A Percolating Hard Sphere Model

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Abstract

Given a homogeneous Poisson point process in $\mathbb{R}^d$, Häggström and Meester [2] asked whether it is possible to place spheres (of differing radii) centred at the points, in a translation-invariant way, so that the spheres do not overlap but there is an unbounded component of touching spheres. We prove that the answer is yes in sufficiently high dimension.

1 Introduction

A **sphere process** is a simple point process Λ on $\mathbb{R}^d \times [0, \infty)$. The support of Λ is the random set $[\Lambda] = \{(x, r) \in \mathbb{R}^d \times [0, \infty) : \Lambda(\{(x, r)\}) = 1\}$. If $(x, r) \in [\Lambda]$, we say that there is a sphere of radius $r \in [0, \infty)$ at $x \in \mathbb{R}^d$. The **centre process** $\widetilde{\Lambda}$ is the point process on $\mathbb{R}^d$ given by the projection $\widetilde{\Lambda}(\cdot) = \Lambda(\cdot \times [0, \infty))$. We say that Λ is a **Poisson sphere process** if $\widetilde{\Lambda}$ is a homogeneous Poisson process.

A **hard sphere process** is one in which the interiors of the spheres do not overlap; that is, almost surely

$$\rho(x, y) \geq r + s \text{ for any distinct } (x, r), (y, s) \in [\Lambda],$$

where ρ denotes Euclidean distance. (Note in particular that, since all the radii are non-negative, in a hard sphere process Λ no sphere may contain any points of $\widetilde{\Lambda}$ other than its own centre.) For $z \in \mathbb{R}^d$ the shifted sphere process $\Lambda + z$ is defined by $(\Lambda + z)(A) = \Lambda(A - z)$, where $A - z := \{(x - z, r) : (x, r) \in A\}$. The sphere process Λ is (translation-)**invariant** if $\Lambda + z$ and Λ are equal in law for all $z \in \mathbb{R}^d$.

Let $G(\Lambda) \subseteq \mathbb{R}^d$ be the random set covered by all the spheres:

$$G(\Lambda) = \big\{ y \in \mathbb{R}^d : \rho(y, x) \leq r \text{ for some } (x, r) \in [\Lambda] \big\}.$$

The connected components of $G(\Lambda)$ are called **clusters**. We say that Λ **percolates** if there is an unbounded cluster. Our main result is the following.

Theorem 1 *For all $d \geq 45$, there exists an invariant Poisson hard sphere process Λ which percolates almost surely.*

Häggström and Meester [2] studied several invariant continuum percolation processes, and proved that the "dynamic lily-pond model" does not percolate. This is a Poisson hard sphere process in which spheres grow from all Poisson points at the same rate, and whenever two spheres touch, they both stop growing. Häggström and Meester asked whether there exists an invariant Poisson hard sphere process which percolates.

In dimension $d = 1$ it is easy to see that any Poisson hard sphere process almost surely does not percolate. For consider 4 consecutive points $x_1 < x_2 < x_3 < x_4$ of the Poisson process which satisfy $x_3 - x_2 > (x_2 - x_1) + (x_4 - x_3)$; then the spheres centred at x_2 and x_3 cannot touch. Since such a configuration of 4 points appears infinitely often in a Poisson process, no percolation is possible. It is unknown whether there exists a percolating Poisson hard sphere process in dimensions $2 \leq d \leq 44$ (even without the requirement of invariance).

Our explicit bound of 45 in Theorem 1 could probably be reduced with some effort. Our construction certainly cannot be adapted to work in dimension less than 3 (and probably not in dimension that low).

Jonasson [3] showed that in the hyperbolic plane H^2 there exists a hard sphere process that percolates when the Poisson process has sufficiently low intensity, but was unable to determine what happens for high intensity. (In $\mathbb{R}^d$, scaling shows that the intensity of the process is immaterial).

Note that our definition of a hard sphere process allows spheres of radius zero. Our proof of Theorem 1 can in fact be adapted to prove the existence

of an invariant percolating Poisson hard sphere process in which the spheres all have positive radii. We explain this in a remark at the end of Section 4.

Our proof of Theorem 1 is in two parts. First we construct a non-invariant hard sphere process Γ which percolates. Then we convert this to an invariant process Λ by "stationarizing" - applying a uniform random translation in a large ball, and taking a limit. The non-invariant construction of Γ proceeds as follows. Starting from a Poisson process, we attempt to grow an unbounded cluster iteratively. At each step we try to choose a radius for a new sphere (centred at some Poisson point) so that it touches the cluster constructed so far. This will be possible only if the proposed new sphere contains no other Poisson points, and we need to ensure this happens sufficiently often that the cluster can continue to grow. We do this by comparison with a certain two-dimensional percolation process; in sufficiently high dimension we can arrange that the probability of success at each step exceeds the relevant critical probability.

To ensure that the stationarized version Λ also percolates, the unbounded clusters of Γ should occupy a positive fraction of space – see below for a precise statement. We will achieve this by repeating the essentially two-dimensional construction described above in infinitely many "layers" throughout $\mathbb{R}^d$.

Let $\mathcal{L}$ denote Lebesgue measure on $\mathbb{R}^d$, and denote the Euclidean ball $B(x,r) = B_d(x,r) := \{y \in \mathbb{R}^d : \rho(x,y) < r\}$, and the origin $0 = (0, \ldots, 0)$. Define the **lower density** of a set $A \subseteq \mathbb{R}^d$ to be

$$\mathcal{D}(A) = \liminf_{r \to \infty} \frac{\mathcal{L}\big(A \cap B(0,r)\big)}{\mathcal{L}B(0,r)}.$$

For $s > 0$ define the **s-neighbourhood** of a set $A \subseteq \mathbb{R}^d$ to be

$$A^{\{s\}} = \bigcup_{x \in A} B(x,s).$$

Using the construction sketched above, we shall prove the following result, from which Theorem 1 will be deduced.

