Comparing point and interval estimates in the bivariate t-copula model with application to financial data

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Abstract

The paper considers joint maximum likelihood (ML) and semiparametric (SP) estimation of copula parameters in a bivariate t-copula. Analytical expressions for the asymptotic covariance matrix involving integrals over special functions are derived, which can be evaluated numerically. These direct evaluations of the Fisher information matrix are compared to Hessian evaluations based on numerical differentiation in a simulation study showing a satisfactory performance of the computationally less demanding Hessian evaluations. Individual asymptotic confidence intervals for the t-copula parameters and the corresponding tail dependence coefficient are derived. For two financial datasets these confidence intervals are calculated using both direct evaluation of the Fisher information and numerical evaluation of the Hessian matrix. These confidence intervals are compared to parametric and nonparametric BCA bootstrap intervals based on ML and SP estimation, respectively, showing a preference for asymptotic confidence intervals based on numerical Hessian evaluations.

Classification codes and Keywords: C13, C14, C16, Fisher information, bivariate t-copula, Hessian, maximum likelihood, semiparametric estimation, efficiency.

1 Introduction

A popular approach for modelling multivariate dependence, which was proposed by Sklar (1959), is based on copulas. A copula is a distribution function of a random vector in $\mathbb{R}^d$ whose margins are uniformly distributed. Estimation of the joint distribution in both bivariate and multivariate cases is of fundamental importance in statistical data analysis, in particular in areas such as risk management (see for example Embrechts et al. (2002), Embrechts et al. (2003), McNeil et al. (2005)). An often used model in applications is the Gaussian copula model, which is historically the most established due to its simplicity (see for example Joe (1997)). However, recent studies indicate that in many situations occurring in practice the Gaussian model does not provide one with adequate fit of the data because of its inability to capture the dependence in extreme values, so-called tail-dependence, which is often observed in financial data (see for example Embrechts et al. (2003), Lindskog (2000), McNeil et al. (2005)). A possibility to overcome this deficiency is to use a more flexible class of copulas such as t-copulas which are simple for estimation and calibration. However, while properties of traditional estimation methods in the multivariate normal case are well established, no analytical expression the asymptotic covariance of the maximum likelihood estimates (MLE) of both t-copula parameters are known to the authors.

The purpose of this paper is to derive an analytical expression for the Fisher information matrix of bivariate t-copula in terms of integrals of special functions, and to examine whether the Hessian evaluation using numerical differentiation is an appropriate substitute for the Fisher information in small samples. Besides ML estimates we also investigate semiparametric estimates. In the semiparametric framework the association parameter ρ is estimated nonparametricaly using Kendal's τ and the resulting estimate is denoted as $\hat{\rho}_{SP}$. The corresponding degrees of freedom (df) parameter maximizes the profile likelihood for given $\hat{\rho}_{SP}$. Further we compare the results obtained from the theoretical evaluation of the Fisher information matrix and its inverse to their numerical approximation provided by the Hessian evaluation. A quasi-Newton algorithm for solving nonlinear optimization problems with upper and lower bounds on the variables developed in Byrd et al. (1995) is used to obtain the MLE and approximate numerically the Hessian matrix resulting from the likelihood optimization. We show that the SP estimates are slightly less efficient than MLE's. The implementation of our methods is illustrated on two financial datasets which exhibit high tail dependence. We estimate correlation and the df parameter together with the tail dependence coefficient and construct confidence intervals based on six different estimation methods. Interval estimates for the MLE of the df parameter, association and tail dependence coefficient are constructed using the asymptotic distribution arising from the central limit theorem, nonparametric and parametric bootstrap (see Efron and Tibshirani (1993)). For the SP estimates bootstrapping is used for the construction of confidence intervals.

The paper is organized as follows. In Section 2 we give the basic properties of

t-copulas including tail dependence. Section 3 is devoted to the theoretical derivation of the Fisher information matrix for bivariate t-copula needed for interval estimates of the t-copula parameters. In Section 4 the asymptotic variance of the MLE of the tail dependence coefficient is derived and the corresponding interval estimates are given. Section 5 contains a simulation study which determines the efficiency of the MLE's compared to SP estimates as well as evaluates the precision of the asymptotic variance and correlation estimates of the MLE's. Section 6 contains two applications involving financial data and Section 7 gives conclusions and outlines suggestions for further analysis.

2 The t-copula

The t-copula has attracted much attention due to its flexibility in modeling the dependence of extreme values, particularly in the framework of financial markets and risk analysis. In this section we outline some basic facts about the multivariate t-copula. For further details we refer the reader to Demarta and McNeil (2005), Embrechts et al. (2003) and Kotz and Nadarajah (2004).

The d-dimensional t-copula with ν degrees of freedom and association matrix Σ is the probability distribution on $[0,1]^d$ whose distribution function is given by

$$C_{\nu,\Sigma}(\boldsymbol{u}) = \int_{-\infty}^{t_\nu^{-1}(u_1)} \cdots \int_{-\infty}^{t_\nu^{-1}(u_d)} \frac{\Gamma\left(\frac{\nu+d}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{(\pi\nu)^d|\Sigma|}} \left(1 + \frac{\boldsymbol{x}'\Sigma^{-1}\boldsymbol{x}}{\nu}\right)^{-\frac{\nu+d}{2}} d\boldsymbol{x}, \tag{1}$$

where $t_\nu(\cdot)$ is the distribution function of a univariate t-distribution with ν degrees of freedom. The probability density function corresponding to (1) equals to

$$c_{\nu,\Sigma}(\boldsymbol{u}) = \frac{dt_{\nu,\Sigma}(t_\nu^{-1}(u_1),\ldots,t_\nu^{-1}(u_d))}{\prod_{i=1}^d dt(t_\nu^{-1}(u_i),\nu)}, \qquad \boldsymbol{u} \in [0,1]^d, \tag{2}$$

where $dt_{\nu,\Sigma}(\cdot)$ and $dt(\cdot,\cdot)$ are the densities of multivariate and univariate t-distribution, respectively.

Since the semiparametric method that we employ in our simulation study and applications involves estimation of Kendal's τ, we outline the basic properties of this commonly used measure of dependence between two random variables. For more details the reader is referred to Embrechts et al. (2002), Kruskal (1958) and Schweizer and Wolff (1981). Consider a random vector $(U_1,\ldots,U_d)$ whose distribution function coincides with (1). The Kendal's τ rank correlation for a pair of random variables $(U_i,U_j), 1 \le i < j \le n$, denoted as τ_{ij}, can be obtained from association matrix $\Sigma = (\rho_{ij})_{1\le i,j\le d}$ using the relationship (see for example McNeil et al. (2005))

$$\tau_{ij} = \frac{2}{\pi}\arcsin\rho_{ij}. \tag{3}$$

Note (3) holds more general for bivariate distributions. The corresponding tail dependence coefficient λ_{ij} for the bivariate t-copula (see for example Joe (1997), Chapter 2 for general definition) equals to

$$\lambda_{ij} = \lambda(\rho_{ij}, \nu) = 2t_{\nu+1}\left(-\sqrt{\nu+1}\sqrt{\frac{1-\rho_{ij}}{1+\rho_{ij}}}\right) \tag{4}$$

Many extensions of the multivariate t-copula concept are developed in order to allow for more heterogeneity in modeling dependent observations. These include skewed t-copulas (see Kotz and Nadarajah (2004)), grouped t-, t-extreme value and t-lower tail copula (Demarta and McNeil (2005)), pair copula constructions applied to bivariate t-copulas (see Aas et al. (2007)), and many others.

Estimation of t-copula parameters is usually performed using ML where the likelihood is maximized with respect to Σ and ν jointly, or the method of moments based on non-parametric estimation of Kendal's τ, relationship (3) and additional likelihood maximization over the remaining parameter ν. In the following we refer to the second method as semiparametric (SP) method. The SP method has been proposed in Lindskog (2000) and Lindskog et al. (2003) and it guarantees no positive definiteness of the estimator $\hat{\Sigma}_{SP}$ of the correlation matrix Σ, but it results in estimates close to the full MLE as demonstrated in Mashal and Zeevi (2002). Though both methods are widely implemented in practice, theoretical properties of the estimates obtained seem to be uncovered in the literature. In the next section we derive the expression for the Fisher information matrix and elaborate on properties of the MLE in the bivariate t-copula case.

3 Interval estimates for bivariate t-copula parameters

In this section we derive expressions for the entries of the Fisher information matrix in terms of integrals of special functions, describe how we evaluate them and provide illustrations of their functional shape. We restrict our attention to the bivariate case, although generalizations to higher dimensions are possible.

The density of the bivariate t-copula with parameters $\rho \in (-1, 1)$ and $\nu > 0$ is given by

$$c(u_1, u_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} \frac{1}{dt(x_1, \nu)dt(x_2, \nu)} \left(1 + \frac{x_1^2 + x_2^2 - 2\rho x_1 x_2}{\nu(1-\rho^2)}\right)^{-\frac{\nu+2}{2}}, \tag{5}$$

where

$$dt(x, \nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

is the density of univariate t-distribution with ν degrees of freedom. Here

$$x_1 := t_\nu^{-1}(u_1), \qquad u_1 \in (0,1),$$
$$x_2 := t_\nu^{-1}(u_2), \qquad u_2 \in (0,1),$$

with $t_\nu^{-1}(\cdot)$ being the quantile function of univariate t-distribution with ν degrees of freedom.

