the relation between the sampled and continuous-time processes we then obtain corresponding parameter estimates for the CAR(1) process. The idea of using the sampled process to estimate the parameters of the underlying continuous-time process was first used by Phillips (1959), but in our case the non-decreasing property of the driving Lévy process and the non-negativity of the corresponding discrete-time increments permits a very large efficiency gain. In Section 4, we show how to recover the driving Lévy process under the assumption that the process is observed continuously and then approximate the results using closely-spaced discrete observations. In Section 5, we derive the asymptotic distribution of the coefficient estimator when the driving Lévy process is a gamma process and illustrate with a simulated example the performance of the estimators of both the CAR(1) parameters and the driving Lévy process. When the continuously observed process is available, the autoregression coefficient can be identified with probability 1. This is discussed in Section 6.

2. Stationary Lévy-driven Ornstein-Uhlenbeck processes.

In order to define the stationary Lévy-driven Ornstein Uhlenbeck (or CAR(1)) process, we first record a few essential facts concerning Lévy processes. (For a detailed account of integration with respect to Lévy processes, see Protter (2004)). Suppose we are given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le\infty}, P)$, where $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$ and $(\mathcal{F}_t)$ is right-continuous.

Definition 1 (Lévy Process). $\{L(t), t \ge 0\}$ is an $(\mathcal{F}_t)$-adapted Lévy process if $L(t) \in \mathcal{F}_t$ for all $t \ge 0$, $L(0) = 0$ a.s., $L(t) - L(s)$ is independent of $\mathcal{F}_s$, $0 \le s < t < \infty$, $L(t) - L(s)$ has the same distribution as $L(t - s)$ and $L(t)$ is continuous in probability.

Every Lévy process has a unique modification which is càdlàg (right continuous with left limits) and which is also a Lévy process. We shall therefore assume that our Lévy process has these properties. For non-decreasing Lévy processes the Laplace transform $\tilde{f}_{L(t)}(s) := E(\exp(-sL(t)))$ has the form

$$\tilde{f}_{L(t)}(s) = \exp(-t\Phi(s)),\ \Re(s) \ge 0,$$

where

$$\Phi(s) = m + \int_{(0,\infty)} (1 - e^{-sx})\nu(dx),$$

with $m \ge 0$ and ν a measure on the Borel subsets of $(0, \infty)$ satisfying

$$\int_{(0,\infty)} \frac{u}{1+u}\nu(du) < \infty.$$

The measure ν is called the **Lévy measure** of the process L and m is the **drift**. There exists a wealth of possible marginal distributions for $L(t)$, attainable by suitable choice of m and ν. (See for example Barndorff-Nielsen and Shephard (2001).) For second-order Lévy processes $E(L(1))^2 < \infty$ and there exist real constants μ and σ such that

$$EL(t) = \mu t \text{ and } \mathrm{Var}(L(t)) = \sigma^2 t,\ \text{for}\, t \ge 0.$$

To avoid problems of parameter identifiability in the CAR(1) process defined below we assume throughout that L is scaled so that $\mathrm{Var}(L(1)) = 1$. Then $\mathrm{Var}(L(t)) = t$ for $t \ge 0$ and we shall refer to the process L as a standardized second-order Lévy process. Throughout this paper we shall be concerned with CAR(1) (or stationary Ornstein-Uhlenbeck) processes driven by standardized second-order non-decreasing Lévy processes. The Lévy-driven CAR(1) process is defined as follows.

Definition 2 (**Lévy-driven CAR(1) process**) A CAR(1) process driven by the Lévy process $\{L(t), t \ge 0\}$, with parameters $a \in \mathbb{R}$ and $\sigma > 0$, is defined to be a strictly stationary solution of the stochastic differential equation,

$$dY(t) + aY(t)dt = \sigma dL(t). \tag{2.1}$$

In the special case when $\{L(t)\}$ is Brownian motion, (2.1) is interpreted as an Itô equation with solution $\{Y(t), t \geq 0\}$ satisfying

$$Y(t) = e^{-at}Y(0) + \sigma \int_0^t e^{-a(t-u)} dL(u), \tag{2.2}$$

where the integral is defined as the L^2 limit of approximating Riemann-Stieltjes sums. For any second-order driving Lévy process, $\{L(t)\}$, the process $\{Y(t)\}$ can be defined in the same way, and if $\{L(t)\}$ is non-decreasing (and hence of bounded variation on compact intervals) $\{Y(t)\}$ can also be defined pathwise as a Riemann-Stieltjes integral by (2.2). We can also write

$$Y(t) = e^{-a(t-s)}Y(s) + \sigma \int_s^t e^{-a(t-u)} dL(u), \text{ for all } t > s \geq 0, \tag{2.3}$$

showing, by independence of the increments of $\{L(t)\}$, that $\{Y(t)\}$ is Markov. (For a general theory of integration with respect to semimartingales, and in particular with respect to Lévy processes see Protter (2004).) The following proposition gives necessary and sufficient conditions for stationarity of $\{Y(t)\}$. For a proof see Brockwell and Marquardt (2005).