Theorem 2 *For $d \geq 45$ there exists a (not necessarily invariant) Poisson hard sphere process Γ which percolates almost surely. Furthermore, Γ can be chosen so as to have the following additional properties.*

(i) The union I of all unbounded clusters satisfies $\lim_{a \to \infty} \mathbf{E}\mathcal{D}(I^{\{a\}}) = 1$.
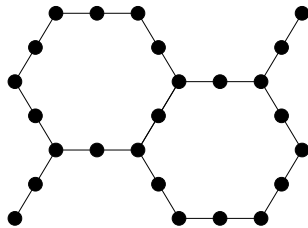
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Figure 1: Part of the lattice $\mathbb{H}^*$

(ii) There exists a constant $K(d) < \infty$ such that $\Gamma(\mathbb{R}^d \times (K, \infty)) = 0$ almost surely; that is, there are no spheres of radius greater than K.

The article is organized as follows. In Section 2 we give the construction that gives rise to the non-invariant process in Theorem 2. In Section 3 we prove that the resulting process does indeed have the properties stated in Theorem 2. In Section 4 we deduce Theorem 1.

2   Construction

In this section we describe the construction of the hard sphere process in Theorem 2. We will prove that it has the required properties in the next section. The construction is given in terms of parameters $\mu = 0.75$, $\delta = 0.1$, $\epsilon = 0.01$ and C, where $C = C(d)$ is a (large) constant to be chosen later; the values for these parameters are chosen so that the construction yields an unbounded cluster. Let $\lambda = \lambda(d) > 0$ be another constant to be chosen later, and let Π be a homogeneous Poisson process of intensity λ, and denote its support $[\Pi]$. We will construct a hard sphere process Γ such that $\widetilde{\Gamma} = \Pi$.

Let $\mathbb{H}$ be the hexagonal lattice in $\mathbb{R}^2$, with edge-length 2. (Thus, the faces are regular hexagons of side 2, and there is a vertex at the origin $0 = (0,0)$, say.) Let $\mathbb{H}^*$ be the graph formed from $\mathbb{H}$ by adding an extra vertex in the middle of each edge; see Figure 1. We will call the vertices of the original lattice $\mathbb{H}$ **site vertices** and the extra vertices of $\mathbb{H}^*$ **bond vertices**. Also fix some arbitrary well-ordering of the vertex set $V(\mathbb{H}^*)$ of $\mathbb{H}^*$.

Fix the dimension $d \geq 3$. For any vertex $v \in V(\mathbb{H}^*)$, define the **cell** centred at v to be the set

$$W(v) := B_2(v, \epsilon) \times B_{d-2}(0, C) \subset \mathbb{R}^d.$$
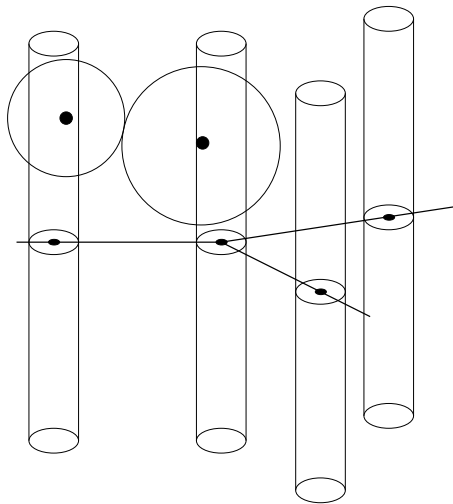
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Figure 2: An example of the construction in dimension $d = 3$. Shown are the cells centred at the origin 0 and at the three neighbouring bond-vertices u, v, w, together with two touching balls $B(x_0, r_0)$ and $B(x_u, r_u)$.

See Figure 2.

First, here is a brief description of the construction. We will attempt to place spheres with their centres in distinct cells in such a way that spheres at adjacent cells touch; see Figure 2. All the spheres will have radii in $[\mu - \delta, \mu + \delta]$. Note that any two non-adjacent vertices of $\mathbb{H}^*$ are at distance at least $\sqrt{3}$. Since $2(\mu + \delta + \epsilon) = 1.72 < \sqrt{3}$, it will be impossible for spheres centred in non-adjacent cells to touch or overlap.

We will explore the lattice iteratively starting from the origin, attempting to construct an unbounded cluster. Each step of this exploration will have two parts. Firstly, for some vertex $w \in V(\mathbb{H}^*)$ we try to find a point $y_w \in [\Pi] \cap W(w)$ such that it is possible to place a sphere of some radius $r_w \in [\mu - \delta, \mu + \delta]$ at y_w that touches one of the existing spheres and does not overlap any of the existing spheres. Secondly, we check to see whether or not there are any other points of the Poisson process within $B(y_w, r_w)$. If there is none, our construction succeeds and we let $x_w = y_w$. If either part fails, we let $x_w = \Delta$.

At each step of the construction, each vertex $v \in V(\mathbb{H}^*)$ will be either **good** (meaning we succeeded in constructing a sphere $B(x_v, r_v)$ at v), **bad**

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(meaning the construction failed), or **unexplored**.

Using $\mathbb{H}^*$ (instead of $\mathbb{H}$, say) will enable us to explore in such a way that every time we explore a new vertex it is adjacent to exactly one good vertex and no bad vertices. This will simplify our arguments. Other choices of lattice are possible, such as the square lattice in place of $\mathbb{H}$ – see the remark at the end of Section 3.

Here is a formal description of the construction. As remarked earlier, we will try to construct a cluster in each of a set of two-dimensional "layers".

First Layer

Start with all vertices in $V(\mathbb{H}^*)$ unexplored. We perform a sequence of steps $0, 1, 2, \ldots$. One new vertex (or sometimes two) will be explored at each step.

Step 0

Let $w_0 = 0$; we will start our exploration at the site vertex w_0. Temporarily denote its cell $W = W(w_0)$. Consider two cases:

Case 1: If $W \cap [\Pi] = \emptyset$, then let $x_0 = y_0 = \Delta$.