The logarithm of the t-copula density equals

$$\begin{aligned}
l(u_1,u_2) = \log c(u_1,u_2) &= -\log(2\pi) - \frac{1}{2}\log(1-\rho^2) \\
&- \left[\log\Gamma\left(\frac{\nu+1}{2}\right) - \frac{1}{2}\log\nu - \frac{1}{2}\log\pi - \log\Gamma\left(\frac{\nu}{2}\right) - \frac{\nu+1}{2}\log\left(1+\frac{x_1^2}{\nu}\right)\right] \\
&- \left[\log\Gamma\left(\frac{\nu+1}{2}\right) - \frac{1}{2}\log\nu - \frac{1}{2}\log\pi - \log\Gamma\left(\frac{\nu}{2}\right) - \frac{\nu+1}{2}\log\left(1+\frac{x_2^2}{\nu}\right)\right] \\
&- \frac{\nu+2}{2}\log\left(1+\frac{x_1^2+x_2^2-2\rho x_1x_2}{\nu(1-\rho^2)}\right) \\
&= -\log 2 - 2\log\Gamma\left(\frac{\nu+1}{2}\right) + 2\log\Gamma\left(\frac{\nu}{2}\right) + \frac{\nu+1}{2}\log(1-\rho^2) - \frac{\nu-2}{2}\log\nu \\
&+ \frac{\nu+1}{2}\left[\log(\nu+x_1^2)+\log(\nu+x_2^2)\right] - \frac{\nu+2}{2}\log\left[\nu(1-\rho^2)+x_1^2+x_2^2-2\rho x_1x_2\right].
\end{aligned}$$

Therefore

$$\frac{\partial l}{\partial\rho}(u_1,u_2) = -(\nu+1)\frac{\rho}{1-\rho^2} + (\nu+2)\frac{\nu\rho+x_1x_2}{\nu(1-\rho^2)+x_1^2+x_2^2-2\rho x_1x_2}. \tag{6}$$

If we denote

$$M(\nu,\rho,x_1,x_2) := \frac{1}{\nu(1-\rho^2)+x_1^2+x_2^2-2\rho x_1x_2}, \tag{7}$$

then

$$\frac{\partial^2 l}{\partial\rho^2}(u_1,u_2) = -(\nu+1)\frac{1+\rho^2}{(1-\rho^2)^2} + \frac{(\nu+2)\nu}{M(\nu,\rho,x_1,x_2)} + 2(\nu+2)\frac{(\nu\rho+x_1x_2)^2}{M(\nu,\rho,x_1,x_2)^2},$$

and

$$\begin{aligned}
I_{\rho\rho} &:= \int_0^1\int_0^1 \frac{\partial^2 l}{\partial\rho^2}(u_1,u_2)c(u_1,u_2)du_1du_2 \qquad (8)\\
&= -(\nu+1)\frac{1+\rho^2}{1-\rho^2} + \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(\nu+2)\nu}{2\pi\sqrt{1-\rho^2}}\frac{[M(\nu,\rho,x_1,x_2))]^{-\frac{\nu+4}{2}}}{[\nu(1-\rho^2)]^{-(\nu+2)/2}}dx_1dx_2 \\
&+ \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\nu+2}{\pi\sqrt{1-\rho^2}}\frac{1}{[\nu(1-\rho^2)]^{-(\nu+2)/2}}(\nu\rho+x_1x_2)^2\left[M(\nu,\rho,x_1,x_2))\right]^{-\frac{\nu+6}{2}}dx_1dx_2.
\end{aligned}$$

In order to obtain partial derivatives of $l(u_1, u_2)$ with respect to ν we first calculate the derivative of $x_i = t_\nu^{-1}(u_i), i = 1, 2$. Namely, we have that

$$\frac{\partial x_i}{\partial \nu}(u_1, u_2) = -\frac{\frac{\partial}{\partial \nu} t_\nu(x_i)}{dt(x_i, \nu)}, \quad i = 1, 2$$

where

$$t_\nu(x) = \begin{cases} 1 - \frac{1}{2} I_{\frac{\nu}{\nu+x^2}}(\frac{\nu}{2}, \frac{1}{2}), & x \geq 0, \\ \frac{1}{2} I_{\frac{\nu}{\nu+x^2}}(\frac{\nu}{2}, \frac{1}{2}), & x < 0, \end{cases} \tag{9}$$

with

$$I_{\frac{\nu}{\nu+x^2}}\left(\frac{\nu}{2}, \frac{1}{2}\right) = \frac{\int_0^{\frac{\nu}{\nu+x^2}} t^{\nu/2-1}(1-t)^{-1/2} dt}{B(\frac{\nu}{2}, \frac{1}{2})}$$

being the regularized β-function (see for example Abramowitz and Stegun (1992), Chapter 6 or Gradshteyn and Ryzhik (1980), Chapter 6). Therefore, for $x \geq 0$ we have that

$$\begin{aligned} \frac{\partial}{\partial \nu} t_\nu(x) =& -\frac{1}{2} \frac{\partial}{\partial \nu} I_{\frac{\nu}{\nu+x^2}}\left(\frac{\nu}{2}, \frac{1}{2}\right) \\ =& -\frac{1}{2} \frac{1}{B(\frac{\nu}{2}, \frac{1}{2})} \left[\left(\frac{1}{x^2+\nu} \right)^{\frac{\nu+1}{2}} \nu^{\frac{\nu}{2}-1} x + \int_0^{\frac{\nu}{\nu+x^2}} t^{\nu/2-1}(1-t)^{-1/2} \log t dt \right] \\ & + \frac{1}{4} I_{\frac{\nu}{\nu+x^2}}\left(\frac{\nu}{2}, \frac{1}{2}\right) \left[\psi\left(\frac{\nu}{2}\right) - \psi\left(\frac{\nu}{2} + \frac{1}{2}\right) \right], \end{aligned} \tag{10}$$

where $\psi(\cdot)$ is the digamma function. Analogously, the derivative for $x < 0$ can be obtained from (9). Therefore it follows that

$$\begin{aligned} \frac{\partial x_i}{\partial \nu}(u_1, u_2) =& \frac{1}{2dt(x_i, \nu)} \frac{1}{B(\frac{\nu}{2}, \frac{1}{2})} \left[\left(\frac{1}{x_i^2+\nu} \right)^{\frac{\nu+1}{2}} \nu^{\frac{\nu}{2}-1} x_i + \int_0^{\frac{\nu}{\nu+x_i^2}} t^{\nu/2-1}(1-t)^{-1/2} \log t dt \right] \\ & - \frac{1}{4dt(x_i, \nu)} I_{\frac{\nu}{\nu+x^2}}\left(\frac{\nu}{2}, \frac{1}{2}\right) \left[\psi\left(\frac{\nu}{2}\right) - \psi\left(\frac{\nu}{2} + \frac{1}{2}\right) \right], \qquad u_i \geq 1/2, i = 1, 2. \end{aligned} \tag{11}$$

Differentiating (6) with respect to ν we obtain

$$\begin{aligned} \frac{\partial^2 l}{\partial \rho \partial \nu}(u_1, u_2) =& -\frac{\rho}{1-\rho^2} + \frac{\nu\rho + x_1 x_2}{M(\nu, \rho, x_1, x_2)} + (\nu+2) \left[\frac{\rho + x_1 \frac{\partial x_2}{\partial \nu} + x_2 \frac{\partial x_1}{\partial \nu}}{M(\nu, \rho, x_1, x_2)} \right. \\ & \left. - (\nu\rho + x_1 x_2) \frac{-2\nu\rho + 2x_1(\frac{\partial x_1}{\partial \nu} - \rho \frac{\partial x_2}{\partial \nu}) + 2x_2(\frac{\partial x_2}{\partial \nu} - \rho \frac{\partial x_1}{\partial \nu})}{M(\nu, \rho, x_1, x_2)^2} \right]. \end{aligned} \tag{12}$$

Thus

$$I_{\rho\nu} := \int_0^1 \int_0^1 \frac{\partial^2 l}{\partial\rho\partial\nu}(u_1,u_2)c(u_1,u_2)du_1du_2, \tag{13}$$

can be determined using $\frac{\partial^2 l}{\partial\rho\partial\nu}(u_1,u_2)$ and $\frac{\partial x_i}{\partial\nu}(u_1,u_2)$ as given in (12) and (11), respectively.

Finally,

$$I_{\nu\nu} := \int_0^1 \int_0^1 \left(\frac{\partial l}{\partial\nu}(u_1,u_2)\right)^2 c(u_1,u_2)du_1du_2 \tag{14}$$

where

$$\frac{\partial l}{\partial\nu}(u_1,u_2) = -\,\psi\left(\frac{\nu+1}{2}\right) + \psi\left(\frac{\nu}{2}\right) + \frac{1}{2}\log(1-\rho^2) - \frac{\nu+2}{2\nu} - \frac{1}{2}\log\nu \tag{15}$$

$$+\,(\nu+1)\left[\frac{x_1\frac{\partial x_1}{\partial\nu}}{\nu+x_1^2} + \frac{x_2\frac{\partial x_2}{\partial\nu}}{\nu+x_2^2}\right] \tag{16}$$

$$-\,\frac{\nu+2}{2}\frac{1-\rho^2+2x_1\frac{\partial x_1}{\partial\nu}+2x_2\frac{\partial x_2}{\partial\nu}-2\rho(x_1\frac{\partial x_2}{\partial\nu}+x_2\frac{\partial x_1}{\partial\nu})}{M(\nu,\rho,x_1,x_2)} \tag{17}$$

and $\frac{\partial x_i}{\partial\nu}(u_1,u_2)$ is given in (11).

