Conditional characteristic functions of processes related to fractional Brownian motion

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Abstract

Fractional Brownian motion (fBm) can be introduced by a moving average representation driven by standard Brownian motion, which is an affine Markov process. Motivated by this we aim at results analogous to those achieved in recent years for affine models. Using a simple prediction formula for the conditional expectation of a fBm and its Gaussianity, we calculate the conditional characteristic functions of fBm and related processes, including important examples like fractional Ornstein-Uhlenbeck- or Cox-Ingersoll-Ross processes. As an application we propose a fractional Vasicek bond market model and compare prices of zero coupon bonds to those achieved in the classical Vasicek case.

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1 Introduction

Prediction problems arise in many financial and technical applications. For example in a bond market driven by an adapted short rate process $r = (r(v))_{v\in[0,T]}$ the price of a non-defaultable zero-coupon bond with maturity $T \geq 0$ at time $0 \leq s \leq t \leq T$ is given by the conditional expectation

$$B(s,t) = E^{\mathbb{Q}}\left[e^{-\int_s^t r(v)dv}\middle| r(v), v \in [0,s]\right] \tag{1.1}$$

under some risk-neutral measure $\mathbb{Q}$.

For a broad class of stochastic processes, in particular affine models (see e.g. Duffie [4] and Duffie, Filipovic and Schachermayer [5]), such predictions are easy to calculate and do only depend on the level of the process at time t due to their Markov property. However, staying in the bond framework above, Markov models may not be sufficient to catch the real market structure as was shown by the ongoing financial crisis. One reason behind this is that short rates, which are driven by macroeconomic variables like domestic gross products, supply and demand rates or volatilities, exhibit long range dependence, which cannot be captured by Markov models. Empirical evidence has been reported over the years and we refer to Henry and Zaffaroni [9] for details and further references.

When it comes to modeling such structures, fractional processes like fractional Brownian motion (fBm) have been at the core of most models.Their non-Markovianity, however, makes prediction more complicated, since all past information will play a role. Since fBm can be introduced by a moving average representation of a standard Brownian motion (which is itself an affine process) certain structures remain when predicting.

In this paper we calculate the conditional characteristic function of fBm driven integrals and, in general, of solutions to fBm driven stochastic differential equations (sde's) which have been considered in Buchmann and Klüppelberg [2] based on previous work by Zähle [14]. Our results include important models like fractional Ornstein-Uhlenbeck (fOU) or Cox-Ingersoll-Ross (fCIR) processes. An application is given by a bond market model, where the short rate is described by a fractional Vasicek sde.

The results of our work are based on a formula for the conditional expectation of a fBm, which has been derived by Gripenberg and Norros [8] and Pipiras and Taqqu [11].

Our paper is organized as follows. Section 2 will briefly recall integration with respect to fBm and state a prediction formula for its conditional expectation. In section 3 we will present our main results on conditional characteristic functions of fBm related processes, including important situations like fOU or fCIR models. As an application we shall consider a fractional bond market described by Vasicek dynamics in section 4. Zero coupon bond prices are calculated and compared to the classical Brownian situation. We conclude the paper with section 5 containing the proofs of our results.

We will always assume a given complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and denote by L^2 the space of square integrable random variables. For a family of random variables $(X(i))_{i\in I}$, I some index set, let $\overline{\sigma\{X(i), i \in I\}}$ denote the completion of the generated σ-algebra. The spaces of integrable and square integrable real functions are denoted by $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$, respectively. On a compact interval $[0,T]$, the corresponding function spaces are denoted by $L^1([0,T])$ and $L^2([0,T])$. Furthermore $\|\cdot\|_2$ is the L^2-norm and $\mathbb{R}_+$ ($\mathbb{R}_-$) are the positive (negative) real half lines.

2 Preliminaries

We recall that fractional Brownian motion (fBm) is a zero mean Gaussian process starting in 0 with stationary increments satisfying $(B^H(ct))_{t\geq 0} \stackrel{d}{=} c^H(B^H(t))_{t\geq 0}$ for every $c > 0$. The parameter $H \in (0,1)$ and $\stackrel{d}{=}$ means equality of finite dimensional distributions. We also assume that B^H is standard; i.e. that $E[B^H(1)^2] = 1$. For general background on fBm we refer to Samorodnitsky and Taqqu [12]. For the present paper we shall heavily draw from Pipiras and Taqqu [10] and [11].

It is appropriate in our context to use fractional calculus, which suggests to replace H by the fractional parameter $\kappa = H - \frac{1}{2} \in (-\frac{1}{2}, \frac{1}{2})$. The cases $\kappa \in (0, \frac{1}{2})$ and $\kappa \in (-\frac{1}{2}, 0)$ refer to long and short range dependence. We also recall that $\kappa = 0$ refers to standard Brownian motion and we shall write $B^0 = B$.

Throughout the whole paper we will work on the compact interval $[0,T]$ for some $T > 0$ and define the fractional Riemann-Liouville integral with finite time horizon for $\kappa > 0$,

$$(I^\kappa_{T-}f)(s) = \frac{1}{\Gamma(\kappa)} \int_s^T f(r)(r-s)^{\kappa-1} dr, \quad 0 \leq s \leq T. \tag{2.1}$$

For $f \in L^2(\mathbb{R})$ this always exists. We shall also need the fractional derivative with finite time horizon for $\kappa \in (0,1)$

$$(D^\kappa_{T-}g)(u) = -\frac{1}{\Gamma(1-\kappa)}\Big(\frac{g(u)}{(T-u)^\kappa} + \kappa \int_u^T \frac{g(u)-g(s)}{(s-u)^{\kappa+1}} ds\Big), \quad 0 < u < T. \tag{2.2}$$

As usual, we shall often write $I^{-\kappa}_{T-} = D^\kappa_{T-}$. For $\kappa = 0$ we set $I^\kappa_{T-} = D^\kappa_{T-} = id$.

Possible spaces of integrands for fBm have been introduced by Pipiras and Taqqu [10, 11]:

$$\Lambda^\kappa_T := \begin{cases} \Big\{f : [0,T] \to \mathbb{R} \Big| \int_0^T [s^{-\kappa} I^\kappa_{T-}((\cdot)^\kappa f(\cdot))(s)]^2 ds < \infty\Big\}, & \kappa \in \Big(0, \frac{1}{2}\Big) \\ \Big\{f : [0,T] \to \mathbb{R} \Big| \exists \phi_f \in L^2[0,T] : f(s) = s^{-\kappa} I^{-\kappa}_{T-}((\cdot)^\kappa \phi_f(\cdot))(s)\Big\}, & \kappa \in \Big(-\frac{1}{2}, 0\Big) \end{cases}$$

For $\kappa = 0$ both spaces fall together and are equal to $L^2[0,T]$. For $\kappa \in (-\frac{1}{2}, \frac{1}{2})$ and $f, g \in \Lambda^\kappa_T$ define the scalar product

$$\langle f, g\rangle_{\kappa,T} \;\; := \;\; \frac{\pi\kappa(2\kappa+1)}{\Gamma(1-2\kappa)\sin(\pi\kappa)} \int_0^T s^{-2\kappa}[I^\kappa_{T-}((\cdot)^\kappa f(\cdot))(s)][I^\kappa_{T-}((\cdot)^\kappa g(\cdot))(s)] ds,$$

where we set $\langle f, g\rangle_{\kappa,T} = \langle f, g\rangle_{L^2}$ for $\kappa = 0$. Denote the corresponding norm by $||\cdot||_{\kappa,T}$. For $\kappa = 0$ we have $||\cdot||_{0,T} = ||\cdot||_2$. If c is a step function, $\int_0^T c(s) dB^\kappa(s)$ can be reduced to a finite sum.

We then have the isometry

$$\Big\| \int_0^T c(s) dB^\kappa(s) \Big\|_2^2 = \|c(\cdot)\|_{\kappa,T}, \tag{2.3}$$

and by using approximating sequences of step functions, integration for general $c \in \Lambda_T^\kappa$ is defined in the L^2-sense, while (2.3) still holds true, cf. Pipiras and Taqqu [11], Theorems 4.1 and 4.2.

