Technische Universität München
Lehrstuhl für Numerische Mathematik / Steuerungstheorie

On the computation of fuel-optimal paths in time-dependent networks

Sebastian G. F. Kluge

Vollständiger Abdruck der von der Fakultät für Mathematik der Technischen Universität München zur Erlangung des akademischen Grades eines

Doktors der Naturwissenschaften (Dr. rer. nat.)

genehmigten Dissertation.

Vorsitzender:		Univ.-Prof. Dr. Anusch Taraz
Prüfer der Dissertation:	1.	Univ.-Prof. Dr. Martin Brokate
	2.	Prof. Dr. Konrad Reif, Duale Hochschule Baden-Württemberg Ravensburg
	3.	Univ.-Prof. Dr. Dorothea Wagner, Karlsruher Institut für Technologie (nur schriftliche Beurteilung)

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Für Katarzyna

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Abstract

The computation of shortest paths in weighted and directed networks has been subject to research for more than five decades by now, and it has never lost its relevance in up-to-date applications. Recently, there has been increasing interest in incorporating time-dependencies into the modeling of the network. This is motivated by a large field of applications, such as intelligent transportation systems, internet routing, multi-agent-systems and networked control systems.

The topic of this thesis is the computation of optimal paths in time-dependent networks. Although the time-independent optimal path problem is polynomially solvable, the time-dependent optimal path problem is NP-hard if the cost is different from the travel time, or the travel time functions do not fulfill the FIFO-property. After providing some background information on the physical modeling of travel times and travel costs in the time-dependent road network we formally introduce the mathematical model of time-dependent networks which is used in this thesis. In this model, we allow negative cost functions and we incorporate arrival time constraints as well as waiting time constraints into the problem description. Based on the theory of dynamic programming, we prove the existence of optimal paths and the lower semicontinuity of the optimal value function both for the optimal path problem in which all travel time and cost functions are precisely known and for the robust shortest path problem.

We identify necessary and sufficient conditions for the continuity of the optimal value function and discuss the cases of piecewise analytic and piecewise linear functions. In particular, under the assumption that all cost and constraint functions are piecewise analytic, we prove the following assertions: The optimal value function is directionally differentiable, the set of points in which the optimal value function is not differentiable is discrete and the optimal value function is analytic in an open neighborhood of any other point. We also carry out a complexity analysis for the case in which all travel time, cost and constraint functions are piecewise linear.

We then consider a problem setting in which the waiting time constraints are formulated in such a way that admissible paths are constrained to stay close to fastest paths (which are computable in polynomial time in FIFO-networks). Assuming that waiting is forbidden everywhere in the network, we discuss the impact of the arrival time constraints on the complexity of the time-dependent optimal path problem with fixed departure time.

We develop two algorithms, which efficiently solve the problem of computing the optimal value function and the corresponding optimal paths in time-dependent networks. The first approach is a generalization of a class of previously published solution methods (decreasing order of time methods) to heuristic search. In the second approach, assuming that the time-dependent network has the FIFO-property, we generate an admissible initial solution in polynomial time and then iteratively and monotonically approximate the optimal value

function. Since an upper bound on the error of the current iterate is maintained throughout the algorithm, this method allows a trade-off between computation time and accuracy of the found solution.
Finally, we demonstrate the application of the above theoretical results and algorithmical solutions in a case study in the time-dependent road network of Ingolstadt.

Contents

1. Introduction

1.1. Motivation

This thesis has been prepared in a cooperative project of the Technische Universität München and HARMAN International. The aim of the project was the development of a well-founded and mathematically correct framework for the computation of fuel-optimal routes in a road network. This task includes the incorporation of a physical model of vehicle dynamics, the consideration of the influence of the traffic flow, the identification of a small but meaningful set of parameters, which describe the influence of the driver behavior on the fuel consumption, a theoretical analysis of the existence and properties of solutions and the generalization of previous routing methods to the resulting problem. Recently, increasing importance has been dedicated to all means of reducing the fuel consumption produced by vehicular traffic, urged by the shortage of the reserves of fossil fuel and the global discussion on climate changes. The minimization of the fuel consumption, or, more generally speaking, the energy consumption associated with traveling from a given location to a given destination is however not only relevant for vehicles with combustion engines, but also constitutes a crucial challenge for the comprehensive launch of electric vehicles.

Since recent research in the field of intelligent transportation systems indicates that the fuel consumption associated with a given route in the road network significantly varies according to changing traffic conditions it is natural to incorporate time-dependencies into the model. The incorporation of time-dependencies is not only a challenging aspect from the modeling point of view but also leads to an optimization problem, which is much harder from a mathematical point of view than its static counterpart. Indeed, it has been shown that the time-dependent minimum-cost path problem is NP-hard if the cost is different to the travel time, or the travel time functions do not fulfill the FIFO-property (i.e., if it is possible to arrive earlier by leaving later).

In order to emphasize that we are considering the minimization with respect to a cost criterion, which may differ from the traveled distance (i.e., the shortest path problem) and the travel time (i.e., the fastest path problem), we will call the problem under consideration the time-dependent optimal path problem. Unless otherwise stated, optimality in this thesis will always be equivalent to the minimization of the cost value, i.e., a generalization of the (time-independent) shortest path problem.

With respect to the discrete-time time-dependent optimal path problem, the literature on the continuous-time time-dependent optimal path problem is sparse. This is because a discrete-time model allows the reduction of the time-dependent optimal path problem to a time-independent optimal path problem in the so-called time-expanded network in which the problem can be easily solved. However, the time-expanded network is very large (especially if a large time frame has to be considered or the original network is already large, such as the German road network for example, which contains more than 2 million road segments), which counteracts the development of efficient solution techniques. Moreover, the consideration of a continuous time variable reflects reality in more detail.

Traditionally, in navigation systems, the optimal path problem is formulated for a fixed location, a fixed departure time and a fixed destination. In time-independent optimal path problems and in some time-dependent optimal path problems it is then sufficient to consider only one point in time at each node in the optimization procedure. In this case, a fixed departure time is associated with the subproblem of computing an optimal path from some intermediate location to the destination. However, although waiting is never beneficial in travel-time weighted FIFO-networks, it can be advantageous at certain nodes in general. Moreover, if model inaccuracies or perturbations shall be incorporated into the solution strategy, the computation of optimal paths for fixed departure times is not sufficient. For this reason, most of this thesis focuses on the problem of computing optimal paths for a fixed location, a fixed destination and varying departure times.
Since there may be only a few places in the road network, such as, e.g., parking lots or gas stations, which allow the driver to wait for a certain amount of time before continuing his travel, we incorporate waiting time restrictions into the model of the time-dependent network. In order to allow a trade-off between the maximal travel time and the fuel saving associated with the choice of a route from a given location to a given destination we also incorporate arrival time constraints into the network model.
In this thesis, we will not take into account the feedback of the route chosen by an individual commodity on the state of the underlying routing network. This simplification is reasonable as long as only a small fraction of the commodities traveling in the network choose their routes according to the decision rules described below, or as long as the choice of a specific route has no impact on the state of the network. Note however that the routing of all vehicles in the road network according to the current traffic situation may lead to a new traffic situation in which the cost associated with the route of each vehicle is increased [31].
We now proceed to listing the contributions and to explaining the organization of this thesis. More background information and an overview on the related work will be provided in the respective sections.

1.2. Contributions

In the following, we provide a brief overview on the main contributions of this thesis.
We provide a general framework for the consideration of time-dependent optimal path problems, which comprises arrival time restrictions, waiting time restrictions and allows the incorporation of negative travel cost functions. We prove the existence of optimal paths in networks with certain problem data and extend these results to the robust optimal path problem. For the robust optimal path problem we also provide sufficient conditions on the set of possible network scenarios, which allow the application of dynamic programming in the time-dependent network without extending the state space. When reduced to a time-independent network, these conditions generalize the concept of interval data, which is commonly used in order to ensure the applicability of the dynamic programming approach. The treatment of the optimal path problem with certain data is carried out in Chapter 4, the robust optimal path problem is dealt with in Chapter 6.
Assuming that all travel time, cost and constraint functions in the network are piecewise analytic, we prove that the optimal value function is directionally differentiable, the set of points in which the optimal value function is not differentiable is discrete and the optimal value function is analytic in a neighborhood of any other point. This proof uses

Lojaciewicz's structure theorem for real analytic varieties and Hironaka's theorem on the resolution of singularities and generalizes previous results to the case in which waiting in the network is allowed and subject to optimization. Since in our problem setting a sequence of continuous parametric optimization problems must be solved along each edge sequence, the result implicitly yields that the optimal solution of a 1-dimensional parametric optimization problem with analytic data is analytic in an open neighborhood of almost every point. The crucial property of the optimal value function is however that for every singular point (i.e., for every point for which there exists no open neighborhood in which the optimal value function is analytic), there exist a one-sided open neighborhood and an analytic parametrization such that the composition of the parametrization and the optimal value function can be analytically extended to an open neighborhood, which contains the singular point. The related results are contained in Section 5.2.

Until now it has been known that, if all travel time and cost functions are piecewise linear, the number of linear pieces of which the optimal value function of the unconstrained time-dependent optimal path problem consists grows at least exponentially with the size of the network in the worst case. Nevertheless, a more detailed analysis of this matter has been an open question. We have carried out a detailed complexity analysis of the piecewise linear time-dependent optimal path problem, taking into account both the impact of the constraints and the FIFO-property. It turns out that both the type of waiting time constraints and the FIFO property have a crucial impact on the space and time complexity of the computation of the optimal value function. In particular, we prove that, if waiting is prohibited everywhere in a FIFO-network, then the space complexity of the optimal value function is polynomial in the size of the network, the number of linear pieces of the network functions and the number of edge sequences which induce optimal paths. Since the latter is exponential in the size of the network in the worst case, the optimal path problem is still NP-hard. However, in a general network with general piecewise linear waiting time constraints, the space complexity of the optimal value function is double exponential in the size of the network even if there exists a unique edge sequence traversed by all optimal paths. The complexity analysis of the piecewise linear optimal path problem is carried out in Section 5.3.

Considering the optimal path problem with fixed departure time in a time-dependent FIFO network in which waiting is prohibited, we derive a pruning technique, which leads to a significant reduction of the search space for any optimal path algorithm. This pruning technique is based on the Lipschitz continuity of the optimal value function and can be applied both in the cases of constrained and unconstrained arrival times. We then formulate the arrival time constraints in such a way that admissible paths are constrained to stay close to fastest paths and carry out a complexity analysis of the time-dependent optimal path problem with fixed departure time. We prove that the resulting discrete-time problem can be solved in polynomial time in FIFO-networks if the arrival time constraints are tight enough, whereas the continuous-time problem is NP-hard unless the arrival time constraints allow only fastest paths. These results are presented in Chapter 7.

We develop two algorithms which efficiently solve the problem of computing the optimal value function and the corresponding optimal paths in time-dependent networks. The first algorithm is presented in Chapter 8 and belongs to the class of decreasing order of time algorithms. It improves over the methods published in the past by working for arbitrary continuous travel time and lower semicontinuous cost functions (as opposed to only working for piecewise linear functions) and by allowing the incorporation of elements of heuristic

search. The second algorithm is presented in Chapter 9 and is designed for FIFO-networks in which the arrival time constraints do not conflict with the earliest arrival and latest departure times. In this case, we first construct an upper bound of the optimal value function, the associated admissible control policy as well as a lower bound for the optimal value function in polynomial time. The control policy is then iterated in such a way that a monotone decreasing sequence of upper bounds and a monotone increasing sequence of lower bounds are generated, which converge to the optimal value function after a finite number of steps. Since an upper bound of the error of the current solution is maintained throughout the algorithm this method allows a trade-off between the computation time and the accuracy of the found solution.
In order to show the applicability of the presented approach we have carried out a case study using real-world data from the German city of Ingolstadt. The results imply that, at least for a moderately-sized network such as the road network of Ingolstadt, continuous-time time-dependent optimal paths can be computed in reasonable computation time. The proposed approach also suggests that the potential for savings of the energy consumption is in the range of 10 % for the considered electric vehicle with respect to fastest paths.
Parts of this thesis have been published or submitted for publication in the following articles: [107], [110], [108].

1.3. Organization of the Thesis

The content of this thesis is structured in three parts. The first part is concerned with the modeling of the fuel consumption, which must be associated with the edges of the time-dependent road network in order to enable the computation of fuel-optimal routes. In the second part, the mathematical formulation of the time-dependent optimal path problem is provided, the existence of optimal paths is proved, properties of the optimal value function are derived and a complexity analysis of the time-dependent optimal path problem is carried out. Based on these theoretical results, algorithmic solutions of the time-dependent optimal path problem are developed in the third part of this thesis.

In particular, Part I contains the description of a physical consumption model, a derivation of speed distributions on urban roads based on traffic theory, a generalization of the approach to load-/time-dependent traffic data and the resulting properties of the network functions.

Part II is concerned with the theoretical treatment of the time-dependent optimal path problem. In Chapter 3 we introduce the notion of time-dependent networks, time-dependent paths, the dual of the time-dependent network and the concept of reachability. We formally define the different types of time-dependent optimal path problems in Chapter 4, prove the existence of optimal paths and the lower semicontinuity of the optimal value function and derive the dynamic programming equations, which the optimal value function satisfies. We conclude this chapter by briefly discussing order relations in time-dependent networks.
In Chapter 5 we address the properties of the optimal value function. In particular, under weak assumptions, we prove the (Lipschitz-) continuity of the optimal value function in Section 5.1. We then prove that the optimal value function is directionally differentiable everywhere and analytic in a neighborhood of almost every point, if the travel time, cost and constraint functions are analytic, cf. Section 5.2. In Section 5.3, we provide a framework

for the description of one- and two-dimensional piecewise linear functions and carry out a complexity analysis of the piecewise linear time-dependent optimal path problem.
We generalize the results of the preceding chapters to the robust optimal path problem in Chapter 6. In particular, we consider a problem setting in which the travel time and cost functions of the network are not precisely known, but only a certain range of values is given in which the particular realization must be contained. In this scenario, we prove the existence of optimal paths, derive properties of the optimal value function and carry out a complexity analysis of the piecewise linear optimal path problem. These results are based on the identification of appropriate assumptions under which the principle of dynamic programming can be used to compute the optimal value function without extending the state space by the set of possible scenarios.
In Chapter 7 we consider the computation of optimal paths for a fixed departure time in FIFO-networks in which waiting is prohibited. We prove the correctness of a pruning principle, which allows a significant reduction of the search space and carry out a complexity analysis for the case in which the admissible arrival times are constrained to stay close to the earliest arrival times.

Part III is concerned with the algorithmic solution of the time-dependent optimal path problem. In Chapter 8 we extend the class of decreasing order of time algorithms to heuristic search by presenting a new point of view on the manner in which a node and time interval can be chosen in one iteration of a decreasing order of time algorithm: In previous publications, the piecewise linear structure of the network had to be assumed in order to enable this choice. Our approach is similar to the choice of the iteration node in Dijkstra's shortest path algorithm or the A* algorithm. We prove the correctness and termination of the resulting algorithm and illustrate its progression with a simple numerical example.
We present a second algorithmic approach to the solution of the time-dependent optimal path problem in Chapter 9, which is applicable in FIFO-networks and allows a trade-off between the computation time and the accuracy of the found solution. We first compute an admissible initial control policy as well as a lower and an upper bound of the optimal value function in polynomial time. We then repeatedly iterate the control policy, thereby generating a sequence of monotone decreasing upper bounds and monotone increasing lower bounds of the optimal value function. We prove the correctness and termination of the resulting algorithm and illustrate its progression with a simple numerical example.
Finally, we conclude our discussion of the time-dependent optimal path problem in Chapter 10 by briefly summarizing our results and indicating directions for future research.

In the appendix (i.e., Appendix A), we carry out a case study in the road network of Ingolstadt, using the traffic data collected in the framework of INI.TUM in the project "Verkehrslage Ingolstadt" [5] and the parameters of the electric vehicle which is being developed in the MUTE project [4]. The results confirm the practicality of the approach which we have chosen in this thesis.

Part I.

Physical Background

2. Computation of the Fuel Consumption

In this chapter we provide some background information on the physical modeling of the travel times and travel costs in the time-dependent road network. In particular, we consider the travel time and the amount of fuel required by a motor vehicle to travel through the road segments and junctions of the road network. The discussion of the underlying physical model is rather meant to be illustrative than restrictive for the definition of time-dependent networks. The results which we derive for the fuel consumption of combustion-engined vehicles can easily be extended to the energy consumption of hybrid and electric vehicles. Indications to the differences in the modeling of the energy balance are provided wherever they occur.

The computation of the fuel consumption of a motor vehicle or a fleet of vehicles in the road network is not only relevant in the field of automotive engineering [129, Chapter V.33], [19, p.416 et seq.], [164] but has also aroused interest in the fields of traffic engineering [103], [76] and politics [21], [1] due to its environmental and economical impact. Considering only the major urban areas of the United States of America, traffic congestion caused a total cost of 47.5 billion $, and a waste of 14.35 billion liters of fuel and 2.7 billion hours of work time per year in the 1990s [177, p.1]. Since the impact of traffic congestion is increasing, the reserves of fossil fuel are finite and the global discussion on climate changes enforces the restrictions on the emission of greenhouse gases, the consideration of fuel consumption in traffic planning has gained increasing importance in the last two decades.

The deployment of a model of the fuel consumption as the basis of decision making for route-planning problems in the road network enables the computation of ecological routes. At this, a route can be classified as ecological if it minimizes the emission of pollutants (CO_2, CO, HC, NO_x, carbon black), which are generated by a vehicle on its drive from a given point of departure to a given destination. Since the amount of emitted pollutants is minimized by different optimization strategies, the simultaneous minimization of all pollutants is not possible in general. Henceforth, we shall be concerned with the minimization of CO_2, which is equivalent to the minimization of the fuel consumption, because the amount of emitted CO_2 is directly proportional to the amount of consumed fuel [129, p.137]. Consequently, the resulting routes can also be considered as optimal from an economical point of view. However, the minimization of the amount of fuel (or, more generally, energy) is not only important for vehicles with combustion engines. At present, one of the core problems of the development of electric vehicles is the conflict of objectives between the cruising range and the size and cost of the battery. As a consequence, energy-optimal route planning is likely to even gain importance in the context of electric mobility [1], see also Appendix A.

The determination of the optimal operating strategy of the vehicle, including the engine management, gearbox control and determination of optimal acceleration and deceleration behavior are important factors that influence the reduction of the fuel consumption. Recently, there has been increasing interest in assessing the usability of navigation data for the purpose of engine management, gearbox control [161],[95] and the determination of optimal acceleration and deceleration behavior [113]. However, since these quantities are

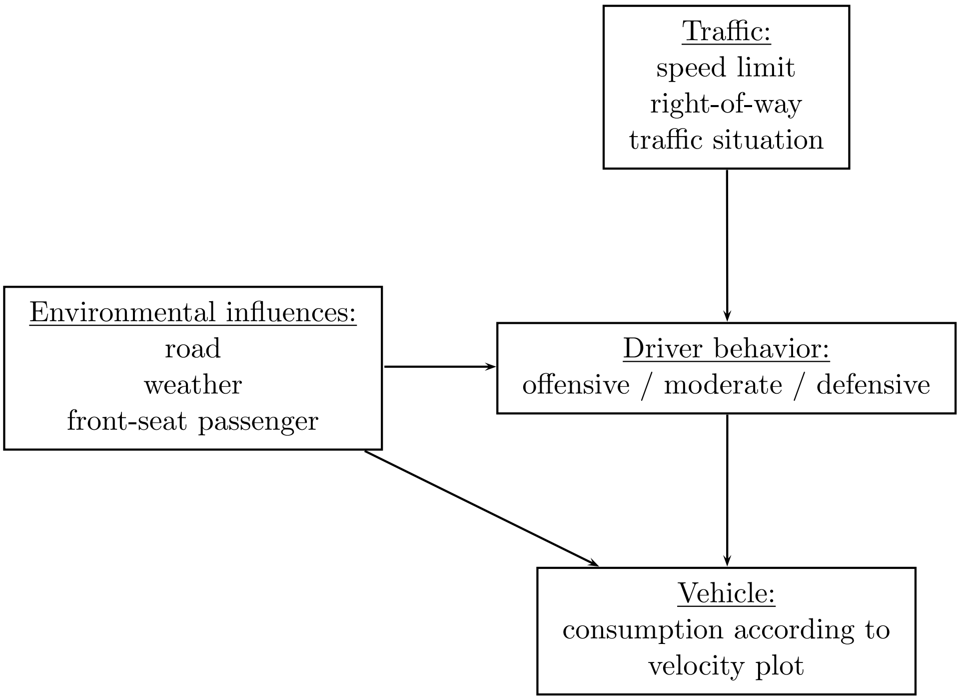


Figure 2.1.: Influences on the fuel consumption of a vehicle which is traveling in the road network.

not subject to optimization in the context of route planning at present, we shall not be further concerned with these topics. For the same reason, the possibility of reducing the fuel consumption by an optimization of traffic control strategies [89] will be excluded from the considerations in this thesis.

In order to use fuel consumption as a basis of decision making for route-planning problems in the road network, a cost value must be associated with each road segment (and possibly with each junction) of the road network at each point in time. This cost value must be proportional to the amount of fuel required for passing through the respective road segment (or junction), and must therefore depend on a variety of influencing factors such as the vehicle model [81], [103], the traffic situation [103], [76] the driver of the vehicle [90], [150] and environmental conditions [174]. These influencing factors are illustrated in Figure 2.1.

Considering only the longitudinal dynamics of the vehicle and assuming that the velocity plots of the drive are given, we compute the resulting fuel consumption in Section 2.1. In Section 2.2, we introduce a model for determining the characteristics of velocity plots based on a recently developed model of urban traffic. We motivate a time-dependent generalization of these results which is based on historical traffic data in Section 2.3 and derive some properties of the resulting travel time and travel cost functions in Section 2.4.

2.1. Energy Balance in the Case of Given Velocity Plots

Although the amount of fuel a vehicle requires for driving through a given road segment or junction depends on a variety of influencing factors, the physical model of the vehicle constitutes the end of the functional chain, cf. Figure 2.1. In particular, given any velocity

plot depending on environmental and traffic conditions as well as driver characteristics, the introduction and parametrization of the physical model of the vehicle permit the computation of the total fuel consumption associated with the respective velocity plot [129], [103], [19], [81]. For this reason we assume in this section that a velocity plot $v \in C^1([0,T],\mathbb{R})$ (in longitudinal direction of the vehicle) with time domain $[0,T] \subset \mathbb{R}$, $T > 0$, is given, and we aim at computing the total fuel consumption B in the time interval $[0,T]$.
For a given velocity plot v we define the acceleration plot $a \in C^0([0,T],\mathbb{R})$ (in longitudinal direction of the vehicle) according to $a(t) = \mathrm{d}/\mathrm{d}t\, v(t)$ for all $t \in [0,T]$ and the distance plot $d \in C^2([0,T],\mathbb{R})$ (in longitudinal direction of the vehicle) according to

$$d(t) = \int_0^t v(s)\mathrm{d}s, \qquad \text{for all } t \in [0,T].$$

Moreover, we assume that an angle of elevation in longitudinal direction of the road $\alpha(l)$ is given for all $l \in [0,L]$, $L = d(T)$.
In order to keep the vehicle moving at a distance $l \in [0,L]$ of the initial point of the drive and in accordance with the velocity plot v the vehicle must countervail a sum of driving resistances. These driving resistances consist of [129, p. 74] the climbing resistance F_c, the rolling resistance F_r, the aerodynamic resistance F_a and the inertial resistance F_i. In order to model the influence of the braking system of the vehicle, we also introduce the braking resistance F_b, cp. [19, p. 417]. Introducing the lower heating value H_l of the utilized fuel, the engine efficiency η_e and the transmission efficiency η_t, the total fuel consumption in the time interval $[0,T]$ is given by [129, Chapter V.33], [19, p.416 et seq.], [103]

$$B = \int_0^T \max\left\{0, \frac{\left[F_c(t) + F_r(t) + F_a(t) + F_i(t) + F_b(t)\right]v(t)}{H_l \eta_e(t)\eta_t(t)}\right\} \mathrm{d}t. \tag{2.1}$$

In order to ease the notation we define the sum $F(t)$ of the driving resistances at time $t \in [0,T]$ by $F(t) = F_c(t) + F_r(t) + F_a(t) + F_i(t) + F_b(t)$. Note that the quantity $F(t)v(t)$ equals the mechanical power $P(t)$ which must be provided by the motor of the vehicle in order to countervail the driving resistances. Since no fuel can be generated in an combustion-engined vehicle we use the fuel flow rate $Q(t) = \max\left\{0, P(t)/\big(H_l\eta_e(t)\eta_t(t)\big)\right\}$ in (2.1) which was termed the 'throttle-cutoff' scenario in [103]. However, in hybrid and electric vehicles negative power $P(t) < 0$ can be used to recharge the battery. Consequently, the total amount of energy required by a hybrid or electric vehicle for traversing a road segment can be negative if, e.g., $\alpha(l) \ll 0$ for all $l \in [0,L]$.
Denoting by g the gravitational acceleration, by m_v the total mass of the vehicle and by c_{rr} the rolling resistance coefficient, the climbing resistance $F_c(t)$ and rolling resistance $F_r(t)$ at time $t \in [0,T]$ are given by

$$F_c(t) = m_v g \sin\Big(\alpha\big(d(t)\big)\Big), \tag{2.2}$$

$$F_r(t) = c_{rr} m_v g \cos\Big(\alpha\big(d(t)\big)\Big). \tag{2.3}$$

At this, the rolling resistance coefficient c_{rr} generally depends on the tire and the road surface. In the above equation (2.3) we have not accounted for the increase of the rolling resistance during cornering, and we have neglected the dependence of the rolling resistance

on the longitudinal velocity since both effects have been classified as negligible in [129, p.7, p.15]. Furthermore, we have assumed that the vehicle mass m_v is constant throughout the drive, thus ignoring small variations due to the filling level of the fuel tank.
Denoting by ρ_a the air density, by A_f the frontal area of the vehicle and by c_d the drag coefficient, the aerodynamic resistance $F_a(t)$ at time $t \in [0, T]$ is given by

$$F_a(t) = \frac{\rho_a A_f c_d}{2} v(t)^2. \tag{2.4}$$

Note that both the variability of the air density along $[0, L]$ and the influence of the wind velocity in longitudinal direction of the vehicle have been neglected in (2.4). Both quantities are hard to predict in applications, and may be considered as system noise in our context.
Denoting by $n(t)$ the gear at time $t \in [0, T]$ and by $\lambda(n)$ the gear-dependent molding body surcharge factor, the inertial resistance $F_i(t)$ at time $t \in [0, T]$ is given by

$$F_i(t) = \lambda\big(n(t)\big) m_v a(t). \tag{2.5}$$

The molding body surcharge factor models the fraction of the moments of inertia with respect to the vehicle mass m_v. Let I_w denote the moment of inertia of the wheels of the vehicle, let r_w denote the radius of the wheels of the vehicle, let I_m denote the moment of inertia of the flywheel in the motor and let $gr(n)$ denote the gear ratio between the rotational speed of the motor and the wheel axle of the vehicle if the gear n is engaged. Neglecting the dynamic deformation of the tires and assuming that all wheels have the same radius, we obtain

$$F_i(t) = m_v a(t) + \frac{I_w}{r_w^2} a(t) + \frac{I_m}{r_w^2} gr\big(n(t)\big)^2 a(t), \tag{2.6}$$

since both the angular velocity and the accelerating moment of the flywheel are scaled by $gr(n)$, cp. [129, eq. (21.5)]. A more detailed discussion, including the moment of inertia of the powertrain, can be found in [129, Chapter III.21] but will not be annotated here. Note that $\lambda(n)$ is uniquely determined by (2.5) and (2.6).
Denoting by $p(t)$ the break pressure at time $t \in [0, T]$ and by $c_p(n)$ a gear-dependent constant of proportionality, the braking resistance $F_b(t)$ at time $t \in [0, T]$ is given by

$$F_i(t) = c_p\big(n(t)\big) p(t). \tag{2.7}$$

The braking resistance has been introduced in order to distinguish between deceleration caused by the motor brake and deceleration caused by an actuation of the brake pedal. Since kinetic energy is converted to heat when actuating the brake pedal, the total deceleration $a(t)$ overestimates the effective deceleration $a_m(t)$ which is relevant for the computation of the power generated by the motor of the vehicle. Consequently, $c_p(n) < 0$ in (2.7). Since the brake causes a deceleration of both rectilinear and circular motions, the constant of proportionality $c_p(n)$ at time $t \in [0, T]$ depends on the gear ratio gr and hence on the gear $n(t)$ at time $t \in [0, T]$, cp. (2.6).
Both the engine efficiency η_m and the transmission efficiency η_t generally depend on the rotational speed $\omega(t)$ of the crankshaft at time $t \in [0, T]$ and the power $P(t)$. While the variation of the transmission efficiency is minor [106, Table 3.1] (between 0.93 and 0.98), the dependence of the engine efficiency η_m typically varies between 0.15 and 0.40 and is described by the so-called engine characteristic map [129, Chapter IV.25], [103]. Sometimes

an equivalent description of the engine-characteristic map is used which depends on the rotational speed ω of the crankshaft and the torque τ on the crankshaft, cf. [129, Figure 25.7].

In addition to the above driving resistances the power required by a number of auxiliary consumers, such as air conditioning, light or car radio must be provided by the motor of the vehicle. Denoting the power of the auxiliary consumers at time $t \in [0, T]$ by $P_0(t)$, the so-called basic power consumption, we obtain

$$B = \int_0^T \max\left\{0, \frac{F(t)v(t) + P_0(t)}{H_l \eta_e(t)\eta_t(t)}\right\} \mathrm{d}t. \tag{2.8}$$

Assuming that all vehicle parameters as well as the velocity plot, the gear selection $n(t)$ at time t and the brake pressure $p(t)$ at time t are given for all $t \in [0, T]$, equation (2.8) allows the computation of the total fuel consumption along $[0, L]$ for $v \gg 0$. For $v \to 0$, the fuel flow rate $Q(t) = \max\big\{0, \big(F(t)v(t) + P_0(t)\big)/\big(H_l\eta_e(t)\eta_t(t)\big)\big\}$ tends to 0 if $P_0 = 0$, which would require the motor to halt. Since under real operating conditions the clutch is released and the motor proceeds to move with a certain idle speed ω_0, (2.8) must be replaced by

$$B = \int_0^T \max\left\{0, \frac{2\pi\tau_0\omega_0 + P_0(t)}{H_l \eta_e(t)\eta_t(t)}\right\} \mathrm{d}t \tag{2.9}$$

if $v = 0$, where τ_0 denotes the torque generated by internal friction. If $v(t) = 0$ for some $t \in [0, T]$ and $v(t) \gg 0$ for some $t \in [0, T]$, a transition between (2.8) and (2.9) must be realized which models the engaging and disengaging of the clutch. We will not discuss this topic in detail, but suggest a linear transition as a simple solution. Due to its minor impact on the fuel consumption, the engaging and disengaging of the clutch is usually left out of consideration [129, Chapter V.33], [19, p.416 et seq.], [103], and has been discussed here only for the sake of completeness.

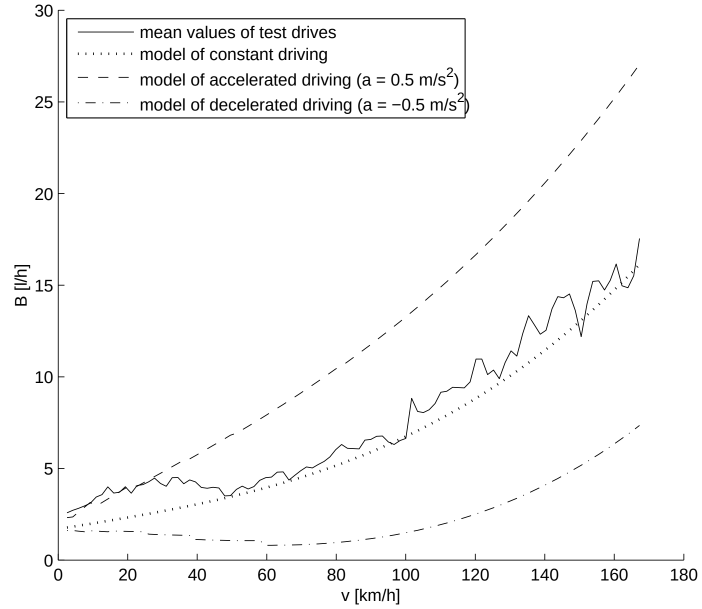
Note that, beside the choice of the velocity plot v, the only influence of the driver on the fuel consumption in (2.8) are the gear-changing behavior and the braking behavior (i.e., to which extent the deceleration of the vehicle is caused by the actuation of the brake pedal).

For an electric vehicle, (2.8) and (2.9) become

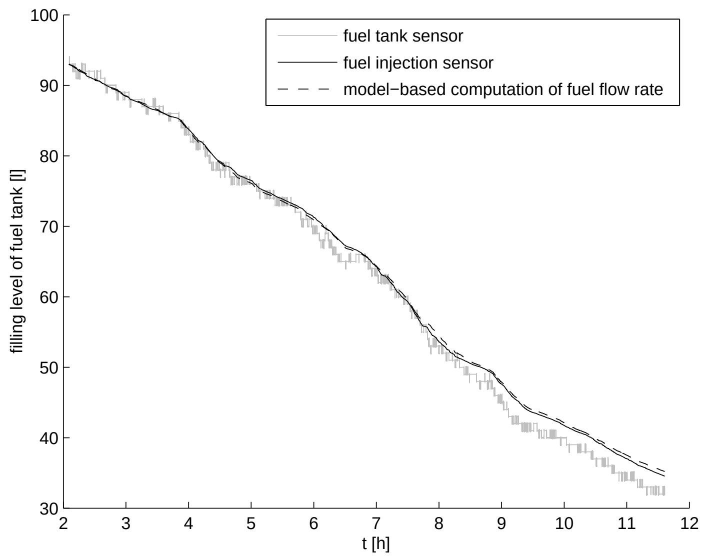
$$E = \int_0^T \frac{F(t)v(t)}{\eta_e(t)\eta_t(t)} + P_0(t)\mathrm{d}t, \qquad E = \int_0^T \max\left\{0, \frac{2\pi\tau_0\omega_0}{\eta_e(t)\eta_t(t)} + P_0(t)\right\} \mathrm{d}t, \tag{2.10}$$

since the power required for the auxiliary consumers can be directly taken from the battery and the recovery of energy is possible during phases of negative energy demands.

Within the framework of this thesis, this method of computing the fuel consumption of a motor vehicle has been implemented in the project work at HARMAN International. The above vehicle parameters have been determined from a series of measurements by using linear regression [109]. The test drives were carried out in a BMW 325i touring [3] and logged by a blue PiraT [2]. The resulting plots were kindly provided by HARMAN International for publication and are depicted in Figure 2.2.



(a) Average measured fuel flow rate in the test drives and model based fuel flow rates.



(b) Comparison of the filling levels of the fuel tank, based on measurements and the vehicle model.

Figure 2.2.: Average fuel flow rate and filling level of the fuel tank, logged during 733 km of test drives with a BMW 325i. As dependent variables in the regression analysis of the vehicle parameters the fuel injection sensor, the rotational speed of the motor and the torque applied by the motor were used. A constant engine characteristic map was assumed since the measurement noise and the quantization errors were of the same magnitude as the variations in the engine characteristic map.

2.2. Characteristic Speed Distributions on Road Segments

In this section, we derive a probability distribution of the vehicle speed on an urban road segment which agrees with the data published in [60] and a recently developed traffic theory for urban traffic flow [87].
In order to describe the driving behavior in urban areas, several driving patterns (including, e.g., average speeds, speed variations, acceleration behavior, gear-changing behavior) have been introduced and measured [61], [60], [34]. The results have been used to describe in which manner the driving patterns are influenced by properties of the driver and the surrounding infrastructure, and in which manner the driving patterns influence the vehicle exhaust emissions [62]. In accordance with the data published in [60], the acceleration and deceleration behavior of a vehicle in urban traffic (within each speed range of $[10k\,\mathrm{km/h}, 10(k+1)\,\mathrm{km/h}[,\ k = 0, ..., 11)$ has been described by half-normal probability distributions in [8]. The probability distributions of the vehicle speeds [60], accelerations [60], [8], and decelerations [60], [8], were derived from repeated measurements of the vehicle speed and acceleration according to a fixed measurement rate $1/\Delta t$ ($\Delta t = 0.1\,\mathrm{s}$ in [60] and $\Delta t = 1\,\mathrm{s}$ in [8]). Such speed and acceleration data can be used as the input for physical consumption models such as the model described in Section 2.1, [34], [103].
Let us consider a road segment of length L on which the free flow velocity plots result in a normal distribution of the measured vehicle speeds $V_\mathrm{f} \sim \mathcal{N}(\mu_v, \sigma_v^2)$, $\mu_v, \sigma_v \in \mathbb{R}_0^+$. Postulating that the vehicle speeds are normally distributed is a common assumption in traffic theory and has been empirically verified for highway traffic, see, e.g., [84]. For urban traffic this is certainly not true (cp., e.g., the data in [60] and the traffic flow model [87]), for which reason we only assume the free flow speed to be normally distributed. Let us assume that each of the following events $E = E_k$, $k = 1, ..., 4$, can occur on this road segment [87]:

- E_1: A car drives freely throughout the road segment.
- E_2: A car accelerates from the speed 0 to some speed V_0^+ and continues driving freely until the end of the road segment.
- E_3: A car drives freely until it is forced to decelerate from some speed V_0^- to the speed 0 at the end of the road segment and then halts for some time T_h.
- E_4: A car accelerates from the speed 0 to some speed V_0^+ in the first half of the road segment (i.e., $[0, L/2]$), continues driving freely until it is forced to decelerate in the second half of the road segment (i.e., $[L/2, L]$) from some speed V_0^- to the speed 0 at the end of the road segment and then halts for some time T_h.

Clearly, this is a simplification of the velocity plots which occur on a road segment in reality. However, it captures the presence of junctions in urban traffic [87], and particularizes the model in [87] by taking into account the acceleration and deceleration processes. We assume that $V_0^+, V_0^-, T_\mathrm{h}$ are independent random variables and that $V_0^+, V_0^- \sim \mathcal{N}(\mu_v, \sigma_v^2)$ and $T_\mathrm{h} \sim \mathcal{U}\big([0, \overline{T}]\big)$ for some $\overline{T} \in \mathbb{R}_0^+$ (which can be interpreted as the duration of the red phase at a signalized junction). Note that the waiting time at a junction is indeed uniformly distributed, provided that the traffic is undersaturated and reaches the junction in a uniformly distributed manner. We further assume that the necessity of accelerating and decelerating is modeled by two independent and identically distributed (i.i.d.) Bernoulli-distributed random variables, and that the necessity of each such transition from free flow

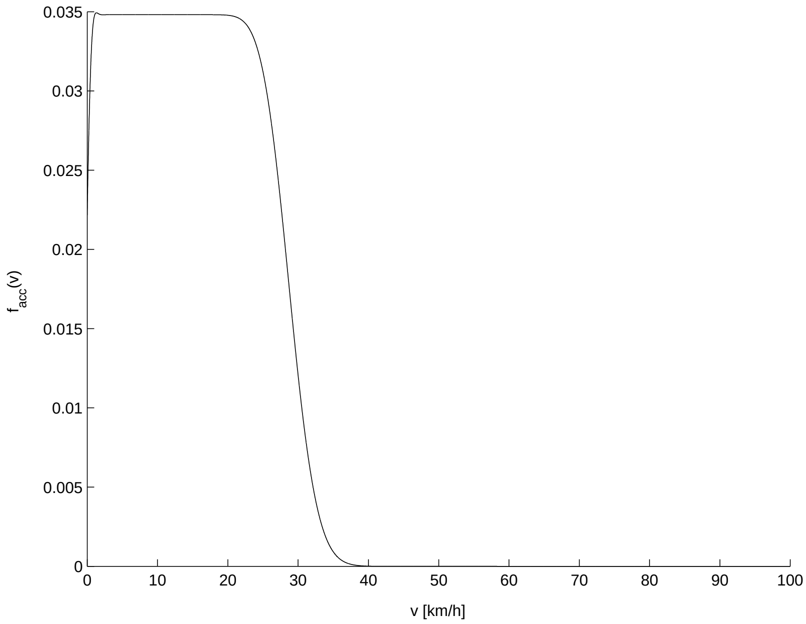


Figure 2.3.: Probability density function f_{acc} of the vehicle speed in an acceleration phase of fixed duration $T^+ = 5\,\text{s}$. The used parameters were $\Delta t = 0.1\,\text{s}$ and $\sigma_a = 2\,\text{m/s}^2$.

to stop (or from stop to free flow) is associated with the probability $p_{\text{t}} \in [0, 1]$. Hence, $\mathbb{P}\{E_1\} = (1 - p_{\text{t}})^2$, $\mathbb{P}\{E_2\} = \mathbb{P}\{E_3\} = p_{\text{t}}(1 - p_{\text{t}})$ and $\mathbb{P}\{E_4\} = p_{\text{t}}^2$. In the derivation of the probability distribution of the vehicle speeds we also assume that the travel time of a freely moving vehicle on a portion $[0, l]$, $l \leq L$, of the road segment is given by l/μ_v. (Note that this equation is almost surely exact for $l \to \infty$ if the free flow speed process is ergodic.) Let us finally assume that the acceleration plots of a vehicle on the road segment result in a half-normal distribution of the measured vehicle accelerations A^+, $A^+ \sim \mathcal{HN}(0, \sigma_a^2)$, and decelerations A^-, $-A^- \sim \mathcal{HN}(0, \sigma_a^2)$, $\sigma_a \in \mathbb{R}^+$. This assumption constitutes a further simplification with respect to [8] which we undertake for the clarity of the presentation.

We clearly have $\mathbb{P}\{V \leq v | E = E_1\} \propto P_1(v, L) = (L/\mu_v)\mathbb{P}\{V_{\text{f}} \leq v\}$. Here, the constant of proportionality has been introduced in order to reflect the measurement process, in which the number of measurements is proportional to the duration of the measurement time interval [60], [8]. The travel time associated with the event E_1 on the road segment is denoted by $T_1(L) = L/\mu_v$.

Let us now consider the case $\{E = E_2\}$. Consider an acceleration phase of fixed duration $T^+ = K\Delta t$, $K \in \mathbb{N}$, an acceleration plot $a : [0, T^+] \to \mathbb{R}$ and an associated velocity plot $v : [0, T^+] \to \mathbb{R}$. Assuming that the acceleration values $(a_k)_{k=1,\ldots,K}$, $a_k = a(k\Delta t)$, are independent and half-normally distributed throughout the acceleration phase, and that a is constant between two consecutive measurement points, the speeds $(v_k)_{k=1,\ldots,K}$, $v_k = v(k\Delta t)$, are given by $v_k = \sum_{i=1}^{k} a_i \Delta t$. The corresponding probability density function of V is depicted in Figure 2.3. In the following, we assume that the speed of the vehicle at time $t \leq T^+$ during the acceleration phase is given by $\mathbb{E}[A^+]t = \sqrt{2/\pi}\sigma_a t$. (Note that this equation is almost surely exact for $t \to \infty$ if the acceleration process is ergodic.) If we assume that the measurement times are uniformly distributed in the acceleration time interval then this corresponds to the approximation of the probability density function in

Figure 2.3 by a characteristic function.
In order to derive the distribution of the vehicle speed V_2 in the case $\{E = E_2\}$, we assume that the driver chooses some $V_0^+ \sim \mathcal{N}(\mu_v, \sigma_v^2)$, accelerates from $v(0) = 0$ until either $v(t) \geq V_0^+$ or the end of the road segment is reached, and continues driving freely until the end of the road segment is reached. We denote the total travel time for a fixed $V_0^+ = v_0^+$ by $T_2(L, v_0^+)$. We further assume that the probability density function f_2 of V_2 is given by the equation

$$f_2(v) \propto \int_{-\infty}^{\infty} T_2(L, v_0^+) f_0(v_0^+) f_{V_2|V_0^+}(v|V_0^+ = v_0^+)\, \mathrm{d}v_0^+, \tag{2.11}$$

where $f_0 : \mathbb{R} \to \mathbb{R}$ denotes the probability density function of V_f, V_0^+ and $f_{V_2|V_0^+}$ denotes the conditional probability density function of V_2 given V_0^+. Note that this corresponds to the measurement process which has led to the probability distribution of the vehicle speed in [60] and the probability distribution of the vehicle acceleration in [8].
Let us assume that some $v_0^+ > 0$ has been fixed and let us first consider the acceleration phase. The equations of motion and the bounded length of the road segment yield $v(t) = \mathbb{E}[A^+]t$ and $d(t) = \mathbb{E}[A^+]t^2/2$ for all $t \leq T_\mathrm{t}(v_0^+) = \max\left\{0, \min\{v_0^+/\mathbb{E}[A^+], \sqrt{2L/\mathbb{E}[A^+]}\}\right\}$. Hence, if $0 \leq v_0^+ \leq \sqrt{2L\mathbb{E}[A^+]}$, then the traveling through the remaining distance $L - (v_0^+)^2/(2\mathbb{E}[A^+])$ takes the time $T_\mathrm{f}(v_0^+) = L/\mu_v - (v_0^+)^2/(2\mathbb{E}[A^+]\mu_v)$. If $v_0^+ < 0$ then the remaining travel time is $T_\mathrm{f}(v_0^+) = L/\mu_v$ and if $v_0^+ > \sqrt{2L\mathbb{E}[A^+]}$ then the remaining travel time is $T_\mathrm{f}(v_0^+) = 0$. In any case we have $T_2(L, v_0^+) = T_\mathrm{t}(v_0^+) + T_\mathrm{f}(v_0^+)$.
Considering measurement times which are uniformly distributed in $[0, T_2(L, v_0^+)]$, we obtain

$$\mathbb{P}\{V_2 \leq v | V_0^+ = v_0^+\} = \begin{cases} \mathbb{P}\{V_\mathrm{f} \leq v\}, & \text{if } v_0^+ \leq 0 \\ \dfrac{T_\mathrm{t}(v_0^+)\min\{1, v/v_0^+\} + T_\mathrm{f}(v_0^+)\mathbb{P}\{V_\mathrm{f} \leq v\}}{T_2(L, v_0^+)}, & \text{if } 0 < v_0^+ \leq \sqrt{2L\mathbb{E}[A^+]}. \\ \min\{1, v/\sqrt{2L\mathbb{E}[A^+]}\}, & \text{if } v_0^+ > \sqrt{2L\mathbb{E}[A^+]} \end{cases}$$

Hence,

$$f_{V_2|V_0^+}(v|V_0^+ = v_0^+) = \begin{cases} f_0(v), & \text{if } v_0^+ \leq 0 \\ \dfrac{T_\mathrm{t}(v_0^+)\chi_{[0,v_0^+]}(v)/v_0^+ + T_\mathrm{f}(v_0^+) f_0(v)}{T_2(L, v_0^+)}, & \text{if } 0 < v_0^+ < \sqrt{2L\mathbb{E}[A^+]} \\ \chi_{[0,\sqrt{2L\mathbb{E}[A^+]}]}(v)/\sqrt{2L\mathbb{E}[A^+]}, & \text{if } v_0^+ > \sqrt{2L\mathbb{E}[A^+]} \end{cases}.$$

Using (2.11), we establish

$$\mathbb{P}(V \leq v | E = E_2) \propto P_2(v, L) = \int_{-\infty}^{v} c_1(v')\chi_{[0,\sqrt{\sqrt{8/\pi}L\sigma_a}]}(v') + c_2 f_0(v')\, \mathrm{d}v',$$

where

$$c_1(v) = \int_v^\infty \frac{1}{\sqrt{2/\pi}\sigma_a} f_0(v_0^+)\, \mathrm{d}v_0^+,$$

$$c_2 = \int_{-\infty}^0 \frac{L}{\mu_v} f_0(v_0^+)\, \mathrm{d}v_0^+ + \int_0^{\sqrt{\sqrt{8/\pi}L\sigma_a}} \frac{\sqrt{8/\pi}L\sigma_a - (v_0^+)^2}{\sqrt{8/\pi}\sigma_a\mu_v} f_0(v_0^+)\, \mathrm{d}v_0^+.$$

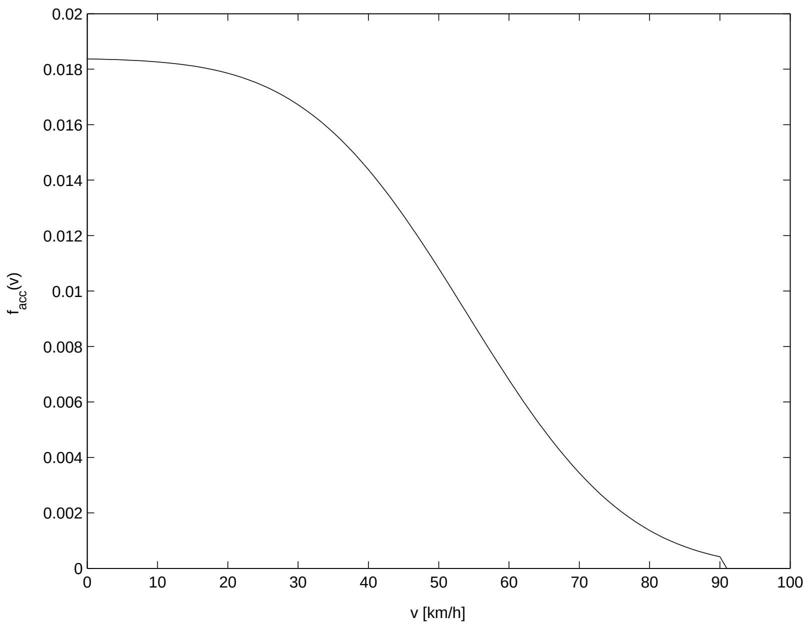


Figure 2.4.: Probability density function f_{acc} of the vehicle speed in an acceleration phase of variable duration. The used parameters were $L = 200\,\mathrm{m}$, $\sigma_a = 2\,\mathrm{m/s^2}$, $\mu_v = 15\,\mathrm{m/s}$, $\sigma_v = 5\,\mathrm{m/s}$.