Integration in (8), (13) and (14) is not possible in analytical form. We have used the symbolic software `Mathematica` to approximate the integrals in (8), (13) and (14). We allow ρ and ν to vary over intervals $[-0.95, 0.95]$ and $[2.1, 10]$, respectively. From general properties of the MLE (see for example Lehmann and Casella (1998), Chapter 6) we have that MLE $(\hat\rho_n,\hat\nu_n)'$ of $(\rho_0,\nu_0)'$ based on n observations $(u_1^1,u_2^1),\ldots,(u_1^n,u_2^n)$ from bivariate t-copula satisfies

$$\sqrt{n}\left[(\hat\rho_n,\hat\nu_n)' - (\rho_0,\nu_0)'\right] \xrightarrow{\mathscr{D}} \mathscr{N}_2(\mathbf{0},\Sigma(\rho_0,\nu_0)), \qquad n\to\infty, \tag{18}$$

where

$$\Sigma(\rho_0,\nu_0) := \begin{pmatrix} \sigma^2_{\rho_0\rho_0} & \sigma_{\rho_0\nu_0} \\ \sigma_{\rho_0\nu_0} & \sigma^2_{\nu_0\nu_0} \end{pmatrix} = \begin{pmatrix} I_{\rho_0\rho_0} & I_{\rho_0\nu_0} \\ I_{\rho_0\nu_0} & I_{\nu_0\nu_0} \end{pmatrix}^{-1} \tag{19}$$

is the inverse of the Fisher information matrix. Further $\mathscr{N}_2(\boldsymbol{\mu},\Sigma)$ denotes the bivariate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix Σ. The shape of $\sigma^2_{\rho_0\rho_0}, \sigma_{\rho_0\nu_0}, \sigma^2_{\nu_0\nu_0}$ is given in Figures 1, 2 and 3, respectively. The $100(1-\alpha)\%$ confidence intervals for parameters ρ and ν are $(\hat\rho_n - z_{1-\alpha/2}\hat\sigma_{\hat\rho_n\hat\rho_n}/\sqrt{n}, \hat\rho_n + z_{1-\alpha/2}\hat\sigma_{\hat\rho_n\hat\rho_n}/\sqrt{n})$ and $(\hat\nu_n - z_{1-\alpha/2}\hat\sigma_{\hat\nu_n\hat\nu_n}/\sqrt{n}, \hat\rho_n + z_{1-\alpha/2}\hat\sigma_{\hat\nu_n\hat\nu_n}/\sqrt{n})$ respectively, where $z_{1-\alpha}$ is the $(1-\alpha)100$ % quantile of the standard normal distribution. Here are $\hat\sigma_{\hat\rho_n\hat\rho_n}$ and $\hat\sigma_{\hat\nu_n\hat\nu_n}$ are suitable estimates of the variances $\sigma_{\rho_0\rho_0}$ and $\sigma_{\nu_0\nu_0}$, respectively.

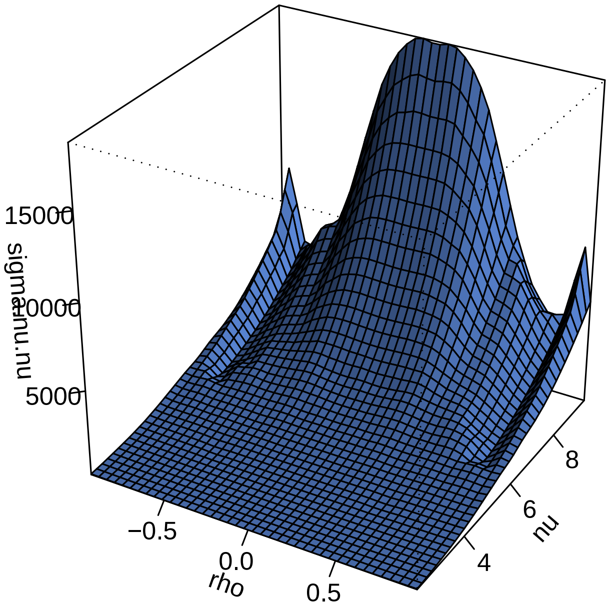

Figure 1: The shape of $\sigma^2_{\rho_0\rho_0}$ as ρ_0 and ν_0 vary.

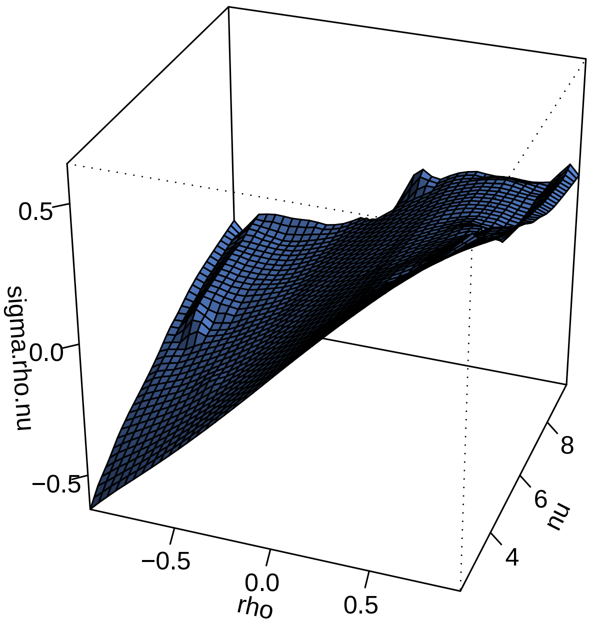

Figure 2: The shape of $r(\rho_0\nu_0) = \sigma_{\rho_0\nu_0}/(\sigma_{\rho_0\rho_0}\sigma_{\nu_0\nu_0})$ as ρ_0 and ν_0 vary.

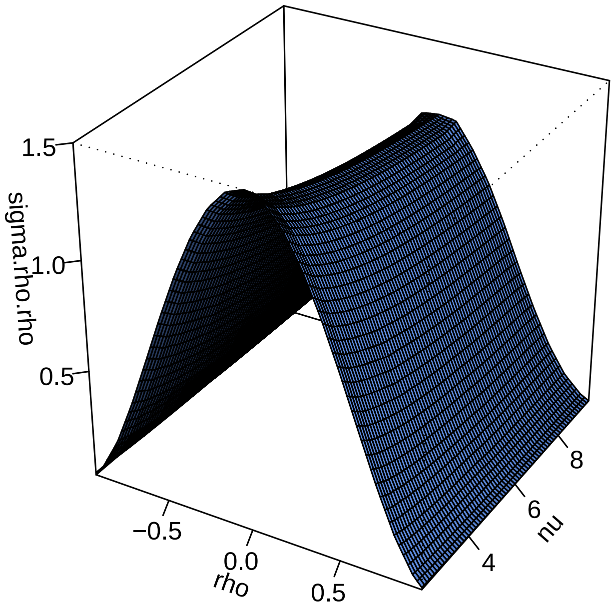

Figure 3: The shape of $\sigma^2_{\nu_0\nu_0}$ as ρ_0 and ν_0 vary.

4 Interval estimates for the tail dependence coefficient

In this section we derive the asymptotic distribution of the MLE for the tail dependence coefficient λ in the bivariate t-copula case and construct the corresponding interval estimate.

As noted in Section 2, for the bivariate t-copula the tail dependence coefficient λ can be expressed in terms of parameters ρ and ν, namely

$$\lambda = \lambda(\rho, \nu) = 2t_{\nu+1}\left(-\sqrt{\nu+1}\sqrt{\frac{1-\rho}{1+\rho}}\right). \tag{20}$$

For details regarding the derivation of (20), the reader is referred to Demarta and McNeil (2005). The MLE of the true tail dependence coefficient $\lambda_0 = \lambda(\rho_0, \nu_0)$ is $\hat{\lambda}_n = \lambda(\hat{\rho}_n, \hat{\nu}_n)$, and its asymptotical properties follow from general point estimation theory (see for example Lehmann and Casella (1998), Chapter 3). Namely,

$$\sqrt{n}(\hat{\lambda}_n - \lambda_0) \xrightarrow{\mathscr{D}} \mathscr{N}(0, \sigma^2_{\lambda_0}), \qquad n \to \infty, \tag{21}$$

where

$$\sigma^2_{\lambda_0} := \left(\frac{\partial\lambda}{\partial\rho}(\rho_0,\nu_0)\right)^2 \sigma^2_{\rho_0\rho_0} + 2\left(\frac{\partial\lambda}{\partial\rho}(\rho_0,\nu_0)\right)\left(\frac{\partial\lambda}{\partial\nu}(\rho_0,\nu_0)\right)\sigma_{\rho_0\nu_0} + \left(\frac{\partial\lambda}{\partial\nu}(\rho_0,\nu_0)\right)^2 \sigma^2_{\nu_0\nu_0}. \tag{22}$$

From (20) we have that

$$\frac{\partial\lambda}{\partial\rho}(\rho,\nu) = 2dt\left(-\sqrt{\nu+1}\sqrt{\frac{1-\rho}{1+\rho}}, \nu+1\right)\sqrt{\nu+1}\sqrt{\frac{1+\rho}{1-\rho}}\frac{1}{(1+\rho)^2} \tag{23}$$

$$\begin{aligned}\frac{\partial\lambda}{\partial\nu}(\rho,\nu) =& dt\left(-\sqrt{\nu+1}\sqrt{\frac{1-\rho}{1+\rho}}, \nu+1\right)\frac{1}{\sqrt{\nu+1}}\sqrt{\frac{1-\rho}{1+\rho}} \\ &+2\frac{\partial}{\partial\nu}t_{\nu+1}\left(-\sqrt{\nu+1}\sqrt{\frac{1-\rho}{1+\rho}}\right),\end{aligned} \tag{24}$$

where $\frac{\partial}{\partial\nu}t_\nu(\cdot)$ is as in (10). The $(1-\alpha)100$ % confidence interval for λ has the form $(\lambda_n - z_{1-\alpha}\hat{\sigma}_{\hat{\lambda}_n}/\sqrt{n}, \lambda_n + z_{1-\alpha}\hat{\sigma}_{\hat{\lambda}_n}/\sqrt{n})$, where $\hat{\sigma}_{\hat{\lambda}_n}$ is a suitable estimate of σ_{λ_0}.