Recall that $\overline{\mathrm{sp}}_{[0,T]}(B^\kappa)$ is the closure in $L^2(\Omega)$ of all possible linear combinations of the increments of fBm on $[0,T]$. Assume we want to calculate an expression for the prediction

$$X_t(s,\kappa) := E[B^\kappa(t) \mid B^\kappa(v), v \in [0,s]], \quad 0 \le s \le t.$$

If $X_t(s,\kappa) \in \overline{\mathrm{sp}}_{[0,s]}(B^\kappa)$, we would hope that there exists some function $c \in \Lambda_T^\kappa$ such that $X_t(s,\kappa) = \int_0^s c(v) dB^\kappa(v)$. This is not clear immediately because it has been shown in [11] that, while for $\kappa \in (-\frac{1}{2}, 0]$ the space $(\Lambda_T^\kappa, \langle , \rangle_{\kappa,T})$ is complete, i.e. a Hilbert space, for $\kappa \in (0, \frac{1}{2})$ this is not true.

However, it has been derived by Gripenberg and Norros [8], Theorem 3.1, that such a suitable c still exists for $\kappa \in (0, \frac{1}{2})$. An explicit formula for c has been calculated. In fact Theorem 7.1 of Pipiras and Taqqu [11] shows that the same formula holds true for $\kappa \in (-\frac{1}{2}, 0]$:

Lemma 2.1. *Let $0 \le s \le t \le T$ and $\kappa \in (-\frac{1}{2}, \frac{1}{2})$. Then*

$$E[B^\kappa(t) \mid B^\kappa(v), v \in [0,s]] = B^\kappa(s) + \int_0^s \Psi^\kappa(s,t,v) dB^\kappa(v)\,, \tag{2.4}$$

where for $v \in (0,t)$,

$$\begin{aligned} \Psi^\kappa(s,t,v) &= v^{-\kappa}(I_{s-}^{-\kappa}(I_{t-}^{\kappa}(\cdot)^\kappa \mathbf{1}_{[s,t]}(\cdot)))(v) \\ &= \frac{\sin(\pi\kappa)}{\pi} v^{-\kappa}(s-v)^{-\kappa} \int_s^t \frac{z^\kappa (z-s)^\kappa}{z-v} dz \end{aligned} \tag{2.5}$$

and for $v \in \{0, s\}$, we have that $\Psi^\kappa(s,t,v) = 0$.

Remark 2.2. For $\kappa = 0$, Lemma 2.1 states that

$$E[B^0(t) \mid B^0(v), v \in [0,s]] = B^0(s)\,, \tag{2.6}$$

which is the known martingale property for standard Brownian motion.

If we write now

$$E[B^\kappa(t) - B^\kappa(s) \mid B^\kappa(v), v \in [0,s]] = \int_0^t \Psi^\kappa(s,t,v) dB^\kappa(v)$$

it is immediately clear, that this prediction formula can be extended to integrals of fBm which has been done in Lemma 1 of Duncan [6]:

Proposition 2.3. *For $0 \le s \le t \le T$ and $\kappa \in (-\frac{1}{2}, \frac{1}{2})$ let $c \in \Lambda_T^\kappa$. Then*

$$E\left[\int_0^t c(v)dB^\kappa(v)\Big|B^\kappa(v), v \in [0,s]\right] = \int_0^s c(v)dB^\kappa(v) + \int_0^s \Psi_c^\kappa(s,t,v)dB^\kappa(v),$$

where for $v \in (0,s)$,

$$\begin{aligned}\Psi_c^\kappa(s,t,v) &= v^{-\kappa}(I_{s-}^{-\kappa}(I_{t-}^\kappa z^\kappa c(z)I_{[s,t]}(z)))(v) \\ &= \frac{\sin(\pi\kappa)}{\pi}v^{-\kappa}(s-v)^{-\kappa}\int_s^t \frac{z^\kappa(z-s)^\kappa}{z-v}c(z)dz. \qquad (2.7)\end{aligned}$$

and for $v \in \{0,s\}$, we have that $\Psi_c^\kappa(s,t,v) = 0$.

In fact all our results in this paper are a consequence of Lemma 2.1, as will be seen later.

3 Main results

Calculating conditional characteristic functions means essentially predicting exponentials. A possible way to approach this problem for fBm driven integrals has been considered in Duncan [6] by transforming the exponential function to a Wick exponential. While this idea works well for $\kappa \in (0, \frac{1}{2})$, Proposition 2 of that paper is not correct. This can be seen immediately, because its result suggests that the prediction is deterministic. The correct version can be found in the Appendix of Biagini, Fink and Klüppelberg [1]. This has also been confirmed by our chosen approach in the present paper, which is based on the simple prediction formula of Lemma 2.1 and classical results on conditional Gaussian distributions. We want to emphasize that our approach also covers the range $\kappa \in (-\frac{1}{2}, 0)$, while the idea in Duncan [6] does not work here.

For notational convenience we fix for the rest of this section a fBm $(B_t^\kappa)_{t\in[0,T]}$ with $\kappa \in (-\frac{1}{2}, \frac{1}{2})$. Denote further $\mathcal{F}_s := \sigma\overline{\{B^\kappa(v), v \in [0,s]\}}$ for $0 \le s \le T$.

Theorem 3.1. *Let $c \in \Lambda_T^\kappa$ and $0 \le s \le t \le T$. Then we have for $u \in \mathbb{R}$*

$$\begin{aligned}E\Big[e^{iu\int_0^t c(v)dB^\kappa(v)}\Big|\mathcal{F}_s\Big] &= \exp\left\{iu\Big[\int_0^s c(v)dB^\kappa(v) + \int_0^s \Psi_c^\kappa(s,t,v)dB^\kappa(v)\Big]\right\} \\ &\quad \times \exp\left\{-\frac{u^2}{2}\left[\|c(\cdot)\mathbf{1}_{[s,t]}(\cdot)\|_{\kappa,T}^2 - \|\Psi_c^\kappa(s,t,\cdot)\mathbf{1}_{[0,s]}(\cdot)\|_{\kappa,T}^2\right]\right\},\end{aligned}$$

i.e. $\int_0^t c(u)dB^\kappa(u)|\mathcal{F}_s$ is normally distributed with

$$\begin{aligned}E\Big[\int_0^t c(v)dB^\kappa(v)\Big|\mathcal{F}_s\Big] &= \int_0^s c(v)dB^\kappa(v) + \int_0^s \Psi_c^\kappa(s,t,v)dB^\kappa(v) \\ Var\Big[\int_0^t c(v)dB^\kappa(v)\Big|\mathcal{F}_s\Big] &= \|c(\cdot)\mathbf{1}_{[s,t]}(\cdot)\|_{\kappa,T}^2 - \|\Psi_c^\kappa(s,t,\cdot)\mathbf{1}_{[0,s]}(\cdot)\|_{\kappa,T}^2.\end{aligned}$$

We compare this result now to the classical Brownian case with its Markov property.

Remark 3.2. By Theorem 3.1 we get for the conditional characteristic function of an fBm

$$\begin{aligned} E\Big[e^{iuB^\kappa(t)}\Big|\mathcal{F}_s\Big] &= \exp\Big\{iu\Big[B^\kappa(s)+\int_0^s \Psi^\kappa(s,t,v)dB^\kappa(v)\Big]\Big\} \\ &\quad\times \exp\Big\{-\frac{u^2}{2}\Big[\|\mathbf{1}_{[s,t]}(\cdot)\|^2_{\kappa,T}-\|\Psi^\kappa(s,t,\cdot)\mathbf{1}_{[0,s]}(\cdot)\|^2_{\kappa,T}\Big]\Big\}, \quad u\in\mathbb{R}. \end{aligned}$$

If we compare that to the standard Brownian motion case, i.e. setting $\kappa=0$, we get

$$E\Big[e^{iuB^0(t)}\Big|\mathcal{F}_s\Big] = \exp\Big\{iuB^0(s)-\frac{u^2}{2}\|\mathbf{1}_{[s,t]}(\cdot)\|_2^2\Big\}, \quad u\in\mathbb{R}.$$

It is not surprising that for $\kappa\neq 0$ the whole past path plays now a role in the predication. Theorem 3.1 and the equations above show that the conditional expectation changes by the term $\int_0^s \Psi^\kappa(s,t,v)dB^\kappa(v)$. As will be seen in the proof it is a consequence of the projection property of the conditional expectation, that the conditional variance equals the unconditional one reduced exactly by

$$Var\Big[\int_0^s \Psi^\kappa(s,t,v)dB^\kappa(v)\Big] = \|\Psi^\kappa(s,t,\cdot)\mathbf{1}_{[0,s]}(\cdot)\|^2_{\kappa,T}.$$