Proposition 1. If $Y(0)$ is independent of $\{L(t), t \geq 0\}$ and $E(L(1)^2) < \infty$, then $Y(t)$ is strictly stationary if and only if $a > 0$ and $Y(0)$ has the distribution of $\sigma \int_0^\infty e^{-au} dL(u)$.

Remark 1. By introducing a second Lévy process $\{M(t), 0 \leq t < \infty\}$, independent of L and with the same distribution, we can extend $\{Y(t), t \geq 0\}$ to a process with index set $(-\infty, \infty)$. Define the following extension of L:

$$L^*(t) = L(t)I_{[0,\infty)}(t) - M(-t-)I_{(-\infty,0]}(t), \quad -\infty < t < \infty.$$

Then, provided $a > 0$, the process $\{Y(t)\}$ defined by

$$Y(t) = \sigma \int_{-\infty}^t e^{-a(t-u)} dL^*(u), \tag{2.4}$$

is a strictly stationary process satisfying equation (2.3) (with L replaced by L^*) for all $t > s$ and $s \in (-\infty, \infty)$. Henceforth, we refer to L^* as the *background driving Lévy process* (BDLP) and denote it by L for simplicity.

Remark 2. From (2.4) we have the relation

$$Y(t) = e^{-a(t-s)}Y(s) + \sigma \int_s^t e^{-a(t-u)} dL(u), \quad t \geq s > -\infty. \tag{2.5}$$

Taking $s = 0$ and using the pathwise interpretation of the integral in (2.5), we can also write

$$Y(t) = e^{-at}Y(0) + \sigma L(t) - a\sigma \int_0^t e^{-a(t-u)}L(u)du, \ t \geq 0, \tag{2.6}$$

where the last integral is a Riemann integral and the equality holds for all finite $t \geq 0$ with probability 1.

3. Parameter estimation via the sampled process.

Setting $t = nh$ and $s = (n-1)h$ in equation (2.5), we see at once that for any $h > 0$, the sampled process $\{Y_n^{(h)}, n = 0, 1, 2, \dots\}$ is the discrete-time AR(1) process satisfying

$$Y_n^{(h)} = \phi\, Y_{n-1}^{(h)} + Z_n, \ n = 0, 1, 2, \dots, \tag{3.1}$$

where

$$\phi = e^{-ah}, \tag{3.2}$$

and

$$Z_n = \sigma \int_{(n-1)h}^{nh} e^{-a(nh-u)}dL(u). \tag{3.3}$$

The noise sequence $\{Z_n\}$ is i.i.d. and positive since L has stationary, independent and positive increments.

If the process $\{Y(t), 0 \leq t \leq T\}$ is observed at times $0, h, 2h, \dots, Nh$, where $N = [T/h]$, i.e., N is the integer part of T/h, then, since the innovations Z_n of the process $\{Y_n^{(h)}\}$ are non-negative and $0 < \phi < 1$, we can use the highly efficient Davis-McCormick estimator of ϕ, namely

$$\hat{\phi}_N^{(h)} = \min_{1 \leq n \leq N} \frac{Y_n^{(h)}}{Y_{n-1}^{(h)}}. \tag{3.4}$$

This estimator has previously been proposed by Jongbloed et al. (2005), who showed the weak consistency of the estimator as $N \to \infty$ with h fixed. To obtain the asymptotic distribution of $\hat{\phi}_N^{(h)}$ we shall suppose that the distribution function F of Z_n satisfies $F(0) = 0$ and that F is regularly varying at zero with exponent α, i.e., that there exists $\alpha > 0$ such that

$$\lim_{t \downarrow 0} \frac{F(tx)}{F(t)} = x^\alpha \quad \text{for all } x > 0.$$

(These conditions are satisfied by the gamma-driven CAR(1) process as we shall show in Section 5.) Under these conditions on F, the results of Davis and McCormick (1989) imply that $\hat{\phi}_N^{(h)} \to \phi$ *a.s.* as $N \to \infty$ with h fixed and that

$$\lim_{N \to \infty} P\left[k_N^{-1}(\hat{\phi}_N^{(h)} - \phi)c_\alpha \leq x\right] = G_\alpha(x), \tag{3.5}$$

where $k_N = F^{-1}(N^{-1})$, $c_\alpha = (EY_1^{(h)\alpha})^{1/\alpha}$ and G_α is the Weibull distribution function,

$$G_\alpha(x) = \begin{cases} 1 - \exp\{-x^\alpha\}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases} \tag{3.6}$$