5 Simulation study

In this section we present the results of a simulation study which compares the asymptotic covariance matrix of $(\hat{\rho}_{ML}, \hat{\nu}_{ML})$ obtained from a bilinear interpolation of numerically evaluated integrals in (8), (13) and (14) and its counterpart determined by

Hessian evaluation using numerical differentiation at $(\hat{\rho}_{ML}, \hat{\nu}_{ML})$. A quasi-Newton algorithm of Byrd et al. (1995) for solving nonlinear optimization problems with upper and lower bounds on the variables is used to obtain MLE's. We obtain the approximation of the Hessian matrix using the `R` routine `optim`. Furthermore, we confirm the phenomenon already observed in Mashal and Zeevi (2002) that estimates obtained using ML method are very close to the SP estimates.

In order to compare the finite sample properties we simulate 200 times a dataset consisting of n independent realizations from a bivariate t-copula distribution with parameters $\rho = 0, 0.5, 0.7$ and $\nu = 3$ and 5, respectively. For $n = 500, 1500, 4500$ and each $i = 1, \ldots, 200$ we obtain the ML estimate $(\hat{\rho}^{(i)}_{ML}, \hat{\nu}^{(i)}_{ML})$ and SP estimate $(\hat{\rho}^{(i)}_{SP}, \hat{\nu}^{(i)}_{SP})$ of (ρ, ν). Moreover, the tail dependence coefficient estimates $\hat{\lambda}^{(i)}_{ML}$ and $\hat{\lambda}^{(i)}_{SP}$ are calculated using the relationship (20).

In order to present results of our simulation we introduce the following notation. Let θ_0 be the true parameter value, and $\hat{\theta}^{(i)}, i = 1, \ldots, 200$ be the estimate of θ_0 from the i^{th} simulation step. We define the overall estimate

$$\hat{\theta} := \sum_{i=1}^{200} \hat{\theta}^{(i)}/200,$$

its empirical standard deviation

$$s(\hat{\theta}) := \frac{1}{\sqrt{200}} \sqrt{\frac{\sum_{i=1}^{200} (\hat{\theta}^{(i)} - \hat{\theta})^2}{199}},$$

the estimated bias

$$\widehat{\text{Bias}}(\hat{\theta}) := \hat{\theta} - \theta_0,$$

the estimated mean squared error

$$\widehat{\text{MSE}}(\hat{\theta}) := \frac{\sum_{i=1}^{200} (\hat{\theta}^{(i)} - \theta_0)^2}{200},$$

and the estimated efficiency of estimate $\hat{\theta}_2$ with respect to estimate $\hat{\theta}_1$ as

$$\widehat{\text{Eff}}(\hat{\theta}_1, \hat{\theta}_2) = \frac{\widehat{\text{MSE}}(\hat{\theta}_1)}{\widehat{\text{MSE}}(\hat{\theta}_2)}.$$

In Tables 1 and 2 we report the overall estimate, its empirical standard deviation and bias for $\hat{\theta}$ being $\hat{\rho}_{SP}, \hat{\rho}_{ML}, \hat{\nu}_{SP}, \hat{\nu}_{ML}, \hat{\lambda}_{SP}, \hat{\lambda}_{ML}$, respectively. Additionally, we give the efficiency $\widehat{\text{Eff}}(\hat{\rho}_{SP}, \hat{\rho}_{ML})$($\widehat{\text{Eff}}(\hat{\nu}_{SP}, \hat{\nu}_{ML})$ and $\widehat{\text{Eff}}(\hat{\lambda}_{SP}, \hat{\lambda}_{ML})$) of $\hat{\rho}_{ML}$ with respect to $\hat{\rho}_{SP}$ (of $\hat{\nu}_{ML}$ with respect to $\hat{\nu}_{SP}$ and $\hat{\lambda}_{ML}$ with respect to $\hat{\lambda}_{SP}$). Efficiency greater than 1 indicates that ML estimate performs better than SP estimate. The MLE are slightly more efficient in general, but computational simplicity of SP method can be seen as a big advantage especially in the case of very large datasets. The performance of ML method is slightly worse when $\nu = 5$ in comparison to $\nu = 3$. We notice that as correlation coefficient ρ increases, the ML becomes more efficient in comparison to SP. In particular, we observe that the MLE of ρ is up to 50% more efficient than the corresponding SP estimate. There is little gain in efficiency for the MLE of ν and λ compared to the SP estimate of ν.

In Tables 3 and 4 the estimated variance of $\hat{\rho}_{ML}, \hat{\nu}_{ML}$ and $\hat{\lambda}_{ML}$ as well as the estimated correlation between $\hat{\rho}_{ML}$ and $\hat{\nu}_{ML}$ based on asymptotic theory and direct numerical Hessian evaluation are compared. In the first column of the tables we give the true asymptotic standard deviations as well as the correlation for the finite sample size n defined by (19)

$$\sigma_n(\rho_0) := \frac{[\Sigma(\rho_0, \nu_0)]_{11}^{1/2}}{\sqrt{n}}$$

$$\sigma_n(\nu_0) := \frac{[\Sigma(\rho_0, \nu_0)]_{22}^{1/2}}{\sqrt{n}}$$

$$r(\rho_0, \nu_0) := \frac{[\Sigma(\rho_0, \nu_0)]_{12}^{1/2}}{[\Sigma(\rho_0, \nu_0)]_{11}^{1/2}[\Sigma(\rho_0, \nu_0)]_{22}^{1/2}},$$

where $[\Sigma(\rho_0, \nu_0)]_{ij}$ denotes the (ij)th element of the matrix $\Sigma(\rho_0, \nu_0)$ defined in (19). Note that the finite sample asymptotic approximation to the correlation coefficient of $(\hat{\rho}_n, \hat{nu}_n)$ is independent of the sample size. The finite sample asymptotic standard deviation of the tail dependence coefficient $\hat{\lambda}_n$ denoted by $\sigma_n(\lambda_0)$, is defined using the relationship (22).

Furthermore, for each $i = 1, \ldots, 200$ the corresponding standard deviations using interpolation (I) and numerical Hessian (H) evaluation are defined as

$$\sigma_{I,n}^{(i)}(\hat{\rho}) := \frac{[\Sigma_I(\hat{\rho}_{ML}^{(i)}, \hat{\nu}_{ML}^{(i)})]_{11}^{1/2}}{\sqrt{n}}, \quad \sigma_{H,n}^{(i)}(\hat{\rho}) := \frac{[\Sigma_H(\hat{\rho}_{ML}^{(i)}, \hat{\nu}_{ML}^{(i)})]_{11}^{1/2}}{\sqrt{n}}$$

$$\sigma_{I,n}^{(i)}(\hat{\nu}) := \frac{[\Sigma_I(\hat{\rho}_{ML}^{(i)}, \hat{\nu}_{ML}^{(i)})]_{22}^{1/2}}{\sqrt{n}}, \quad \sigma_{H,n}^{(i)}(\hat{\nu}) := \frac{[\Sigma_H(\hat{\rho}_{ML}^{(i)}, \hat{\nu}_{ML}^{(i)})]_{22}^{1/2}}{\sqrt{n}}$$

as well as the corresponding correlation coefficients

$$r_I^{(i)}(\hat{\rho}, \hat{\nu}) := \frac{[\Sigma_I(\hat{\rho}_{ML}^{(i)}, \hat{\nu}_{ML}^{(i)})]_{12}}{[\Sigma_I(\hat{\rho}_{ML}^{(i)}, \hat{\nu}_{ML}^{(i)})]_{11}^{1/2}[\Sigma_I(\hat{\rho}_{ML}^{(i)}, \hat{\nu}_{ML}^{(i)})]_{22}^{1/2}},$$

$$r_H^{(i)}(\hat{\rho}, \hat{\nu}) := \frac{[\Sigma_H(\hat{\rho}_{ML}^{(i)}, \hat{\nu}_{ML}^{(i)})]_{12}}{[\Sigma_H(\hat{\rho}_{ML}^{(i)}, \hat{\nu}_{ML}^{(i)})]_{11}^{1/2}[\Sigma_H(\hat{\rho}_{ML}^{(i)}, \hat{\nu}_{ML}^{(i)})]_{22}^{1/2}}$$

are obtained from numerical interpolation of the theoretical asymptotic covariance matrix (Σ_I) and direct Hessian evaluation (Σ_H). Analogously, $\sigma^{(i)}_{I,n}(\hat{\lambda})$ and $\sigma^{(i)}_{H,n}(\hat{\lambda})$ are defined using the relationship (22). The comparison of the two methods is performed using the true finite sample asymptotic value, its estimate and estimated standard errors for both methods. We notice that the numerical interpolation gives slightly more stable estimates since its estimated standard error is uniformly below the corresponding estimated standard error of the Hessian method. However, both evaluation methods give nearly unbiased estimates over all simulation setups. Thus the Hessian evaluation based on numerical differentiation is as precise as the numerical interpolation of the asymptotic covariance matrix.