In a next step we want to predict OU-type processes driven by fBm. Therefore, we consider the sde with time dependent coefficient functions

$$dX(t)=(k(t)-a(t)X(t))dt+\sigma(t)dB^\kappa(t)\,, \quad X(0)\in\mathbb{R}, \quad t\in[0,T], \tag{3.1}$$

where integration is defined in Section 2. Here $k(\cdot),a(\cdot)$ are locally integrable and continuous on $\mathbb{R}_+$, $\sigma(\cdot)\neq 0$, continuous and $\sigma(\cdot)\in\Lambda_T^\kappa$. Then the unique solution to (3.1) is given by the process $X=(X(t))_{t\in[0,T]}$, defined by

$$X(t)=X(0)e^{-\int_0^t a(s)ds}+\int_0^t e^{-\int_s^t a(u)du}k(s)ds+\int_0^t e^{-\int_s^t a(u)du}\sigma(s)dB^\kappa(s)\,, \quad t\in[0,T]. \tag{3.2}$$

Because σ does not hit zero, we have the equality $\mathcal{F}_s=\overline{\sigma\{X(v),v\in[0,s]\}}$ for $0\le s\le T$.

Theorem 3.3. *Let $0\le s\le t\le T$. Then we have for $u\in\mathbb{R}$*

$$\begin{aligned} E\Big[e^{iuX(t)}\Big|\mathcal{F}_s\Big] &= \exp\Big\{iu\Big[X(s)e^{-\int_s^t a(v)dv}+\int_s^t e^{-\int_v^t a(w)dw}k(v)dv+\int_0^s \Psi_c^\kappa(s,t,v)dB^\kappa(v)\Big]\Big\} \\ &\quad\times \exp\Big\{-\frac{u^2}{2}\Big[\|c(\cdot)\mathbf{1}_{[s,t]}(\cdot)\|^2_{\kappa,T}-\|\Psi_c^\kappa(s,t,\cdot)\mathbf{1}_{[0,s]}(\cdot)\|^2_{\kappa,T}\Big]\Big\}, \end{aligned}$$

with $c(\cdot)=e^{-\int_\cdot^t a(w)dw}\sigma(\cdot)$, i.e. $X(t)|\mathcal{F}_s$ is normally distributed with

$$\begin{aligned} E[X(t)\mid\mathcal{F}_s] &= X(s)e^{-\int_s^t a(v)dv}+\int_s^t e^{-\int_v^t a(w)dw}k(v)dv+\int_0^s \Psi_c^\kappa(s,t,v)dB^\kappa(v) \\ Var[X(t)\mid\mathcal{F}_s] &= \|c(\cdot)\mathbf{1}_{[s,t]}(\cdot)\|^2_{\kappa,T}-\|\Psi_c^\kappa(s,t,\cdot)\mathbf{1}_{[0,s]}(\cdot)\|^2_{\kappa,T}. \end{aligned}$$

If we assume further that $\sigma(\cdot)$ and $1/\sigma(\cdot)$ are of bounded p-variation for some $0 < p < 1/(1-\kappa)$, cf. Young[13], we can consider (3.1) as a pathwise sde and the fBm driven integral in (3.2) exists further as pathwise limit of Riemann-Stieltjes sums. We therefore are in the context of Buchmann and Klüppelberg [2] and the solutions to very general sde's can be written as a monotone transformation of a certain OU process of the type in (3.2).

Another advantage of these stronger assumptions on $\sigma(\cdot)$ is, that we are now able to invert the sde (3.2) and rewrite the prediction in terms of X:

Proposition 3.4. *In the situation of Theorem 3.3 assume that $\sigma(\cdot)$ and $1/\sigma(\cdot)$ are of bounded p-variation for some $0 < p < 1/(1-\kappa)$. Let $0 \le s \le t \le T$. Then we have for $u \in \mathbb{R}$*

$$\begin{aligned} E\Big[e^{iuX(t)}\Big|\mathcal{F}_s\Big] &= \exp\Big\{iu\Big[X(s)e^{-\int_s^t a(v)dv} + \int_s^t e^{-\int_v^t a(w)dw}k(v)dv - \int_0^s \Psi_c^\kappa(s,t,v)\frac{k(v)}{\sigma(v)}dv \\ &\quad + \int_0^s \Psi_c^\kappa(s,t,v)\frac{a(v)}{\sigma(v)}X(v)dv + \int_0^s \Psi_c^\kappa(s,t,v)\frac{1}{\sigma(v)}dX(v)\Big]\Big\} \\ &\quad \times \exp\Big\{-\frac{u^2}{2}\Big[\|c(\cdot)\mathbf{1}_{[s,t]}(\cdot)\|^2_{\kappa,T} - \|\Psi_c^\kappa(s,t,\cdot)\mathbf{1}_{[0,s]}(\cdot)\|^2_{\kappa,T}\Big]\Big\}, \end{aligned}$$

with $c(\cdot) = e^{-\int_\cdot^t a(v)dv}\sigma(\cdot)$, i.e. $X(t)|\mathcal{F}_s$ is normally distributed with

$$\begin{aligned} E[X(t) \mid \mathcal{F}_s] &= X(s)e^{-\int_s^t a(v)dv} + \int_s^t e^{-\int_v^t a(w)dw}k(v)dv - \int_0^s \Psi_c^\kappa(s,t,v)\frac{k(v)}{\sigma(v)}dv \\ &\quad + \int_0^s \Psi_c^\kappa(s,t,v)\frac{a(v)}{\sigma(v)}X(v)dv + \int_0^s \Psi_c^\kappa(s,t,v)\frac{1}{\sigma(v)}dX(v) \\ Var[X(t) \mid \mathcal{F}_s] &= \|c(\cdot)\mathbf{1}_{[s,t]}(\cdot)\|^2_{\kappa,T} - \|\Psi_c^\kappa(s,t,\cdot)\mathbf{1}_{[0,s]}(\cdot)\|^2_{\kappa,T}. \end{aligned}$$

Of special importance concerning sde's is the OU process with time-independent coefficient functions, as is explained later.

Corollary 3.5. *Consider in the sde (3.1) with $k(\cdot) = 0$, $a(\cdot) = a > 0$ and $\sigma(\cdot) = 1$. Then the solution X is given by*

$$X(t) = X(0)e^{-at} + \int_0^t e^{-a(t-s)}dB^\kappa(s)\,, \quad t \in [0,T]. \tag{3.3}$$

For $0 \le s \le t \le T$ and $u \in \mathbb{R}$ we have

$$\begin{aligned} E\Big[e^{iuX(t)}\Big|\mathcal{F}_s\Big] &= \exp\Big\{iu\Big[X(s)e^{-a(t-s)} + a\int_0^s \Psi_c^\kappa(s,t,v)X(v)dv + \int_0^s \Psi_c^\kappa(s,t,v)dX(v)\Big]\Big\} \\ &\quad \times \exp\Big\{-\frac{u^2}{2}\Big[\|c(\cdot)\mathbf{1}_{[s,t]}(\cdot)\|^2_{\kappa,T} - \|\Psi_c^\kappa(s,t,\cdot)\mathbf{1}_{[0,s]}(\cdot)\|^2_{\kappa,T}\Big]\Big\}, \end{aligned} \tag{3.4}$$

with $c(\cdot) = e^{-a(t-\cdot)}$, i.e. $X(t)|\mathcal{F}_s$ is normally distributed with

$$\begin{aligned} E[X(t) \mid \mathcal{F}_s] &= X(s)e^{-a(t-s)} + a\int_0^s \Psi_c^\kappa(s,t,v)X(v)dv + \int_0^s \Psi_c^\kappa(s,t,v)dX(v) \\ Var[X(t) \mid \mathcal{F}_s] &= \|c(\cdot)\mathbf{1}_{[s,t]}(\cdot)\|^2_{\kappa,T} - \|\Psi_c^\kappa(s,t,\cdot)\mathbf{1}_{[0,s]}(\cdot)\|^2_{\kappa,T}. \end{aligned}$$