From the observations $\{Y_n^{(h)}, n = 0, 1 \ldots, N\}$ we thus obtain the estimator $\hat{\phi}_N^{(h)}$ and, from (3.2), the corresponding estimator,

$$\hat{a}_N^{(h)} = -h^{-1} \log \hat{\phi}_N^{(h)} \tag{3.7}$$

of the CAR(1) coefficient a. Provided the distribution function F of the noise terms Z_n in the discrete-time sampled process satisfies the conditions indicated above, we can also determine the asymptotic distributions of this estimator. In particular, using a Taylor series approximation, we find that

$$\lim_{N\to\infty} P\left[(-h)e^{-ah} c_\alpha k_N^{-1} \left(\hat{a}_N^{(h)} - a\right) \leq x\right] = G_\alpha(x), \tag{3.8}$$

where G_α is given in (3.6). Since $\text{var}(Y^{(h)}) = \sigma^2/(2a)$, we use the estimator,

$$\hat{\sigma}_N^2 = \frac{2\hat{a}_N^{(h)}}{N} \sum_{i=0}^{N} (Y_i^{(h)} - \overline{Y}_N^{(h)})^2, \tag{3.9}$$

where $\overline{Y}_N^{(h)} = \sum_{i=0}^N Y_i^{(h)}/(N+1)$, to estimate σ^2.

4. Estimating the Lévy increments.

So far, we have made no assumptions about the driving Lévy process except for non-negativity and existence of $E(L(1)^2)$. In order to suggest an appropriate parametric model for L and to estimate the parameters, it is important to recover an approximation to L from the observed data. If the CAR(1) process is continuously observed on $[0, T]$, then the integrated form of (2.1) immediately gives

$$L(t) = \sigma^{-1}\left[Y(t) - Y(0) + a\int_0^t Y(s)ds\right]. \tag{4.1}$$

From (4.1), the increment of the driving Lévy process on the interval $((n-1)h, nh]$ is given by

$$\triangle\, L_n^{(h)} := L(nh) - L((n-1)h) = \sigma^{-1}\left[Y(nh) - Y((n-1)h) + a\int_{(n-1)h}^{nh} Y(u)du\right].$$

Replacing the CAR(1) parameters by their estimators and the integral by a trapezoidal approximation, we obtain the estimated increments,

$$\triangle\, \hat{L}_n^{(h)} = \hat{\sigma}_N^{-1}\left[Y_n^{(h)} - Y_{n-1}^{(h)} + \hat{a}_N^{(h)} h (Y_n^{(h)} + Y_{n-1}^{(h)})/2\right]. \tag{4.2}$$

5. The gamma-driven CAR(1) process

In this section, we illustrate the preceding estimating procedure in the case when L is a standardized gamma process. Thus $L(t)$ has the gamma density $f_{L(t)}$ with exponent γt, scale-parameter $\gamma^{-1/2}$, mean $\gamma^{1/2}t$ and variance t. The Laplace transform of $L(t)$ is

$$\tilde{f}_{L(t)}(s) := E\exp(-sL(t)) = \exp\{-t\Phi(s)\}, \ \Re(s) \geq 0, \tag{5.1}$$

where $\Phi(s) = \gamma\log(1+\beta s)$, $\beta = \gamma^{-1/2}$ and $\gamma > 0$.

Based on the h-spaced observations $\{Y_n^{(h)}, n = 0, 1, \ldots, N\}$, we estimate the discrete-time autoregression coefficient ϕ and the CAR(1) parameters a and σ^2 using (3.4), (3.7) and (3.9) respectively. We then estimate the Lévy increments as in (4.2) and use them to estimate the parameter γ of the standardized gamma process L. To obtain the asymptotic distributions of $\phi_N^{(h)}$ and $\hat{a}_N^{(h)}$ as $N \to \infty$ with h fixed, we first show that the distribution function F of Z_n in (3.1) is regularly varying at zero with exponent γh and then determine the coefficients $k_N = F^{-1}(N^{-1})$ and $c_\alpha = (EY_1^{(h)\alpha})^{1/\alpha}$ in (3.5). To do so, we use the Laplace transform (5.1) to investigate the behavior of the density of $Z_1 = \sigma\int_0^h e^{-a(h-t)}dL(t)$ near zero.

