

A FRACTIONALLY INTEGRATED ECOGARCH PROCESS

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ABSTRACT

In this paper we introduce a fractionally integrated exponential continuous time GARCH(p, d, q) process. It is defined in such a way that it is a continuous time extension of the discrete time FIEGARCH(p, d, q) process. We investigate stationarity and moment properties of the new model. It is also shown that the long memory effect introduced in the log-volatility propagates to the volatility process.

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1. INTRODUCTION

GARCH type processes have become very popular in financial econometrics to model returns of stocks, exchange rates and other series observed at equidistant time points. They have been designed (see Engle [12] and Bollerslev [6]) to capture so-called *stylized facts* of such data, which are e.g. volatility clustering, dependence without correlation and tail heaviness. Several authors have found empirical evidence for the existence of long-run volatility persistence in financial data. Among these are e.g. Andersen and Bollerslev [1], who analysed high-frequency foreign exchange data, Baillie et al. [5], Bollerslev and Mikkelsen [7] and Baillie [4], who gave an overview over long memory processes in econometrics. These findings have to be treated carefully since certain empirical evidence, like a slow decay of the empirical autocorrelation function, could also be due to non-stationarity of the data. This was e.g. shown by Mikosch and Stărică [19] for a long time series of S&P 500 log-returns. In the following this problem will not be our subject.

Since there are different ways to characterise long range dependence, we first want to recall the definition of a long memory process as we will use it before we go on.

Definition 1.1. *Let Z be a stationary stochastic process and $\gamma_Z(h) = \text{Cov}(Z_{t+h}, Z_t)$, $h \in \mathbb{R}$, be its autocovariance function. If there exists $0 < d < 0.5$ and a constant*

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$c_Z > 0$ such that

$$\lim_{h \rightarrow \infty} \frac{\gamma_Z(h)}{h^{2d-1}} = c_Z, \quad (1.1)$$

then Z is called a stationary process with long memory.

In the discrete time GARCH framework there are various models with long range dependence in the volatility process. Among these are the IGARCH(p, q) process of Engle and Bollerslev [13], the FIGARCH(p, d, q) process proposed by Baillie et al. [5] or the fractionally integrated EGARCH(p, d, q) process of Bollerslev and Mikkelsen [7]. The FIGARCH process has to be treated carefully since the existence of a stationary version is not clear; see section 4 in Mikosch and Stărică [19] and Remark 3.2 in Kazakevičius and Leipus [15]. The FIEGARCH(p, d, q) process is a modification of the EGARCH model of Nelson [21] in the sense that the log-volatility process is modeled by a fractionally integrated ARMA(p, d, q) process instead of a short memory ARMA process. This long memory effect introduced in the log-volatility process propagates to the volatility and the squared return process. This was shown by Surgailis and Viano [25].

The availability of high frequency data, which increased enormously in the last years, is one reason to consider continuous time models with similar behaviour as discrete time GARCH models. The reason for this is ofcourse that at the highest available frequency the observations of the price process occur at irregularly spaced time points and therefore it is kind of natural to assume an underlying continuous time model. Different approaches have been taken to set up a continuous time model, which has the same features as discrete time GARCH processes. Recently Klüppelberg et al. [16] developed a continuous time GARCH(1, 1) model, which was extended by Brockwell et al. [9] to a continuous time GARCH(p, q) process for general orders $p, q \in \mathbb{N}$, $q \geq p$, henceforth called COGARCH(p, q). Their approach differs fundamentally from previous attempts, which could be summarized as diffusion approximations (see e.g. Nelson [20]), by the fact that their model is driven by only one source of randomness (like discrete time GARCH) instead of two (like in the diffusion approximations). Haug and Czado [14] have defined an exponential continuous time GARCH process, which is a continuous time extension to the EGARCH process. All these models exhibit a short memory in the volatility process. To incorporate a long memory effect into a continuous time model Comte and Renault [11] defined a continuous time stochastic volatility (SV) model by specifying the log-volatility process as an OU process driven by a fractional Brownian motion. Brockwell and Marquardt [10] proposed to model the stochastic volatility as a non-negative fractionally integrated CARMA process. Another non-Gaussian continuous time SV model with long memory was introduced by Anh et al. [2], where they define their model via the Green function solution of a fractional differential equation driven by a Lévy process. Since this shows considerable interest in continuous time models with long memory in the volatility process, we now want to show in this paper how to extend the ECOGARCH(p, q) in such a way.

The paper is now organized as follows. In section 2 we define the *fractionally integrated exponential continuous time* GARCH(p, d, q) process after a short review of elementary properties of Lévy processes and analyse stationarity conditions. The second order behaviour of the volatility process is investigated in section 3, while section 4 deals with second order behaviour of the return process.

2. FRACTIONALLY INTEGRATED EXPONENTIAL COGARCH

In this section we want to construct a continuous time analogue of the discrete time fractionally integrated EGARCH(p, d, q) process, which is defined in the following way:

Let $p, q \in \mathbb{N}, \mu, \alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_p \in \mathbb{R}$, suppose $\alpha_q \neq 0, \beta_p \neq 0$ and that the autoregressive polynomial

$$\phi(z) := 1 - \alpha_1 z - \dots - \alpha_q z^q$$

and the moving average polynomial

$$\psi(z) := \beta_1 + \beta_2 z + \dots + \beta_p z^{p-1}$$

have no common zeros and that $\phi(z) \neq 0$ on $\{z \in \mathbb{C} \mid |z| \leq 1\}$. Let $(\epsilon_n)_{n \in \mathbb{Z}}$ be an i.i.d. sequence with $\mathbb{E}(\epsilon_1) = 0$ and $\text{Var}(\epsilon_1) = 1$ and $-0.5 < d < 0.5$. Define the measurable function $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) := \theta x + \gamma[|x| - \mathbb{E}(|x|)], \quad x \in \mathbb{R}, \quad (2.1)$$

with real coefficients θ and γ . Then we call $(X_n)_{n \in \mathbb{Z}}$, where $X_n = \sigma_n \epsilon_n$, a FIE-GARCH(p, d, q) process if

$$\log(\sigma_n^2) = \mu + \phi(B)^{-1}(1 - B)^{-d}(1 + \psi(B))f(\epsilon_{n-1}), \quad (2.2)$$

where B is the backward shift operator, $BX_n = X_{n-1}$.