Case 2: If $W \cap [\Pi] \neq \emptyset$, then pick y_0 uniformly at random from the set $W \cap [\Pi]$ (conditional on $W \cap [\Pi]$). Take $r_0 = \mu$. Now take

$$x_0 = \begin{cases} y_0 & \text{if} \quad B(y_0, r_0) \cap [\Pi] = \{y_0\}; \\ \Delta & \text{if} \quad B(y_0, r_0) \cap [\Pi] \supsetneq \{y_0\}. \end{cases}$$

Declare $w_0 = 0$ to be bad if $x_0 = \Delta$, and good otherwise.

Step n (n ≥ 1)

Step n consists of parts (a)–(c).

(a) We first choose an unexplored vertex w_n to explore, according to the following rules.

    (i) If there is an unexplored site vertex w adjacent to some good bond vertex and two unexplored bond vertices, choose w_n to be the first such w in the ordering on $V(\mathbb{H}^*)$.

(ii) If there is no w as in (i), but there is an unexplored bond vertex w adjacent to a good site vertex and an unexplored site vertex, choose w_n to be the first such w.

(iii) If neither (i) nor (ii) hold then **stop**, and proceed to 'Subsequent Layers' below.

(b) Temporarily write $w = w_n$ for the vertex chosen in (a). The rules in (a) ensure that w is unexplored, and it has exactly one good neighbour, v say, while all the other neighbours of w are unexplored. We will try to construct $x_w \in W(w)$, the centre of a sphere tangent to $B(x_v, r_v)$.

Define
$$U(v) = B(x_v, r_v + \mu + \delta) \setminus B(x_v, r_v + \mu - \delta);$$

this is the set of possible centres for a sphere of radius $\in [\mu - \delta, \mu + \delta]$ that touches $B(x_v, r_v)$. Let

$$S = W(w) \cap U(v).$$

Now consider two cases:

Case 1: If $S \cap [\Pi] = \emptyset$, then let $x_w = y_w = \Delta$.

Case 2: If $S \cap [\Pi] \neq \emptyset$, then pick y_w uniformly at random from $S \cap [\Pi]$ (conditional on $S \cap [\Pi]$), and let

$$r_w = \rho(x_v, y_w) - r_v$$

(by the definition of $V(v)$ we have $r_w \in [\mu - \delta, \mu + \delta]$). Now take
$$x_w = \begin{cases} y_w & \text{if } B(y_w, r_w) \cap [\Pi] = \{y_w\}; \\ \Delta & \text{if } B(y_w, r_w) \cap [\Pi] \supsetneq \{y_w\}. \end{cases}$$

(c) Declare the vertex w to be bad if $x_w = \Delta$, and good otherwise.

In addition, if w is a bond vertex and is declared bad, then declare its remaining unexplored site vertex neighbour to be bad also.

Continue either for an infinite sequence of steps, or until we **stop** in (a)(iii) above.

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Subsequent Layers

All the points x_w constructed above (for good vertices w) lie in the "layer" $\mathbb{R}^2 \times B_{d-2}(0, C)$, and the radii r_w are at most $\mu + \delta$. We want to repeat the construction in other layers, ensuring that the spheres in different layers do not overlap. Therefore let $L = 2(C + \mu + \delta + 1)$, and consider the lattice of points $L\mathbb{Z}^{d-2} \subset \mathbb{R}^{d-2}$. For each $z \in L\mathbb{Z}^{d-2}$, repeat the "first layer" construction above, but now in the layer $\mathbb{R}^2 \times B_{d-2}(z, C)$ (using the Poisson points in cells of the form $B_2(v, \epsilon) \times B_{d-2}(z, C)$). This results in independent identically distributed clusters in each of the layers.

Definition of Γ

Finally, construct the hard sphere process Γ by placing a sphere centred at x_w with radius r_w, for each good vertex w, in every layer. Also place a sphere of radius zero centred at each remaining Poisson point $z \in [\Pi]$.

3 Proof of Construction

In this section we prove Theorem 2, using the construction of Section 2. We start by assembling some tools.

Percolation

Let $\mathbb{H}$ be the hexagonal lattice in $\mathbb{R}^2$ with side-length 2. Consider Bernoulli site percolation with parameter p on $\mathbb{H}$. That is, each vertex is open with probability p and closed otherwise, independently for different vertices. There exists a critical probability $p_c = p_c^{\text{site}}(\mathbb{H}) < 1$, with the property that if $p > p_c$ then there is a.s. a unique infinite connected cluster of open vertices (see [1]). It is proved in [7] that

$$p_c < 0.794.$$

Now suppose $p > p_c$ and let $K \subset \mathbb{R}^2$ be the vertex set of the infinite open cluster if contains 0, and let $K = \emptyset$ otherwise. Suppose $d \geq 3$ and let $(K_z)_{z \in \mathbb{Z}^{d-2}}$ be a family of i.i.d. random sets each with the same law as K. Fix any $L > 0$, and define the random set

$$Y = \bigcup_{z \in \mathbb{Z}^{d-2}} K_z \times \{Lz\} \subset \mathbb{R}^d.$$

Lemma 3 *For $p > p_c^{\mathrm{site}}(\mathbb{H})$ and any $L > 0$, the set Y defined above satisfies $\mathbf{E}\mathcal{D}(Y^{\{a\}}) \to 1$ as $a \to \infty$.*