Hence, the probability density function of the vehicle speeds in the case $\{E = E_2\}$ is the superposition of the probability density function of the free flow speed and a weighted characteristic function. The form of the weighted characteristic function (i.e., the contribution of the acceleration phase) is depicted in Figure 2.4. The existence of the above integrals is easily verified.

In the case $\{E = E_3\}$, if $\overline{T} = 0$, then we obtain the same probability distribution of the vehicle speed V_3 as in the case $\{E = E_2\}$, since we have assumed that the terminal speed is fixed to 0. Taking into account the independence of V_0^-, T_{h}, we obtain

$$\mathbb{P}\{V \leq v | E = E_3\} \propto P_3(v, L) = \begin{cases} P_2(v, L), & \text{if } v < 0 \\ P_2(v, L) + \overline{T}/2, & \text{if } v \geq 0 \end{cases}$$

We denote the travel time associated with the event $\{E = E_3, V_0^- = v_0^-, T_{\mathrm{h}} = t_{\mathrm{h}}\}$ on the road segment by $T_3(L, v_0^-, t_{\mathrm{h}}) = T_2(L, v_0^-) + t_{\mathrm{h}}$.
In order to obtain the probability distribution of the vehicle speed V_4 in the case $\{E = E_4\}$, reasoning again with the symmetry of the acceleration and deceleration process, we decompose the given road segment into two road segments of length $L/2$ on one of which E_2 occurs and on the other of which E_3 occurs, i.e., $\mathbb{P}\{V \leq v | E = E_4\} \propto P_4(v, L) = P_2(v, L/2) + P_3(v, L/2)$. Using the same arguments, we obtain the travel time $T_4(L, v_0^+, v_0^-, t_{\mathrm{h}}) = T_2(L/2, v_0^+) + T_3(L/2, v_0^-, t_{\mathrm{h}})$ associated with the event $\{E = E_4, V_0^+ = v_0^+, V_0^- = v_0^-, T_{\mathrm{h}} = t_{\mathrm{h}}\}$ on the road segment .

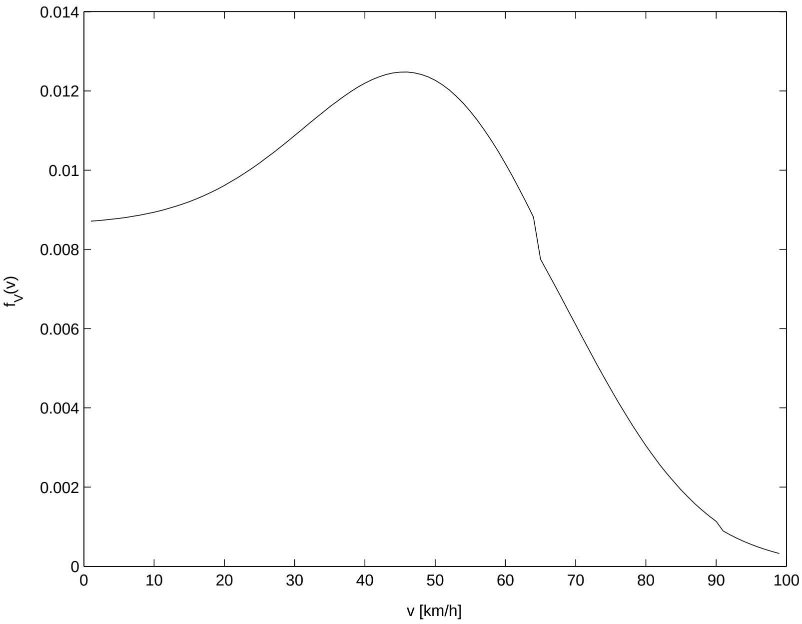


Figure 2.5.: Density function f_V on a road segment of length $L = 200\,\mathrm{m}$. The jumps at $v \approx 65\,\mathrm{km/h}$ and $v \approx 90\,\mathrm{km/h}$ result from the finite length of the road segment. The probability of finding the vehicle at a stop is $\mathbb{P}(V = 0) \approx 0.215$ and $\mathbb{P}(V < 0) \approx 4 \cdot 10^{-4}$. The used parameters were $\sigma_a = 2\,\mathrm{m/s}^2$, $\mu_v = 15\,\mathrm{m/s}$, $\sigma_v = 5\,\mathrm{m/s}$, $p_\mathrm{t} = 0.3$, $\overline{T} = 60\,\mathrm{s}$.

Using the mutual independence of all random variables, we finally obtain

$$\mathbb{P}\{V \leq v\} = \frac{(1-p_\mathrm{t})^2 P_1(v,L) + (1-p_\mathrm{t})p_\mathrm{t}\big(P_2(v,L) + P_3(v,L)\big) + p_\mathrm{t}^2 P_4(v,L)}{\lim_{v\to\infty}(1-p_\mathrm{t})^2 P_1(v,L) + (1-p_\mathrm{t})p_\mathrm{t}\big(P_2(v,L) + P_3(v,L)\big) + p_\mathrm{t}^2 P_4(v,L)}. \tag{2.12}$$

Here, it is easily verified that the limit in the denominator of (2.12) exists and is finite. Let δ_0 denote the Dirac measure concentrated at 0 and let $p_\mathrm{s} = \mathbb{P}\{V = 0\}$ denote the probability of finding the vehicle at a stop. We decompose the probability measure p_V associated with V according to $p_V = p_\mathrm{s}\delta_0 + \widetilde{p}_V$, where $\widetilde{p}_V$ possesses the Radon-Nikodym-derivative f_V with respect to the Lebesgue measure on $\mathbb{R}$ (i.e., f_V is the density function associated with $\widetilde{p}_V$). Hence, V can be modeled by an equation of the type

$$V = (1 - d_\mathrm{s})\widetilde{V}, \tag{2.13}$$

where $d_\mathrm{s} \in \{0, 1\}$ is a random variable with $\mathbb{P}\{d_\mathrm{s} = 1\} = p_\mathrm{s}$ and $\widetilde{V}$ is a random variable with probability density function $(1 - p_\mathrm{s})^{-1} f_V$. An example of this probability distribution is shown in Figure 2.5.

Note that in general the probability of a stop at the beginning and at the end of a road segment cannot be determined from the road segment alone. Suppose that the considered road segment connects the junction J_2 to the junction J_1. Then the probability of stopping at the end of the road segment may depend on the maneuver at J_2, i.e., whether the car drives straight ahead or performs a right, left or U-turn. Similarly, the probability of stopping at the beginning of a road segment may depend on the maneuver at J_1. In order to model these dependencies, the differences between the speed distributions (and

cost values, respectively) associated with the different maneuvers must be associated with the junctions. The resulting optimal path problem must then be solved in the dual road network, cf. Section 3.4.
We will use the above methodology to compute the fuel consumption of a vehicle traveling through a given road segment in Subsection 2.4.2. Let us conclude this section by computing the total expected travel time T associated with the road segment. The independence of the random variables yields

$$T = (1-p_{\mathrm{t}})^2\mathbb{E}\big[T_1(L)\big] + (1-p_{\mathrm{t}})p_{\mathrm{t}}\Big(\mathbb{E}\big[T_2(L,V_0^+)\big] + \mathbb{E}\big[T_3(L,V_0^-,T_{\mathrm{h}})\big]\Big) \\ + p_{\mathrm{t}}^2\mathbb{E}\big[T_4(L,V_0^+,V_0^-,T_{\mathrm{h}})\big]. \tag{2.14}$$

Above, we have already seen that $T_1(L) = L/\mu_v$. Next we compute

$$\mathbb{E}\big[T_2(L,V_0^+)\big] = \int_{-\infty}^{\infty} T_2(L,v_0^+)f_0(v_0^+)\,\mathrm{d}v_0^+$$
$$= \frac{L}{\mu_v} + \int_0^{\sqrt{\sqrt{8/\pi}L\sigma_a}} \frac{2v_0^+\mu_v - (v_0^+)^2}{\sqrt{8/\pi}\sigma_a\mu_v} f_0(v_0^+)\,\mathrm{d}v_0^+ + \int_{\sqrt{\sqrt{8/\pi}L\sigma_a}}^{\infty} \left(\frac{\sqrt{2\pi}L}{\sigma_a} - \frac{L}{\mu_v}\right) f_0(v_0^+)\,\mathrm{d}v_0^+.$$

Using the symmetry of the acceleration and the deceleration process as well as the mutual independence of all random variables we obtain $\mathbb{E}\big[T_3(L,V_0^-,T_{\mathrm{h}})\big] = \mathbb{E}\big[T_2(L,V_0^+)\big] + \overline{T}/2$ and $\mathbb{E}\big[T_4(L,V_0^+,V_0^-,T_{\mathrm{h}})\big] = 2\mathbb{E}\big[T_2(L/2,V_0^+)\big] + \overline{T}/2$.

2.3. Historical Traffic Data

In the last two decades there has been increasing interest in incorporating time-dependencies into the modeling of the road network. This is due to the fact that both travel times [87], [13], and fuel consumption [21], [103], [76] vary strongly under different traffic conditions. The most common approach to describing the time-dependency of the road network is the measurement of the average vehicle speeds, using, e.g., floating-car data [119], [155], [173], [57], inductive loops [41], [55], [141], or airborne systems [64]. Another widely spread method is the simulation of traffic flow [33], e.g., with software tools, such as VISSIM [6], [115]. An example plot of such average speeds (which is contained in a commercially available digital map of Germany) has been kindly provided by HARMAN International for publication, see Figure 2.6.

The measurement of the traffic density, scattering parameters of the vehicle speeds or parameters describing vehicle accelerations requires a large amount of data and an appropriate measurement infrastructure to be available in large parts of the road network. Since a spatially inclusive and comprehensive construction of such infrastructure has not yet been completed, such measurements are only sporadically available and rarely contained in the digital maps which are used for infrastructure and route planning purposes. In order to use the methodology which we have developed in the preceding section, we must therefore derive a method for estimating the distribution parameters $\mu_v, \sigma_v, \sigma_a, \overline{T}, p_{\mathrm{t}}$ for varying traffic conditions.
Let us assume that a measured (space- and time-averaged, cp. Subsection 2.4.1) speed $\bar{v}(t)$ is given for each time $t \in \mathbb{R}$, from which the (average) travel time $T(t)$ (cf. (2.14)) at time $t \in \mathbb{R}$ can be computed as $T(t) = L/\bar{v}(t)$. Assuming that the traffic remains undersaturated

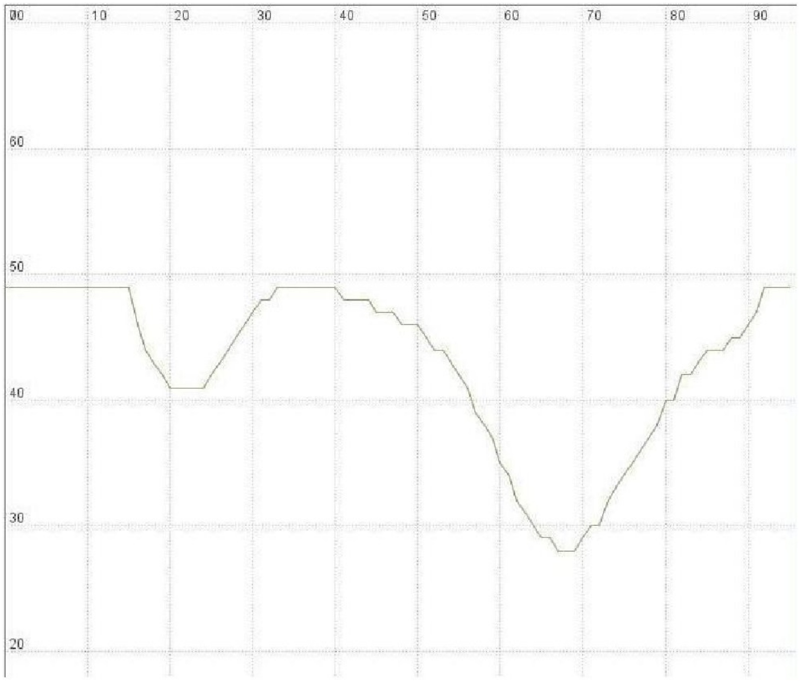


Figure 2.6.: Evolution of the average speed on an urban road segment throughout one day. The unit of the horizontal axis is 20min, the unit of the vertical axis is km/h.

throughout the day, we may use [87, Eqs.(74),(76)], i.e.,

$$\bar{v}(u) = \frac{v_0 \ln\Big(1 + \big(1 - (1+\delta)u\big)\frac{T_{\text{los}} v_0}{\big(1-s(1+\delta)u\big)L}\Big)}{\frac{(1-u)T_{\text{los}} v_0}{\big(1-s(1+\delta)u\big)L}} + v_0 \frac{\delta u}{1-u}, \qquad \rho(u) = \frac{u\hat{Q}}{\bar{v}(u)}, \tag{2.15}$$

in order to estimate the traffic density $\rho(t)$ associated with $\bar{v}(t)$. Here, u denotes the utilization of the outflow capacity, and the fit parameters in the above equations are the lost service time T_{los}, the length of the road segment L, the speed limit v_0, the outflow capacity $\hat{Q}$ and the safety factor δ (which has been introduced in order to cope with variations in the inflow), as well as the the number of signal phases s at the junction (which is supposed to be situated at the end of the road segment). Although this is a simplification (if the average speed $\bar{v}(t)$ is small, then the traffic is usually saturated) this approach has been empirically verified, see [87, Fig. 6].

The traffic density ρ which results from (2.15) corresponds to the average density on the considered road segment, including all vehicles which are queued at the junction. Using [87, Eq.(34)] we compute the average number of delayed vehicles ΔN to obtain the average density ρ_{f} of freely moving vehicles

$$\rho_{\text{f}}(u) = \frac{L\rho(u) - \Delta N(u)}{L - \Delta N(u) l_{\text{veh}}}, \quad \text{where} \quad \Delta N(u) = \left(\frac{L}{\bar{v}(u)} - \frac{L}{v_0}\right) u\hat{Q}. \tag{2.16}$$

Here, l_{veh} denotes the average vehicle length. We suppose a vehicle whose drive is not affected by the junction to be driving through traffic with the density ρ_{f}.

Let us now consider the free flow speeds, which we have assumed to be normally-distributed in the last section. This assumption does not only coincide with empirical observations but also with a traffic theory which is based on a gas-kinetic foundation. The theoretical results not only suggest that the speed distribution is normal (in the absence of junctions), but also provide relations between the traffic density, the average speed and the variance of the average speed. Taking into account that these derivations are based on the absence of junctions we apply this methodology only to the estimation of μ_v, σ_v from ρ_f, i.e., for a traffic situation in which we have excluded the influence of junctions.
In a traffic model which is based on the application of the theory of dense gases and granular materials to the Boltzmann-like traffic model by Paveri-Fontana, the equilibrium relations [83, Eq. (103), (104)], [85, Eq. (20.60), (20.62)] have been suggested between (the equilibrium variables) $\rho_\mathrm{f}, \mu_v, \sigma_v$. Note that the formulas in [85] also take speed and variance variations between the lanes into account which we neglect here for simplicity. Setting $\beta = 0$ [83] (resp., $B = 0$ [85]), i.e., neglecting the effect which results from an overbraking caused by a slower vehicle in front, we obtain:

$$\mu_v(\rho_\mathrm{f}) = v_0 - \tau(\rho_\mathrm{f})(1 - p(\rho_\mathrm{f}))\rho_\mathrm{f}\chi(\rho_\mathrm{f})\sigma_v(\rho_\mathrm{f}), \qquad \sigma_v(\rho_\mathrm{f}) = \sqrt{\frac{A(\rho_\mathrm{f})\mu_v(\rho_\mathrm{f})}{1 - A(\rho_\mathrm{f})}}. \tag{2.17}$$

Here, v_0 is the average speed associated with the density $\rho_\mathrm{f} = 0$ (i.e., the speed limit in urban traffic), χ is a factor which takes into account the increase of the rate of interaction between vehicles caused by their finite space requirements (i.e., a space requirement > 0), τ denotes the effective relaxation time, p denotes the probability of overtaking and A denotes the individual fluctuation strength [85, Chapter 20.2]. Although the asymptotic behavior of these model functions can be derived, empirical data must generally be used to determine χ, τ, p, A. It has been verified that this approach allows a very precise description of the measured speed data, cp. [83], [84], [85].
While ρ, ρ_f can be determined from $\bar{v}$ using a few fit parameters (cf. (2.15) and (2.16)), the determination of μ_v, σ_v from (2.17) requires a number of model functions. We now describe a simplistic approach to determining these model functions from a few fit parameters. At this, we take the asymptotic and qualitative behavior into account which has been observed in empirical data. In order to maintain $\mu_v(0) = v_0$, $(\mathrm{d}\mu_v/\mathrm{d}\rho_\mathrm{f})(0) = 0$, $\mu_v(\rho_\mathrm{f}) \leq (\rho_\mathrm{max} - \rho_\mathrm{f})/(\rho_\mathrm{max}\rho_\mathrm{f}T_r)$ as well as $\mu_v(\rho_\mathrm{f}) \approx (\rho_\mathrm{max} - \rho_\mathrm{f})/(\rho_\mathrm{max}^2 T_r)$ for $\rho_\mathrm{f} \approx \rho_\mathrm{max}$ [85, Chapter 20.3], we choose the ansatz

$$\mu_v(\rho_\mathrm{f}) = \begin{cases} v_0 + v_2\rho_\mathrm{f}^2 + v_3\rho_\mathrm{f}^3, & \text{if } 0 < \rho_\mathrm{f} \leq \rho_0 \\ \dfrac{\rho_\mathrm{max} - \rho_\mathrm{f}}{\rho_\mathrm{max}^2 T_r}, & \text{if } \rho_0 < \rho_\mathrm{f} \leq \rho_\mathrm{max} \end{cases}. \tag{2.18}$$

Here, ρ_max is the maximal density, T_r is the reaction time of the driver [85, Chapter 20.3] and ρ_0 is the density at which the transition between the two asymptotic behaviors occurs. Once we have fixed $\rho_\mathrm{max}, T_r, v_0$, we determine ρ_0, v_2, v_3 in such a way that $\rho_\mathrm{f} \mapsto \mu_v(\rho_\mathrm{f})$ is continuously differentiable and $\mu_v(\hat{\rho}_\mathrm{f}) = \hat{\mu}_v$ for one measured pair $(\hat{\rho}_\mathrm{f}, \hat{\mu}_v)$.
Furthermore, using [85, Eq. (20.88)], we obtain

$$\sigma_v(\rho_\mathrm{f}) = \sqrt{\frac{\big(v_0 - \mu_v(\rho_\mathrm{f})\big)\big(1 - \rho_\mathrm{f}/\rho_\mathrm{max} - \rho_\mathrm{f}T_r\mu_v(\rho_\mathrm{f})\big)}{\tau(\rho_\mathrm{f})\big(1 - p(\rho_\mathrm{f})\big)\rho}}. \tag{2.19}$$

Note that we can now determine the fluctuation strength A from (2.18), (2.19) if the product $\tau(1-p)$ is known. In order to obtain the correct asymptotic behavior (cp. [85, Chapter 20.3]) of σ_v we must take into account that $p \in [0,1]$ and the relaxation time remains finite for $\rho_{\mathrm{f}} \to \rho_{\max}$. Taking into account [85, Eq. (20.112), Fig. 20.9], a quadratic interpolation of $\rho_{\mathrm{f}} \mapsto \tau(\rho_{\mathrm{f}})\big(1-p(\rho_{\mathrm{f}})\big)$ constitutes the lowest-order polynomial interpolation which yields a reasonable result. In particular, we suggest a fit function

$$\tau(\rho_{\mathrm{f}})\big(1-p(\rho_{\mathrm{f}})\big) \approx a_0(0.0143 + 0.0843\rho_{\mathrm{f}} + 0.0014\rho_{\mathrm{f}}^2) \tag{2.20}$$

with the parameter a_0 which can be chosen in such a way that $\sigma_v(\hat{\rho}_{\mathrm{f}}) = \hat{\sigma}_v$ for one measured pair $(\hat{\rho}_{\mathrm{f}}, \hat{\sigma}_v)$. The parameter a_0 corresponds to the scaling of the relaxation time in the absence of obstacles [85, Chapter 20.2].

Clearly, if more measurement data is available, more sophisticated methods can be applied to determine the model functions which are contained in (2.17). However, if few empirical data is available, the estimation of μ_v, σ_v from (2.18)-(2.20) seems to be a reasonable approach.

Assuming that σ_a and p_{t} do not depend on the time of day, we determine the value of $\overline{T}$ from (2.14), using the values of $\mu_v(t), \sigma_v(t)$ which we have determined from (2.15)-(2.20). Note that, following the derivation in [87], p_{t} and $\overline{T}$ depend on the utilizations of all road segments which discharge into the junction situated at the end of the considered road segment. However, if we assume that the utilization of all road segments which discharge into the junction increases in a similar manner, we may approximate the resulting effect by a variation of $\overline{T}$ only. An example of such distributions for varying average speeds is given in Appendix A, in which we apply the described methodology to traffic data measured in the German city of Ingolstadt.

2.4. Properties of the Network Functions

In this section, we derive some properties of the travel time and travel cost functions associated with the road segments in the road network.

2.4.1. Travel Times

The models which we have introduced in the preceding sections of this chapter lead to a stochastic description of the velocity plots of a vehicle passing through a road segment. An even more general approach is used in gas-kinetic traffic models [86, Section III.E], [40, Chapter 5], in which the velocity v of a vehicle on a road segment is a random variable depending on the position d on the road segment and the time t. In other traffic models (see also [97], for a review, see, e.g., [86], [40]) the focus is on the macroscopic characteristics of the traffic flow rather than on the characteristics of the vehicles which form the traffic flow. In these models, the velocity of the traffic flow on a road segment is commonly a deterministic variable depending on the position d on the road segment and the time t. Following these approaches (and thereby considering an even more general framework than in the preceding sections), we show that a physical interpretation of the travel time function which is derived from the velocity plots on a road segment satisfies the non-passing property [162], [44].

We first consider the deterministic case.

Lemma 2.4.1 *Let $v : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}_0^+$ be a continuous function which is globally Lipschitz-continuous in the first variable and let $L > 0$. For each $t \in \mathbb{R}$ define $d_t : [t, \infty) \to \mathbb{R}_0^+$,*

$$\begin{aligned} d_t'(s) &= v\big(d_t(s), s\big), \quad s \in [t, \infty), \\ d_t(t) &= 0. \end{aligned}$$

and denote $\tau : \mathbb{R} \to \mathbb{R}_0^+$,

$$\tau(t) = \min\Big\{\theta \in \mathbb{R}_0^+ : d_t(t+\theta) \geq L\Big\}. \tag{2.21}$$

Then, for all $t_1, t_2 \in \mathbb{R}$, there holds

$$t_2 \geq t_1 \quad \Longrightarrow \quad t_2 + \tau(t_2) \geq t_1 + \tau(t_1). \tag{2.22}$$

Proof Suppose that there exist $t_1, t_2 \in \mathbb{R}$ with $t_2 > t_1$, such that (2.22) does not hold. As v is nonnegative and $d_t(t) = 0$ for all $t \in \mathbb{R}$, we have $d_{t_i}(s) \geq 0$ for all $s \in [t_i, \infty)$, $i = 1, 2$. In particular, we have $d_{t_1}(t_2) \geq 0$. Next, according to (2.21), there exists a $s^+ \in (t_2, \infty)$ such that $d_{t_1}(s^+) > d_{t_2}(s^+)$. Consequently, there exists a $s_0 \in [t_2, s^+)$, such that $d_{t_1}(s_0) = d_{t_2}(s_0) = l_0$. Now, $d_{t_1}|_{[s_0,s^+]}$ and $d_{t_2}|_{[s_0,s^+]}$ satisfy the initial value problems

$$\begin{aligned} d_{t_i}'(s) &= v\big(d_{t_i}(s), s\big), \quad s \in [s_0, s^+], \\ d_{t_i}(s_0) &= l_0, \end{aligned}$$

$i = 1, 2$. Since v is continuous and globally Lipschitz-continuous in the first variable, the Picard-Lindelöf theorem [116, p.140] implies that $d_{t_1}|_{[s_0,s^+]} = d_{t_2}|_{[s_0,s^+]}$, a contradiction. □

Interpreting the quantities v, L in Lemma 2.4.1 as the velocity plot on and the length of the road segment, respectively, and interpreting τ as the travel time function, Lemma 2.4.1 implies that the physical modeling of the time-dependent road network implies the FIFO-property, cp. Definition 3.2.7.
We now address the stochastic modeling of the velocity plots. The following result implies the stochastic consistency of the (physically modeled) travel time functions in a stochastic time-dependent road network, cp. (6.1).

Lemma 2.4.2 *Denote by $\mathcal{C}_v$ the set of continuous functions mapping $\mathbb{R} \times \mathbb{R}_0^+$ to $\mathbb{R}_0^+$ which are globally Lipschitz-continuous in the first variable, denote by $\mathcal{C}_\tau$ the set of continuous functions mapping $\mathbb{R}$ to $\mathbb{R}_0^+$ and denote by $\mathcal{C}_d^1\big([t,\infty)\big)$ the set of continuously differentiable functions mapping $[t, \infty)$ to $\mathbb{R}_0^+$, $t \in \mathbb{R}$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $L > 0$, let $v : \Omega \to \mathcal{C}_v$ be a random variable and for each $t \in \mathbb{R}$ define $d_t : \Omega \to \mathcal{C}_d^1\big([t,\infty)\big)$,*

$$\begin{aligned} \frac{\mathrm{d}d_t(\omega)}{\mathrm{d}s}(s) &= v\big(d_t(\omega)(s), s\big), \quad s \in [t, \infty), \\ d_t(\omega)(t) &= 0. \end{aligned}$$

and denote $\tau : \Omega \to \mathcal{C}_\tau$,

$$\tau(\omega)(t) = \min\Big\{\theta \in \mathbb{R}_0^+ : d_t(\omega)(t+\theta) \geq L\Big\}. \tag{2.23}$$

Then, for all $t_1, t_2 \in \mathbb{R}$, there holds

$$t_2 \geq t_1 \quad \Longrightarrow \quad \mathbb{P}\big\{t_2 + \tau(t_2) \leq c\big\} \leq \mathbb{P}\big\{t_1 + \tau(t_1) \leq c\big\}, \quad \forall c \in \mathbb{R}. \tag{2.24}$$

Proof Since, for each $\omega \in \Omega$, $\tau(\omega)$ satisfies (2.22) according to Lemma 2.4.1, we obtain $\{\omega \in \Omega : t_2 + \tau(\omega)(t_2) \leq c\} \subset \{\omega \in \Omega : t_1 + \tau(\omega)(t_1) \leq c\}$ for all $c \in \mathbb{R}$. This implies (2.24). □

Let us now come back to the non-stochastic modeling of the travel times and relate the above results to the quantities which are contained in digital maps of the road network. Since the measurement of the position- and time-dependent vehicle speeds would require an appropriate measurement infrastructure to be available in large parts of the road network and an extensive amount of memory in order to be stored, such precise speed data are usually not contained in the digital maps which are available for routing purposes. Instead, a position-independent mean speed $\bar{v} : \mathbb{R} \to \mathbb{R}_0^+$ is commonly introduced, which is (in case of a non-stochastic modeling and using the notation of Lemma 2.4.1) defined as

$$\bar{v}(t) = \frac{L}{\tau(t)} = \frac{1}{\tau(t)} \int_t^{t+\tau(t)} v\big(d_t(s), s\big)\, \mathrm{d}s. \tag{2.25}$$

Hence, the time-dependent speed in the digital map is commonly a space- and time-average of the position- and time-dependent vehicle speeds on the road segment. Clearly, under the assumptions of Lemma 2.4.2, the travel time function $\tau(t) = L/\bar{v}(t)$ still satisfies the FIFO-property.

We finally discuss some of the impacts of the modeling in Section 2.2 on the travel time function τ. Let us assume that the distribution parameters $\mu_v, \sigma_v, \sigma_a, p_\mathrm{t}, \overline{T}$ have been fixed. Observe that, assuming that the free flow speed process is ergodic, the definition of the travel time $T_1(L)$ almost surely equals the average travel time of traveling infinitely often through the road segment on which E_1 occurs every time. In a more realistic modeling of τ, some probabilistic variation should be associated with $T_1(L)$ in order to take into account that the road segment is only traveled through once when the travel time τ is incurred. Similar considerations hold for $T_2(L, v_0), T_3(L, v_0, t_\mathrm{h}), T_4(L, v_0^+, v_0^-, t_\mathrm{h})$. Disregarding this observation, we obtain a compact positive range of the mapping

$$(k, v_0^+, v_0^-, t_\mathrm{h}) \mapsto \begin{cases} T_1(L), & \text{if } k = 1 \\ T_2(L, v_0^+), & \text{if } k = 2 \\ T_3(L, v_0^-, t_\mathrm{h}), & \text{if } k = 3 \\ T_4(L, v_0^+, v_0^-, t_\mathrm{h}), & \text{if } k = 4 \end{cases}. \tag{2.26}$$

Observe that, if the parameters $L = 150\,\mathrm{m}$, $\sigma_a = 1.5\,\mathrm{m/s^2}$, $\mu_v = 15\,\mathrm{m/s}$, $\overline{T} = 60\,\mathrm{s}$ have been fixed, then $T_1(L) = 10\,\mathrm{s}$ and $T_4(L, \mu_v, \mu_v, \overline{T}) \approx 82\,\mathrm{s}$. Hence, a very large variation of the travel times is caused by the presence of junctions in urban traffic. Although such variations are likely to average out over long routes, their magnitude renders a precise prediction of the travel time of the entire route almost impossible.

While we assume that the travel times are given by one value (e.g., defined by (2.14)) in the Chapters 3-5, we understand the travel time rather as a set-valued mapping (e.g., defined by (2.26)) in Section 6.

2.4.2. Travel Cost

In the following, we describe in which manner the models introduced in the Sections 2.1-2.3 can be combined in order to associate a fuel consumption with each road segment at each time. Let us assume that some parameters $L, p_{\mathrm{t}}, \overline{T}, \sigma_a, \mu_v, \sigma_v$ have been determined according to Section 2.3 for some road segment at some point in time t, and that these parameters correspond to a space- and time-average of the velocity plots of a vehicle departing on the road segment at time t. Let us further assume that the basic power consumption P_0 of the vehicle and the angle of elevation in longitudinal direction of the road segment α are constant, and that a gear changing behavior $(v,a) \mapsto \widetilde{n}(v,a)$ and a braking behavior $(v,a) \mapsto \widetilde{p}(v,a)$ have been fixed. Note that this implies that the effective acceleration (resp., deceleration) which must be applied by the motor is given by $\widetilde{e}(v,a)a$ for some $\widetilde{e}(v,a) \in \mathbb{R}$, cp. Section 2.1. Furthermore, we assume that the vehicle acceleration and deceleration A is normally distributed during the free flow behavior , i.e., $A \sim \mathcal{N}(0, \sigma_a^2)$. We denote by f_0, f_A, f_{A^+} the probability density functions associated with $\mathcal{N}(\mu_v, \sigma_v^2), \mathcal{N}(0, \sigma_a^2), \mathcal{HN}(0, \sigma_a^2)$, respectively.
In order to adapt the vehicle model introduced in Section 2.1, we must take into account that the power P which can be generated by a motor is bounded by some $P_{\min}, P_{\max}$, i.e., $P_{\min} \leq P \leq P_{\max}$. (For a combustion engine there holds $P_{\min} = 0$.) This results in a restriction $0 \leq v \leq v_{\max}$ of the vehicle speeds, $v_{\max} \in \mathbb{R}^+$, and a speed-dependent restriction of the effective accelerations $a_{\min}(v) \leq \widetilde{e}(v,a)a \leq a_{\max}(v)$, $a_{\min}(v), a_{\max}(v) : [0, v_{\max}] \to \mathbb{R}$ [129, Chapters 29, 32]. Since any reasonable set of parameters $\sigma_a, \mu_v, \sigma_v$ leads to a very small probability of violating these constraints we have neglected the thereby defined driving limits in the derivation of the speed- and acceleration distributions. In the following, we take the driving limits into account by setting

$$\widetilde{P}(v,a,\alpha) = \begin{cases} \dfrac{2\pi\tau_0\omega_0 + P_0}{\widetilde{\eta}(0,0,0)}, & \text{if} \quad v \leq 0 \\ \dfrac{\widetilde{F}(v,a,\alpha)v + P_0}{\widetilde{\eta}(v,a,\alpha)}, & \text{if} \quad v \in (0, v_{\max}] \text{ and } \widetilde{e}(v,a)a \in [a_{\min}(v), a_{\max}(v)] \\ P_{\min}, & \text{if} \quad v \in (0, v_{\max}] \text{ and } \widetilde{e}(v,a)a < a_{\min}(v) \\ P_{\max}, & \text{if} \quad v \in (0, v_{\max}] \text{ and } \widetilde{e}(v,a)a > a_{\max}(v) \\ & \quad\quad \text{or } v > v_{\max} \end{cases},$$

where $\widetilde{F}(v,a,\alpha)$ is the sum of the driving resistances and $\widetilde{\eta}(v,a,\alpha)$ is the product of the engine and the transmission efficiency, cp. (2.8). Note that the definitions in Section 2.1 imply that both quantities can be written as functions of the vehicle speed and acceleration if a gear-changing behavior and a braking behavior have been fixed.
With the event $\{E = E_1\}$ we associate the fuel consumption

$$B_1(L) = \int_{t=0}^{L/\mu_v} \int_{v=-\infty}^{\infty} \int_{a=-\infty}^{\infty} \frac{\widetilde{P}(v,a,\alpha)}{H_l} f_A(a) f_0(v) \, \mathrm{d}a \, \mathrm{d}v \, \mathrm{d}t,$$

Observe that, assuming that the free flow speed process is ergodic, the definition of the fuel consumption $B_1(L)$ almost surely equals the average fuel consumption associated with an infinitely often repeated travel through the road segment on which E_1 occurs every time.
Next, we associate the fuel consumption $B_2(L, v_0^+)$ with the event $\{E = E_2, V^+ = v_0^+\}$ on

the road segment. If $v_0^+ \leq 0$, we obtain

$$B_2(L, v_0^+) = \int_{t=0}^{L/\mu_v} \int_{v=-\infty}^{\infty} \int_{a=-\infty}^{\infty} \frac{\widetilde{P}(v, a, \alpha)}{H_l} f_A(a) f_0(v) \, \mathrm{d}a \, \mathrm{d}v \, \mathrm{d}t,$$

if $0 < v_0^+ \leq \sqrt{\sqrt{8/\pi} L \sigma_a}$, then

$$\begin{aligned} B_2(L, v_0^+) = & \int_{t=0}^{v_0^+/(\sqrt{2/\pi}\sigma_a)} \int_{a=0}^{\infty} \frac{\widetilde{P}(\sqrt{2/\pi}\sigma_a t, a, \alpha)}{H_l} f_{A^+}(a) \, \mathrm{d}a \, \mathrm{d}t \\ & + \int_{t=0}^{L/\mu_v - (v_0^+)^2/(\sqrt{8/\pi}\sigma_a \mu_v)} \int_{v=-\infty}^{\infty} \int_{a=-\infty}^{\infty} \frac{\widetilde{P}(v, a, \alpha)}{H_l} f_A(a) f_0(v) \, \mathrm{d}a \, \mathrm{d}v \, \mathrm{d}t, \quad (2.27) \end{aligned}$$

and if $v_0^+ > \sqrt{\sqrt{8/\pi} L \sigma_a}$, then

$$B_2(L, v_0^+) = \int_{t=0}^{\sqrt{\sqrt{2\pi} L/\sigma_a}} \int_{a=0}^{\infty} \frac{\widetilde{P}(\sqrt{2/\pi}\sigma_a t, a, \alpha)}{H_l} f_{A^+}(a) \, \mathrm{d}a \, \mathrm{d}t. \quad (2.28)$$

Note that, assuming again that both the acceleration and the free flow speed process are ergodic, the definition of the fuel consumption $B_2(L, v_0^+)$ almost surely equals the average fuel consumption associated with an infinitely often repeated travel through the road segment on which E_2, v_0^+ occur every time. Similarly, $\mathbb{E}[B_2(L, V_0^+)]$ can be interpreted as an infinitely often repeated travel through the road segment on which E_2 occurs every time. By replacing $\widetilde{P}(\sqrt{2/\pi}\sigma_a t, a, \alpha)$ by $\widetilde{P}(\sqrt{2/\pi}\sigma_a t, -a, \alpha)$ in (2.27) and (2.28) and by adding $t_\mathrm{h}\widetilde{P}(0, 0, 0)/H_l$ to the resulting fuel consumption (cp. (2.9)), we obtain the fuel consumption $B_3(L, v_0^-, t_\mathrm{h})$ for the event $\{E = E_3, V_0^- = v_0^-, T_\mathrm{h} = t_\mathrm{h}\}$. Finally, we establish the fuel consumption $B_4(L, v_0^+, v_0^-, t_\mathrm{h}) = B_2(L/2, v_0^+) + B_3(L/2, v_0^-, t_\mathrm{h})$ for the event $\{E = E_4, V_0^+ = v_0^+, V_0^- = v_0^-, T_\mathrm{h} = t_\mathrm{h}\}$. Disregarding the observation that each of the values $B_1(L), B_2(L, v_0^+), B_3(L, v_0^-, t_\mathrm{h}), B_4(L, v_0^+, v_0^-, t_\mathrm{h})$ corresponds to the average fuel consumption over an infinite number of times the vehicle is traveling through the road segment, we obtain a compact range of the mapping

$$(k, v_0^+, v_0^-, t_\mathrm{h}) \mapsto \begin{cases} B_1(L), & \text{if } k = 1 \\ B_2(L, v_0^+), & \text{if } k = 2 \\ B_3(L, v_0^-, t_\mathrm{h}), & \text{if } k = 3 \\ B_4(L, v_0^+, v_0^-, t_\mathrm{h}), & \text{if } k = 4 \end{cases} . \quad (2.29)$$

In case of a combustion engine we have $P_\mathrm{min} = 0$ which results in a compact nonnegative range of the fuel consumption defined by (2.29). Let us illustrate this range by considering the parameters of the electric vehicle which are provided in Table A.1, assuming that the brake pedal is never actuated, $\alpha, \tau_0 = 0$, and assuming that $L = 150\,\mathrm{m}$, $\sigma_a = 1.5\,\mathrm{m/s^2}$, $\mu_v = 15\,\mathrm{m/s}$, $\sigma_v = 4\,\mathrm{m/s^2}$, $\overline{T} = 60\,\mathrm{s}$ have been fixed. Then $B_1(L) \approx 0.06\,\mathrm{kWh}$ and $B_4(L, \mu_v, \mu_v, \overline{T}) \approx 0.14\,\mathrm{kWh}$. The same magnitude of variation of the fuel consumption was observed in [103, Table 2] between free and congested traffic.

Finally, we define

$$\begin{aligned} B = & (1-p_\mathrm{t})^2\mathbb{E}\big[B_1(L)\big] + (1-p_\mathrm{t})p_\mathrm{t}\Big(\mathbb{E}\big[B_2(L,V_0^+)\big] + \mathbb{E}\big[B_3(L,V_0^-,T_\mathrm{h})\big]\Big) \\ & + p_\mathrm{t}^2\mathbb{E}\big[B_4(L,V_0^+,V_0^-,T_\mathrm{h})\big]. \end{aligned} \tag{2.30}$$

While we assume that the travel costs β are given by one value (e.g., defined by (2.30)) in the Chapters 3-5, we understand the travel costs rather as a set valued-mapping (e.g., defined by (2.29)) in Chapter 6. By choosing the distribution parameters, the gear changing behavior and the braking behavior in such a way that the characteristics of the traffic situation and the driver are taken into account, we can therefore associate a road-, traffic- and driver-dependent consumption value to each road segment in the road network. (Note that driving patterns related to speed, acceleration and gear-changing behavior were identified as the five most significant influence factors on the exhaust emissions in [34].)
Depending on whether the motor is switched off, idling, or fuel is being tanked up while the vehicle is at rest, we define the cost $\delta(\Delta t)$ of waiting for some time $\Delta t \in \mathbb{R}_0^+$ according to $\delta(\Delta t) = 0$, according to (2.9) or according to $\delta(\Delta t) = q\Delta t$, $q < 0$, respectively. (Here, q can be interpreted as the fuel flow rate during the tanking process.)
We conclude this section by discussing the cost of driving in a circle of length L in the road network. In order to derive a lower bound of this cost we assume that $p, P_0 \equiv 0$ (i.e., no energy is lost by an actuation of the brake pedal or auxiliary consumers). We further assume that a gear-changing behavior has been fixed. Given a vehicle speed v and a vehicle acceleration a, we denote by $\widetilde{\lambda}(v,a)$ the resulting molding body surcharge factor, cf. Section 2.1. Let us assume that a differentiable velocity plot $v : [0,T] \to \mathbb{R}^+$, $T \in \mathbb{R}^+$, is given, let $d : [0,T] \to \mathbb{R}_0^+$, $d(t) = \int_0^t v(s)\,\mathrm{d}s$, denote the associated distance plot with $d(T) = L$, and let a differentiable height plot $h : [0,L] \to \mathbb{R}$ be given with $h(0) = h(L)$. Denote $v_d : [0,L] \to \mathbb{R}^+$, $v_d(l) = v(d^{-1}(l))$, $a_d : [0,L] \to \mathbb{R}^+$, $a_d(l) = v'(d^{-1}(l))$. By substituting $l = d(t)$ in (2.8) (which equals the energy consumption in (2.10) if $H_l = 1$) and using (2.2)-(2.6), we obtain

$$B = \int_0^L \frac{m_v g \sin\big(\alpha(l)\big) + m_v g c_{rr}\cos\big(\alpha(l)\big) + 0.5\rho_a A_f c_d v_d(l)^2 + \widetilde{\lambda}(v_d(l), a_d(l)) m_v a_d(l)}{H_l \widetilde{\eta}(v_d(l), a_d(l))}\,\mathrm{d}l,$$

where $\alpha(l) = \arctan(h'(l))$. Taking into account the constraints $h_d(0) = h_d(L)$, $0 \leq v_d \leq v_{\max}$, $a_{\min}(v_d) \leq \widetilde{e}(v_d, a_d)a_d \leq a_{\max}(v_d)$, $-\alpha_{\max} \leq \alpha \leq \alpha_{\max}$ for some $\alpha_{\max} \in \mathbb{R}^+$, one might derive a lower bound for any circle of length L in the road network by minimizing over all function h, v_d out of appropriate function spaces. Closed solutions can easily be derived if $\widetilde{\lambda}, \widetilde{\eta}$ are constant functions, and solutions have also been found for given height plots h [113], [95]. However, this shall not be topic of this thesis. We only note that, since the potential energy in the initial and terminal position in one traversal of the circle coincide and the kinetic energy of the vehicle is bounded, there exists a $N \in \mathbb{N}$, such that the total amount of energy incurred for traveling N times through the circle is positive. For simplicity, we assume that $N = 1$ in this thesis.

Part II.

The Time-Dependent Optimal Path Problem

3. Time-Dependent Networks

Many applications in which networks are used for modeling or optimization purposes, such as, e.g., intelligent transportation systems [101], [38], [153], internet routing [124], multi-agent-systems [136] and networked control systems [17], involve time-dependencies whose incorporation into the network model lead to the notion of time-dependent networks. In these systems, there need to be considered either a time-dependent variation of the travel times and travel costs along the edges of the network, a time-dependent change in the topology and connectivity of the network, or both of these features. This allows not only for a more realistic modeling of the underlying physical systems, but gives also rise to a great number of new phenomena. In Section 3.1 we motivate the introduction of time-dependent networks based on the example of the road network. In the following sections of this chapter, we introduce time-dependent networks, discuss time constraints and turn restrictions, and define certain classes of time-dependent networks which are of particular relevance in practical and theoretical considerations.

3.1. The Road Network as a Time-Dependent Network

The main task of an automotive navigation system is to provide the user with guiding instructions, which conduct the driver of the vehicle from his present location to a predetermined destination. These instructions must take the structure of the road network into account, be consistent with the local traffic rules, and should determine a route of a certain quality. Common measures for quality are the traveled distance, the time required to reach the destination and the number of turn maneuvers involved. In order to compute such a route a digital map is required, which contains all information which is necessary for the determination of the present location, the route and the route guidance. Before focusing on the mathematical description of time-dependent networks, we give a brief overview of the digital map which is the data base according to which the time-dependent network is defined.

Digital maps of the road network usually contain a large number of information which are not relevant for the definition of the time-dependent network. For instance, information about green spaces, lakes, rivers or residential areas are only of geographical interest and are commonly used only for the map display. Similarly, geometrical information about the precise course of a certain road may be useful for the determination of the current position of the vehicle, but are not relevant for the computation of a route, since the vehicle is constrained to stay on the road. The integral component of the digital map which is essential for the computation of a route in the road network is the so-called topological information, containing the description which road segments are connected by junctions. An appropriate mathematical model for this description is a graph, in which the junctions correspond to the nodes and the road segments correspond to the edges of the graph. In order to take into account that certain roads, such as, e.g., one-way roads, can only be traveled in one direction, it is convenient to use a directed graph. In the directed graph of the road network, a directed edge between two nodes is introduced if the road segment which connects the corresponding junctions can be passed through in the respective direction.

In the digital map certain attributes such as distance, travel time or travel cost are associated to each road segment and each junction of the road network. These attributes serve as a means for determining the quality of a route and as a basis of decision-making in the computation of optimal routes. The incorporation of these attributes into the description of the directed graph leads to a weighted and directed graph, in which a cost value (or a vector of cost values) is associated with each node and edge. Weighted and directed graphs are also termed networks [79].

Certain attributes, such as travel time or fuel consumption, depend on the current traffic situation on the respective road segment or at the respective junction, cp. Part I. Since the traffic situation underlies temporal and seasonal changes, the latest digital maps contain a time-dependent description of the corresponding cost values, which are derived from historical traffic data [41], [119], see also Section 2.3. In order to compute routes which are optimal with respect to the predictable variations in the traffic situation, the cost values associated with a node or segment in the network must be replaced by cost functions which depend on the point in time at which the vehicle is predicted to reach the respective node or segment. This leads to the definition of time-dependent networks in Section 3.2. In the context of time-dependent network flows a model has been developed in which the travel time and cost functions depend on the current state of the network, i.e., the current traffic flow in the road network [114]. Since our interest is in the determination of an optimal route for a single vehicle and not a set of optimal routes for a fleet of vehicles, we assume that the network functions are not affected by the routing decision and depend only on the time variable.

The incorporation of time-dependent travel time and travel cost functions allows an anticipatory optimization of routes in the road network. In the past, the computation of optimal routes in the road network was only based on static cost criteria and dynamic traffic information replacing the cost values contained in the digital map. Such dynamic traffic information is for instance provided by the traffic message channel (TMC), transmitting additional attributes for certain road segments in the case of exceptional traffic situations, such as, e.g., traffic jams [177, Chapters 8, 11]. Since these attributes can only be used for the correction of the static cost values when an exceptional situation has already occurred, only near-term reoptimization of the currently traveled route is possible. An anticipatory avoidance of certain road segments which are jammed during each rush-hour is only possible if a time-dependent network is used for the computation of the optimal route. Assuming that all traffic changes are predictable from historical traffic data, the time-dependent approach clearly leads to routes of better quality.

However, not all changes in the traffic situation are predictable. Random - or at least hardly predictable - events, such as accidents and weather conditions significantly codetermine the traffic situation. Considering traffic models which incorporate a human element of uncertainty, even the same conditions may lead to different situations (cf., e.g., [102, Chapter 2.4], [86, p.1088, eq. (15)], [156]). A realistic mapping of the influence of the traffic situation on the computation of optimal routes in the road network must therefore result in a time-dependent and uncertain model. We introduce such a model in Chapter 6. Of course, dynamic traffic information can be used to further improve the network description based on current exceptional events.

3.2. Notation

In this section, we introduce time-dependent networks as well as some basic notation from graph theory and control theory. Moreover, we present a network transformation which allows the decoupling of the discrete and continuous state transitions in the time-dependent network. Finally, we introduce the FIFO-property of time-dependent networks, which has a strong impact on the complexity of computing optimal paths. The following definition of a continuous-time time-dependent network is similar to the definition of a discrete-time time-dependent network in [36].

Definition 3.2.1 *A time-dependent network is a quintuple $G = (V, E, \tau; \beta, \delta)$, where V denotes the (finite) set of nodes, E the (finite) set of directed edges (i.e., E is a multiset over the cartesian product $V \times V$), $\tau : E \times \mathbb{R} \to \mathbb{R}_0^+$ the travel time function, $\beta : E \times \mathbb{R} \to \mathbb{R}$ the travel cost function and $\delta : V \times \mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}$ the waiting cost function.*

Remark 3.2.2 *The second argument of τ, β denotes the departure time on the edge determined by the first argument. The third argument of δ denotes the waiting time at the node specified by the first argument, whereas the waiting time interval begins at the time determined by the second argument.*

The definition of the time-dependent network, Definition 3.2.1, allows the directed graph (V, E) to be a multigraph. In particular, there may be more than one edge $e \in E$ connecting the same two nodes $v, v' \in V$. In order to express that $e \in E$ is one element of the multiset corresponding to $(v, v') \in V \times V$, we denote $e \sim (v, v')$. Furthermore, we denote the cardinality of a set S by $|S|$, and the power set of S by $\mathcal{P}(S)$. The following definition introduces some basic objects of graph theory.

Definition 3.2.3 *The head function is denoted by $\omega : E \to V$ and the tail function is denoted by $\alpha : E \to V$ (i.e., if $e \sim (v, v')$, then $\alpha(e) = v$ and $\omega(e) = v'$).*
For each $v \in V$, the set $E^-(v)$ of edges terminating in v and the set $E^+(v)$ of edges emanating from v are defined as

$$E^-(v) = \{e \in E : e \sim (v', v), v' \in V\},$$
$$E^+(v) = \{e \in E : e \sim (v, v'), v' \in V\},$$

respectively. By $\deg^+(v) = |E^+(v)|$, we denote the outdegree of v, by $\deg^-(v) = |E^-(v)|$ we denote the indegree of v.
For each $v \in V$, the set $V^-(v)$ of the predecessors of v and the set $V^+(v)$ of the successors of v are defined as

$$V^-(v) = \{v' \in V : \exists e \in E, e \sim (v', v)\},$$
$$V^+(v) = \{v' \in V : \exists e \in E, e \sim (v, v')\},$$

respectively.