We will define the process using the idea of Klüppelberg et al. [16] to replace the innovations ϵ_n of the discrete time model by the jumps of a Lévy process $L = (L_t)_{t \geq 0}$. Any Lévy process L on \mathbb{R} has a characteristic function of the form

$$\mathbb{E}(e^{iuL_t}) = \exp\{t\psi_L(u)\}, \quad t \geq 0,$$

with

$$\psi_L(u) := i\gamma_L u - \frac{\tau_L^2}{2} u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux\chi_{B_1}(x))\nu_L(dx), \quad u \in \mathbb{R},$$

where $B_1 := \{x \in \mathbb{R} : |x| \leq 1\}$, $\tau_L^2 \geq 0$, $\gamma_L \in \mathbb{R}$, the measure ν_L satisfies

$$\nu_L(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}} \min(x^2, 1)\nu_L(dx) < \infty$$

and $\chi_A(\cdot)$ denotes the indicator function of the set $A \subset \mathbb{R}$. The measure ν_L is called the Lévy measure of L and the triplet $(\gamma_L, \tau_L^2, \nu_L)$ is called the characteristic triplet of L . The map ψ_L is called the Lévy symbol or Lévy exponent. For L to have finite mean and variance it is necessary and sufficient that

$$\int_{|x|>1} |x|\nu_L(dx) < \infty \quad \text{and} \quad \int_{|x|>1} x^2\nu_L(dx) < \infty,$$

respectively (Sato [23], Example 25.12). For more details on Lévy processes we refer to Sato [23] or Applebaum [3].

We consider zero mean Lévy processes L defined on a probability space (Ω, \mathcal{F}, P) with jumps $\Delta L_t := L_t - L_{t-}$. Since $\mathbb{E}(L_t) = t(\gamma_L + \int_{|x|>1} x\nu_L(dx))$, a zero mean implies that $\gamma_L = -\int_{|x|>1} x\nu_L(dx)$ and hence the corresponding Lévy symbol is of the form

$$\psi_L(u) = -\tau_L^2 \frac{u^2}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux)\nu_L(dx), \quad (2.3)$$

and the Lévy-Itô decomposition (see e.g. Theorem 2.4.16 of Applebaum [3]) of L is

$$L_t = B_t + \int_{\mathbb{R}-\{0\}} x \tilde{N}_L(t, dx), \quad t \geq 0, \quad (2.4)$$

where B is a Brownian motion with variance τ_L^2 and $\tilde{N}_L(t, dx) = N_L(t, dx) - t\nu_L(dx)$, $t \geq 0$, is the compensated random measure associated to the Poisson random measure

$$N_L(t, A) = \#\{0 \leq s < t; \Delta L_s \in A\} = \sum_{0 < s \leq t} \chi_A(\Delta L_s), \quad A \in \mathcal{B}(\mathbb{R} - \{0\}),$$

on $\mathbb{R}_+ \times \mathbb{R} - \{0\}$, which is independent of B .

The Lévy process L can be extended to a Lévy process L^* defined on the whole real line by choosing a second Lévy process $(\tilde{L}_t)_{t \geq 0}$ independent of L and with the same distribution as L and specifying

$$L_t^* := L_t \chi_{[0, \infty)}(t) - \tilde{L}_{-t} \chi_{(-\infty, 0)}(t), \quad t \in \mathbb{R},$$

where $\chi_A(\cdot)$ denotes the indicator function of the set A . In the following we will work with L^* but write for simplicity L instead of L^* .

Now we are able to define the *fractionally integrated exponential continuous time GARCH*(p, d, q) process, shortly called FIECOGARCH(p, d, q). The stationary log-volatility process will be modeled by a fractionally integrated continuous time ARMA($q, d, p-1$) process, henceforth called FICARMA($q, d, p-1$) process (see e.g. Brockwell and Marquardt [10] for details on FICARMA processes). The driving noise process of the log-volatility process will be defined similarly to (2.1).

Definition 2.1. Let $L = (L_t)_{t \geq 0}$ be a Lévy process with $\mathbb{E}(L_1) = 0$, $\text{Var}(L_1) = 1$ and Lévy measure ν_L and let the $q \times q$ matrix A and vectors $\mathbf{b} \in \mathbb{R}^q$ and $\mathbf{1}_q \in \mathbb{R}^q$ be defined by

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_q & -a_{q-1} & -a_{q-2} & \cdots & -a_1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{q-1} \\ b_q \end{bmatrix}, \quad \mathbf{1}_q = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

with coefficients $a_1, \dots, a_q, b_1, \dots, b_p \in \mathbb{R}$, where $a_q \neq 0, b_p \neq 0$, and $b_{p+1} = \dots = b_q = 0$. Then for $0 < d < 0.5$ we define the fractionally integrated exponential COGARCH(p, d, q) process G_d as the stochastic process satisfying,

$$dG_{d,t} = \sigma_{d,t} dL_t, \quad t > 0, \quad G_0 = 0, \quad (2.5)$$

where the log-volatility process is given by

$$\log(\sigma_{d,t+}^2) = \mu + \int_{-\infty}^t g_d(t-u) dM_u, \quad t > 0, \quad (2.6)$$

with mean $\mu \in \mathbb{R}$ and initial value $\log(\sigma_{d,0}^2)$ independent of the driving Lévy process L . The process

$$M_t := \int_{\mathbb{R} - \{0\}} h(x) \tilde{N}_L(t, dx), \quad t > 0, \quad (2.7)$$

is a zero mean Lévy process (see Remark 2.2) with

$$h(x) := \theta x + \gamma |x|$$

and parameters $\theta, \gamma \in \mathbb{R}$. The kernel function

$$g_d(t) = \int_0^t g(t-u) \frac{u^{d-1}}{\Gamma(d)} du, \quad 0 < d < 0.5, \quad (2.8)$$

is the Riemann-Liouville fractional integral of order d (see Definition 2.1 in Sako et al. [22]) of the kernel function $g(t) = \mathbf{b}^T e^{At} \mathbf{1}_q \chi_{(0, \infty)}(t)$.