PROOF. Let I be the infinite open cluster, and let $\theta = \mathbf{P}(0 \in I) > 0$. Define a random set $\widetilde{K}$ as follows. Flip a θ-coin independently of I, and let $\widetilde{K} = I$ with probability θ, and $\widetilde{K} = \emptyset$ with probability $1 - \theta$. By the Harris-FKG inequality (see for example [1]) we see that K stochastically dominates $\widetilde{K}$. Now let $(\widetilde{K}_z)_{z \in \mathbb{Z}^{d-2}}$ be a family of i.i.d. random sets with the same law as $\widetilde{K}$, and let $\widetilde{Y} = \bigcup_{z \in \mathbb{Z}^{d-2}} \widetilde{K}_z \times \{Lz\} \subset \mathbb{R}^d$. Then Y dominates $\widetilde{Y}$, so $\mathcal{D}(Y^{\{a\}})$ dominates $\mathcal{D}(\widetilde{Y}^{\{a\}})$. The random set $\widetilde{Y}^{\{a\}}$ is invariant in law under isometries of $\mathbb{H} \times \mathbb{Z}^{d-2}$, so Fubini's theorem implies

$$\mathbf{E}\mathcal{D}(\widetilde{Y}^{\{a\}}) \geq \inf_{x \in \mathbb{R}^d} \mathbf{P}(x \in \widetilde{Y}^{\{a\}}) = \inf_{x \in \mathbb{R}^d} \mathbf{P}(B(x,a) \cap \widetilde{Y} \neq \emptyset) \to \mathbf{P}(\widetilde{Y} \neq \emptyset) = 1$$

as $a \to \infty$. $\qquad\square$

Random points

Let Π be a point process in $\mathbb{R}^d$. For $r > 0$ we call a point $x \in [\Pi]$ **r-isolated** if there is no other point of $[\Pi]$ within distance r of x.

Lemma 4 *Let Π be a homogeneous Poisson point process of intensity λ in $\mathbb{R}^d$. Let $S \subseteq \mathbb{R}^d$ be a Borel set with $\mathcal{L}S \in (0, \infty)$. Let X be a point chosen uniformly at random from the set $[\Pi] \cap S$ (provided it is non-empty), conditional on Π. Then*

$$\mathbf{P}(X \text{ exists and is } r\text{-isolated}) \geq e^{-\lambda \mathcal{L} B(0,r)} - e^{-\lambda \mathcal{L} S}.$$

Lemma 5 *Under the assumptions of Lemma 4, let $Z \subseteq \mathbb{R}^d$ be disjoint from S. Then*

$$\mathbf{P}(X \text{ exists and is } r\text{-isolated} \mid \Pi(Z) = 0) \geq \mathbf{P}(X \text{ exists and is } r\text{-isolated}).$$

PROOF OF LEMMA 4. For convenience, let $X = \Delta$ if $\Pi(S) = 0$, and take Δ to be not r-isolated. Denote the random sets $B = B(X, r)$, $U = B \cap S$, and $V = B \setminus S$. Then we have

$$
\begin{aligned}
&\mathbf{P}(X \text{ is } r\text{-isolated} \mid \Pi(S), X) \\
&= \mathbf{P}(\Pi(U) = 1, \Pi(V) = 0 \mid \Pi(S), X)\mathbf{1}[\Pi(S) > 0] \\
&= \left(1 - \frac{\mathcal{L}U}{\mathcal{L}S}\right)^{\Pi(S)-1} e^{-\lambda \mathcal{L} V}\mathbf{1}[\Pi(S) > 0].
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathbf{P}(X \text{ is } r\text{-isolated}) &= \mathbf{E}\mathbf{P}(X \text{ is } r\text{-isolated} \mid \Pi(S), X) \\
&= \mathbf{E}\sum_{k=1}^{\infty}\left(1 - \frac{\mathcal{L}U}{\mathcal{L}S}\right)^{k-1} e^{-\lambda\mathcal{L}V} e^{-\lambda\mathcal{L}S} \frac{(\lambda\mathcal{L}S)^k}{k!} \\
&= \mathbf{E}\left[\left(1 - \frac{\mathcal{L}U}{\mathcal{L}S}\right)^{-1} e^{-\lambda\mathcal{L}V - \lambda\mathcal{L}S}\left(e^{\lambda\mathcal{L}S - \lambda\mathcal{L}U} - 1\right)\right] \\
&= \mathbf{E}\left[\left(1 - \frac{\mathcal{L}U}{\mathcal{L}S}\right)^{-1}\left(e^{-\lambda\mathcal{L}B} - e^{-\lambda\mathcal{L}V - \lambda\mathcal{L}S}\right)\right] \\
&\geq e^{-\lambda\mathcal{L}B} - e^{-\lambda\mathcal{L}S}.
\end{aligned}
$$

$\square$

Let $\Pi|_S$ denote Π restricted to S; that is the point process with support $[\Pi] \cap S$.

PROOF OF LEMMA 5.   We have

$$
\begin{aligned}
\mathbf{P}&\big(X \text{ is } r\text{-isolated}, \Pi(Z) = 0 \mid \Pi|_S, X\big) \\
&= \mathbf{1}\big[\Pi(B(X,R) \cap S) = 1\big]\mathbf{P}\big(\Pi(B(X,R) \setminus S) = 0, \Pi(Z) = 0 \mid X\big) \\
&\geq \mathbf{1}\big[\Pi(B(X,R) \cap S) = 1\big]\mathbf{P}\big(\Pi(B(X,R) \setminus S) = 0 \mid X\big)\mathbf{P}\big(\Pi(Z) = 0\big) \\
&= \mathbf{P}\big(X \text{ is } r\text{-isolated} \mid \Pi|_S, X\big)\mathbf{P}\big(\Pi(Z) = 0\big).
\end{aligned}
$$

Taking expectations yields the result. $\square$

Volume bound

We write $\omega_d := \mathcal{L}B_d(0,1)$ for the volume of the unit ball. Fix $d \geq 3$ and $\epsilon, C > 0$. Suppose $w, w' \in \mathbb{R}^2$ are such that $\rho(w, w') = 1$. Define the sets

$$
W = B_2(w, \epsilon) \times B_{d-2}(0, C) \quad \text{and} \quad W' = B_2(w', \epsilon) \times B_{d-2}(0, C).
$$

Lemma 6 *Fix* $\mu = 0.75$, $\delta = 0.1$ *and* $\epsilon = 0.01$. *Let* $d \geq 11$. *Let* W, W' *be as above. There exists* $C' = C'(d)$ *such that if* $C \geq C'$, *then for all* $x \in W$ *and* $r \in [\mu - \delta, \mu + \delta]$, *writing*

$$S = W' \cap \big(B(x, r + \mu + \delta) \setminus B(x, r + \mu - \delta)\big),$$

we have

$$\mathcal{L}S \geq \frac{\omega_2 \omega_{d-2} \epsilon^2}{3} \left(1.2^{\frac{d-2}{2}} - 1\right).$$

In order to prove Lemma 6, we use some further geometric results.

