STOCHASTIC CALCULUS FOR CONVOLUTED LÉVY PROCESSES

CHRISTIAN BENDER*† AND TINA MARQUARDT‡

Abstract. We develop a stochastic calculus for processes which are built by convoluting a pure jump, zero expectation Lévy process with a Volterra-type kernel. This class of processes contains, for example, fractional Lévy processes as studied in Marquardt (2006b). The integral which we introduce is a Skorohod integral. Nonetheless we avoid the technicalities from Malliavin calculus and white noise analysis, and give an elementary definition based on expectations under change of measure. As a main result we derive an Itô formula, which separates the different contributions from the memory due to the convolution and from the jumps.

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1. Introduction. In recent years fractional Brownian motion and other Gaussian processes, which are obtained by convolution of an integral kernel with a Brownian motion, have been widely studied as a noise source with memory effects, see e.g. Alòs et al. (2001), Bender (2003b), Biagini et al. (2004), and the survey article by Nualart (2003). Potential applications for noise sources with memory are in such diverse fields as telecommunication, hydrology, and finance, to mention a few.

In Marquardt (2006b) fractional Lévy processes were introduced. While capturing memory effects in a similar fashion as a Brownian motion does, the convolution with a Lévy process provides more flexibility concerning the distribution of the noise, e.g. heavy tails. In this paper we consider a larger class of processes by convolution of a rather general Volterra type kernel with a centered pure jump Lévy process. These convoluted Lévy process may have jumps and/or memory effects depending on the choice of the kernel. Following the elementary S-transform approach, developed by Bender (2003b) for fractional Brownian motion, we motivate and construct a stochastic integral with respect to convoluted Lévy processes. The integral is of Skorohod type, and so its zero expectation property makes it a possible choice to model an additive noise. As a main result we derive Itô formulas for these integrals. The Itô formulas clarify the different influences of jumps and memory effects, which are captured in different terms.

*CORRESPONDING AUTHOR
†INSTITUTE FOR MATHEMATICAL STOCHASTICS, TU BRAUNSCHWEIG, POCKELSSTR. 14, D-38106 BRAUNSCHWEIG, GERMANY, EMAIL: C.BENDER@TU-BS.DE
‡CENTER OF MATHEMATICAL SCIENCES, MUNICH UNIVERSITY OF TECHNOLOGY, D-85747 GARCHING, GERMANY, EMAIL: MARQUARD@MA.TUM.DE

The only other paper, we are aware of, that treats integration for a similar class of processes is Decreusefond & Savy (2006). The class of filtered Poisson processes considered in their paper is analogously defined by replacing the Lévy process by a marked point process in the convolution. They define a Skorohod integral and a Stieltjes integral (the Stieltjes integral exists in their framework as marked point processes have finite activity only). Their Skorohod integral is essentially equivalent to ours, if both are defined. However, we emphasize that our approach allows the Lévy process to be of infinite variation and that our Itô formula for the Skorohod integral is quite different from the one Decreusefond & Savy (2006) derive for the Stieltjes integral only.

The paper is organized as follows: After some preliminaries on Lévy processes and convoluted Lévy processes in Section 2, we discuss the S-transform in Section 3. The results from Section 3 motivate a definition for a Skorohod integral with respect to convoluted Lévy processes which is given in Section 4. In this section some basic properties of this integral are discussed as well. Section 5 is devoted to the derivation of the Itô formulas, while some results are specialized to fractional Lévy processes in Section 6.

2. Preliminaries.

2.1. Basic Facts on Lévy Processes.

We state some elementary properties of Lévy processes that will be needed below. For a more general treatment and proofs we refer to Cont & Tankov (2004) and Sato (1999). For notational convenience we abbreviate $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$. Furthermore, $||f||$ is the ordinary L^2-norm of the function $f : \mathbb{R} \to \mathbb{R}$ and the corresponding inner product is denoted by $(f, g)_{L^2(\mathbb{R})}$. In this paper we assume as given an underlying complete probability space $(\Omega, \mathcal{F}, P)$. Since the distribution of a Lévy processes L on $(\Omega, \mathcal{F}, P)$ is infinitely divisible, L is determined by its characteristic function in the Lévy-Khintchine form $E\left[e^{iuL(t)}\right] = \exp\{t\psi(u)\}, \quad t \geq 0$, where

$$\psi(u) = i\gamma u - \frac{1}{2}u^2\sigma^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux1_{\{|x|\leq 1\}})\,\nu(dx), \quad u \in \mathbb{R}, \tag{2.1}$$

$\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$ and ν is a measure on $\mathbb{R}$ that satisfies

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}} (x^2 \wedge 1)\,\nu(dx) < \infty.$$

The measure ν is referred to as the Lévy measure of L. Notice that conversely, given a generating triplet (γ, σ, ν) satisfying (2.1), the corresponding Lévy process is unique in distribution.

It is a well-known fact that one can associate to every càdlàg Lévy process L on $\mathbb{R}$ a random measure N on $\mathbb{R}_0 \times \mathbb{R}$ describing the jumps of L. For any measurable set $B \subset \mathbb{R}_0 \times \mathbb{R}$,

$$N(B) = \sharp\{s \geq 0 : \ (L_s - L_{s-}, s) \in B\}.$$

The jump measure N is a Poisson random measure on $\mathbb{R}_0 \times \mathbb{R}$ (see e.g. Cont & Tankov (2004, Definition 2.18)) with intensity measure $n(dx, ds) = \nu(dx)\, ds$. By the Lévy-Itô decomposition there exists a Brownian motion $\{B_t\}_{t\geq 0}$ on $\mathbb{R}$ with variance σ^2 such that we can rewrite L almost surely as

$$L(t) = \gamma t + B_t + \int\limits_{|x|\geq 1, s\in[0,t]} x\, N(dx, ds) + \lim_{\varepsilon\downarrow 0} \int\limits_{\varepsilon\leq|x|\leq 1,\, s\in[0,t]} x\tilde{N}(dx, ds), \quad t \geq 0. \quad (2.2)$$