Define $W_h := Z_1/\sigma$. The Laplace transform of W_h is

$$\begin{aligned}\tilde{f}_{W_h}(s) &= \exp\left[-\int_0^h \Phi(se^{-at})dt\right] \\ &= \exp\left[-\int_0^h \gamma\log(1+\beta se^{-at})dt\right].\end{aligned} \tag{5.2}$$

The exponent in (5.2) has the power series expansion,

$$\begin{aligned}-\int_0^h \gamma\log(1+\beta se^{-at})dt &= -\gamma\int_0^h \log\left[s\beta e^{-at}(1+\frac{1}{\beta se^{-at}})\right]dt \\ &\approx \log(s\beta)^{-\gamma h} + \frac{1}{2}\gamma ah^2 + \gamma\left[\frac{1}{\beta sa}(1-e^{ah}) - \frac{1}{4\beta^2 s^2 a}(1-e^{2ah}) + \cdots\right],\end{aligned}$$

as $s \to \infty$. Hence $\tilde{f}_{W_h}(s)$ has the corresponding expansion,

$$\tilde{f}_{W_h}(s) \approx \frac{\beta^{-\gamma h}}{s^{\gamma h}}e^{\frac{1}{2}\gamma ah^2} + \frac{C_1}{s^{\gamma h+1}} + \frac{C_2}{s^{\gamma h+2}} + \cdots,$$

where $C_1, C_2, \ldots$ are constants depending on γ, β, h and a. Since $\tilde{f}_{Z_1}(s) = \tilde{f}_{W_h}(\sigma s)$,

$$s^{\gamma h}\tilde{f}_{Z_1}(s) \to (\sigma\beta)^{-\gamma h}e^{\frac{1}{2}\gamma ah^2}, \text{ as } s \to \infty.$$

By Theorem 30.2 of Doetsch (1974), the density f_{Z_1} of Z_1 has the expansion, in a neighbourhood of zero,

$$f_{Z_1}(x) = \frac{(\sigma\beta)^{-\gamma h}x^{\gamma h-1}}{\Gamma(\gamma h)}e^{\frac{1}{2}\gamma ah^2} + \frac{(\sigma x)^{\gamma h}C_1}{\sigma\Gamma(\gamma h+1)} + \frac{(\sigma x)^{\gamma h+1}C_2}{\sigma\Gamma(\gamma h+2)} + \cdots.$$

So

$$\frac{f_{Z_1}(x)}{x^{\gamma h-1}} \to (\sigma\beta)^{-\gamma h} e^{\frac{1}{2}\gamma a h^2}/\Gamma(\gamma h),\ \text{as } x \to 0,$$

and

$$F_{Z_1}(x) \sim x^{\gamma h}(\sigma\beta)^{-\gamma h} e^{\frac{1}{2}\gamma a h^2}/\Gamma(\gamma h+1),\ \text{as } x \to 0. \tag{5.3}$$

Thus the distribution F of Z_n is regularly varying at zero with exponent γh.

From the definition of k_N in (3.5) we have $\frac{1}{N} = \int_0^{k_N} F_{Z_1}(du)$. This equation, together with (5.3), gives

$$k_N^{-1} \sim (\sigma\beta)^{-1}\left[\Gamma(\gamma h+1)\right]^{-1/(\gamma h)} e^{\frac{1}{2}ah} N^{1/(\gamma h)},\ \text{as } N \to \infty. \tag{5.4}$$

In order to calculate $c_{\gamma h}$, we need to find $E[Y_n^{(h)}]^{\gamma h}$, where $Y_n^{(h)} = \sum_{j=0}^{\infty} \phi^j Z_{n-j}$. The Laplace transform of $Y_n^{(h)}$ is

$$\tilde{f}_{Y_n^{(h)}}(s) = E\, e^{-sY_n^{(h)}} = \prod_{j=0}^{\infty} E\, e^{-s\phi^j Z_{n-j}}.$$

So

$$\begin{aligned}
\log \tilde{f}_{Y_n^{(h)}}(s) &= \sum_{j=0}^{\infty} \log \tilde{f}_{Z_1}(s\phi^j) \\
&= \sum_{j=0}^{\infty} \log \tilde{f}_{W_h}(s\sigma\phi^j) \\
&= -\gamma \sum_{j=0}^{\infty} \int_0^h \log(1+\beta s\sigma\phi^j e^{-ay})dy,
\end{aligned}$$

and hence

$$\begin{aligned}
\tilde{f}_{Y_n^{(h)}}(s) &= \exp\left[\frac{\gamma}{a}\sum_{j=0}^{\infty}\left[\text{dilog}(1+\beta s\sigma\phi^j) - \text{dilog}(1+\beta s\sigma\phi^j e^{-ah})\right]\right] \\
&= \exp\left(\frac{\gamma}{a}\text{dilog}(1+\beta s\sigma)\right),
\end{aligned}$$

where dilog is the dilogarithm function, $\text{dilog}(x) = \int_1^x \log(u)/(1-u)du$. Using Theorem 2.1 of Brockwell and Brown (1978), we find, for $\gamma h < 1$,

$$\begin{aligned}
E[Y_n^{(h)}]^{\gamma h} =& \frac{1}{\Gamma(1-\gamma h)} \int_0^{\infty} s^{-\gamma h} \left| D\tilde{f}_{Y_n^{(h)}}(s)\right| ds \\
=& \frac{\gamma}{a\Gamma(1-\gamma h)} \int_0^{\infty} s^{-\gamma h-1} \exp\left(\frac{\gamma}{a}\text{dilog}(1+\beta s\sigma)\right) \log(1+\beta s\sigma) ds,
\end{aligned} \tag{5.5}$$

where Df denotes the derivative of f. Then $c_{\gamma h} = \left[E[Y_n^{(h)}]^{\gamma h}\right]^{1/(\gamma h)}$ can be numerically evaluated from (5.5) for fixed h. Theorem 2.1 also covers the case $\gamma h \geq 1$ but our prime concern here is with small values of h.