Lemma 7 *Fix* d *and* $0 < R_- \leq R_+ < \infty$. *There exists* $C' = C'(d, R_-, R_+)$ *such that if* $C \geq C'$, *for all* $x \in B(0, C)$ *and* $R \in (R_-, R_+)$ *we have*

$$\mathcal{L}\big(B(0, C) \cap B(x, R)\big) \geq \frac{1}{3}\mathcal{L}\big(B(x, R)\big).$$

PROOF. First suppose that $\rho(0, x) = C$ (so that x is on the surface of the ball $B(0, C)$). Let

$$f(C, R) = \frac{\mathcal{L}(B(0, C) \cap B(x, R))}{\mathcal{L}(B(x, R))},$$

and note that f depends only on the ratio R/C, while for fixed R, the function f is increasing and continuous in C, and converges to $1/2$ as $C \to \infty$ (since near x, the ball $B(0, C)$ approaches a half space). Therefore by the intermediate value theorem the claimed result holds for the case $\rho(0, x) = C$.

Suppose now that $\rho(0, x) < C$. The result is trivial when $B(x, R) \subseteq B(0, C)$. If not we can replace $B(0, C)$ with $B(0, \widetilde{C})$, where $\rho(0, x) = \widetilde{C}$ (and so $\widetilde{C} \in [C - R_+, C))$, and appeal to the case already proved. $\qquad \square$

Lemma 8 *Let* μ, δ, ϵ *and* W, W' *be as in Lemma 6. There exists* $C'(d) < \infty$ *such that if* $C \geq C'$, *then for all* $x \in W$ *and* $R \in [2(\mu - \delta), 2(\mu + \delta)]$ *we have*

$$\frac{1}{3}\omega_2 \omega_{d-2} \epsilon^2 R_1^{d-2} \leq \mathcal{L}\big(W' \cap B(x, R)\big) \leq \omega_2 \omega_{d-2} \epsilon^2 R_2^{d-2},$$

where $R_1 = \sqrt{R^2 - (1 + 2\epsilon)^2}$ *and* $R_2 = \sqrt{R^2 - (1 - 2\epsilon)^2}$.

PROOF.   For $z = (z_1, z_2, \ldots z_d) \in \mathbb{R}^d$ we write $\underline{z} = (z_1, z_2)$ and $\overline{z} = (z_3, z_4, \ldots z_d)$, and we write $\underline{\rho}$ and $\overline{\rho}$ for Euclidean distances on $\mathbb{R}^2$ and $\mathbb{R}^{d-2}$ respectively.

We claim that for any $x \in W$,

$$B_2(w', \epsilon) \times \left[B_{d-2}(0, C) \cap B_{d-2}(\overline{x}, R_1) \right]$$
$$\subseteq W' \cap B(x, R) \subseteq$$
$$B_2(w', \epsilon) \times \left[B_{d-2}(0, C) \cap B_{d-2}(\overline{x}, R_2) \right].$$

To prove this, first note that for $\underline{x} \in B_2(w, \epsilon)$ and $\underline{z} \in B_2(w', \epsilon)$, we have

$$\underline{\rho}(w, w') - \underline{\rho}(\underline{z}, w') - \underline{\rho}(\underline{x}, w) \leq \underline{\rho}(\underline{z}, \underline{x}) \leq \underline{\rho}(\underline{z}, w') + \underline{\rho}(w, w') + \underline{\rho}(w', \underline{x}),$$

which gives

$$1 - 2\epsilon \leq \underline{\rho}(\underline{z}, \underline{x}) \leq 1 + 2\epsilon.$$

Take any $z \in W' \cap B(x, R)$. Then $\underline{z} \in B_2(w', \epsilon)$ and $\underline{x} \in B_2(w, \epsilon)$. Since $z \in B(x, R)$, we have

$$\overline{\rho}(\overline{z}, \overline{x}) \leq \sqrt{R^2 - \underline{\rho}(\underline{z}, \underline{x})^2} \leq \sqrt{R^2 - (1 - 2\epsilon)^2} = R_2;$$

that is $\overline{z} \in B_{d-2}(\overline{x}, R_2)$. The second inclusion of the claim follows.

On the other hand, if $z \in B_2(w', \epsilon) \times (B_{d-2}(0, C) \cap B_{d-2}(\overline{x}, R_1))$, then $\underline{\rho}(\underline{z}, \underline{x}) \leq 1 + 2\epsilon$ and $\overline{\rho}(\overline{z}, \overline{x}) \leq R_1$. This gives $\rho(z, x) \leq R$, so $z \in W' \cap B(x, R)$. Thus the first inclusion of the claim is proved.