Here $\tilde{N}(dx, ds) = N(dx, ds) - \nu(dx)\, ds$ is the compensated jump measure, the terms in (2.2) are independent and the convergence in the last term is a.s. and locally uniform in $t \geq 0$. Assuming that ν satisfies additionally

$$\int\limits_{|x|>1} x^2\, \nu(dx) < \infty, \quad (2.3)$$

L has finite mean and variance given by

$$\mathrm{var}(L(1)) = \int\limits_{\mathbb{R}} x^2\, \nu(dx). \quad (2.4)$$

If in (2.1) $\sigma = 0$ and hence $B_t = 0$ for all $t \geq 0$, we call L a Lévy process without Brownian component. In what follows we will always assume that the Lévy process L has no Brownian part. Furthermore we suppose that $E[L(1)] = 0$, hence $\gamma = -\int_{|x|>1} x\, \nu(dx)$. Thus, (2.1) can be written in the form

$$\psi(u) = \int\limits_{\mathbb{R}} (e^{iux} - 1 - iux)\, \nu(dx), \quad u \in \mathbb{R}, \quad (2.5)$$

and (2.2) simplifies to

$$L(t) = \int\limits_0^t \int\limits_{\mathbb{R}_0} x\tilde{N}(dx, ds), \quad t \in \mathbb{R}. \quad (2.6)$$

In this case $L = \{L(t)\}_{t\geq 0}$ is a martingale. In the sequel we will work with a two-sided Lévy process $L = \{L(t)\}_{t\in\mathbb{R}}$, constructed by taking two independent copies $\{L_1(t)\}_{t\geq 0}$, $\{L_2(t)\}_{t\geq 0}$ of a one-sided Lévy process and setting

$$L(t) = \begin{cases} L_1(t) & \text{if} \quad t \geq 0 \\ L_2(-t-) & \text{if} \quad t < 0. \end{cases} \quad (2.7)$$

From now on we will suppose that $\mathcal{F}$ is the completion of the σ-algebra generated by the two-sided Lévy process L and denote $L^2(\Omega) := L^2(\Omega, \mathcal{F}, P)$.

2.2. Convoluted and Fractional Lévy Processes.

We call a stochastic process $M = \{M(t)\}_{\in\mathbb{R}}$ given by

$$M(t) = \int_{\mathbb{R}} f(t,s)\, L(ds), \qquad t \in \mathbb{R}, \tag{2.8}$$

a *convoluted Lévy process* with kernel f. Here, $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a measurable function satisfying the following properties

(i) $f(t,\cdot) \in L^2(\mathbb{R})$ for all $t \in \mathbb{R}$,

(ii) for every $t \geq 0$ the mapping $s \mapsto f(t,s)$ is left-continuous and it is right-continuous for every $t < 0$,

(iii) $f(t,s) = 0$ whenever $s > t \geq 0$, i.e. the kernel is of Volterra type.

(iv) $f(0,s) = 0$ for almost every s, hence $M(0) = 0$.

Furthermore, we suppose that $L = \{L(t)\}_{t\in\mathbb{R}}$ is a Lévy process without Brownian component satisfying $E[L(1)] = 0$ and $E[|L(t)|^m] < \infty$ for all $m \in \mathbb{N}$. Hence the process M can be rewritten as

$$M(t) = \int_{\mathbb{R}} \int_{\mathbb{R}_0} f(t,s)x\, \tilde{N}(dx, ds), \qquad t \in \mathbb{R}, \tag{2.9}$$

and has absolute moments of arbitrary order.

Since $f(t,\cdot) \in L^2(\mathbb{R})$, the integral (2.9) exists in $L^2(\Omega, P)$ and

$$E[M(t)^2] = E[L(1)^2] \int_{\mathbb{R}} f^2(t,s)\, ds = E[L(1)^2] \|f(t,\cdot)\|^2_{L^2(\mathbb{R})}. \tag{2.10}$$

As an important example for convoluted Lévy processes we now consider univariate fractional Lévy processes. The name "fractional Lévy process" already suggests that it can be regarded as a generalization of fractional Brownian motion (FBM). We review the definition of a one-dimensional fractional Lévy process (FLP). For further details on FLPs see Marquardt (2006a) and Marquardt (2006b).

Definition 2.1 (Fractional Lévy Process (FLP)). *Let $L = \{L(t)\}_{t\in\mathbb{R}}$ be a Lévy process on $\mathbb{R}$ with $E[L(1)] = 0$, $E[L(1)^2] < \infty$ and without Brownian component. For fractional integration parameter $-0.5 < d < 0.5$ a stochastic process*

$$M_d(t) = \frac{1}{\Gamma(d+1)} \int_{-\infty}^{\infty} \left[(t-s)_+^d - (-s)_+^d\right] L(ds), \quad t \in \mathbb{R}, \tag{2.11}$$

is called a fractional Lévy process *(FLP).*

Note that the kernel (2.11) given by

$$f_t(s) = \frac{1}{\Gamma(1+d)} [(t-s)_+^d - (-s)_+^d], \quad s \in \mathbb{R}, \tag{2.12}$$

satisfies conditions (i)–(iv). Thus, fractional Lévy processes are well-defined and belong to $L^2(\Omega)$ for fixed t.

Moreover, the kernel can be represented by fractional integrals, resp. derivatives of the indicator function. Recall, for $0 < \alpha < 1$ the fractional integral of Riemann-Liouville type $I^\alpha_\pm$ is defined by

$$(I^\alpha_- f)(x) = \frac{1}{\Gamma(\alpha)} \int\limits_x^\infty f(t)(t-x)^{\alpha-1}\,dt,$$

$$(I^\alpha_+ f)(x) = \frac{1}{\Gamma(\alpha)} \int\limits_{-\infty}^x f(t)(x-t)^{\alpha-1}\,dt,$$

if the integrals exist for almost all $x \in \mathbb{R}$.

The fractional derivative of Marchaud's type $I^{-\alpha}_\pm$ of order $0 < \alpha < 1$ is given by $(\epsilon > 0)$

$$(I^{-\alpha}_{\pm,\epsilon} f)(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int\limits_\epsilon^\infty \frac{f(x) - f(x \mp t)}{t^{1+\alpha}}\,dt$$

and

$$(I^{-\alpha}_\pm f) = \lim_{\epsilon \to 0+} (I^{-\alpha}_{\pm,\epsilon} f),$$

if the limit exists in $L^p(\mathbb{R})$ for some $p > 1$.