Remark 3.6. If we choose in the situation of Corollary 3.5 $\kappa = 0$ we get the classical Brownian case

$$E\Big[e^{iuX(t)}\Big|\mathcal{F}_s\Big] = \exp\Big\{iuX(s)e^{-a(t-s)} - \frac{u^2}{2}\|c(\cdot)\mathbf{1}_{[s,t]}(\cdot)\|_2^2\Big\}, \quad u \in \mathbb{R}. \tag{3.5}$$

We want to emphasize again that for all $\kappa \in (-\frac{1}{2}, \frac{1}{2})$ we have

$$Var[X(t) \mid \mathcal{F}_s] = Var[X(t)] - Var[E[X(t) \mid \mathcal{F}_s]].$$

When calculating prices in a bond market the situation arises that not the short rate process r has to be predicted, but the integrated process like stated in (1.1). The next proposition will handle this case. For notational convenience we set

$$D(\cdot, t) = \int_{\cdot}^{t} e^{-\int_{\cdot}^{v} a(w)dw} dv, \quad t \in [0, T]. \tag{3.6}$$

Proposition 3.7. *Denote by X the process given in (3.2). Then for $0 \leq t \leq T$ and $u \in \mathbb{R}$ we have*

$$\begin{aligned} E\Big[e^{iu\int_0^t X(v)dv}\Big|\mathcal{F}_s\Big] &= \exp\Big\{iu\Big[\int_0^s X(v)dv + D(s,t)X(s) + \int_s^t D(v,t)k(v)dv \\ &\qquad + \int_0^s \Psi_c^\kappa(s,t,v)dB^\kappa(v)\Big]\Big\} \\ &\qquad \times \exp\Big\{-\frac{u^2}{2}\Big[\|c(\cdot)\mathbf{1}_{[s,t]}(\cdot)\|_{\kappa,T}^2 - \|\Psi_c^\kappa(s,t,\cdot)\mathbf{1}_{[0,s]}(\cdot)\|_{\kappa,T}^2\Big]\Big\}, \end{aligned}$$

with $c(\cdot) = D(\cdot, t)\sigma(\cdot)$, i.e. $\int_0^t X(v)dv\Big|\mathcal{F}_s$ is normally distributed with

$$\begin{aligned} E\Big[\int_0^t X(v)dv\Big|\mathcal{F}_s\Big] &= \int_0^s X(v)dv + D(s,t)X(s) + \int_s^t D(v,t)k(v)dv \\ &\qquad + \int_0^s \Psi_c^\kappa(s,t,v)dB^\kappa(v) \\ Var\Big[\int_0^t X(v)dv\Big|\mathcal{F}_s\Big] &= \|c(\cdot)\mathbf{1}_{[s,t]}(\cdot)\|_{\kappa,T}^2 - \|\Psi_c^\kappa(s,t,\cdot)\mathbf{1}_{[0,s]}(\cdot)\|_{\kappa,T}^2. \end{aligned}$$

If we assume further that $\sigma(\cdot)$ and $1/\sigma(\cdot)$ are of bounded p-variation for some $0 < p < 1/(1-\kappa)$, then we have

$$\begin{aligned} E\Big[e^{iu\int_0^t X(v)dv}\Big|\mathcal{F}_s\Big] &= \exp\Big\{iu\Big[\int_0^s X(v)dv + D(s,t)X(s) + \int_s^t D(v,t)k(v)dv \\ &\qquad - \int_0^s \Psi_c^\kappa(s,t,v)\frac{k(v)}{\sigma(v)}dv + \int_0^s \Psi_c^\kappa(s,t,v)\frac{a(v)}{\sigma(v)}X(v)dv \\ &\qquad + \int_0^s \Psi_c^\kappa(s,t,v)\frac{1}{\sigma(v)}dX(v)\Big]\Big\} \\ &\qquad \times \exp\Big\{-\frac{u^2}{2}\Big[\|c(\cdot)\mathbf{1}_{[s,t]}(\cdot)\|_{\kappa,T}^2 - \|\Psi_c^\kappa(s,t,\cdot)\mathbf{1}_{[0,s]}(\cdot)\|_{\kappa,T}^2\Big]\Big\}. \end{aligned}$$

Consider now a general sde with fractional Brownian noise, i.e.

$$dZ(t) = \mu(Z(t))dt + \sigma(Z(t))dB^{\kappa}(t), \quad Z(0) \in \mathbb{R}, \quad t \in [0,T], \tag{3.7}$$

for suitable coefficient functions $\mu(\cdot)$ and $\sigma(\cdot)$. Buchmann and Klüppelberg [2] have shown that for $\kappa \in (0, \frac{1}{2})$, and under certain conditions on $\mu(\cdot)$ and $\sigma(\cdot)$, solutions to (3.8) are given by

$$\begin{aligned} Z(t) &= f(X(t)) & (3.8) \\ dX(t) &= -aX(t)dt + dB^{\kappa}(t)\,, \quad X(0) = f^{-1}(Z(0)), \quad t \in [0,T], & (3.9) \end{aligned}$$

for some suitable monotone and differentiable $f : \mathbb{R} \to \mathbb{R}$ and $a > 0$. The next theorem considers now the conditional characteristic function of such a solutions. We remark that because of the assumptions on f, we have $\mathcal{F}_s = \sigma\overline{\{Z(v), v \in [0,s]\}}$ for $0 \le s \le T$.

Theorem 3.8. *Let the process Z be given by (3.8) and $0 \le s \le t \le T$. Then we have for $u \in \mathbb{R}$*

$$E\left[e^{iuZ(t)}\middle|\mathcal{F}_s\right] = \int_{\mathbb{R}} \left(E\left[e^{(i\xi+1)X(t)}\middle|\mathcal{F}_s\right]g_+(\xi,u) + E\left[e^{(i\xi-1)X(t)}\middle|\mathcal{F}_s\right]g_-(\xi,u)\right) d\xi$$

with $g_\star(\xi,u) = (2\pi)^{-1}\int_{\mathbb{R}_\star} e^{-(i\xi\star 1)x+iuf(x)}dx$, $\star \in \{+,-\}$, where $E[e^{(i\xi+1)X(t)}|\mathcal{F}_s]$ is given by the continuation of (3.4) to $\mathbb{C}$.

Example 3.9 (CIR process)**.** We consider for $\kappa \in (0, \frac{1}{2})$ a CIR model given by the pathwise solution to the sde

$$dZ(t) = -\lambda Z(t)dt + \sigma\sqrt{|Z(t)|}dB^{\kappa}(t), \quad Z(0) \in \mathbb{R}, \quad t \in [0,T],$$

for some $\lambda, \sigma > 0$. Then with Proposition 5.7 of Buchmann and Klüppelberg [2] we know that a solution is given by

$$\begin{aligned} Z(t) &= f(X(t)) \\ dX(t) &= -\frac{\lambda}{2}X(t)dt + dB^{\kappa}(t)\,, \quad X(0) = f^{-1}(Z(0)), \quad t \in [0,T], \end{aligned}$$

where $f(x) = \text{sign}(x)\frac{\sigma^2}{4}x^2$. Considering the Fourier transforms of Theorem 3.8 we get for $u \in \mathbb{R}/\{0\}$ after a lengthy, but straightforward calculation

$$\begin{aligned} g_+(\xi,u) &= \frac{1}{2\pi}\int_{\mathbb{R}_+} e^{-(i\xi+1)x+iuf(x)}dx = \Phi\left(\frac{\sigma\sqrt{u}}{2}(1-i)(\xi+2-i)\right), \\ g_-(\xi,u) &= \frac{1}{2\pi}\int_{\mathbb{R}_-} e^{-(i\xi-1)x+iuf(x)}dx = \Phi\left(\frac{\sigma\sqrt{u}}{2}(1+i)(\xi(2-i)+1)\right), \end{aligned}$$

where Φ denotes the distribution function of a standard normal distribution. The existence of this expression for the complex arguments above is ensured by the existence of the left hand side of the equation.

We want to emphasize that this solution is in contrast to the classical Brownian case ($\kappa = 0$) not unique; for further details we refer to Fink and Klüppelberg [7], Proposition 5.1. We considered there a similar case for fractional Lévy processes.

In the next chapter we will consider an application of our results to bond markets. Recall that in many cases a characteristic function can be extended to $\mathbb{C}$.

4 Application: fractional bond market

As mentioned in the introduction, an application for our theory is the modeling of bond markets, because of empirical evidence of long range dependence in short rates. In particular, we consider a fractional version of the classical Brownian Vasicek model. As we also allow $\kappa = 0$, the Brownian case is included.