In the context of static networks, the set of nodes V can be understood as the set of possible states, whereas the set of edges E specifies the possible state transitions. Since in a time-dependent network the travel time function τ and the cost functions β, δ depend on the time variable, it is convenient to define the state space $X = V \times \mathbb{R}$. In particular, each state

$x \in X$ is a pair (v, t), consisting of a node $v \in V$ and a time $t \in \mathbb{R}$.
In many applications, such as, e.g., railway networks, it may be necessary and optimal to wait at certain states in the network, before traversing the next edge. For this reason, we specify the state transitions in the time-dependent network by a pair $u = (\Delta t, e)$, consisting of a waiting time $\Delta t \in \mathbb{R}_0^+$ and an edge $e \in E$. In view of the control-theoretic formulation of the optimal path problem, we call the pair $u = (\Delta t, e)$ a control action. Similar to static networks, a control action $u = (\Delta t, e)$ can only be applied at states $x = (v, t) \in X$ with $e \in E^+(v)$. The state transition resulting from the application of such a control action u in the state x is specified by the control-to-state mapping $\varphi : \bigcup_{(v,t)\in V\times\mathbb{R}} \big\{\{(v,t)\} \times \{\mathbb{R}_0^+ \times E^+(v)\}\big\} \to V \times \mathbb{R}$,

$$\varphi\big((v,t),(\Delta t, e)\big) = \big(\omega(e), t + \Delta t + \tau(e, t + \Delta t)\big). \tag{3.1}$$

In particular, the application of $(\Delta t, e)$ in (v, t) corresponds to the actions of waiting at v during the time interval $[t, t + \Delta t]$ and traversing e at time $t + \Delta t$. Note that this definition excludes the possibility of waiting for several time intervals at some node v before departing on some edge $e \in E^+(v)$. In case of cumulative waiting cost functions, i.e., if the cost of waiting is defined as the integral over a time-dependent waiting cost potential, the exclusion of multiple waiting is no restriction, since the cost of multiple waiting equals the cost of waiting once for the entire time interval [47, p.21], [138]. This approach is commonly used in the context of time-dependent network flows [158], [133] and the characterization of optimal time-dependent paths by linear programming [142].
Our presentation of time-dependent networks is closely related to hybrid control theory [32], [82]. In time-dependent networks, both the state variable $x = (v, t)$ and the control variable $u = (\Delta t, e)$ are composed of a discrete variable ($v \in V$ and $e \in E$, respectively) and a continuous variable ($t \in \mathbb{R}$ and $\Delta t \in \mathbb{R}_0^+$, respectively). The continuous time variable t fulfills the initial value problem $\dot{t} = 1$, $t(0) = 0$. In the framework of hybrid control, the computation of optimal paths corresponds to the computation of optimal control strategies, and the cost values of these paths define the optimal value function. To the best of our knowledge, the control-theoretic approach has not yet been applied in the literature of time-dependent networks. However, since it allows for a simple notation and an clearer presentation of the dynamic programming equations, we will pursue this approach here.
One important question in mathematical systems theory is in which manner the system behaves, depending on a variation of the initial state or a variation of the system input [88, Chapter 3]. The most elementary concept on which such considerations are based is the notion of continuity, i.e., whether the resulting state depends continuously on the initial state and the control input, cf. [88, Definition 3.1.1.]. We equip the discrete sets V, E with the discrete topology and (subsets of) $\mathbb{R} \cup \{\pm\infty\}$ with the order topology generated by the open intervals and open rays. It is easily seen that φ is continuous if τ is continuous, at which the continuity of τ is equivalent to the continuity of the partial functions $t \mapsto \tau(e, t)$ for all $e \in E$.
In the following, we will consider the problem of computing optimal paths in a time-dependent network with state space and control constraints. Such constraints arise in many applications. For example, taking into account an earliest departure time and a latest arrival time in route-planning applications already constrain the set of admissible arrival times at the nodes of the road network. Moreover, there may be only a few places in the road network, such as, e.g., parking lots, which allow the driver to wait for a certain amount

of time before continuing his travel. Consequently, both the arrival times and the waiting times at the nodes are subject to constraints in certain applications. Note that turn restrictions constitute a third kind of constraint, which we will deal with in Section 3.4.
We now precisely formulate the state space and control constraints. To each $v \in V$, we associate a set $T(v) \subset \mathbb{R}$ of admissible arrival times. The resulting state space is

$$X = \bigcup_{v \in V} \big\{\{v\} \times T(v)\big\}. \tag{3.2}$$

We associate a nonempty set $\Delta T(x) \subset \mathbb{R}_0^+$ of admissible waiting times to each state $x \in X$. Note that $v \mapsto T(v)$ defines a point-to-set mapping $T : V \to \mathcal{P}(\mathbb{R})$ and $x \mapsto \Delta T(x)$ defines a point-to-set mapping $\Delta T : X \to \mathcal{P}(\mathbb{R}_0^+)$.

Definition 3.2.4 *A control action $u \in \mathbb{R}_0^+ \times E$ is admissible for a given state $x = (v, t) \in X$, if*

$$u \in \Delta T(x) \times E^+(v) \quad \text{ and } \quad \varphi(x, u) \in X. \tag{3.3}$$

A mapping $\mu : X \to \mathbb{R}_0^+ \times E$ is called an admissible control policy if, for all $x \in X$, the control action $u = \mu(x)$ is admissible for x.

Remark 3.2.5 *In view of equation (3.3), a control u which is admissible for $x = (v, t) \in X$ must account for the control constraint $u \in \Delta T(x) \times E^+(v)$, as well as the state space constraint $\varphi(x, u) \in X$. This allows a flexible network description, since both the control constraints and the state space constraints may change independently of one another in different applications.*

Remark 3.2.6 *The incorporation of the state space constraints allows an implicit incorporation of a varying network topology as follows: Suppose that associated with each edge in the network there is a set of points in time at which the traversal of the respective edge is admissible. We then replace each edge $e \in E$ by a virtual node v_e, a virtual edge $e_1 \sim (\alpha(e), v_e)$ and the edge $e_2 \sim (v_e, \omega(e))$ as depicted in Figure 3.1. The travel time and cost on e_1, e_2 are given by*

$$\tau(e_1, t) = 0, \qquad \tau(e_2, t) = \tau(e, t),$$
$$\beta(e_1, t) = 0, \qquad \beta(e_2, t) = \beta(e, t),$$

for all $t \in \mathbb{R}$, respectively. Moreover $\Delta T(v_e, t) = \{0\}$ and $\delta(v_e, t, 0) = 0$ for all $t \in \mathbb{R}$. We define $T(v_e)$ as the set of all points in time at which the traversal of the edge e is admissible. In the resulting network, all constraints on the traversal times of the edges are formulated as state space constraints. We will henceforth only consider networks with a fixed topology.

We associate the cost $\delta(v, t, \Delta t) + \beta(e, t + \Delta t)$ with the application of a control action $(\Delta t, e) \in \mathbb{R}_0^+ \times E$ in $(v, t) \in V \times \mathbb{R}$. This sum corresponds to the cost of waiting at v during the time interval $[t, t + \Delta t]$ in addition to the cost of traversing e at time $t + \Delta t$. In order to illustrate the sequential structure of the corresponding state transitions, the split network has been introduced in [47, p.27]. Since the split network turns out to be very useful for algorithmic purposes, we now recapitulate its construction. For simplicity, we assume that the directed graph (V, E) contains no multiple edges.

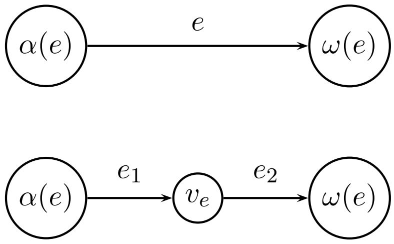


Figure 3.1.: Replacing the edge e by the virtual node v_e and the edges e_1, e_2.

Each original node $v \in V$ is split into two virtual nodes v_{w} and v_{nw} with

$$E^-(v_{\mathrm{w}}) = \{(v'_{\mathrm{nw}}, v_{\mathrm{w}}) : \exists e \in E, e \sim (v', v)\}, \qquad E^+(v_{\mathrm{w}}) = \{(v_{\mathrm{w}}, v_{\mathrm{nw}})\},$$
$$E^-(v_{\mathrm{nw}}) = \{(v_{\mathrm{w}}, v_{\mathrm{nw}})\}, \qquad E^+(v_{\mathrm{nw}}) = \{(v_{\mathrm{nw}}, v'_{\mathrm{w}}) : \exists e \in E, e \sim (v, v')\}.$$

The node set V_{s} and edge set E_{s} of the transformed graph is defined accordingly. Next,

$$T_{\mathrm{s}}(v_{\mathrm{w}}) = T(v), \qquad \Delta T_{\mathrm{s}}(v_{\mathrm{w}}, t) = \Delta T(v, t),$$
$$T_{\mathrm{s}}(v_{\mathrm{nw}}) = \mathbb{R}, \qquad \Delta T_{\mathrm{s}}(v_{\mathrm{nw}}, t) = \{0\}.$$

Note that the arrival time at v now corresponds to the arrival time at v_{w}, and all resulting time restrictions at v_{nw} are implicitly taken into account by the sets of admissible waiting times associated with v_{w}. Clearly, the definition of the control constraints implies that waiting is only possible at v_{w}. The set of all waiting nodes of the split network is denoted by V_{w} and the set of all nodes of the split network at which waiting is prohibited is denoted by V_{nw}. Finally, we define the travel time function τ_{s} and the cost functions $\beta_{\mathrm{s}}, \delta_{\mathrm{s}}$ of the split network for all $t \in \mathbb{R}$.

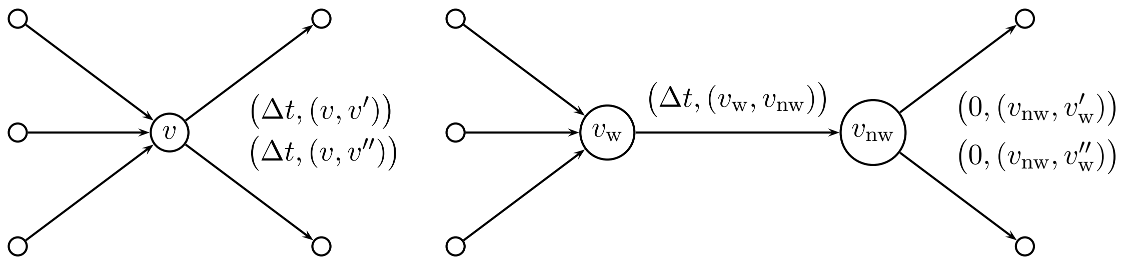
$$\tau_{\mathrm{s}}(e_{\mathrm{s}}, t) = \begin{cases} 0, & \text{if } e_{\mathrm{s}} \sim (v_{\mathrm{w}}, v_{\mathrm{nw}}) \text{ for some } v \in V \\ \tau(e, t), & \text{if } \exists e \in E, e \sim (v, v') \text{ with } e_{\mathrm{s}} \sim (v_{\mathrm{nw}}, v'_{\mathrm{w}}) \end{cases},$$
$$\beta_{\mathrm{s}}(e_{\mathrm{s}}, t) = \begin{cases} 0, & \text{if } e_{\mathrm{s}} \sim (v_{\mathrm{w}}, v_{\mathrm{nw}}) \text{ for some } v \in V \\ \beta(e, t), & \text{if } \exists e \in E, e \sim (v, v') \text{ with } e_{\mathrm{s}} \sim (v_{\mathrm{nw}}, v'_{\mathrm{w}}) \end{cases},$$
$$\delta_{\mathrm{s}}(v_{\mathrm{w}}, t, \Delta t) = \delta(v, t, \Delta t), \qquad \forall \Delta t \in \mathbb{R}_0^+, \qquad \delta_{\mathrm{s}}(v_{\mathrm{nw}}, t, 0) = 0.$$

This transformation is illustrated in Figure 3.2. It is easily seen that the split network is a time-dependent network in the sense of Definition 3.2.1 and that there is a one-to-one correspondence between $x \in X$, $u \in U(x)$ and the corresponding state and control pairs in the split network. Note that the split network never has to be explicitly generated, it rather serves as an illustrative model on which the algorithms in this thesis are based, cf. Chapter 8 and Chapter 9. For simplicity of notation, we will henceforth omit the subscript "s" when discussing the split-network.

In the following definition, we introduce a class of time-dependent networks, which has been subject to extensive research, especially in the context of the minimization of the travel time [137], [39], [49].

Definition 3.2.7 *A function $f : \mathbb{R} \to \mathbb{R}$, satisfies the FIFO-property, if for all $t, t' \in \mathbb{R}$, there holds*

$$t' \geq t \quad \Longrightarrow \quad t' + f(t') \geq t + f(t). \tag{3.4}$$



(a) Original node and control actions (b) Virtual nodes and control actions

Figure 3.2.: Splitting of the node v into the virtual nodes $v_{\mathrm{w}}, v_{\mathrm{nw}}$. The labels of the arrows correspond to the control actions, which are applied at their tails, respectively.

f satisfies the strong FIFO-property, if for all $t, t' \in \mathbb{R}$, there holds

$$t' > t \quad \Longrightarrow \quad t' + f(t') > t + f(t). \tag{3.5}$$

A time-dependent network $G = (V, E, \tau; \beta, \delta)$ satisfies the FIFO-property, if for all $e \in E$ the partial mappings $t \mapsto \tau(e, t)$ satisfy (3.4). G satisfies the strong FIFO-property, if for all $e \in E$ the partial mappings $t \mapsto \tau(e, t)$ satisfy (3.5). Time-dependent networks which satisfy the (strong) FIFO-property are called (strong) FIFO-networks.

It has been shown that the FIFO-property has a strong impact on the structure of fastest paths and on the complexity of the computation of fastest paths in time-dependent networks [137]. We will generalize these results to the case in which the optimization is subject to state space and control constraints in Section 3.5. Moreover, we will prove that the FIFO-property has a strong impact on the complexity of computing the optimal value function if all network functions are piecewise linear, cf. Section 5.3. Note that Lemma 2.4.1 implies that the road network satisfies the FIFO-property, which is also known as the non-passing property in the field of intelligent transportation systems [162].

3.3. Paths in Time-Dependent Networks

In graph theory, a path is defined as a concatenated sequence of nodes of the graph [92]. In a time-dependent context, this sequence must not only consist of nodes, but also of the corresponding points in time. In our setting, we may identify these paths with the trajectories in the state space, when a sequence of admissible controls is applied in an initial state $x_0 \in X$.

Definition 3.3.1 *A sequence of controls $u = (u_k)_{k=1,2,...}$ is called admissible for a given state $x_0 \in X$, if u_k is admissible for x_{k-1}, $k = 1, 2, ...$, where*

$$x_k = \varphi(x_{k-1}, u_k), \quad k = 1, 2, \dots . \tag{3.6}$$

The set of all finite control sequences which are admissible for x_0 is denoted by $U(x_0)$.
A path p is a sequence of states $p = (x_0, x_1, ...)$, such there exists a sequence of controls $u = (u_k)_{k=1,2,...} \in U(x_0)$, which fulfills (3.6).

Remark 3.3.2 *Obviously, a path p is uniquely determined by the control-to-state mapping, if an initial state x_0 and an admissible control sequence $u \in U(x_0)$ are given, cf. (3.1), (3.6). In the following, we denote $p = \Phi(x_0, u)$, and call Φ the control-to-path mapping. The set of all paths in G subject to the time constraints $T, \Delta T$ is denoted by*

$$P = X \cup \Phi \left(\bigcup_{x \in X} \{\{x\} \times U(x)\} \right). \tag{3.7}$$

Note that P includes paths p of length 0, which correspond to the states in the time-dependent network.

Let $x_0 \in X$, $u \in U(x_0)$ and $p = \Phi(x_0, u)$ with $u = (u_k)_{k=1,...,n}$, $p = (x_k)_{k=0,...,n}$. We define the length of the control sequence u by $|u| = n$ and denote the k-th control action u_k of u by u_k, $k = 1, ..., n$. Similarly, we define the (topological) length of the path p by $|p| = n$ and denote the k-th state x_k of p by p_k, $k = 0, ..., n$. Moreover, we denote the subsequences $(u_k)_{k=i,...,j}$, $1 \leq i \leq j \leq |u|$, and $(p_k)_{k=i,...,j}$, $0 \leq i \leq j \leq |p|$, by $u_{i:j}$ and $p_{i:j}$, respectively. For any path $p \in P$ and any $i, j \in \mathbb{N}$ with $0 \leq i \leq j \leq |p|$, $p_{i:j}$ is called a subpath of p. Finally, with a slight abuse of notation, for $u_{|u|} = (\Delta t, e)$ we denote $\omega(u) = \omega(e)$.
In a similar manner as in time-independent networks we define simple time-dependent paths and circles.

Definition 3.3.3 *If $p = \big((v_k, t_k)\big)_{k=0,...,n} \in P$ is a time-dependent path, then the sequence of nodes $(v_k)_{k=0,...,n}$ is called the topological path associated with p.*
p is called simple, if $(v_k)_{k=0,...,n}$ is simple, i.e., if $v_k \neq v_l$ for all $k, l \in \{0, ..., n\}$ with $k \neq l$.
p is called a circle, if $(v_k)_{k=0,...,n}$ is a circle, i.e., if $v_0 = v_n$.

As in the time-independent case, a time-dependent path p is simple if and only if no subpath of p is a circle.
With the application of the finite control sequence $u = \big((\Delta t_k, e_k)\big)_{k=1,...,|u|}$ in x_0, we associate the sum of the costs of the application of each control action in the corresponding state on the path $\Phi(x_0, u) = \big((v_k, t_k)\big)_{k=0,...,|u|}$. In order to ease the notation, we introduce the path cost function $\mathcal{B} : \bigcup_{x \in X} \big\{\{x\} \times U(x)\big\} \to \mathbb{R}$,

$$\mathcal{B}(x_0, u) = \sum_{k=1}^{|u|} \big[\delta(v_{k-1}, t_{k-1}, \Delta t_k) + \beta(e_k, t_{k-1} + \Delta t_k)\big]. \tag{3.8}$$

Definition 3.3.4 *A path p^* is called optimal with respect to an initial state $x_0 \in X$ and a goal node $v' \in V$, if there exists a finite control sequence $u^* \in U(x_0)$ with $p^* = \Phi(x_0, u^*)$, $\omega(u^*) = v'$ and*

$$\mathcal{B}(x_0, u^*) = \inf \big\{\mathcal{B}(x_0, u) : u \in U(x_0), \omega(u) = v'\big\}. \tag{3.9}$$

In this case, u^* is called an optimal control sequence (with respect to x_0 and v'). For a fixed goal node $v' \in V$, we define the optimal value function $b^* : X \to \mathbb{R} \cup \{\pm\infty\}$ by

$$b^*(x_0) = \inf \big\{\mathcal{B}(x_0, u) : u \in U(x_0), \omega(u) = v'\big\}. \tag{3.10}$$

Remark 3.3.5 *By convention, we set* $\inf_{u\in U} f(u) = \infty$ *and* $\sup_{u\in U} f(u) = -\infty$ *if* $U = \emptyset$ *and* f *is a real-valued function depending on* u*. This implies, that* $b^*(x_0) = \infty$ *if and only if there exists no finite path from* x_0 *to* $v' \times T(v')$*.*

Remark 3.3.6 *In order to obtain* $b^*(v', t) = 0$ *for all* $t \in T(v')$*, we assume that there is one additional control action "termination" in* $U(v', t)$*. The application of this control action generates no additional cost and leads to a terminal state in which no further control action can be applied, cp. [25, Chapter 2.1].*

In Definition 3.3.4, the optimality relates to a given goal node and a varying initial state. In the same manner, we define an optimal value function with respect to a given source node and a varying terminal state. In order to facilitate the notation, we introduce the path travel time function $\mathcal{T} : \bigcup_{x\in X} \big\{\{x\} \times U(x)\big\} \to \mathbb{R}_0^+$,

$$\mathcal{T}(x_0, u) = \sum_{k=1}^{|u|} \big[\Delta t_k + \tau(e_k, t_{k-1} + \Delta t_k)\big], \tag{3.11}$$

where $x_0 \in X$, $u = \big((\Delta t_k, e_k)\big)_{k=1,\dots,|u|} \in U(x_0)$ and $\Phi(x_0, u) = \big((v_k, t_k)\big)_{k=0,\dots,|u|}$. Note that $\mathcal{T}$ and $\mathcal{B}$ coincide if $\beta \equiv \tau$ and $\delta(v, t, \Delta t) = \Delta t$ for all $v \in V$, $t \in \mathbb{R}$, $\Delta t \in \mathbb{R}_0^+$. Using the function $\mathcal{T}$, the arrival time at the terminal node $v_{|u|} \in V$ of the path $p = \Phi((v_0, t_0), u)$ is given by $t_0 + \mathcal{T}((v_0, t_0), u)$.

Definition 3.3.7 *A path* p_* *is called optimal with respect to a source node* $v_0 \in V$ *and a terminal state* $(v', t') \in X$*, if there exist a departure time* $t_0 \in T(v_0)$ *and a finite control sequence* $u_* \in U(v_0, t_0)$ *with* $p_* = \Phi((v_0, t_0), u_*)$*,* $p_{*|p_*|} = (v', t')$ *and*

$$\mathcal{B}\big((v_0, t_0), u_*\big) = \inf\Big\{\mathcal{B}\big((v_0, t), u\big) : u \in U(v_0, t), \omega(u) = v', t + \mathcal{T}\big((v_0, t), u\big) = t'\Big\}. \tag{3.12}$$

In this case, u_* *is called an optimal control sequence (with respect to* v_0 *and* (v', t')*). For a fixed source node* $v_0 \in V$*, we define the reverse optimal value function* $b_* : X \to \mathbb{R} \cup \{\pm\infty\}$ *by*

$$b_*(v', t') = \inf\Big\{\mathcal{B}\big((v_0, t), u\big) : u \in U(v_0, t), \omega(u) = v', t + \mathcal{T}\big((v_0, t), u\big) = t'\Big\}. \tag{3.13}$$

In the sequel, we will mostly consider the optimization with respect to a given goal node and varying initial state. (See Section 4.1 for an overview over possible variants of the optimal path problem.) In order to distinguish whether the optimality relates to a given goal node or a given source node, we will use '$*$' as a superscript or subscript, respectively (cp., Definition 3.3.4 and Definition 3.3.7). Note that if G is a strong FIFO network and either waiting is forbidden everywhere or it is never optimal to wait, then both problems are conjugate and can be transformed to each other by reversing the time axis and appropriately transforming the travel time and cost functions [45].
The existence of a path connecting any given pair of nodes in a static directed graph can be guaranteed if the graph is strongly connected [79, Chapter 16], [135, Chapter 3]. In particular, if v, v' are any two nodes in the graph, then there exists a path from v to v' and a path from v' to v if and only if both nodes are contained in the same strongly connected component of the graph. As a consequence, a variety of solution techniques has

been developed to compute the maximal connected components of a graph, see, e.g., [135, Algorithm 3.1, Algorithm 5.3], [79, Corollary 16.8], [9, Chapter 8.6].
The determination of connected components of a time-dependent network is complicated by the augmentation of the state space by the time variable and the state transitions specified by the travel time function τ. Since $\tau \geq 0$ it is impossible to travel backwards in time, and consequently there can be no pair of states which are reachable one from another if $\tau > 0$. Moreover, the definition of state space and control constraints causes further difficulties in the computation of the set of states which are reachable from a given initial state (or a set of initial states). Since $b^*(x) = \infty$ if and only if $v' \times T(v')$ is not reachable from x, the computation of the optimal value function is intimately related with the determination of reachable subsets of the state space. We will consider this task in more detail in Section 3.5. It is clear that the computation of reachable subsets of the state space requires the computation of earliest arrival times and latest departure times, and hence the computation of fastest paths [142].
Fastest paths are optimal paths in a time-dependent network in which the travel costs equal the travel times and the waiting costs equal the waiting times. In order to avoid the definition of a second time-dependent network we replace $\mathcal{B}$ by $\mathcal{T}$ in Definition 3.3.4 and Definition 3.3.7 to obtain the optimal travel time function $t^* : X \to \mathbb{R}_0^+ \cup \{\infty\}$

$$t^*(x_0) = \inf \left\{\mathcal{T}(x_0, u) : u \in U(x_0), \omega(u) = v'\right\} \tag{3.14}$$

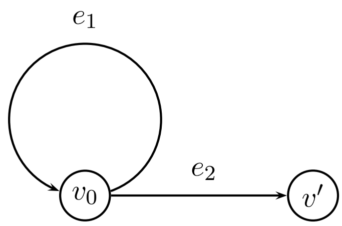
with respect to a given goal node $v' \in V$, and the reverse optimal travel time function $t_* : X \to \mathbb{R}_0^+ \cup \{\infty\}$

$$t_*(v', t') = \inf \Big\{\mathcal{T}\big((v_0, t), u\big) : u \in U(v_0, t), \omega(u) = v', t + \mathcal{T}\big((v_0, t), u\big) = t'\Big\} \tag{3.15}$$

with respect to a given source node $v_0 \in V$. In the following, the corresponding control sequences will be called time-optimal control sequences, and the corresponding paths will be called fastest paths. As before, in order to distinguish whether the optimality relates to a given goal node or a given source node, we will use '$*$' as a superscript or subscript, respectively.
In some applications the computation of optimal paths is subject to additional constraints which are based on optimal travel time, or on a combination of both cost criteria. For example, the travel time associated with a fuel-optimal route might be constrained to be no longer than 110% of the travel time of a fastest route. In order to adapt such constraints to our model, we can define the set of admissible arrival times $T(v)$ depending on the optimal travel time. We will explicitly deal with such problems in Section 7.3.
Let us conclude this section with the discussion of a simple example. Consider the time-dependent network in Figure 3.3, which is a FIFO-network. Let $T(v_0) = T(v') = \mathbb{R}$ and consider the departure time $t_0 = 0$ at the source node v_0. If waiting is forbidden at v_0, i.e., if $\Delta T(v_0, t_0) = \{0\}$ for all $t_0 \in \mathbb{R}$, then the cost associated with the control sequence $u = (u_k)_{k=1,\ldots,n}$,

$$u_k = \begin{cases} (0, e_1), \; k < n \\ (0, e_2), \; k = n \end{cases}, \tag{3.16}$$

is $2 + 2^{1-n}$. Since all admissible control sequences are of the form (3.16), there exists no finite optimal control sequence and no finite optimal path in this example. This situation



$$\tau(e_1,t) = \frac{|2-t|}{2}, \qquad \tau(e_2,t) = 1,$$
$$\beta(e_1,t) = \frac{|2-t|}{2}, \qquad \beta(e_2,t) = |4-2t|,$$
$$\delta(v_0,t,\Delta t) = \Delta t.$$

Figure 3.3.: Time-dependent network in which no finite optimal path needs to exist.

contrasts to the static case, in which (given that the goal node is reachable from the source node) a simple optimal path exists if all costs are nonnegative, cp. [9, Section 4.1]. In order to deal with such situations, a concept of infinite paths has been developed which defines an infinite path as the limit of an invariant generation sequence (see [138] for details). Considering a sequence of control sequences of the form (3.16) with increasing length n, the corresponding sequence of paths would be invariant and their limit would be optimal in the sense of [138]. Since infinite paths are not practicable in applications, we will focus on finite paths in the sequel. In Section 4.2 we will prove the existence of (finite) optimal paths under fairly weak assumptions, and in Chapter 6 we will extend these results to time-dependent networks with uncertain travel times.
Now consider the case in which unrestricted waiting is allowed at the source node v_0, i.e., $\Delta T(v_0,t_0) = \mathbb{R}_0^+$ for all $t_0 \in \mathbb{R}$. In this case, it is easily seen that an optimal control sequence is given by $u = (2, e_2)$, resulting in the travel cost 2. Hence, in contrast to the computation of fastest paths in FIFO-networks, any optimization procedure must take the hybrid structure of the time-dependent network into account, optimizing with respect to both the optimal edge and the optimal waiting time. We will discuss two solution techniques in the Chapters 8 and 9.

3.4. Turn Restrictions and the Dual Network

In the preceding sections of this chapter we have introduced time-dependent networks with state space and control constraints. In this section we additionally consider turn restrictions, which can be understood as constraints on the set of admissible control sequences. A description of such turn restrictions is contained in most digital maps of the road network and must be incorporated into the routing process in automotive navigation systems. Since certain turn restrictions may be relevant only for a small group of vehicles (e.g., trucks), they are usually defined separately from the road network. Motivated by the practical relevance of these constraints, we extend the results in [105], [7], [172], [11] to time-dependent networks with state-space and control constraints. In particular, starting with a time-dependent network with turn restrictions, we construct a time-dependent network without turn restrictions and prove the one-to-one correspondence of the paths in both networks.

Definition 3.4.1 *A time-dependent network with turn restrictions G_ρ is an octupel $G_\rho =$*

$(V, E, \rho, \tau, \sigma; \beta, \delta, \iota)$, where $G = (V, E, \tau; \beta, \delta)$ is a time-dependent network and

$$\rho : \bigcup_{v \in V} \{E^-(v) \times E^+(v)\} \to \{0, 1\},$$
$$\sigma : \bigcup_{v \in V} \{E^-(v) \times E^+(v)\} \times \mathbb{R} \to \mathbb{R}_0^+,$$
$$\iota : \bigcup_{v \in V} \{E^-(v) \times E^+(v)\} \times \mathbb{R} \to \mathbb{R},$$

are a turn restriction function, a turn travel time function and a turn cost function, respectively. A control sequence $u = \big((\Delta t_k, e_k)\big)_{k=1,\ldots,n}$ is said to respect the turn restrictions ρ if $\rho(e_k, e_{k+1}) = 1$ for all $k = 1, \ldots, n-1$.

Remark 3.4.2 *The first argument $(e^-, e^+) \in E \times E$ of σ, ι denotes the turn maneuver at the node $v = \omega(e^-) = \alpha(e^+)$ which is effected at the point in time specified by the second argument.*

As in Section 3.2, we associate a set of admissible arrival times $T_\rho(v) \subset \mathbb{R}$ to each $v \in V$, define the state space $X_\rho = \bigcup_{v \in V} \big\{\{v\} \times T_\rho(v)\big\}$ and a set of admissible waiting waiting times $\Delta T_\rho(x) \subset \mathbb{R}_0^+$ for each $x \in X_\rho$.
The functions σ, ι allow an explicit modeling of the time and cost which are necessary to perform a turn maneuver in the network. They are commonly used in the field of automotive navigation systems since they allow a larger set of criteria with respect to which a path can be optimized [172]. Moreover, they allow a more realistic modeling of a moving object in the network, since straight-ahead driving is usually associated with a lower travel time and cost than a turn maneuver. The incorporation of the turn travel time and turn cost require a slight modification of the definition of the state transitions (3.1) and the associated cost (3.8), which are defined as follows for time-dependent networks with turn restrictions. Let $(v_0, t_0) \in X_\rho$ and $u = (u_k)_{k=1,\ldots,n}$ with $u_k = (\Delta t_k, e_k) \in \mathbb{R}_0^+ \times E$. The path $p = \big((v_k, t_k)\big)_{k=0,\ldots,n}$ associated with the application of u in (v_0, t_0) is given by

$$\begin{aligned}
v_k &= \omega(e_k), && k = 1, \ldots, n,\\
t_k &= t_{k-1} + \Delta t_k + \tau(e_k, t_{k-1} + \Delta t_k)\\
&\qquad + \sigma\big(e_k, e_{k+1}, t_{k-1} + \Delta t_k + \tau(e_k, t_{k-1} + \Delta t_k)\big), && k = 1, \ldots, n-1,\\
t_n &= t_{n-1} + \Delta t_n + \tau(e_n, t_{n-1} + \Delta t_n).
\end{aligned}$$

As a consequence of this definition, waiting in the road network takes place before traversing a road segment and after crossing a junction. The control sequence u is said to be admissible for $x \in X_\rho$ in G_ρ, if $\Delta t_k \in \Delta T_\rho(v_{k-1}, t_{k-1})$, $e_k \in E^+(v_{k-1}$, $(v_k, t_k) \in X_\rho$ for all $k = 1, \ldots, n$, and u respects the turn restrictions. The set of all control sequences which are admissible for $x \in X_\rho$ in G_ρ is denoted by $U_\rho(x)$. In a similar manner as in Section 3.3 we define $\mathcal{B}_\rho$,

$$\begin{aligned}\mathcal{B}_\rho\big((v_0,t_0),u\big) = \sum_{k=1}^{n-1} \Big[&\delta(v_{k-1},t_{k-1},\Delta t_k) + \beta(e_k, t_{k-1}+\Delta t_k) \\ &+ \iota\big(e_k, e_{k+1}, t_{k-1}+\Delta t_k + \tau(e_k, t_{k-1}+\Delta t_k)\big)\Big] \\ &+ \delta(v_{n-1},t_{n-1},\Delta t_n) + \beta(e_n, t_{n-1}+\Delta t_n), \end{aligned} \tag{3.17}$$

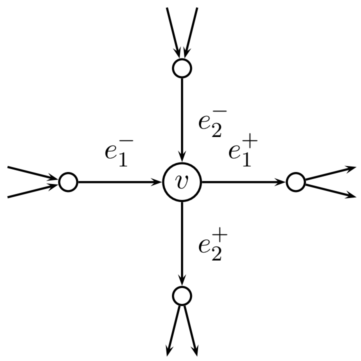
where v_k, t_k are defined above for $k = 1, ..., n$.
Turn restrictions and turn costs complicate the application of the principle of dynamical programming, which is the basic principle of almost all practically relevant solution techniques for the computation of optimal paths such as, e.g., [66], [56], [80], [123], [138], [127], [143], [144], [38], [39], [47], [54]. Since turn restrictions are no state-dependent constraints, the optimal control sequence associated with an intermediate state depends on the manner in which this state has been reached from the initial state. This violates the principle of optimality [22]. In order to avoid this problem, the state space must be modified appropriately. Among the possible transformations of static networks, the dual network has proven to be the most suitable [7], [172]. Note that modeling each junction as two sets of nodes (i.e., access nodes to the junction and departure nodes from the junction) and a set of edges (i.e., the possible turns in the junction) leads to a significant increase in the number of nodes and edges of the resulting network. Since the worst-case time complexity of all optimal path algorithms is $O(f(|E|,|V|))$, where f is a superlinear function of the number of nodes and edges, this network transformation has proven to be inefficient. Both network transformations are illustrated in Figure 3.4.
There are several notions of duality in graph theory, including combinatorial and geometric duality [79, Chapter 11]. We will use a concept of duality which differs from those definitions and which follows the concepts introduced in [105] and used in [7], [172], [91]. Note that the dual network was termed auxiliary network in [11].

Definition 3.4.3 *Let $G_\rho = (V, E, \rho, \tau, \sigma; \beta, \delta, \iota)$ denote a time-dependent network with turn restrictions. For each $v \in V$, let $T_\rho(v) \subset \mathbb{R}$ denote the set of admissible arrival times at v and, for each $x \in X_\rho$, let $\Delta T_\rho(x) \subset \mathbb{R}_0^+$ denote the set of admissible waiting times at x. We define the time-dependent network with turn restrictions $G_\rho^T = (V^T, E^T, \rho^T, \tau^T, \sigma^T; \beta^T, \delta^T, \iota^T)$ by*

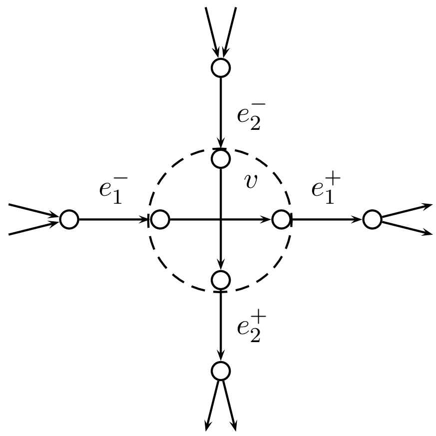
$$V^T = E,$$
$$E^T = \big\{(e^-, e^+) \in V^T \times V^T : e^- \in E^-(v), e^+ \in E^+(v), \rho(e^-, e^+) = 1\big\},$$

and

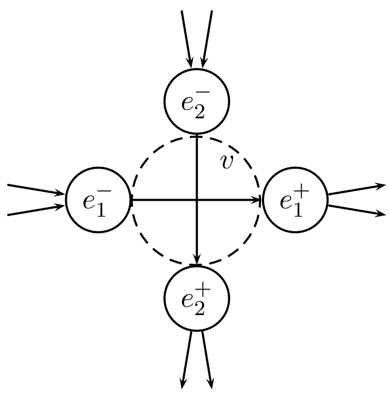
$$\begin{aligned} \tau^T(e^T,t) &= \tau(e^-,t) + \sigma(e^T, t+\tau(e^-,t)), && \text{if } e^T = (e^-,e^+) \in E \times E,\\ \beta^T(e^T,t) &= \beta(e^-,t) + \iota(e^T, t+\tau(e^-,t)), && \text{if } e^T = (e^-,e^+) \in E \times E,\\ \delta^T(v^T,t,\Delta t) &= \delta(\alpha(e),t,\Delta t), && \text{if } v^T = e \in E, \end{aligned}$$



(a) Original graph (V, E), only straight-ahead driving is allowed by a turn restriction function ρ.



(b) Transformed graph (V', E') resulting from the splitting of the junction into access nodes, departure nodes and turns.
$|V'| = \sum_{v \in V}(\deg^-(v) + \deg^+(v)) = 2|E|, \quad |E'| \leq |E| + \sum_{v \in V}(\deg^-(v) \cdot \deg^+(v))$



(c) Dual graph (V^T, E^T) in which the primal edges are interpreted as nodes.
$|V^T| = |E|, \quad |E^T| \leq \sum_{v \in V}(\deg^-(v) \cdot \deg^+(v))$

Figure 3.4.: Illustration of two possible network transformations which allow the incorporation of turn restrictions into dynamic-programming-based algorithms.

as well as $\rho^T \equiv 1$, $\sigma^T \equiv 0$, $\iota^T \equiv 0$. *Moreover, we define*

$$\begin{aligned} T_\rho^T(v^T) &= T_\rho(\alpha(e)), && \textit{if } v^T = e \in E, \\ \Delta T_\rho^T(v^T, t) &= \Delta T_\rho(\alpha(e), t), && \textit{if } v^T = e \in E. \end{aligned}$$

G_ρ^T *is called the dual of* G_ρ, *and* $T_\rho^T, \Delta T_\rho^T$ *are called the dual constraints of* $T_\rho, \Delta T_\rho$.

Remark 3.4.4 *Note that the introduction of time-dependent turn restrictions would result in constraints on the traversal times of the edges in the dual network. Such constraints might be used, e.g., to model the impact of traffic lights in the road network [11]. As we have mentioned in Remark 3.2.6, the incorporation of such constraints into the concept of time-dependent networks is possible but will not be pursued in this thesis.*

It is easily seen that G_ρ^T is indeed a time-dependent network with turn restrictions in the sense of Definition 3.4.1. We will denote the state space of G_ρ^T by X_ρ^T, the set of admissible controls for $x^T \in X_\rho^T$ by $U_\rho^T(x^T)$ and the path cost function will be denoted by $\mathcal{B}_\rho^T$, cf. (3.17).
Note, that the turn restrictions and turn costs in the dual network are trivial and allow the definition of the equivalent time-dependent network $G^T = (V^T, E^T, \tau^T; \beta^T, \delta^T)$ with constraints $T^T = T_\rho^T, \Delta T^T = \Delta T_\rho^T$, in which the set of admissible control sequences $U^T(x^T)$ for $x^T \in X^T = X_\rho^T$ satisfies $U^T(x^T) = U_\rho^T(x^T)$. A similar equivalence of the time-dependent network with and without turn restrictions holds for the second dual. In the second dual of the road network each node corresponds to an unrestricted turn at a junction and each edge corresponds to the same road segment as in the primal network.
In order to use the dual network for the computation of optimal paths in the primal network, we need a further extension of Definition 3.4.3, which allows a one-to-one correspondence between the initial and terminal states of the paths in the primal and the dual network [7], [172], [11]. Let $V_0 \subset V$ be a set of source nodes and let $V' \subset V$ be a set of goal nodes with $V_0 \cap V' = \emptyset$. We define the time-dependent network $\widetilde{G} = (\widetilde{V}, \widetilde{E}, \widetilde{\tau}; \widetilde{\beta}, \widetilde{\delta})$ by

$$\widetilde{V} = V^T \cup V_0 \cup V', \tag{3.18}$$

$$\widetilde{E} = E^T \cup \bigcup_{v_0 \in V_0} \big\{\{v_0\} \times E^+(v_0)\big\} \cup \bigcup_{v' \in V'} \big\{E^-(v') \times \{v'\}\big\}, \tag{3.19}$$

$$\widetilde{\tau}(\widetilde{e}, t) = \begin{cases} \tau^T(e^T, t), & \text{if } \widetilde{e} = e^T \in E^T \\ 0, & \text{if } \widetilde{e} \in V_0 \times E \\ \tau(e^-, t), & \text{if } \widetilde{e} = (e^-, v') \in E \times V' \end{cases}, \tag{3.20}$$

$$\widetilde{\beta}^T(\widetilde{e}, t) = \begin{cases} \beta(e^T, t), & \text{if } \widetilde{e} = e^T \in E^T \\ 0, & \text{if } \widetilde{e} \in V_0 \times E \\ \beta(e^-, t), & \text{if } \widetilde{e} = (e^-, v') \in E \times V' \end{cases}, \tag{3.21}$$

$$\widetilde{\delta}^T(\widetilde{v}, t, \Delta t) = \begin{cases} \delta^T(v^T, t, \Delta t), & \text{if } \widetilde{v} = v^T \in V^T \\ 0, & \text{if } \widetilde{v} \in V_0 \cup V' \end{cases}. \tag{3.22}$$

In $\widetilde{G}$, we consider the state space and control constraints given by

$$\widetilde{T}(\widetilde{v}) = \begin{cases} T^T(v^T), & \text{if } \widetilde{v} = v^T \in V^T \\ T_\rho(v), & \text{if } \widetilde{v} \in V_0 \cup V' \end{cases}, \tag{3.23}$$

$$\widetilde{\Delta T}(\widetilde{v}, t) = \begin{cases} \Delta T^T(v^T, t), & \text{if } \widetilde{v} = v^T \in V^T \\ \{0\}, & \text{if } \widetilde{v} \in V_0 \cup V' \end{cases}. \tag{3.24}$$

It is easily seen, that $\widetilde{G}$ is indeed a time-dependent network in the sense of Definition 3.2.1. The associated state space $\widetilde{X}$, admissible control sequences $\widetilde{U}$ and path cost function $\widetilde{\mathcal{B}}$ are defined as in (3.2) and Section 3.3.

Theorem 3.4.5 *Let $G_\rho = (V, E, \rho, \tau, \sigma; \beta, \delta, \iota)$ be a time-dependent network with turn restrictions and let the arrival time and waiting time restrictions be given by $T_\rho, \Delta T_\rho$. Let $\widetilde{X}, \widetilde{U}, \widetilde{\mathcal{B}}$ be defined as above and denote*

$$\Upsilon_\rho = \bigcup_{x=(v,t)\in X_\rho : v\in V_0} \{(x,u) : u \in U_\rho(x), \omega(u) \in V'\},$$

$$\widetilde{\Upsilon} = \bigcup_{\widetilde{x}=(\widetilde{v},t)\in \widetilde{X} : \widetilde{v}\in V_0} \{(\widetilde{x},\widetilde{u}) : \widetilde{u} \in \widetilde{U}(\widetilde{x}), \omega(\widetilde{u}) \in V'\}.$$

There exists a bijective mapping $\Psi : \Upsilon_\rho \to \widetilde{\Upsilon}$ satisfying $\mathcal{B}_\rho(x,u) = \widetilde{\mathcal{B}}\big(\Psi(x,u)\big)$.

Proof Let $(x,u) \in \Upsilon_\rho$, $x = (v,t)$, $u = \big((\Delta t_k, e_k)\big)_{k=1,\dots,n}$ and $v' = \omega(u)$. We define $\widetilde{x} = x$ and $\widetilde{u} = (\widetilde{u}_k)_{k=1,\dots,n+1}$ by

$$\begin{aligned} \widetilde{u}_1 &= \big(0, (v, e_1)\big), \\ \widetilde{u}_k &= \big(\Delta t_{k-1}, (e_{k-1}, e_k)\big), \quad k = 2, ..., n, \\ \widetilde{u}_{n+1} &= \big(\Delta t_n, (e_n, v')\big). \end{aligned}$$

As an immediate consequence of the above construction (3.18)-(3.24) we obtain $\widetilde{x} \in \widetilde{X}$ and $\widetilde{u} \in \widetilde{U}(\widetilde{x})$. Clearly, $\omega(\widetilde{u}) \in V'$. An easy computation yields $\mathcal{B}_\rho(x,u) = \mathcal{B}^T_\rho(x^T, u^T)$.
Now let $(\widetilde{x}, \widetilde{u}) \in \widetilde{\Upsilon}$, $\widetilde{u} = \big((\Delta t_k, \widetilde{e}_k)\big)_{k=1,\dots,n}$. We define $x = \widetilde{x}$ and $u = (u_k)_{k=1,\dots,n-1}$ by

$$u_{k-1} = \big(\Delta t_k, \alpha(\widetilde{e}_k)\big), \quad k = 2, ..., n.$$

As an immediate consequence of the above construction (3.18)-(3.24) we obtain $x \in X_\rho$ and $u \in U_\rho(x)$. An easy computation yields $\mathcal{B}_\rho(x,u) = \widetilde{\mathcal{B}}(\widetilde{x}, \widetilde{u})$. □

Based on the result in Theorem 3.4.5, we can compute optimal paths in the extended dual network $\widetilde{G}$ instead of computing optimal paths in the primal network with turn restrictions G_ρ. The correspondence of the paths in G_ρ and $\widetilde{G}$ is illustrated in Figure 3.5. Henceforth, we will only consider time-dependent networks without turn restrictions.

3.5. Reachability in Time-Dependent Networks

The computation of optimal paths in time-dependent networks is intimately related to the computation of the reachable subsets of the state space, since $b^*(x) < \infty$ if and only if

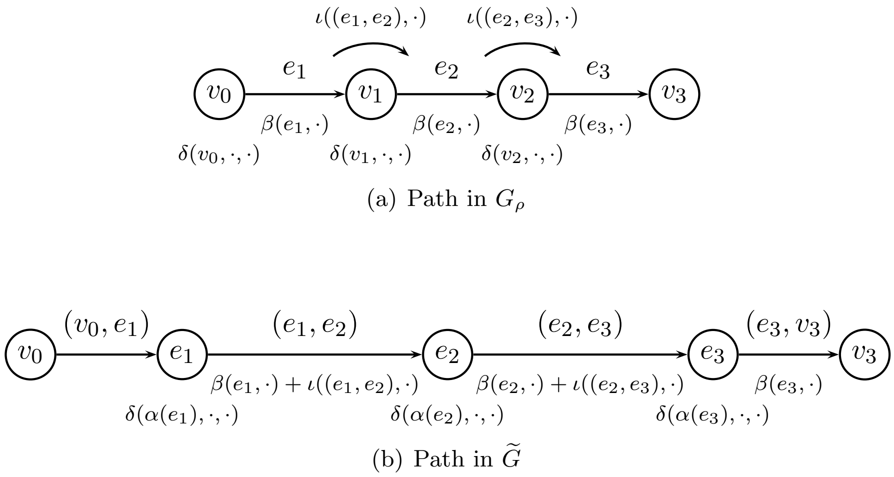


Figure 3.5.: Correspondence of the paths and costs in G_ρ and $\widetilde{G}$.

there exists an admissible control sequence $u \in U(x)$ with $\omega(u) = v'$. Let us consider a time-dependent network $G = (V, E, \tau; \beta, \delta)$ with arrival time constraints T and waiting time constraints ΔT. Let us further suppose that a source node $v_0 \in V$ and a goal node $v' \in V$ are given. When computing an optimal path from some $x_0 \in \{v_0\} \times T(v_0)$ to $\{v'\} \times T(v')$, we only need to take into account those states $x_R \in X$ which are reachable from x_0, and for which there exists a $u \in U(x_R)$ with $\omega(u) = v'$. Keeping this simple observation in mind when constructing solution methods can lead to a significant decrease in computational complexity.
In this section, we introduce the sets of reachable points in time associated with the nodes of the network. We generalize the concept of strong connectivity to time-dependent networks and prove that the sets of reachable points in time can be computed in polynomial time under appropriate assumptions.

Definition 3.5.1 *Let a time-dependent network $G = (V, E, \tau; \beta, \delta)$ with arrival time constraints T and waiting time constraints ΔT as well as a source node $v_0 \in V$ and a goal node $v' \in V$ be given. For $v \in V$, we define the set of reachable points in time $T_R(v)$ by*

$$T_R(v) = \big\{t \in T(v) : \exists p \in P \text{ with } p_0 \in \{v_0\} \times T(v_0),\ p_{|p|} \in \{v'\} \times T(v') \text{ and } p_k = (v,t) \text{ for some } k \in \{0, ..., |p|\}\big\}. \tag{3.25}$$

The reachable part of the state space is denoted by

$$X_R = \bigcup_{v \in V} \big\{\{v\} \times T_R(v)\big\}. \tag{3.26}$$

Observe that, according to the above definition, a source node $v_0 \in V$ and a goal node $v' \in V$ must be fixed in order to define $T_R(v)$ for $v \in V$.
The following lemma generalizes the concept of strong connectivity [9, p.27] to time-dependent networks.

Lemma 3.5.2 *Let a time-dependent network $G = (V, E, \tau; \beta, \delta)$ with arrival time constraints T and waiting time constraints ΔT as well as a source node $v_0 \in V$ and a goal node $v' \in V$ be given. Suppose that τ is continuous, $T(v) = \mathbb{R}$ for all $v \in V$ and (V, E) is strongly connected. If*

$$\lim_{t \to -\infty} t + \tau(e,t) = -\infty, \quad \forall e \in E, \tag{3.27}$$

and there exists a continuous function $\underline{\Delta t} : X \to \mathbb{R}_0^+$ with $\underline{\Delta t}(x) \in \Delta T(x)$ for all $x \in X$ and

$$\lim_{t \to -\infty} t + \underline{\Delta t}(v,t) = -\infty, \quad \forall v \in V, \tag{3.28}$$

then $T_R(v) = \mathbb{R}$ for all $v \in V$.