In terms of these fractional operators fractional Lévy processes can be rewritten as

$$M_d(t) = \int\limits_{-\infty}^\infty (I^d_- \chi_{[0,t]})(s)\,L(ds), \quad t \in \mathbb{R}, \tag{2.13}$$

where the indicator $\chi_{[a,b]}$ is given by $(a, b \in \mathbb{R})$:

$$\chi_{[a,b]}(t) = \begin{cases} 1, \text{ if } a \leq t < b \\ -1, \text{ if } b \leq t < a \\ 0, \text{ otherwise.} \end{cases} \tag{2.14}$$

Remark 2.2. The distribution of $M_d(t)$ is infinitely divisible for all $t \in \mathbb{R}$,

$$E[M_d(t)^2] = t^{2d+1} E[L(1)^2], \quad t \in \mathbb{R}, \qquad \text{and}$$

$$E[\exp\{izM_d(t)\}] = \exp\left\{ \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \left(e^{izf_t(s)x} - 1 - izf_t(s)x \right) \nu(dx)\,ds \right\},\ t, z \in \mathbb{R}. \tag{2.15}$$

3. The Lévy Wick Exponential and the S-transform. One of our aims is to introduce a Hitsuda-Skorohod integral for convoluted Lévy processes without touching the technicalities from the Malliavin calculus and white noise analysis. Our approach is based upon the S-transform, which uniquely determines a square integrable random variable by its expectation under an appropriately rich class of probability measures. As a preparation and motivation we compute the S-transform of Itô integrals with respect to the compensated jump measure $\tilde{N}$ in this section. This result then yields a simple definition for an anticipative integrals with respect to $\tilde{N}$.

We begin with some definitions:

Definition 3.1 (Lévy Wick Exponential). *Let $\mathcal{S}(\mathbb{R}^2)$ denote the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}^2$. For $\eta \in \Xi$, where*

$$\Xi = \left\{ \eta \in S(\mathbb{R}^2) : \ \eta(x,t) > -1, \ \ \eta(0,t) = 0, \ \ \frac{d}{dx}\eta(0,t) = 0, \ \ \textit{for all}\, t, x \in \mathbb{R} \right\},$$

the Wiener integral *is defined by*

$$I_1(\eta) = \int_{\mathbb{R}} \int_{\mathbb{R}_0} \eta(x,s)\, \tilde{N}(dx,ds), \tag{3.1}$$

and the Wick exponential *of $I_1(\eta)$ by*

$$\exp^{\diamond}(I_1(\eta)) = \exp\left\{ \int_{\mathbb{R}} \int_{\mathbb{R}_0} \log(1+\eta(x,t)) N(dx,dt) - \int_{\mathbb{R}} \int_{\mathbb{R}_0} \eta(x,t)\, \nu(dx,dt) \right\}. \tag{3.2}$$

Remark 3.2.

(1) By Theorem 3.1 in Lee & Shih (2004),

$$\exp^{\diamond}(I_1(\eta)) = \sum_{n=0}^{\infty} \frac{I_n(\eta^{\otimes n})}{n!}. \tag{3.3}$$

where I_n denotes the multiple Wiener integral of order n with respect to the compensated Lévy measure. This respresentation justifies the name Wick exponential.

(2) Since $\exp^{\diamond}(I_1(\eta))$ coincides with the Doléans-Dade exponential of $I_1(\eta)$ at $t = \infty$ it is straightforward that for $\eta, \tilde{\eta} \in \Xi$ we have

$$E[\exp^{\diamond}(I_1(\eta))] = 1 \quad \text{and} \quad E[\exp^{\diamond}(I_1(\eta)) \cdot \exp^{\diamond}(I_1(\tilde{\eta}))] = \exp\{(\eta, \tilde{\eta})_{L^2(\nu \times \lambda)}\},$$

where λ denotes the Lebesgue measure.

We can now define the S-transform.

Definition 3.3 (S-transfrom). *For $X \in L^2(\Omega, P)$ the S-transform SX of X is an integral transform defined on the set Ξ by*

$$(SX)(\eta) = E^{Q_\eta}[X], \tag{3.4}$$

where

$$dQ_\eta = \exp^\diamond(I_1(\eta))\, dP.$$

Various definitions of the S-transform can be found in the literature, which differ by the chosen subset of deterministic integrands. Our choice of Ξ is particularly convenient because of the smoothness of its members. Moreover, it is a sufficiently rich set, as is demonstrated by the following theorem. It states that every square integrable random variable is uniquely determined by its S-transform.

PROPOSITION 3.4. *The S-transform is injective, i.e. if $S(X)(\eta) = S(Y)(\eta)$ for all $\eta \in \Xi$, then $X = Y$.*

Proof. The assertion is proven in Løkka & Proske (2006, Theorem 5.3) by reformulating a more general result from Albeverio et al. (1996, Theorem 5). □

For later reference we introduce the Wick product, which can be defined in terms of the S-transform.

DEFINITION 3.5 (Wick product). *Let $X, Y \in L^2(\Omega, P)$ and assume that there is an element $X \diamond Y \in L^2(\Omega)$ such that*

$$S(X \diamond Y)(\eta) = S(X)(\eta) S(Y)(\eta), \quad \textit{for all} \quad \eta \in \Xi.$$

Then $X \diamond Y$ is referred to as the Wick product of X and Y.

EXAMPLE 3.6. *Let $\eta, \tilde{\eta} \in \Xi$. Then*

$$\exp^\diamond(I_1(\eta)) \diamond \exp^\diamond(I_1(\tilde{\eta})) \;=\; \exp^\diamond(I_1(\eta + \tilde{\eta}))\,.$$

This product rule is another justification for the terminology 'Wick exponential'.

We shall now calculate the S-transform of an Itô integral w.r.t. the compensated jump measure $\tilde{N}$. To this end let $T > 0$ and $X : \mathbb{R}_0 \times [0, T] \times \Omega \to \mathbb{R}$ a predictable random field (with respect to the filtration $\mathcal{F}_t$ generated by the Lévy process $L(s),\ 0 \le s \le t$) satisfying

$$E\left[\int_0^T \int_{\mathbb{R}_0} |X(y,t)|^2\, \nu(dy)\, dt\right] < \infty.$$

Then the compensated Poisson integral $\int_0^T \int_{\mathbb{R}_0} X(y,t)\, \tilde{N}(dy, dt)$ exists in $L^2(\Omega, P)$.

The following theorem characterizes this integral in terms of the S-transform. The result was derived by Løkka & Proske (2006, Corollary 7.4) by lengthy calculations involving multiple Wiener integrals. We here provide a short proof which only makes use of classical tools such as the Girsanov theorem.