Returns over a time interval of length $r > 0$ are described by the increments of G_d

$$G_{d,t}^{(r)} := G_{d,t} - G_{d,t-r} = \int_{(t-r, t]} \sigma_{d,s} dL_s, \quad t \geq r > 0. \quad (2.9)$$

On the other hand an equidistant sequence of such non-overlapping returns of length r is given by $(G_{nr}^{(r)})_{n \in \mathbb{N}}$. Thus this gives us the possibility to model ultra high frequency data, which consists of returns over varying time intervals.

In the rest of the paper the following terminology will be used:

G_d	(log-) price process
$G_d^{(r)}$	(log-) return process
σ_d^2	volatility process
$\log(\sigma_d^2)$	log-volatility process.

Remark 2.2. (i) The log-volatility process (2.6) is well-defined and stationary if the real part of the eigenvalues of A is negative, since then

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |g_d(t-s)x|^2 \nu_L(dx) ds, \quad \forall t \geq 0,$$

and we can apply Theorem 4.3.4 and 4.3.16 in Applebaum [3] from which the assumptions follow. In this case the log-volatility process is a FICARMA(p, d, q) process.

(ii) The process M defined by (2.7) is by construction a process with independent and stationary increments and by Theorem 4.3.4 in Applebaum [3] well defined if

$$\int_{\mathbb{R}} |h(x)|^2 \nu_L(dx) < \infty. \quad (2.10)$$

Condition (2.10) is satisfied since ν_L is a Lévy measure and L has finite variance. By equation (2.9) of Applebaum [3] the characteristic function of M at time $t \geq 0$ is given by

$$\begin{aligned} \mathbb{E}(e^{iuM_t}) &= \exp\left(t \int_{\mathbb{R}} [e^{iux} - 1 - iux] \nu_M(dx)\right) \\ &= \exp\left(t \left\{ iu\gamma_M + \int_{\mathbb{R}} [e^{iux} - 1 - iux\chi_{|x|\leq 1}] \nu_M(dx) \right\}\right) \\ &=: \exp(t\psi_M(u)), \end{aligned}$$

where $\nu_M := \nu_L \circ h^{-1}$ is the Lévy measure of M and $\gamma_M := -\int_{|x|>1} x\nu_M(dx)$. The concrete form of ν_M depends on the sign and size of θ and γ and is given in the following:

$$\nu_M((-\infty, -x]) = \begin{cases} \nu_L([\frac{-x}{\theta+\gamma}, \infty)) + \nu_L((-\infty, -\frac{x}{\theta-\gamma}]), & \theta + \gamma < 0 \text{ and } \theta - \gamma > 0 \\ \nu_L((-\infty, -\frac{x}{\theta-\gamma}]), & \theta - \gamma > 0 \text{ and } \theta + \gamma > 0 \\ \nu_L([\frac{-x}{\theta+\gamma}, \infty)), & \theta + \gamma < 0 \text{ and } \theta - \gamma < 0 \\ 0 & \theta + \gamma > 0 \text{ and } \theta - \gamma < 0 \end{cases}$$

and

$$\nu_M([x, \infty)) = \begin{cases} \nu_L([\frac{x}{\theta+\gamma}, \infty)) + \nu_L((-\infty, \frac{x}{\theta-\gamma}]), & \theta + \gamma > 0 \text{ and } \theta - \gamma < 0 \\ \nu_L((-\infty, \frac{x}{\theta-\gamma}]), & \theta - \gamma < 0 \text{ and } \theta + \gamma < 0 \\ \nu_L([\frac{x}{\theta+\gamma}, \infty)), & \theta + \gamma > 0 \text{ and } \theta - \gamma > 0 \\ 0 & \theta + \gamma < 0 \text{ and } \theta - \gamma > 0 \end{cases}$$

for $x > 0$. One recognises that for $\theta + \gamma < 0 \vee \theta - \gamma > 0$ M is a spectrally negative Lévy process, i.e. M has only negative jumps, and for $\theta + \gamma > 0 \vee \theta - \gamma < 0$ M is a spectrally positive Lévy process. Therefore M has the characteristic triplet $(\gamma_M, 0, \nu_M)$.

(iii) The model can of course also be defined for a different choice of h , as long as condition (2.10) is satisfied.

Alternatively the log-volatility process can be defined in terms of the fractional Lévy process M_d associated with M . We recall the definition of a fractional Lévy process from Marquardt [18].

Remark 2.3. Let $M = (M_t)_{t \in \mathbb{R}}$ be a Lévy process on \mathbb{R} with $\mathbb{E}M_1 = 0$, $\mathbb{E}M_1^2 < \infty$ and without Brownian component. For the fractional integration parameter $0 < d < 0.5$ the stochastic process

$$M_{d,t} = \frac{1}{\Gamma(d+1)} \int_{\mathbb{R}} [(t-s)_+^d - (-s)_+^d] dM_s, \quad t \in \mathbb{R}, \quad (2.11)$$

is called a fractional Lévy process.

The strictly stationary log-volatility process (2.6) is then equal to

$$\mu + \int_{-\infty}^t \mathcal{D}_+^d g_d(t-u) dM_{d,u}, \quad t > 0, \quad (2.12)$$

in the L^2 -sense, where $\mathcal{D}_+^d g_d(x) = \frac{1}{\Gamma(1-d)} \frac{d}{dx} \int_{-\infty}^x \frac{g_d(u)}{(x-u)^d} du$ is the Riemann-Liouville fractional derivative of g_d of order d (see Definition 2.2 in Sako et al. [22]). Since $g \in L^1(\mathbb{R})$ we get from Theorem 2.4 in Sako et al. [22] that $\mathcal{D}_+^d g_d = g$. The proof of the equivalence of (2.6) and (2.12) can be found in Marquardt [18], Theorem 6.5.