Theorem 1. For a sequence of observations $\{Y_n^{(h)}, n = 0, 1, \dots, N\}$ from a gamma-driven CAR(1) process, we have $\hat{a}_N^{(h)} \to a$ a.s. and

$$\lim_{N\to\infty} P\left[(-h)e^{-ah}k_N^{-1}(\hat{a}_N^{(h)} - a)c_\alpha \leq x\right] = G_\alpha(x),$$

where G_α is as in (3.6), $\alpha = \gamma h$, $\hat{a}_N^{(h)}$ is defined in (3.7), k_N^{-1} is given in (5.4), and c_α is evaluated through (5.5).

Proof. At the beginning of Section 3, we have shown that $Y_n^{(h)}$ is a stationary discrete-time AR(1) with autoregression coefficient $\phi \in (0, 1)$ and driven by i.i.d. noise $\{Z_n\}$. According to (5.3), the distribution function F of Z_n is regularly varying at zero with exponent $\alpha = \gamma h$ and satisfies the condition $F(0) = 0$. Since $0 \leq Z_n \leq \sigma(L(nh) - L((n-1)h)$, $\int u^\xi F(du) < \infty$ for all $\xi > 0$. By Corollary 2.4 of Davis and McCormick (1989), we have $\hat{\phi}_N^{(h)} \to \phi$ a.s., which implies $\hat{a}_N^{(h)} \to a$ a.s.. From the same corollary, we also conclude that

$$\lim_{N\to\infty} P\left[k_N^{-1}(\hat{\phi}_N^{(h)} - \phi)c_{\gamma h} \leq x\right] = G_{\gamma h}(x),$$

where $\hat{\phi}_N^{(h)}$ and k_N^{-1} are given in (3.4) and (5.4) respectively, and $c_{\gamma h}$ is evaluated through (5.5). Using a Taylor series expansion, we find from this result that

$$\lim_{N\to\infty} P\left[(-h)e^{-ah}k_N^{-1}(\hat{a}_N^{(h)} - a)c_{\gamma h} \leq x\right] = G_{\gamma h}(x). \qquad \square$$

Theorem 1 gives the limiting distribution of $\phi_N^{(h)}$ for fixed h as $N \to \infty$. It is of interest also to consider the behaviour of the estimator as h also goes to zero. For any non-negative random variable Y with density function $f(u)$, we have

$$\begin{aligned}[EY^s]^{1/s} &= \left[\int_0^\infty u^s f(u)du\right]^{1/s} = \left[1 + s\int_0^\infty u^{s-1} f(u)du\right]^{1/s} \\ &\to \exp\left(\int_0^\infty u^{-1} f(u)du\right) = \exp\left(EY^{-1}\right) \quad \text{as} \quad s \to 0,\end{aligned}$$

as long as EY^{-1} is finite. Applying this result to $Y_n^{(h)}$, recalling the stationarity of the sequence $\{Y_n^{(h)}\}$ and using Theorem 2.1 of Brockwell and Brown (1978), we obtain

$$\begin{aligned}\lim_{h\to 0} c_{\gamma h} &= \exp\left(E(Y_n^{(h)})^{-1}\right) = \exp\left(\int_0^\infty \tilde{f}_{Y_n^{(h)}}(s)ds\right) \\ &= \exp\left(\int_0^\infty e^{\frac{\gamma}{a}\text{dilog}(1+\beta s\sigma)}ds\right). \qquad (5.6)\end{aligned}$$

The behaviour of k_N^{-1}, defined in (5.4), is more complicated. Using l'Hôpital's Rule, we have

$$\lim_{s\to 0} -\frac{\log\Gamma(s+1)}{s} = -\lim_{s\to 0}\frac{\Gamma'(s+1)}{\Gamma(s+1)} = -\Gamma'(1) = \gamma_E,$$

where γ_E is the Euler-Mascheroni constant, with numerical value of $0.5772\cdots$. Hence $\lim_{h\to 0}[\Gamma(\gamma h+1)]^{-1/(\gamma h)} = e^{\gamma_E}$ and

$$\lim_{h\to 0}\lim_{N\to\infty} N^{-1/(\gamma h)} k_N^{-1} = (\sigma\beta)^{-1} e^{\gamma_E} \tag{5.7}$$