THEOREM 3.7. *Let X denote a predictable random field satisfying the above integrability condition. Then $\int_0^T \int_{\mathbb{R}_0} X(y,t)\, \tilde{N}(dy, dt)$ is the unique square integrable random variable with S-transform given by*

$$\int_0^T \int_{\mathbb{R}_0} S(X(y,t))(\eta)\, \eta(y,t)\, \nu(dy)\, dt, \quad \eta \in \Xi. \tag{3.5}$$

Proof. Applying Girsanov's Theorem for random measures (Jacod & Shiryaev (2003, Theorem 3.17)) we obtain that under the measure Q_η the compensator of $N(dy, dt)$ is given by $(1 + \eta(y,t))\nu(dy)\,dt$. Hence,

$$\int_0^T \int_{\mathbb{R}_0} X(y,t)\,\tilde{N}(dy,dt) - \int_0^T \int_{\mathbb{R}_0} X(y,t)\eta(y,t)\,\nu(dy)\,dt \tag{3.6}$$

is a Q_η- local martingale. In particular, if $0 = \tau_1 \leq \ldots \tau_N < \infty$ is a localizing sequence of stopping times with $\lim_{N\to\infty} \tau_N = \infty$ a.s., then

$$\begin{aligned}
&\lim_{N\to\infty} E^{Q_\eta}\left[\int_0^{T\wedge\tau_N} \int_{\mathbb{R}_0} X(t,y)\,\tilde{N}(dy,dt)\right] \\
&= \lim_{N\to\infty} E^{Q_\eta}\left[\int_0^{T\wedge\tau_N} \int_{\mathbb{R}_0} X(t,y)\,\eta(y,t)\nu(dy)\,dt\right] \\
&= E^{Q_\eta}\left[\int_0^{T} \int_{\mathbb{R}_0} X(t,y)\,\eta(y,t)\nu(dy)\,dt\right]
\end{aligned}$$

by a straightforward application of the dominated convergence theorem.

To treat the limit in the first line, note that

$$E^{Q_\eta}\left[\int_0^{T\wedge\tau_N} \int_{\mathbb{R}_0} X(y,t)\,\tilde{N}(dy,dt)\right] = E^P\left[\exp^\diamond(I_1(\eta)) \int_0^{T\wedge\tau_N} \int_{\mathbb{R}_0} X(y,t)\,\tilde{N}(dy,dt)\right].$$

The integrand on the right hand side is dominated by

$$\exp^\diamond(I_1(\eta)) \sup_{0\leq u\leq T}\left|\int_0^{u} \int_{\mathbb{R}_0} X(y,t)\,\tilde{N}(dy,dt)\right|,$$

which is P-integrable by Hölder's inequality, Doob's inequality and the assumed integrability of the random field. Thus,

$$E^{Q_\eta}\left[\int_0^{T} \int_{\mathbb{R}_0} X(y,t)\,\tilde{N}(dy,dt)\right] = E^{Q_\eta}\left[\int_0^{T} \int_{\mathbb{R}_0} X(y,t)\,\eta(y,t)\nu(dy)\,dt\right],$$

and the assertion follows by applying Fubini's theorem.

Note, that the last identity shows that the local Q_η-local martingale (3.6) is a Q_η-martingale indeed. □

EXAMPLE 3.8. *From the previous theorem, applied to both sides of the two-sided Lévy process separately, we derive, for $t \geq 0$,*

$$S(M(t))(\eta) = \int_{-\infty}^{t} \int_{\mathbb{R}_0} f(t,s) y\,\eta(y,s)\,\nu(dy)\,ds,$$

since

$$M(t) = \int_{-\infty}^{t} f(t,s) L(ds) = \int_{-\infty}^{t} \int_{\mathbb{R}_0} f(t,s) y \, \tilde{N}(dy, ds).$$

The S-transform characterization in the previous theorem gives rise to a straightforward extension to anticipative random fields.

DEFINITION 3.9. *Suppose X is a random field.*
(i) The Hitsuda-Skorohod integral *of X with respect to the compensated jump measure $\tilde{N}$ is said to exist in $L^2(\Omega)$, if there is a random variable $\Phi \in L^2(\Omega)$ such that for all $\eta \in \Xi$*

$$S\Phi(\eta) = \int_{\mathbb{R}} \int_{\mathbb{R}_0} S(X(y,t))(\eta) \, \eta(y,t) \, \nu(dy) \, dt.$$

It is denoted by $\Phi = \int_{\mathbb{R}} \int_{\mathbb{R}_0} X(y,t) \, \tilde{N}^{\diamond}(dy, dt)$.
(ii) The Hitsuda-Skorohod integral *of X with respect to the jump measure N is defined as*

$$\int_{\mathbb{R}} \int_{\mathbb{R}_0} X(y,t) \, N^{\diamond}(dy, dt) := \int_{\mathbb{R}} \int_{\mathbb{R}_0} X(y,t) \, \tilde{N}^{\diamond}(dy, dt) + \int_{\mathbb{R}} \int_{\mathbb{R}_0} X(y,t) \, \nu(dy) dt$$

if both integrals on the right hand side exist in $L^2(\Omega)$.

REMARK 3.10. From the previous definition we get immediately that

$$S\left(\int_{0}^{T} \int_{\mathbb{R}_0} X(y,t) \, N^{\diamond}(dy, dt) \right)(\eta) = \int_{0}^{T} \int_{\mathbb{R}_0} S(X(y,t))(\eta) \, (1 + \eta(y,t)) \, \nu(dy) \, dt.$$

Clearly, if the integrand is predictable, this Skorohod integral reduces to the ordinary stochastic integral for random measures, and the diamond can be omitted in this case.

REMARK 3.11. Theorem 3.7 implies

$$S\left(\tilde{N}(A, [0,t])\right)(\eta) = t \int_{A} \eta(y,t) \, \nu(dy).$$

Hence, we can write in a suggestive notation

$$S\left(\int_{0}^{T} \int_{\mathbb{R}_0} X(y,t) \, \tilde{N}^{\diamond}(dy, dt) \right)(\eta) = \int_{0}^{T} \int_{\mathbb{R}_0} S(X(y,t))(\eta) S\left(\tilde{N}(dy, dt)\right)(\eta).$$

In view of Example 3.8, Theorem 3.7 can be specialized to integrals with respect to the Lévy process L as follows.

Corollary 3.12. *Let $0 \le a \le b$ and $X : [a,b] \times \Omega \to \mathbb{R}$ be a predictable process such that $E[\int_a^b |X(t)|^2\, dt] < \infty$. Then $\int_a^b X(s)\, L(ds)$, is the unique square integrable random variable with S-transform given by*

$$\int_a^b \int_{\mathbb{R}_0} S(X(t))(\eta)\, \frac{d}{dt} S(L(t))(\eta)\, dt, \quad \eta \in \Xi.$$

4. A Skorohod Integral for Convoluted Lévy Processes. In this section we define the Skorohod integral for convoluted Lévy processes and state some basic properties. The definition is strongly motivated by Corollary 3.12 above.