The required statement now follows from the claim. For the lower bound we use Lemma 7, noting that

$$R_1 \in \left[\sqrt{4(\mu - \delta)^2 - (1 + 2\epsilon)^2} \, , \, \sqrt{4(\mu + \delta)^2 - (1 + 2\epsilon)^2} \right] \subset (0, \infty).$$

For the upper bound we discard the intersection with $B_{d-2}(0, C)$.   □

PROOF OF LEMMA 6.   We have

$$\mathcal{L}S = \mathcal{L}\big(W' \cap B(x, r + \mu + \delta)\big) - \mathcal{L}\big(W' \cap B(x, r + \mu - \delta)\big).$$

Therefore, choosing C' according to Lemma 8, for $C \geq C'$ we have by that lemma,

$$\mathcal{L}S \geq \omega_2 \omega_{d-2} \epsilon^2 \left[\overline{r}^{d-2}/3 - \underline{r}^{d-2} \right],$$

where $\bar{r} := \sqrt{(r+\mu+\delta)^2 - (1+2\epsilon)^2}$ and $\underline{r} := \sqrt{(r+\mu-\delta)^2 - (1-2\epsilon)^2}$. For C chosen as above, we want to find a uniform lower bound on $\mathcal{LS}$ for all possible $r \in [\mu - \delta, \mu + \delta]$. We claim that for $d \geq 11$, the function

$$g(r) := \bar{r}^{d-2}/3 - \underline{r}^{d-2}$$

is increasing on $r \in [\mu - \delta, \mu + \delta]$. Once this is proved we obtain

$$g(r) \geq g(\mu - \delta) \geq (\sqrt{1.2})^{d-2}/3 - (\sqrt{0.73})^{11-2} \geq \tfrac{1}{3}\left(1.2^{\frac{d-2}{2}} - 1\right)$$

on this interval, and the result follows.

To prove the above claim note that

$$g'(r) = \frac{d-2}{3}\left[(r+\mu+\delta)\bar{r}^{d-4} - 3(r+\mu-\delta)\underline{r}^{d-4}\right],$$

so it is enough to check that

$$\left(\frac{\bar{r}^2}{\underline{r}^2}\right)^{\frac{d-4}{2}} > 3$$

for the required values of r and d. But is is straightforward to check that the quadratic function $\bar{r}^2 - 1.4\underline{r}^2$ is positive for $r \in [\mu - \delta, \mu + \delta]$, hence $\bar{r}^2/\underline{r}^2 > 1.4$; and we have $1.4^{\frac{11-4}{2}} > 3$. $\qquad\square$

Proof of Theorem 2

PROOF OF THEOREM 2. Let $d \geq 45$, and construct the hard sphere process Γ as in Section 2, where the constant C is chosen according to Lemma 6, and the intensity λ of the Poisson process will be chosen later.

Consider the first layer of the construction, and recall that at step n, vertex w_n is explored, and a sphere of radius r_{w_n} is placed with centre $x_{w_n} = y_{w_n} \in W(w_n)$, provided the vertex is found to be good. As a notational convenience, if the algorithm stops during step N we write $w_n = \Delta$ for all $n \geq N$, and call Δ a good vertex. Let $\mathcal{F}_n$ be the σ-algebra generated by all of the random variables

$$\left(w_i, y_{w_i}, x_{w_i}, r_{w_i}, \Pi|_{W(w_i)}, \Pi|_{B(x_{w_i}, r_{w_i})}\right)_{i=0,\dots,n};$$

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that is "the information known up to and including step n". Let $\mathcal{F}_{-1}$ be the trivial σ-algebra.

We will compare the set of good vertices with a percolation cluster. Suppose that for some q we can show that for all $n \geq 0$,

$$\mathbf{P}(w_n \text{ is good} \mid \mathcal{F}_{n-1}) \geq q \quad \text{almost surely} \tag{1}$$

(that is, each newly explored vertex is good with probability at least q uniformly in the past). Then the random set of good *site* vertices stochastically dominates $C_{q^2}^{\mathbb{H}}(0)$, the open cluster at the origin for site percolation with parameter q^2 on $\mathbb{H}$. This is because, as long as it is possible to add a new site vertex to the cluster of good site vertices at the origin, the algorithm attempts to do so by first exploring the intervening bond vertex, and then immediately exploring the new site vertex. The probability that both steps succeed is at least q^2. Therefore we can compare with a cluster-growing algorithm for $C_{q^2}^{\mathbb{H}}(0)$.

Also recall that the construction gives, in each layer, a cluster of touching spheres (with radii $\in [\mu - \delta, \mu + \delta]$) including a sphere with its centre in the cell of each good vertex. Note therefore that if J is such a cluster of spheres then $J^{\{C+\epsilon\}}$ contains all good vertices. Therefore, since the layers of the construction are independent, if we can establish (1) with $q^2 > p_c^{\text{site}}(\mathbb{H})$ then the statements of the theorem will follow by Lemma 3. Thus all that remains is to prove (1) for with $q \geq 0.892 > \sqrt{0.794} \geq \sqrt{p_c^{\text{site}}(\mathbb{H})}$.

Recall that in step n of the construction we choose a random Poisson point $y = y_{w_n}$ (if any exists) in a certain $\mathcal{F}_{n-1}$-measurable set S, and check to see whether a certain ball $B(y, r_{w_n})$ is free of other Poisson points. If both steps succeed then the vertex is declared good. The radius r_{w_n} depends on which point y is chosen, but since it can be at most $\mu + \delta$ we can bound the required probability by the probability that the larger ball $B(y, \mu + \delta)$ contains no other points:

$$\mathbf{P}(w_n \text{ is good} \mid \mathcal{F}_{n-1}) \geq \mathbf{P}\big(y \neq \Delta, \text{ and } y \text{ is } (\mu + \delta)\text{-isolated} \mid \mathcal{F}_{n-1}\big).$$

The latter event depends on Π only through its restriction to $W(w_n)^{\{\mu+\delta\}}$. The conditioning on $\mathcal{F}_{n-1}$ does not affect the process $\Pi|_{W(w_n)}$, while the fact that one vertex v_n adjacent to w_n is good tells us only that a certain $\mathcal{F}_{n-1}$-measurable set Z not intersecting S contains no points of Π. (Recall that $2(\mu + \delta + \epsilon) < \sqrt{3}$, so the conditioning on non-adjacent vertices has no effect).