The overall state of our stochastic system is described by the process $r = (r(t))_{t\geq 0}$ on the probability space endowed with the filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by r, representing the complete market information and satisfying the usual conditions of completeness and right continuity. The stochastic process r models the *short rate*.

We are aware of the fact that, as a Gaussian process, the short rate can also takes negative values. However it is always possible to shift, and perhaps also scale, the model such that the probability of becoming negative is arbitrarily small. On the other hand it is attractive and useful to have a benchmark model where quantities can be calculated explicitly.

Remaining in this framework and given a maturity date $T > 0$ we consider now a multivariate fBm given by $\mathbf{B}^\kappa(t) = (B^{\kappa(1)}_{(1)}(t), \ldots, B^{\kappa(d)}_{(d)}(t))^T$, $t \in [0,T]$, for some $d \in \mathbb{N}$ and $\kappa \in (-\frac{1}{2}, \frac{1}{2})^d$. The components $B^{\kappa(i)}_{(i)}$, $1 \leq i \leq d$, are assumed to be independent. We remark that although empirical evidence shows long range dependence in short rates, our model also includes the perturbance case $\kappa(i) \in (-\frac{1}{2}, 0)$.

Consider now for $\mathbf{X}(0) = (X^{(1)}(0), \ldots, X^{(n)}(0))^T \in \mathbb{R}^d$ a system of d fractional Vasicek sde's given for $1 \leq i \leq d$ by

$$dX^{(i)}(t) = (k^{(i)}(t) - a^{(i)}(t)X^{(i)}(t))dt + \sigma^{(i)}(t)dB^{\kappa(i)}_{(i)}(t)\,, \quad t \in [0,T]. \tag{4.1}$$

We assume that $k^{(i)}(\cdot), a^{(i)}(\cdot)$ are locally integrable and continuous on $\mathbb{R}_+$, $\sigma^{(i)}(\cdot) \neq 0$, continuous with $\sigma^{(i)}(\cdot) \in \Lambda^{\kappa(i)}_T$. Furthermore, let $\sigma^{(i)}(\cdot)$ and $1/\sigma^{(i)}(\cdot)$ are of bounded $p(i)$-variation for some $0 < p(i) < 1/(1-\kappa(i))$. Considering (3.1) we see that the unique solution of (4.1) is given by $\mathbf{X}(t) = (X^{(1)}(t), \ldots, X^{(d)}(t))^T$, where $X^{(i)}$ is defined as in (3.2).

Now for $b \in \mathbb{R}^d$ fixed, define for $t \in [0,T]$,

$$r(t) = b^T\mathbf{X}(t).$$

Then $\mathcal{F}_s = \sigma\overline{\{r(v), v \in [0,s]\}}$ for $0 \leq s \leq T$. The price of a zero coupon bond is calculated in the next theorem.

Theorem 4.1. *Using the assumptions above, let $0 \leq s \leq t \leq T$. Then the price of a zero coupon bond $B(s,t)$ at time s with maturity t is given by*

$$
\begin{aligned}
B(s,t) &= E\Big[e^{-\int_s^t r(v)dv}\Big|\mathcal{F}_s\Big] \\
&= \prod_{i=1}^{d} \exp\Big\{-b^{(i)}\Big[D^{(i)}(s,t)X^{(i)}(s) + \int_s^t D^{(i)}(v,t)k^{(i)}(v)dv \\
&\qquad - \int_0^s \Psi_{c^{(i)}}^{\kappa(i)}(s,t,v)\frac{k^{(i)}(v)}{\sigma^{(i)}(v)}dv + \int_0^s \Psi_{c^{(i)}}^{\kappa(i)}(s,t,v)\frac{a^{(i)}(v)}{\sigma^{(i)}(v)}X^{(i)}(v)dv \\
&\qquad + \int_0^s \Psi_{c^{(i)}}^{\kappa(i)}(s,t,v)\frac{1}{\sigma^{(i)}(v)}dX^{(i)}(v)\Big]\Big\} \\
&\quad \times \exp\Big\{\frac{(b^{(i)})^2}{2}\Big[\|c^{(i)}(\cdot)\mathbf{1}_{[s,t]}(\cdot)\|_{\kappa(i),T}^2 - \|\Psi_{c^{(i)}}^{\kappa(i)}(s,t,\cdot)\mathbf{1}_{[0,s]}(\cdot)\|_{\kappa(i),T}^2\Big]\Big\}
\end{aligned}
$$

with $c^{(i)}(\cdot) = D^{(i)}(\cdot,t)\sigma^{(i)}(\cdot)$, $1 \leq i \leq n$.

Proof. We calculate

$$
B(s,t) = E\Big[e^{-\int_s^t r(v)dv}\Big|\mathcal{F}_s\Big] = E\Big[e^{-\int_s^t b^T\mathbf{X}(v)dv}\Big|\mathcal{F}_s\Big] = \prod_{i=1}^{d} E\Big[e^{\int_s^t b^{(i)}X^{(i)}(v)dv}\Big|\mathcal{F}_s\Big], \tag{4.2}
$$

where we used the independence of the $X^{(i)}$ in the last equality. The result follows now by an application of Proposition 3.7. The extension of the conditional characteristic function to the whole of the complex plane $\mathbb{C}$ exists because of Gaussianity. $\square$

Example 4.2 (Fractional one-factor model)**.** We want to compare prices in our fractional model to the classical Brownian case, i.e. $\kappa = 0$. For simplicity we assume constant coefficient functions in (4.1) and set $d = 1$ with $b = 1$. Today's prices of the zero coupon bonds are given by

$$
B(0,t) = \exp\Big\{-D(0,t)X(0) - k\int_0^t D(v,t)dv + \frac{\sigma^2}{2}\|D(\cdot,t)\mathbf{1}_{[0,t]}(\cdot)\|_{\kappa,T}^2\Big\}, \quad t \geq 0.
$$

Since negative κ is not relevant as explained before, we allow only $\kappa \in [0, \frac{1}{2})$. Because of the singularities in the norms in (4.3) standard numerical methods are dangerous. Therefore we apply the following discretization scheme for $\kappa \in (0, \frac{1}{2})$ and $t \in [0,T]$: we have

$$
\|D(\cdot,t)\mathbf{1}_{[0,t]}(\cdot)\|_{\kappa,T}^2 = \frac{\pi\kappa(2\kappa+1)}{\Gamma(1-2\kappa)\sin(\pi\kappa)(\Gamma(\kappa))^2}\int_0^T s^{-2\kappa}\Big(\int_s^T \frac{r^\kappa D(r,t)\mathbf{1}_{[0,t]}(r)}{(r-s)^{1-\kappa}}dr\Big)^2 ds.
$$

In a first step we decompose the outer integral for $n \in \mathbb{N}$ and $0 = s_0 \leq s_1 \leq \cdots \leq s_n = T$

$$
\int_0^T s^{-2\kappa}\Big(\int_s^T \frac{r^\kappa D(r,t)\mathbf{1}_{[0,t]}(r)}{(r-s)^{1-\kappa}}dr\Big)^2 ds = \sum_{i=0}^{n-1}\int_{s_i}^{s_{i+1}} s^{-2\kappa}\Big(\int_s^T \frac{r^\kappa D(r,t)\mathbf{1}_{[0,t]}(r)}{(r-s)^{1-\kappa}}dr\Big)^2 ds.
$$

For sufficiently small intervals $[s_i, s_{i+1}]$ we get a reasonable approximation by

$$
\int_s^T \frac{r^\kappa D(r,t)\mathbf{1}_{[0,t]}(r)}{(r-s)^{1-\kappa}}dr \approx \int_{s_i}^T \frac{r^\kappa D(r,t)\mathbf{1}_{[0,t]}(r)}{(r-s_i)^{1-\kappa}}dr.
$$