Definition 4.1. *Suppose that the mapping*

$$t \mapsto S(M(t))(\eta)$$

is differentiable for every $\eta \in \Xi$. Let $X : B \times \Omega \to L^2(\Omega)$ ($B \subset \mathbb{R}$ a Borel set). Then X is said to have a Hitsuda-Skorohod integral *with respect to M if*

$$S(X(\cdot))(\eta) \frac{d}{dt} S(M(\cdot))(\eta) \in L^1(B) \quad \textit{for any } \eta \in \Xi$$

and there is a $\Phi \in L^2(\Omega)$ such that for all $\eta \in \Xi$,

$$S(\Phi)(\eta) = \int_B S(X(t))(\eta) \frac{d}{dt} S(M(t))(\eta)\, dt.$$

In that case Φ is uniquely determined by the injectivity of the S-transform and we denote

$$\Phi = \int_B X(t)\, M^\diamond(dt).$$

Remark 4.2. (i) The definition of the Skorohod integral does not require measurability conditions such as predictability or progressive measurability. Hence, it also generalizes the Itô integral w.r.t the underlying Lévy process to anticipative integrands.

(ii) Since the Lévy process itself is stochastically continuous, the S-transform cannot distinguish between $L(t)$ and $L(t-)$ for fixed t. Consequently, we obtain e.g.

$$\int_0^t L(s)\, L^\diamond(ds) = \int_0^t L(s-)\, L^\diamond(ds) = \int_0^t L(s-)\, L(ds),$$

where the last integral is the classical Itô integral.

The following properties of the Skorohod integral are an obvious consequence of the definition:

Proposition 4.3.

(i) For all $a < b \in \mathbb{R}$,

$$M(b) - M(a) = \int_a^b M^\diamond(dt).$$

(ii) Let $X : B \times \Omega \to L^2(\Omega)$ be Skorohod integrable. Then

$$\int_B X(t)\, M^\diamond(dt) = \int_{\mathbb{R}} 1_B(t) X(t)\, M^\diamond(dt),$$

where 1_B denotes the indicator function of the set B.

(iii) Let $X : B \times \Omega \to L^2(\Omega)$ be Skorohod integrable. Then

$$E\left[\int_B X(t)\, M^\diamond(dt)\right] = 0.$$

We note, that (iii) holds since the expectation coincides with the S-transform at $\eta = 0$. The zero expectation property makes the integral a promising candidate for modeling an additive noise.

Example 4.4. *As an example we show how to calculate*

$$\int_0^T M(t)\, M^\diamond(dt).$$

In the following manipulations $\tilde{N}^\eta$ denotes the compensated jump measure under the probability measure $Q_\eta = \exp^\diamond(I_1(\eta))dP$. In particular, it follows from Girsanov's theorem, as in the proof of Theorem 3.7, that

$$M(T) = \int_{-\infty}^T \int_{\mathbb{R}_0} f(T,s) y\, \tilde{N}^\eta(dy, ds) + \int_{-\infty}^T \int_{\mathbb{R}_0} f(T,s) y \eta(y,s) \nu(dy)\, ds.$$

By this identity, integration by parts, and Example 3.8, we obtain

$$S\left(2\int_0^T M(t)M^\diamond(dt)\right)(\eta) = 2\int_0^T S(M(t))(\eta)\frac{d}{dt}S(M(t))(\eta)dt$$

$$= \left(\int_0^t S(M(t))(\eta)dt\right)^2 = \left(\int_{-\infty}^T\int_{\mathbb{R}_0} f(T,s)y\eta(y,s)\nu(dy)\,ds\right)^2$$

$$= E^{Q_\eta}\left[\left(\int_{-\infty}^T\int_{\mathbb{R}_0} f(T,s)y\,\tilde{N}^\eta(dy,ds) + \int_{-\infty}^T\int_{\mathbb{R}_0} f(T,s)y\eta(y,s)\nu(dy)\,ds\right)^2\right]$$

$$-E^{Q_\eta}\left[\left(\int_{-\infty}^T\int_{\mathbb{R}_0} f(T,s)y\,\tilde{N}^\eta(dy,ds)\right)^2\right]$$

$$= S(M(T)^2)(\eta) - \int_{-\infty}^T\int_{\mathbb{R}_0} f(T,s)^2y^2(1+\eta(y,s))\nu(dy)\,ds.$$

Here we have used that $\int_{-\infty}^T\int_{\mathbb{R}_0} f(T,s)y\,\tilde{N}^\eta(dy,ds)$ has zero expectation and variance $\int_{-\infty}^T\int_{\mathbb{R}_0} f(T,s)^2y^2(1+\eta(y,s))\nu(dy)\,ds$, since the compensator of N under Q_η is given by $(1+\eta(y,s))\nu(dy,ds)$.

Hence, we derive from Remark 3.11 the identity

$$2\int_0^T M(t)M^\diamond(dt) = M(T)^2 - \int_{-\infty}^T\int_{\mathbb{R}_0} f(T,s)^2y^2\,N(dy,ds)$$

$$= M(T)^2 - \sum_{-\infty<s\leq T} f(T,s)^2(\Delta L(s))^2.$$

The next elementary result states that the Skorohod integral nicely behaves under Wick multiplication.

THEOREM 4.5. *Let $X:\mathbb{R}\times\Omega\to L^2(\Omega)$ and $Y\in L^2(\Omega)$. Then*

$$Y\diamond\int_{\mathbb{R}} X(s)\,M^\diamond(ds) = \int_{\mathbb{R}} Y\diamond X(s)\,M^\diamond(ds),$$

in the sense that if one side is well-defined then so is the other and both coincide.

Proof. The assertion follows by calculating the S-transform of both sides. □

REMARK 4.6. Applying the same techniques as in Example 4.4 one can easily obtain for $a\leq b$

$$M(a)\int_a^b 1\,M^\diamond(dt) = \int_a^b M(a)\,M^\diamond(dt) + \int_0^a\int_{\mathbb{R}_0} f(a,s)(f(b,s)-f(a,s))y^2\,N(dy,ds).$$

Hence ordinary multiplication with a random variable, which is measurable with respect to the information up to the lower integration bound, cannot in general be introduced under the integral sign, if the kernel depends on the past.