If the Lévy process M is of finite activity, i.e. $\nu_M(\mathbb{R}) < \infty$, then the corresponding fractional Lévy process M_d is of finite variation. In this case the integral in (2.12) can be defined as a Riemann-Stieltjes integral. In case where M is not of finite activity the corresponding fractional Lévy process is not a semimartingale, but for a deterministic integrand the integral with respect to M_d can be defined in the L^2 -sense (we refer to section 5 of Marquardt [18] for details). We do not restrict the driving Lévy process to be of finite activity but we only deal with deterministic integrands and hence this turns out to be sufficient for our purpose.

The log-volatility process (2.12) is now the solution of the continuous time state space model

$$\log(\sigma_{d,t}^2) = \mu + \mathbf{b}^T \mathbf{X}_{d,t-}, \quad t > 0, \quad \log(\sigma_{d,0}^2) = \mu + \mathbf{b}^T \mathbf{X}_{d,0} \quad (2.13)$$

$$d\mathbf{X}_{d,t} = A\mathbf{X}_{d,t}dt + \mathbf{1}_q dM_{d,t}, \quad t > 0, \quad (2.14)$$

where $\mathbf{X}_{d,0}$ is independent of $(M_{d,t})_{t \geq 0}$ and A , \mathbf{b} and $\mathbf{1}_q$ are defined in Definition 2.1.

The state space representation of the log-volatility process is also advantageous for the purpose of simulating the log-price process G_d . The simulation procedure is the following:

1. Choose simulation times $0 = t_0 < t_1 < \dots < t_n \leq T$, possibly random.
2. Generate increments $M_{d,t_{i+1}} - M_{d,t_i}$, $i = 0, \dots, n-1$, of the driving fractional Lévy process.
3. Approximate the state process (2.14) of the log-volatility by a stochastic Euler scheme.
4. Compute

$$\log(\widehat{\sigma}_{d,t_i}^2) = \mu + \mathbf{b}^T \widehat{\mathbf{X}}_{d,t_{i-1}}$$

for $i = 1, \dots, n$.

5. Compute an approximation \widehat{G}_d by a stochastic Euler scheme:

$$\widehat{G}_{d,t_i} = \widehat{G}_{d,t_{i-1}} + \widehat{\sigma}_{d,t_{i-1}} W_i + \widehat{\sigma}_{d,t_{i-1}} J_i,$$

where $W_i \sim N(0, t_i - t_{i-1})$ and J_i is an increment of the jump part of L over the time interval $[t_{i-1}, t_i]$.

Since the fractional Lévy process M_d at time t is an integral with respect to the driving Lévy process L it can be approximated by the corresponding Riemann sums. This approximation is explained in chapter 2.4.3 in Marquardt [17].

Defined in this way $\log(\sigma_d^2)$ is not strictly stationary by definition. The conditions for stationarity of $\log(\sigma_d^2)$ the volatility process σ_d^2 and the return process $G_d^{(r)}$ are summarized in the next proposition. The autocovariance function of the log-volatility process and its asymptotic behaviour is also stated. Therefore we will call two functions f_1 and f_2 asymptotically equivalent if $\lim_{x \rightarrow \infty} f_1(x)/f_2(x) = 1$ and denote it by $f_1(x) \sim f_2(x)$ as $x \rightarrow \infty$.

Proposition 2.4. *Let $\log(\sigma_d^2)$ be defined by (2.13) and (2.14) and G_d as in Definition 2.1. If the eigenvalues of A all have negative real parts and $\log(\sigma_{d,0}^2)$ has the same distribution as $\int_0^\infty \mathbf{b}^T e^{As} \mathbf{1}_d dM_{d,s}$, then $\log(\sigma_d^2)$ and σ_d^2 are strictly stationary and G_d has strictly stationary increments. The log-volatility process is weakly stationary and for $t > 0, h \geq 0$ has autocovariance function*

$$\text{Cov}(\log(\sigma_{d,t+h}^2), \log(\sigma_{d,t}^2)) = \mathbb{E}(M_1^2) \int_{\mathbb{R}} g_d(u+h)g_d(u)du, \quad (2.15)$$

$$\sim C_1 h^{2d-1}, \quad \text{as } h \rightarrow \infty, \quad (2.16)$$

where $C_1 := \frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)} \mathbb{E}(M_1^2) \left(\int_{\mathbb{R}} g(s)ds \right)^2$.

The strict stationarity of $\log(\sigma_d^2)$, σ_d^2 and the increments of G_d follows from the same reasoning as in the short memory case (see Proposition 3.3 and Corollary 3.5 in Haug and Czado [14]). The proof of (2.15) and (2.16) is given in Marquardt [18], Theorem 6.7 and 6.6.

Remark 2.5. *The asymptotic behaviour of the autocovariance function of a FICARMA process was derived by Brockwell [8]. The result depends on the asymptotic behaviour of the kernel function g_d*

$$g_d(s) \sim \left(\int_{\mathbb{R}} g(x)dx \right) s^{d-1}, \quad \text{for } s \rightarrow \infty, \quad (2.17)$$

which was shown in Brockwell [8], section 4. In the following the constant in (2.17) will be denoted by C_2 .