Thus we deduce by Lemmas 5 and 4 that

$$\mathbf{P}(w_n \text{ is good} \mid \mathcal{F}_{n-1}) \geq e^{-\lambda \mathcal{L} B(0, \mu + \delta)} - e^{-\lambda \mathcal{L} S}$$
$$\geq 1 - \lambda \mathcal{L} B(0, \mu + \delta) - e^{-\lambda \mathcal{L} S}.$$

But Lemma 6 applies to give

$$\mathcal{L} S \geq \frac{\omega_2 \omega_{d-2} \epsilon^2}{3} \left(1.2^{\frac{d-2}{2}} - 1 \right)$$

almost surely.

Thus, we require that

$$F(\lambda) := 1 - \lambda B - e^{-\lambda A} \geq 0.892,$$

where

$$A := \frac{\omega_2 \omega_{d-2} 10^{-4}}{3} \left(1.2^{\frac{d-2}{2}} - 1 \right) \quad \text{and} \quad B := \omega_d 0.85^d.$$

For each d we can choose the intensity λ so as to get the best bound. Differentiating shows that F has a maximum at

$$\lambda^* = \lambda^*(d) = \frac{1}{A} \log \frac{A}{B},$$

at which

$$F(\lambda^*) = 1 - \frac{\log (A/B) + 1}{A/B}.$$

But $\omega_2 = \pi$ and $\frac{\omega_{d-2}}{\omega_d} = \frac{d}{2\pi}$ (see for example [6]), therefore

$$\frac{A}{B} = \frac{10^{-4} d \left(1.2^{\frac{d-2}{2}} - 1 \right)}{6 \times 0.85^d}.$$

Thus A/B is an increasing function of d, and it is easy to check that for $d \geq 31$ we have $A/B > 1$, and therefore λ^* is positive (which is a requirement for an intensity). Furthermore, $F(\lambda^*)$ is an increasing function of A/B for $A/B > 1$, and therefore an increasing function of $d \geq 31$. Finally it is straightforward to check that $F(\lambda^*(d)) > 0.892$ for $d \geq 45$, as required. $\square$

Remark – choice of lattice. Our construction could be adapted to work for other two-dimensional lattices such as the square lattice in place of the hexagonal lattice, but at the expense of increasing the dimension d. The fundamental requirement is that the diagonals (that is, the distances between non-adjacent vertices of the lattice) must be greater than $2/\sqrt{3}$ times the edge length. This allows the set S (in which we try to find a possible centre for a sphere) to be made much larger in volume than the hard spheres (which must be empty of other Poisson points), while preventing overlap between non-adjacent spheres.

4   Stationarization

In this section we deduce Theorem 1 from Theorem 2.

PROOF OF THEOREM 1.   For each positive integer n, let U_n be a random variable uniformly distributed on $B(0, n)$, and independent of Γ. Define the randomly shifted process $\Gamma_n = \Gamma + U_n$. Clearly Γ_n is a percolating Poisson hard sphere process, and has no spheres larger than K. We shall use Prohorov's Theorem to construct Λ as a weak limit of the sequence (Γ_n), and show that it has all the required properties.

We claim that the sequence of random variables (Γ_n) is tight in the weak topology induced by the vague topology on point measures on $\mathbb{R}^d \times [0, \infty)$. To check this, it is enough to show that the sequence $(\Gamma_n(A))$ is tight for any relatively compact Borel $A \in \mathbb{R}^d \times [0, \infty)$. (See [4] Lemma 16.15). Any such A is a subset of $B \times [0, \infty)$ for some bounded Borel $B \subseteq \mathbb{R}^d$, and we have a.s. $\Gamma_n(A) \leq \Gamma_n(B \times [0, K])$ since Γ_n has no spheres larger than K. But the latter quantity has the same (Poisson) distribution for each n, so the sequence is clearly tight as required.

Now let Λ be any weak subsequential limit of (Γ_n), so

$$\Gamma_{n_k} \Rightarrow \Lambda \text{ as } k \to \infty \tag{2}$$

in the topology referred to above. Clearly Λ is integer-valued and thus a sphere process. Furthermore, it is easily seen that the set of hard sphere processes supported a.s. on $\mathbb{R}^d \times [0, K]$ is weak closed, and therefore Λ is a hard sphere process supported on $\mathbb{R}^d \times [0, K]$. Let $B \subseteq \mathbb{R}^d$ be a bounded Borel set with $\mathcal{L}$-null boundary. By Theorem 16.16 of [4], the convergence in (2) implies the convergence in distribution $\Gamma_{n_k}(B \times [0, K]) \to \Lambda(B \times [0, K])$.

Hence the latter has Poisson distribution with mean $\mathcal{L}B$ and so Λ is a Poisson sphere process.

Next we show that Λ is invariant. It is sufficient to show that for any $z \in \mathbb{R}^d$ and any continuous compactly-supported function $f : \mathbb{R}^d \times [0, \infty) \to [0, 1]$ we have $\mathbf{E} \int f \, d\Lambda = \mathbf{E} \int f \, d(\Lambda + z)$. Recall that $\Gamma_n = \Gamma + U_n$; we shall compare Γ_n and $\Gamma_n + z$ for large n. Fix z and f, and write $J = J(n) = B(0, n) \cap B(z, n)$. Let V be random variable uniformly distributed on J. Also write $\alpha = \alpha(n) = \mathcal{L}(B(0, n) \setminus J)/\mathcal{L}B(0, n)$, and note that $\alpha \le cn^{d-1}/n^d \to 0$ as $n \to \infty$. Recall that U_n was uniform on $B(0, n)$. Observe that conditional on $U_n \in J$ (which occurs with probability $1 - \alpha$), the law of U_n equals that of V. And conditional on $U_n + z \in J$ (which occurs with probability $1 - \alpha$), the law of $U_n + z$ equals that of V. Hence

$$\mathbf{E} \int f \, d\Gamma_n$$

$$= \mathbf{E}\left[\int f \, d(\Gamma + U_n)\middle| U_n \in J\right](1 - \alpha) + \mathbf{E}\left[\int f \, d(\Gamma + U_n)\middle| U_n \in J^c\right]\alpha$$

$$= \mathbf{E} \int f \, d(\Gamma + V)(1 - \alpha) + \mathbf{E}\left[\int f \, d(\Gamma + U_n)\middle| U_n \in J^c\right]\alpha,$$

and similarly

$$\mathbf{E} \int f \, d(\Gamma_n + z)$$

$$= \mathbf{E} \int f \, d(\Gamma + V)(1 - \alpha) + \mathbf{E}\left[\int f \, d(\Gamma + U_n + z)\middle| U_n + z \in J^c\right]\alpha.$$