5. Itô's Formula. In this section we will derive an Itô formula for convoluted Lévy processes. The proof is based on a calculation of the time derivative of $S(G(M(t)))(\eta)$. It may be seen as a generalization of the calculations in Example 4.4. This technique of proof is in the spirit of Kubo (1983), Bender (2003a), and Lee & Shih (2000), where this approach was applied to obtain Itô formulas for generalized functionals of a Brownian motion, a fractional Brownian motion, and a Lévy process with Brownian component respectively.

During the derivation of the Itô formula we have to interchange differentiation and integration several times. Under the following (rather strong) conditions on the convolution kernel these manipulations are easily justified. However, the Itô formulas below may also be viewed as generic results which hold for more general kernels (with the technicalities to be checked on a case-by-case basis).

(**H**) The kernel f has compact support, is bounded and the derivative $\frac{d}{dt}f(t,s)$ is bounded as well.

Example 5.1. *The following prominent examples satisfy condition (H):*
(a) One-sided shot noise processes *defined by the kernel*

$$f(t,s) = \begin{cases} k(t-s), & 0 \le s \le t \le T^* \\ 0, & \text{otherwise} \end{cases}$$

for constants $T^* > 0$ *and* $k \in \mathbb{R}$.
(b) One-sided Ornstein Uhlenbeck type processes *defined by the kernel*

$$f(t,s) = \begin{cases} e^{-k(t-s)}, & 0 \le s \le t \le T^* \\ 0, & \text{otherwise} \end{cases}$$

for constants $T^* > 0$ *and* $k \in \mathbb{R}$.

To state the Itô formulas precisely we finally recall that the *Wiener algebra* is defined as

$$A(\mathbb{R}) := \{G \in L^1(\mathbb{R});\ \mathcal{F}G \in L^1(\mathbb{R})\}$$

where $\mathcal{F}$ denotes the Fourier transform. Note that the space of rapidly decreasing smooth functions is included in the Wiener algebra.

The first Itô formula requires that the underlying Lévy process is a finite variation process.

Theorem 5.2 (Itô formula I). *Let* $T > 0$*, (H) hold and*

$$\int_{\mathbb{R}_0} |x|\nu(dx) < \infty.$$

Furthermore assume that $G \in C^1(\mathbb{R})$ with $G, G' \in A(\mathbb{R})$ bounded. Then,

$$\int_0^T \left(\int_{-\infty}^t \int_{\mathbb{R}_0} G'(M(t) + xf(t,s))x \frac{d}{dt} f(t,s)\; N^\diamond(dx, ds) \right) dt$$

exist in $L^2(\Omega)$ and

$$\begin{aligned} G(M(T)) = G(0) &- \left(\int_{\mathbb{R}_0} x\,\nu(dx) \right) \int_0^T G'(M(t)) \left(f(t,t) + \int_{-\infty}^t \frac{d}{dt} f(t,s)\, ds \right) dt \\ &+ \sum_{0 \le t \le T} G(M(t-) + f(t,t)\Delta L(t)) - G(M(t-)) \\ &+ \int_0^T \left(\int_{-\infty}^t \int_{\mathbb{R}_0} G'(M(t-) + xf(t,s))x \frac{d}{dt} f(t,s)\; N^\diamond(dx, ds) \right) dt. \end{aligned}$$

In the general case the Itô formula reads as follows:

Theorem 5.3 (Itô formula II). *Let $T > 0$, (H) hold. Furthermore assume that $G \in C^1(\mathbb{R})$ with $G, G' \in A(\mathbb{R})$. Then,*

$$\begin{aligned} G(M(T)) &= G(0) + \int_0^T G'(M(t-))\; M^\diamond(dt) \\ &+ \sum_{0 \le t \le T} G(M(t-) + f(t,t)\Delta L(t)) - G(M(t-)) - G'(M(t-)) f(t,t)\Delta L(t) \\ &+ \int_0^T \left(\int_{-\infty}^t \int_{\mathbb{R}_0} (G'(M(t-) + xf(t,s)) - G'(M(t-)))\, x \frac{d}{dt} f(t,s)\; N^\diamond(dx, ds) \right) dt, \end{aligned}$$

provided all terms exist in $L^2(\Omega)$.

The above Itô formulas reduce to well-known formulas for Lévy processes with the choice $f(t,s) = \chi_{(0,t]}(s)$, as in this case the last Skorohod integral with respect to N vanishes. We would like to emphasize that M is continuous, if and only if $f(t,t) = 0$ for all t. Moreover, M has independent increments, if and only if $\frac{d}{dt} f(t,s) = 0$ for all t. Hence the contributions from discontinuities and memory effects are nicely separated in the above Itô formulas. Finally, notice that the formula for $M(t)^2$ from Example 4.4 can be recovered by formally applying the Itô formula II with $G(y) = y^2$.

Remark 5.4. The Itô formula II has the drawback that the conditions do not guarantee that all members of the identity exist in $L^2(\Omega)$. However the manipulations below can be cast into a white noise framework as developed in Øksendal & Proske (2004) in a way that all members exist as generalized random variables.

The remainder of this section is devoted to the proof of the Itô formulas. As a general strategy we wish to show that both sides of the asserted identities have the same S-transform. Indeed, the following calculations show how to identify the right hand side constructively. We first write

$$S(G(M(T)))(\eta) = G(0) + \int_0^T \frac{d}{dt} S(G(M(t)))(\eta) dt$$

and then calculate $\frac{d}{dt} S(G(M(t)))$ explicitly. To achieve this, we apply the inverse Fourier theorem and obtain for $G \in A(\mathbb{R})$

$$S(G(M(t)))(\eta) = E^{Q_\eta}[G[M(t)]] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathcal{F}G(u) E^{Q_\eta}[e^{iuM(t)}] du \tag{5.1}$$

To differentiate this expression we calculate the characteristic function of M under Q_η.