Proof Let $v \in V$. Since (V, E) is strongly connected, there exists a connected edge sequence $(e_k)_{k=1,...,k_0}$ with $\alpha(e_1) = v_0$, $\omega(e_{k_0}) = v$ and a connected edge sequence $(e'_k)_{k=1,...,k'}$ with $\alpha(e'_1) = v$, $\omega(e'_{k'}) = v'$. Let $t_0 \in \mathbb{R}$ be arbitrary but fixed. Since $\underline{\Delta t}(x) \in \Delta T(x)$ for all $x \in X$ and $T(v) = \mathbb{R}$ for all $v \in V$, the control sequence $u(t_0) = \big(u_k(t_0)\big)_{k=1,...,k_0+k'}$ and the path $p(t_0) = \big(x_k(t_0)\big)_{k=0,...,k_0+k'}$ defined recursively by $x_0(t_0) = (v_0, t_0)$,

$$\begin{aligned}
u_k(t_0) &= \big(\underline{\Delta t}\big(x_{k-1}(t_0)\big), e_k\big), & x_k(t_0) &= \varphi\big(x_{k-1}(t_0), u_k(t_0)\big), & k &= 1, ..., k_0,\\
u_k(t_0) &= \big(\underline{\Delta t}\big(x_{k-1}(t_0)\big), e'_{k-k_0}\big), & x_k(t_0) &= \varphi\big(x_{k-1}(t_0), u_k(t_0)\big), & k &= k_0+1, ..., k_0+k',
\end{aligned}$$

are admissible. Let $\theta_k : \mathbb{R} \to \mathbb{R}$, $\theta_k(t_0) = t_0 + \mathcal{T}\big((v_0, t_0), u_{1:k}(t_0)\big)$, $k = 1, ..., k_0 + k'$. An induction over k yields the continuity of θ_k and (3.27) and (3.28) imply that $\lim_{t_0 \to -\infty} \theta_k(t_0) = -\infty$. Since $\tau, \underline{\Delta t} \geq 0$ we also have $\lim_{t_0 \to \infty} \theta_k(t_0) = \infty$. As a consequence of the intermediate value theorem [68, p.97, Satz 1], for each $t \in \mathbb{R}$ there exists a $t_0(t) \in \mathbb{R}$ with $\theta_{k_0}\big(t_0(t)\big) = t$. Using $p\big(t_0(t)\big) = \Phi\big(\big(v_0, t_0(t)\big), u\big(t_0(t)\big)\big)$ in (3.25) yields $T_R(v) = \mathbb{R}$. $\square$

In the remainder of this section, we assume that $0 \in \Delta T(x)$ for all $x \in X$. If $0 \notin \Delta T(x)$ for some $x \in X$, but $\Delta T(x)$ is closed for all $x \in X$, then we define the real-valued function $\underline{\Delta T} : X \to \mathbb{R}_0^+$, $\underline{\Delta T}(v,t) = \min \Delta T(v,t)$, denote

$$\begin{aligned}
\widetilde{\Delta T}(v,t) &= \Delta T(v,t) - \underline{\Delta T}(v,t),\\
\widetilde{\tau}(e,t) &= \underline{\Delta T}(\alpha(e),t) + \tau\big(e, t + \underline{\Delta T}(\alpha(e),t)\big),\\
\widetilde{\beta}(e,t) &= \beta\big(e, t + \underline{\Delta T}(\alpha(e),t)\big),\\
\widetilde{\delta}(v,t,\Delta t) &= \delta\big(v, t, \Delta t + \underline{\Delta T}(v,t)\big),
\end{aligned}$$

and consider the transformed time-dependent network $\widetilde{G} = (V, E, \widetilde{\tau}; \widetilde{\beta}, \widetilde{\delta})$ with the constraints specified by $T, \widetilde{\Delta T}$. It is easily seen that $0 \in \widetilde{\Delta T}(x)$ for all $x \in X$. Moreover, if $\tau, \underline{\Delta T}$ are both continuous and satisfy (3.4), then $\widetilde{G}$ is a FIFO network. Note that this construction does not affect the existence of optimal paths if $\underline{\Delta T}$ is continuous, cp. Assumption 4.2.3.

In many applications the computation of optimal paths is subject to an earliest departure time $\underline{t} \in \mathbb{R}$ at the initial location v_0 and a latest arrival time $\overline{t} \in \mathbb{R}$ at the destination v'. We incorporate such (additional) constraints into our model by replacing $T(v_0)$ by $T(v_0) \cap [\underline{t}, \infty)$ and $T(v')$ by $T(v') \cap (-\infty, \overline{t}]$. In order to keep the notation simple, we assume that $\underline{t} \in T(v_0)$, $\overline{t} \in T(v')$ and that these replacements have already been performed

when defining the sets of admissible arrival times $T(v_0), T(v')$. The earliest departure time at v_0 and the latest arrival time at v' are then given by $\underline{t} = \min T(v_0)$ and $\bar{t} = \max T(v')$, respectively.
The earliest arrival time at $v \in V$ with respect to the initial state (v_0, t_0), $t_0 \in T(v_0)$, is defined as

$$\underline{t}_{t_0}(v) = \begin{cases} t_0, & \text{if } v = v_0 \\ \inf\{t_0 + \mathcal{T}\big((v_0, t_0), u\big) : u \in U(v_0, t_0), \omega(u) = v\}, & \text{if } v \neq v_0 \end{cases},$$

the latest departure time at $v \in V$ with respect to the terminal state (v', t'), $t' \in T(v')$, is defined as

$$\bar{t}_{t'}(v) = \begin{cases} t', & \text{if } v = v' \\ \sup\{t \in T(v) : \exists u \in U(v, t), \omega(u) = v', t + \mathcal{T}\big((v, t), u\big) = t'\}, & \text{if } v \neq v' \end{cases}.$$

By allowing the variation of the possible departure and arrival times we define the earliest arrival time $\underline{t}_R(v)$ at $v \in V$ with respect to the source node v_0 and the latest departure time $\bar{t}_R(v)$ at $v \in V$ with respect to the goal node v':

$$\underline{t}_R(v) = \begin{cases} \underline{t}, & \text{if } v = v_0 \\ \inf_{t_0 \in T(v_0)} \underline{t}_{t_0}(v), & \text{if } v \neq v_0 \end{cases}, \tag{3.29}$$

$$\bar{t}_R(v) = \begin{cases} \bar{t}, & \text{if } v = v' \\ \sup_{t' \in T(v')} \bar{t}_{t'}(v), & \text{if } v \neq v' \end{cases}. \tag{3.30}$$

Using the FIFO-property and an inductive argument, it is easily seen that $\underline{t}_R(v) = \underline{t}_{\underline{t}}(v)$ and $\bar{t}_R(v) = \bar{t}_{\bar{t}}(v)$ in a time-dependent FIFO-network without arrival time and waiting time constraints. In the next lemma, we prove the equivalence of the fastest path problem in time-dependent FIFO-networks with and without certain time constraints.

Assumption 3.5.3 *Let $G = (V, E, \tau; \beta, \delta)$ be a time-dependent FIFO-network with continuous travel time function. Let a source node $v_0 \in V$ and a goal node $v' \in V$ be given and denote $\underline{t} = \min T(v_0)$, $\bar{t} = \max T(v')$. Denote by $\underset{\sim}{\underline{t}}_{\underline{t}}(v)$ the earliest arrival time of a path from $(v_0, \underline{t}\,)$ to v and by $\widetilde{\bar{t}}_{\bar{t}}(v)$ the latest departure time of a path from v to $(v', \bar{t}\,)$ in the unconstrained network, let $\widetilde{T}_R(v) = [\underset{\sim}{\underline{t}}_{\underline{t}}(v), \infty) \cap (-\infty, \widetilde{\bar{t}}_{\bar{t}}(v)]$ for $v \in V$ and suppose that $\widetilde{T}_R(v_0) \neq \emptyset$, $\widetilde{T}_R(v') \neq \emptyset$,*

$$0 \in \Delta T(v, t), \qquad \forall (v, t) \in X, \tag{3.31}$$

$$T(v) \supset \widetilde{T}_R(v), \qquad \forall v \in V. \tag{3.32}$$

Lemma 3.5.4 *Let $G = (V, E, \tau; \beta, \delta)$ be a time-dependent network in which Assumption 3.5.3 holds and let $v \in V$ with $\widetilde{T}_R(v) \neq \emptyset$. Then there exist a simple fastest path without waiting from $(v_0, \underline{t})$ to $\big(v, \underset{\sim}{\underline{t}}_{\underline{t}}(v)\big)$ and a simple fastest path without waiting from $\big(v, \widetilde{\bar{t}}_{\bar{t}}(v)\big)$ to $(v', \bar{t})$. The computation of these paths can be carried out in $\mathcal{O}\big(|E| + |V| \log |V|\big)$ time.*

Proof It is sufficient to prove the assertion for time-optimal control sequences and fastest paths emanating from v_0. The result for time-optimal control sequences and fastest paths terminating in v' then follows in a similar manner as in [45]. According to [137, Corollary 1 and Section 3.2], there exists a simple fastest path $p = \big((v_k, t_k)\big)_{k=0,\ldots,n}$ from

(v_0, t_0) to $\big(v, \underset{\sim}{t}_{\underline{t}}(v)\big)$ in the unconstrained network which is generated by some $u(v_0, v) = \big((0, e_k)\big)_{k=1,...,n}$ at (v_0, t_0), $t_0 = \underline{t}$. We now prove that $u(v_0, v) \in U(v_0, t_0)$ in the constrained network. Since $\underset{\sim}{t}_{\underline{t}}(v) \leq \underline{t}_{\underline{t}}(v)$, this implies that $\underset{\sim}{t}_{\underline{t}}(v) = \underline{t}_{\underline{t}}(v)$, p is a fastest path and $u(v_0, v)$ is a time-optimal control sequence in the constrained case.
We clearly have $u_k(v_0, v) \in \Delta T(v_{k-1}, t_{k-1}) \times E^+(v_{k-1})$, since $0 \in \Delta T(v, t)$ for all $(v, t) \in X$. It remains to show that $(v_k, t_k) \in X$ for all $k = 1, ..., n$. Since the FIFO-property implies that $t \mapsto t + \tau(e, t)$ is monotone increasing in $t \in \mathbb{R}$ for all $e \in E$, $t_0 = \underline{t}$ and an induction over k immediately yield $t_k \geq \underset{\sim}{t}_{\underline{t}}(v_k)$ for all $k = 1, ..., n$. Now, suppose that $t_K > \widetilde{t}_{\bar{t}}(v_K)$ for some $K \in \{1, ..., n-1\}$. Let $u(v, v')$ denote a control sequence connecting $\big(v, \widetilde{t}_{\bar{t}}(v)\big)$ to $(v', \bar{t})$ in the unconstrained network. Then the concatenation of $u_{K+1:n}(v_0, v)$ and $u(v, v')$ defines a control sequence whose application at (v_K, t_K) generates a path which terminates at some (v', t'). Since $t_n < \widetilde{t}_{\bar{t}}(v_n)$ by assumption, the FIFO-property and an inductive argument yield $t' \leq \bar{t}$. This contradicts the fact that $\widetilde{t}_{\bar{t}}(v_K)$ is the latest departure time at v_K. Consequently, $t_k \in \widetilde{T}_R(v_k) \subset T(v_k)$ for all $k = 1, ..., n$, and hence $u(v_0, v) \in U(v_0, t_0)$.
A slightly modified version of Dijkstra's shortest path algorithm [56] can be used to compute simple fastest paths without waiting in an unconstrained FIFO network [10]. Using Fibonacci heap implementation [69], this algorithm can be implemented in $\mathcal{O}(|E| + |V| \log |V|)$ time. □

Remark 3.5.5 *Note that in sparse networks, i.e., networks in which $|E| = \mathcal{O}(|V|)$, the complexity bound of Lemma 3.5.4 becomes $\mathcal{O}(|V| \log |V|)$. The road network, in which the number of roads emanating from any junction is bounded, is a sparse network and satisfies the FIFO-property according to Lemma 2.4.1.*

The preceding Lemma shows that, under Assumption 3.5.3, $\underline{t}_R(v)$ and $\bar{t}_R(v)$ can be determined from fastest paths with fixed departure time $\underline{t}$ at v_0 and fixed arrival time $\bar{t}$ at v' in the unconstrained network, respectively. In particular, $\underline{t}_R(v) = \underset{\sim}{t}_{\underline{t}}(v)$ and $\bar{t}_R(v) = \widetilde{t}_{\bar{t}}(v)$ for all $v \in V$. The following result provides the structure of the set of reachable points in time $T_R(v)$, cf. Definition 3.5.1.

Lemma 3.5.6 *Let $G = (V, E, \tau; \beta, \delta)$ be a time-dependent network in which Assumption 3.5.3 holds. Then $T_R(v) = [\underline{t}_R(v), \infty) \cap (-\infty, \bar{t}_R(v)]$ for all $v \in V$.*

Proof Let $(v, t) \in X$ be arbitrary but fixed. Clearly, if $t < \underline{t}_R(v)$, there is no feasible path from v_0 to (v, t), which departs at v_0 at or after $\underline{t}_R(v_0)$. Similarly, if $t > \bar{t}_R(v)$, there is no feasible path from (v, t) to v', which terminates at v' at or before $\bar{t}_R(v')$. Suppose that $\underline{t}_R(v) \leq \bar{t}_R(v)$ and let $t \in [\underline{t}_R(v), \bar{t}_R(v)]$. We now construct a feasible path from v_0 through (v, t) to v', which departs at or after $\underline{t}_R(v_0)$ and terminates at or before $\bar{t}_R(v')$. Denote by $u(v_0, v), u(v, v')$ control sequences corresponding to fastest paths from $\big(v_0, \underline{t}_R(v_0)\big)$ to v, and from v to $\big(v', \bar{t}_R(v')\big)$, respectively. Without loss of generality, we may assume that $u(v_0, v), u(v, v')$ satisfy $u_i(v_0, v), u_j(v, v') \in \{0\} \times E$ for all $i = 1, ..., |u(v_0, v)|$, $j = 1, ..., |u(v, v')|$. As a consequence of the FIFO-property and the same reasoning as in Lemma 3.5.4, we obtain $u(v, v') \in U(v, t)$ for all $t \in [\underline{t}_R(v), \bar{t}_R(v)]$. Similarly, we obtain the existence of a $\bar{t}_0 \in T_R(v_0)$ such that

$$u(v_0, v) \in U(v_0, t_0) \quad \Longleftrightarrow \quad t_0 \in [\underline{t}, \bar{t}_0].$$

Suppose that $\bar{t}_0 + \mathcal{T}\big((v_0, \bar{t}_0), u(v_0, v)\big) < \bar{t}_R(v)$, denote $n = |u(v_0, v)|$, and denote $p = \Phi\big((v_0, \bar{t}_0), u(v_0, v)\big) = \big((v_k, t_k)\big)_{k=0,...,n}$. Then we must have $t_K = \bar{t}_R(v_K)$ for some $K < n$.

However, this contradicts the fact that $\bar{t}_R(v_K)$ is the latest departure time at v_K, since

$$t_K + \mathcal{T}\big((v_K, t_K), u_{K+1:n}(v_0, v)\big) < \underline{t}_R(v),$$

and hence there exists an $\widetilde{t}_K > t_K$ such that

$$\widetilde{t}_K + \mathcal{T}\Big((v_K, \widetilde{t}_K), \big(u_{K+1:n}(v_0, v), u(v, v')\big)\Big) \leq \bar{t}_R(v) + \mathcal{T}\big((v, \bar{t}_R(v)), u(v, v')\big) = \bar{t}.$$

Hence, $\bar{t}_0 + \mathcal{T}\big((v_0, \bar{t}_0), u(v_0, v)\big) = \bar{t}_R(v)$. Since τ is continuous, for any $t \in T_R(v)$, [68, p.97, Satz 1] yields the existence of a $t_0(t) \in T_R(v_0)$ such that $t_0(t) + \mathcal{T}\big(\big(v_0, t_0(t)\big), u(v_0, v)\big) = t$ and $\big(u(v_0, v), u(v, v')\big) \in U\big(v_0, t_0(t)\big)$. □

By combining Lemma 3.5.4 and Lemma 3.5.6 we obtain the following corollary:

Corollary 3.5.7 *Let $G = (V, E, \tau; \beta, \delta)$ be a time-dependent network in which Assumption 3.5.3 holds. Then $T_R(v)$ can be determined in $\mathcal{O}\big(|E| + |V| \log |V|\big)$ time for all $v \in V$.*

4. Optimal Paths in Time-Dependent Networks

In this chapter, we first provide an overview over different problem variants of the time-dependent optimal path problem, cf. Section 4.1. In Section 4.2, we prove the lower semicontinuity of the optimal value function and the existence of optimal paths in time-dependent networks with arrival time and waiting time constraints. This result can easily be extended to the case in which the travel time function may assume negative values, which is of particular interest in the context of time-dependent optimal flow problems. We derive the dynamic programming equations which are associated with the time-dependent optimal path problem in Section 4.3 and conclude this chapter by briefly discussing order relations on time-dependent networks, cf. Section 4.4.

4.1. Problem Variants and Literature Overview

The computation of shortest paths in weighted and directed networks has been subject to research for more than five decades by now, but it has never lost its relevance in up-to-date applications. Although the main ideas for solving the problem date back to the 1950ies, cf. [56] and [23], there has been a great number of improvements in the fields of algorithm engineering, discrete optimization and operations research, see, e.g., [80], [69], [162], [20]. Besides these direct applications, shortest path problems also arise as subproblems in network optimization [92], [158], [133], and even - as a consequence of certain discretization procedures - in the field of continuous control theory [96].

In [42], the optimal path problem in time-dependent networks has been explicitly introduced after it had been indirectly mentioned in the context of maximal flows in [67]. Based on these early results a variety of different network models and optimization problems have been introduced. There are two approaches to modeling the time variable of the time-dependent network, i.e., the discrete-time approach followed in, e.g., [38], [10], [35], [36] and the continuous-time approach followed in, e.g., [138], [47], [54], [142]. While the problem of computing optimal paths in a discrete-time time-dependent network can be analyzed and solved with the methods of graph theory and discrete mathematics, both discrete and continuous methods must generally be involved if the time-variable is continuous and the resulting time-dependent network is a hybrid mathematical system. Furthermore, the optimization problems can be distinguished between the optimization with respect to the travel time [101], [137], [10], [47] and the optimization with respect to a cost different from travel time [39], [48], [10], [38], [138].

The time-dependent optimal path problem can be formulated both as a linear program in the space of positive Borel measures [142] and based on the theory of dynamic programming [138]. While we pursue the latter approach, the characterization of optimal paths by optimal flows in the network allows a number of interesting theoretical results such as the development of a duality theory [133], [158]. However, the restriction of the travel times to be constant functions and the size of the associated linear program essentially limit the applicability of the linear programming formulation of the time-dependent optimal path problem in applications in which the underlying network is large or the assumption of constant travel times is too restrictive.

Due to the relevance of the travel time for the determination of the cost of a path, optimal path problems in which the cost is different from travel time bear a certain resemblance to multicriterial optimal path problems [25, Section 2.3.4], [163]. Thus it is not surprising that the optimal path problem in time-dependent networks is generally NP-hard if the cost is different from travel time. This can be proved by a reduction to the number partition problem [10, Theorem 2]. Considering a time-dependent network with a discrete time variable in which variations of the travel time and cost functions take place only at a finite number of points in time, an algorithm has been developed which computes optimal paths in pseudo-polynomial time [38]. However, no similar results hold if the time variable is continuous, cf. Theorem 7.3.3. If the time-dependent network satisfies the FIFO property (cf. Definition 3.2.7) and waiting is either unrestricted or forbidden everywhere in the network, the time-dependent fastest path problem is of the same order of complexity as the optimal path problem in static networks [137]. In particular, the worst-case time complexity of the fastest path problem is $O(|V| \log |V| + |E|)$. However, if the FIFO-property is violated and waiting is forbidden, then the fastest path problem is NP-hard [137].

As a consequence of these hardness results, few computational experiments have been carried out concerning the computation of optimal paths in time-dependent networks with cost different from travel time. By contrast, considering the computation of fastest paths in FIFO-networks, a variety of speed-up techniques have been developed, which allow the computation of fastest paths in a split second even in very large networks such as the European road network [132], [54], [53]. Our computational results in Appendix A show that such query times are not yet attainable even in the small network of the German city of Ingolstadt if the cost is different from travel time.

As we have motivated in Chapter 2 and Section 3.1 by means of the road network, the travel times and travel costs cannot be determined or predicted with certainty in many applications. In order to cope with this situation the optimal path problem has been extended to stochastic networks [134], [139], [123], and time-dependent stochastic networks [167], [70], [128], [71], [72], [104]. Since the stochastic time-dependent optimal path problem does not allow closed analytic solutions in general [70], we are considering the uncertain time-dependent optimal path problem in the framework of worst-case-optimization, namely in the framework of min-max-control [25, Section 1.6], [176], [27], [99], [179], [152], in Chapter 6.

In any of the above problem settings there are again a number of different optimal path problems. Without raising the claim of completeness, we have illustrated the most common problem variants of the time-dependent optimal path problem in Figure 4.1. The one-to-one optimal path problem consists of the computation of optimal paths from one source node to one goal node, whereas the one-to-all optimal path problem consists of the computation of optimal paths from one source node to all other nodes in the network. Since most optimal path algorithms are based on the principle of dynamic programming, most one-to-one solution methods rely on one-to-all solution methods which are terminated as soon as the goal node is reached with the optimal cost. The all-to-one optimal path problem consists of the computation of optimal paths to one goal node from all other nodes in the network and is equivalent to the one-to-all optimal path problem if the direction of optimization

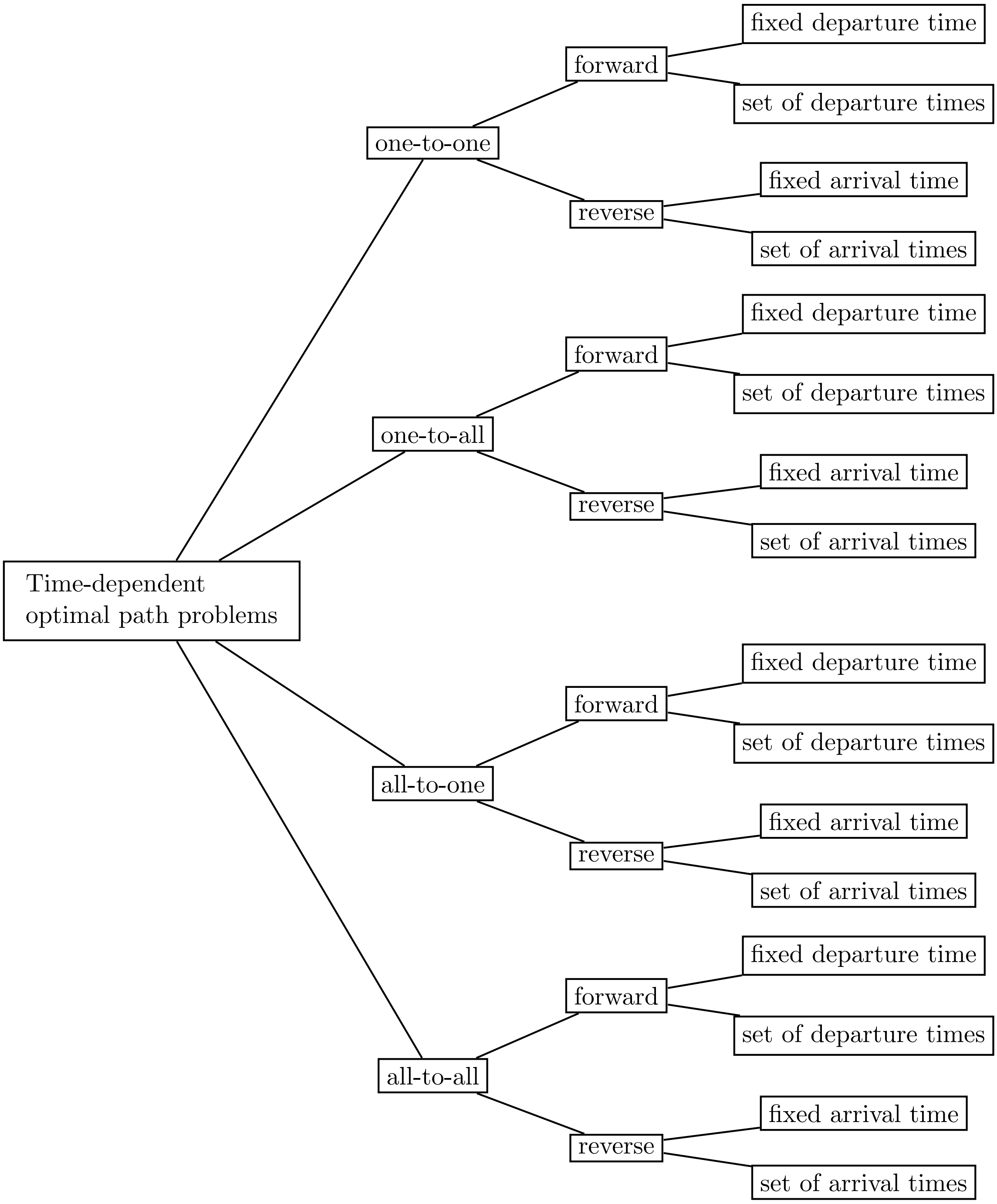


Figure 4.1.: Problem variants of the time-dependent optimal path problem.

can be reversed without changing the mathematical structure of the optimization problem. Finally, the all-to-all optimal path problem consists of the computation of optimal paths between any pair of nodes in the network.
In time-dependent networks, the direction of optimization cannot be generally reversed without changing the mathematical structure of the optimization problem. The computation of fastest paths in FIFO-networks constitutes an exception of this situation. Yet, if the travel time function is not bijective or waiting times must be considered in the optimization procedure, we must distinguish between the forward (cp. Definition 3.3.4) and the reverse optimal path problem (cp. Definition 3.3.7). Let us assume that one source node $v_0 \in V$ to one goal node $v' \in V$ have been fixed (i.e., let us consider the one-to-one optimal path problem), which can be seen as a problem variant from which the other variants can be constructed. In the forward problem, we then further distinguish between a fixed departure time and a varying departure time (taking values in a certain set of departure times). Here, for each departure time t_0 under consideration, the optimal path from (v_0, t_0) to v' must be determined, whereat the arrival time $t' \in T(v')$ is arbitrary. Similarly, in the reverse optimal path problem, we distinguish between a fixed arrival time and a varying arrival time (taking values in a certain set of arrival times). At this, for each arrival time t' under consideration, the optimal path from v_0 to (v', t') must be determined, whereat the departure time $t_0 \in T(v_0)$ at v_0 is arbitrary.
As we have anticipated in Section 3.3, we will mostly consider the optimization with respect to a given goal node and varying initial state. In particular, we will develop solution techniques for the one-to-one and the all-to-one forward optimal path problem with fixed and varying departure time, cf. Chapters 8, 9. In the remainder of this chapter, we will derive the theoretical foundation of these methods, which we extend to the situation of uncertain travel times and costs in Chapter 6.

4.2. Existence of Optimal Paths

After recalling some properties of point-to-set mappings at the beginning of this section, we will address the question under which assumptions finite optimal paths exist. We will consider both the case of forward optimal paths (cf. Theorem 4.2.4) and the case of reverse optimal paths (cf. Theorem 4.2.9). The results which we present in this section extend the state of the art both by considering a more general network model and by relaxing some of the usually imposed assumptions [138].
In the unconstrained and continuous case, using cumulative waiting functions, there always exists a finite optimal path if the cost functions fulfill certain growth conditions [138]. This result is based on the continuity of the optimal value function and on the non-negativity of the cost functions. In [143] the continuity of the cost functions was relaxed to lower semicontinuity, but the positivity of the cost functions had to be guaranteed starting from a certain point in time. Considering cost values which are related to energy consumption, we believe that this assumption is too restrictive, cf. Section 2.1 and Subsection 2.4.2. It may generally be possible to gain energy by traveling from a state of high potential to a state of low potential. Nevertheless, if both the potential energy and the kinetic energy of a commodity traveling in the network are bounded, it is not possible to gain an infinite amount of energy. Hence, we will assume that the cost associated with each control action is bounded from below, and that the cost along each circle in the network is strictly positive.

(By a circle, we mean a control sequence u with $\omega(u) = \alpha(u)$. Note that the corresponding edge sequence is a circle in the graph (V, E).) Similar assumptions have been used to prove the finiteness of optimal paths in static networks [92, Lemma 2.132] and in a network-flow formulation of the time-dependent optimal path problem [112]. We will now extend this result to time-dependent networks with state space and control constraints.
For this purpose, we first recall some properties of lower-semicontinuous functions. Let Y be a topological space. An extended real-valued function $f : Y \to \mathbb{R} \cup \{\pm\infty\}$ is called lower semicontinuous, if for any $y \in Y$ and any $\epsilon > 0$ there exists a neighborhood U_Y of y, such that $f(U_Y) \subset [f(y) - \epsilon, \infty]$ [30]. It is well known (see, e.g., [30][Theorem 2.6]) that lower semicontinuous functions attain their minimum on compact sets. The following definition is adopted from [65, Definition 2.2.1, Definition 2.2.2, p.13] for completeness:

Definition 4.2.1 *Let T, Y be topological spaces and Γ be a point-to-set mapping from T to subsets of Y.*

(i) Γ is upper semicontinuous at $t_0 \in T$ if, for each open set $U_Y \subset Y$ containing $\Gamma(t_0)$ there exists a neighborhood $U_T \subset T$ of t_0, such that for each $t \in U_T$, $\Gamma(t) \subset U_Y$. If Γ is upper semicontinuous at each point of T with $\Gamma(t)$ compact for each $t \in T$, then Γ is said to be upper semicontinuous.

(ii) Γ is lower semicontinuous at $t_0 \in T$ if, for each open set $U_Y \subset Y$ satisfying $U_Y \cap \Gamma(t_0) \neq \emptyset$ there exists a neighborhood $U_T \subset T$ of t_0, such that for each $t \in U_T$, $\Gamma(t) \cap U_Y \neq \emptyset$. If Γ is lower semicontinuous at each point of T, then Γ is said to be lower semicontinuous.

(iii) Γ is continuous in T if it is both upper and lower semicontinuous.

(iv) Let $t_n \in \mathbb{N}$ for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty} t_n = t_0$ and $t_0 \in T$. Γ is closed at t_0 if $y_n \in \Gamma(t_n)$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} y_n = y_0$ together imply $y_0 \in \Gamma(t_0)$. If Γ is closed at each point of T, then Γ is said to be closed in T.

(v) Γ is uniformly compact near $t_0 \in T$ if there exists an open neighborhood $U_T \subset T$ of t_0 such that $\mathrm{cl}\big(\bigcup_{t\in U_T} \Gamma(t)\big)$ is compact. If Γ is uniformly compact near each point of T, then Γ is said to be uniformly compact.

By $\mathrm{graph}(\Gamma) = \{(t, y) \in T \times Y : y \in \Gamma(t)\}$ *we denote the graph of Γ and by* $\mathrm{supp}(\Gamma) = \{t \in T : \Gamma(t) \neq \emptyset\}$ *we denote the support of Γ.*

It is easily seen that the graph and the support of an upper semicontinuous point-to-set mapping are closed. Moreover, it is well-known that upper semicontinuous point-to-set mappings are closed [65, p.13]. The following lemma will be used to combine the waiting time restrictions with the state space constraints.

Lemma 4.2.2 *Let T, Y, Z denote topological spaces, $f : T \times Y \to Z$ a continuous function, $\Lambda : T \to \mathcal{P}(Y)$ an upper semicontinuous point-to-set mapping, and suppose that Y is locally compact and satisfies the second axiom of countability. If $C \subset Z$ is a closed set, then the point-to-set mapping $\Gamma : T \to \mathcal{P}(Y)$, $\Gamma(t) = \{y \in \Lambda(t) : f(t, y) \in C\}$ is upper semicontinuous.*

Proof Suppose that there is a $t_0 \in T$, such that Γ is not upper semicontinuous in t_0. Then, there exists an open set $U_Y \subset Y$ containing $\Gamma(t_0)$, such that there is no open neighborhood U_T of t_0 with $\Gamma(t) \subset U_Y$ for all $t \in U_T$. Hence, there is a sequence $(t_n)_{n\in\mathbb{N}}$, $\lim_{n\to\infty} t_n = t_0$, such that for each $n \in \mathbb{N}$ there exists at least one $y_n \in \Gamma(t_n)\backslash U_Y$. Since $(t_n)_{n\in\mathbb{N}}$ is convergent, $\{t_n\}_{n\in\mathbb{N}}$ is compact. As Λ is upper semicontinuous, for every $n \in \mathbb{N}$ and for any open set U_Y^n containing $\Lambda(t_n)$, there exists an open neighborhood U_T^n of t_n, such that $\Lambda(t) \subset U_Y^n$ for all $t \in U_T^n$. Since Y is locally compact and $\Lambda(t_n)$ is compact for all $n \in \mathbb{N}$, we may choose the $(U_Y^n)_{n\in\mathbb{N}}$ in such a way, that the closure $\mathrm{cl}(U_Y^n)$ of U_Y^n is compact. Furthermore, as $\{t_n\}_{n\in\mathbb{N}}$ is compact, we may choose a finite number of the $(U_T^n)_{n\in\mathbb{N}}$, say $(U_T^n)_{n=1,...,N}$, $N \in \mathbb{N}$, with $\{t_n\}_{n\in\mathbb{N}} \subset \bigcup_{n=1}^N U_T^n$. This implies, that $K_Y = \bigcup_{n=1}^N \mathrm{cl}(U_Y^n)$ is compact and contains $\Lambda(t_n)$ for all $n \in \mathbb{N}$. Hence, as Y satisfies the second axiom of countability [93, Definition 10.10], we may (possibly after choosing a subsequence) assume that $\lim_{n\to\infty} y_n = y_0 \in K_Y$. Since Λ is upper semicontinuous, it is also closed, and we obtain $y_0 \in \Lambda(t_0)$. As f is continuous and C is closed, we also obtain $f(t_0, y_0) \in C$, and hence $y_0 \in \Gamma(t_0)$. On the other hand, we have assumed that $y_n \in Y \setminus U_Y$, which implies $y_0 \in Y \setminus U_Y$ and thus $y_0 \notin \Gamma(t_0)$, because $Y \setminus U_Y$ is closed. This is a contradiction. As $\Lambda(t_0)$ is closed, C is closed and f is continuous, $\Gamma(t_0) = \Lambda(t_0) \cap \{y \in Y : f(t_0, y) \in C\}$ is closed. Since $\Lambda(t_0)$ is compact, $\Gamma(t_0)$ is compact. □

In the remainder of this section, we assume that a time-dependent network $G = (V, E, \tau; \beta, \delta)$ with arrival time constraints T and waiting time constraints ΔT is given. We will also shortly denote this triple by $(G, T, \Delta T)$. Note that, if a goal node has been fixed, the optimal value function is uniquely determined by $(G, T, \Delta T)$.
In the following theorem we establish the existence of the solution to the forward optimal path problem.

Assumption 4.2.3 *Let $G = (V, E, \tau; \beta, \delta)$ denote a time-dependent network. Suppose that τ is continuous and β, δ are lower semicontinuous. Suppose that $T(v)$ is a closed set for all $v \in V$ and that the point-to-set mapping ΔT is upper semicontinuous. Further, suppose that there exist $\underline{\mathcal{B}}, \underline{\mathcal{B}}^\circ \in \mathbb{R}$, $\underline{\mathcal{B}}^\circ > 0$, such that*

$$\mathcal{B}\big((v,t),u\big) \geq \underline{\mathcal{B}}, \quad \forall u \in U(v,t) \text{ with } |u| = 1, \tag{4.1}$$

$$\mathcal{B}\big((v,t),u\big) \geq \underline{\mathcal{B}}^\circ, \quad \forall u \in U(v,t) \text{ with } \omega(u) = \alpha(u). \tag{4.2}$$

Theorem 4.2.4 *Let $G = (V, E, \tau; \beta, \delta)$ denote a time-dependent network in which Assumption 4.2.3 holds and let $v_0, v' \in V$. Then, for any $t_0 \in T_R(v_0)$, there exists a finite optimal path from (v_0, t_0) to v' and the partial function $t_0 \mapsto b^*(v_0, t_0)$ is lower semicontinuous on $T_R(v_0)$.*

Proof Let $(e_1, ..., e_n)$ denote a finite, connected edge sequence from v_0 to v'. Denote $v_{k-1} = \alpha(e_k)$ for $k = 1, ..., n$, and $\omega(e_n) = v_n = v'$. Note that the choice of this specific edge sequence imposes additional constraints on the set of admissible controls at each $(v_k, t) \in X$, $k = 0, ..., n-1$. We denote $\widetilde{T}_n = T(v_n)$, and for $k = 0, ..., n-1$, we define $\widetilde{\Delta T}_k : T(v_k) \to \mathcal{P}(\mathbb{R}_0^+)$,

$$\widetilde{\Delta T}_k(t) = \big\{\Delta t \in \Delta T(v_k, t) : t + \Delta t + \tau(e_k, t + \Delta t) \in \widetilde{T}_{k+1}\big\}, \tag{4.3}$$

$$\widetilde{T}_k = \mathrm{supp}(\widetilde{\Delta T}_k). \tag{4.4}$$

By backwards induction, as τ is continuous, ΔT is upper semicontinuous and $\widetilde{T}_n$ is closed, Lemma 4.2.2 implies that $\widetilde{\Delta T}_k$ is an upper semicontinuous point-to-set mapping and $\widetilde{T}_k$ is closed for all $k = 0, ..., n-1$. Note that the set of admissible control actions at $(v_k, t) \in X$ along $(e_1, ..., e_n)$ is given by $\widetilde{\Delta T}_k(t) \times \{e_k\}$, cf. (3.3).
We now analyze the optimal-cost function $\widetilde{b}^*$ along this edge sequence by backwards induction. Since the cost of each circle is strictly positive, we have

$$\widetilde{b}^*(v_n, t) = \widetilde{b}^*(v', t) = b^*(v', t) = 0$$

for all $t \in \widetilde{T}_n = T(v')$. Clearly, $\widetilde{b}^*(v_n, \cdot)$ is lower semicontinuous. As the determination of the optimal cost function along $(e_1, ..., e_n)$ is a decision problem over a finite number of stages, [25, Proposition 1.3.1] yields

$$\widetilde{b}^*(v_k, t) = \inf_{\Delta t \in \widetilde{\Delta T}_k(t)} b_k(t, \Delta t), \quad k = 0, ..., n-1, \tag{4.5}$$

where

$$b_k(t, \Delta t) = \delta(v_k, t, \Delta t) + \beta(e_k, t + \Delta t) + \widetilde{b}^*\big(v_{k+1}, t + \Delta t + \tau(e_k, t + \Delta t)\big).$$

Since b_k is a real-valued lower semicontinuous function and $t \mapsto \widetilde{\Delta T}_k(t)$ is an upper semicontinuous point-to-set mapping, [65, Theorem 2.2.1] implies that $t \mapsto \widetilde{b}^*(v_k, t)$ is lower semicontinuous on $\widetilde{T}_k$. Moreover, as $\widetilde{\Delta T}_k(t)$ is compact and nonempty for each $t \in \widetilde{T}_k$, the minimum in (4.5) is attained by some $\Delta t_k^*(t)$, $k = 0, ..., n-1$.
Next, if $N \in \mathbb{N}$, we observe that any control sequence u with $|u| \geq N|V| + |V| - 1$ contains at least N circles, which implies

$$\mathcal{B}\big((v_0, t_0), u\big) \geq (|V| - 1)\underline{\mathcal{B}} + N\underline{\mathcal{B}}^\circ. \tag{4.6}$$

Now, for $t_0 \in T_R(v_0)$, there exists a (finite) control sequence $u_0 \in U(v_0, t_0)$ with $\omega(u_0) = v'$. Let $\overline{\mathcal{B}}_0 = \mathcal{B}\big((v_0, t_0), u_0\big)$. (4.6) implies that the length of an optimal control sequence $u^* \in U(v_0, t_0)$ is bounded from above by

$$|u^*| \leq |V| - 1 + \frac{\overline{\mathcal{B}}_0 - (|V| - 1)\underline{\mathcal{B}}}{\underline{\mathcal{B}}^\circ}|V|. \tag{4.7}$$

As there is only a finite number of edge sequences of bounded length, there exists an optimal path from (v_0, t_0) to v'.
We now prove the lower semicontinuity of the partial function $t \mapsto b^*(v_0, t)$ at $t_0 \in T_R(v_0)$. Let $t_k \in T_R(v)$ for $k \in \mathbb{N}$ with $\lim_{k\to\infty} t_k = t_0$. For each $k \in \mathbb{N}$ there exists a $u_k \in U(v_0, t_k)$ with $\mathcal{B}\big((v_0, t_k), u_k\big) = b^*(v_0, t_k)$. Let $(e_1^k, ..., e_{n(k)}^k)$ be the edge sequence applied by the optimal control sequence u_k, $k \in \mathbb{N}$. As there is only a finite number of edge sequences of bounded length, there exists a $k_0 \in \mathbb{N}$ such that for all $K > k_0$ there either holds

(i) $n(K) > |V| - 1 + |V|\big(\overline{\mathcal{B}}_0 - (|V| - 1)\underline{\mathcal{B}}\big)/\underline{\mathcal{B}}^\circ$, or

(ii) there exists a $k \leq k_0$ such that $(e_1^k, ..., e_{n(k)}^k) = (e_1^K, ..., e_{n(K)}^K)$.

(4.6) implies that

$$n(K) > |V| - 1 + \frac{\overline{\mathcal{B}}_0 - (|V|-1)\underline{\mathcal{B}}}{\underline{\mathcal{B}}^\circ}|V| \quad \Longrightarrow \quad b^*(v_0, t_k) > b^*(v_0, t_0). \tag{4.8}$$

Let $\mathcal{E}$ denote the (finite) set of all edge sequences $(e_1, ..., e_n)$ with $n \leq |V|-1+|V|\big(\overline{\mathcal{B}}_0-(|V|-1)\underline{\mathcal{B}}\big)/\underline{\mathcal{B}}^\circ$ and $(e_1, ..., e_n) = (e_1^k, ..., e_{n(k)}^k)$ for infinitely many $k \in \mathbb{N}$. For each $(e_1, ..., e_n) \in \mathcal{E}$, let

$$\widetilde{T}_{(e_1,...,e_n)} = \big\{t \in T(v_0) : \exists (\Delta t_k)_{k=1,...,n} \text{ such that } \big((\Delta t_k, e_k)\big)_{k=1,...,n} \in U(v_0, t)\big\}.$$

For each $(e_1, ..., e_n) \in \mathcal{E}$, we obtain that $t_0 \in \widetilde{T}_{(e_1,...,e_n)}$ since $\widetilde{T}_{(e_1,...,e_n)}$ is the support of an upper semicontinuous point-to-set mapping and hence closed. The lower semicontinuity of the optimal value function along each $(e_1, ..., e_n) \in \mathcal{E}$ together with (4.8) yields the lower semicontinuity of the partial function $t \mapsto b^*(v_0, t)$ at $t_0 \in T_R(v_0)$. □

Remark 4.2.5 *In Theorem 4.2.4, we have implicitly assumed that the set of admissible waiting times $\Delta T(v, t)$ is compact for each $(v, t) \in X$ (cf. Definition 4.2.1). This assumption can be relaxed if the waiting cost function is uniformly divergent, i.e.,*

$$\lim_{\Delta t \to \infty} \delta(v, t, \Delta t) = \infty, \quad \textit{uniformly in } (v, t) \in X,$$

and $\Delta T(v, t)$ is closed for each $(v, t) \in X$. (In this case, for each $\overline{\mathcal{B}}_0 > 0$, there exists a $\overline{\Delta t} > 0$ such that $\delta(v, t, \Delta t) > \overline{\mathcal{B}}_0$ for all $\Delta t > \overline{\Delta t}$ and we may consider the modified waiting time restrictions $\Delta T(v, t) \cap [0, \overline{\Delta t}]$ without changing the optimal value function.) Note, that a similar growth condition has been assumed in [138], in order to show that each optimal path has an associated finite waiting policy.

Remark 4.2.6 *We are only considering nonnegative travel times in our model, cf. Definition 3.2.1. However, in the context of optimal flows in time-dependent networks it is essential to allow the travel times to assume negative values when computing optimal paths in the residual network [112]. It can be seen from the proofs of Lemma 4.2.2 and Theorem 4.2.4, that the result of Theorem 4.2.4 also holds for a continuous travel time function $\tau : E \times \mathbb{R} \to \mathbb{R}$.*

The existence of optimal paths from v_0 to v' is only guaranteed if the departure time at v_0 satisfies $t_0 \in T_R(v_0)$. Using the results of Section 3.5 we immediately obtain the following corollary.

Corollary 4.2.7 *If Assumption 4.2.3 and the assumptions of Lemma 3.5.2 hold, then for any $(v_0, t_0) \in X$ and any $v' \in V$ there exists an optimal path from (v_0, t_0) to v'.*

Proof From Lemma 3.5.2 we obtain $t_0 \in T_R(v_0)$ for all $t_0 \in T(v_0) = \mathbb{R}$. The result now follows from Theorem 4.2.4. □

We now address the question under which assumptions an optimal solution of the reverse optimal path problem exists. Here, the situation is complicated by the fact that for each traversal of an edge $e \in E$, we have to consider the set of all possible departure times. In particular, instead of considering the time transition $t \mapsto t + \tau(e,t)$, we have to consider the mapping $t' \mapsto \{t \in \mathbb{R} : t + \tau(e,t) = t'\}$. In order to analyze the reverse optimal path problem, we introduce the point-to-set mapping $\tau^{-1} : E \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$,

$$\tau^{-1}(e,t') = \{t \in \mathbb{R} : t + \tau(e,t) = t'\}. \tag{4.9}$$

Note, that τ^{-1} is not the inverse function of τ, but rather a notation. However, if G satisfies the strong FIFO property, then $t' \mapsto \tau^{-1}(e,t')$ is indeed the inverse of $t \mapsto t + \tau(e,t)$ for each $e \in E$. In order to establish the existence of reverse optimal paths and the lower semicontinuity of the reverse optimal value function we first prove the following preliminary lemma.

Lemma 4.2.8 *Suppose that the travel time function τ is continuous, $T(v)$ is a closed set for all $v \in V$ and that the point-to-set mapping ΔT is upper semicontinuous. If either*

(i) $\lim_{t\to-\infty} \tau(e,t) = -\infty$ for all $e \in E$, and there exists a real-valued function $\overline{\Delta T} : X \to \mathbb{R}_0^+$ such that, for each $v \in V$, $\Delta T(v,\widetilde{t}) \subset [0, \overline{\Delta T}(v,t)]$ for all $\widetilde{t} \in T(v)$ with $\widetilde{t} \le t$, or

(ii) $T(v)$ is bounded from below for all $v \in V$,

then the point-to-set mapping $\Lambda : E \times \mathbb{R} \to \mathcal{P}(\mathbb{R}^2)$,

$$\Lambda(e,t') = \Big\{(t,\Delta t) \in \mathbb{R}^2 : t \in T\big(\alpha(e)\big), \Delta t \in \Delta T\big(\alpha(e),t\big), t + \Delta t \in \tau^{-1}(e,t')\Big\} \tag{4.10}$$

is upper semicontinuous.

Proof First, we prove that $\Lambda(e,t')$ is compact for each $(e,t') \in E \times \mathbb{R}$. Let $e \in E$, $s : \mathbb{R}^2 \to \mathbb{R}$, $s(t,\Delta t) = t + \Delta t$, and $\Delta T_e : T\big(\alpha(e)\big) \to \mathcal{P}(\mathbb{R}_0^+)$, $\Delta T_e(t) = \Delta T\big(\alpha(e),t\big)$. Obviously, s is continuous and ΔT_e is upper semicontinuous. Since $\operatorname{graph}(\Delta T_e)$ is closed and s, τ are continuous $\Lambda(e,t') = \operatorname{graph}(\Delta T_e) \cap s^{-1}(\tau^{-1}(e,t'))$ is closed. Suppose that (i) holds. Then there exists a $\underline{t} \in T\big(\alpha(e)\big)$ such that $t + \Delta t + \tau(e, t+\Delta t) < t'$ for all $(t,\Delta t) \in \mathbb{R} \times \mathbb{R}_0^+$ with $t + \Delta t < \underline{t}$. Since $\tau \ge 0$ there holds $t + \Delta t \le t'$ for all $(t,\Delta t) \in \tau^{-1}(e,t')$. Using $0 \le \Delta t \le \overline{\Delta T}(t')$, we obtain $\underline{t} - \overline{\Delta T}(t') \le t \le t'$ for all $(t,\Delta t) \in \Lambda(e,t')$. Now, suppose that (ii) holds. Then $\min T(\alpha(e)) \le t \le t'$ and $0 \le \Delta t \le t' - \min T\big(\alpha(e)\big)$ for all $(t,\Delta t) \in \Lambda(e,t')$. In either case, $\Lambda(e,t')$ is bounded and hence compact.
Next, we prove that τ^{-1} is closed. Let $e \in E$ and $t'_n \in T\big(\omega(e)\big)$ for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty} t'_n = t'_0$ and $t'_0 \in T\big(\omega(e)\big)$. Let $t_n \in \tau^{-1}(e,t'_n)$ for $n \in \mathbb{N}$ with $\lim_{n\to\infty} t_n = t_0$, $t_0 \in \mathbb{R}$. The continuity of τ implies $t'_0 = \lim_{n\to\infty} \tau(e,t_n) = \tau(e,t_0)$ and hence $t_0 \in \tau^{-1}(e,t'_0)$.
We now prove that Λ is upper semicontinuous. Suppose that there is a $e \in E$ and a $t'_0 \in T\big(\omega(e)\big)$ such that Λ is not upper semicontinuous in (e,t'_0). Then there exists an open set $U_\Lambda \subset \mathbb{R}^2$ with $\Lambda(e,t'_0) \subset U_\Lambda$ and a sequence of real numbers $\{t'_n\}_{n\in N}$ with $t'_n \in T\big(\omega(e)\big)$ and $\lim_{n\to\infty} t'_n = t'_0$, as well as a sequence $\{(t_n,\Delta t_n)\}_{n\in\mathbb{N}}$ satisfying $(t_n,\Delta t_n) \in \Lambda(e,t'_n)$ and $(t_n,\Delta t_n) \notin U_\Lambda$ for all $n \in \mathbb{N}$. As $\{t'_n\}_{n\in\mathbb{N}}$ is convergent, from a similar reasoning as above, we obtain the existence of a compact set $K \subset \mathbb{R}^2$ with $\Lambda(t'_n) \subset K$ for all $n \in \mathbb{N}$. Consequently, $\{(t_n,\Delta t_n)\}_{n\in\mathbb{N}}$ must contain a convergent subsequence, which (without loss of generality) we again denote by $\{(t_n,\Delta t_n)\}_{n\in\mathbb{N}}$. Since $T\big(\alpha(e)\big)$ is closed and $t_n \in T\big(\alpha(e)\big)$ for all $n \in \mathbb{N}$, we obtain $t_0 = \lim_{n\to\infty} t_n \in T\big(\alpha(e)\big)$. Similarly, since $\Delta t_n \in \Delta T_e(t_n)$ and ΔT_e is upper semicontinuous (and hence closed) we obtain $\Delta t_0 = \lim_{n\to\infty} \Delta t_n \in \Delta T_e(t_0)$. Finally,

since s is continuous and τ^{-1} is closed we obtain $t_0 + \Delta t_0 \in \tau^{-1}(e, t_0')$, and consequently $(t_0, \Delta t_0) \in \Lambda(e, t_0') \subset U_\Lambda$. This is a contradiction, as U_Λ is open and $(t_n, \Delta t_n) \notin U_\Lambda$ for all $n \in \mathbb{N}$. □

We are now ready to prove the analogon of Theorem 4.2.4 for the reverse optimal path problem.