6 Applications

In this section we illustrate the performance of our estimation methods for two financial datasets which display heavy tail dependence. We focus especially on interval estimates for the tail dependence coefficient λ. For the SP estimation method bootstrapping (see for example Efron and Tibshirani (1993)) is required to evaluate the precision of the estimates and constructing confidence intervals.

In particular, we are interested in comparing the performance of six different variance estimation methods for ρ, ν and λ estimates, respectively. The first two methods are based on the asymptotic theory for MLEs, as before one uses interpolation of the Fisher information (MLE.TH) and the other one uses the numerical evaluation of the Hessian matrix (MLE.HESS). The next two methods use parametric and nonparametric bootstrap based on ML estimation. They are denoted by MLE.PB and MLE.NPB, respectively. The final two methods correspond also to parametric and nonparametric bootstrap, but this time based on the SP estimation method for (ρ, ν). We abreviate these final methods by SP.PB and SP.NPB, respectively. We consider the MLE.TH as benchmark estimation method ignoring the estimation error of the parameters. The performance of the remaining five methods are evaluated against this benchmark. Since ρ and ν are jointly estimated we also derive estimated correlations between ρ and ν estimates for all six methods. Finally, we also present 95% confidence intervals for the tail dependence coefficient λ for all six estimation methods. For the bootstrap methods we use BCA confidence intervals (see Efron and Tibshirani (1993)).

6.1 Euro swap rates

The data contains three time-series of daily Euro swap rates for $2, 3$ and 10 year maturity over a time period from December $7, 1988$ to May, $21, 2001$. We investigate the dependence between the swap rates for 2 and 3, and 2 and 10 years maturity, respectively. Since the daily observations exhibit high serial correlation, we fit an

ARMA$(1,1)$-GARCH$(1,1)$ model and use the standardized residuals for our analysis. The standardized residuals are transformed using their empirical distribution function to uniform marginals and a bivariate t-copula model is fitted to the transformed data. When the copula is fitted to the transformed swap rates for 2 year and 3 year maturity we obtain $\hat{\rho}_{SP} = 0.938, \hat{\nu}_{SP} = 2.827, \hat{\lambda}_{SP} = 0.745, \hat{\rho}_{MLE} = 0.937, \hat{\nu}_{MLE} = 2.758$ and $\hat{\lambda}_{MLE} = 0.745$, while the coresponding estimates for the transformed swap rates for 2 and 10 years maturity swap rates are $\hat{\rho}_{SP} = 0.785, \hat{\nu}_{SP} = 5.162, \hat{\lambda}_{SP} = 0.421, \hat{\rho}_{MLE} = 0.780, \hat{\nu}_{MLE} = 5.054$ and $\hat{\lambda}_{MLE} = 0.420$.

We see that SP and ML estimates are close as expected from the simulation study. Further we see as expected stronger correlation and high tail dependence between swap rates with close maturities. To evaluate the precision of the different estimation methods we estimate standard errors of ρ, ν and λ estimates as well as the correlation between ρ and ν estimates for the different estimation methods. The results for the 2 and 3 year maturity swap rates are presented in Table 5, while the ones for 2 and 10 years are presented in Table 6. For the bootstrap methods we used 500 bootstrap replications. Comparing MLE.HESS to the benchmark MLE.TH we see that MLE.HESS performs well with regard to the evaluation of precision of $\hat{\nu}_{ML}$ and $\hat{\rho}_{ML}$ for both swap rate datasets, while the correlation between $\hat{\nu}_{ML}$ and $\hat{\rho}_{ML}$ is underestimated implying also slight underestimation of the variability of $\hat{\lambda}_{ML}$. The bootstrap methods MLE.PB and MLE.NPB perform reasonable compared to the benchmark with regard to variability of $\hat{\nu}_{ML}$ and $\hat{\rho}_{ML}$, but less so with regard to $\hat{r}(\hat{\rho}_{ML}, \hat{\nu}_{ML})$ and $\hat{\sigma}(\hat{\lambda}_{ML})$. Especially, MLE.NPB underestimates the correlation between $\hat{\nu}_{ML}$ and $\hat{\rho}_{ML}$. For the bootstrap methods based on SP estimation of the variability of $\hat{\rho}_{SP}$ and $\hat{\nu}_{SP}$ is larger than the corresponding one for $\hat{\rho}_{ML}$ and $\hat{\nu}_{ML}$, which is indicative of the higher efficiency of the ML over the SP estimates. The estimated correlation between $\hat{\rho}_{SP}$ and $\hat{\nu}_{SP}$ are quite far away from the corresponding benchmark values. Overall the nonparametric bootstrap methods are quite variable with regard to assess the correlation between ρ and ν estimates.

Next we present 95% confidence intervals for ρ, ν and λ for all estimation methods. The individual interval estimate of ρ and ν is influenced by the bias of the estimate and the estimated standard error for the estimate. For the confidence interval of λ the correlation between ρ and ν has an additional influence. For ρ, the methods MLE.HESS, MLE.PB and MLE.NPB give similar confidence intervals as the benchmark MLE.TH method for both swap rate datasets. For ν only MLE.HESS is close to the benchmark for both swap rate datasets. For λ the comparisons are not that simple. This is to be expected since the estimation methods are quite variable in their assessment of the correlation between the ρ and ν estimates. Overall MLE.HESS provides good agreement with benchmark intervals for ρ and ν, while the methods based on SP estimates are less satisfactory.

6.2 Hong Kong spot and futures prices

In this section we analyze the daily stock returns from the spot and futures market of Hong Kong. The spot price indices (Hang Seng Price Index of Hong Kong) and the corresponding futures price indices (Hang Seng Futures Exchange Index of Hong Kong) are collected from January $1, 1998$ to June, $10, 2005$. The procedure described in Section 6.1 is applied to the daily log-returns. Table 7 contains estimated standard errors for ρ, ν and λ estimates as well as the estimated correlation between ρ and ν estimates for the different estimation methods. The conclusion from this analysis is similar to that obtained for the swap rates. Namely, nonparametric bootstrap when combined with SP estimation has poor performance. The SP estimation results in wider individual interval estimates for ρ and ν, while ML estimation combined with both parametric and nonparametric bootstrapping results in confidence intervals close to those one can expect from the theory. The confidence intervals for the tail dependence coefficient obtained from nonparametric bootstrap using MLE are the closest to those obtained using asymptotic result in (18) with the exact Fisher information matrix interpolated at the ML estimate. Notice that this phenomenon is clearly visible from Table 7.

7 Summary and conclusions

This paper considers ML and the popular semiparametric approach to estimate bivariate t-copula parameters and their asymptotic variances. We use a simulation study to illustrate that the MLE is slightly more efficient than semiparametric estimate based on Kendal's τ and its relationship to correlation coefficient ρ in the t-copula framework. We theoretically derive the Fisher information matrix and estimate the asymptotic variance of $\hat{\rho}_{\mathrm{ML}}$ and $\hat{\nu}_{\mathrm{ML}}$ using direct numerical integration or Hessian evaluation using numerical differentiation. The procedure based on integration is shown to be slightly more stable. We further apply both approaches to two financial datasets where in addition to ρ and ν the tail-dependence coefficient λ is also estimated. We construct individual confidence intervals for the three parameters of interest using our asymptotic results, parametric and non-parametric bootstrap. It is shown that using the numerical evaluation of the Hessian matrix to assess the variability of the MLE for λ is preferable over bootstrapping the corresponding semiparametric estimate of λ.