PROPOSITION 5.5. *Let $M = \{M(t)\}_{t\in\mathbb{R}}$ be a convoluted Lévy process as defined in (2.8) with kernel function f. Then, for $t \geq 0$,*

$$S(e^{iuM(t)})(\eta) = E^{Q_\eta}[e^{iuM(t)}]$$
$$= \exp\left\{ \int_{-\infty}^{t} \int_{\mathbb{R}_0} \Big[\Big(e^{iuxf(t,s)} - 1 - iuxf(t,s)\Big)(1 + \eta(x,s)) + iuS(M(t))(\eta)\Big] \nu(dx)\, ds \right\}$$

Proof. It follows from the proof of Theorem 3.7 that

$$L^Q(t) := L(t) - \int_0^t \int_{\mathbb{R}_0} x\eta(x,s)\, \nu(dx)\, ds$$

is a Q_η-martingale with zero mean. Applying Girsanov's Theorem for semimartingales (Jacod & Shiryaev (2003, Theorem 3.7)) yields that L^Q has semimartingale characteristics $(\gamma_s^Q, 0, \nu_s^Q)$, where

$$\gamma_s^Q = - \int_{|x|>1} x(1 + \eta(x,s))\, \nu(dx)$$

and

$$\nu_s^Q(dx) = (1 + \eta(x,s))\, \nu(dx).$$

Hence

$$S(\exp\{iuL^Q(t)\})(\eta) = E^{Q_\eta}[e^{iuL^Q(t)}]$$
$$= \exp\left\{ \int_0^t \int_{\mathbb{R}_0} \left[e^{iux} - 1 - iux\right] \left[1 + \eta((x,s)\right] \nu(dx)\, ds \right\}$$

Finally,

$$
\begin{aligned}
&S(\exp\{iuM(t)\})(\eta) = E^{Q_\eta}\left[\exp\left\{iu\int_{-\infty}^{t} f(t,s)\,L(ds)\right\}\right]\\
=&E^{Q_\eta}\left[\exp\left\{iu\int_{-\infty}^{t} f(t,s)\,L^Q(ds) + iu\int_{-\infty}^{t} f(t,s)\int_{\mathbb{R}_0} x\eta(x,s)\,\nu(dx)\,ds\right\}\right]\\
=&\exp\left\{\int_{-\infty}^{t}\int_{\mathbb{R}_0}\left[e^{iuxf(t,s)} - 1 - iuxf(t,s)\right][1+\eta(x,s)]\;\nu(dx)\,ds\right\}\\
&\quad\times\exp\left\{\int_{-\infty}^{t}\int_{\mathbb{R}_0} iuxf(t,s)\eta(x,s)\,\nu(dx)\,ds\right\}\\
=&\exp\left\{\int_{-\infty}^{t}\int_{\mathbb{R}_0}\left\{\left[e^{iuxf(t,s)} - 1\right][1+\eta(x,s)] - iuxf(t,s)\right\}\;\nu(dx)\,ds\right\}.
\end{aligned}
$$

Taking the S-transform of M into account, which was calculated in Example 3.8, the assertion follows. □

By introducing the derivative under the integral sign, we get

$$
\begin{aligned}
&\frac{d}{dt}E^{Q_\eta}[e^{iuM(t)}]\\
=&E^{Q_\eta}[e^{iuM(t)}]\int_{\mathbb{R}_0}\left[\left(e^{iuxf(t,t)} - 1 - iuxf(t,t)\right)(1+\eta(x,t))\right]\nu(dx)\\
&\quad + E^{Q_\eta}[e^{iuM(t)}]\int_{-\infty}^{t}\int_{\mathbb{R}_0}\left[x\frac{d}{dt}f(t,s)\left(e^{iuxf(t,s)} - 1\right)(1+\eta(x,t))\right]\nu(dx)ds\\
&\quad + E^{Q_\eta}[e^{iuM(t)}]iuS(M(t))(\eta) \qquad (5.2)
\end{aligned}
$$

Combining (5.1) with (5.2), and interchanging differentiation and integration again, (which can be justified under (H), since $G, G' \in A(\mathbb{R})$), we obtain

$$
\begin{aligned}
&\frac{d}{dt}S(G(M(t)))(\eta) \\
=&\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}(\mathcal{F}G)(u)E^{Q_\eta}[e^{iuM(t)}]\int_{\mathbb{R}_0}\Big[\Big(e^{iuxf(t,t)}-1-iuxf(t,t)\Big)(1+\eta(x,t))\Big]\nu(dx)du \\
&+\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}(\mathcal{F}G)(u)E^{Q_\eta}[e^{iuM(t)}] \\
&\qquad\times\int_{-\infty}^{t}\int_{\mathbb{R}_0}\Big[x\frac{d}{dt}f(t,s)\Big(e^{iuxf(t,s)}-1\Big)(1+\eta(x,t))\Big]\nu(dx)ds\,du \\
&+\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}(\mathcal{F}G)(u)E^{Q_\eta}[e^{iuM(t)}]iuS(M(t))(\eta)du \\
=:&(I)+(II)+(III)
\end{aligned}
$$

Now standard manipulations of the Fourier transform together with (5.1) yield

$$
\begin{aligned}
(I)=&\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\int_{\mathbb{R}_0}\Big[(\mathcal{F}G(\cdot+xf(t,t))(u)-(\mathcal{F}G)(u)-xf(t,t)(\mathcal{F}G')(u)\Big] \\
&\times E^{Q_\eta}[e^{iuM(t)}]\,(1+\eta(x,t))\,\nu(dx)du \\
=&\int_{\mathbb{R}_0}S\left(G(M(t-)+xf(t,t))-G(M(t-))-xf(t,t)G'(M(t-))\right)(\eta) \\
&\times(1+\eta(x,t))\,\nu(dx).
\end{aligned}
$$

The second term can be treated analogously and thus,

$$
\begin{aligned}
(II)=&\int_{-\infty}^{t}\int_{\mathbb{R}_0}x\frac{d}{dt}f(t,s)S\left(G'(M(t-)+xf(t,s))-G'(M(t-))\right)(\eta) \\
&\times(1+\eta(x,t))\,\nu(dx)ds.
\end{aligned}
$$

Finally,

$$
(III)=S\left(G'(M(t-))\right)(\eta)\frac{d}{dt}S(M(t))(\eta).
$$

We now collect terms and integrate t from 0 to T, whence

$$\begin{aligned}
&S(G(M(T)))(\eta) - G(0) \\
=&\int_0^T \int_{\mathbb{R}_0} S\left(G(M(t-) + xf(t,t)) - G(M(t-)) - xf(t,t)G'(M(t-))\right)(\eta) \\
&\quad \times (1 + \eta(x,t))\, \nu(dx)dt \\
&\quad + \int_0^T \int_{-\infty}^t \int_{\mathbb{R}_0} x\frac{d}{dt}f(t,s)S\left(G'(M(t-) + xf(t,s)) - G'(M(t-))\right)(\eta) \\
&\quad \times (1 + \eta(x,t))\, \nu(dx)dsdt \\
&\quad + \int_0^T S\left(G'(M(t-))\right)(\eta)\frac{d}{dt}S(M(t))(\eta)dt \\
=:&(i) + (ii) + (iii).
\end{aligned} \tag{5.3}$$