Theorem 4.2.9 *Let $G = (V, E, \tau; \beta, \delta)$ be a time-dependent network in which Assumption 4.2.3 holds and suppose that either*

(i) $\lim_{t \to -\infty} \tau(e, t) = -\infty$ *for all* $e \in E$*, and there exists a real-valued function* $\overline{\Delta T} : X \to \mathbb{R}_0^+$ *such that, for each* $v \in V$*,* $\Delta T(v, \widetilde{t}) \subset [0, \overline{\Delta T}(v, t)]$ *for all* $\widetilde{t} \in T(v)$ *with* $\widetilde{t} \leq t$*, or*

(ii) $T(v)$ *is bounded from below for all* $v \in V$*.*

Let a source node $v_0 \in V$ and a goal node $v' \in V$ be given. Then, for any $t' \in T_R(v')$, there exists a (finite) reverse optimal path from v_0 to (v', t') and the partial function $t' \mapsto b_*(v', t')$ is lower semicontinuous on $T_R(v')$.

Proof In a similar manner as in the proof of Theorem 4.2.4 we consider a finite connected edge sequence $(e_1, ..., e_n)$ from v_0 to v'. Along this edge sequence we use the principle of dynamic programming [25, Proposition 1.3.1], which yields for the optimal cost function $\widetilde{b}_*$ of this subproblem

$$\widetilde{b}_*(\alpha(e_1), t) = 0, \tag{4.11}$$

$$\widetilde{b}_*(\omega(e_k), t) = \inf_{(t, \Delta t) \in \Lambda(e_k, t)} b_k(t, \Delta t), \quad k = 1, ..., n\,, \tag{4.12}$$

where Λ is defined in Lemma 4.10 and

$$b_k(t, \Delta t) = \delta(\alpha(e_k), t, \Delta t) + \beta(e_k, t + \Delta t) + \widetilde{b}_*\big(\alpha(e_k), t + \Delta t + \tau(e_k, t + \Delta t)\big).$$

Since $\widetilde{b}_*(\alpha(e_1), t)$ is lower semicontinuous and Λ is upper semicontinuous, by forward induction [65, Theorem 2.2.1] yields the lower semicontinuity of $\widetilde{b}_*$ and the existence of a minimizing argument in (4.12). The rest of the proof follows as in the proof of Theorem 4.2.4. □

Corollary 4.2.10 *If the assumptions of Theorem 4.2.9 with (i) rather than (ii) and the assumptions of Lemma 3.5.2 hold, then for any $v_0 \in V$ and any $(v', t') \in X$ there exists an optimal path from v_0 to (v', t').*

Proof From Lemma 3.5.2 we obtain $t' \in T_R(v')$ for all $t' \in T(v') = \mathbb{R}$. The result now follows from Theorem 4.2.9. □

4.3. Dynamic Programming in Time-Dependent Networks

Similar to the time-independent case, the algorithmic solution of time-dependent optimal path problems is usually based on the principle of dynamic programming [137], [138], [143],

[144], [35], [38], [47], [39], [49], [132], [54]. Based on the results in the previous section, we hereafter present the dynamic programming equations both for the forward and the reverse optimal path problem. Due to the hybrid structure of time-dependent networks these equations are of a hybrid nature, requiring the simultaneous optimization with respect to the optimal edge and the optimal waiting time. We show how these equations can be decoupled by using the split network which we have introduced in Section 3.2.
We begin by considering the forward optimal path problem. As in the previous section, we assume that a time-dependent network $G = (V, E, \tau; \beta, \delta)$ with arrival time constraints T and waiting time constraints ΔT is given.

Proposition 4.3.1 *Suppose that Assumption 4.2.3 holds and that a goal node $v' \in V$ is given. The optimal value function b^* defined in (3.10) satisfies the following dynamic programming equations:*

$$b^*(v', t) = 0, \qquad \forall t \in T(v'), \tag{4.13}$$

$$b^*(v, t) = \min_{\substack{u \in U(v,t) \\ u = (\Delta t, e)}} \Big[b^*\big(\varphi((v,t),u)\big) + \delta(v, t, \Delta t) + \beta(e, t + \Delta t) \Big], \qquad \forall v \in V \setminus \{v'\}, t \in T(v). \tag{4.14}$$

Proof As a consequence of (4.2) and since $\underline{\mathcal{B}}^\circ > 0$, we observe that $b^*(v', t) \geq 0$ for all $t \in T(v)$, and that the termination of the path from v' to v' in the initial state leads to the optimal cost $b^*(v', t) = 0$ for all $t \in T(v')$. Since we have already proved the existence of optimal paths in Theorem 4.2.4, the result follows from standard arguments, see, e.g., [26, Proposition 3.1.1]. □

Referring to the result of Proposition 4.3.1, we define the optimal control policy $\mu^* : X_R \to (\mathbb{R}_0^+ \times E) \cup \{\text{"termination"}\}$,

$$\mu^*(v', t) = \text{"termination"}, \qquad \forall t \in T_R(v'), \tag{4.15}$$

$$\mu^*(v, t) = \arg\min_{\substack{u \in U(v,t) \\ u = (\Delta t, e)}} \Big[b^*\big(\varphi((v,t),u)\big) + \delta(v, t, \Delta t) + \beta(e, t + \Delta t) \Big], \qquad \forall v \in V \setminus \{v'\}, t \in T_R(v). \tag{4.16}$$

The optimal control policy can be used to construct the optimal control sequence u^* with respect to a given initial state $x_0 \in X$ and the terminal node $v' \in V$ as follows:

$$u_k^* = \mu^*(x_{k-1}), \qquad k = 1, 2, \dots, \tag{4.17}$$

$$x_k = \varphi(x_{k-1}, u_k^*), \qquad k = 1, 2, \dots. \tag{4.18}$$

The above recursion terminates as soon as $\mu^*(x_k) =$ "termination" for some $k \in \mathbb{N}$. Note that due to the definition of the optimal control policy in (4.15)-(4.16), there holds $\mathcal{B}(x_0, u^*) = b^*(x_0)$. Since Theorem 4.2.4 implies that infinite control sequences cause infinite cost the recursion terminates after a finite number of steps. The characterization (4.17)-(4.18) of optimal control sequences is commonly used for the algorithmic reconstruction of the optimal path $p^* = (x_k)_{k=0,1,\dots}$, cp. Chapter 8 and Chapter 9.
Since the simultaneous optimization of Δt and e in (4.14) is not convenient for algorithmic

purposes, we will also use an alternative version of the optimality equations, which is based on the split network, cf. Section 3.2. In the split network, the optimality equation (4.14) is decoupled and takes the form

$$b^*(v_{\mathrm{w}}, t_{\mathrm{w}}) = \min_{\Delta t \in \Delta T(v_{\mathrm{w}}, t_{\mathrm{w}})} \Big[b^*(v_{\mathrm{nw}}, t_{\mathrm{w}} + \Delta t) + \delta(v_{\mathrm{w}}, t_{\mathrm{w}}, \Delta t) \Big], \tag{4.19}$$

$$b^*(v_{\mathrm{nw}}, t_{\mathrm{nw}}) = \min_{\substack{e \in E^+(v_{\mathrm{nw}}): \\ t_{\mathrm{nw}} + \tau(e, t_{\mathrm{nw}}) \in T(\omega(e)_{\mathrm{w}})}} \Big[b^*\big(\omega(e)_{\mathrm{w}}, t_{\mathrm{nw}} + \tau(e, t_{\mathrm{nw}})\big) + \beta(e, t_{\mathrm{nw}}) \Big]. \tag{4.20}$$

Remark 4.3.2 *In order to solve (4.20) at a given node $v \in V$, the minimum function of the cost functions $\{b^*\big(\omega(e), t + \tau(e,t)\big) + \beta(e,t)\}_{e \in E^+(v)}$ must be computed for all $t \in \mathbb{R}$. Note that this involves the computation of the intersections of two functions, a task which is not explicitly solvable in the nonlinear case, and which may be adherent to an unknown number of arithmetic operations in general. Yet, if all network functions (i.e., $\tau, \beta, \delta, \Delta T$) are piecewise linear, the minimum function can be efficiently and explicitly computed [47]. In order to solve (4.19), a parametric optimization problem must be solved [18], [78]. Although efficient solution strategies have been developed (see, e.g., [78], [145]), this is not a simple task in general. Let us again consider the case in which the network functions are piecewise linear (see, e.g., [149] for properties of piecewise linear programs): In this case, the problem of determining the optimal waiting policy breaks down to the problem of choosing one of the line segments resulting from the time constraints and the piecewise linear structure of $(t, \Delta t) \mapsto b^*(v_{nw}, t + \Delta t) + \delta(v_w, t, \Delta t)$ in the t-Δt-plane. (If the waiting cost function and the optimal value function are linear on a polygon contained in the t-Δt-plane, then the edges of the polygon contain an optimal solution for all t, cp. Section 5.3.) Therefore, the problem of computing the optimal waiting policy is equivalent to finding the line segment with minimal associated cost. This task is equivalent to determining the optimal edge policy in (4.20).*

Referring to (4.19), (4.20), the optimal control policy μ^* for $(v,t) \in X$, $v \neq v'$ can be determined from the optimal waiting policy at v_{w}

$$\mu^*_{\mathrm{w}} : \bigcup_{v_{\mathrm{w}} \in V_{\mathrm{w}}} \big\{ \{v_{\mathrm{w}}\} \times T_R(v_{\mathrm{w}}) \big\} \to \mathbb{R}_0^+,$$

$$\mu^*_{\mathrm{w}}(v_{\mathrm{w}}, t_{\mathrm{w}}) = \operatorname*{arg\,min}_{\Delta t \in \Delta T(v_{\mathrm{w}}, t_{\mathrm{w}})} \Big[b^*(v_{\mathrm{nw}}, t_{\mathrm{w}} + \Delta t) + \delta(v_{\mathrm{w}}, t_{\mathrm{w}}, \Delta t) \Big], \tag{4.21}$$

and the optimal edge policy at v_{nw}

$$\mu^*_{\mathrm{nw}} : \bigcup_{v_{\mathrm{nw}} \in V_{\mathrm{nw}}} \big\{ \{v_{\mathrm{nw}}\} \times T_R(v_{\mathrm{nw}}) \big\} \to E, \tag{4.22}$$

$$\mu^*_{\mathrm{nw}}(v_{\mathrm{nw}}, t_{\mathrm{nw}}) = \operatorname*{arg\,min}_{\substack{e \in E^+(v_{\mathrm{nw}}): \\ t_{\mathrm{nw}} + \tau(e, t_{\mathrm{nw}}) \in T(\omega(e)_{\mathrm{w}})}} \Big[b^*\big(\omega(e)_{\mathrm{w}}, t_{\mathrm{nw}} + \tau(e, t_{\mathrm{nw}})\big) + \beta(e, t_{\mathrm{nw}}) \Big], \tag{4.23}$$

as follows (cp. (3.1) and Figure 4.2):

$$\mu^*(v,t) = \Big(\mu^*_{\mathrm{w}}(v_{\mathrm{w}}, t), \mu^*_{\mathrm{nw}}\big(v_{\mathrm{nw}}, t + \mu^*_{\mathrm{w}}(v_{\mathrm{w}}, t)\big) \Big). \tag{4.24}$$

For completeness, we now briefly consider the reverse optimal path problem.

Proposition 4.3.3 *Suppose that Assumption 4.2.3 holds and that a source node $v_0 \in V$ is given. Let Λ denote the point-to-set mapping defined in (4.10). The optimal value function b_* defined in (3.13) satisfies the following dynamic programming equations:*

$$b_*(v_0, t') = 0, \qquad \forall t' \in T(v_0), \tag{4.25}$$

$$b_*(v, t') = \min_{\substack{(t,\Delta t,e) \in \mathbb{R}^2 \times E^-(v): \\ (t,\Delta t) \in \Lambda(e,t')}} \Big[b_*(\alpha(e), t) + \delta(\alpha(e), t, \Delta t) + \beta(e, t + \Delta t) \Big],$$
$$\forall v \in V \setminus \{v_0\}, t' \in T(v). \tag{4.26}$$

Proof As a consequence of (4.2) and since $\underline{\mathcal{B}}^\circ > 0$, we observe that $b^*(v_0, t') \geq 0$ for all $t' \in T(v_0)$, and that the termination of the path from v_0 to v_0 in the initial state leads to the optimal cost $b^*(v_0, t') = 0$ for all $t' \in T(v_0)$. Since we have already proved the existence of optimal paths in Theorem 4.2.9, the result follows from standard arguments, see, e.g., [26, Proposition 3.1.1]. □

In general, an optimal control policy for the reverse optimal path problem must not only specify the control action which must have been implied in a preceding state $x \in X$ to reach a certain state $x' \in X$ from the source node v_0, but due to the fact that τ^{-1} is set-valued, the preceding state must also be specified. This situation is illustrated by formulating the equivalent of (4.26) in the split network:

$$b_*(v_\mathrm{w}, t'_\mathrm{w}) = \min_{\substack{(t_\mathrm{nw},e) \in \mathbb{R} \times E^-(v_\mathrm{w}): \\ t_\mathrm{nw} \in \tau^{-1}(e,t'_\mathrm{w})}} \Big[b_*(\alpha(e)_\mathrm{nw}, t_\mathrm{nw}) + \beta(e, t_\mathrm{nw}) \Big], \tag{4.27}$$

$$b_*(v_\mathrm{nw}, t_\mathrm{nw}) = \min_{\substack{(t_\mathrm{w},\Delta t) \in T(v_\mathrm{w}) \times \mathbb{R}_0^+: \\ \Delta t \in \Delta T(v_\mathrm{w},t_\mathrm{w}), t_\mathrm{w} + \Delta t = t_\mathrm{nw}}} \Big[b_*(v_\mathrm{w}, t_\mathrm{w}) + \delta(v_\mathrm{w}, t_\mathrm{w}, \Delta t) \Big]. \tag{4.28}$$

The state transitions in the split network are depicted in Figure 4.2. Note that t_nw is subject to optimization in (4.27), whereas the choice of t_w already determines the waiting time $\Delta t = t_\mathrm{nw} - t_\mathrm{w}$ in (4.28). Since, for some $x = (v, t) \in X$, the specification of the edge terminating in v uniquely determines the preceding node on a path through x, the specification of the corresponding departure time suffices for the unique reconstruction of reverse optimal paths. This approach has been carried out in [138] but will not be pursued here, since we will only be concerned with the algorithmic solution of the forward optimal path problem.

Since the computation of optimal waiting policies is costly in terms of computation time (cf. Theorem 5.3.21), it is of particular importance in real-time applications to identify network structures and problem variants in which this optimization can be avoided. Clearly, this is the case if $\Delta T(x)$ contains only one element for all $x \in X$. In this case, we may assume that $\Delta T(x) = \{0\}$ and $\delta(x, 0) = 0$ by appropriately transforming the network functions, cp. Section 3.5. Suppose that it can be shown that $\mu^*(x) \in \{0\} \times E$ for all $x \in X$, i.e., that it is never optimal to wait. This holds, e.g., for the computation of fastest paths under Assumption 3.5.3. In this case the dynamic programming equations are crucially simplified

$$(\alpha(e)_{\mathrm{w}}, t_{\mathrm{w}}) \xrightarrow{\Delta t} (\alpha(e)_{\mathrm{nw}}, t_{\mathrm{nw}}) \xrightarrow{e} (\omega(e)_{\mathrm{nw}}, t'_{\mathrm{w}})$$

Figure 4.2.: State transitions in the split network. In the forward problem, the optimization first takes place with respect to the edge e, then with respect to the waiting time Δt. In the reverse problem, the optimization first takes place with respect to the waiting time Δt, then with respect to both the departure time t_{nw} and the edge e.

and take the form:

$$\begin{aligned} b^*(v', t) &= 0, && \forall t \in T(v'), \\ b^*(v, t) &= \inf_{\substack{e \in E^+(v): \\ t+\tau(e,t) \in T(\omega(e))}} \Big[b^*\big(\omega(e), t + \tau(e,t)\big) + \beta(e,t) \Big], && \forall v \in V \setminus \{v'\}, t \in T(v). \end{aligned}$$

If a source node $v_0 \in V$ and a fixed departure time t_0 at v_0 are given, according to the above equations, the optimal path problem can be solved by slightly modified static optimal path algorithms [137], [39], [54]. Similarly, the equations for the reverse optimal value function become:

$$\begin{aligned} b_*(v_0, t') &= 0, && \forall t' \in T(v_0), \\ b_*(v, t') &= \inf_{\substack{e \in E^-(v): \\ \tau^{-1}(e,t') \in T(\alpha(e))}} \Big[b_*\big(\alpha(e), \tau^{-1}(e,t')\big) + \beta\big(e, \tau^{-1}(e,t')\big) \Big], && \forall v \in V \setminus \{v_0\}, t' \in T(v). \end{aligned}$$

Based on the similarity of the above dynamic programming equations, a network transformation has been introduced in [45], which allows the solution of the reverse optimal path problem using algorithms designed for the solution of the forward optimal path problem.

4.4. Order Relations on Time-Dependent Networks

The purpose of this section is mainly the introduction of a concept of comparability of time-dependent networks. After providing the definition, we will prove a basic result, which we will use in the derivation of the solution method in Chapter 9.

Definition 4.4.1 *Let $G_1 = (V_1, E_1, \tau_1; \beta_1, \delta_1), G_2 = (V_2, E_2, \tau_2; \beta_2, \delta_2)$ be time-dependent networks with arrival time constraints T_1, T_2 and waiting time constraints $\Delta T_1, \Delta T_2$, respectively. Denote by X_1 the state space associated with $(G_1, T_1, \Delta T_1)$, by X_2 the state space associated with $(G_2, T_2, \Delta T_2)$ and let $V = V_1 \cap V_2$, $X = X_1 \cap X_2$.*

We denote $G_1 \overset{*}{\leq} G_2$ if, for any fixed goal node $v' \in V$, the optimal value function b_1^* associated with $(G_1, T_1, \Delta T_1)$ and the optimal value function b_2^* associated with $(G_2, T_2, \Delta T_2)$ satisfy

$$b_1^*(x_0) \leq b_2^*(x_0), \qquad \forall x_0 \in X. \tag{4.29}$$

If $G_1 \overset{*}{\leq} G_2$ and $G_2 \overset{*}{\leq} G_1$, we denote $G_1 \overset{*}{=} G_2$.

Theorem 4.4.2 *Let $G_1 = (V_1, E_1, \tau_1; \beta_1, \delta_1)$ and $G_2 = (V_2, E_2, \tau_2; \beta_2, \delta_2)$ be time-dependent networks with arrival time constraints T_1, T_2 and waiting time constraints $\Delta T_1, \Delta T_2$, respectively. Denote by X_1 the state space associated with $(G_1, T_1, \Delta T_1)$, by X_2 the state space associated with $(G_2, T_2, \Delta T_2)$. If*

$$V_1 \supset V_2, \qquad E_1 \supset E_2,$$
$$T_1(v) \supset T_2(v), \quad \forall v \in V, \qquad \Delta T_1(x) \supset \Delta T_2(x), \quad \forall x \in X,$$

and

$$\tau_1(e_2, t_2) = \tau_2(e_2, t_2), \qquad \forall e_2 \in E_2, t_2 \in T_2\big(\alpha(e_2)\big), \tag{4.30}$$
$$\beta_1(e_2, t_2) \le \beta_2(e_2, t_2), \qquad \forall e_2 \in E_2, t_2 \in T_2\big(\alpha(e_2)\big), \tag{4.31}$$
$$\delta_1(x_2, \Delta t_2) \le \delta_2(x_2, \Delta t_2), \qquad \forall x_2 \in X_2, \Delta t_2 \in \Delta T_2(v_2, t_2), \tag{4.32}$$

then $G_1 \overset{*}{\le} G_2$.

Proof Let a goal node $v' \in V = V_1 \cap V_2$ and an initial state $x_0 \in X = X_1 \cap X_2$ be given. Let $U_1(x_0)$ denote the set of admissible control sequences in G_1 subject to $T_1, \Delta T_1$, let $U_2(x_0)$ denote the set of admissible control sequences in G_2 subject to $T_2, \Delta T_2$, and let the functions $\mathcal{B}, \Phi$ in G_1, G_2 be distinguished by the index $1, 2$, respectively. Since $V_1 \supset V_2$ we obtain $V = V_2$, and since $T_1(v) \supset T_2(v)$ for all $v \in V$ we obtain $X = X_2 \subset X_1$. Now, $\Delta T_1(x) \supset \Delta T_2(x)$ for all $x \in X$ implies that $U_1(x_0) \supset U_2(x_0)$ for all $x_0 \in X$. Moreover, (4.30) implies that $\Phi_1(x_0, u) = \Phi_2(x_0, u)$ and (4.31), (4.32) imply that $\mathcal{B}_1(x_0, u) \le \mathcal{B}_2(x_0, u)$ for all $u \in U_2(x_0)$ and all $x_0 \in X$. Let $x_0 \in X$ and define

$$u_2^* = \underset{u \in U_2(x_0) : \omega(u) = v'}{\arg\min} \mathcal{B}_2(x_0, u).$$

Then $u_2^* \in U_1(x_0)$ and we obtain

$$b_2^*(x_0) = \mathcal{B}_2(x_0, u_2^*) \ge \mathcal{B}_1(x_0, u_2^*) \ge \min_{u \in U_1(x_0) : \omega(u) = v'} \mathcal{B}_1(x_0, u) = b_1^*(x_0).$$

□

Remark 4.4.3 *Theorem 4.4.2 can be used to construct contraction hierarchies for the optimal path problem. This approach has led to a significant reduction of the computation time of time-independent optimal path problems [154], [74], and time-dependent fastest path problems [51], [54], [52].*

5. Properties of the Optimal Value Function

In this chapter, we only consider the forward optimal path problem and suppose that a time-dependent network $G = (V, E, \tau; \beta, \delta)$ with arrival time constraints T and waiting time constraints ΔT is given. We also assume that a goal node $v' \in V$ has been fixed.
In Section 5.1, we prove that the optimal value function is (Lipschitz-) continuous under some additional assumptions. We impose more mathematical structure on the definition of the waiting time restrictions in Section 5.2 and prove that the optimal value function is piecewise analytic if the network functions are. In Section 5.3 we are concerned with time-dependent networks in which all network functions are piecewise linear. Due to the explicit solvability (with a finite number of arithmetic operations) of all involved subproblems, such networks are of particular importance in practical applications. At this, we consider both the lower semicontinuous case of Section 4.2 and the continuous case of Section 5.1. We show that both the type of waiting time constraints, the piecewise linear structure of the network functions and the FIFO property of the time-dependent network have a strong impact on the time and space complexity of computing the optimal value function. Finally, we conclude this chapter by briefly discussing periodical time-dependent networks in Section 5.4.

5.1. Continuity

Before proving the continuity of the optimal value function in Theorem 5.1.3, we first present a result on the set of reachable states which is used in the proof of the theorem. We then prove in Theorem 5.1.10 that the optimal value function is even Lipschitz-continuous under some additional assumptions.

Assumption 5.1.1 *Let $v_0 \in V$ be a fixed source node, suppose that $\tau, \Delta T$ are continuous and suppose that $T(v_0) = [\underline{t}, \infty)$ for some $\underline{t} \in \mathbb{R}$. Denote $\widetilde{T}(v_0) = T(v_0)$ and $\widetilde{T}(v) = \mathbb{R}$ for all $v \in V \setminus \{v_0\}$. For each $(v,t) \in X$, let $\widetilde{\underline{t}}_R$ denote the earliest arrival time at v in $(G, \widetilde{T}, \Delta T)$, and let $\widetilde{U}(v,t)$ denote the set of control sequences which are admissible for (v,t) in $(G, \widetilde{T}, \Delta T)$. Suppose that $T(v) \supset [\widetilde{\underline{t}}_R(v), \infty)$ for all $v \in V$.*

Lemma 5.1.2 *Let $v_0 \in V$ be a fixed source node and suppose that Assumption 5.1.1 holds.*

(i) For all $v \in V$ and all $t \geq \widetilde{\underline{t}}_R(v)$, there holds $U(v,t) = \widetilde{U}(v,t)$. In particular, if $(v', t') = \varphi\big((v,t), u\big)$ for some $u \in \widetilde{U}(v,t)$, $t \geq \widetilde{\underline{t}}_R(v)$, then $t' \geq \widetilde{\underline{t}}_R(v')$.

(ii) If, in addition to the above assumptions, there exists a $\underline{\mathcal{T}}^\circ \in \mathbb{R}^+$ such that

$$\mathcal{T}\big((v,t), u\big) \geq \underline{\mathcal{T}}^\circ, \quad \forall u \in U(v,t) \text{ with } \omega(u) = \alpha(u), \tag{5.1}$$

then $T_R(v) = [\widetilde{\underline{t}}_R(v), \infty)$ for all $v \in V$.

Proof *(i)* We first show that $\widetilde{U}(v_0, t_0) = U(v_0, t_0)$ for all $t_0 \in T(v_0)$. Let $t_0 \in T(v_0)$. We clearly have $U(v_0, t_0) \subset \widetilde{U}(v_0, t_0)$. Let $u \in \widetilde{U}(v_0, t_0)$ and denote $p = \big((v_k, t_k)\big)_{k=0,...,n} = \Phi\big((v_0, t_0), u\big)$. According to the definition of $\widetilde{\underline{t}}_R(v)$, $v \in V$, we obtain $t_k \geq \widetilde{\underline{t}}_R(v_k)$, $k = 0, ..., n$. As $T(v) \supset [\widetilde{\underline{t}}_R(v), \infty)$ for all $v \in V$, this implies that $u \in U(v_0, t_0)$.
Now, let $v \in V$ and $t \geq \widetilde{\underline{t}}_R(v)$. Clearly, we have $t \in T(v)$. Suppose that $u \in \widetilde{U}(v, t) \backslash U(v, t)$ is such that $u_{1:|u|-1} \in U(v, t)$, and denote $v' = \omega(u)$. Then $t' = t + \mathcal{T}\big((v, t), u\big) \notin T(v')$, which implies that $t' < \widetilde{\underline{t}}_R(v')$. By definition of $\widetilde{\underline{t}}_R(v)$ there exists a sequence $(t_i)_{i\in\mathbb{N}}$ with $t_i \in T(v_0)$ for all $i \in \mathbb{N}$, and for each $i \in \mathbb{N}$, there exists a sequence $(u_{i,j})_{j\in\mathbb{N}}$ with $u_{i,j} \in \widetilde{U}(v_0, t_i)$ for all $j \in \mathbb{N}$, such that

$$\widetilde{\underline{t}}_R(v) = \lim_{i\to\infty} \lim_{j\to\infty} t_i + \mathcal{T}\big((v_0, t_i), u_{i,j}\big).$$

Let $(u_{i,j}, u) \in \widetilde{U}(v_0, t_i)$ be the control sequence which results from the concatenation of $u_{i,j}$ and u. If $t = \widetilde{\underline{t}}_R(v)$, then the continuity of $\mathcal{T}$ implies that

$$t' = \lim_{i\to\infty} \lim_{j\to\infty} t_i + \mathcal{T}\big((v_0, t_i), (u_{i,j}, u)\big) \geq \widetilde{\underline{t}}_R(v').$$

This contradicts the fact that $t' < \underline{t}_R(v')$. If $t > \widetilde{\underline{t}}_R(v)$, then the continuity of $\mathcal{T}$ implies that there exist $i_0, j_0 \in \mathbb{N}$, such that

$$t > t_{i_0} + \mathcal{T}\big((v_0, t_{i_0}), u_{i_0,j_0}\big).$$

Let $u_{i_0,j_0} = \big((\Delta t_k, e_k)\big)_{k=1,...,n}$. Since ΔT is continuous there exist continuous functions $\mu_{\mathrm{w},1}, ..., \mu_{\mathrm{w},n} : T(v_0) \to \mathbb{R}_0^+$ satisfying $\mu_{\mathrm{w},k}(t_{i_0}) = \Delta t_k$ and

$$\widetilde{u}(t_0) = \big((\mu_{\mathrm{w},1}(t_0), e_1), ..., (\mu_{\mathrm{w},n}(t_0), e_n)\big) \in \widetilde{U}(v_0, t_0), \quad \forall t_0 \in T(v_0).$$

Let $\theta : T(v_0) \to \mathbb{R}$, $\theta(t_0) = t_0 + \mathcal{T}\big((v_0, t_0), \widetilde{u}(t_0)\big)$. Since $\tau, \mu_{\mathrm{w},1}, ..., \mu_{\mathrm{w},n}$ are continuous, θ is continuous. As $\theta(t_0) \geq t_0$ and $\theta(t_{i_0}) < t$, [68, p.97, Satz 1] implies that there exists a $t_0(t) \in T(v_0)$ such that $\theta\big(t_0(t)\big) = t$. Let $\big(\widetilde{u}\big(t_0(t)\big), u\big) \in \widetilde{U}\big(v_0, t_0(t)\big)$ be the control sequence which results from the concatenation of $\widetilde{u}\big(t_0(t)\big)$ and u. Then,

$$\widetilde{\underline{t}}_R(v') > t' = t_0(t) + \mathcal{T}\Big(\big(v_0, t_0(t)\big), \big(\widetilde{u}\big(t_0(t)\big), u\big)\Big),$$

contradicting the definition of $\widetilde{\underline{t}}_R(v')$.
(ii) The assertion follows in a similar manner as above by using the fact that the earliest arrival times are attained due to the existence of fastest paths and the lower semicontinuity of the earliest arrival time function (cf. Theorem 4.2.4 using $\underline{\mathcal{T}}^\circ \in \mathbb{R}^+$ and $\tau \geq 0$). □

Theorem 5.1.3 *Suppose that ΔT is a continuous point-to-set mapping, τ, β, δ are continuous and there exist $\underline{\mathcal{B}}, \underline{\mathcal{B}}^\circ \in \mathbb{R}$, $\underline{\mathcal{B}}^\circ > 0$, such that (4.1) and (4.2) hold.*

(i) Let a source node $v_0 \in V$ be given and let Assumption 5.1.1 hold, then the partial mapping $t_0 \mapsto b^*(v_0, t_0)$ is continuous on $T(v_0)$.

(ii) If $X = V \times \mathbb{R}$ and (V, E) is strongly connected, then b^* is continuous.

Proof *(i)* Let $(e_1, ..., e_n)$ be an edge sequence with $\alpha(e_1) = v_0$ and $\omega(e_n) = v'$. Denote $v_k = \omega(e_k)$, $k = 1, ..., n$, and $\widetilde{\Delta T}_k, \widetilde{T}_k$ as in (4.3), (4.4), $k = 0, ..., n-1$. Lemma 5.1.2 implies

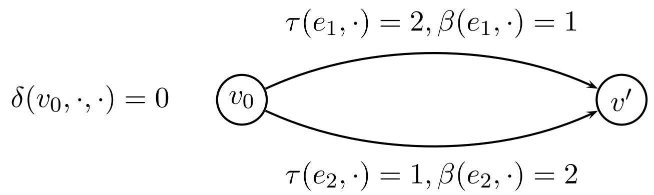


Figure 5.1.: Example network in which the optimal value function is not continuous. All network functions are constant, assuming the values depicted in the drawing. Moreover, $T(v_0)=\mathbb{R}$, $T(v')=(-\infty,0]$ and $\Delta T(v_0,\cdot)=\{0\}$.

that $\widetilde{\Delta T}_k(t)=\Delta T(v_k,t)$ for all $t\in\widetilde{T}_k$ and hence $\widetilde{T}_k=\operatorname{supp}(\widetilde{\Delta T}_k)=T(v_k)$, $k=0,...,n-1$. Using [65, Theorem 2.2.2] instead of [65, Theorem 2.2.1] in the backward induction along the edge sequence $(e_1,...,e_n)$, we obtain the continuity of the optimal-cost function along $(e_1,...,e_n)$. In a similar manner as in Theorem 4.2.4 we then obtain the continuity of the partial mapping $t_0\mapsto b^*(v_0,t_0)$ on $T_R(v_0)=T(v_0)$.
(ii) For any source node $v_0\in V$, as (V,E) is strongly connected, there exists a $n\in\mathbb{N}$ and at least one connected edge sequence $(e_1,...,e_n)\in E^n$ such that $\alpha(e_1)=v_0$ and $\omega(e_n)=v'$. Lemma 5.1.2 implies that $T_R(v_0)=\mathbb{R}$ for any source node $v_0\in V$ and we may repeat the reasoning of *(i)* for any $v_0\in V$. Consequently, b^* is continuous. □

Remark 5.1.4 *It is not possible to renounce Assumption 5.1.1 in order to establish the continuity of the optimal value function, as can be seen from the simple example depicted in Figure 5.1. Obviously, the optimal value function at v_0 is discontinuous at $t_0=-2$.*

Remark 5.1.5 *Similar results to Lemma 5.1.2 and Theorem 5.1.3 can be established for the reverse optimal path problem considering $T(v')=(-\infty,\bar{t}\,]$ for some $\bar{t}\in\mathbb{R}$, cp. Theorem 4.2.9.*

We will now establish the Lipschitz-continuity of the optimal value function under some additional assumptions. This property will be used in Section 7.2 to construct a pruning criterion for the forward optimal path problem. In particular, based on the result of Theorem 5.1.10, we will formulate decision rules in Lemma 7.2.1 and Corollary 7.2.5, which allow the identification of certain suboptimal paths in the search tree of an arbitrary optimal path algorithm. In order to prove the Lipschitz-continuity of the optimal value function we introduce the notion of uniformly linearly continuous point-to-set mappings, cp. [65, Definition 2.2.4].

Definition 5.1.6 *Let Γ be a point-to-set mapping from $T\subset\mathbb{R}^m$ to subsets of $\mathbb{R}^n$. Then Γ is said to be uniformly linearly continuous on $T_0\subset T$ if there exists a value $L_\Gamma>0$ such that for all $t,t'\in T_0$ there holds*

$$\inf_{x'\in\Gamma(t')}\|x-x'\|\leq L_\Gamma\|t-t'\|,\quad \forall x\in\Gamma(t). \tag{5.2}$$

For simplicity, in the next two results we utilize the 1-norm on $\mathbb{R}^n$, i.e., $\|x\|=\sum_{i=1}^n|x_i|$ for $x=(x_1,...,x_n)\in\mathbb{R}^n$. Note that the triangle inequality in $(\mathbb{R},|\cdot|)$ implies $|x+y|\leq\|(x,y)\|$

for $x, y \in \mathbb{R}$.
The following Lemma is a generalization of [65, Theorem 2.2.12], in which a more general form of the constraints is allowed and in which the parameter occurs both in the definition of the constraints and the objective function of the parametric optimization problem.

Lemma 5.1.7 *Let $f : \mathbb{R}^{m+n} \to \mathbb{R}$ be Lipschitz-continuous with constant L_f and let $\Gamma : \mathbb{R}^m \to \mathcal{P}(\mathbb{R}^n)$ be a uniformly linearly continuous point-to-set mapping with constant L_Γ. Then the function $f^* : \mathbb{R}^m \to \mathbb{R}$,*

$$f^*(t) = \inf_{x\in\Gamma(t)} f(t, x), \tag{5.3}$$

is Lipschitz-continuous with constant $L_f(1 + L_\Gamma)$.

Proof First we observe that as a consequence of [65, Theorem 2.2.8], for each $t \in \mathbb{R}^m$, the minimum in (5.3) is attained by some $x^* \in \Gamma(t)$. Next, for $t, t' \in \mathbb{R}^m$ there holds

$$\begin{aligned}\|f^*(t) - f^*(t')\| \le &\left\| \min_{x\in\Gamma(t)} f(t, x) - \min_{x'\in\Gamma(t')} f(t, x')\right\| \\ &+ \left\| \min_{x'\in\Gamma(t')} f(t, x') - \min_{x'\in\Gamma(t')} f(t', x')\right\|. \end{aligned}\tag{5.4}$$

Let

$$x_* \in \operatorname*{arg\,min}_{x\in\Gamma(t)} f(t, x), \qquad x'_* \in \operatorname*{arg\,min}_{x'\in\Gamma(t')} f(t, x'), \qquad x''_* \in \operatorname*{arg\,min}_{x'\in\Gamma(t')} f(t', x').$$

Since Γ is uniformly linearly continuous, there exists a $\widetilde{x}' \in \Gamma(t')$ such that $\|x_* - \widetilde{x}'\| \le L_\Gamma\|t - t'\|$. The Lipschitz-continuity of f implies

$$\begin{aligned} f(t, x_*) &\ge f(t, \widetilde{x}') - L_f L_\Gamma\|t - t'\| \ge \min_{x'\in\Gamma(t')} f(t, x') - L_f L_\Gamma\|t - t'\| \\ &= f(t, x'_*) - L_f L_\Gamma\|t - t'\|. \end{aligned}\tag{5.5}$$

Moreover, there exists a $\widetilde{x} \in \Gamma(t)$ such that $\|\widetilde{x} - x'_*\| \le L_\Gamma\|t - t'\|$, which yields

$$\begin{aligned} f(t, x'_*) &\ge f(t, \widetilde{x}) - L_f L_\Gamma\|t - t'\| \ge \min_{x\in\Gamma(t)} f(t, x) - L_f L_\Gamma\|t - t'\| \\ &= f(t, x_*) - L_f L_\Gamma\|t - t'\|. \end{aligned}\tag{5.6}$$

Combining (5.5) and (5.6), we obtain

$$\left\| \min_{x\in\Gamma(t)} f(t, x) - \min_{x'\in\Gamma(t')} f(t, x')\right\| \le L_f L_\Gamma\|t - t'\|. \tag{5.7}$$

Next, the Lipschitz-continuity of f implies

$$\begin{aligned} f(t, x'_*) &\ge f(t', x'_*) - L_f\|t - t'\| \ge \min_{x'\in\Gamma(t')} f(t', x') - L_f\|t - t'\| \\ &= f(t', x''_*) - L_f\|t - t'\| \end{aligned}\tag{5.8}$$

and

$$f(t', x''_*) \geq f(t, x''_*) - L_f \|t - t'\| \geq \min_{x' \in \Gamma(t')} f(t, x') - L_f \|t - t'\|$$
$$= f(t, x'_*) - L_f \|t - t'\|. \tag{5.9}$$

Combining (5.8) and (5.9), we obtain

$$\left\| \min_{x' \in \Gamma(t')} f(t, x') - \min_{x' \in \Gamma(t')} f(t', x') \right\| \leq L_f \|t - t'\|. \tag{5.10}$$

The assertion now follows from (5.4), (5.7) and (5.10). □

Lemma 5.1.8 *Suppose that τ, β, δ are Lipschitz-continuous with constants $L_\tau, L_\beta, L_\delta$, and ΔT is a uniformly linearly continuous point-to-set mapping with constant $L_{\Delta T}$. Let a source node $v_0 \in V$ be given and suppose that either Assumption 5.1.1 holds or $X = V \times \mathbb{R}$ and (V, E) is strongly connected. Let $(e_1, ..., e_n)$ be a connected edge sequence with $\alpha(e_1) = v_0$ and $\omega(e_n) = v'$, and denote $v_k = \omega(e_k)$, $k = 1, ..., n$. Then the optimal cost function $\widetilde{b}^* : \bigcup_{k=0}^{n} \big(\{v_k\} \times T(v_k)\big) \to \mathbb{R} \cup \{\infty\}$ along $(e_1, ..., e_n)$,*

$$\widetilde{b}^*(v_k, t_k) = \inf \Big\{ \mathcal{B}\big((v_k, t_k), u\big) : u = \big((\Delta t_i, e_i)\big)_{i=k+1,...,n} \in U(v_k, t_k) \Big\},$$

is Lipschitz-continuous with constant

$$L = (1 + L_{\Delta T})(L_\beta + L_\delta) \frac{(1 + L_{\Delta T} + L_\tau + L_{\Delta T} L_\tau)^n - 1}{L_{\Delta T} + L_\tau + L_{\Delta T} L_\tau}. \tag{5.11}$$

Proof Denote $\widetilde{\Delta T}_k, \widetilde{T}_k$ as in (4.3), (4.4), $k = 0, ..., n-1$. Lemma 5.1.2 implies that $\widetilde{\Delta T}_k(t) = \Delta T(v_k, t)$ for all $t \in \widetilde{T}_k$ and hence $\widetilde{T}_k = \operatorname{supp}(\widetilde{\Delta T}_k) = T(v_k)$, $k = 0, ..., n-1$. Consequently, $\widetilde{\Delta T}_k$ is uniformly linearly continuous with constant $L_{\Delta T}$. [25, Proposition 1.3.1] yields (cp. the proof of Theorem 5.1.3) for all $t \in T(v_k)$

$$\widetilde{b}^*(v_k, t) = \inf_{\Delta t \in \Delta T(v_k, t)} b_k(t, \Delta t), \quad k = 0, ..., n-1,$$

where we used the function $b_k : \operatorname{graph}(\widetilde{\Delta T}_k) \to \mathbb{R}$,

$$b_k(t, \Delta t) = \delta(v_k, t, \Delta t) + \beta(e_{k+1}, t + \Delta t) + \widetilde{b}^*\big(v_{k+1}, t + \Delta t + \tau(e_{k+1}, t + \Delta t)\big),$$

$k \in \{0, ..., n-1\}$, to simplify the notation. We now prove by backwards induction that $t \mapsto \widetilde{b}^*(v_k, t)$ is Lipschitz-continuous with constant L_k for each $k = 0, ..., n$, whereat $L_n = 0$ and

$$L_k = \big[L_\delta + L_\beta + (1 + L_\tau) L_{k+1}\big](1 + L_{\Delta T}). \tag{5.12}$$

Clearly, $t \mapsto \widetilde{b}^*(v_n, t)$ is Lipschitz-continuous with constant L_n since $\widetilde{b}^*(v_n, \cdot) \equiv 0$. Now, let $k \in \{0, ..., n-1\}$ and suppose that $t \mapsto \widetilde{b}^*(v_{k+1}, t)$ is Lipschitz-continuous with constant L_{k+1}. The Lipschitz-continuity of τ, β, δ implies that for all $(t, \Delta t), (t', \Delta t') \in \operatorname{graph}(\widetilde{\Delta T}_k)$

there holds

$$
\begin{aligned}
\big|b_k(t,\Delta t) - b_k(t',\Delta t')\big| &\le \big|\delta(v_k,t,\Delta t) - \delta(v_k,t',\Delta t')\big| \\
&\quad + \big|\beta(e_{k+1},t+\Delta t) - \beta(e_{k+1},t'+\Delta t')\big| \\
&\quad + \Big|\widetilde{b}^*\big(v_{k+1},t+\Delta t+\tau(e_{k+1},t+\Delta t)\big) - \widetilde{b}^*\big(v_{k+1},t'+\Delta t'+\tau(e_{k+1},t'+\Delta t')\big)\Big| \\
&\le L_\delta \big\|(t,\Delta t) - (t',\Delta t')\big\| + L_\beta\big(|t-t'| + |\Delta t - \Delta t'|\big) \\
&\quad + L_{k+1}\Big(|t-t'| + |\Delta t - \Delta t'| + \big|\tau(e_{k+1},t+\Delta t) - \tau(e_{k+1},t'+\Delta t')\big|\Big) \\
&\le (L_\delta + L_\beta)\big\|(t,\Delta t) - (t',\Delta t')\big\| + \\
&\quad + L_{k+1}\Big(\big\|(t,\Delta t) - (t',\Delta t')\big\| + L_\tau\big\|(t,\Delta t) - (t',\Delta t')\big\|\Big).
\end{aligned}
$$

We now obtain (5.12) from the above inequality and Lemma 5.1.7. Using (5.12), an easy inductive calculation shows that the L_k, $k = 0, ..., n$, are given by

$$
L_k = (1 + L_{\Delta T})(L_\beta + L_\delta) \sum_{j=k}^{n-1} \big[(1 + L_{\Delta T})(1 + L_\tau)\big]^{n-j-1}. \tag{5.13}
$$

Substituting $i = n - j - 1$ in (5.13) and using the formula for the geometric series we establish

$$
\begin{aligned}
L_k &= (1 + L_{\Delta T})(L_\beta + L_\delta) \sum_{i=0}^{n-1-k} \big[(1 + L_{\Delta T})(1 + L_\tau)\big]^{i} \\
&= (1 + L_{\Delta T})(L_\beta + L_\delta) \frac{\big[(1 + L_{\Delta T})(1 + L_\tau)\big]^{n-k} - 1}{(1 + L_{\Delta T})(1 + L_\tau) - 1}.
\end{aligned}
$$

Since $L_{\Delta T}, L_\tau, L_\beta, L_\delta \ge 0$ we obtain $L \ge L_k$ for all $k = 0, ..., n$. □

In order to generalize the result of Lemma 5.1.8 to the optimal value function of the time-dependent network, we must establish a bound on the topological length of optimal paths. The following Lemma shows that such a bound exists under mild assumptions which are satisfied in most practical applications.

Lemma 5.1.9 *(i) Let Assumption 4.2.3 hold, suppose that $T(v)$ is compact for all $v \in V$ and $\tau : E \times \mathbb{R} \to \mathbb{R}^+$. Then for any $v_0, v' \in V$ the topological length N of an optimal path from v_0 to v' is bounded from above by*

$$
N \le \frac{\max T(v') - \min T(v_0)}{\underline{\mathcal{T}}}, \tag{5.14}
$$

where

$$
\underline{\mathcal{T}} = \min_{(v,t)\in X} \min_{(\Delta t,e)\in U(v,t)} \mathcal{T}\big((v,t),(\Delta t,e)\big) > 0.
$$

(ii) Let $v_0, v' \in V$ be given and suppose that either Assumption 5.1.1 holds or $X = V \times \mathbb{R}$ and (V,E) is strongly connected. Suppose that there exists a $\overline{\mathcal{B}} \in \mathbb{R}^+$ such that, for

all $(v,t) \in X$, there holds

$$\mathcal{B}\big((v,t),u\big) \le \overline{\mathcal{B}}, \quad \forall u \in U(v,t) \text{ with } |u| = 1. \tag{5.15}$$

Then the topological length N of an optimal path from v_0 to v' is bounded from above by

$$N \le |V| - 1 + \frac{|V|\overline{\mathcal{B}} - (|V|-1)\underline{\mathcal{B}}}{\underline{\mathcal{B}}^{\circ}}|V|. \tag{5.16}$$

Proof *(i)* Since τ is continuous and X is compact, graph(ΔT) is compact and hence $\underline{\mathcal{T}} > 0$. Hence, for each admissible control sequence u of length n we obtain $\mathcal{T}\big((v,t),u\big) \ge n\underline{\mathcal{T}}$. Since a control sequence $u \in U(v_0,t_0)$, $t_0 \in T(v_0)$, with $\omega(u) = v'$ must satisfy $\mathcal{T}\big((v,t),u\big) \le \max T(v') - \min T(v_0)$ we obtain (5.14).
(ii) If either Assumption 5.1.1 holds or $X = V \times \mathbb{R}$ and (V,E) is strongly connected, then there exists a simple connected edge sequence $(e_1,...,e_n) \in E^n$ with $n \le |V|$ and continuous functions $\mu_{\mathrm{w},k} : T(v_0) \to \mathbb{R}_0^+$, $k = 1,...,n$, with $u(t_0) = \big((\mu_{\mathrm{w},k}(t_0), e_k)\big)_{k=1,...,n} \in U(v_0,t_0)$ for all $t_0 \in T(v_0)$. Now, (5.15) yields $\mathcal{B}\big((v_0,t_0),u(t_0)\big) \le n\overline{\mathcal{B}} \le |V|\overline{\mathcal{B}}$. Inserting this result into (4.7) we obtain (5.16). □

Theorem 5.1.10 *Suppose that τ, β, δ are Lipschitz-continuous with constants $L_\tau, L_\beta, L_\delta$, ΔT is a uniformly linearly continuous point-to-set mapping with constant $L_{\Delta T}$ and there exist $\underline{\mathcal{B}}, \underline{\mathcal{B}}^{\circ}, \overline{\mathcal{B}} \in \mathbb{R}$, $\underline{\mathcal{B}}^{\circ}, \overline{\mathcal{B}} > 0$, such that (4.1), (4.2) and (5.15) hold. Denote*

$$L^* = (1 + L_{\Delta T})(L_\beta + L_\delta)\frac{(1 + L_{\Delta T} + L_\tau + L_{\Delta T}L_\tau)^N - 1}{L_{\Delta T} + L_\tau + L_{\Delta T}L_\tau},$$

with

$$N = |V| - 1 + \frac{|V|\overline{\mathcal{B}} - (|V|-1)\underline{\mathcal{B}}}{\underline{\mathcal{B}}^{\circ}}|V|.$$

(i) If Assumption 5.1.1 holds, then the partial mapping $t_0 \mapsto b^*(v_0,t_0)$ is Lipschitz-continuous on $T(v_0)$ with constant L^*.