3. SECOND ORDER PROPERTIES OF THE VOLATILITY PROCESS

Second order properties are now derived under the assumption that the log-volatility process is strictly stationary. The integral in (2.6) is well defined and from (2.5) in Sato and Yamazato [24] it follows that the characteristic function of $\log(\sigma_{d,t+}^2)$ is given

by

$$\begin{aligned}
 \mathbb{E}(e^{iu \log(\sigma_{d,t+}^2)}) &= e^{iu\mu} \exp \left\{ \int_0^\infty \psi_M(g_d(s)u) ds \right\} \\
 &= e^{iu\mu} \exp \left\{ iu \left[\int_0^\infty g_d(s) \gamma_M ds + \int_0^\infty \int_{\mathbb{R}} (\chi_{B_1}(g_d(s)x) - \chi_{B_1}(x)) \nu_M(dx) ds \right] \right. \\
 &\quad \left. + \int_0^\infty \int_{\mathbb{R}} (e^{iug_d(s)x} - 1 - iug_d(s)x \chi_{B_1}(g_d(s)x)) \nu_M(dx) ds \right\} \\
 &= e^{iu\mu} \exp \left\{ iu\gamma_\infty + \int_{\mathbb{R}} (e^{iux} - 1 - iux \chi_{B_1}(x)) \nu_\infty(dx) \right\},
 \end{aligned}$$

since $\log(\sigma_{d,t}^2) \stackrel{d}{=} \mu + \int_0^\infty g_d(s) dM_s$.

The stationary distribution F_d of $\log(\sigma_d^2)$ is therefore infinitely divisible with characteristic triplet $(\gamma_{d,\infty}, 0, \nu_{d,\infty})$, where

$$\begin{aligned}
 \gamma_{d,\infty} &= \int_0^\infty g_d(s) \gamma_M ds \\
 &\quad + \int_0^\infty \int_{\mathbb{R}} g_d(s)x (\chi_{\{B_1\}}(g_d(s)x) - \chi_{B_1}(x)) \nu_M(dx) ds \quad (3.1)
 \end{aligned}$$

$$\nu_{d,\infty}(B) = \int_0^\infty \int_{\mathbb{R}} \chi_B(g_d(s)x) \nu_M(dx) ds, \quad B \in \mathcal{B}(\mathbb{R}). \quad (3.2)$$

The second order behaviour is now summarised in the following proposition.

Proposition 3.1. *Let $\log(\sigma_d^2)$ be strictly stationary with marginal distribution F_d , where F_d is infinitely divisible with characteristic triplet $(\gamma_{d,\infty}, 0, \nu_{d,\infty})$. The k -th moment of $\sigma_{d,t}^2$ is finite, if*

$$k \in K_{d,\infty} = \{s \in \mathbb{R} : \int_{|x|>1} e^{sx} \nu_{d,\infty}(dx) < \infty\}.$$

In this case

$$\Psi_{d,\infty}(k) = \int_0^\infty \Psi_M(g_d(s)k) ds, \quad (3.3)$$

where $\Psi_M(u) := \psi_M(-iu)$, $u \in \mathbb{R}$, is well defined and

$$\mathbb{E}(\sigma_{d,t}^{2k}) = e^{\mu k} e^{\Psi_{d,\infty}(k)}, \quad \forall t \geq 0. \quad (3.4)$$

Assume that $\mathbb{E}(\sigma_{d,t}^4) < \infty$. Let $\Psi_{d,\infty}^h(k)$ and $\Psi_d^h(k)$ be defined by (3.3) with kernel function g_d replaced by

$$g_{d,\infty}^h(s) := g_d(s) + g_d(s+h) \quad \text{and} \quad g_d^h(s) := g_d(s) \chi_{(0,h)}(s)$$

respectively. Then the autocovariance function of σ_d^2 is given by

$$\text{Cov}(\sigma_{d,t+h}^2, \sigma_{d,t}^2) = e^{2\mu} (e^{\Psi_{d,\infty}^h(1)} e^{\Psi_d^h(1)} - e^{2\Psi_{d,\infty}(1)}). \quad (3.5)$$

If we replace the kernel functions in the proof of Proposition 4.3 and 4.5 in Haug and Czado [14] appropriately with the kernel functions $g_d, g_{d,\infty}^h$ and g_d^h , then the result follows.

Next we will show that the long memory property introduced in the log volatility process implies also a long memory effect in the volatility process. The proof is based on a result for the FICARMA(p, d, q) process which can be found in Lemma 1.23 in Marquardt [17].

Theorem 3.2. *Let $\log(\sigma_d^2)$ be the strictly stationary long memory process (2.6) with long memory parameter $0 < d < 0.5$ and assume that $2 \in K_{d,\infty}$. Then $\mathbb{E}(\sigma_{d,t}^4) < \infty$, $\forall t \geq 0$, and*

$$\text{Cov}(\sigma_{d,t+h}^2, \sigma_{d,t}^2) \sim e^{2(\mu + \Psi_{d,\infty}(1))} C_1 h^{2d-1}, \quad \text{as } h \rightarrow \infty, \quad (3.6)$$

where $C_1 = \frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)} \mathbb{E}(M_1^2) \left(\int_{\mathbb{R}} g(s) ds \right)^2$.

Proof: From equation (3.5) it follows that

$$\begin{aligned} \text{Cov}(\sigma_{d,t+h}^2, \sigma_{d,t}^2) &= e^{2\mu} (e^{\Psi_{d,\infty}^h(1)} e^{\Psi_d^h(1)} - e^{2\Psi_{d,\infty}(1)}) \\ &= e^{2\mu} e^{2\Psi_{d,\infty}(1)} (e^{\Psi_{d,\infty}^h(1) + \Psi_d^h(1) - 2\Psi_{d,\infty}(1)} - 1) \\ &= e^{2\mu} e^{2\Psi_{d,\infty}(1)} [\Psi_{d,\infty}^h(1) + \Psi_d^h(1) - 2\Psi_{d,\infty}(1) \\ &\quad + O((\Psi_{d,\infty}^h(1) + \Psi_d^h(1) - 2\Psi_{d,\infty}(1))^2)]. \end{aligned}$$