From Remark 3.11 we get

$$\begin{aligned}
(i) =&S\left(\int_0^T \int_{\mathbb{R}_0} G(M(t-) + xf(t,t)) - G(M(t-)) - xf(t,t)G'(M(t-))N^{\diamond}(dx,ds)\right)(\eta) \\
=&S\left(\sum_{0 \le t \le T} G(M(t-) + f(t,t)\Delta L(t)) - G(M(t-)) - G'(M(t-))f(t,t)\Delta L(t)\right)(\eta),
\end{aligned}$$

where the second identity holds, because the Skorohod integral is an Itô integral by predictability. Similarly,

$$(ii) = S\left(\int_0^T \int_{-\infty}^t \int_{\mathbb{R}_0} x\frac{d}{dt}f(t,s)\left[G'(M(t-) + xf(t,s)) - G(M(t-))\right] N^{\diamond}(dx,ds)\right)(\eta).$$

Finally, by the definition of the Skorohod integral with respect to M,

$$(iii) = S\left(\int_0^T G'(M(t-))\, M^{\diamond}(dt)\right)(\eta).$$

Hence, both sides of the Itô formula II have the same S-transform, which proves this formula.

To get Itô formula I, we rearrange the terms in (5.3). By Example 3.8 and differentiating under the integral sign again, we have

$$\frac{d}{dt}S(M(t))(\eta) = f(t,t)\int_{\mathbb{R}_0} x\eta(x,t)\, \nu(dx) + \int_{-\infty}^t \frac{d}{dt}f(t,s)\int_{\mathbb{R}_0} x\eta(x,s)\, \nu(dx)\, ds.$$

Thus, by (5.3) and similar considerations as above,

$$\int_0^T \int_{-\infty}^t \int_{\mathbb{R}_0} x \frac{d}{dt} f(t,s) S\left(G'(M(t-) + x f(t,s))\right)(\eta)\left(1 + \eta(x,t)\right) \nu(dx) ds dt$$

$$= S\Bigg(G(M(T)) - G(0) + \left(\int_{\mathbb{R}_0} x\, \nu(dx) \right) \int_0^T G'(M(t)) \left(f(t,t) + \int_{-\infty}^t \frac{d}{dt} f(t,s)\, ds \right) dt$$

$$- \sum_{0 \le t \le T} G(M(t-) + f(t,t)\Delta L(t)) - G(M(t-)) \Bigg)(\eta)$$

The expression under the S-transform on the right hand side clearly belongs to $L^2(\Omega)$ under the assumptions of Itô formula I. Then, by Remark 3.11, the Skorohod integral

$$\int_0^T \int_{-\infty}^t \int_{\mathbb{R}_0} x \frac{d}{dt} f(t,s) G'(M(t-) + x f(t,s))\, N(dx, ds) dt$$

exists in $L^2(\Omega)$ and coincides with the expression under the S-transform on the right hand side. This proves Itô formula I.

6. Stochastic Calculus for Fractional Lévy Processes. We shall now specialize from a convoluted Lévy process to a fractional one. In Marquardt (2006b) a Wiener type integral with respect to a fractional Lévy process is defined for deterministic integrands. Its domain is the space of functions g such that

$$I_-^d g \in L^2(\mathbb{R}),$$

and it can be characterized by the property

$$\int_{\mathbb{R}} g(s)\, M_d(ds) = \int_{\mathbb{R}} (I_-^d g)(s)\, L(ds).$$

The following theorem shows that a similar characterization holds for Skorohod integrals with respect to fractional Lévy processes. It, hence, also proves as a by-product that the Wiener type integral is a special case of the Skorohod integral.

As a preparation, note that

$$S(M_d(t))(\eta) = \int_{\mathbb{R}} \int_{\mathbb{R}_0} I_-^d \chi_{[0,t]}(s)\, y\eta(s,y)\, \nu(dy)\, ds.$$

Hence, by Fubini's theorem and fractional integration by parts we obtain the following theorem.

THEOREM 6.1. *Suppose M_d is a fractional Lévy process with $-0.5 < d < 0.5$. Then, for all $\eta \in \Xi$,*

$$\frac{d}{dt} S(M_d(t))(\eta) = \int_{\mathbb{R}_0} (I_+^d \eta)(t,y) y \nu(dy),$$

where, by convention, fractional integral and differential operators are applied only to the time variable t.

Furthermore, suppose that $X \in L^p(\Omega)$ *with* $1 < p < 1/d$ *when* $d > 0$ *and that* $I_-^d X \in L^2(\Omega)$ *and* $X \in L^p(\Omega)$ *with* $1 \le p < \infty$ *when* $d < 0$. *Then*

$$\int_{\mathbb{R}} X(t)\, M_d^{\diamond}(dt) = \int_{\mathbb{R}} (I_-^d X)(t)\, L(dt)$$

in the usual sense, that is if one of the integrals exists then so does the other and both coincide.

Proof. The proof follows the same lines as the one of Theorem 3.4. in Bender (2003b). □

Note that only Itô formula II makes sense for fractional Lévy processes. When we apply this Itô formula formally, the following observation is noteworthy. For $d > 0$ the process M_d is continuous and has memory, whence

$$\begin{aligned} G(M_d(T)) &= G(0) + \int_0^T G'(M_d(t-))\; M^{\diamond}(dt) \\ &+ \int_0^T \Bigg(\int_{-\infty}^{t} \int_{\mathbb{R}_0} \Big(G'(M_d(t-) + \frac{x}{\Gamma(d+1)}((t-s)_+^d - (-s)_+^d)) \\ &\qquad - G'(M_d(t-))\Big) \frac{x}{\Gamma(d)} (t-s)_+^{d-1}\; N^{\diamond}(dx, ds) \Bigg)\, dt. \end{aligned}$$

However, the Lévy process L itself comes up as limit of M_d, when d tends to 0. As this process has independent increments and jumps, its well-known Itô formula reads

$$\begin{aligned} G(L(T)) &= G(0) + \int_0^T L'(M(t-))\; L(dt) \\ &+ \sum_{0 \le t \le T} G(L(t)) - G(L(t-)) - G'(L(t-))\Delta L(t). \end{aligned}$$

So apparently, the Itô formulas do not transform continuously into each other when passing to this limit. This is in contrast to the Gaussian case, in which the Itô formula for Brownian motion is recovered by plugging $H = 1/2$ (the Hurst parameter corresponds to d by $d = H - 1/2$) into the Itô formula for fractional Brownian motions, see e.g. Bender (2003b).

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