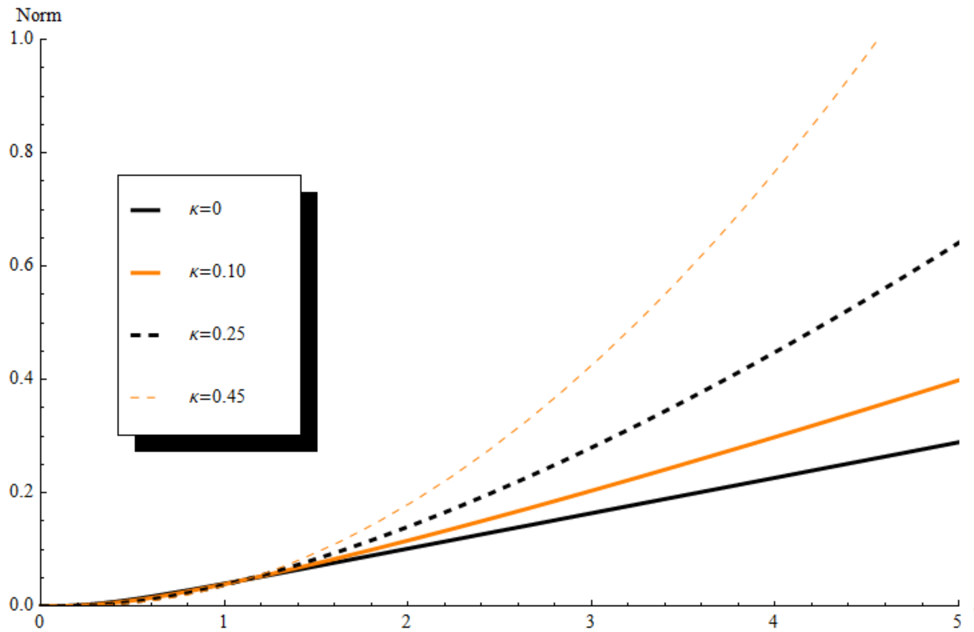

Figure 1: Calculation of $\|D(\cdot,t)\mathbf{1}_{[0,t]}(\cdot)\|^2_{\kappa,T}$ in the fractional one-factor model for varying κ and maturity t, using $a=4$. The case $\kappa=0$ has been calculated analytically.

Now we take for $i=0,\dots,n-1$ a partition $s_i=u_0^i\le u_1^i\le\dots\le u^i_{m_i}=s_{i+1}$ for some $m_i\in\mathbb{N}$

$$\int_{s_i}^T \frac{r^\kappa D(r,t)\mathbf{1}_{[0,t]}(r)}{(r-s_i)^{1-\kappa}}dr = \sum_{j=0}^{m_i-1}\int_{u_j^i}^{u_{j+1}^i}\frac{r^\kappa D(r,t)\mathbf{1}_{[0,t]}(r)}{(r-s_i)^{1-\kappa}}dr$$

$$\approx \quad \frac{1}{\kappa}\sum_{j=0}^{m_i-1}[(u^i_{j+1}-s_i)^\kappa-(u^i_j-s_i)^\kappa]\frac{(u^i_j)^\kappa D(u^i_j,t)+(u^i_{j+1})^\kappa D(u^i_{j+1},t)}{2}$$

Putting everything together and using $\Gamma(\kappa)\cdot\kappa=\Gamma(\kappa+1)$, we obtain

$$\|D(\cdot,t)\mathbf{1}_{[0,t]}(\cdot)\|^2_{\kappa,T} = \frac{\pi\kappa(2\kappa+1)}{\Gamma(2-2\kappa)\sin(\pi\kappa)(2\Gamma(\kappa+1))^2}\sum_{i=0}^{n-1}[s_{i+1}^{1-2\kappa}-s_i^{1-2\kappa}]$$
$$\times\Big[\sum_{j=0}^{m_i-1}[(u^i_{j+1}-s_i)^\kappa-(u^i_j-s_i)^\kappa](u^i_j)^\kappa D(u^i_j,t)+(u^i_{j+1})^\kappa D(u^i_{j+1},t)\Big]^2.$$

Choosing now $s_i=0.01i$, $i=0,\dots,100t$, and $u^i_j=0.01(i+j)$, $j=0,\dots,100t-i$, we obtain

$$\|D(\cdot,t)\mathbf{1}_{[0,t]}(\cdot)\|^2_{\kappa,T}$$
$$\approx \frac{\pi\kappa(2\kappa+1)}{\Gamma(2-2\kappa)\sin(\pi\kappa)2\Gamma(\kappa+1)^2}0.01^{1+2\kappa}\sum_{i=0}^{100t-1}\Big([(i+1)^{1-2\kappa}-i^{1-2\kappa}] \tag{4.3}$$
$$\times\sum_{j=0}^{100t-i-1}[(j+1)^\kappa-j^\kappa][(i+j)^\kappa D(0.01(i+j),t)+(i+j+1)^\kappa D(0.01(i+j+1),t)]\Big).$$

Finally, examples of the norms and bond prices can be found in Figure 1 and Figure 2.

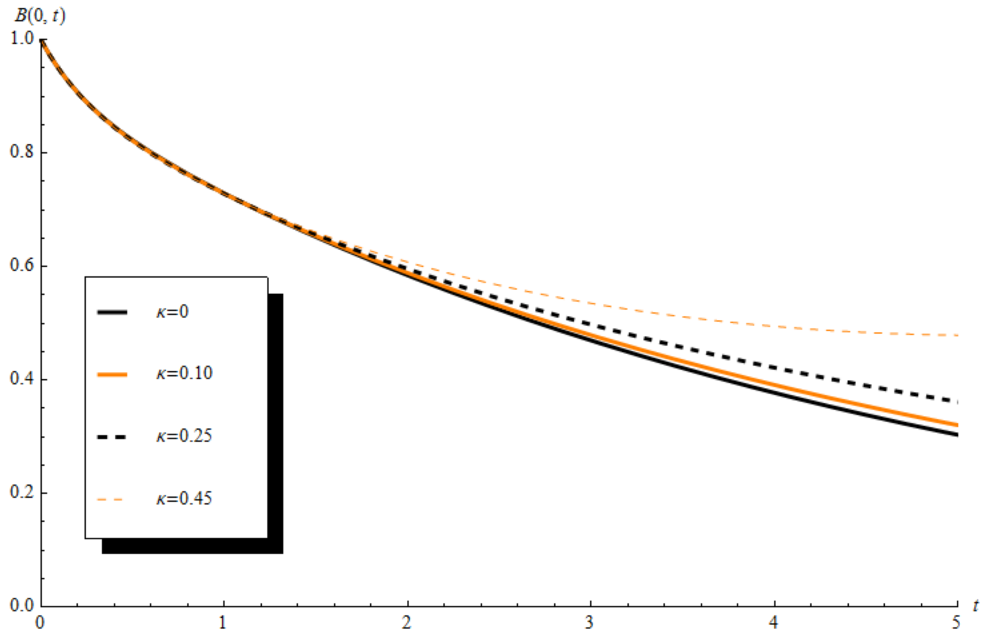

Figure 2: Bond prices $B(0,t)$ in the fractional one-factor Vasicek model (4.2) for varying $\kappa \geq 0$ and maturity t, using constant coefficients $a = 4$, $k = 1$ and $\sigma = 1$. Negative κ is not relevant as explained in the introduction to this section. Recall that $\kappa = 0$ corresponds to the Brownian Vasicek model. Prices increase with κ as a consequence of long range dependence.

5 Proofs

Before we start with the proof of the main results in Section 3, we will state a well-known property of the multivariate normal distribution.

Lemma 5.1. *Let $Z \sim N(\mu, \Sigma)$, i.e. $Z = (z_1, \ldots, z_d)^T$ is multivariate normally distributed with mean $\mu \in \mathbb{R}^n$ and variance-covariance matrix $\Sigma \in \mathbb{S}^{d\times d}$. For $k \in \{1, \ldots, d-1\}$, set $X = (z_1, \ldots, z_k)^T$ and $Y = (z_{k+1}, \ldots, z_d)$. Partition*

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

with $\mu_1 \in \mathbb{R}^k$, $\mu_2 \in \mathbb{R}^{d-k}$, $\Sigma_{11} \in \mathbb{S}^{k\times k}$, $\Sigma_{22} \in \mathbb{S}^{(d-k)\times(d-k)}$ and $\Sigma_{12}^T = \Sigma_{21} \in \mathbb{R}^{d-k}$. Then we have

$$X|\{Y = y\} \sim N(\mu_1 + \Sigma_{21}\Sigma_{22}^{-1}(y - \mu_2), \Sigma_{11} - \Sigma_{21}\Sigma_{22}^{-1}\Sigma_{12}).$$

Now we can start with the proofs.