(ii) If $X = V \times \mathbb{R}$ and (V,E) is strongly connected, then, for each $v_0 \in V$, the partial mapping $t_0 \mapsto b^*(v_0,t_0)$ is Lipschitz-continuous on $\mathbb{R}$ with constant L^*.

Proof The result follows directly from Theorem 5.1.3, Lemma 5.1.8 and Lemma 5.1.9. □

5.2. Directional Differentiability

The results of the previous section on the properties of the optimal value function require very little mathematical structure in the problem formulation. We are next concerned with matters of differentiability which require the time-dependent network to be specified more precisely. In particular, we need to further specify the structure of the point-to-set mapping ΔT. In the remainder of this chapter, we assume that

$$\Delta T(x) = [\,\underline{\Delta T}(x), \overline{\Delta T}(x)], \quad \forall x \in X, \tag{5.17}$$

where $\underline{\Delta T}, \overline{\Delta T} : X \to \mathbb{R}_0^+$ with $\underline{\Delta T}(x) \le \overline{\Delta T}(x)$ for all $x \in X$. We could similarly assume that $\Delta T(x)$ consists of a finite union of sets of the form (5.17). All results derived below also hold in this more general setting. However, in order to keep the notation as simple as possible, we have introduced the particular form of the waiting time constraints (5.17). In order to derive the directional differentiability of the optimal value function, we assume that the partial mappings $t \mapsto \underline{\Delta T}(v,t)$ and $t \mapsto \overline{\Delta T}(v,t)$, $v \in V$, $t \in T(v)$, as well as the network functions τ, β, δ have additional properties. Note, that due to the switching between edge sequences traversed by optimal paths at different points in time we cannot expect the optimal value function to be differentiable on X_R. Such breakpoints not only occur as a result of the switching of an edge sequence but also as a consequence of the parametric optimization problems (4.5) which must be solved along each edge sequence, cp. [30, Example 4.11]. In order to obtain the differentiability of the optimal value function, a certain class of differentiability of the cost and constraint functions (we can interpret $\Delta t \in \Delta T(x)$ as the inequality constraints $-\Delta t \leq \underline{\Delta T}(x)$ and $\Delta t \leq \overline{\Delta T}(x)$), certain constraint qualifications (see, e.g., [65, Section 2.3], [78, Section 2.4], [30, Section 2.3.4]) and certain regularity assumptions are usually assumed [65, Section 2.4, Chapter 3], [78, Section 2.2]. Although the constraint qualifications are trivially satisfied by the simple structure of the inequality constraints given by (5.17) and the set of problems for which the regularity assumption holds is dense in the strong topology on $C^3(\mathbb{R}^2, \mathbb{R})$ [78, Theorem 2.2.9] (see [78, Theorem 2.5.1] for a statement including the constraints), we do not pursue this approach here, since the eventual switching of the optimal edge sequence obstructs the global differentiability of the optimal value function. The best we can generally expect is to obtain the directional differentiability of the partial functions $t \mapsto b^*(v,t)$ for each $v \in V$ and $t \in T_R(v)$, cp. [30, Section 4.3].

Using implicit function theorem results (cp. [65, Section 2.4]), even if the cost and constraint functions are n times continuously differentiable and the constraint qualifications and regularity assumptions hold, the optimal value function can only be shown to be $(n-1)$ times continuously differentiable in general. This is due to the characterization of the optimal waiting policy as a zero of the derivative of the cost function at interior points of the domain of admissibility. Consequently, along an edge sequence of (topological) length k, the optimal value function can only be shown to be $\max\{0, n-k\}$ times continuously differentiable. Next, it can be seen from the proof of Theorem 4.2.4 that the optimal value function is the pointwise minimum of the optimal cost functions along the edge sequences connecting the terminal node to the source node. Consequently, even if an upper bound for the topological length of optimal paths is known, the set of intersection points between these cost functions can contain accumulation points, thereby foiling even the directional differentiability of the optimal value function. In order to avoid this situation, we consider piecewise analytic functions in the following. According to the above discussion, they constitute the largest class of functions which allow the directional differentiability of the optimal value function in general. Similar assumptions were imposed on the network functions in the context of time-dependent optimal flows [133], [112].

Note that the restriction of $\Delta T(x)$ to a finite number of connected components for each $x \in X$ is necessary in order to establish that the optimal value function is piecewise analytic. (Consider $\Gamma(t) = \{0\} \cup \bigcup_{n \in \mathbb{N}} \{\frac{1}{n}\}$, $f(t,x) = \sin(tx)$ and $f^*(t) = \min_{x \in \Gamma(t)} f(t,x)$. Then the intersections between the partial functions $t \mapsto f(t, \frac{1}{n})$, $n \in \mathbb{N}$, and $t \mapsto f(t,0)$ contain the accumulation point π.)

Since we must consider (lower semi-) continuous functions on closed domains in order to obtain the existence of optimal solutions we must first be concerned with an appropriate

definition of differentiability on closed sets. After introducing two concepts of piecewise analytic functions and proving a number of preliminary lemmas we present the main result of this section, Theorem 5.2.14.
Let $n \in \mathbb{N}$ and $\Theta \subset \mathbb{R}^n$ be an open set. We denote the set of k-times continuously differentiable functions $f : \Theta \to \mathbb{R}$ by $\mathcal{C}^k(\Theta)$. We call $f : \Theta \to \mathbb{R}$ real analytic if, for each $y \in \Theta$, the function f may be represented by a convergent power series in some neighborhood of y, cp., [118, Definition 1.6.1]. The set of all real analytic functions $f : \Theta \to \mathbb{R}$ are denoted by $\mathcal{C}^\omega(\Theta)$.
Let us now consider the case in which Θ is a closed set. In case of pathological closed sets, a very general definition of k-times continuously differentiable functions has been formulated by Whitney, see, e.g., [168], [118, Definition 2.3.5]. We are henceforth concerned with closed connected sets which have a smooth boundary and consist mostly of inner points. For such a set $\Theta \subset \mathbb{R}^n$, we may also consider a k-times continuously differentiable (resp., real analytic) function f as the restriction of a function $\widetilde{f}$ which is k-times continuously differentiable (resp., real analytic) on an open set $\widetilde{\Theta}$ containing Θ.

Definition 5.2.1 *Let $n \in \mathbb{N}$ and $\Theta \subset \mathbb{R}^n$ be a closed connected set, such that $\Theta = \mathrm{cl}(\mathrm{int}(\Theta))$ and such that there exists a $(n-1)$-dimensional analytic variety M_Θ over an open set $U_\Theta \subset \mathbb{R}^n$ containing Θ with $\mathrm{bd}(\Theta) \subset M_\Theta$. Let $f : \Theta \to \mathbb{R}$.*
We say that $f \in \mathcal{PC}^\omega(\Theta)$, if there exists a $(n-1)$-dimensional analytic variety M over an open set $U_\Theta \subset \mathbb{R}^n$ containing Θ such that $\mathrm{bd}(\Theta) \subset M$ and for each connected component Θ' of $\Theta \setminus M$ there holds $f|_{\mathrm{cl}(\Theta')} \in \mathcal{C}^\omega\big(\mathrm{cl}(\Theta')\big)$.

Remark 5.2.2 *It is easily seen that the sum of two piecewise analytic functions f, g with appropriate domain is a again a piecewise analytic function by using [118, Proposition 1.1.4] and observing that for $M_f = \{x \in \mathbb{R}^n : z_f(x) = 0\}$ and $M_g = \{x \in \mathbb{R}^n : z_g(x) = 0\}$ we obtain $M_f \cap M_g = \{x \in \mathbb{R}^n : z_f(x) \cdot z_g(x) = 0\}$.*

Even when considering parametric optimization problems in which only analytic functions occur in the problem formulation, the optimal value function is not necessarily piecewise analytic. Consider the function $f(t,x) = \int_0^x \sin(\xi^2)\mathrm{d}\xi - x\sin(t)$ for $(t,x) \in [0,1] \times \mathbb{R}$ and the optimization problem

$$f^*(t) = \min_{x \in [t,1]} f(t,x). \tag{5.18}$$

Clearly $f \in \mathcal{C}^\omega(\mathbb{R}^2)$, $t, 1 \in \mathcal{C}^\omega(\mathbb{R})$, and we have

$$\partial_x f(t,x) = \sin(x^2) - \sin(t), \qquad \partial_x^2 f(t,x) = 2x\cos(x^2).$$

It is easy to verify that for $t > 0$ the curve $t \mapsto (t, \sqrt{t})$ contains local minima of (5.18). However, the function $f^*(t) = f(t, \sqrt{t})$ is not in $\mathcal{C}^\omega\big([0,1]\big)$ as $(f^*)'(t) = -\sqrt{t}\cos(t)$ is not differentiable at $t = 0$. Nevertheless, the function $t \mapsto f^*(t^2)$ is in $\mathcal{C}^\omega\big([0,1]\big)$. In order to cope with such situations we introduce the following type of functions:

Definition 5.2.3 *Let $T \subset \mathbb{R}$ be a closed interval with non-empty interior and $f : T \to \mathbb{R}$. We say that $f \in \mathcal{C}^{1,\omega}(T)$, if $f \in \mathcal{C}^1\big(T\big)$, $f|_{\mathrm{int}(T)} \in \mathcal{C}^\omega\big(\mathrm{int}(T)\big)$, and if T has a left boundary point $\underline{t}$, then there exists $\underline{\gamma} \in \mathcal{C}^\omega\big([0,1]\big)$ satisfying*

$$\underline{\gamma}\big([0,1]\big) \subset T, \qquad \underline{\gamma}(0) = \underline{t}, \qquad \underline{\gamma}'(s) > 0, \quad \forall s \in (0,1], \tag{5.19}$$

$$f \circ \underline{\gamma} \in \mathcal{C}^\omega\big([0,1]\big), \tag{5.20}$$

and if T has a right boundary point $\bar{t}$, then there exists $\overline{\gamma} \in \mathcal{C}^\omega\big([-1,0]\big)$ satisfying

$$\overline{\gamma}\big([-1,0]\big) \subset T, \qquad \overline{\gamma}(0) = \bar{t}, \qquad \overline{\gamma}'(s) > 0, \quad \forall s \in [-1,0), \tag{5.21}$$
$$f \circ \overline{\gamma} \in \mathcal{C}^\omega\big([-1,0]\big). \tag{5.22}$$

For $T \subset \mathbb{R}$ and $f : T \to \mathbb{R}$, we say that $f \in \mathcal{PC}^{1,\omega}(T)$ if there exists a 0-dimensional analytic variety M over an open set $U_T \subset \mathbb{R}$ containing T such that $\mathrm{bd}(T) \subset M$ and for each connected component T' of $T \setminus M$ there holds $f|_{\mathrm{cl}(T')} \in \mathcal{C}^{1,\omega}\big(\mathrm{cl}(T')\big)$.

Remark 5.2.4 *If $U_T \subset \mathbb{R}$ is an open set and $h \in \mathcal{C}^\omega(U_T)$ is not the zero-function, then the zero set $M = \{t \in U_T : h(t) = 0\}$ of h contains no accumulation point in U_T, cp. [118, Corollary 1.2.6]. Hence instead of restricting the breakpoints of f to the zero set of an analytic function as in Definition 5.2.1, we could have required $M \cap K$ to be finite for each compact $K \subset T$ in Definition 5.2.3.*

Remark 5.2.5 *In [133], [112], a piecewise analytic function $f : T \to \mathbb{R}$, $T \subset \mathbb{R}$, was defined as a lower semicontinuous function which is analytic on each subset of its domain on which it is continuous. At this, it was assumed that T is a closed interval which contains only a finite number of breakpoints of f. It has then been shown that the optimal value function is piecewise analytic and lower semicontinuous (in the above sense) if the network functions are piecewise analytic and lower semicontinuous (in the above sense). Continuing the approach we have chosen in Section 5.1, we are able to prove an even stronger result, i.e., the continuity and the directional differentiability of the optimal value function at any point of its domain. Note that, if $T \subset \mathbb{R}$ is a closed interval and $f \in \mathcal{PC}^{1,\omega}(T)$, then f is directionally differentiable at any $t \in T$, cp. Remark 5.2.4.*

Remark 5.2.6 *In order to ease the notation, if Θ is as in Definition 5.2.1, D is a discrete set and $f : \Theta \times D \to \mathbb{R}$, we will denote $f \in \mathcal{PC}^\omega(\Theta \times D)$ if the partial mappings $y \mapsto f(y,d)$ are in $\mathcal{PC}^\omega(\Theta)$ for all $d \in D$. Similarly, if T is as in Definition 5.2.3, D is a discrete set and $f : T \times D \to \mathbb{R}$, we will denote $f \in \mathcal{PC}^{1,\omega}(T \times D)$ if the partial mappings $t \mapsto f(t,d)$ are in $\mathcal{PC}^{1,\omega}(T)$ for all $d \in D$.*

In a similar manner as in Remark 5.2.2, if $T \subset \mathbb{R}$, $f \in \mathcal{PC}^{1,\omega}(T)$ and $g \in \mathcal{PC}^\omega(T)$, we obtain $f + g \in \mathcal{PC}^{1,\omega}(T)$ by using [118, Proposition 1.6.7] and the local reparametrizations of f specified in Definition 5.2.3.

Lemma 5.2.7 *Let $U_\Theta \subset \mathbb{R}^2$ be an open neighborhood of the origin, $f \in \mathcal{C}^1(U_\Theta)$ and suppose that 0 is a local and isolated minimum of the optimization problem $\min_{x \in \mathbb{R}:(0,x)\in\Theta} f(0,x)$. Then, for any $\epsilon > 0$, there exists an $\epsilon_t > 0$, such that for each $t \in (-\epsilon_t, \epsilon_t)$ there exists a $x^*(t) \in (-\epsilon, \epsilon)$ such that $x^*(t)$ is a local minimum of the optimization problem* $\min_{x \in \mathbb{R}:(t,x)\in\Theta} f(t,x)$.

Proof Let $\epsilon > 0$. Since $f \in \mathcal{C}^1(U_\Theta)$ and 0 is a local and isolated minimum, there holds $\partial_x f(0,0) = 0$, and there exists an $\epsilon_x > 0$ with $2\epsilon_x < \epsilon$, such that $\partial_x f(0,x) < 0$ for all $x \in (-2\epsilon_x, 0)$ and $\partial_x f(0,x) > 0$ for all $x \in (0, 2\epsilon_x)$. As $\partial_x f(0,-\epsilon_x) < 0$, $\partial_x f(0,\epsilon_x) > 0$ and $f \in \mathcal{C}^1(U_\Theta)$, there exists an $\epsilon_t > 0$, such that

$$\partial_x f(t, -\epsilon_x) < 0, \quad \forall t \in (-\epsilon_t, \epsilon_t), \tag{5.23}$$
$$\partial_x f(t, \epsilon_x) > 0, \quad \forall t \in (-\epsilon_t, \epsilon_t). \tag{5.24}$$

Since the set $C_x = [-\epsilon_x, \epsilon_x]$ is compact, the optimization problem $\min_{x \in C_x} f(t, x)$ contains a solution for each $t \in (-\epsilon_t, \epsilon_t)$. As a consequence of (5.23) and (5.24), the solution cannot be a boundary point of C_x. □

Lemma 5.2.8 *Let $\epsilon_f, \epsilon_g > 0$ and $f \in \mathcal{C}^\omega([0, \epsilon_f])$, $g \in \mathcal{C}^\omega([0, \epsilon_g])$ with $f(0) = g(0) = 0$ and $f'(t) > 0$ for all $t \in (0, \epsilon_f]$, $g'(t) > 0$ for all $t \in (0, \epsilon_g]$. Denote by n the multiplicity of the zero 0 of f, i.e., $n = \min\{k \in \mathbb{N} : f^{(k)}(0) \neq 0\}$, and denote by N the multiplicity of the zero 0 of g, i.e., $N = \min\{k \in \mathbb{N} : g^{(k)}(0) \neq 0\}$. Suppose that there exists a $m \in \mathbb{N}$ such that $N = nm$. Then there exists an $\epsilon_h > 0$, such that $h : [0, \epsilon_h] \to \mathbb{R}$, $h(t) = f^{-1}\big(g(t)\big)$ satisfies $h \in \mathcal{C}^\omega([0, \epsilon_h])$, $h(0) = 0$ and $h'(t) > 0$ for all $t \in (0, \epsilon_h]$.*

Proof Let $\{a_k\}_{k \in \mathbb{N}}$ denote the coefficients of the power series expansion of f in 0, i.e., $f(t) = \sum_{k=0}^\infty a_k t^k$ for all $t \in [0, R)$, where $R > 0$ denotes the radius of convergence of this power series expansion. Since 0 is a zero with multiplicity n, we have $a_k = 0$ for $k = 0, ..., n-1$, and consequently we may denote $f(t) = t^n a(t)$, whereat $a(t) = a_n + \sum_{k=n+1}^\infty a_k t^{k-n}$. [118, Corollary 1.2.3] implies that $a \in \mathcal{C}^\omega\big((-R, R)\big)$. By definition of n, there holds $a_n \neq 0$, and since $f'(t) > 0$ for all $t \in (0, \epsilon_f]$, we have $a_n > 0$. Denote $p(t) = t^n$. As $a_n > 0$, there exists a $b > 0$, such that $a_n = a(0) = p(b) = b^n$, and there holds $p'(b) = nb^{n-1} \neq 0$. As a consequence of the inverse function theorem [118, Theorem 1.4.3], there exists an open neighborhood U_p of b, on which there exists an analytic inverse function p^{-1} of p. We choose $\widetilde{\epsilon}_f > 0$ in such a way that $\widetilde{\epsilon}_f < \min\{R, \epsilon_f\}$, $0 \notin a\big((-\widetilde{\epsilon}_f, \widetilde{\epsilon}_f)\big)$, $a\big((-\widetilde{\epsilon}_f, \widetilde{\epsilon}_f)\big) \subset U_p$, and define $r_f : (-\widetilde{\epsilon}_f, \widetilde{\epsilon}_f) \to \mathbb{R}$, $r_f(t) = tp^{-1}\big(a(t)\big)$. According to [118, Proposition 1.6.7], there holds $r_f \in \mathcal{C}^\omega\big((-\widetilde{\epsilon}_f, \widetilde{\epsilon}_f)\big)$,

$$r_f(t)^n = t^n \Big[p^{-1}\big(a(t)\big)\Big]^n = t^n a(t) = f(t), \quad \forall t \in [0, \widetilde{\epsilon}_f)$$

and $r'_f(0) = p^{-1}\big(a(0)\big) = b > 0$. Clearly, there holds $r_f(0) = 0$ and, possibly after choosing a smaller $\widetilde{\epsilon}_f > 0$, we may assume that $r'_f(t) > 0$ for all $t \in (-\widetilde{\epsilon}_f, \widetilde{\epsilon}_f)$. In a similar manner we obtain the existence of $\widetilde{\epsilon}_g \in (0, \epsilon_g]$ and $r_g \in \mathcal{C}^\omega\big((-\widetilde{\epsilon}_g, \widetilde{\epsilon}_g)\big)$ satisfying $r_g(t)^N = g(t)$ for all $t \in [0, \widetilde{\epsilon}_g)$, $r_g(0) = 0$ and $r'_g(t) > 0$ for all $t \in (-\widetilde{\epsilon}_g, \widetilde{\epsilon}_g)$.
As a consequence of the inverse function theorem [118, Theorem 1.4.3], since $r'_f(0) > 0$, there exists an open neighborhood U_f of $r_f(0) = 0$, on which there exists an analytic inverse function r_f^{-1} of r_f. We choose $\epsilon_r > 0$ in such a way that $[0, \epsilon_r] \subset U_f$, and for $s \in [0, \epsilon_r^n]$ we define $\hat{f}(s) = r_f^{-1}(s^{1/n})$. Then

$$(\hat{f} \circ f)(t) = r_f^{-1}\big(f(t)^{1/n}\big) = r_f^{-1}\Big(\big(r_f(t)^n\big)^{1/n}\Big) = t, \quad \forall t \in r_f^{-1}\big([0, \epsilon_r]\big)$$
$$(f \circ \hat{f})(s) = r_f\big(\hat{f}(s)\big)^n = r_f\Big(\big(r_f^{-1}(s^{1/n})\big)\Big)^n = s, \quad \forall s \in [0, \epsilon_r^n].$$

Hence $\hat{f}$ is the inverse function of f. Since $r_g(0) = 0$ and $r'_g(t) > 0$ for all $t \in (-\widetilde{\epsilon}_g, \widetilde{\epsilon}_g)$, there exists an $\epsilon_h \in (0\widetilde{\epsilon}_g]$ such that $r_g([0, \epsilon_h]) \subset [0, \epsilon_r^{1/m}]$. Finally, for $t \in [0, \epsilon_h]$, there holds

$$h(t) = f^{-1}\big(g(t)\big) = r_f^{-1}\Big(\big(r_g(t)^N\big)^{1/n}\Big) = r_f^{-1}\big(r_g(t)^m\big).$$

Since $r_g \in \mathcal{C}^\omega\big((-\widetilde{\epsilon}_g, \widetilde{\epsilon}_g)\big)$, $(-\widetilde{\epsilon}_g, \widetilde{\epsilon}_g)$ is an open superset of $[0, \epsilon_h]$, $r_f^{-1} \in \mathcal{C}^\omega(U_f)$ and U_f is an open superset of $[0, \epsilon_r] \supset r_g\big([0, \epsilon_h]\big)^m$, we obtain $h \in \mathcal{C}^\omega\big([0, \epsilon_h]\big)$. Moreover, as $r_g(0) = 0$

and $r_f^{-1}(0) = 0$, there holds $h(0) = 0$ and using [68, Kapitel 15, Satz 3] we establish

$$h'(t) = \big(r_f^{-1}\big)'\big(r_g(t)^m\big) \cdot m r_g(t)^{m-1} \cdot r_g'(t) = \frac{m r_g(t)^{m-1} r_g'(t)}{r_f'\Big(r_f^{-1}\big(r_g(t)^m\big)\Big)} > 0, \qquad \forall t \in (0, \epsilon_h],$$

since each of the factors is positive for $t \in (0, \epsilon_h]$. □

Lemma 5.2.9 *Let $T \subset \mathbb{R}$ be a compact interval with non-empty interior and let $f_1, f_2 \in \mathcal{C}^{1,\omega}(T)$. Then the function $f^* : T \to \mathbb{R}$,*

$$f^*(t) = \min\big\{f_1(t), f_2(t)\big\}$$

satisfies $f^* \in \mathcal{PC}^{1,\omega}(T)$.

Proof Let $T_{12} = \{t \in T : f_1(t) = f_2(t)\}$. We first show that either (1) $T_{12} = \emptyset$ or (2) $T_{12} = T$ or (3) T_{12} is a finite set.
Suppose that neither of the cases (1), (2), (3) has occurred. Since T_{12} contains an infinite number of points and T is compact, T_{12} contains an accumulation point t_0. If $t_0 \in \operatorname{int}(T)$ then the identity theorem for real analytic functions [118, Corollary 1.2.6] yields $f_1 = f_2$ on $\operatorname{int}(T)$. Since $f_1, f_2 \in \mathcal{C}^1(T)$ we must even have $f_1 = f_2$ on T, i.e., $T_{12} = T$ which we have precluded. Hence $t_0 \in \operatorname{bd}(T)$. Without loss of generality we may assume that t_0 is the left boundary point of T and $t_0 = 0$. Let $\underline{\gamma}_i \in \mathcal{C}^\omega\big([0,1]\big)$ be such that (5.19) and (5.20) hold for f_i, $i = 1, 2$. Let n_i denote the multiplicity of the zero 0 of $\underline{\gamma}_i$, $i = 1, 2$, and denote $N = n_1 n_2$, $U_0 = \underline{\gamma}_1\big([0,1]\big) \cap \underline{\gamma}_2\big([0,1]\big)$. Since $\underline{\gamma}_i(0) = 0$, $\underline{\gamma}_i'(s) > 0$ for all $s \in (0,1]$ and $\underline{\gamma}_i$ is continuous, $i = 1, 2$, there exists an $\epsilon_0 > 0$ such that $[0, \epsilon_0] \subset U_0$. We now define

$$\widetilde{\gamma}_i(t) = \underline{\gamma}_i^{-1}(t^N), \quad t \in [0, \epsilon_0^{1/N}],\ i = 1, 2.$$

Lemma 5.2.8 implies that there exist $\epsilon_i > 0$ such that $\widetilde{\gamma}_i \in \mathcal{C}^\omega([0, \epsilon_i])$, $\widetilde{\gamma}_i(0) = 0$ and $\widetilde{\gamma}_i'(t) > 0$ for all $t \in (0, \epsilon_i]$, $i = 1, 2$. Consequently, using [118, Proposition 1.6.7], $t \mapsto f_i\Big(\underline{\gamma}_i\big(\widetilde{\gamma}_i(t)\big)\Big)$ is in $\mathcal{C}^\omega([0, \epsilon_i])$ for $i = 1, 2$, and

$$t \mapsto f_i\Big(\underline{\gamma}_i\big(\widetilde{\gamma}_i(t)\big)\Big) = f_i\Big(\underline{\gamma}_i\big(\underline{\gamma}_i^{-1}(t^N)\big)\Big) = f_i(t^N), \quad i = 1, 2.$$

Since T_{12} contains the accumulation point 0 and $t \mapsto t^N$ is bijective on $\mathbb{R}_0^+$, [118, Corollary 1.2.6] yields $f_1(t^N) = f_2(t^N)$ for all $t \in [0, \epsilon_0]$. Since $f_i|_{\operatorname{int}(T)} \in \mathcal{C}^\omega\big(\operatorname{int}(T)\big)$ for $i = 1, 2$, we even obtain $f_1 = f_2$ on $\operatorname{int}(T) \cup \{t_0\}$. A similar reasoning for the right boundary point yields $f_1 = f_2$ on T, i.e., $T_{12} = T$.
We now consider the cases (1), (2), (3) separately. If either (1) or (2) has occurred then $f^*(t) = f_i(t)$ for an $i^* \in \{1, 2\}$ and the assertion follows. Let us assume that the case (3) has occurred. Let $[\,\underline{t}, \overline{t}\,]$ denote the closure of an arbitrary connected component of $T \setminus T_{12}$. It is sufficient to prove that $f^*|_{[\underline{t},\overline{t}]} \in \mathcal{C}^{1,\omega}\big([\,\underline{t}, \overline{t}\,]\big)$.
Since $f_1, f_2 \in \mathcal{C}^1(T)$ we immediately obtain $f^*|_{[\underline{t},\overline{t}]} \in \mathcal{C}^1\big([\,\underline{t}, \overline{t}\,]\big)$. Moreover, since $f^*(t) = f_{i^*}(t)$ for all $t \in [\,\underline{t}, \overline{t}\,]$ and an $i \in \{1, 2\}$, there holds $f^*|_{(\underline{t},\overline{t})} \in \mathcal{C}^\omega\big((\underline{t}, \overline{t})\big)$. Let us now construct a curve $\underline{\gamma}^* \in \mathcal{C}^\omega\big([0,1]\big)$ such that (5.19) and (5.20) hold for f^*. If $\underline{t}$ is the left boundary point of T then we may set $\underline{\gamma}^* = \underline{\gamma}_{i^*}$. Otherwise there exists an $\epsilon > 0$ such that $[\,\underline{t}, \underline{t} + 2\epsilon) \subset T$. We may then set $\underline{\gamma}^*(s) = \underline{t} + \epsilon s$. The assertion follows in a similar manner for $\overline{t}$. □

Lemma 5.2.10 *Let $\underline{\theta}, \overline{\theta} \in \mathbb{R}$ with $\underline{\theta} < \overline{\theta}$ and let $T \subset \mathbb{R}$ be a compact interval. Let $\gamma \in \mathcal{C}^\omega(T)$ with $\underline{\theta} \le t + \gamma(t) \le \overline{\theta}$ for all $t \in T$ and $f \in \mathcal{C}^{1,\omega}\big([\underline{\theta}, \overline{\theta}]\big)$. Then the function $f^* : T \mapsto \mathbb{R}$, $f^*(t) = f\big(t + \gamma(t)\big)$, satisfies $f^* \in \mathcal{PC}^{1,\omega}(T)$.*

Proof Let $T_{\underline{\theta}} = \{t \in T : t + \gamma(t) = \underline{\theta}\}$. Since $\gamma \in \mathcal{C}^\omega(T)$ and T is compact, it follows from the identity theorem for real analytic functions [118, Corollary 1.2.6] that there are three possibilities for the structure of $T_{\underline{\theta}}$: (1) $T_{\underline{\theta}} = \emptyset$, (2) $T_{\underline{\theta}} = T$ and (3) $T_{\underline{\theta}}$ is a finite set. A similar assertion holds for $T_{\overline{\theta}} = \{t \in T : t + \gamma(t) = \overline{\theta}\}$. We first assume that $T_{\overline{\theta}} = \emptyset$.
In the case of (1), since $t \mapsto t + \gamma(t)$ is analytic on an open superset of T, the mapping $t \mapsto f\big(t+\gamma(t)\big)$ is analytic on an open superset of T. In the case of (2), since $t \mapsto t+\gamma(t) = \underline{\theta}$ is constant, the mapping $t \mapsto f\big(t + \gamma(t)\big)$ is analytic on an open superset of T. In the case of (3), let T' be the closure of a connected component of $T \setminus T_{\underline{\theta}}$. As $t + \gamma(t) \in (\underline{\theta}, \overline{\theta})$ for all $t \in \operatorname{int}(T')$, we obtain that $f^*|_{\operatorname{int}(T')} \in \mathcal{C}^\omega\big(\operatorname{int}(T')\big)$. Since f^* is the composition of functions which are the restriction of continuously differentiable functions on open supersets of their respective domains, we also obtain $f^*|_{T'} \in \mathcal{C}^1(T')$. Let $t_{\text{bd}} \in \operatorname{bd}(T')$. Without loss of generality we assume that t_{bd} is the left boundary point of T' and that $t_{\text{bd}} = \underline{\theta} = 0$. We now construct a curve $\underline{\gamma}^* \in \mathcal{C}^\omega\big([0,1]\big)$ satisfying $\underline{\gamma}^*\big([0,1]\big) \subset T'$, $\underline{\gamma}^*(0) = 0$ with $(\underline{\gamma}^*)'(s) > 0$ for all $s \in (0,1]$, such that the mapping $t \mapsto f\big(\underline{\gamma}^*(t) + \gamma(\underline{\gamma}^*(t))\big)$ is in $\mathcal{C}^\omega\big([0,1]\big)$. By assumption, the mapping $t \mapsto d(t) = t + \gamma(t)$ is in $\mathcal{C}^\omega\big(T'\big)$, $d(0) = 0$ and there exists an $\epsilon > 0$ such that $d'(t) > 0$ for all $t \in (0, \epsilon]$. Let $\underline{\gamma} \in \mathcal{C}^\omega\big([0,1]\big)$ be such that (5.19) and (5.20) hold for f, let n denote the multiplicity of the zero 0 of d (recall that we have assumed that d is non-constant in the case (3)) and define

$$\widetilde{\underline{\gamma}}^*(\widetilde{s}) = d^{-1}\big(\underline{\gamma}(\widetilde{s}^n)\big).$$

From the power series expansion of $s \mapsto \underline{\gamma}(s)$ we immediately obtain that 0 is a zero of $s \mapsto \underline{\gamma}(s^n)$ with multiplicity $N = nm$ for some $m \in \mathbb{N}$. Lemma 5.2.8 implies that there is an $\widetilde{\epsilon} > 0$ such that $\widetilde{\underline{\gamma}}^* \in \mathcal{C}^\omega([0,\widetilde{\epsilon}])$, $\widetilde{\underline{\gamma}}^*(0) = 0$ and $(\widetilde{\underline{\gamma}}^*)'(\widetilde{s}) > 0$ for all $\widetilde{s} \in (0, \widetilde{\epsilon}]$. Consequently, using [118, Proposition 1.6.7],

$$\widetilde{s} \mapsto f^*\big(\widetilde{\underline{\gamma}}^*(\widetilde{s})\big) = f\Big(d\big(\widetilde{\underline{\gamma}}^*(\widetilde{s})\big)\Big) = f\bigg(d\Big(d^{-1}\big(\underline{\gamma}(\widetilde{s}^n)\big)\Big)\bigg) = f\big(\underline{\gamma}(\widetilde{s}^n)\big)$$

is in $\mathcal{C}^\omega([0,\widetilde{\epsilon}])$. By defining $\underline{\gamma}^*(s) = \widetilde{\underline{\gamma}}^*(\widetilde{\epsilon} s)$ we obtain the desired curve $\underline{\gamma}^* \in \mathcal{C}^\omega\big([0,1]\big)$ satisfying $\underline{\gamma}^*\big([0,1]\big) \subset T$, $\underline{\gamma}^*(0) = t_{\text{bd}}$ with $(\underline{\gamma}^*)'(s) > 0$ for all $s \in (0,1]$, such that $f^* \circ \widetilde{\underline{\gamma}}^* \in \mathcal{C}^\omega\big([0,1]\big)$.
In a similar manner we can construct such a curve for each $t \in T_{\overline{\theta}}$ if $T_{\overline{\theta}}$ is a finite set. If $T_{\overline{\theta}} = T$ then $T_{\underline{\theta}} = \emptyset$ and the assertion follows in a similar manner as if $T_{\overline{\theta}} = \emptyset$ and $T_{\underline{\theta}} = T$.
□

Lemma 5.2.11 *Let $T \subset \mathbb{R}$ be a compact interval with non-empty interior and let $\underline{\Gamma}, \overline{\Gamma} \in \mathcal{C}^\omega(T)$. Suppose that $\underline{\Gamma}(t) \le \overline{\Gamma}(t)$ for all $t \in T$ and denote $\Gamma : T \to \mathcal{P}(\mathbb{R})$, $\Gamma(t) = [\underline{\Gamma}(t), \overline{\Gamma}(t)]$, $\Theta = \operatorname{graph}(\Gamma)$ and*

$$\underline{\theta} = \min\big\{t + x \in \mathbb{R} : (t,x) \in \Theta\big\}, \qquad \overline{\theta} = \max\big\{t + x \in \mathbb{R} : (t,x) \in \Theta\big\}.$$

Let $g \in \mathcal{C}^\omega(\Theta)$, $f \in \mathcal{C}^{1,\omega}\big([\underline{\theta}, \overline{\theta}]\big)$, denote $h : \Theta \to \mathbb{R}$, $h(t,x) = g(t,x) + f(t+x)$, and suppose *that*

$$M = \big\{(t,x) \in \mathrm{int}(\Theta) : \partial_x h(t,x) = 0\big\}$$

is a 1*-dimensional real analytic variety. Then for each* $(t_{\mathrm{bd}}, x_{\mathrm{bd}}) \in \mathrm{bd}(M)$*, there exists an open neighborhood* U_{bd} *of* $(t_{\mathrm{bd}}, x_{\mathrm{bd}})$*, such that* $M \cap U_{\mathrm{bd}} = M_0 \cap M_1$*, where* M_i *is the finite disjoint union of connected i-dimensional real analytic manifolds,* $i = 0, 1$*, with* $M_0 \subset \mathrm{bd}(M_1)$*. Moreover, for each connected component* M_1' *of* M_1 *with* $(t_{\mathrm{bd}}, x_{\mathrm{bd}}) \in \mathrm{bd}(M_1')$*, there exists a curve* $\gamma = (\gamma_1, \gamma_2)$ *with* $\gamma_1, \gamma_2 \in \mathcal{C}^\omega\big([0,1]\big)$ *such that*

$$\gamma(0) = (t_{\mathrm{bd}}, x_{\mathrm{bd}}), \qquad \gamma\big((0,1]\big) \subset M_1', \qquad D\gamma(s) \neq 0, \quad \forall s \in (0,1], \tag{5.25}$$

$$h \circ \gamma \in \mathcal{C}^\omega\big([0,1]\big). \tag{5.26}$$

Proof We separately consider different characterizations of boundary points. First, we assume that $t_{\mathrm{bd}} + x_{\mathrm{bd}} \notin \{\underline{\theta}, \overline{\theta}\}$.
Since there exists a real analytic continuation $\widehat{h}$ of h to an open neighborhood U_{bd} of $(t_{\mathrm{bd}}, x_{\mathrm{bd}})$ (cp. [118, Definition 1.2.7]), the set

$$M_{\mathrm{bd}} = \big\{(t,x) \in U_{\mathrm{bd}} : \partial_x \widehat{h}(t,x) = 0\big\}$$

is an analytic variety of top dimension $d \in \{0,1\}$, cp. [118, Chapter 5]. Note that, if M_{bd} were an analytic variety of dimension 2, then the power series expansion of h in an inner point of M_{bd} would yield $\partial_x h \equiv 0$ and hence M could not be a 1-dimensional variety. The Lojaciewicz Structure Theorem for analytic varieties [118, Theorem 5.2.3] yields that $M_{\mathrm{bd}} = M_0 \cup M_1$, where M_i is the finite disjoint union of connected i-dimensional real analytic manifolds, $i = 0, 1$, with $M_0 \subset \mathrm{bd}(M_1)$. Note that, if M_{bd} were a 0-dimensional analytic variety $M_{\mathrm{bd}} = M_0$, since $U_{\mathrm{bd}} \cap M \neq \emptyset$ and $M_0 \subset \mathrm{bd}(M_1)$, M could not be a 1-dimensional analytic variety. Let M_1' be a connected component of $M \cap U_{\mathrm{bd}}$ with $(t_{\mathrm{bd}}, x_{\mathrm{bd}}) \in \mathrm{bd}(M_1')$.
If $(t_{\mathrm{bd}}, x_{\mathrm{bd}}) \in M_1$, then we may choose $\gamma : [-1,1] \to M_{\mathrm{bd}}$ as a real analytic local parametrization of the connected component of M_1 containing $M_1' \cup \{(t_{\mathrm{bd}}, x_{\mathrm{bd}})\}$. Without loss of generality, possibly after an appropriate transformation of the curve parameter, we may assume that $\gamma(0) = (t_{\mathrm{bd}}, x_{\mathrm{bd}})$ and $\gamma\big((0,1]\big) \subset M_1'$. Since γ is a local parametrization of a real analytic manifold, we have $D\gamma(s) \neq 0$ for all $s \in [-1,1]$, cp. [118, Definition 1.9.1]. Since $t + x \notin \{\underline{\theta}, \overline{\theta}\}$ for all $(t,x) \in M \subset \mathrm{int}(\Theta)$ we obtain $(h \circ \gamma)|_{[0,1]} \in \mathcal{C}^\omega\big([0,1]\big)$.
If $(t_{\mathrm{bd}}, x_{\mathrm{bd}}) \in M_0$, then Hironaka's theorem on the resolution of singularities of real analytic varieties [118, Theorem 5.1.6] yields that there is a blowup $\pi : Y \to U_{\mathrm{bd}}$, such that the proper transform M_Y of $\big\{(t,x) \in U_{\mathrm{bd}} : \partial_x \widehat{h}(t,x) = 0\big\}$ in Y is a real analytic manifold. Let $\gamma_Y : [-1,1] \to Y$ be a real analytic local parametrization of M_Y with $\pi\big(\gamma_Y(0)\big) = (t_{\mathrm{bd}}, x_{\mathrm{bd}})$ and $\pi\Big(\gamma_Y\big((0,1]\big)\Big) \subset M_1'$. Since γ_Y is a local parametrization of a real analytic manifold we have $D\gamma_Y(s) \neq 0$ for all $s \in [-1,1]$, and since π is a projection, π is real analytic. Without loss of generality, possibly after restricting the domain of γ_Y to a smaller interval and rescaling the curve parameter, we may assume that π is injective on $\gamma_Y\big((0,1]\big) \subset \pi^{-1}(M_1')$. By setting $\gamma(s) = \big(\gamma_1(s), \gamma_2(s)\big) = \pi\big(\gamma_Y(s)\big)$ for $s \in [0,1]$, we obtain $\gamma_1, \gamma_2 \in \mathcal{C}^\omega\big([0,1]\big)$. Possibly after restricting s to a smaller parameter interval and rescaling the curve γ, we obtain that $D\gamma(s) \neq 0$ for all $s \in (0,1]$. Since $t + x \notin \{\underline{\theta}, \overline{\theta}\}$ for all $(t,x) \in M \subset \mathrm{int}(\Theta)$ we obtain $(h \circ \gamma)|_{[0,1]} \in \mathcal{C}^\omega\big([0,1]\big)$.
We now consider the case in which $t_{\mathrm{bd}} + x_{\mathrm{bd}} \in \{\underline{\theta}, \overline{\theta}\}$. Without loss of generality we only consider the case in which $t_{\mathrm{bd}} + x_{\mathrm{bd}} = \underline{\theta}$ and $\underline{\theta} = 0$. Let $\underline{\gamma} \in \mathcal{C}^\omega\big([0,1]\big)$ be such that (5.19)

and (5.20) hold for f and let $\Theta_{[0,1]} = \{(t, x) \in \Theta : 0 \leq t + x \leq 1\}$, $\psi : \Theta_{[0,1]} \to \mathbb{R}^2$,

$$\psi(t, x) = \big(\psi_1(t, x), \psi_2(t, x)\big) = \big(t, \underline{\gamma}(t + x) - t\big) \tag{5.27}$$

and $\widetilde{\Theta}_{[0,1]} = \psi(\Theta_{[0,1]})$. Observe that $\widetilde{\Theta}_{[0,1]} \subset \big\{(\widetilde{t}, \widetilde{x}) \in \mathbb{R}^2 : 0 \leq \widetilde{t} + \widetilde{x}\big\}$. Obviously, ψ is continuous, and since $\underline{\gamma}'(s) > 0$ for all $s \in (0, 1]$, $\psi : \Theta_{[0,1]} \to \widetilde{\Theta}_{[0,1]}$ is bijective. Hence, there exists a continuous bijective inverse ψ^{-1} of ψ on $\widetilde{\Theta}_{[0,1]}$. A simple calculation yields

$$\psi^{-1}(\,\widetilde{t}, \widetilde{x}\,) = \big(\,\widetilde{t}, \underline{\gamma}^{-1}(\,\widetilde{t} + \widetilde{x}\,) - \widetilde{t}\,\big). \tag{5.28}$$

Moreover, $\psi(t_{\text{bd}}, x_{\text{bd}}) = (t_{\text{bd}}, x_{\text{bd}})$ and $\psi_1(t, x) + \psi_2(t, x) = \underline{\gamma}(t+x)$, and since $\underline{\gamma} \in \mathcal{C}^\omega\big([0, 1]\big)$ there exists a real analytic continuation of ψ to an open neighborhood of $(t_{\text{bd}}, x_{\text{bd}})$. By the assumptions on f and $\underline{\gamma}$, using [118, Proposition 1.6.7], there also exists an analytic continuation $(\,\widetilde{t}, \widetilde{x}\,) \mapsto \widetilde{h}(\,\widetilde{t}, \widetilde{x}\,)$ of the mapping $(t, x) \mapsto h\big(\psi(t, x)\big)$ to an open neighborhood U_ψ of $(t_{\text{bd}}, x_{\text{bd}})$. We choose U_ψ in such a way that there also exists an analytic continuation of ψ to U_ψ. Now we have

$$\partial_{\widetilde{x}}\widetilde{h}(t, x) = \partial_x h\big(\psi(t, x)\big)\underline{\gamma}'(t + x), \quad \forall (t, x) \in U_\psi \cap \Theta$$

and $\underline{\gamma}'(t+x) > 0$ for all $(t, x) \in U_\psi \cap \text{int}(\Theta)$. The Lojaciewicz Structure Theorem for analytic varieties [118, Theorem 5.2.3] implies that there is an open neighborhood $\widetilde{U}_\psi \subset U_\psi$ of $(t_{\text{bd}}, x_{\text{bd}})$ such that $\big\{(\,\widetilde{t}, \widetilde{x}\,) \in \widetilde{U}_\psi : \partial_{\widetilde{x}}\widetilde{h}(\,\widetilde{t}, \widetilde{x}\,) = 0\big\} = \widetilde{M}_0 \cup \widetilde{M}_1$, where $\widetilde{M}_i$ is the finite disjoint union of connected i-dimensional real analytic manifolds, $i = 0, 1$, and $\widetilde{M}_0 \subset \text{bd}(\widetilde{M}_1)$. Let $\widetilde{U}_> = \big\{(\,\widetilde{t}, \widetilde{x}\,) \in \widetilde{U}_\psi : \widetilde{t} + \widetilde{x} > \underline{\theta} = 0\big\}$. Observe that $(\widetilde{M}_0 \cup \widetilde{M}_1) \cap \widetilde{U}_>$ also is the disjoint union of a finite disjoint union of connected i-dimensional real analytic manifolds, $i = 0, 1$. For the open neighborhood $U_{\text{bd}} = \psi(\widetilde{U}_\psi)$ of $(t_{\text{bd}}, x_{\text{bd}})$ we obtain

$$M \cap U_{\text{bd}} = \psi\big((\widetilde{M_0} \cup \widetilde{M_1}) \cap \widetilde{U}_>\big) = \psi(\widetilde{M_0} \cap \widetilde{U}_>) \cup \psi(\widetilde{M_1} \cap \widetilde{U}_>),$$

whereat $\psi(\widetilde{M_0} \cup \widetilde{U}_>) \subset M$ is the finite disjoint union of connected 0-dimensional real analytic manifolds, $\psi(\widetilde{M_1} \cup \widetilde{U}_>)$ is the finite disjoint union of connected 1-dimensional real analytic manifolds and $\psi(\widetilde{M_0} \cap \widetilde{U}_>) \subset \text{bd}\big(\psi(\widetilde{M_1} \cap \widetilde{U}_>)\big)$. Let $M_1' \subset M \cap U_{\text{bd}}$ be a connected component of $M \cap U_{\text{bd}}$ with $(t_{\text{bd}}, x_{\text{bd}}) \in \text{bd}(M_1')$. Then there exists a connected real analytic manifold $\widetilde{M_1'} \subset \widetilde{M}_1$ with $\psi(\widetilde{M_1'} \cap \widetilde{U}_>) = M_1'$. If $\psi^{-1}(t_{\text{bd}}, x_{\text{bd}}) \in \widetilde{M_1'}$ then there exists a real analytic local parametrization $\widetilde{\gamma} : [-1, 1] \to \widetilde{M}$ of $\widetilde{M_1'}$ with

$$\widetilde{\gamma}(0) = (t_{\text{bd}}, x_{\text{bd}}), \qquad \widetilde{\gamma}\big((0, 1]\big) \subset \psi^{-1}(M_1'), \qquad D\widetilde{\gamma}(s) \neq 0, \quad \forall s \in [-1, 1]. \tag{5.29}$$

If $\psi^{-1}(t_{\text{bd}}, x_{\text{bd}}) \in \widetilde{M_0}$ then [118, Theorem 5.1.6] yields in a similar manner as above that there exist $\widetilde{\gamma}_1, \widetilde{\gamma}_2 \in \mathcal{C}^\omega[-1, 1]$, such that $\widetilde{\gamma} = (\widetilde{\gamma}_1, \widetilde{\gamma}_2)$ satisfies $\widetilde{\gamma}(0) = (t_{\text{bd}}, x_{\text{bd}})$, $\widetilde{\gamma}\big((0, 1]\big) \subset \psi^{-1}(M_1')$ and $D\widetilde{\gamma}(s) \neq 0$ for all $s \in (0, 1]$. Since $\widetilde{h}$ is real analytic on $\widetilde{U}_\psi$ we obtain $\widetilde{h} \circ \widetilde{\gamma} \in \mathcal{C}^\omega\big([-1, 1]\big)$. We now set $\gamma : [0, 1] \to \text{cl}(M_1')$, $\gamma = (\gamma_1, \gamma_2)$,

$$\gamma(s) = \psi\big(\widetilde{\gamma}(s)\big). \tag{5.30}$$

Since there exists an analytic continuation of ψ to $\widetilde{U}_\psi$, [118, Proposition 1.6.7] implies that

$\gamma_1, \gamma_2 \in \mathcal{C}^\omega\big([0,1]\big)$. Clearly, $\gamma(0) = (t_{\mathrm{bd}}, x_{\mathrm{bd}})$ and $\gamma\big((0,1]\big) \subset M_1'$. Moreover,

$$D\gamma(s) = \begin{pmatrix} \widetilde{\gamma}_1'(s) \\ \underline{\gamma}'\big(\widetilde{\gamma}_1(s) + \widetilde{\gamma}_2(s)\big)\big[\widetilde{\gamma}_1'(s) + \widetilde{\gamma}_2'(s)\big] - \widetilde{\gamma}_1'(s) \end{pmatrix}.$$

It remains to prove that $D\gamma(s) \neq 0$ for all $s \in (0,1]$. Let $s \in (0,1]$ and suppose that $\widetilde{\gamma}_1'(s) = 0$. Note that this implies $\widetilde{\gamma}_2'(s) \neq 0$. Since $\widetilde{\gamma}(\,\widetilde{s}\,) \in \widetilde{U}_>$, there holds $\widetilde{\gamma}_1(s) + \widetilde{\gamma}_2(s) > 0$, and we obtain $\underline{\gamma}'\big(\widetilde{\gamma}_1(s) + \widetilde{\gamma}_2(s)\big) > 0$, which yields

$$\underline{\gamma}'\big(\widetilde{\gamma}_1(s) + \widetilde{\gamma}_2(s)\big)\big[\widetilde{\gamma}_1'(s) + \widetilde{\gamma}_2'(s)\big] - \widetilde{\gamma}_1'(s) = \underline{\gamma}'\big(\widetilde{\gamma}_1(s) + \widetilde{\gamma}_2(s)\big)\widetilde{\gamma}_2'(s) \neq 0,$$

and hence $D\gamma(s) \neq 0$ for all $s \in (0,1]$. Finally, we observe that

$$(h \circ \gamma)(s) = h\Big(\psi\big(\widetilde{\gamma}(s)\big)\Big) = (\,\widetilde{h} \circ \widetilde{\gamma}\,)(s), \qquad \forall s \in [0,1].$$

Since $t + x \notin \{\underline{\theta}, \overline{\theta}\}$ for all $(t,x) \in M \subset \operatorname{int}(\Theta)$ this implies $h \circ \gamma \in \mathcal{C}^\omega\big([0,1]\big)$. □

Lemma 5.2.12 *Let $T \subset \mathbb{R}$ be a compact interval and let $\underline{\Gamma}, \overline{\Gamma} \in \mathcal{C}^\omega(T)$. Suppose that $\underline{\Gamma}(t) \leq \overline{\Gamma}(t)$ for all $t \in T$ and denote $\Gamma : T \to \mathcal{P}(\mathbb{R})$, $\Gamma(t) = [\,\underline{\Gamma}(t), \overline{\Gamma}(t)]$, $\Theta = \operatorname{graph}(\Gamma)$ and*

$$\underline{\theta} = \min\{t + x \in \mathbb{R} : (t,x) \in \Theta\}, \qquad \overline{\theta} = \max\{t + x \in \mathbb{R} : (t,x) \in \Theta\}.$$

Let $g \in \mathcal{C}^\omega(\Theta)$, $f \in \mathcal{C}^{1,\omega}\big([\,\underline{\theta}, \overline{\theta}\,]\big)$ and denote $h : \Theta \to \mathbb{R}$, $h(t,x) = g(t,x) + f(t+x)$. Then the optimal value function $f^* : T \to \mathbb{R}$,

$$f^*(t) = \min_{x \in [\,\underline{\Gamma}(t), \overline{\Gamma}(t)]} h(t,x) \tag{5.31}$$

satisfies $f^* \in \mathcal{PC}^{1,\omega}(T)$.