When h is small, k_N^{-1} and $c_{\gamma h}$ can be well approximated by (5.6) and (5.7). Since the rate of convergence in Theorem 1, as indicated by k_N^{-1}, increases as h decreases and since the limiting distribution $G_{\gamma h}$ becomes degenerate as $h \to 0$, this suggests the possibility of super-convergence of $\hat{a}_N^{(h)}$ to a as $N\to\infty$ and $h\to 0$. In fact, in Section 6, we show that for any fixed $T>0$, $\hat{a}_{T/h}^{(h)} \to a$ a.s. as $h\to 0$.

Example 1. We now illustrate the estimation procedure with a simulated example. The gamma-driven CAR(1) process defined by,

$$dY(t) + 0.6Y(t)\ dt = dL(t), \qquad t\in[0,5000], \tag{5.8}$$

was simulated at times $0, 0.001, 0.002, \ldots, 5000$, using an Euler approximation. The parameter γ of the standardized gamma process was 2. The process was then sampled at intervals $h=0.01$, $h=0.1$ and $h=1$ by selecting every 10^{th}, 100^{th} and 1000^{th} value respectively. We generated 100 such realizations of the process and applied the above estimation procedure to generate 100 independent estimates, for each h, of the parameters a and σ. The sample means and standard deviations of these estimators are shown in Table 1, which illustrates the remarkable accuracy of the estimators.

Table 1. Estimated parameters based on 100 replicates on $[0,5000]$ of the gamma-driven CAR(1) process (5.8) with $\gamma=2$, observed at times $nh, n=0,\ldots,\lfloor T/h\rfloor$.

Spacing	Parameter	Gamma increments	
		Sample mean of estimators	Sample std deviation of estimators
h=1	a	0.59269	0.00381
	σ	0.99796	0.01587
h=0.1	a	0.59999	0.00000
	σ	1.00011	0.01281
h=0.01	a	0.60000	0.00000
	σ	0.99990	0.01175

To estimate the parameter γ of the driving standardized gamma process, the following procedure was used. For each h and each realization, the estimated CAR(1) parameters were used in (4.2) to generate the estimated increments $\Delta L_n^{(h)},\ n = 1, \ldots, 5000/h$. These were then added in blocks of length $1/h$ to obtain 5000 independent estimated increments of L in one time unit. The histogram of the increments for one realization with $h = .01$ is shown, together with the true probability density of $L(1)$, in Figure 1. Even if we did not know that the background driving Lévy process is a gamma process, the histogram strongly suggests that this is the case. For each h and for each realization of the process, the sample mean $\hat{\gamma}$ of the estimated increments per unit time was then used to estimate the parameter γ of the driving standardized Lévy process, giving a set of 100 independent estimates of γ for each h. The sample means and standard deviations of these estimators are shown in Table 2.