Suggestions for further research on issues addressed in this paper are varied. First, extensions to higher dimensions are to be examined. Secondly, the properties of estimators in the pair-copula construction case (see Aas et al. (2007)) are to be established and compared to the results obtained in a Bayesian framework (see Min and Czado (2008) and Dalla Valle (2007)). Moreover, models with parametric rather than uniform margins are to be considered. Here we plan to investigate theoretically properties of

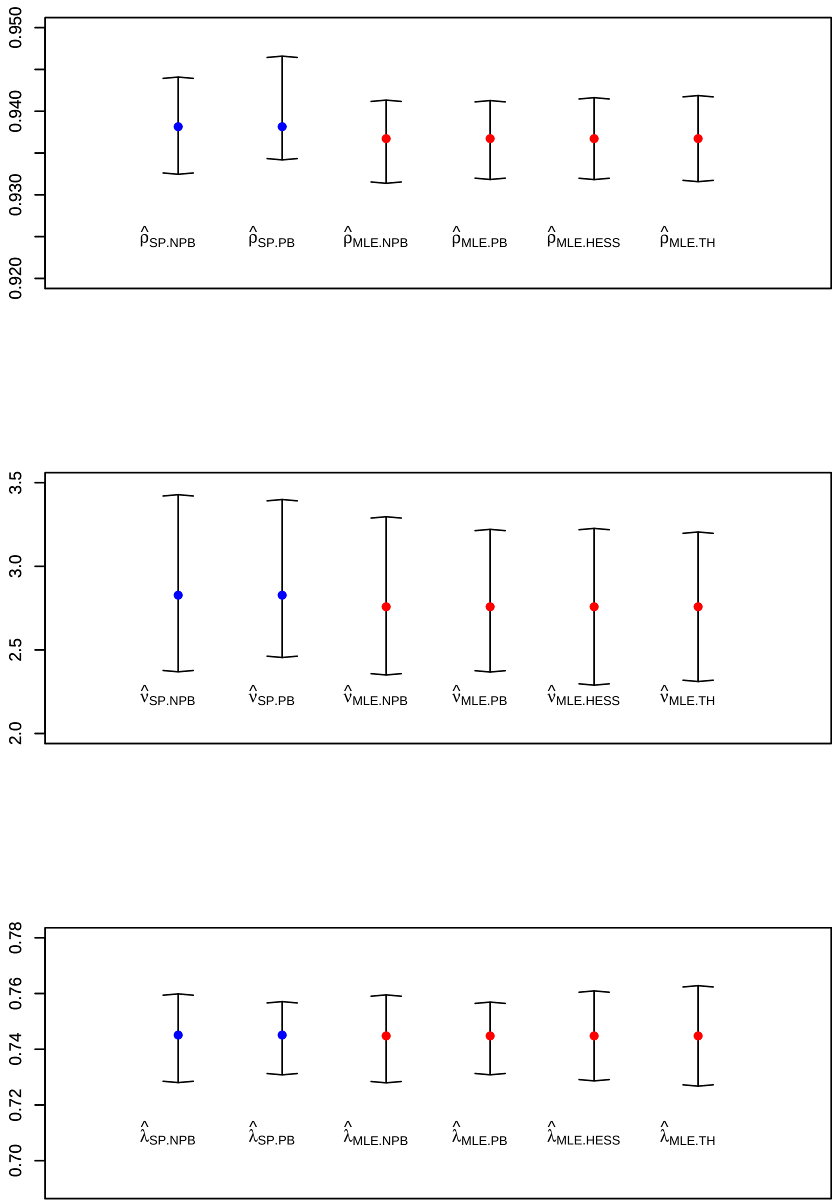

Figure 4: 95% individual confidence intervals for ρ, ν and λ ($\hat{\rho}_{SP} = 0.938, \hat{\nu}_{SP} = 2.827, \hat{\lambda}_{SP} = 0.745, \hat{\rho}_{MLE} = 0.937, \hat{\nu}_{MLE} = 2.758, \hat{\lambda}_{MLE} = 0.745$) for swap rates with 2 and 3 years maturity.

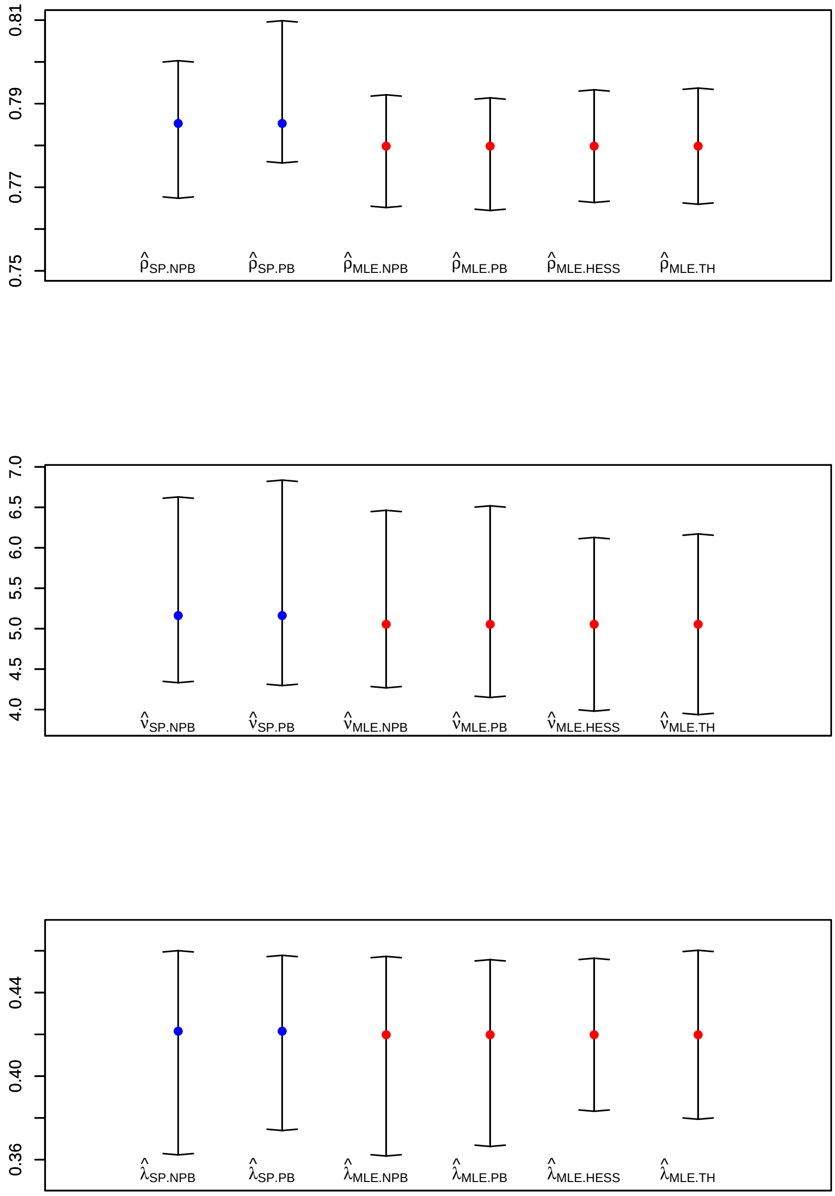

Figure 5: 95% individual confidence intervals for ρ, ν and λ ($\hat{\rho}_{SP} = 0.785, \hat{\nu}_{SP} = 5.162, \hat{\lambda}_{SP} = 0.421, \hat{\rho}_{MLE} = 0.780, \hat{\nu}_{MLE} = 5.054, \hat{\lambda}_{MLE} = 0.420$) for swap rates with 2 and 10 years maturity.

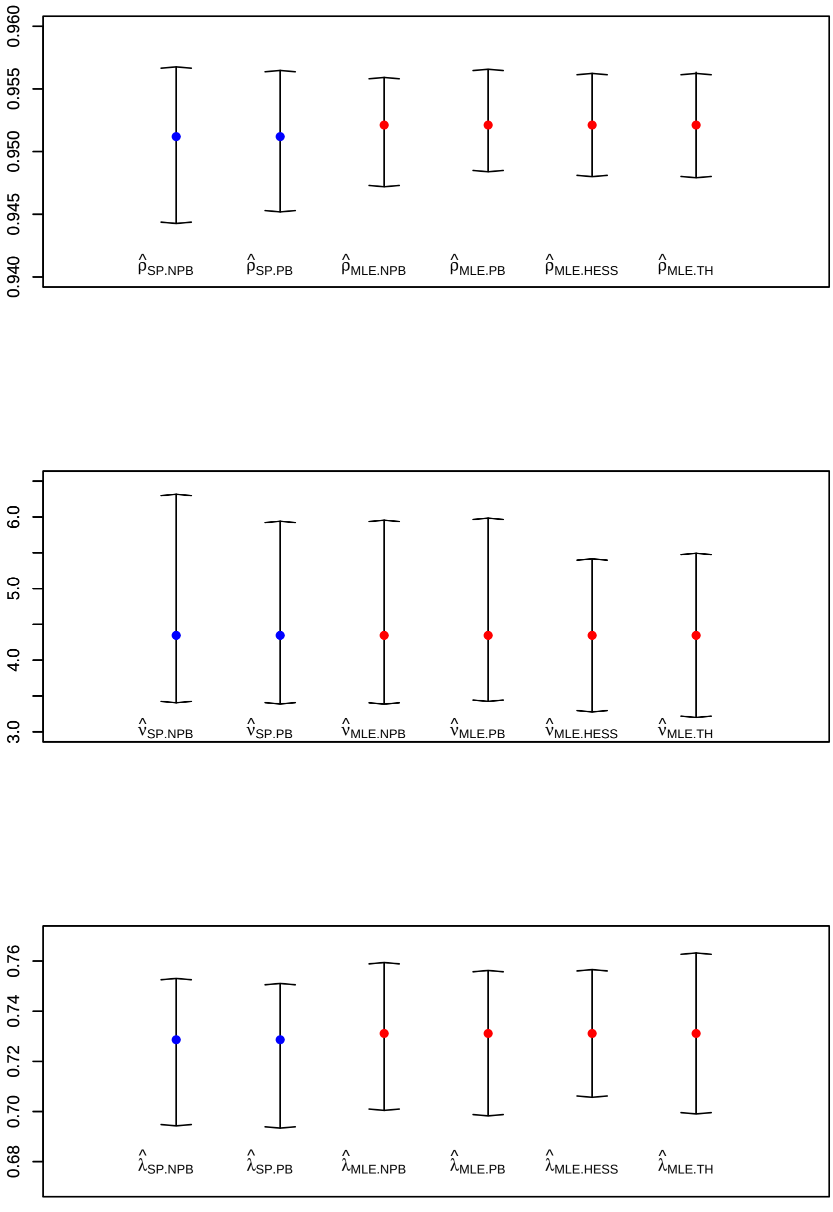

Figure 6: 95% individual confidence intervals for ρ, ν and λ ($\hat{\rho}_{SP} = 0.951, \hat{\nu}_{SP} = 4.346, \hat{\lambda}_{SP} = 0.728, \hat{\rho}_{MLE} = 0.952, \hat{\nu}_{MLE} = 4.346, \hat{\lambda}_{MLE} = 0.731$) for Hong Kong spot future returns.

estimators obtained from simultaneous estimation of both marginal and copula parameters. Finally, applications to various multivariate datasets are to be discussed.