If we can show that $\Psi_{d,\infty}^h(1) + \Psi_d^h(1) - 2\Psi_{d,\infty}(1) \sim C_1 h^{2d-1}$, as $h \rightarrow \infty$, the result follows. Consider therefore

$$\begin{aligned} &\Psi_{d,\infty}^h(1) - \Psi_{d,\infty}(1) + \Psi_d^h(1) - \Psi_{d,\infty}(1) \\ &= \int_0^\infty \int_{\mathbb{R}} \left\{ e^{g_{d,\infty}^h(s)x} - 1 - g_{d,\infty}^h(s)x - \left[e^{g_d(s)x} - 1 - g_d(s)x \right] \right\} \nu_M(dx) ds \\ &\quad + \int_0^\infty \int_{\mathbb{R}} \left\{ e^{g_d^h(s)x} - 1 - g_d^h(s)x - \left[e^{g_d(s)x} - 1 - g_d(s)x \right] \right\} \nu_M(dx) ds \\ &= \int_0^\infty \int_{\mathbb{R}} \left\{ e^{g_{d,\infty}^h(s)x} - e^{g_d(s)x} + e^{g_d^h(s)x} - e^{g_{d,\infty}^h(s)x} \right\} \nu_M(dx) ds \\ &= \int_0^\infty \int_{\mathbb{R}} e^{g_d(s)x} \left\{ e^{g_d(s+h)x} - 1 \right\} \nu_M(dx) ds - \int_h^\infty \int_{\mathbb{R}} \left\{ 1 - e^{g_d(s)x} \right\} \nu_M(dx) ds. \end{aligned}$$

Series expansion of the exponential function yields

$$\begin{aligned}
& \Psi_{d,\infty}^h(1) - \Psi_{d,\infty}(1) + \Psi_d^h(1) - \Psi_d(1) \\
&= \int_0^\infty \int_{\mathbb{R}} \left[\sum_{k=1}^\infty \frac{(g_d(s+h)x)^k}{k!} + g_d(s)x \sum_{k=1}^\infty \frac{(g_d(s+h)x)^k}{k!} \right. \\
&\quad \left. + \sum_{m=2}^\infty \frac{(g_d(s)x)^m}{m!} \sum_{k=1}^\infty \frac{(g_d(s+h)x)^k}{k!} - \sum_{k=1}^\infty \frac{(g_d(s+h)x)^k}{k!} \right] \nu_M(dx) ds \\
&= \int_0^\infty \int_{\mathbb{R}} \left[x g_d(s) \sum_{m=1}^\infty \frac{(g_d(s+h)x)^m}{m!} \right. \\
&\quad \left. + \sum_{m=2}^\infty \frac{(g_d(s)x)^m}{m!} \sum_{k=1}^\infty \frac{(g_d(s+h)x)^k}{k!} \right] \nu_M(dx) ds \\
&= \int_0^\infty \int_{\mathbb{R}} \left[x^2 g_d(s) g_d(s+h) + g_d(s)x \sum_{m=2}^\infty \frac{(g_d(s+h)x)^m}{m!} \right. \\
&\quad \left. + g_d(s+h)x \sum_{m=2}^\infty \frac{(g_d(s)x)^m}{m!} + \sum_{m=2}^\infty \frac{(g_d(s)x)^m}{m!} \sum_{k=2}^\infty \frac{(g_d(s+h)x)^k}{k!} \right] \nu_M(dx) ds \\
&= \int_0^\infty \int_{\mathbb{R}} \left[g_d(s+h)x \sum_{m=1}^\infty \frac{(g_d(s)x)^m}{m!} \right. \\
&\quad \left. + \sum_{m=1}^\infty \frac{(g_d(s)x)^m}{m!} \sum_{k=2}^\infty \frac{(g_d(s+h)x)^k}{k!} \right] \nu_M(dx) ds.
\end{aligned}$$

Define $M_j := \int_{\mathbb{R}} x^j \nu_M(dx)$, $j \in \mathbb{N}$. Since $\int_{|x|>1} e^x \nu_{d,\infty}(dx) < \infty$ implies that $\int_{\mathbb{R}} |x|^k \nu_M(dx) < \infty$, $k \geq 2$, we get that all moments M_j , $j \geq 2$, of the Lévy measure ν_M are finite. Consider now the integral

$$I_1(h) := \int_0^\infty g_d(s+h)g_d(s) \left[M_2 + \sum_{m=2}^\infty \frac{(g_d(s))^{m-1}}{m!} M_{m+1} \right] ds.$$

We want to show that

$$I_1(h) \sim M_2 \int_0^\infty G_d(s+h)G_d(s)ds =: I_G(h), \quad \text{as } h \rightarrow \infty, \quad (3.7)$$

with $G_d(s) := C_2 s^{d-1}$, since $I_G(h) \sim C_1 h^{2d-1}$, $h \rightarrow \infty$. We show first, that

$$\begin{aligned}
& g_d(s+h+h^{d/2})g_d(s+h^{d/2}) \left[M_2 + \sum_{m=2}^\infty \frac{(g_d(s+h^{d/2}))^{m-1}}{m!} M_{m+1} \right] \\
&\quad \sim M_2 G_d(s+h+h^{d/2})G_d(s+h^{d/2}),
\end{aligned}$$

if $h \rightarrow \infty$. Consider therefore the limit

$$\begin{aligned} & \lim_{s \geq 0, h \rightarrow \infty} \frac{g_d(s+h+h^{d/2})g_d(s+h^{d/2}) \left[M_2 + \sum_{m=2}^{\infty} \frac{(g_d(s+h^{d/2}))^{m-1}}{m!} M_{m+1} \right]}{M_2 G_d(s+h+h^{d/2}) G_d(s+h^{d/2})} \\ &= 1 + \lim_{s \geq 0, h \rightarrow \infty} M_2^{-1} \sum_{m=2}^{\infty} \frac{(g_d(s+h^{d/2}))^{m-1}}{m!} M_{m+1}, \end{aligned}$$

which is equal to 1 because of (2.17),

$$\begin{aligned} & \left| g_d(s+h^{d/2})^{-1} \sum_{m=2}^{\infty} \frac{(g_d(s+h^{d/2}))^m}{m!} M_{m+1} \right| \\ & \leq M^* |g_d(s+h^{d/2})|^{-1} \left(e^{|g_d(s+h^{d/2})|} - |g_d(s+h^{d/2})| - 1 \right), \end{aligned}$$

with $M^* := \sup_{j \in \mathbb{N}} |M_j| < \infty$, and $\lim_{x \geq 0, x \rightarrow 0} x^{-1}(e^x - x - 1) = 0$. From Lemma 1.22 in Marquardt [17] it follows that