Proof of Theorem 3.1. Let $0 \leq s \leq t \leq T$. To calculate the conditional characteristic function of $\int_0^t c(v)dB^\kappa(v)$ we invoke the fact that by Gaussianity and Lemma 5.1, $\int_0^t c(v)dB^\kappa(v)|\mathcal{F}_s$ is again normally distributed. Since $\int_0^s c(v)dB^\kappa(v)$ is $\mathcal{F}_s$-measurable, it suffices to consider $\int_s^t c(v)dB^\kappa(v)|\mathcal{F}_s$.

As we know from Proposition 2.3,

$$E\left[\int_s^t c(v)dB^\kappa(v)\Big|\mathcal{F}_s\right] = \int_0^s \Psi_c^\kappa(s,t,v)dB^\kappa(v)$$

and, therefore, we need only to calculate the conditional variance $Var[\int_s^t c(v)dB^\kappa(v)|\mathcal{F}_s]$.

Choose a sequence of partitions $(\pi_n)_{n\in\mathbb{N}}$ of $[0,s]$ such that for $n \in \mathbb{N}$ we have $\pi_n = (s_i^n)_{i=0,\dots,m_n}$ for $m_n \in \mathbb{N}$ with

$$0 = s_0^n < s_1^n < \dots < s_{m_n}^n \le s \quad \text{and} \quad \sup_{i=1,\dots,m_n} |s_i^n - s_{i-1}^n| \to 0 \text{ as } n \to \infty.$$

Using this notation we know by Lemma 5.1 for $n \in \mathbb{N}$

$$E\left[\int_s^t c(v)dB^\kappa(v)\Big|B^\kappa(s_i^n) - B^\kappa(s_{i-1}^n), i = 1,\dots,m_n\right] = \Sigma_{21}^n(\Sigma_{22}^n)^{-1}\begin{pmatrix} \vdots \\ B^\kappa(s_i^n) - B^\kappa(s_{i-1}^n) \\ \vdots \end{pmatrix} \tag{5.1}$$

and

$$Var\left[\int_s^t c(v)dB^\kappa(v)\Big|B^\kappa(s_i^n) - B^\kappa(s_{i-1}^n), i = 1,\dots,m_n\right] = \Sigma_{11}^n - \Sigma_{21}^n(\Sigma_{22}^n)^{-1}\Sigma_{12}^n$$

where $\Sigma_{11}^n = Var[\int_s^t c(v)dB^\kappa(v)]$,

$$(\Sigma_{12}^n)^T = \Sigma_{21}^n = \begin{pmatrix} \vdots \\ Cov\left[\int_s^t c(v)dB^\kappa(v), B^\kappa(s_i^n) - B^\kappa(s_{i-1}^n)\right] \\ \vdots \end{pmatrix} \in \mathbb{R}^{m_n}$$

and

$$\Sigma_{22}^n = \left(Cov\left[B^\kappa(s_i^n) - B^\kappa(s_{i-1}^n), B^\kappa(s_j^n) - B^\kappa(s_{j-1}^n)\right]\right)_{i,i=1,\dots,m_n} \in \mathbb{S}^{m_n \times m_n}.$$

By Lemma 5.1 and p. 290 of Dudley [3] follows that as $n \to \infty$,

$$\begin{aligned} E\left[\int_s^t c(v)dB^\kappa(v)\Big|B^\kappa(s_i^n) - B^\kappa(s_{i-1}^n), i = 1,\dots,m_n\right] &\to E\left[\int_s^t c(v)dB^\kappa(v)\Big|\mathcal{F}_s\right], \quad (5.2) \\ Var\left[\int_s^t c(v)dB^\kappa(v)\Big|B^\kappa(s_i^n) - B^\kappa(s_{i-1}^n), i = 1,\dots,m_n\right] &\to Var\left[\int_s^t c(v)dB^\kappa(v)\Big|\mathcal{F}_s\right]. \end{aligned}$$

This implies by (5.1) and Proposition 2.3 that as $n \to \infty$,

$$\begin{aligned} \Sigma_{21}^n(\Sigma_{22}^n)^{-1}\begin{pmatrix} \vdots \\ B^\kappa(s_i^n) - B^\kappa(s_{i-1}^n) \\ \vdots \end{pmatrix} &= \sum_{i=1}^{m_n}(\Sigma_{21}^n(\Sigma_{22}^n)^{-1})_i[B^\kappa(s_i^n) - B^\kappa(s_{i-1}^n)] \\ \to \int_0^s \Psi^\kappa(s,t,v)dB^\kappa(v). \end{aligned}$$

Therefore it follows as $n \to \infty$,

$$\sum_{i=1}^{m_n} (\Sigma_{21}^n (\Sigma_{22}^n)^{-1})_i \mathbf{1}_{[s_{i-1}^n, s_i^n]}(\cdot) \to \Psi^\kappa(s,t,\cdot).$$

With this result we can now calculate the conditional variance, since using the isometry (2.3)

$$\begin{aligned}
\Sigma_{21}^n (\Sigma_{22}^n)^{-1} \Sigma_{12}^n &= \Sigma_{21}^n (\Sigma_{22}^n)^{-1} \begin{pmatrix} \vdots \\ Cov\Big[\int_s^t c(v) dB^\kappa(v), B^\kappa(s_i^n) - B^\kappa(s_{i-1}^n)\Big] \\ \vdots \end{pmatrix} \\
&= \sum_{i=1}^{m_n} (\Sigma_{21}^n (\Sigma_{22}^n)^{-1})_i Cov\left[\int_s^t c(v) dB^\kappa(v), B^\kappa(s_i^n) - B^\kappa(s_{i-1}^n)\right] \\
&= \sum_{i=1}^{m_n} (\Sigma_{21}^n (\Sigma_{22}^n)^{-1})_i < c(\cdot), \mathbf{1}_{[s_{i-1}^n, s_i^n]}(\cdot) >_{\kappa,T} \\
&= < c(\cdot), \sum_{i=1}^{m_n} (\Sigma_{21}^n (\Sigma_{22}^n)^{-1})_i \mathbf{1}_{[s_{i-1}^n, s_i^n]}(\cdot) >_{\kappa,T} \\
&\to < c(\cdot), \Psi^\kappa(s,t,\cdot) >_{\kappa,T}, \qquad \text{as } n \to \infty.
\end{aligned}$$

where we used in the last line the continuity of the scalar product.

It remains to observe that again by the isometry (2.3)

$$\begin{aligned}
&< c(\cdot), \Psi^\kappa(s,t,\cdot) >_{\kappa,T} = E\left[\int_s^t c(v) dB^\kappa(v) \int_0^s \Psi^\kappa(s,t,v) dB^\kappa(v)\right] \\
= \; & E\left[\Big(\int_s^t c(v) dB^\kappa(v) - E\left[\int_s^t c(v) dB^\kappa(v) \Big| \mathcal{F}_s\right]\Big) E\left[\int_s^t c(v) dB^\kappa(v) \Big| \mathcal{F}_s\right]\right] \\
& + E\left[E\left[\int_s^t c(v) dB^\kappa(v) \Big| \mathcal{F}_s\right]^2\right] \\
= \; & E\left[E\left[\int_s^t c(v) dB^\kappa(v) \Big| \mathcal{F}_s\right]^2\right] = \|\Psi^\kappa(s,t,\cdot)\|_{\kappa,T}^2
\end{aligned}$$

by the projection property of the conditional expectation in L^2.

Finally we conclude that

$$\begin{aligned}
Var\left[\int_s^t c(v) dB^\kappa(v) \Big| \mathcal{F}_s\right] &= \lim_{n\to\infty} Var\left[\int_s^t c(v) dB^\kappa(v) \Big| B^\kappa(s_i^n) - B^\kappa(s_{i-1}^n), i = 1, \ldots, m_n\right] \\
&= \lim_{n\to\infty} (\Sigma_{11}^n - \Sigma_{21}^n (\Sigma_{22}^n)^{-1} \Sigma_{12}^n) \\
&= \|c(\cdot) \mathbf{1}_{[s,t]}(\cdot)\|_{\kappa,T} - < c(\cdot), \Psi^\kappa(s,t,\cdot) >_{\kappa,T} \\
&= \|c(\cdot) \mathbf{1}_{[s,t]}(\cdot)\|_{\kappa,T} - \|\Psi^\kappa(s,t,\cdot)\|_{\kappa,T}^2.
\end{aligned}$$

□

Remark 5.2. The proof of Theorem 3.1 can be shortened by the following: since (5.2) and Lemma 5.1 imply that the conditional variance is deterministic we can apply the well-known formula

$$Var[X] = E[Var[X|\mathcal{F}_s]] + Var[E[X|\mathcal{F}_s]]$$

for $X = \int_s^t c(v)dB^\kappa(v)$. However by our chosen approach we get the convergence results of $(\Sigma_{22}^n)^{-1}\Sigma_{12}^n$ and $\Sigma_{21}^n(\Sigma_{22}^n)^{-1}\Sigma_{12}^n$ which are interesting in their own right.