Since U_n is independent of Γ, the processes $\Gamma + U_n$ and $\Gamma + U_n + z$ appearing on the right sides of the above equations are Poisson sphere processes, even when conditioned on U_n. Now for any Poisson sphere process Υ say, $\mathbf{E} \int f d\Upsilon$ is bounded by the expected number of Poisson points in the projection of the support of f onto $\mathbb{R}^d$, that is the Lebesgue measure of that projection, C say. Hence, subtracting the two equations above gives

$$\left|\mathbf{E} \int f \, d\Gamma_n - \mathbf{E} \int f \, d(\Gamma_n + z)\right| \le 0 + \alpha C \to 0 \text{ as } n \to \infty.$$

Taking weak limits as $k \to \infty$ of $\Gamma_{n_k}, \Gamma_{n_k} + z$ we deduce from (2) that

$$\left|\mathbf{E} \int f \, d\Lambda - \mathbf{E} \int f \, d(\Lambda + z)\right| = 0.$$

Hence Λ is invariant as required.

Finally we must show that Λ percolates almost surely. For a hard sphere process Υ and for $0 < a < b$ let $H_{a,b}(\Upsilon)$ be the event that $G(\Upsilon)$ has a connected set of spheres with radii at most K which intersects both $B(0,a)$ and $B(0,b)^C$. Also let $H_{a,\infty}(\Upsilon)$ be the event that $G(\Upsilon)$ has an unbounded connected set of spheres with radii at most K which intersects $B(0,a)$. Note that $H_{a,\infty}(\Upsilon)$ is the decreasing limit of the events $H_{a,b}(\Upsilon)$ as $b \to \infty$. Also denote by $I(\Upsilon)$ the union of all infinite clusters of the hard sphere process Υ. Recalling the definition of Γ_n above, we have

$$
\begin{aligned}
\mathbf{P}(H_{a,b}(\Gamma_n)) \;&\geq\; \mathbf{P}(H_{a,\infty}(\Gamma_n)) \\
&=\; \mathbf{P}(0 \in I(\Gamma)^{\{a\}}) \\
&=\; \mathbf{E}\frac{\mathcal{L}(I(\Gamma)^{\{a\}} \cap B(0,n))}{\mathcal{L}B(0,n)}.
\end{aligned}
$$

Hence by Fatou's Lemma and the definition of lower density,

$$
\liminf_{n\to\infty} \mathbf{P}(H_{a,b}(\Gamma_n)) \geq \mathbf{E}\mathcal{D}(I(\Gamma)^{\{a\}}).
$$

Note that the event $H_{a,b}(\Upsilon)$ depends only on the process Υ restricted to the compact set $B(0, b+K) \times [0, K]$. Furthermore it is straightforward to see that the event is closed in the vague topology on point measures. It follows from (2) and the Portmanteau Theorem ([4] Theorem 4.25) that

$$
\mathbf{P}(H_{a,b}(\Lambda)) \geq \limsup_{k\to\infty} \mathbf{P}(H_{a,b}(\Gamma_{n_k})).
$$

From the last two inequalities we have

$$
\mathbf{P}(H_{a,b}(\Lambda)) \geq \mathbf{E}\mathcal{D}(I(\Gamma)^{\{a\}}),
$$

and hence letting $b \to \infty$ we deduce

$$
\mathbf{P}(\Lambda \text{ percolates}) \geq \mathbf{P}(H_{a,\infty}(\Lambda)) \geq \mathbf{E}\mathcal{D}(I(\Gamma)^{\{a\}}).
$$

Finally letting $a \to \infty$, Theorem 2 gives

$$
\mathbf{P}(\Lambda \text{ percolates}) = 1.
$$

$\square$

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Remark – positive radii. As noted earlier, our proof may be adapted so that all the spheres have positive radii. To achieve this, take a small parameter $\eta > 0$, and modify the construction in Section 2 as follows: having chosen a potential centre y_w for a sphere, we declare the vertex good only if the larger ball $B(y_w, r_w + \eta)$ contains no other Poisson points (rather than the ball $B(y_w, r_w)$). If η is small enough then this does not affect the later computations, and we still obtain a non-invariant percolating hard sphere process Γ for $d \geq 45$. But now Γ has the additional property that no zero-radius sphere is within distance η of any nonzero-radius sphere. Therefore the stationarized version Λ will inherit the same property. Finally, we modify Λ as follows. If there is a zero-radius sphere centred at z, replace it with a sphere of radius r, where r is $1/2$ of the distance from z to the closest other sphere of Λ (including other zero-radius spheres). This r is always positive because the radii of the existing spheres are uniformly bounded above, and any bounded region of $\mathbb{R}^d$ contains only finitely many Poisson points.

Open Problems

(i) Does there exist a percolating Poisson hard sphere process (invariant or non-invariant) in dimensions $2 \leq d \leq 44$? The case $d = 2$ seems particularly interesting.

(ii) In any dimension, does there exist a percolating, invariant Poisson hard sphere process which is a deterministic function of the Poisson process?

(iii) Do percolating hard sphere processes exist for other point processes, such as Gaussian zeros processes [5].

Acknowledgements

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