$$\tilde{I}_1(h) \sim \tilde{I}_G(h), \quad \text{for } h \rightarrow \infty, \quad (3.8)$$

where

$$\tilde{I}_1(h) := \int_{h^{d/2}}^{\infty} g_d(s+h)g_d(s) \left[M_2 + \sum_{m=2}^{\infty} \frac{(g_d(s))^{m-1}}{m!} M_{m+1} \right] ds$$

and

$$\tilde{I}_G(h) := M_2 \int_{h^{d/2}}^{\infty} G_d(s+h)G_d(s)ds.$$

Now (3.7) follows if we can show

$$\frac{|I_1(h) - I_G(h)|}{|I_G(h)|} \leq \frac{|I_1(h) - \tilde{I}_1(h)|}{|I_G(h)|} + \frac{|\tilde{I}_1(h) - \tilde{I}_G(h)|}{|I_G(h)|} + \frac{|\tilde{I}_G(h) - I_G(h)|}{|I_G(h)|} \rightarrow 0$$

for $h \rightarrow \infty$. This can be done in a similar way as in the proof of Lemma 1.23 in Marquardt [17]. In particular, since $|I_G(h)| \geq |\tilde{I}_G(h)|$ it follows from (3.8) that

$$\frac{|\tilde{I}_1(h) - \tilde{I}_G(h)|}{|I_G(h)|} \leq \frac{|\tilde{I}_1(h) - \tilde{I}_G(h)|}{|\tilde{I}_G(h)|} \rightarrow 0.$$

For $d < 0.5$ we get $|I_G(h)| \geq |C_2|^2 \frac{h^{2d-1}}{1-2d}$ and for all $h \geq K$, K large enough, we have $|g_d(s+h)| \leq 2|C_2|h^{d-1}$. There exists also a constant $C_g > 0$ with $\sup_{s \geq 0} |g_d(s)| \leq C_g$. This yields for $h \geq K$

$$\begin{aligned} |I_1(h) - \tilde{I}_1(h)| &= \left| \int_0^{h^{d/2}} g_d(s+h)g_d(s) \left[M_2 + \sum_{m=2}^{\infty} \frac{(g_d(s))^{m-1}}{m!} M_{m+1} \right] ds \right| \\ &\leq \int_0^{h^{d/2}} 2|C_2|h^{d-1}C_g [M_2 + M^*C_g^{-1}(e^{C_g} - C_g - 1)] ds \\ &\leq 2|C_2|C_g [M_2 + M^*C_g^{-1}(e^{C_g} - C_g - 1)] h^{2d-1-d/2} \end{aligned}$$

and hence

$$\frac{|I_1(h) - \tilde{I}_1(h)|}{|I_G(h)|} \leq \frac{2C_g [M_2 + M^*C_g^{-1}(e^{C_g} - C_g - 1)] h^{2d-1-d/2}}{|C_2|^{\frac{h^{2d-1}}{1-2d}}} \rightarrow 0 \quad \text{for } h \rightarrow \infty.$$

Similarly we get

$$\frac{|\tilde{I}_G(h) - I_G(h)|}{|I_G(h)|} \leq \frac{C_g h^{2d-1-d/2}}{|C_2|^{\frac{h^{2d-1}}{1-2d}}} \quad \text{for } h \rightarrow \infty,$$

from which the result follows. Analogously we get with

$$\int_0^\infty g_d^k(s+h)g_d(s)ds \sim C_3 h^{(k+1)d-k}, \quad k \geq 2,$$

that

$$I_k(h) := \int_0^\infty \frac{1}{k!} g_d^k(s+h)g_d(s) \left[M_{k+1} + \sum_{m=2}^\infty \frac{(g_d(s))^{m-1}}{m!} M_{m+k} \right] ds = o(h^{2d-1}).$$

and hence it follows that

$$\Psi_{d,\infty}^h(1) + \Psi_d^h(1) - 2\Psi_{d,\infty}(1) \sim C_1 h^{2d-1}, \quad \text{for } h \rightarrow \infty,$$

which proves the assertion. \square

Example 3.3. In this example we consider a fractionally integrated ECOGARCH(1, 0.4, 1) process driven by a Lévy process L with Lévy symbol

$$\psi_L(u) = -\frac{u^2}{2} + \int_{\mathbb{R}} (e^{iux} - 1) \lambda \Phi_{0,1/\lambda}(dx),$$

where $\Phi_{0,1/\lambda}(\cdot)$ is the distribution function of a normal distribution with mean 0 and variance $1/\lambda$. This means that L is the sum of a standard Brownian motion W and the compound Poisson process

$$J_t = \sum_{k=1}^{N_t} Z_k, \quad J_{-t} = \sum_{k=1}^{-N_{-t}} Z_{-k}, \quad t \geq 0,$$

where $(N_t)_{t \in \mathbb{R}}$ is an independent Poisson process with intensity $\lambda > 0$ and jump times $(T_k)_{k \in \mathbb{Z}}$. The Poisson process N is also independent from the i.i.d. sequence of jump sizes $(Z_k)_{k \in \mathbb{Z}}$, with $Z_1 \sim N(0, 1/\lambda)$. The Lévy process M is in this case given by the following expression

$$M_t = \sum_{k=1}^{N_t} [\theta Z_k + \gamma |Z_k|] - Ct, \quad t > 0,$$

with $C = \gamma \int_{\mathbb{R}} |x| \lambda \Phi_{0,1/\lambda}(dx) = \sqrt{\frac{2\lambda}{\pi}} \gamma$. M_{-t} , $t \geq 0$ is defined analogously. The parameter θ is equal to -0.15 and γ is equal to 0.1 . The stationary log-volatility process is of the form

$$\log(\sigma_{d,t+}^2) = \mu + \int_{-\infty}^t b_1 e^{-a_1(t-s)} dM_{d,s}, \quad t > 0,$$

where $\mu = -5$, $a_1 = 0.5$ and $b_1 = 1$. In Figure 1 we plotted 3000 observations of the sample path of the simulated log-volatility $\log(\sigma_d^2)$ and volatility σ_d^2 process observed at 10000 equidistant time points in the first row. The long memory parameter d in this example is 0.4 . Hence we will expect a slow decay of the autocorrelation function (acf). This is indeed the case. The empirical autocorrelation function of the volatility σ_d^2 and log-volatility $\log(\sigma_d^2)$ process are shown in the bottom left panel of Figure 1 as a dashed and solid line, respectively. One observes that the empirical acf of both series show similar asymptotic behaviour as indicated from Theorem 3.2. In the bottom right panel the corresponding return process