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		n=500				n=1500				n=4500			
		$\hat{\theta}$	$s(\hat{\theta})$	$\widehat{\text{Bias}}(\hat{\theta})$	$\widehat{\text{Eff}}(\hat{\theta}_{SP}, \hat{\theta}_{ML})$	$\hat{\theta}$	$s(\hat{\theta})$	$\widehat{\text{Bias}}(\hat{\theta})$	$\widehat{\text{Eff}}(\hat{\theta}_{SP}, \hat{\theta}_{ML})$	$\hat{\theta}$	$s(\hat{\theta})$	$\widehat{\text{Bias}}(\hat{\theta})$	$\widehat{\text{Eff}}(\hat{\theta}_{SP}, \hat{\theta}_{ML})$
Parameter ρ													
$\rho = 0$	SP	−0.0045	0.0042	−0.0045	1.0396	−0.0038	0.0022	−0.0038	1.0166	−0.0006	0.0012	−0.0006	1.0136
	ML	−0.0039	0.0041	−0.0039	1.0000	−0.0039	0.0022	−0.0039	1.0000	−0.0008	0.0012	−0.0008	1.0000
$\rho = 0.5$	SP	0.4958	0.0032	−0.0042	1.1464	0.4966	0.0017	−0.0034	1.0427	0.4996	0.0009	−0.0004	1.0999
	ML	0.4961	0.0030	−0.0039	1.0000	0.4968	0.0017	−0.0032	1.0000	0.4997	0.0009	−0.0003	1.0000
$\rho = 0.7$	SP	0.6971	0.0022	−0.0029	1.2477	0.6978	0.0012	−0.0022	1.1206	0.6998	0.0006	−0.0002	1.1903
	ML	0.6972	0.0020	−0.0028	1.0000	0.6980	0.0011	−0.0020	1.0000	0.6999	0.0006	−0.0001	1.0000
Parameter ν													
$\rho = 0$	SP	3.1315	0.0419	0.1315	1.0006	3.0307	0.0227	0.0307	1.0008	3.0199	0.0128	0.0199	0.9993
	ML	3.1290	0.0419	0.1290	1.0000	3.0297	0.0226	0.0297	1.0000	3.0197	0.0128	0.0197	1.0000
$\rho = 0.5$	SP	3.1327	0.0455	0.1327	1.0214	3.0358	0.0244	0.0358	0.9404	3.0243	0.0136	0.0243	0.9979
	ML	3.1345	0.0450	0.1345	1.0000	3.0399	0.0251	0.0399	1.0000	3.0250	0.0136	0.0250	1.0000
$\rho = 0.7$	SP	3.1409	0.0519	0.1409	1.0584	3.0429	0.0261	0.0429	0.9234	3.0277	0.0146	0.0277	1.0230
	ML	3.1451	0.0503	0.1451	1.0000	3.0503	0.0271	0.0503	1.0000	3.0291	0.0144	0.0291	1.0000
Parameter λ													
$\rho = 0$	SP	0.1128	0.0021	−0.0033	0.9975	0.1150	0.0013	−0.0011	0.9997	0.1154	0.0007	−0.0007	1.0005
	ML	0.1131	0.0021	−0.0030	1.0000	0.1151	0.0013	−0.0010	1.0000	0.1154	0.0007	−0.0008	1.0000
$\rho = 0.5$	SP	0.3056	0.0027	−0.0069	1.0006	0.3095	0.0017	−0.0030	1.0003	0.3111	0.0009	−0.0014	1.0000
	ML	0.3054	0.0027	−0.0071	1.0000	0.3094	0.0017	−0.0031	1.0000	0.3111	0.0009	−0.0014	1.0000
$\rho = 0.7$	SP	0.4411	0.0026	−0.0070	1.0014	0.4448	0.0016	−0.0033	1.0006	0.4466	0.0008	−0.0015	1.0001
	ML	0.4405	0.0026	−0.0076	1.0000	0.4445	0.0016	−0.0036	1.0000	0.4466	0.0008	−0.0015	1.0000

Table 1: Parameter estimates, estimated standard errors and estimated efficiency of ML and SP methods for $\nu = 3$ and $\rho = 0, 0.5, 0.7$ respectively.

		n=500				n=1500				n=4500			
		$\hat{\theta}$	$s(\hat{\theta})$	$\widehat{\text{Bias}}(\hat{\theta})$	$\widehat{\text{Eff}}(\hat{\theta}_{SP}, \hat{\theta}_{ML})$	$\hat{\theta}$	$s(\hat{\theta})$	$\widehat{\text{Bias}}(\hat{\theta})$	$\widehat{\text{Eff}}(\hat{\theta}_{SP}, \hat{\theta}_{ML})$	$\hat{\theta}$	$s(\hat{\theta})$	$\widehat{\text{Bias}}(\hat{\theta})$	$\widehat{\text{Eff}}(\hat{\theta}_{SP}, \hat{\theta}_{ML})$
Parameter ρ													
$\rho = 0$	SP	−0.0026	0.0040	−0.0026	1.0083	−0.0031	0.0021	−0.0031	0.9878	−0.0006	0.0012	−0.0006	0.9911
	ML	−0.0016	0.0040	−0.0016	1.0000	−0.0031	0.0021	−0.0031	1.0000	−0.0008	0.0012	−0.0008	1.0000
$\rho = 0.5$	SP	0.4968	0.0031	−0.0032	1.2255	0.4974	0.0016	−0.0026	1.1751	0.4994	0.0009	−0.0006	1.0836
	ML	0.4971	0.0028	−0.0029	1.0000	0.4974	0.0015	−0.0026	1.0000	0.4994	0.0008	−0.0006	1.0000
$\rho = 0.7$	SP	0.6978	0.0021	−0.0022	1.5026	0.6979	0.0011	−0.0021	1.3728	0.6996	0.0006	−0.0004	1.1927
	ML	0.6981	0.0018	−0.0019	1.0000	0.6982	0.0010	−0.0018	1.0000	0.6996	0.0006	−0.0004	1.0000
Parameter ν													
$\rho = 0$	SP	5.1341	0.0925	0.1341	0.9925	5.0901	0.0597	0.0901	0.9963	5.0392	0.0338	0.0392	0.9988
	ML	5.1376	0.0929	0.1376	1.0000	5.0910	0.0598	0.0910	1.0000	5.0395	0.0338	0.0395	1.0000
$\rho = 0.5$	SP	5.1610	0.0978	0.1610	1.0086	5.0987	0.0632	0.0987	0.9596	5.0283	0.0341	0.0283	1.0048
	ML	5.1793	0.0972	0.1793	1.0000	5.1118	0.0644	0.1118	1.0000	5.0316	0.0340	0.0316	1.0000
$\rho = 0.7$	SP	5.2304	0.1058	0.2304	1.0281	5.1020	0.0650	0.1020	0.9575	5.0250	0.0347	0.0250	1.0423
	ML	5.2595	0.1040	0.2595	1.0000	5.1253	0.0662	0.1253	1.0000	5.0299	0.0340	0.0299	1.0000
Parameter λ													
$\rho = 0$	SP	0.0539	0.0019	0.0041	0.9978	0.0539	0.0019	0.0041	0.9978	0.0539	0.0019	0.0041	0.9978
	ML	0.0540	0.0019	0.0042	1.0000	0.0540	0.0019	0.0042	1.0000	0.0540	0.0019	0.0042	1.0000
$\rho = 0.5$	SP	0.2077	0.0036	0.0007	1.0035	0.2077	0.0036	0.0007	1.0035	0.2077	0.0036	0.0007	1.0035
	ML	0.2070	0.0036	−0.0000	1.0000	0.2070	0.0036	−0.0000	1.0000	0.2070	0.0036	−0.0000	1.0000
$\rho = 0.7$	SP	0.3400	0.0039	−0.0032	1.0049	0.3400	0.0039	−0.0032	1.0049	0.3400	0.0039	−0.0032	1.0049
	ML	0.3383	0.0040	−0.0049	1.0000	0.3383	0.0040	−0.0049	1.0000	0.3383	0.0040	−0.0049	1.0000

Table 2: Parameter estimates, estimated standard errors and estimated efficiency of ML and SP methods for $\nu = 5$ and $\rho = 0, 0.5, 0.7$ respectively.