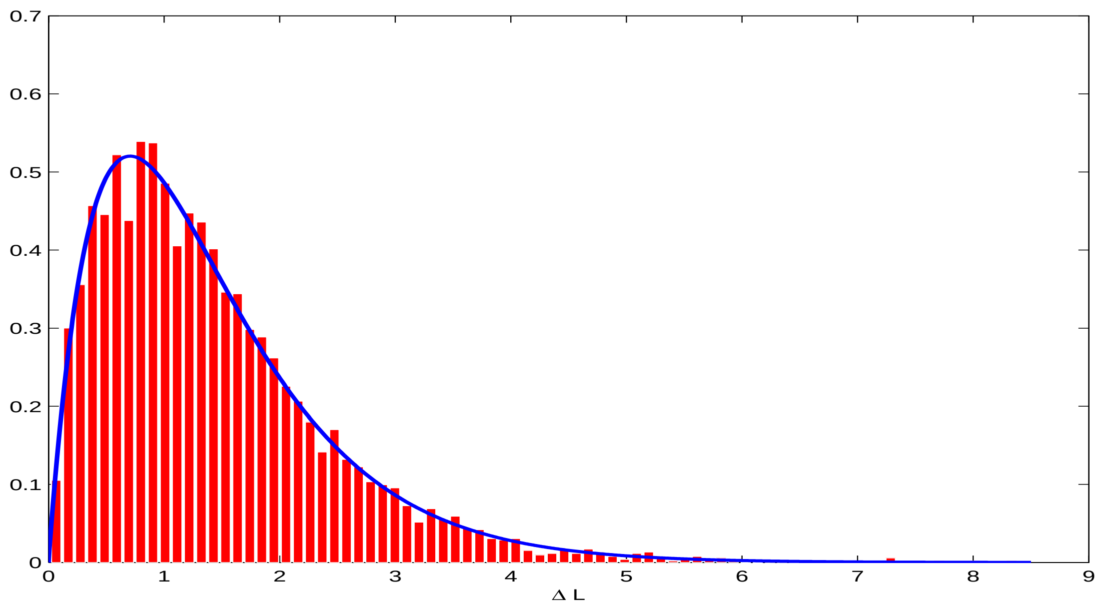

Figure 1: The probability density of the increments per unit time of the standardized Lévy process with $\gamma = 2$ and the histogram of the estimated increments from a realization of the CAR(1) process (5.8), obtained by computing $\Delta L_n^{(.01)},\ n = 1, \ldots, 500000$, from (4.2) and adding successive values in blocks of 100 to give estimated increments per unit time.

Table 2. Estimated parameter of the standardized driving Lévy process.

Spacing	Parameter	Sample mean of estimators	Sample std deviation of estimators
$h = 1$	γ	1.99598	0.05416
$h = 0.1$	γ	2.00529	0.03226
$h = 0.01$	γ	2.00547	0.02762

6. Estimation for the continuously observed process.

It is interesting to note that from a continuously observed realization on $[0, T]$ of a CAR(1) process driven by a non-decreasing Lévy process with drift $m = 0$, the value of a can be identified exactly with probability 1. This contrasts strongly with the case of a Gaussian CAR(1) process. The result is a corollary of the following theorem.

Theorem 2. If the CAR(1) process $\{Y(t), t \geq 0\}$ defined by (2.1) is driven by a non-decreasing Lévy process L with drift m and Lévy measure ν, then for each fixed t,

$$\frac{Y(t+h) - Y(t)}{h} + aY(t) \ \rightarrow \ m\sigma \text{ a.s. as } h \downarrow 0.$$

Proof. From (2.6) we find that

$$Y(t+h) - Y(t) = Y(0)(e^{-a(t+h)} - e^{-at}) + \sigma(L(t+h) - L(t)) \\ -a\sigma \int_0^t e^{-a(t-u)}(e^{-ah} - 1)L(u)du - a\sigma \int_t^{t+h} e^{-a(t+h-u)}L(u)du.$$

Dividing each side by h, letting $h \downarrow 0$, and using the fact that $\lim_{h\downarrow 0}(L(t+h)-L(t))/h = m$ (Shtatland (1965)), we see that

$$\frac{Y(t+h) - Y(t)}{h} \rightarrow m\sigma - aY(0)e^{-at} + a^2\sigma \int_0^t e^{-a(t-u)}L(u)du - a\sigma L(t) = m\sigma - aY(t). \quad \square$$

Corollary 1. If $m = 0$ in Theorem 2 (this is the case if the point zero belongs to the closure of the support of $L(1)$), then for each fixed t, with probability 1,

$$a = \lim_{h\downarrow 0} \frac{\log Y(t) - \log Y(t+h)}{h}. \tag{6.1}$$

For each fixed $T > 0$, a is also expressible, with probability 1, as

$$a = \sup_{0\leq s<t\leq T} \frac{\log Y(s) - \log Y(t)}{t - s}. \tag{6.2}$$

Proof. By setting $L(t) = 0$ for all t in the defining equation (2.1) we obtain the inequality, for all s and t such that $0 \le s < t \le T$,

$$\log Y(s) - \log Y(t) \le a(t-s),$$

from which it follows that

$$a \ge \sup_{0 \le s < t \le T} \frac{\log Y(s) - \log Y(t)}{t-s}. \tag{6.3}$$

From Theorem 2 with $m = 0$ we find that

$$\frac{Y(t) - Y(t+h)}{hY(t)} \to a \text{ as } h \downarrow 0.$$

From the inequalities (6.3) and $1 - x \le -\log x$ for $0 < x \le 1$, we obtain the inequalities,

$$\frac{Y(t) - Y(t+h)}{hY(t)} \le \frac{\log Y(t) - \log Y(t+h)}{h} \le a,$$

and letting $h \downarrow 0$ gives (6.1). But this implies that

$$a \le \sup_{0 \le s < t \le T} \frac{\log Y(s) - \log Y(t)}{t-s},$$

which, with (6.3), gives (6.2). □

Remark 3. If observations are available only at times $\{nh : n = 0, 1, 2, \dots, [T/h]\}$, and if the driving Lévy process has zero drift, Corollary 1 suggests the estimator,

$$\hat{a}_T^{(h)} = \sup_{0 \le n < [T/h]} \frac{\log Y(nh) - \log Y((n+1)h)}{h}.$$

This estimator is precisely the same as the estimator (3.7). Its remarkable accuracy has already been illustrated in Table 1. The analogous estimator, based on closely but irregularly spaced observations at times $t_1, t_2, \dots, t_N$ such that $0 \le t_1 < t_2 < \cdots < t_N \le T$, is

$$\hat{a}_T = \sup_n \frac{\log Y(t_n) - \log Y(t_{n+1})}{t_{n+1} - t_n}.$$

By Corollary 1, both estimators converge almost surely to a as the maximum spacing between successive observations converges to zero.