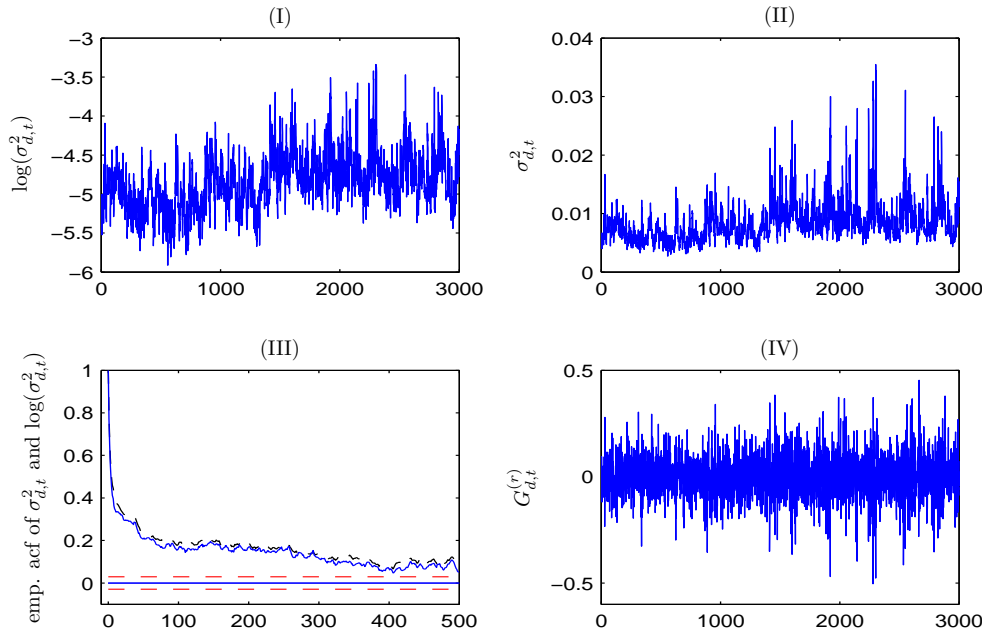


FIGURE 1: (I) The log-volatility process $\log(\sigma_d^2)$ and (II) volatility process σ_d^2 of a FIECOGARCH(1,0.4,1) with parameters $a_1 = 0.5, b_1 = 1, \mu = -5, \theta = -0.15, \gamma = 0.1$ and $d = 0.4$. (III) The empirical autocorrelation function of σ_d^2 (solid line) and $\log(\sigma_d^2)$ (dashed line). (IV) 3000 observations of the return process $G_d^{(1)}$. The jumps of the compound Poisson process are $N(0, 1/2)$ distributed.

Remark 3.4. In the last Theorem we have shown, that the autocovariance function of the volatility process decays at a hyperbolic rate. For the discrete time EGARCH process this was shown by Surgailis and Viano [25].

In the continuous time setting Comte and Renault [11] showed this effect for a long memory stochastic volatility model, where the log-volatility process was modeled as an OU process driven by a fractional Brownian motion. Hence our result can also be applied to a continuous-time stochastic volatility model, where the log-price process $Y = (Y_t)_{t \geq 0}$ satisfies

$$dY_t = \sigma_t dW_t, \quad t \geq 0, \quad (3.9)$$

with a Brownian motion W , and the log-volatility process $\log(\sigma^2)$ is described by a FICARMA(p, d, q), $p \geq q$, process, where the Lévy measure of the driving noise process has finite moments of all orders $k \geq 2$.

4. SECOND ORDER PROPERTIES OF THE RETURN PROCESS

Second order properties are now derived under the assumption that the log-volatility process is strictly stationary. The structure of the price process G_d is the same as that of an ECGARCH(p, q) process. Therefore the result concerning the first and second moment, as well as the autocovariance function, of the return process is analogously to the result in Proposition 5.1 in Haug and Czado [14].

Proposition 4.1. Let L be a Lévy process with $\mathbb{E}(L_1) = 0$ and $\mathbb{E}(L_1^2) < \infty$. Assume that $\log(\sigma_d^2)$ is strictly stationary with marginal distribution F_d , where F_d is infinitely divisible with characteristic triplet $(\gamma_{d,\infty}, 0, \nu_{d,\infty})$ and $1 \in K_{d,\infty}$. Then $\mathbb{E}(G_{d,t}^2) < \infty$ for all $t \geq 0$, and for every $t, h \geq r > 0$ it holds

$$\mathbb{E}G_{d,t}^{(r)} = 0 \quad (4.1)$$

$$\mathbb{E}(G_{d,t}^{(r)})^2 = e^{\mu + \Psi_{d,\infty}(1)r} \mathbb{E}(L_1^2) \quad (4.2)$$

$$\text{Cov}(G_{d,t}^{(r)}, G_{d,t+h}^{(r)}) = 0. \quad (4.3)$$

If further $\mathbb{E}(L_1^4) < \infty$ and the volatility process has finite second moment, then $\mathbb{E}(G_{d,t}^4) < \infty$ for all $t \geq 0$ and for every $t, h \geq r > 0$ it holds

$$\text{Cov}((G_{d,t}^{(r)})^2, (G_{d,t+h}^{(r)})^2) = \mathbb{E}(L_1^2) \int_h^{h+r} \text{Cov}(G_{d,r}^2, \sigma_s^2) ds. \quad (4.4)$$

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