Proof The candidates for a minimizer of (5.31) at $t \in T$ are $\underline{\Gamma}(t), \overline{\Gamma}(t)$ and $x \in \big(\underline{\Gamma}(t), \overline{\Gamma}(t)\big)$ satisfying $\partial_x h(t,x) = 0$. We first discuss the mappings $t \mapsto h\big(t, \underline{\Gamma}(t)\big)$ and $t \mapsto h\big(t, \overline{\Gamma}(t)\big)$, cf. [78, eq. (2.2.5), p.24].
Clearly, the mappings $t \mapsto g\big(t, \underline{\Gamma}(t)\big)$ and $t \mapsto g\big(t, \overline{\Gamma}(t)\big)$ are in $\mathcal{C}^\omega(T)$, cf. [118, Proposition 1.6.7]. Lemma 5.2.10 implies that $t \mapsto f\big(t + \underline{\Gamma}(t)\big)$ and $t \mapsto f\big(t + \underline{\Gamma}(t)\big)$ are in $\mathcal{PC}^{1,\omega}(T)$. Consequently, using [118, Proposition 1.6.7], we obtain that $t \mapsto h\big(t, \underline{\Gamma}(t)\big)$ and $t \mapsto h\big(t, \overline{\Gamma}(t)\big)$ are in $\mathcal{PC}^{1,\omega}(T)$.
We next consider candidates for local minima $x \in \big(\underline{\Gamma}(t), \overline{\Gamma}(t)\big)$, $t \in T$. Let

$$S = \Big\{(t,x) \in T \times \mathbb{R} : x \in \big(\underline{\Gamma}(t), \overline{\Gamma}(t)\big)\Big\}.$$

If $S = \emptyset$ then we obtain

$$f^*(t) = \min\Big\{h\big(t, \underline{\Gamma}(t)\big), h\big(t, \overline{\Gamma}(t)\big)\Big\}. \tag{5.32}$$

Lemma 5.2.9 implies that in this case $f^* \in \mathcal{PC}^{1,\omega}(T)$. Let us assume that $S \neq \emptyset$. The set

$$M = \Big\{(t,x) \in S : \partial_x h(t,x) = 0\Big\}$$

is either empty or an analytic variety of dimension $d \in \{0,1,2\}$, cp. [118, Chapter 5], and contains all candidates for local minima in S. If $M=\emptyset$ then the assertion follows from (5.32). In the following we assume that $M \neq \emptyset$. We will now separately consider each possible dimension d of M.
$\underline{d=0}$: The Lojaciewicz Structure Theorem for analytic varieties [118, Theorem 5.2.3] states that each $(t,x) \in M$ is isolated, hence Lemma 5.2.7 implies that M contains no minimum of (5.31). Consequently, the assertion follows from (5.32).
$\underline{d=2}$: Since $\operatorname{int}(M) \neq \emptyset$ all derivatives of h vanish at some $(t,x) \in \operatorname{int}(M) \subset S$. The power series expansion of h then yields $\partial_x h \equiv 0$ on S. As $h \in \mathcal{C}^1(\Theta)$ and $\Theta = \operatorname{cl}(S)$ we even have $\partial_x h \equiv 0$ on Θ. Consequently, the partial function $x \mapsto h(t,x)$ is constant for each $t \in T$ and the assertion follows from (5.32).
$\underline{d=1}$: We first show that M is locally the finite union of 0- and 1-dimensional real analytic manifolds.
Let $(t_0,x_0) \in S$. If $(t_0,x_0) \notin M$, since M is closed in S, there exists an open neighborhood $U^{(t_0,x_0)}$ of (t_0,x_0), such that $U^{(t_0,x_0)} \cap M = \emptyset$. If $(t_0,x_0) \in M$, the Lojaciewicz structure theorem for analytic varieties [118, Theorem 5.2.3] implies that there exists an open neighborhood $U^{(t_0,x_0)}$ of (t_0,x_0), such that $M \cap U^{(t_0,x_0)} = M_0^{(t_0,x_0)} \cup M_1^{(t_0,x_0)}$, where $M_i^{(t_0,x_0)}$ is the finite disjoint union of connected i-dimensional real analytic manifolds, $i=0,1$, and $M_0^{(t_0,x_0)} \subset \operatorname{bd}\big(M_1^{(t_0,x_0)}\big)$.
Let $(t_0,x_0) \in \operatorname{graph}(\underline{\Gamma}) \cup \operatorname{graph}(\overline{\Gamma})$. If $(t_0,x_0) \notin \operatorname{bd}(M)$, since $\operatorname{cl}(M)$ is closed in Θ, there exists an open neighborhood $U^{(t_0,x_0)}$ of (t_0,x_0), such that $U^{(t_0,x_0)} \cap M = \emptyset$. If $(t_0,x_0) \in \operatorname{bd}(M)$, then Lemma 5.2.11 implies that there exists an open neighborhood $U^{(t_0,x_0)}$ of (t_0,x_0), such that $M \cap U^{(t_0,x_0)} = M_0^{(t_0,x_0)} \cup M_1^{(t_0,x_0)}$, where $M_i^{(t_0,x_0)}$ is the finite disjoint union of connected i-dimensional real analytic manifolds, $i=0,1$, and $M_0^{(t_0,x_0)} \subset \operatorname{bd}\big(M_1^{(t_0,x_0)}\big)$.
Since T is compact and $\underline{\Gamma}, \overline{\Gamma}$ are continuous, Θ is compact. Consequently, the open cover $\{U^{(t_0,x_0)}\}_{(t_0,x_0)\in\Theta}$ of Θ contains a finite subcover. This implies that $M = M_0 \cup M_1$, where M_i is the finite disjoint union of connected i-dimensional real analytic manifolds, $i=0,1$, and $M_0 \subset \operatorname{cl}(M_1)$. Hence, there are only a finite number $I \in \mathbb{N}_0$ of $t_i^{\mathrm{c}} \in T$, $i \in \{1,...,I\}$, such that $x \mapsto h(t_i^{\mathrm{c}},x)$ is constant. Let $J \in \mathbb{N}_0$ be the number of $t_j^0 \in T$, $j \in \{1,...,J\}$, such that $\big(\{t_j^0\} \times \mathbb{R}\big) \cap M_0 \neq \emptyset$. Since $I,J \in \mathbb{N}_0$, it is sufficient to prove that f^* has the desired properties on the closure of each connected component of $T \setminus \big(\bigcup_{i=1}^{I}\{t_i^{\mathrm{c}}\} \cup \bigcup_{j=1}^{J}\{t_j^0\}\big)$.
Let T' be the closure of such a connected component, and denote

$$M \cap \operatorname{int}\big(\operatorname{graph}(\Gamma|_{T'})\big) = \bigcup_{k=1}^{K} M'_k, \tag{5.33}$$

where $K \in \mathbb{N}_0$ and M'_k is a connected 1-dimensional real analytic manifold for each k. If $K=0$ then the assertion follows from (5.32). In the following we assume that $K \in \mathbb{N}$, and for each $k \in \{1,...,K\}$, we denote

$$T'_k = \big\{t \in T' : \exists\, x \in \Gamma(t) \text{ such that } (t,x) \in \operatorname{cl}(M'_k)\big\}.$$

Obviously, T'_k is a closed interval. Moreover, as a consequence of the identity theorem for real analytic functions [118, Corollary 1.2.6], for each $k \in \{1,...,K\}$ and each $t \in \operatorname{int}(T'_k)$, the set $\big\{x \in \Gamma(t) : (t,x) \in \operatorname{cl}(M'_k)\big\}$ is finite, since we have assumed that the mapping

$x \mapsto h(t,x)$ is non-constant for all $t \in \text{int}(T'_k)$. We define $\widetilde{f}_k^* : T'_k \to \mathbb{R}$,

$$f_k^*(t) = \min_{(t,x)\in\text{cl}(M'_k)} h(t,x), \tag{5.34}$$

and claim that

$$f^*(t) = \min\Big\{h\big(t,\underline{\Gamma}(t)\big), h\big(t,\overline{\Gamma}(t)\big), \min_{\substack{k\in\{1,..,K\}:\\ t\in T'_k}} f_k^*(t)\Big\}, \quad \forall t \in T'. \tag{5.35}$$

This characterization of f^* is obviously correct for $t \in \text{int}(T')$, cf. (5.33), (5.34). Moreover, [65, Theorem 2.2.8] implies that f^* is continuous. Hence, (5.35) also holds at the boundary points of T'. We next analyze M'_k and f_k^* in more detail in order to establish that $f^*|_{T'} \in \mathcal{PC}^{1,\omega}(T')$.
As $M'_k \subset M_1 \subset M$, M is closed in S, $M_0 \subset \text{bd}(M_1)$ and $\text{graph}(\Gamma|_{\text{int}(T')}) \cap M_0 = \emptyset$, there holds $\text{bd}(M'_k) \subset \text{bd}\big(\text{graph}(\Gamma|_{T'})\big)$. Moreover, since M_0 is the finite disjoint union of connected 0–dimensional real analytic manifolds, the set of boundary points $\text{bd}(M'_k)$ is finite. (If the set of boundary points were not finite, since $\text{bd}\big(\text{graph}(\Gamma|_{T'})\big)$ is compact, the set of boundary points would contain an accumulation point. Using real analytic local parametrizations of the boundary curve and M'_k in a neighborhood of the accumulation point, the identity theorem [118, Corollary 1.2.6] would yield $M'_k \subset \text{bd}\big(\text{graph}(\Gamma|_{T'})\big)$, a contradiction.) We denote $\text{bd}(M'_k) = \bigcup_{l=1}^{L}\{(t_l^{\text{bd}}, x_l^{\text{bd}})\}$ for some $L \in \mathbb{N}_0$.
We next observe that $\text{cl}(M'_k)$ is compact since $\text{graph}(\Gamma|_{T'})$ is compact. Suppose that there is an infinite number of $(t^{\text{t}}, x^{\text{t}}) \in M'_k$ such that the tangent space to M'_k at $(t^{\text{t}}, x^{\text{t}})$ is given by $\{(t,x) \in \mathbb{R}^2 : t = 0\}$. Since $\text{cl}(M'_k)$ is compact, the set of all such points contains an accumulation point $(t_0^{\text{t}}, x_0^{\text{t}})$. Assume that $(t_0^{\text{t}}, x_0^{\text{t}}) \in M'_k$. As $x \mapsto \partial_x h(t,x)$ is non-constant for each $t \in \text{int}(T')$, using a real analytic parameterization of M'_k in an open neighborhood of the accumulation point, [118, Corollary 1.2.6] implies that $M'_k \subset \{t_0^{\text{t}}\}\times\mathbb{R}$, a contradiction to the construction of T'. We now separately discuss the different possibilities for $(t_0^{\text{t}}, x_0^{\text{t}}) \in \text{bd}(M'_k)$.
Lemma 5.2.11 implies that, for each for each $(t_0^{\text{t}}, x_0^{\text{t}}) \in \text{bd}(M'_k)$, there exists a curve $\gamma = (\gamma_1, \gamma_2)$ with $\gamma_1, \gamma_2 \in \mathcal{C}^\omega\big([0,1]\big)$, $\gamma(0) = (t_0^{\text{t}}, x_0^{\text{t}})$, $\gamma\big((0,1]\big) \subset M'_k$ and $D\gamma(s) \neq 0$ for all $s \in (0,1]$. Now, [118, Corollary 1.2.6] implies that γ_1 is constant. However, if γ_1 is constant, then $x \mapsto \partial_x h(t_0^{\text{t}}, x)$ must be constant and hence $M'_k \subset \{t_0^{\text{t}}\} \times \mathbb{R}$, a contradiction to the construction of T'.
Let $(t_n^{\text{t}}, x_n^{\text{t}})$, $n = 1, ..., N$, $N \in \mathbb{N}_0$, denote the points of M'_k at which the tangent space to M'_k is given by $\{(t,x) \in \mathbb{R}^2 : t = 0\}$. Furthermore, let $P_k \in \mathbb{N}$ denote the number of connected components of $T'_k \setminus \big(\bigcup_{l=1}^{L}\{t_l^{\text{bd}}\} \cup \bigcup_{n=1}^{N}\{t_n^{\text{t}}\}\big)$ and let $T''_{k,p}$ denote the closure of the p-th such connected component, $p = 1, ..., P_k$. Observe that the number $Q_{k,p}$ of connected components of $M'_k \cap \text{graph}(\Gamma|_{\text{int}(T''_{k,p})})$ is finite, $Q_{k,p} \in \mathbb{N}_0$, and let $M''_{k,pq}$ denote the q-th such connected component, $q = 1, ..., Q_{k,p}$. Note that, for each $t \in \text{int}(T''_{k,p})$, there exists exactly one $x \in \Gamma(t)$ with $(t,x) \in M''_{k,pq}$, and for each $t \in \text{bd}(T''_{k,p})$, there exists exactly one $x \in \Gamma(t)$ with $(t,x) \in \text{bd}(M''_{k,pq})$. We denote this unique $x \in \Gamma(t)$ by $x_{pq}(t)$ in the following, $p = 1, ..., P_k$, $q = 1, ..., Q_{k,p}$. On $T''_{k,p}$ we then obtain the following characterization of f_k^*:

$$f_k^*(t) = \min_{q\in\{1,...,Q_{k,p}\}} h\big(t, x_{pq}(t)\big). \tag{5.36}$$

We now prove that $t \mapsto f^*_{k,pq}(t) = h\big(t, x_{pq}(t)\big)$ satisfies $f^*_{k,pq} \in \mathcal{C}^{1,\omega}(T''_{k,p})$. By inserting (5.36) into (5.35) for all $k \in \{1, ..., K\}$ and all $p \in \{1, ..., P_k\}$, and by applying Lemma 5.2.9 to (5.35) on every non-empty set $\bigcap_{(k,p)\in\mathcal{K}_p} T''_{k,p}$ with

$$\mathcal{K}_p = \Big\{(k_r, p_r)_{r=1,\dots,R} : (k_r)_{r=1,\dots,R} \in \mathcal{P}\big(\{1, ..., K\}\big), p_r \in \{1, ..., P_{k_r}\}\Big\},$$

it then follows that $f^* \in \mathcal{PC}^{1,\omega}(T')$ and hence $f^* \in \mathcal{PC}^{1,\omega}(T)$.
Since $M''_{k,pq}$ is connected $f^*_{k,pq}$ is continuous. Let $t \in \operatorname{int}(T''_{k,p})$. Then there exists a real analytic local parametrization $\gamma = (\gamma_1, \gamma_2)$ of $M''_{k,pq}$ in a neighborhood U''_{pq} of $\big(t, x_{pq}(t)\big)$ with $\gamma_1, \gamma_2 \in \mathcal{C}^\omega\big([-1, 1]\big)$, $\gamma(0) = \big(t, x_{pq}(t)\big)$ and $\gamma_1'(s) \neq 0$ for all $s \in [-1, 1]$. The real analytic inverse function theorem [118, Theorem 1.4.3] implies that there exists an inverse function γ_1^{-1} of γ_1 in a neighborhood U''_0 of $\gamma_1(0)$, such that $\gamma_1^{-1} \in \mathcal{C}^\omega(U''_0)$. We obtain

$$f^*_{k,pq}(t) = h\Big(t, \gamma_2\big(\gamma_1^{-1}(t)\big)\Big),$$

which is in $\mathcal{C}^\omega(U''_0)$ according to [118, Proposition 1.6.7]. Furthermore,

$$\begin{aligned}(f^*_{k,pq})'(t) &= \partial_t h\Big(t, \gamma_2\big(\gamma_1^{-1}(t)\big)\Big) + \partial_x h\Big(t, \gamma_2\big(\gamma_1^{-1}(t)\big)\Big)\gamma_2'\big(\gamma_1^{-1}(t)\big)(\gamma_1^{-1})'(t)\\ &= \partial_t h\Big(t, \gamma_2\big(\gamma_1^{-1}(t)\big)\Big),\end{aligned} \tag{5.37}$$

as $\partial_x h\big(t, x_{pq}(t)\big) = 0$.
Let $t_{\mathrm{bd}} \in \operatorname{bd}(T''_{k,p})$. Without loss of generality we assume that t_{bd} is the left boundary point of $T''_{k,p}$. We have already shown that there exists a curve $\gamma = (\gamma_1, \gamma_2)$ with $\gamma_1, \gamma_2 \in \mathcal{C}^\omega\big([0, 1]\big)$, $\gamma(0) = \big(t_{\mathrm{bd}}, x_{pq}(t_{\mathrm{bd}})\big)$ and $D\gamma(s) \neq 0$ for all $s \in (0, 1]$, such that $\gamma(s) \in \operatorname{cl}(M''_{k,pq})$ for all $s \in [0, 1]$ and $h \circ \gamma \in \mathcal{C}^\omega\big([0, 1]\big)$. As a consequence of the construction of $T''_{k,p}$, we obtain that $\gamma_1'(s) > 0$ for all $s \in (0, 1]$. Since $\gamma(s) \in \operatorname{cl}(M''_{k,pq})$ for all $s \in [0, 1]$, there holds $\partial_x h\big(\gamma(s)\big) = 0$ for all $s \in [0, 1]$. Consequently, for each $s \in (0, 1]$, there holds

$$\frac{\mathrm{d}}{\mathrm{d}s}(f^*_{k,pq} \circ \gamma_1)(s) = \partial_t h\big(\gamma(s)\big)\gamma_1'(s) + \partial_x h\big(\gamma(s)\big)\gamma_2'(s) = \partial_t h\big(\gamma(s)\big)\gamma_1'(s). \tag{5.38}$$

If the directional derivative in direction 1 of $f^*_{k,pq}$ at t_{bd}, $D_1 f^*_{k,pq}(t_{\mathrm{bd}})$, exists, then it is given by, cf. [65, Definition 2.3.2],

$$\begin{aligned}D_1 f^*_{k,pq}(t_{\mathrm{bd}}) &= \lim_{t\to 0^+} \frac{f^*_{k,pq}\big(t_{\mathrm{bd}} + t\big) - f^*_{k,pq}\big(t_{\mathrm{bd}}\big)}{t}\\ &= \lim_{s\to 0^+} \frac{f^*_{k,pq}\big(\gamma_1(s)\big) - f^*_{k,pq}\big(\gamma_1(0)\big)}{\gamma_1(s) - \gamma_1(0)}\\ &= \lim_{s\to 0^+} \frac{\big[f^*_{k,pq}\big(\gamma_1(s)\big) - f^*_{k,pq}\big(\gamma_1(0)\big)\big]/\big[s - 0\big]}{\big[\gamma_1(s) - \gamma_1(0)\big]/\big[s - 0\big]},\end{aligned}$$

where we have used that $\gamma_1(s) \to {t_{\mathrm{bd}}}^+$ for $s \to 0^+$. As

$$\lim_{s\to 0^+} \frac{f^*_{k,pq}\big(\gamma_1(s)\big) - f^*_{k,pq}\big(\gamma_1(0)\big)}{s - 0} = \frac{\mathrm{d}}{\mathrm{d}s}(f^*_{k,pq} \circ \gamma_1)(0), \qquad \lim_{s\to 0^+} \frac{\gamma_1(s) - \gamma_1(0)}{s - 0} = \gamma_1'(0),$$

using (5.38), we obtain the existence of $D_1 f^*_{k,pq}(t_{\mathrm{bd}})$. Using (5.37), we establish

$$D_1 f^*_{k,pq}(t_{\mathrm{bd}}) = \partial_t h\big(t_{\mathrm{bd}}\big) = \lim_{t \to t_{\mathrm{bd}}{}^+} (f^*_{k,pq})'(t).$$

Now Whitney's extension theorem [118, Theorem 2.3.6] yields the existence of $\widetilde{f}^*_{k,pq} \in \mathcal{C}^1(\mathbb{R})$ with $f^*_{k,pq} = \widetilde{f}^*_{k,pq}|_{T''_{k,p}}$. □

Lemma 5.2.13 *Let $T, T' \subset \mathbb{R}$ be compact intervals, let $\underline{\Gamma}, \overline{\Gamma} \in \mathcal{C}^\omega(T)$ with $\underline{\Gamma}(t) \leq \overline{\Gamma}(t)$ for all $t \in T$ and denote $\Gamma(t) = [\underline{\Gamma}(t), \overline{\Gamma}(t)]$. Let*

$$\underline{\theta} = \min\{t + x \in \mathbb{R} : t \in T, x \in \Gamma(t)\}, \qquad \overline{\theta} = \max\{t + x \in \mathbb{R} : t \in T, x \in \Gamma(t)\}$$

and $f \in \mathcal{C}^\omega\big([\underline{\theta}, \overline{\theta}]\big)$. If $\{(t,x) \in \mathrm{graph}(\Gamma) : f(t+x) \in T'\} \neq \emptyset$, then there exists a $N \in \mathbb{N}$, compact intervals $T_n \subset T$ and functions $\underline{\Gamma}_n, \overline{\Gamma}_n \in \mathcal{C}^\omega(T_n)$, $n = 1, ..., N$, such that

$$\{x \in \Gamma(t) : f(t+x) \in T'\} = \bigcup_{\substack{n \in \{1,...,N\}: \\ t \in T_n}} [\underline{\Gamma}_n(t), \overline{\Gamma}_n(t)], \quad \forall t \in T. \tag{5.39}$$

Proof If f is constant then we have $\{x \in \Gamma(t) : f(t+x) \in T'\} = \Gamma(t)$. Hence, in this case the assertion holds. Let us assume that f is not constant. Since f is continuous, the boundary points of $f^{-1}(T')$ are given by $f^{-1}\big(\mathrm{bd}(T')\big)$. Since $[\underline{\theta}, \overline{\theta}]$ is compact and $f \in \mathcal{C}^\omega\big([\underline{\theta}, \overline{\theta}]\big)$, [118, Corollary 1.2.6] yields that the set $f^{-1}\big(\mathrm{bd}(T')\big)$ is finite. Consequently, $f^{-1}(T')$ consists of a finite number $K \in \mathbb{N}$ of compact intervals, say H_k, $k = 1, ..., K$. We define the functions $\underline{\Lambda}_k, \overline{\Lambda}_k : T \to \mathbb{R}$ by $\underline{\Lambda}_k(t) = \max\{\underline{\Gamma}(t), \min H_k - t\}$ and $\overline{\Lambda}_k(t) = \min\{\overline{\Gamma}(t), \max H_k - t\}$, $k = 1, ..., K$. Obviously, $\underline{\Lambda}_k, \overline{\Lambda}_k \in \mathcal{PC}^\omega(T)$. Next we define $T'_k = \{t \in T : \underline{\Lambda}_k(t) \leq \overline{\Lambda}_k(t)\}$ and $\underline{\Psi}_k = \underline{\Lambda}_k|_{T'_k}$, $\overline{\Psi}_k = \overline{\Lambda}_k|_{T'_k}$. Since $\underline{\Lambda}_k, \overline{\Lambda}_k \in \mathcal{PC}^\omega(T)$, T'_k is the finite union of compact intervals. Moreover, T'_k can be decomposed into a finite number $N_k \in \mathbb{N}$ of intervals on each of which $\underline{\Psi}_k, \overline{\Psi}_k$ are analytic, $k = 1, ..., K$. For $k \in \{1, ..., K\}$ denote the closure of the n-th such interval by T_j, $j = \sum_{i=1}^{k-1} N_i + n$, $n = 1, ..., N_k$, and let $\underline{\Gamma}_j, \overline{\Gamma}_j$ be the restriction of $\underline{\Psi}_k, \overline{\Psi}_k$ on the respective connected component T_j of T'_k. This construction yields (5.39). □

Theorem 5.2.14 *Suppose that $\tau, \beta \in \mathcal{PC}^\omega(E \times \mathbb{R})$, $\delta \in \mathcal{PC}^\omega(V \times \mathbb{R} \times \mathbb{R}_0^+)$, $\underline{\Delta T}, \overline{\Delta T} \in \mathcal{PC}^\omega(X)$ and there exist $\underline{\mathcal{B}}, \underline{\mathcal{B}}^\circ \in \mathbb{R}$, $\underline{\mathcal{B}}^\circ > 0$, such that (4.1) and (4.2) hold.*

(i) Let a source node $v_0 \in V$ be given and let Assumption 5.1.1 hold, then the partial mapping $t_0 \mapsto b^*(v_0, t_0)$ is in $\mathcal{PC}^{1,\omega}\big(T(v_0)\big)$.

(ii) If $X = V \times \mathbb{R}$ and (V, E) is strongly connected, then $b^* \in \mathcal{PC}^{1,\omega}(X)$.

Proof We first assume that $\tau, \beta \in \mathcal{C}^\omega(E \times \mathbb{R})$, $\delta \in \mathcal{C}^\omega(V \times \mathbb{R} \times \mathbb{R}_0^+)$, $\underline{\Delta T}, \overline{\Delta T} \in \mathcal{C}^\omega(X)$. We proceed in a similar manner as in the proof of Theorem 4.2.4. Let $(e_1, ..., e_n)$ denote a finite, connected edge sequence from v_0 to v'. Denote $v_k = \omega(e_k)$, $k = 1, ..., n$, and $\widetilde{\Delta T}_k, \widetilde{T}_k$ as in (4.3), (4.4), $k = 0, ..., n-1$. Lemma 5.1.2 implies that $\widetilde{\Delta T}_k(t) = \Delta T(v_k, t)$ for all $t \in \widetilde{T}_k$ and hence $\widetilde{T}_k = \mathrm{supp}(\widetilde{\Delta T}_k) = T(v_k)$, $k = 0, ..., n-1$. Along the edge sequence $(e_1, ..., e_n)$, for each $k = 0, ..., n-1$, we must solve the following parametric optimization problem, cp. (4.5):

$$\widetilde{b}^*(v_k, t) = \inf_{\Delta t \in \widetilde{\Delta T}_k(t)} b_k(t, \Delta t), \qquad t \in \widetilde{T}_k,$$

where

$$b_k(t,\Delta t) = \delta(v_k,t,\Delta t) + \beta(e_k,t+\Delta t) + \widetilde{b}^*\big(v_{k+1}, t+\Delta t + \tau(e_k,t+\Delta t)\big)$$

and $\widetilde{b}^*(v_n,t) \equiv 0$ is in $\mathcal{C}^\omega(\widetilde{T}_n) \subset \mathcal{PC}^{1,\omega}(\widetilde{T}_n)$.
We will now prove by backwards induction that $t \mapsto \widetilde{b}^*(v_k,t)$ is in $\mathcal{PC}^{1,\omega}(\widetilde{T}_k)$. According to Remark 5.2.4, it is sufficient to prove that, for an arbitrary compact interval $K \subset \widetilde{T}_k$, $t \mapsto \widetilde{b}^*(v_k,t)$ is in $\mathcal{PC}^{1,\omega}(K)$, $k = 0,...,n-1$. Let $k \in \{0,...,n-1\}$ and $K \subset \widetilde{T}_k$ be a compact interval. Since ΔT is continuous, $\mathrm{graph}(\Delta T|_{\{v_k\}\times K})$ is compact, the set

$$T'_{k+1} = \{t+\Delta t+\tau(e_k,t+\Delta t) \in \mathbb{R} : (t,\Delta t) \in \mathrm{graph}(\Delta T|_{\{v_k\}\times K})\}$$

is a compact interval. (Recall that Lemma 5.1.2 implies that $T'_{k+1} \subset T(v_{k+1})$.) Using the induction hypothesis, i.e., using the fact that $t \mapsto \widetilde{b}^*(v_{k+1},t)$ is in $\mathcal{PC}^{1,\omega}(\widetilde{T}_{k+1})$, there exists a decomposition of T'_{k+1} into $I_k \in \mathbb{N}$ compact intervals $T'_{k+1,i}$, $i = 1,...,I_k$, such that $t \mapsto \widetilde{b}^*(v_{k+1},t)$ is in $\mathcal{C}^{1,\omega}(T'_{k+1,i})$ for each $i \in \{1,...,I_k\}$. By construction, we have $\{(t,\Delta t) \in \mathrm{graph}(\Delta T|_{\{v_k\}\times K}) : t+\Delta t+\tau(e_k,t+\Delta t) \in T'_{k+1,i}\} \neq \emptyset$ for each $i \in \{1,...,I_k\}$. Lemma 5.2.13 implies that, for each $i \in \{i,...,I_k\}$, there are $J_{k,i} \in \mathbb{N}$ and compact intervals $T_{k,ij} \subset K$, $j = 1,...,J_{k,i}$, as well as analytic functions $\underline{\Delta T}_{k,ij}, \overline{\Delta T}_{k,ij} \in \mathcal{C}^\omega(T_{k,ij})$, $j = 1,...,J_{k,i}$, satisfying

$$\{\Delta t \in \Delta T(v_k,t) : t+\Delta t+\tau(e_k,t+\Delta t) \in T'_{k+1,i}\} = \bigcup_{\substack{j\in\{1,...,J_{k,i}\}:\\ t\in T_{k,ij}}} [\underline{\Delta T}_{k,ij}(t), \overline{\Delta T}_{k,ij}(t)],$$

for all $t \in K$. We define the point-to-set mapping $\Delta T_{k,ij} : T_{k,ij} \to \mathcal{P}(\mathbb{R}_0^+)$,

$$\Delta T_{k,ij}(t) = [\,\underline{\Delta T}_{k,ij}(t), \overline{\Delta T}_{k,ij}(t)].$$

By construction of $T'_{k+1}, T_{k,ij}, \Delta T_{k,ij}$, $i = 1,...,I_k$, $j = 1,...,J_{k,i}$, we have

$$K = \bigcup_{i=1}^{I_k}\bigcup_{j=1}^{J_{k,i}} T_{k,ij}, \qquad \widetilde{\Delta T}_k(t) = \bigcup_{i=1}^{I_k} \bigcup_{\substack{j\in\{1,...,J_{k,i}\}:\\ t\in T_{k,ij}}} \Delta T_{k,ij}(t), \quad \forall t \in K.$$

Lemma 5.2.12 implies that the function $f_{k,ij} : T_{k,ij} \to \mathbb{R}$,

$$f_{k,ij}(t) = \min_{\Delta t\in\Delta T_{k,ij}(t)} b_k(t,\Delta t), \tag{5.40}$$

satisfies $f_{k,ij} \in \mathcal{PC}^{1,\omega}(T_{k,ij})$. Now

$$\widetilde{b}^*(v_k,t) = \min_{\substack{i\in\{1,...,I_k\}, j\in\{1,...,J_{k,i}\}:\\ t\in T_{k,ij}}} f_{k,ij}(t).$$

Using Lemma 5.2.9 on every nonempty set of the form $\bigcap_{i\in\mathcal{I}}\bigcap_{j\in\mathcal{J}_i} T_{k,ij}$ with $\mathcal{I} \subset \{1,...,I_k\}$ and $\mathcal{J}_i \subset \{1,...,J_{k,i}\}$, we obtain that $t \mapsto \widetilde{b}^*(v_k,t)$ is in $\mathcal{PC}^{1,\omega}(\widetilde{T}_k)$. Consequently, by backwards induction, $t \mapsto \widetilde{b}^*(v_0,t)$ is in $\mathcal{PC}^{1,\omega}(\widetilde{T}_0)$.

Let $K_i = T(v_0) \cap [i, i+1]$ for $i \in \mathbb{Z}$. Since the optimal cost function along each connected edge sequence from v_0 to v' is in $\mathcal{PC}^{1,\omega}\big(T(v_0)\big)$, the partial function $t_0 \mapsto b^*(v_0, t_0)$ is bounded on each non-empty set K_i, $i \in \mathbb{Z}$. Let $i \in \mathbb{Z}$ with $K_i \neq \emptyset$. As in the proof of Theorem 4.2.4 we see that the number of edge sequences which may possibly be traversed by an optimal path from v_0 to v' with departure time $t_0 \in K_i$ is finite. Let $\mathcal{E}$ denote this finite set of edge sequences. For each $\epsilon \in \mathcal{E}$ let $\widetilde{b}^*_\epsilon(v_0, t_0)$ denote the optimal cost along ϵ with departure time $t_0 \in K_i$ at v_0. As

$$b^*(v_0, t_0) = \min_{\epsilon \in \mathcal{E}} \widetilde{b}^*_\epsilon(v_0, t_0),$$

the assertion follows from Lemma 5.2.9.
Let us now assume that $\tau, \beta \in \mathcal{PC}^\omega(E \times \mathbb{R})$, $\delta \in \mathcal{PC}^\omega(V \times \mathbb{R} \times \mathbb{R}_0^+)$ and $\underline{\Delta T}, \overline{\Delta T} \in \mathcal{PC}^\omega(X)$. By further decomposing, for each $k \in \{0, ..., n-1\}$, $T_{k,ij}$ with respect to the breakpoints of $\tau, \beta, \underline{\Delta T}, \overline{\Delta T}$ and $\Delta T_{k,ij}$ with respect to the breakpoints of δ, the assertion follows in a similar manner as above. □

Remark 5.2.15 *We conjecture that the result of Theorem 5.2.14 can be slightly generalized by letting $\tau, \beta \in \mathcal{PC}^{1,\omega}(E \times \mathbb{R})$, $\delta_1 \in \mathcal{PC}^\omega(V \times \mathbb{R} \times \mathbb{R}_0^+)$, $\delta_2 \in \mathcal{PC}^{1,\omega}(V \times \mathbb{R})$ and defining $\delta(v, t, \Delta t) = \delta_1(v, t, \Delta t) + \delta_2(v, t + \Delta t)$. In order to prove this conjecture it would be necessary to show that the sum and concatenation of two $\mathcal{PC}^{1,\omega}$-functions of appropriate domain are again a $\mathcal{PC}^{1,\omega}$-functions by using a similar technique as in the proof of Lemma 5.2.9.*

We conjecture that similar assertions as in Lemma 5.2.8 - Theorem 5.2.14 can be proved if the word “analytic” is replaced by the word “algebraic”. Furthermore, using the above techniques, it seems possible to prove similar results as in [133], [112]. In this case it would be possible to replace the assumption on the continuity of the piecewise analytic functions by an appropriate concept of lower semicontinuous piecewise analytic functions. We will pursue none of these approaches here and leave them as a topic for further research.

5.3. Piecewise Linearity

In this section we consider a problem setting which is of particular importance in practical applications. Although we have shown that the optimal value function is (almost, cp. Definition 5.2.3) piecewise analytic if the network functions are, the computation of the optimal value function is rather nontrivial in this general case. This is due to the characterization of the candidates for optimal waiting times as the zeros of certain nonlinear mappings, which is itself a nontrivial problem, regardless of the parametric structure which is additionally imposed by the variation of the departure times.
For this reason we now consider the case in which the network functions are piecewise linear, and we prove that the optimal value function and the optimal control policy are both piecewise linear. Piecewise linear network models are not only used in the consideration of the time-dependent optimal path problem [47], [162], [98], [51], but are also a common means of describing electrical networks [120]. After introducing a suitable concept of piecewise linear functions in Subsection 5.3.1, we carry out a complexity analysis in Subsection 5.3.2, which is based on the structure of the set of breakpoints of the piecewise linear functions. Such an analysis has been suggested in [54], but to the best of our knowledge no results have been published on this topic so far.

Since the solution of a one-dimensional linear equation requires a constant number of arithmetic operations, measuring the complexity of piecewise linear function by the structure of the set of breakpoints is not only natural when considering the space complexity, but also when considering the time complexity of the computation of the optimal value function. We show in Subsection 5.3.2, that both the manner in which the waiting times are constrained, the specific form of the waiting cost function and the FIFO-property of the travel time function have a crucial impact on the complexity of the computation of the optimal value function in time-dependent networks.

5.3.1. Piecewise Linear Functions

Usually, a piecewise linear function $f : \mathbb{R}^m \to \mathbb{R}$ is defined via a set of linear functions $f_i : \mathbb{R}^m \to \mathbb{R}$, $i \in I$, $I \subset \mathbb{N}$, such that $f(x) = f_i(x)$ for some $i \in I$ and f is continuous, see, e.g., [29], [120]. The continuity of f implies that the set of breakpoints of f, i.e., the set of all $x \in \mathbb{R}^m$ for which there exists no open neighborhood on which f is linear, has a linear structure. Since the optimal value function is only lower semicontinuous in general, cf. Theorem 4.2.4, we henceforth also consider discontinuous piecewise linear functions. However, we retain the assumption that the set of breakpoints has a linear structure. Our approach is similar to the one followed recently in [165]. Since we require a finite representation of f (in view of space complexity) and a finite number of operations to compute the zeros of f (in view of time complexity), we restrict ourselves to the following definition.

Definition 5.3.1 *Let $T \subset \mathbb{R}$ be the finite union of closed intervals and points, and let $f : T \to \mathbb{R}$. Suppose that there exist $N_0, N_1 \in \mathbb{N}$, points $t_{n_0} \in T$, $n_0 = 1, ..., N_0$, and open intervals $T_{n_1} \subset T$ (open in $\mathbb{R}$), $n_1 = 1, ..., N_1$, such that*

$$T = \left(\bigcup_{n_0=1}^{N_0} \{t_{n_0}\} \right) \cup \left(\bigcup_{n_1=1}^{N_1} T_{n_1} \right)$$

and $f|_{T_{n_1}}$ is linear, $n_1 = 1, ..., N_1$. We say that $f \in \mathcal{PL}^1_c(T)$, $f \in \mathcal{PL}^1_{lsc}(T)$, $f \in \mathcal{PL}^1_{usc}(T)$ if f is continuous, lower semicontinuous, upper semicontinuous, respectively. We denote $\mathcal{PL}^1(T) = \mathcal{PL}^1_c(T) \cup \mathcal{PL}^1_{lsc}(T) \cup \mathcal{PL}^1_{usc}(T)$.

Remark 5.3.2 *If T is unbounded, Definition 5.3.1 precludes periodical piecewise linear functions (except for constant functions) from the following discussion. However, we will see in Section 5.4 that, under weak assumptions, it is sufficient to consider only a compact time interval in order to compute the optimal value function in a periodical time-dependent network.*

Let $T \subset \mathbb{R}$ be the finite union of closed intervals and points and let $f \in \mathcal{PL}^1(T)$. We now show that there are minimal numbers $N_0, N_1 \in \mathbb{N}$ and a unique minimal decomposition of T into points $t_{n_0} \subset T$, $n_0 = 1, ..., N_0$, and open sets $T_{n_1} \subset T$, $n_1 = 1, ..., N_1$, which satisfy Definition 5.3.1: Let T' be the set of all $t \in T$ such that there exists no open neighborhood $U_T \subset T$ of t (open in $\mathbb{R}$) on which f is linear. In any decomposition of T into open intervals and points according to Definition 5.3.1, no point of T' can be contained in an open interval by construction. Since $f \in \mathcal{PL}^1(T)$, T' is finite, say $|T'| = N_0 \in \mathbb{N}$, and by construction N_0 is minimal. Now, by construction, the set $T \setminus T'$ contains a finite and minimal number $N_1 \in \mathbb{N}$ of connected components, each of which is an open interval.

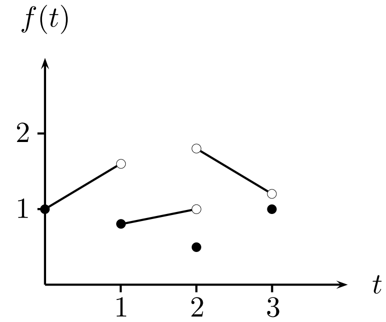


Figure 5.2.: $f \in \mathcal{PL}^1_{lsc}([0,3])$ with $\#f = (2,2,3)$. Values which are attained are marked by filled circles, values which are not attained are drawn as unfilled circles.

These minimal numbers play an important role in the consideration of the complexity of concatenating, summing and computing the pointwise minimum of a finite number of $\mathcal{PL}^1(T)$-functions. We also need the following number, in the definition of which we assume that the minimal unique decomposition $\{t_{n_0}\}_{n_0=1,...,N_0}$, $\{T_{n_1}\}_{n_1=1,...,N_1}$ is given: Let $N_{0,1}$ be the number of points t_{n_0}, $n_0 \in \{1,...,N_0\}$, for which there exists a $n_1 \in \{1,...,N_1\}$ such that $t_{n_0} \in \mathrm{bd}(T_{n_1})$ and $f|_{T_{n_1}\cup\{t_{n_0}\}}$ is linear. Without loss of generality we assume that these points are $\{t_{n_0}\}_{n_0=1,...,N_{0,1}}$. We further denote $N_{0,0} = N_0 - N_{0,1}$.

Definition 5.3.3 *Let $T \subset \mathbb{R}$ be the finite union of closed intervals and points and let $f \in \mathcal{PL}^1(T)$. Let $\{t_{n_0}\}_{n_0=1,...,N_0}$, $\{T_{n_1}\}_{n_1=1,...,N_1}$, $N_{0,0}, N_{0,1}, N_1 \in \mathbb{N}$ be as defined above. We define*

$$\#f = (N_{0,0}, N_{0,1}, N_1),$$

call the points $\{t_{n_0}\}_{n_0=1,...,N_{0,1}}$ half-isolated of (T,f) and call the points $\{t_{n_0}\}_{n_0=N_{0,1}+1,...,N_0}$ isolated of (T,f).

Remark 5.3.4 *In the following, for $N, N' \in \mathbb{N}_0^k$, $k \in \mathbb{N}$, with $N = (N_1,...,N_k)$, $N' = (N_1',...,N_k')$ we denote $N \leq N'$ if $N_i \leq N_i'$ for all $i = 1,...,k$. This eases the notation of complexity bounds for sums, concatenations and pointwise minima of $\mathcal{PL}^1$-functions.*

We have illustrated the measure of complexity of a $\mathcal{PL}^1_{lsc}([0,3])$-function f in Figure 5.2. Obviously, for a linear function $f : \mathbb{R} \to \mathbb{R}$ we have $\#f = (0,0,1)$, and for a piecewise linear and continuous function $f : \mathbb{R} \to \mathbb{R}$ consisting of N_1 linear pieces, there holds $\#f = (0, N_1 - 1, N_1)$.

In the following lemmas we have summarized some simple results on the decomposition of the domain and on the complexity of $\mathcal{PL}^1$-functions. These results include the number of arithmetic operations which are necessary to carry out certain computations. At this, we assume that the $\mathcal{PL}^1$-functions are stored in the following manner:
Let T be a finite union of closed intervals and points, let $f \in \mathcal{PL}^1(T)$ and let $\{t'_{n_0}\}_{n_0=1,...,N_0}$ be the points in the minimal decomposition of (T,f). We uniquely characterize f by the array of sextuples $(t_{n_0}, Df^-_{n_0}, f^-_{n_0}, f_{n_0}, f^+_{n_0}, Df^+_{n_0})_{n_0=1,...,N_0}$, which has the property that

$t_{n_0} < t_{n_0+1}$ for all $n_0 = 1, ..., N_0 - 1$ and $\bigcup_{n_0=1}^{N_0}\{t'_{n_0}\} = \bigcup_{n_0=1}^{N_0}\{t_{n_0}\}$. Here,

$$f_{n_0}^- = \begin{cases} \lim_{t\to t_{n_0}^-} f(t), & \text{if } (t_{n_0}-\epsilon, t_{n_0}] \subset T \text{ for some } \epsilon > 0 \\ \infty, & \text{otherwise} \end{cases},$$
$$f_{n_0} = f(t_{n_0}),$$
$$f_{n_0}^+ = \begin{cases} \lim_{t\to t_{n_0}^+} f(t), & \text{if } [t_{n_0}, t_{n_0}+\epsilon) \subset T \text{ for some } \epsilon > 0 \\ \infty, & \text{otherwise} \end{cases},$$

and

$$Df_{n_0}^- = \begin{cases} \lim\limits_{t\to t_{n_0}^-} \dfrac{f(t)-f_{n_0}^-}{t-t_{n_0}}, & \text{if } (t_{n_0}-\epsilon, t_{n_0}] \subset T \text{ for some } \epsilon > 0 \\ \infty, & \text{otherwise} \end{cases},$$
$$Df_{n_0}^+ = \begin{cases} \lim\limits_{t\to t_{n_0}^+} \dfrac{f(t)-f_{n_0}^+}{t-t_{n_0}}, & \text{if } [t_{n_0}, t_{n_0}+\epsilon) \subset T \text{ for some } \epsilon > 0 \\ \infty, & \text{otherwise} \end{cases}.$$

The function $f \in \mathcal{PL}_{lsc}^1([0,3])$ in Figure 5.2 would then be characterized by

$$\begin{bmatrix} 0 & \infty & \infty & 1 & 1 & 0.6 \\ 1 & 0.6 & 1.6 & 0.8 & 0.8 & 0.2 \\ 2 & 0.2 & 1 & 0.5 & 1.8 & -0.6 \\ 3 & -0.6 & 1.2 & 1 & \infty & \infty \end{bmatrix},$$

where each line stands for one sextuple. This representation simplifies the derivation of the number of arithmetic operations which are necessary to carry out the computations in the following lemmas.

Lemma 5.3.5 *Let $T \subset \mathbb{R}$ be the finite union of closed intervals and points and let $f \in \mathcal{PL}^1(T)$ with $\#f = (N_{0,0}, N_{0,1}, N_1)$. Then $N_1 \le N_{0,0} + N_{0,1} + 1$.*
If f is continuous, then $N_1 \le N_{0,1} + 1$ and $N_{0,1} \le 2N_1$.

Proof Let the decomposition of T according to the discussion preliminary to Definition 5.3.3 be given by the N_1 open intervals T_{n_1}, $n_1 = 1, ..., N_1$, and $N_0 = N_{0,0} + N_{0,1}$ points t_{n_0}, $n_0 = 1, ..., N_0$. By induction over N_1 it is easily seen that the intervals T_{n_1}, $n_1 = 1, ..., N_1$, contain at least $N_1 - 1$ disjoint boundary points. At this, the minimum is attained if each boundary point of each T_{n_1}, $n_1 = 1, ..., N_1$, is an inner point of $T = \mathbb{R}$. Since $\bigcup_{n_1=1}^{N_1} \mathrm{bd}(T_{n_1}) \subset \bigcup_{n_0=1}^{N_0}\{t_{n_0}\}$, this implies $N_1 \le N_0 + 1$. Moreover, if f is continuous, then the boundary points of the T_{n_1}, $n_1 = 1, ..., N_1$, are given by t_{n_0}, $n_0 = 1, ..., N_{0,1}$, and we obtain $N_1 \le N_{0,1} + 1$. Conversely, since each open interval is bounded by at most two boundary points, we obtain $N_{0,1} \le 2N_1$. □

Lemma 5.3.6 *Let $T, T' \subset \mathbb{R}$ be the finite union of closed intervals and points, let $f, g \in \mathcal{PL}^1(T)$ with $\#f = (N_{0,0}^f, N_{0,1}^f, N_1^f)$, $\#g = (N_{0,0}^g, N_{0,1}^g, N_1^g)$ and let $h \in \mathcal{PL}_c^1(T')$ with $\#h = (N_{0,0}^h, N_{0,1}^h, N_1^h)$. Let $N_0^f = N_{0,0}^f + N_{0,1}^f$, $N_0^g = N_{0,0}^g + N_{0,1}^g$, $N_0^h = N_{0,0}^h + N_{0,1}^h$.*

(i) If $c \in \mathbb{R}\setminus\{0\}$ then $\#(cf) = \#f$ and cf can be computed in $\mathcal{O}(N_0^f)$ arithmetic operations.

(ii) *Let $\star \in \{+, \cdot\}$. Then*

$$\#(f\star g) \leq \big(N^f_{0,0}+N^g_{0,0}+\min\{N^f_{0,1}, N^g_{0,1}\},\ N^f_{0,1}+N^g_{0,1}+\min\{N^f_{0,0}, N^g_{0,0}\},\ N^f_0+N^g_0+1 \big),$$

and if f, g are continuous, then $N^f_{0,0} = N^g_{0,0}$ and

$$\#(f \star g) \leq \big(N^f_{0,0},\ N^f_{0,1} + N^g_{0,1},\ N^f_{0,1} + N^g_{0,1} + 1 \big).$$

Moreover, if the decomposition of T according to the discussion preliminary to Definition 5.3.3 is identical for f and g, and g is continuous, then $\#(f \star g) = \#f$.
In any case, $f \star g$ can be computed in $\mathcal{O}\big(N^f_0 + N^g_0\big)$ arithmetic operations.

(iii) *Let $\widetilde{T} \subset T$ be a closed interval or a point. Then the set $h^{-1}(\widetilde{T})$ consists of at most $N^h_0 + 1$ connected components and can be computed in $\mathcal{O}(N^h_0)$ arithmetic operations. If $\widetilde{T}$ is a point, then $h^{-1}(\widetilde{T})$ contains at most $N^h_0 + 1$ boundary points,*
If h is monotonically increasing and T' is a closed interval, then the set $h^{-1}(\widetilde{T})$ consists of at most one connected component and can be computed in $\mathcal{O}\big(\log(N^h_0)\big)$ arithmetic operations.