n	ρ_0	$\sigma_n(\rho_0)$	$\sigma_{I,n}(\hat{\rho})$	$s(\sigma_{I,n}(\hat{\rho}))$	$\sigma_{H,n}(\hat{\rho})$	$s(\sigma_{H,n}(\hat{\rho}))$
	0	5.292	5.254	0.006	5.263	0.014
500	0.5	3.899	3.896	0.019	3.905	0.021
	0.7	2.613	2.631	0.018	2.626	0.019
	0	3.055	3.046	0.002	3.051	0.004
1500	0.5	2.251	2.257	0.006	2.259	0.007
	0.7	1.509	1.519	0.006	1.516	0.006
	0	1.764	1.761	0.001	1.763	0.001
4500	0.5	1.300	1.299	0.002	1.300	0.002
	0.7	0.871	0.872	0.002	0.870	0.002

n	ρ_0	$\sigma_n(\nu_0)$	$\sigma_{I,n}(\hat{\nu})$	$s(\sigma_{I,n}(\hat{\nu}))$	$\sigma_{H,n}(\hat{\nu})$	$s(\sigma_{H,n}(\hat{\nu}))$
	0	0.521	0.587	0.015	0.586	0.015
500	0.5	0.555	0.616	0.017	0.625	0.019
	0.7	0.585	0.652	0.019	0.663	0.024
	0	0.301	0.313	0.004	0.310	0.004
1500	0.5	0.320	0.333	0.005	0.332	0.005
	0.7	0.338	0.354	0.005	0.352	0.006
	0	0.174	0.178	0.001	0.177	0.001
4500	0.5	0.185	0.190	0.001	0.188	0.002
	0.7	0.195	0.200	0.002	0.199	0.002

n	ρ_0	$r(\rho_0,\nu_0)$	$r_I(\hat{\rho},\hat{\nu})$	$s(r_I)$	$r_H(\hat{\rho},\hat{\nu})$	$s(r_H)$
	0	0.000	−0.002	0.003	−0.006	0.005
500	0.5	0.315	0.306	0.002	0.309	0.004
	0.7	0.419	0.410	0.002	0.413	0.004
	0	0.000	−0.002	0.001	−0.003	0.003
1500	0.5	0.315	0.311	0.001	0.311	0.003
	0.7	0.419	0.416	0.001	0.415	0.002
	0	0.000	−0.000	0.001	0.001	0.001
4500	0.5	0.315	0.313	0.001	0.315	0.001
	0.7	0.419	0.418	0.001	0.419	0.001

n	ρ_0	$\sigma_n(\lambda_0)$	$\sigma_{I,n}(\hat{\lambda})$	$s(\sigma_{I,n}(\hat{\lambda}))$	$\sigma_{H,n}(\hat{\lambda})$	$s(\sigma_{H,n}(\hat{\lambda}))$
	0	0.016	0.016	0.0022	0.016	0.0024
500	0.5	0.033	0.033	0.0032	0.033	0.0043
	0.7	0.036	0.037	0.0046	0.037	0.0064
	0	0.009	0.009	0.0007	0.009	0.0008
1500	0.5	0.019	0.019	0.0011	0.019	0.0014
	0.7	0.021	0.021	0.0016	0.021	0.0019
	0	0.005	0.005	0.0002	0.005	0.0002
4500	0.5	0.011	0.011	0.0003	0.011	0.0004
	0.7	0.012	0.012	0.0005	0.012	0.0005

Table 3: True finite sample asymptotic variances and correlation as well as their estimates and their estimated standard errors based on interpolation (I) and numerical Hessian evaluation (H) for $\nu = 3$ and $\rho = 0, 0.5, 0.7$, respectively.

n	ρ_0	$\sigma_n(\rho_0)$	$\sigma_{I,n}(\hat{\rho})$	$s(\sigma_{I,n}(\hat{\rho}))$	$\sigma_{H,n}(\hat{\rho})$	$s(\sigma_{H,n}(\hat{\rho}))$
	0	5.073	5.049	0.007	5.064	0.014
500	0.5	3.618	3.634	0.019	3.645	0.022
	0.7	2.353	2.375	0.016	2.377	0.018
	0	2.929	2.920	0.003	2.926	0.005
1500	0.5	2.089	2.095	0.006	2.100	0.007
	0.7	1.359	1.368	0.006	1.368	0.006
	0	1.691	1.688	0.001	1.690	0.002
4500	0.5	1.206	1.207	0.002	1.208	0.002
	0.7	0.784	0.787	0.002	0.786	0.002
n	ρ_0	$\sigma_n(\nu_0)$	$\sigma_{I,n}(\hat{\nu})$	$s(\sigma_{I,n}(\hat{\nu}))$	$\sigma_{H,n}(\hat{\nu})$	$s(\sigma_{H,n}(\hat{\nu}))$
	0	1.354	1.574	0.065	1.519	0.055
500	0.5	1.361	1.524	0.052	1.566	0.055
	0.7	1.373	1.481	0.043	1.623	0.058
	0	0.782	0.840	0.022	0.833	0.020
1500	0.5	0.786	0.839	0.020	0.856	0.022
	0.7	0.793	0.824	0.016	0.864	0.022
	0	0.451	0.463	0.006	0.464	0.006
4500	0.5	0.454	0.465	0.006	0.469	0.006
	0.7	0.458	0.466	0.005	0.472	0.006
n	ρ_0	$r(\rho_0,\nu_0)$	$r_I(\hat{\rho},\hat{\nu})$	$s(r_I)$	$r_H(\hat{\rho},\hat{\nu})$	$s(r_H)$
	0	0.000	−0.001	0.002	−0.006	0.005
500	0.5	0.250	0.246	0.002	0.250	0.004
	0.7	0.324	0.311	0.005	0.324	0.005
	0	0.000	−0.002	0.001	−0.002	0.003
1500	0.5	0.250	0.246	0.001	0.251	0.003
	0.7	0.324	0.316	0.003	0.323	0.003
	0	0.000	−0.000	0.001	−0.001	0.002
4500	0.5	0.250	0.249	0.001	0.252	0.002
	0.7	0.324	0.323	0.002	0.325	0.002
n	ρ_0	$\sigma_n(\lambda_0)$	$\sigma_{I,n}(\hat{\lambda})$	$s(\sigma_{I,n}(\hat{\lambda}))$	$\sigma_{H,n}(\hat{\lambda})$	$s(\sigma_{H,n}(\hat{\lambda}))$
	0	0.013	0.013	0.0029	0.013	0.0031
500	0.5	0.041	0.041	0.0053	0.042	0.0067
	0.7	0.050	0.049	0.0066	0.053	0.0121
	0	0.007	0.007	0.0010	0.007	0.0011
1500	0.5	0.024	0.024	0.0021	0.024	0.0027
	0.7	0.029	0.029	0.0025	0.030	0.0045
	0	0.004	0.004	0.0004	0.004	0.0004
4500	0.5	0.014	0.014	0.0007	0.014	0.0008
	0.7	0.017	0.017	0.0010	0.017	0.0013

Table 4: True finite sample asymptotic variances and correlation as well as their estimates and their estimated standard errors based on interpolation (I) and numerical Hessian evaluation (H) for $\nu = 5$ and $\rho = 0, 0.5, 0.7$, respectively.

Method	$\hat{\sigma}(\hat{\rho})$	$\hat{\sigma}(\hat{\nu})$	$\hat{r}(\hat{\rho}, \hat{\nu})$	$\hat{\sigma}(\hat{\lambda})$
SP.NPB	0.0030	0.2628	0.4847	0.0084
SP.PB	0.0029	0.2355	0.6259	0.0068
MLE.NPB	0.0025	0.2412	0.3696	0.0084
MLE.PB	0.0024	0.2241	0.5795	0.0067
MLE.HESS	0.0025	0.2392	0.5063	0.0082
MLE.TH	0.0026	0.2282	0.5530	0.0092

Table 5: Estimated standard errors for ρ, ν and λ estimates as well as estimated correlation between ρ and ν estimates using different estimation methods for swap rates of 2 and 3 years maturity.

Method	$\hat{\sigma}(\hat{\rho})$	$\hat{\sigma}(\hat{\nu})$	$\hat{r}(\hat{\rho}, \hat{\nu})$	$\hat{\sigma}(\hat{\lambda})$
SP.NPB	0.0082	0.6014	0.2120	0.0235
SP.PB	0.0082	0.5998	0.4020	0.0221
MLE.NPB	0.0071	0.5743	0.0645	0.0238
MLE.PB	0.0068	0.5970	0.3780	0.0222
MLE.HESS	0.0069	0.5481	0.2582	0.0187
MLE.TH	0.0071	0.5702	0.3503	0.0206

Table 6: Estimated standard errors for ρ, ν and λ estimates as well as estimated correlation between ρ and ν estimates using different estimation methods for swap rates of 2 and 10 years maturity.

Method	$\hat{\sigma}(\hat{\rho})$	$\hat{\sigma}(\hat{\nu})$	$\hat{r}(\hat{\rho}, \hat{\nu})$	$\hat{\sigma}(\hat{\lambda})$
SP.NPB	0.0031	0.7034	0.5396	0.0151
SP.PB	0.0028	0.6585	0.5237	0.0144
MLE.NPB	0.0022	0.6799	0.4451	0.0155
MLE.PB	0.0022	0.6424	0.4320	0.0150
MLE.HESS	0.0021	0.5452	0.4231	0.0130
MLE.TH	0.0021	0.5847	0.4323	0.0164

Table 7: Estimated standard errors for ρ, ν and λ estimates as well as estimated correlation between ρ and ν estimates using different estimation methods for Hong Kong spot-future returns.