7. Conclusions

Under the conditions specified in Section 3, we have examined the asymptotic properties of a highly efficient method, using observations at times $0, h, 2h, \dots, Nh$, for estimating the parameters of a stationary Ornstein-Uhlenbeck process $\{Y(t)\}$ driven by a non-decreasing Lévy process $\{L(t)\}$. For h small, we used a discrete approximation to the exact integral representation of $L(t)$ in terms of $\{Y(s), s \le t\}$ to estimate the increments of the driving Lévy process, and hence to estimate the parameters of the Lévy process. Under the specified conditions we obtained the asymptotic distribution of the estimator of the CAR(1) coefficient as $N \to \infty$ with h fixed. The accuracy of the procedure was illustrated with a simulated example of a gamma-driven process. We also showed that the CAR(1) coefficient a is determined almost surely by a continuously observed realization of Y on any interval $[0, T]$. The expression for a suggests an estimator based on discrete observations of Y which, for uniformly spaced observations, is the same as the estimator developed in Section 3.

In Section 6 we found that if L has zero drift the estimator $\hat{a}_T^{(h)}$ based on h-spaced observations on $[0, T]$ is almost surely consistent as $h \to 0$ for any fixed $T > 0$. The asymptotic distribution, as $T \to \infty$ with h fixed, was computed explicitly in Section 5 in the case when L is a gamma process. The critical step was the establishment of the regular variation of the distribution function of $\int_0^h e^{-au} dL(u)$ at 0. More generally if $Ee^{-sL(t)} = e^{-t\Phi(s)}$ where $\Phi(s) = c\log(s) + \sum_{j=0}^{\infty} c_j s^{-j}$ for $|s| > R$, then the same argument as in Section 5 shows that the distribution of the integral is again regularly varying at 0 and that the exponent is ch. In fact the regular variation (RV) condition depends only on the behaviour of the Lévy measure ν near zero. This can be shown as follows. For any given $\epsilon > 0$ the process $L(t)$ can be written as the sum of two independent Lévy processes,

$$L(t) = S(t) + B(t),$$

where S has Lévy measure $\nu((0, \epsilon] \cap \cdot)$ and B has the finite Lévy measure $\nu([\epsilon, \infty) \cap \cdot)$. Provided $x < \epsilon e^{-ah}$, the distribution function of $\int_0^h e^{-au} dL(u)$ can then be expressed as

$$F(x) = P\left(\int_0^h e^{-au} dL(u) \le x\right) = P\left(\int_0^h e^{-au} dS(u) \le x\right) e^{-h\nu((\epsilon,\infty))}.$$

This shows that $F(cx)/F(x)$is independent of $\nu((\epsilon, \infty) \cap \cdot)$ for $x < \epsilon e^{-ah}$. If, in particular, the Lévy density coincides with that of the gamma process $(\gamma x^{-1} e^{-\beta x})$ on some interval $(0, \epsilon]$ then the RV condition is satisfied. This leads to the conjecture that a Lévy density of the form $\gamma x^{-1}(1 + o(x))$ ensures satisfaction of the RV condition with exponent γh.

If the Lévy density of L has the form $\gamma x^{-1-\beta}$, with $\gamma > 0$ and $0 < \beta < 1$, then $L(h)$ and $Z_1 := \sigma \int_0^h e^{-a(h-t)} dL(t)$ are both positive stable random variables with exponent β and the RV condition is not satisfied. This is readily seen when $\beta = 1/2$, in which case the density of Z_1 has the form $f(x) = \sqrt{c/(2\pi)} x^{-1.5} e^{-c/(2x)}$. (For the general case there is no concise explicit form of the density function, but a series expansion can be found in Brockwell and Brown (1978).) The density $f(x)$ approaches zero at a much faster rate as $x \to 0$ than in the gamma case, and since the success of our estimator depends upon the existence of an interval of length h with a very small increment in L, it should be less efficient in this case than in the gamma case.

If on the other hand the Lévy density of L is less than or equal to $\gamma x^{-1-\beta}$, with $\beta < 0$, in some neighbourhood of zero then L has finite Lévy measure and the distributions of $L(h)$ and Z_1 both have positive mass at 0. Hence, with probability one, $\min\{n > 0 : Y(nh) = e^{-ah}Y((n-1)h)\}$ is finite for any fixed $h > 0$ and the estimator $\hat{a}_T^{(h)}$ is *equal* to a for sufficiently large T.

The generalization of the procedure to non-negative Lévy-driven continuous-time ARMA processes is currently in progress.

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