Theorem 3.3 is now a consequence of Theorem 3.1. Its proof follows below.

Proof. [Theorem 3.3]
By (3.2) we see that for $0 \le s \le t \le T$ it follows that

$$X(t) = X(s)e^{-\int_s^t a(v)dv} + \int_s^t e^{-\int_v^t a(w)dw}k(v)dv + \int_s^t e^{-\int_v^t a(w)dw}\sigma(v)dB^\kappa(v). \tag{5.3}$$

Therefore $X(t)|\mathcal{F}_s$ is again Gaussian distributed. Since $X(s)$ is $\mathcal{F}_s$-measurable, a direct consequence is now that

$$\begin{aligned} E\Big[X(t)\Big|\mathcal{F}_s\Big] &= X(s)e^{-\int_s^t a(v)dv} + \int_s^t e^{-\int_v^t a(w)dw}k(v)dv + E\Big[\int_s^t e^{-\int_v^t a(w)dw}\sigma(v)dB^\kappa(v)\Big|\mathcal{F}_s\Big], \\ Var\Big[X(t)\Big|\mathcal{F}_s\Big] &= Var\Big[\int_s^t e^{-\int_v^t a(w)dw}\sigma(v)dB^\kappa(v)\Big|\mathcal{F}_s\Big]. \end{aligned}$$

Invoking Theorem 3.1 with $c(\cdot) = e^{-\int_\cdot^t a(w)dw}\sigma(\cdot)$ concludes the proof. □

The main step in the proof of Proposition 3.4 is an application of a density formula for Riemann-Stieltjes integrals.

Proof. [Theorem 3.3]
By assumption on the coefficient functions, all appearing integrals in this proof can be considered in the pathwise Riemann-Stieltjes sense, cf. Young [13].

Our goal is now to invert (3.1). By (3.2) we have for $0 \le s \le t \le T$

$$\int_s^t e^{-\int_v^t a(w)dw}\sigma(v)dB^\kappa(v) = X(t) - X(s)e^{-\int_v^t a(w)dw} - \int_s^t e^{-\int_v^t a(w)dw}k(v)dv$$

and, invoking a density formula (which can be applied by Theorem A.4 of Fink and Klüppelberg [7]) we get for

$$\begin{aligned} B^\kappa(t) - B^\kappa(s) &= \int_s^t \frac{e^{\int_v^t a(w)dw}}{\sigma(v)} d\Big(-\int_v^t e^{-\int_z^t a(w)dw}\sigma(z)dB^\kappa(z)\Big) && (5.4) \\ &= \int_s^t \frac{e^{\int_v^t a(w)dw}}{\sigma(v)} d\Big(\int_v^t e^{-\int_z^t a(w)dw}k(z)dz + X(v)e^{-\int_v^t a(w)dw} - X(t)\Big) && (5.5) \\ &= -\int_s^t \frac{k(v)}{\sigma(v)}dv + \int_s^t \frac{a(v)}{\sigma(v)}X(v)dv + \int_s^t \frac{1}{\sigma(v)}dX(v). && (5.6) \end{aligned}$$

It remains to plug this result into the formulas of Theorem 3.3 and the proof is finished. □

Corollary 3.5 is just a special case of Theorem 3.3 and we skip its proof.

Proof. [Proposition 3.7]
Let $0 \leq s \leq t \leq T$. By Gaussianity we see again that $\int_0^t X(v)dv|\mathcal{F}_s$ is normally distributed and as before it remains to calculate its expectation and variance to archive the conditional characteristic function. Since $\int_0^s X(v)dv$ is $\mathcal{F}_s$-measurable we just consider $\int_s^t X(v)dv|\mathcal{F}_s$. From (3.2) we obtain by (5.3) and Fubini's Theorem

$$\begin{aligned}\int_s^t X(v)dv &= \int_s^t \Big\{X(s)e^{-\int_s^v a(w)dw} + \int_s^v e^{-\int_z^v a(w)dw}k(z)dz + \int_s^v e^{-\int_z^v a(w)dw}\sigma(z)dB^\kappa(z)\Big\}dv \\ &= D(s,t)X(s) + \int_s^t D(v,t)k(v)dv + \int_s^t D(v,t)\sigma(v)dB^\kappa(v).\end{aligned}$$

It follows

$$\begin{aligned}E\Big[\int_0^t X(v)dv\Big|\mathcal{F}_s\Big] &= \int_0^s X(v)dv + D(s,t)X(s) + \int_s^t D(v,t)k(v)dv \\ &\quad + E\Big[\int_s^t D(v,t)\sigma(v)dB^\kappa(v)\Big|\mathcal{F}_s\Big], \\ Var\Big[\int_0^t X(v)dv\Big|\mathcal{F}_s\Big] &= Var\Big[\int_s^t D(v,t)\sigma(v)dB^\kappa(v)\Big|\mathcal{F}_s\Big].\end{aligned}$$

Applying Theorem 3.1 with $c(\cdot) = D(\cdot,t)\sigma(\cdot)$ shows the first assertion. The second one follows by applying (5.4). □

The proof of Theorem 3.8 uses Fourier techniques and follows below.

Proof. [Theorem 3.8]
Let $x \in \mathbb{R}$ and set $g(x,u) = \exp(iuf(x))$. First we decompose the transformation g by

$$g(x,u) = e^x[e^{-x}g(x,u)\mathbf{1}_{[0,\infty)}(x)] + e^{-x}[e^x g(x,u)\mathbf{1}_{(-\infty,0)}(x)] =: e^x g_+(x,u) + e^{-x}g_-(x,u).$$

Denote for fixed $u \in \mathbb{R}$ with $\widehat{g_+}(\cdot,u)$ and $\widehat{g_-}(\cdot,u)$ the Fourier transforms of $g_+(\cdot,u)$ and $g_-(\cdot,u)$ respectively. Using classical Fourier analysis we obtain for $x,\xi \in \mathbb{R}$ and $\star \in \{+,-\}$

$$\begin{aligned}\widehat{g_\star}(\xi,u) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\xi x}g_\star(x,u)dx = \frac{1}{2\pi}\int_{\mathbb{R}_\star} e^{-(i\xi\star 1)x+iuf(x)}dx, \\ g_\star(x,u) &= \int_{\mathbb{R}} e^{i\xi x}\widehat{g_\star}(\xi,u)d\xi,\end{aligned}$$

where we used the fact that $g_+(\cdot,u)$ and $g_-(\cdot,u)$ are in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Now we obtain

$$\begin{aligned}E\Big[e^{iuZ(t)}\Big|\mathcal{F}_s\Big] &= E[g(X(t)) \mid \mathcal{F}_s] = E[e^{X(t)}g_+(X(t),u) \mid \mathcal{F}_s] + E[e^{-X(t)}g_-(X(t),u) \mid \mathcal{F}_s] \\ &= E\Big[e^{X(t)}\int_{\mathbb{R}} e^{i\xi X(t)}\widehat{g_+}(\xi,u)d\xi\Big|\mathcal{F}_s\Big] + E\Big[e^{-X(t)}\int_{\mathbb{R}} e^{i\xi X(t)}\widehat{g_-}(\xi,u)d\xi\Big|\mathcal{F}_s\Big]\end{aligned}$$

Since $E[e^{bX(t)}] < \infty$ for all $b \in \mathbb{C}$ we can interchange conditional expectation and integration and get

$$E\Big[e^{iuZ(t)}\Big|\mathcal{F}_s\Big] = \int_{\mathbb{R}}\Big(E\Big[e^{(i\xi+1)X(t)}\Big|\mathcal{F}_s\Big]g_+(\xi,u) + E\Big[e^{(i\xi-1)X(t)}\Big|\mathcal{F}_s\Big]g_-(\xi,u)\Big)\,d\xi.$$

□

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