(iv) *Let $\widetilde{h} = h|_{T' \cap h^{-1}(T)}$. There holds*

$$\#(f \circ \widetilde{h}\,) \leq \big((N^h_0 + 1)N^f_{0,0} + N^h_0,\ (N^h_0 + 1)N^f_{0,1} + N^h_{0,1},\ (N^h_0 + 1)N^f_0 + N^h_0 + 1 \big),$$

and if f is continuous, then

$$\#(f \circ \widetilde{h}\,) \leq \big((N^h_0 + 1)N^f_{0,0} + N^h_0,\ (N^h_0 + 1)N^f_{0,1} + N^h_{0,1},\ (N^h_0 + 1)N^f_{0,1} + N^h_{0,1} + 1 \big),$$

and $f \circ \widetilde{h}$ can be computed in $\mathcal{O}(N^h_0 N^f_0)$ arithmetic operations.
Assume that h is monotonically increasing and T' is a closed interval. Then

$$\#(f \circ \widetilde{h}\,) \leq (N^f_{0,0} + N^h_{0,0},\ N^f_{0,1} + N^h_{0,1},\ N^f_0 + N^h_{0,1} + 1),$$

and if f is continuous, then

$$\#(f \circ \widetilde{h}\,) \leq (N^f_{0,0} + N^h_{0,0},\ N^f_{0,1} + N^h_{0,1},\ N^f_{0,1} + N^h_{0,1} + 1),$$

and $f \circ \widetilde{h}$ can be computed in $\mathcal{O}\big(\log(N^h_0)N^f_0\big)$ arithmetic operations.

(v) *Suppose that $g \in \mathcal{PL}^1_c(T')$ with $\#g = (N^g_{0,0}, N^g_{0,1}, N^g_1)$, and the decomposition of T' according to the discussion preliminary to Definition 5.3.3 is identical for g and h, then $\#(f \circ h + g) \leq \#(f \circ h)$, and $f \circ h + g$ can be computed in $\mathcal{O}(N^h_0 N^f_0)$ arithmetic operations. If h is monotonically increasing and T' is a closed interval, then $f \circ h + g$ can be computed in $\mathcal{O}\big(\log(N^h_0)N^f_0\big)$ arithmetic operations.*

Proof The result *(i)* is obvious.
(ii) Denote $\#(f \star g) = (N^{f\star g}_{0,0}, N^{f\star g}_{0,1}, N^{f\star g}_1)$. If $t \in T$ is an isolated point of $(T, f \star g)$, then either t is an isolated point of (T, f), or t is an isolated point of (T, g), or t is a half-isolated point of both (T, f) and (T, g) and there only exist linear continuations of f and g to disjoint open intervals. This implies $N^{f\star g}_{0,0} \leq N^f_{0,0} + N^g_{0,0} + \min\{N^f_{0,1}, N^g_{0,1}\}$. If f, g are continuous,

then $f \star g$ is continuous and the isolated points of $(T,f),(T,g),(T,f\star g)$ coincide and are given by the isolated points of T. Consequently, $N_{0,0}^{f} = N_{0,0}^{g} = N_{0,0}^{f\star g}$. Moreover, if the decomposition of T according to the discussion preliminary to Definition 5.3.3 is identical for f and g, and g is continuous, then each isolated point t of (T,g) must be an isolated point of T, and hence t is also an isolated point of (T,f) and $(T,f\star g)$. This implies $N_{0,0}^{f\star g} = N_{0,0}^{f}$. If $t \in T$ is a half-isolated point of $(T,f\star g)$, then there exists no open neighborhood of t on which both f and g are linear. Furthermore, if t is an isolated point of (T,f) then t must also be an isolated point of (T,g). This implies $N_{0,1}^{f\star g} \leq N_{0,1}^{f} + N_{0,1}^{g} + \min\{N_{0,1}^{f}, N_{0,1}^{g}\}$. If f and g are continuous and t is a half-isolated point of $(T,f\star g)$, then t cannot be an isolated point of T, i.e., t cannot be an isolated point of (T,f) or (T,g). This implies $N_{0,1}^{f\star g} \leq N_{0,1}^{f} + N_{0,1}^{g}$. Moreover, if the decomposition of T according to the discussion preliminary to Definition 5.3.3 is identical for f and g, and g is continuous, then we obtain $N_{0,1}^{f\star g} \leq N_{0,1}^{f}$. Now, the remaining inequalities follow from Lemma 5.3.5 and the observation that each isolated or half-isolated point of $(f\star g, T)$ must be an isolated or half-isolated point of either (T,f) or (T,g). In order to determine the function $f\star g$, we parallely run through the array representations of f and g in increasing order of the first elements. For each half-isolated or isolated point of f and g we compute the sextupel associated with $f\star g$ from the representations of f and g in constant time. The resulting sextupel is stored unless the left- and right-sided limits of $f\star g$ coincide with the value $(f\star g)(t)$ and the left- and right-sided derivatives of $f\star g$ coincide. Since this decision requires a constant number of arithmetic operations, $f\star g$ can be computed in $\mathcal{O}\big(N_0^f + N_0^g\big)$ arithmetic operations.

(iii) We first assume that h is monotone increasing and T' is a closed interval. As $\widetilde{T}$ is connected, the set $h^{-1}(\widetilde{T})$ is connected. Let $(t'_{n_0}, Dh^{-}_{n_0}, h^{-}_{n_0}, h_{n_0}, h^{+}_{n_0}, Dh^{+}_{n_0})_{n_0=1,\dots,N_0}$ be the array representation of h. As h is continuous and monotone increasing, there holds $h^{-}_{n_0} = h_{n_0} = h^{+}_{n_0}$ and $h_{n_0} \leq h_{n'_0}$ if $n_0 < n'_0$. In order to determine (the existence of) the boundary points of $h^{-1}(\widetilde{T})$, we first determine the maximal index n_0^- with $h_{n_0^-} < \min\widetilde{T}$ (if no such index exists then either the left boundary point of $h^{-1}(\widetilde{T})$ is $\min T'$, or there exists no left boundary point) and the minimal index n_0^+ with $h_{n_0^+} > \max\widetilde{T}$ (if no such index exists then either the right boundary point of $h^{-1}(\widetilde{T})$ is $\max T'$, or there exists no right boundary point). We can determine these indices by bisection in $\mathcal{O}\big(\log(N_0^h)\big)$ arithmetic operations. The boundary points of $h^{-1}(\widetilde{T})$ are then contained in $(t_{n_0^-}, t_{n_0^-+1}]$ and $(t_{n_0^+-1}, t_{n_0^+}]$, and can be computed in constant time since $h|_{(t_{n_0^-}, t_{n_0^-+1}]}$ and $h|_{(t_{n_0^+-1}, t_{n_0^+}]}$ are linear.

We next assume that h is not monotonically increasing and that $\widetilde{T} = \{\,\widetilde{t}\,\}$ is a point. We now prove that $h^{-1}(\{\,\widetilde{t}\,\})$ contains at most $N_0^h + 1$ boundary points. This also implies that $h^{-1}(\{\,\widetilde{t}\,\})$ consists of at most $N_0^h + 1$ connected components. (Unless $h^{-1}(\{\,\widetilde{t}\,\}) = \mathbb{R}$, each connected component has at least one boundary point. Since $N_0^h + 1 \geq 1$, the assertion also holds if $h^{-1}(\{\,\widetilde{t}\,\}) = \mathbb{R}$.) Let $\{t'_{n_0}\}_{n_0=1,\dots,N_0^h}$, $\{T'_{n_1}\}_{n_1=1,\dots,N_1^h}$ be the decomposition of T' with respect to h as in the discussion preliminary to Definition 5.3.3. The assertion obviously holds if $N_0^h = 0$, because then $T' = \mathbb{R}$ and h is linear. Let us assume that $N_0^h > 0$. If t' is a boundary point of $h^{-1}(\{\,\widetilde{t}\,\})$, then either h properly intersects the constant function $\widetilde{t}$ on some T'_{n_1}, $n_1 \in \{1,\dots,N_1^h\}$, or $t' \in \bigcup_{n_0\in\{1,\dots,N_0^h\}}\{t'_{n_0}\}$. Moreover, if h properly intersects the constant function $\widetilde{t}$ at some $t' \in T'_{n_1}$, $n_1 \in \{1,\dots,N_1^h\}$, then $\mathrm{bd}(T'_{n_1}) \cap \bigcup_{n_0\in\{1,\dots,N_{0,1}^h\}}\{t'_{n_0}\} = \emptyset$ and $h^{-1}(\{\,\widetilde{t}\,\}) \cap T'_{n_1} = \{t'\}$. Hence, as h is continuous, for each such intersection, $|\mathrm{bd}(T'_{n_1})|$ points in $\{t'_{n_0}\}_{n_0=1,\dots,N_0^h}$ cannot be boundary points of $h^{-1}(\{\,\widetilde{t}\,\})$. Since we have assumed

that $N_0^h > 0$, there holds $|\mathrm{bd}(T'_{n_1})| \geq 1$, and hence $h^{-1}(\{\widetilde{t}\})$ contains at most $N_0^h + 1$ boundary points. In order to compute the connected components, we must check whether $h(t'_{n_0}) = \widetilde{t}$ for $n_0 = N_{0,1}^h + 1, ..., N_0^h$. Moreover, for each interval $\mathrm{cl}(T'_{n_1})$, $n_1 = 1, ..., N_1^h$, we must solve one linear equation in order to determine whether $\mathrm{cl}(T'_{n_1}) \cap h^{-1}(\widetilde{t}) \neq \emptyset$ and whether $\mathrm{cl}(T'_{n_1}) \subset h^{-1}(\{\widetilde{t}\})$. Now, using Lemma 5.3.5, the assertion follows as in *(ii)*. If $\widetilde{T}$ is a closed interval with nonempty interior, then the assertion follows in a similar manner by considering the boundary points of $\widetilde{T}$, and by observing that each connected component of $h^{-1}(\widetilde{T})$ has nonempty interior.
(iv) Let $\#(f \circ \widetilde{h}) = (N_{0,0}^{f\circ\widetilde{h}}, N_{0,1}^{f\circ\widetilde{h}}, N_1^{f\circ\widetilde{h}})$. If $t' \in T'$ is an isolated point of $(T' \cap h^{-1}(T), f \circ \widetilde{h})$, then either t' is an isolated point of (T', h), or $t = h(t')$ is an isolated point of (T, f), or t' is a half-isolated point of (T', h), $h(t')$ is a half-isolated point of (T, f) (and a boundary point of a connected component with nonempty interior of T) and t' is an isolated point of $h^{-1}(T)$. It is easily seen that the last case cannot occur if h is monotone increasing. *(iii)* yields $N_{0,0}^{f\circ\widetilde{h}} \leq (N_0^h + 1)N_{0,0}^f + N_0^h$ and, if h is monotonically increasing and T' is a closed interval, $N_{0,0}^{f\circ\widetilde{h}} \leq N_{0,0}^f + N_{0,0}^h$. If $t' \in T'$ is a half-isolated point of $(T' \cap h^{-1}(T), f \circ \widetilde{h})$, then either t' is a half-isolated point of (T', h), or $t = h(t')$ is a half-isolated point of (T, f). *(iii)* yields $N_{0,1}^{f\circ\widetilde{h}} \leq (N_0^h + 1)N_{0,1}^f + N_{0,1}^h$ and, if h is monotonically increasing and T' is a closed interval, $N_{0,1}^{f\circ\widetilde{h}} \leq N_{0,1}^f + N_{0,1}^h$. Since each half-isolated point of (T', h) results either in an isolated or a half-isolated point of $(T' \cap h^{-1}(T), f \circ \widetilde{h})$, there holds $N_{0,0}^{f\circ\widetilde{h}} + N_{0,1}^{f\circ\widetilde{h}} \leq (N_0^h + 1)N_0^f + N_0^h$. Now the remaining inequalities follow from Lemma 5.3.5. The assertions on the order of the number of necessary arithmetic operations follow as in *(ii)*.
Using the same arguments as in *(ii)* and *(iv)*, we obtain *(v)*. □

Lemma 5.3.7 *Let $T \subset \mathbb{R}$ be a closed interval, $N_{0,1} = |\mathrm{bd}(T)|$, let $f_n : T \to \mathbb{R}$, $n = 1, ..., N$, be a family of linear functions and let the function $f^* : T \to \mathbb{R}$ be defined by*

$$f^*(t) = \min_{n=1,...,N} f_n(t), \qquad \forall t \in T. \tag{5.41}$$

Then $f^* \in \mathcal{PL}_c^1(T)$ is continuous and concave with $\#f^* \leq (0, N_{0,1} + N - 1, N)$. Moreover, f^* can be computed from f in $\mathcal{O}(N^2)$ arithmetic operations.

Proof We prove the assertion by induction. Since $\#f_1 = (0, N_{0,1}, 1)$ and each linear function is continuous and concave the assertion follows if $N = 1$. Now suppose that $f_{N-1}^* : T \to \mathbb{R}$, defined by

$$f_{N-1}^*(t) = \min_{n=1,...,N-1} f_n(t), \qquad \forall t \in T$$

is concave and $f_{N-1}^* \in \mathcal{PL}_c^1(T)$ with $\#f_{N-1}^* \leq (0, N_{0,1} + N - 2, N - 1)$. Clearly $f^* \in \mathcal{PL}_c^1(T)$ as the pointwise minimum of two $\mathcal{PL}_c^1(T)$-functions. Next, f^* is concave, because for every

$t, t' \in T$ and every $\lambda \in [0,1]$ there holds

$$\begin{aligned} f^*(\lambda t + (1-\lambda)t') &= \min\Big\{ f^*_{N-1}\big(\lambda t + (1-\lambda)t'\big), f_N\big(\lambda t + (1-\lambda)t'\big)\Big\} \\ &\geq \min\big\{\lambda f^*_{N-1}(t) + (1-\lambda) f^*_{N-1}(t'), \lambda f_N(t) + (1-\lambda) f_N(t')\big\} \\ &\geq \lambda \min\big\{f^*_{N-1}(t), f_N(t)\big\} + (1-\lambda)\min\{f^*_{N-1}(t'), f_N(t')\} \\ &= \lambda f^*(t) + (1-\lambda) f^*(t'). \end{aligned}$$

Since f^*_{N-1} is concave and f_N is linear, there are at most 2 proper intersections of f^*_{N-1} and f_N. If there is no proper intersection then either $f^* = f^*_{N-1}$ or $f^* = f_N$ and the assertion follows. If there is exactly one proper intersection then $\#f^* \leq \#f^*_{N-1} + (0,1,1) \leq (0, N_{0,1} + N - 1, N)$. Finally, if there are exactly two proper intersections, say $t, t' \in T$, then there must be $k \geq 1$ breakpoints of f^*_{N-1} in $[t, t']$. Consequently, $\#f^* = \#f^*_{N-1} + (0,2,2) - (0,k,k) \leq (0, N_{0,1} + N - 1, N)$.
We again proceed inductively in order to prove the result on the number of arithmetic operations. Clearly, if $N = 1$ then f^* can be computed in 0 operations. Now let us suppose that we have already computed f^*_{N-1}. We first compute the the intersections $(t_{N,i}, y_{N,i})$, $i = 1, ..., I$, $I \in \{0,1,2\}$, of f^*_{N-1} and f_N. Since f^*_{N-1} is continuous and f_N is linear, this can be implemented by running through the array representation of f^*_{N-1}, solving $N-1$ linear equations and checking whether the found solutions are contained in the considered intervals. Hence, we can compute all intersections in $\mathcal{O}(N)$ arithmetic operations. On each connected component of $T \setminus \bigcup_{i=1}^{I} \{t_{N,i}\}$ we either have $f^* = f_N \leq f^*_{N-1}$ or $f^* = f^*_{N-1} \leq f_N$. By comparing the directional derivatives of f and f^*_{N-1} at one boundary point of each connected component we determine which function outvalues the other. Since $\mathcal{O}\big(\sum_{n=2}^{N}[(n-1)+1]\big) = \mathcal{O}(N^2)$, the assertion follows. □

It is obvious that the optimal value function is piecewise linear if all network functions are piecewise linear, the admissible and optimal waiting times always equal zero, and no state space constraints are imposed [47]. In Subsection 5.3.2, we will extend this result to the case in which waiting for a waiting time unequal to zero may be necessary and (or) optimal and certain arrival time constraints are imposed at the nodes. Here, the situation is complicated by the partial waiting cost functions $(t, \Delta t) \mapsto \delta(v, t, \Delta t)$, $v \in V$, and by the consideration of the parametric optimization problem (4.5). We now introduce a certain kind of 2-dimensional piecewise linear functions which turn out to be convenient when studying the complexity of the optimal value function.
In the following, a set $S \subset \mathbb{R}^2$ is called a closed line segment if there exist $x, y \in \mathbb{R}^2$ and a closed Interval $I \subset \mathbb{R}$ such that $S = \{s \in \mathbb{R}^2 : s = x + t(y-x), t \in I\}$. Similarly, $S \subset \mathbb{R}^2$ is called an open line segment if there exist $x, y \in \mathbb{R}^2$ and an open interval $I \subset \mathbb{R}$, such that $S = \{s \in \mathbb{R}^2 : s = x + t(y-x), t \in I\}$. Moreover, a set $\Theta \subset \mathbb{R}^2$ is a closed polygon if $\Theta = \mathrm{cl}(\mathrm{int}(\Theta))$ and there exists a finite set of line segments $\{S_n\}_{n=1,...,N}$, $N \in \mathbb{N}$, such that $\mathrm{bd}(\Theta) = \bigcup_{n=1}^{N} S_n$. Note that these definitions also allow line segments of infinite length and unbounded polygons.

Definition 5.3.8 *Let $\Theta \subset \mathbb{R}^2$ be the finite union of closed polygons, closed line segments and points, and let $f : \Theta \to \mathbb{R}$. Suppose that there exist $M_0, M_1 \in \mathbb{N}$, points $(t'_{m_0}, x'_{m_0}) \in \Theta$,*

$m_0 = 1, ..., M_0$, and open line segments $S'_{m_1} \subset \Theta$, $m_1 = 1, ..., M_1$, such that

$$\begin{aligned}&\big\{(t',x') \in \Theta : \nexists U_\Theta \subset \Theta \text{ open in } \mathbb{R}^2, (t',x') \in U_\Theta, \text{ such that } f|_{U_\Theta} \text{ is linear}\big\}\\ &= \left(\bigcup_{m_0=1}^{M_0} \{(t'_{m_0}, x'_{m_0})\}\right) \cup \left(\bigcup_{m_1=1}^{M_1} S'_{m_1}\right) =: \Theta'(f) \qquad (5.42)\end{aligned}$$

and $f|_{S'_{m_1}}$ is linear, $m_1 = 1, ..., M_1$. We say that $f \in \mathcal{PL}^2_c(\Theta)$, $f \in \mathcal{PL}^2_{lsc}(\Theta)$, $f \in \mathcal{PL}^2_{usc}(\Theta)$, if f is continuous, lower semicontinuous, upper semicontinuous, respectively. We denote $\mathcal{PL}^2(\Theta) = \mathcal{PL}^2_c(\Theta) \cup \mathcal{PL}^2_{lsc}(\Theta) \cup \mathcal{PL}^2_{usc}(\Theta)$.

Remark 5.3.9 *Note that $\Theta'(f)$ is a closed set since Θ is closed and the set*

$$\big\{(t',x') \in \Theta : \exists U_\Theta \subset \Theta \text{ open in } \mathbb{R}^2, (t',x') \in U_\Theta, \text{ such that } f|_{U_\Theta} \text{ is linear}\big\}$$

is open in $\mathbb{R}^2$ by construction.

Remark 5.3.10 *It can be easily seen that the generalization of Definition 5.3.1 and Definition 5.3.8 to arbitrary dimensions $n \in \mathbb{N}$ is possible. However, the consideration of the measures of complexity and the notation of these functions become more difficult in higher dimensions. We have separately introduced the classes $\mathcal{PL}^1, \mathcal{PL}^2$ in order to simultaneously introduce a notation which is appropriate for the consideration of the complexity of the time-dependent optimal path problem in Subsection 5.3.2.*

Let $\Theta \subset \mathbb{R}^2$ be the finite union of closed polygons, closed line segments and points and let $f \in \mathcal{PL}^2(\Theta)$. In the following, we will use a particular (not necessarily minimal) decomposition of $\Theta'(f)$ into points and open line segments. First, we prove that there exists a unique minimal decomposition corresponding to Definition 5.3.8, from which we then construct another decomposition which turns out to be convenient when studying the parametric optimization problem (4.5), cp. Figure 5.3. Let Θ_0^* be the set of all points $(t',x') \in \Theta$ for which there exists no open line segment $S' \subset \Theta$ with $(t',x') \in S'$ on which f is linear. This also implies that for each $(t',x') \in \Theta_0^*$, there exists no open neighborhood of (t',x') on which f is linear. Consequently, in any decomposition of $\Theta'(f)$ corresponding to Definition 5.3.8, each point $(t',x') \in \Theta_0^*$ must be in the set of (isolated) points. By construction Θ_0^* is minimal and by assumption $M_0^* = |\Theta_0^*| \in \mathbb{N}$. Let Θ_1^* be the set of all points $(t',x') \in \Theta$ for which there exists an open line segment S' containing (t',x') on which f is linear, but there exists no open neighborhood of (t',x') on which f is linear. By construction we have $\Theta_1^* \cap \Theta_0^* = \emptyset$ and $\Theta'(f) = \Theta_0^* \cup \Theta_1^*$. In any decomposition of $\Theta'(f)$ corresponding to Definition 5.3.8, each $(t',x') \in \Theta_1^*$ must be contained either in a line segment or in the set of (isolated) points, and each connected component of Θ_1^* is the union of open line segments. By construction there is a unique decomposition of each connected component into open line segments whose mutual intersection is either a point or empty. This decomposition defines the minimal number of line segments, thereby simultaneously minimizing the number of (isolated) points. By assumption, the number $M_1^* \in \mathbb{N}$ of such line segments is finite.
From this decomposition we construct the required decomposition as follows: Let Θ_0'' denote the set of all $(t',x') \in \Theta_1^*$ for which there exists a line segment $S' \subset \Theta_1^*$ with $(t',x') \in \mathrm{bd}(S')$ in the minimal decomposition of $\Theta'(f)$, and let $\Theta_0' = \Theta_0^* \cup \Theta_0''$. As the number of open line segments in the minimal decomposition of $\Theta'(f)$ is finite, the set Θ_0'' is finite. We denote

$M_0 = |\Theta_0'|$, $\Theta_0' = \{(t_{m_0}', x_{m_0}')\}_{m_0=1,...,M_0}$ and $\Theta_1' = \Theta_1^* \setminus \Theta_0''$. Let $M_1 \in \mathbb{N}$ denote the number of line segments in the unique minimal decomposition of Θ_1' into open line segments, and let $\{S_{m_1}'\}_{m_1=1,...,M_1}$ denote these line segments. (Note, that the number of these line segments is finite as Θ_0'' is a finite set.)
Due to the structure of the parametric optimization problem (4.5) we distinguish between line segments which are parallel to the t-axis and line segments which are not parallel to the t-axis in the (t, x)-plane $\mathbb{R}^2$. We denote by $\pi : \mathbb{R}^2 \to \mathbb{R}$ the orthogonal projection to the first axis of coordinates, $\pi(t, x) = t$. Let

$$\Theta_{1,1} = \bigcup_{\substack{m_1=1,...,M_1 \\ |\pi(S_{m_1}')|>1}} S_{m_1}', \qquad \Theta_{1,0} = \bigcup_{\substack{m_1=1,...,M_1 \\ |\pi(S_{m_1}')|=1}} S_{m_1}'. \tag{5.43}$$

As for $\mathcal{PL}^1$-functions, we partition the set Θ_0' into two sets: Let $M_{0,1}$ be the number of points (t_{m_0}', x_{m_0}'), $m_0 \in \{1, ..., M_0\}$, for which there exists a $m_1 \in \{1, ..., M_1\}$ such that $(t_{m_0}', x_{m_0}') \in \mathrm{bd}(S_{m_1}')$, $f|_{S_{m_1}' \cup \{(t_{m_0}', x_{m_0}')\}}$ is linear, and $S_{m_1}' \subset \Theta_{1,1}'$. Without loss of generality we assume that these points are $\{(t_{m_0}', x_{m_0}')\}_{m_0=1,...,M_{0,1}}$. We further denote $M_{0,0} = M_0 - M_{0,1}$ and

$$\Theta_{0,1} = \bigcup_{m_0=1}^{M_{0,1}} \{(t_{m_0}', x_{m_0}')\}. \qquad \Theta_{0,0} = \bigcup_{m_0=M_{0,1}+1}^{M_0} \{(t_{m_0}', x_{m_0}')\}, \tag{5.44}$$

In order to define a measure of the complexity of $\mathcal{PL}^2$-functions which is appropriate for the consideration of the parametric optimization problem (4.5), we must introduce some more notation: We denote $\{t_{n_0}\}_{n_0=1,...,N_0} = \pi(\Theta_{0,0} \cup \Theta_{0,1} \cup \Theta_{1,0})$, and by $\{T_{n_1}\}_{n_1=1,...,N_1}$ we denote the connected components of $\pi(\Theta) \setminus \bigcup_{n_0=1}^{N_0} \{t_{n_0}\}$. Note that, by assumption, both sets are finite and, by construction, T_{n_1} is an open interval for each $n_1 \in \{1, ..., N_1\}$. Let $N_{0,1}$ be the number of points t_{n_0}, $n_0 = 1, ..., N_0$, such that either for each $(t_{n_0}, x') \in \Theta'(f)$ there exists an open line segment $S' \subset \pi^{-1}\big((-\infty, t_{n_0})\big) \cap \Theta_{1,1}$ with $(t_{n_0}, x') \in \mathrm{bd}(S')$ such that $f|_{S' \cup \{(t_{n_0}, x')\}}$ is linear, or for each $(t_{n_0}, x') \in \Theta'(f)$ there exists an open line segment $S' \subset \pi^{-1}\big((t_{n_0}, \infty)\big) \cap \Theta_{1,1}$ with $(t_{n_0}, x') \in \mathrm{bd}(S')$ such that $f|_{S' \cup \{(t_{n_0}, x')\}}$ is linear. Without loss of generality we assume that these points are $\{t_{n_0}\}_{n_0=1,...,N_{0,1}}$. We further denote $N_{0,0} = N_0 - N_{0,1}$. Finally, let

$$\begin{aligned}
J_0 &= \max_{n_0=N_{0,1}+1,...,N_0} \Big|\big\{x' \in \mathbb{R} : (t_{n_0}, x') \in \Theta_{0,0} \cup \Theta_{0,1} \cup \Theta_{1,1}\big\}\Big|, \\
J_1 &= \max_{n_1=1,...,N_1} \max_{t \in T_{n_1}} \Big|\big\{x' \in \mathbb{R} : (t, x') \in \Theta_{1,1}\big\}\Big|, \\
I_0 &= \max_{\tilde{t} \in \mathbb{R}} \Big|\big\{(t', x') \in \Theta_{0,1} : t' + x' = \tilde{t}\big\}\Big|, \\
I_1 &= \max_{\tilde{t} \in \mathbb{R}} \Big|\big\{m_1 \in \{1, ..., M_1\} : \exists!\, (t', x') \in S_{m_1}', t' + x' = \tilde{t}\big\}\Big|.
\end{aligned}$$

Definition 5.3.11 *Let $\Theta \subset \mathbb{R}^2$ be the finite union of closed polygons, closed line segments and points and let $f \in \mathcal{PL}^2(\Theta)$. Let $N_{0,0}, N_{0,1}, J_0, I_0, N_1, J_1, I_1$ be as defined above. We*

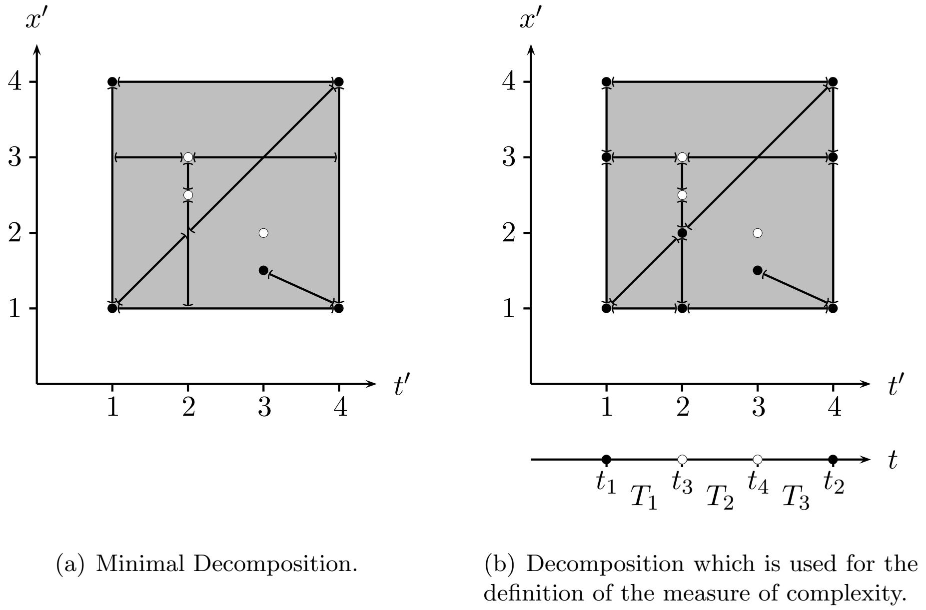


Figure 5.3.: (a) Decomposition of the domain of $f \in \mathcal{PL}^2_{lsc}\big([1,4]^2\big)$ into $M_0^* = 8$ points and $M_1^* = 10$ line segments. (b) Decomposition of the domain of f into $M_0 = 12$ points and $M_1 = 15$ line segments. Points (t', x') for which there exists no adjacent line segment S' such that $f|_{S' \cup \{(t',x')\}}$ is linear are drawn as unfilled circles. There holds $\#f = (2, 2, 5, 2, 3, 5, 6)$.

define

$$\#f = (N_{0,0}, N_{0,1}, J_0, I_0, N_1, J_1, I_1).$$

We have illustrated the measure of complexity of a $\mathcal{PL}^2_{lsc}\big([1,4]^2\big)$-function f in Figure 5.3. The notion of the complexity of a $\mathcal{PL}^2$-function is not quite obvious at first appearance, but it will be easily understood in view of Lemma 5.3.12 and Lemma 5.3.13. As an illustrative example, considering $\Theta = \{(t', x') \in \mathbb{R}^2 : t', x' \geq 0 \wedge |t' + x'| \leq 1\}$ and a continuous linear function $f : \Theta \to \mathbb{R}$, we obtain $\#f = (0, 2, 0, 2, 1, 2, 2)$. Simple expressions for the complexity of functions $f \in \mathcal{PL}^2_c(\Theta)$ can also be derived if $\Theta'(f)$ forms a regular triangular grid.

We now show that the notions of the complexity of $\mathcal{PL}^1$- and $\mathcal{PL}^2$-functions are appropriate when studying parametric optimization problems of the form (4.5). At this, we assume that the $\mathcal{PL}^2$-functions are stored in the following manner:

Let Θ be the finite union of closed polygons, line segments and points, let $f \in \mathcal{PL}^2(\Theta)$ and let $\{(t'_{m_0}, x'_{m_0})\}_{m_0=1,\dots,M_0}$ and $\{S'_{m_1}\}_{m_1=1,\dots,M_1}$ be as described preliminary to Definition 5.3.11. Let further $\{P'_{m_2}\}_{m_2=1,\dots,M_2}$ be the collection of the connected components of $\Theta \setminus \Theta'(f)$, $M_2 \in \mathbb{N}$. Observe that each P_{m_2}, $m_2 \in \{1, \dots, M_2\}$, is an open polygon and that $M_2 \leq J_1 N_1$. Finally, let $\{t_{n_0}\}_{n_0=1,\dots,N_0}$ and $\{T_{n_1}\}_{n_1=1,\dots,N_1}$ be as described preliminary to Definition 5.3.11. We uniquely characterize f by 3 linked lists L_l, $l = 0, 1, 2$, of objects [43, Chapter 10.2]: The first list contains M_0 objects $O^0_{m_0}$, each of which contains the coordinates of one point (t'_{m_0}, x'_{m_0}), $m_0 \in \{1, \dots, M_0\}$, the respective value of f, and one pointer. The second list contains M_1 objects $O^1_{m_1}$, each of which consists of a set of inequalities which uniquely characterize a line segment S'_{m_1}, $m_1 \in \{1, \dots, M_1\}$, the 3 coefficients of the linear function $f|_{S'_{m_1}}$ and a pointer. The third list contains M_2 objects $O^2_{m_2}$, each of which consists

of a set of inequalities which uniquely characterize an open polygon P'_{m_2}, $m_2 \in \{1, ..., M_2\}$, the 3 coefficients of the linear function $f|_{P'_{m_2}}$ and one pointer. (We will describe below in which manner the pointers are set.)
In order to efficiently solve the subproblems that arise in the computation of optimal paths we further store two functions: The first function, `GetPartialFunctions`, is a linked list of objects $O_i^{\mathtt{GPF}}$, $i = 1, ..., I^{\mathtt{GPF}}$, each object consisting of a point set and a linked list of pointers. Let $\mathrm{supp}(O_i^{\mathtt{GPF}})$ denote the point set associated with $O_i^{\mathtt{GPF}}$, $i = 1, ..., I^{\mathtt{GPF}}$. The point sets are given by $\{t_{n_0}\}_{n_0=1,...,N_0}$ and $\{T_{n_1}\}_{n_1=1,...,N_1}$, and the objects $O_i^{\mathtt{GPF}}$, $i = 1, ..., I^{\mathtt{GPF}}$, are linked in such a way that $t_i < t_{i+1}$ for all $t_i \in \mathrm{supp}(O_i^{\mathtt{GPF}})$, $t_{i+1} \in \mathrm{supp}(O_{i+1}^{\mathtt{GPF}})$, $i = 1, ..., I^{\mathtt{GPF}} - 1$. The pointers in $O_i^{\mathtt{GPF}}$, $i \in \{1, ..., I^{\mathtt{GPF}}\}$, point to the objects $O_m^l \subset (\mathrm{supp}(O_i^{\mathtt{GPF}}) \times \mathbb{R})$, i.e., to the objects in the linked lists L_l, $l = 0, 1, 2$. The pointer in the respective object O_m^l in the linked list L_l, $l \in \{0, 1, 2\}$, $m \in \{1, ..., M_l\}$, is set in such a way that it points to $O_i^{\mathtt{GPF}}$.
The second function, `GetDecomposition`, is a linked list of objects $O_i^{\mathtt{GD}}$, $i = 1, ..., I^{\mathtt{GD}}$, each object consisting of a point set and a linked list of pointers. Observe that the mapping

$$\widetilde{t} \mapsto \big\{(l, m) \in \{0, 1, 2\} \times \mathbb{N} : m \leq M_l, \mathrm{supp}(O_m^l) \cap \{(t' + x') \in \mathbb{R}^2 : t' + x' = \widetilde{t}\} \neq \emptyset\big\}, \quad (5.45)$$

is piecewise constant. The point sets associated with the objects $O_i^{\mathtt{GD}}$, $i = 1, ..., I^{\mathtt{GD}}$, are given by the maximal connected components of $\mathbb{R}$ on which the mapping (5.45) is constant. Let $\mathrm{supp}(O_i^{\mathtt{GD}})$ denote the point set associated with $O_i^{\mathtt{GD}}$, $i = 1, ..., I^{\mathtt{GD}}$. The point sets are linked in such a way that $\widetilde{t}_i < \widetilde{t}_{i+1}$ for all $\widetilde{t}_i \in \mathrm{supp}(O_i^{\mathtt{GD}}), \widetilde{t}_{i+1} \in \mathrm{supp}(O_{i+1}^{\mathtt{GD}})$, $i = 1, ..., I^{\mathtt{GD}} - 1$. The pointers in $O_i^{\mathtt{GD}}$, $i \in \{1, ..., I^{\mathtt{GD}}\}$, point to the objects O_m^l with $\mathrm{supp}(O_m^l) \cap \{(t' + x') \in \mathbb{R}^2 : t' + x' = \widetilde{t}\} \neq \emptyset$, $l \in \{0, 1, 2\}, m \in \{0, ..., M_l\}$. Let us denote the pointers associated with some object $O_i^{\mathtt{GD}}$, $i = 1, ..., I^{\mathtt{GD}}$, by $a_{i,j}$, $j = 1, ..., J_i$, $J_i \in \mathbb{N}$ and let $\mathrm{supp}(a_{i,j})$ denote the point set associated with the object the pointer $a_{i,j}$ points to. The pointers are linked in such a way that $\mathrm{supp}(a_{i,j}), \mathrm{supp}(a_{i,j+1})$ are adjacent point sets with the property that $t_j < t_{j+1}$ for all $(t_j, x'_j) \in \mathrm{supp}(a_{i,j})$, $(t_{j+1}, x'_{j+1}) \in \mathrm{supp}(a_{i,j+1})$, $j = 1, ..., J_i - 1$.
We compute the functions `GetPartialFunctions` and `GetDecomposition` in a preprocessing step, and use them to trade off the space complexity of storing $\mathcal{PL}^2$-functions to the time complexity of solving the time-dependent optimal path problem.

Lemma 5.3.12 *Let $\Theta \subset \mathbb{R}^2$ be the finite union of closed polygons, closed line segments and points. Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ denote the orthogonal projection to the first coordinate axis, denote $T = \pi(\Theta)$, define*

$$\Gamma : T \to \mathcal{P}(\mathbb{R}), \qquad \Gamma(t) = \pi^{-1}(t) \cap \Theta,$$

and suppose that $\Gamma(t)$ is compact for each $t \in T$. Let $f \in \mathcal{PL}^2_{lsc}(\Theta)$ with $\#f = (N_{0,0}, N_{0,1}, J_0, I_0, N_1, J_1, I_1)$, and let the function $f^* : T \to \mathbb{R}$ be defined by

$$f^*(t) = \min_{x' \in \Gamma(t)} f(t, x').$$

Let $\Theta_{0,0}, \Theta_{0,1}, \Theta_{1,0}, \Theta_{1,1} \subset \Theta$ be a characterization of the breakpoints of f as in (5.43), (5.44) and define

$$\Lambda : T \to \mathcal{P}(\mathbb{R}), \qquad \Lambda(t) = \pi^{-1}(t) \cap (\Theta_{0,0} \cup \Theta_{0,1} \cup \Theta_{1,1}).$$

Then $f^* \in \mathcal{PL}^1_{lsc}(T)$ *with* $\#f^* \leq (N_{0,0},\, N_{0,1} + N_1(J_1-1),\, N_1 J_1)$ *and*

$$f^*(t) = \min_{x' \in \Lambda(t)} f(t, x'). \tag{5.46}$$

Moreover, f^* *can be computed from* f *in* $\mathcal{O}\big(N_{0,0}J_0 + N_1 J_1^2\big)$ *arithmetic operations.*

Proof Since Θ consists of a finite number of closed polygons, closed line segments and points, $\Gamma(t)$ is closed for each $t \in T$. As $\pi(\Theta_{0,0} \cup \Theta_{0,1} \cup \Theta_{1,0})$ is a finite set which is closed in T, Γ is upper semicontinuous at each $t \in T$. Since $\Gamma(t)$ is compact for each $t \in T$, Γ is upper semicontinuous. Since f is lower semicontinuous, [65, Theorem 2.2.1] implies that f^* is lower semicontinuous. Next, we show that (5.46) holds. Let $t \in T$ and let $x^* \in \arg\min_{x' \in \Gamma(t)} f(t, x')$. Suppose that $x^* \notin \Lambda(t)$. Since $\Theta \setminus (\Theta_{0,0}, \Theta_{0,1}, \Theta_{1,1})$ is open in Θ, there exists a maximal connected component S' of $\Gamma(t) \setminus \Lambda(t)$ with $x^* \in S'$, such that $f|_{S'}$ is linear. As $x^* \in \arg\min_{x' \in \Gamma(t)} f(t, x')$, $f|_{S'}$ must be constant. Let $x'' \in \mathrm{bd}(S')$. Then $x'' \in \Lambda(t)$ and, since f is lower semicontinuous, $f(t, x'') \leq \lim_{x' \to x''} f(t, x') = f(t, x^*)$. Consequently, (5.46) holds.
Let $\{t_{n_0}\}_{n_0=1,\dots,N_0}$ and $\{T_{n_1}\}_{n_1=1,\dots,N_1}$ be as described preliminary to Definition 5.3.11. For each $n_1 \in \{1, \dots, N_1\}$, T_{n_1} is an open interval and $\mathrm{graph}(\Lambda|_{T_{n_1}}) \subset \Theta_{1,1}$ consists of a finite number $J \leq J_1$ of line segments S_j, $j = 1, \dots, J$, on each of which f is linear. By $\gamma_j : T_{n_1} \to \mathbb{R}$ we denote the linear function satisfying $\big(t, \gamma_j(t)\big) \in S_j$ for all $t \in T_{n_1}$ and by $\widetilde{f}_j : \mathrm{cl}(T_{n_1}) \to \mathbb{R}$ we denote the linear continuation of $t \mapsto f\big(t, \gamma_j(t)\big)$ to $\mathrm{cl}(T_{n_1})$. Let $\widetilde{f}^* : \mathrm{cl}(T_{n_1}) \to \mathbb{R}$, $\widetilde{f}^*(t) = \min_{j=1,\dots,J} \widetilde{f}_j(t)$ and observe that $\widetilde{f}^*(t) = f^*(t)$ for all $t \in T_{n_1}$. Lemma 5.3.12 yields that $\widetilde{f}^* \in \mathcal{PL}^1_c\big(\mathrm{cl}(T_{n_1})\big)$, $\widetilde{f}^*$ is concave and satisfies $\#\widetilde{f}^* \leq \big(0, J_1 - 1 + |\mathrm{bd}(T_{n_1})|, J_1\big)$. Note that $\bigcup_{n_1=1}^{N_1} \mathrm{bd}(T_{n_1}) \subset \bigcup_{n_0=1}^{N_0} \{t_{n_0}\}$. For each t_{n_0}, $n_0 = 1, \dots, N_{0,1}$, there exists an adjacent T_{n_1}, $n_1 \in \{1, \dots, N_1\}$, such that f can be continuously continued to each $(t_{n_0}, x') \in \mathrm{graph}(\Lambda|_{\{t_{n_0}\}})$ from some $S' \subset \mathrm{graph}(\Lambda|_{T_{n_1}}) \subset \Theta_{1,1}$ with $(t_{n_0}, x') \in \mathrm{bd}(S')$. This yields $\#f^* \leq (N_{0,0}, N_{0,1} + N_1(J_1 - 1), N_1 J_1)$.
Since the minimum of J real numbers can be computed in J arithmetic operations, and such a minimum has to be computed for each $t \in T' = \bigcup_{N_{0,1}+1}^{N_0} \{t_{n_0}\}$, at which $J \leq J_0$ for all $t \in T'$, the computation of $f^*|_{T'}$ requires $\mathcal{O}(N_{0,0}J_0)$ arithmetic operations. (Here we access the values of f in $\mathcal{O}(N_{0,0}J_0)$ arithmetic operations by using the function `GetPartialFunctions`.) As we may use Lemma 5.3.7 for the closure of each of the N_1 remaining time intervals in $\pi(\Theta)$ (we again use the function `GetPartialFunctions` in order to access the partial linear functions in $\mathcal{O}(N_1 J_1)$ arithmetic operations), the result follows. □

We next prove two simple lemmas which will be useful when applying the piecewise linear approach to the time-dependent optimal path problem.

Lemma 5.3.13 *Let* $\Theta \subset \mathbb{R}^2$ *be the finite union of closed polygons, closed line segments and points and let* $\widetilde{T} \subset \mathbb{R}$ *be the finite union of closed intervals and points. Denote* $\widetilde{\Theta} = \{(t', x') \in \Theta : t' + x' \in \widetilde{T}\}$. *Let* $g \in \mathcal{PL}^2_{lsc}(\Theta)$ *with* $\#g = (N^g_{0,0}, N^g_{0,1}, J^g_0, I^g_0, N^g_1, J^g_1, I^g_1)$, $f \in \mathcal{PL}^1_{lsc}(\widetilde{T})$ *with* $\#f = (N^f_{0,0}, N^f_{0,1}, N^f_1)$ *and let the function* $h : \widetilde{\Theta} \to \mathbb{R}$ *be defined by* $h(t', x') = g(t', x') + f(t' + x')$. *Then* $h \in \mathcal{PL}^2_{lsc}(\widetilde{\Theta})$ *with* $\#h = (N^h_{0,0}, N^h_{0,1}, J^h_0, I^h_0, N^h_1, J^h_1, I^h_1)$, *satisfying*

$$\begin{aligned} N^h_{0,0} &\le N^g_0 + I^g_1 N^f_{0,0}, & N^h_{0,1} &\le N^g_{0,1} + I^g_1 N^f_{0,1}, & J^h_0 &\le J^g_0 + N^f_0, & I^h_0 &\le I^g_0 + I^g_1,\\ N^h_1 &\le N^g_1 + I^g_1 N^f_0, & & & J^h_1 &\le J^g_1 + N^f_0, & I^h_1 &\le I^g_1 + I^g_0,\end{aligned}$$

where $N^f_0 = N^f_{0,0} + N^f_{0,1}$*. Moreover,* h *can be computed from* f, g *in* $\mathcal{O}\big((N^g_0 + I^g_1 N^f_0)(J^g_0 + J^g_1 + 2N^f_0)\big)$ *arithmetic operations.*

Proof First, we observe that $\widetilde{\Theta}$ is the finite union of closed polygons, closed line segments and points since $\{(t', x') \in \mathbb{R}^2 : t' + x' \in \widetilde{T}\}$ is the finite union of closed polygons and closed line segments. Furthermore, if $\widetilde{g}, \widetilde{f} \in \mathcal{PL}^2_{lsc}(\widetilde{\Theta})$, then it is obvious from Definition 5.3.8 that $h = \widetilde{g} + \widetilde{f} \in \mathcal{PL}^2_{lsc}(\widetilde{\Theta})$. Let $\widetilde{g} = g|_{\widetilde{\Theta}}$, let $\widetilde{f} : \widetilde{\Theta} \to \mathbb{R}$ be defined by $\widetilde{f}(t', x') = f(t' + x')$ and let $\{\widetilde{t}_{n_0}\}_{n_0=1,\dots,N^f_0}$ denote the minimal number of breakpoints of f according to Definition 5.3.1. For each $(t', x') \in \operatorname{int}(\widetilde{\Theta})$ with $t' + x' \neq \widetilde{t}_{n_0}$ for all $n_0 = 1, ..., N^f_0$ there exists an open neighborhood of (t', x') on which $\widetilde{f}$ is linear. Moreover, for each $(t', x') \in \operatorname{int}(\widetilde{\Theta})$ with $t' + x' = \widetilde{t}_{n_0}$ for some $n_0 \in \{1, ..., N^g_0\}$ there exists no open neighborhood of (t', x') on which $\widetilde{f}$ is linear, but $\widetilde{f}$ is linear on the connected component C' of $\{(t'', x'') \in \operatorname{int}(\widetilde{\Theta}) : t'' + x'' = \widetilde{t}_{n_0}\}$ which contains (t', x'). Since C' is either an open line segment or a point, we obtain $\widetilde{f} \in \mathcal{PL}^2_{lsc}(\widetilde{\Theta})$.

Let $\Theta^g_{0,0}, \Theta^g_{0,1}, \Theta^g_{1,0}, \Theta^g_{1,1}$ denote the decomposition of Θ with respect to g, let $\widetilde{\Theta}^{\widetilde{g}}_{0,0}, \widetilde{\Theta}^{\widetilde{g}}_{0,1}, \widetilde{\Theta}^{\widetilde{g}}_{1,0}, \widetilde{\Theta}^{\widetilde{g}}_{1,1}$ denote the decomposition of $\widetilde{\Theta}$ with respect to $\widetilde{g}$ and let $\widetilde{\Theta}^h_{0,0}, \widetilde{\Theta}^h_{0,1}, \widetilde{\Theta}^h_{1,0}, \widetilde{\Theta}^h_{1,1}$ denote the decomposition of $\widetilde{\Theta}$ with respect to h according to (5.43) and (5.44).

If $(t', x') \in \widetilde{\Theta}^h_{0,0}$, then either $(t', x') \in \Theta^g_{0,0}$, or $(t', x') \in \Theta^g_{0,1}$ and $t' + x' = \widetilde{t}_{n_0}$ for some $n_0 \in \{1, ..., N^f_0\}$, or $(t', x') \in \Theta^g_{1,1} \cup \Theta^g_{1,0}$ and $t' + x' = \widetilde{t}_{n_0}$ for some $n_0 \in \{N^f_{0,1} + 1, ..., N^f_0\}$. This yields $N^h_{0,0} \le N^g_0 + I^g_1 N^f_{0,0}$. Next, we observe that the lower semicontinuity of f, g implies that, if $(t', x') \in \Theta^g_{0,0}$, then $(t', x') \notin \Theta^h_{0,1}$. Hence, if $(t', x') \in \widetilde{\Theta}^h_{0,1}$, then either $(t', x') \in \Theta^g_{0,1}$, or $(t', x') \in \Theta^g_{1,1} \cup \Theta^g_{1,0}$ and $t' + x' = \widetilde{t}_{n_0}$ for some $n_0 \in \{1, ..., N^f_{0,1}\}$. This yields $N^h_{0,1} \le N^g_{0,1} + I^g_1 N^f_{0,1}$. From the above reasoning it is easily seen that the number of additional points in the decomposition of $\pi(\widetilde{\Theta})$ bounded from above by $I^g_1 N^f_0$. Hence, $N^h_0 \le N^g_0 + I^g_1 N^f_0$ and $N^h_1 \le N^g_1 + I^g_1 N^f_0$.

Let $t' \in \pi(\widetilde{\Theta})$. If $(t', x') \in \pi^{-1}(t') \cap (\widetilde{\Theta}^h_{0,0} \cup \widetilde{\Theta}^h_{0,1} \cup \widetilde{\Theta}^h_{1,1})$ then either $(t', x') \in \pi^{-1}(t') \cap (\Theta^g_{0,0} \cup \Theta^g_{0,1} \cup \Theta^g_{1,1})$ or $x' + t' = \widetilde{t}_{n_0}$ for some $n_0 \in \{1, ..., N^f_0\}$. This yields $J^h_0 \le J^g_0 + N^f_0$ and $J^h_1 \le J^g_1 + N^f_0$.

Let $\widetilde{t} \in \widetilde{T}$. If $(t', x') \in \widetilde{\Theta}^h_{0,1}$ with $t' + x' = \widetilde{t}$, then either $(t', x') \in \Theta^g_{0,1}$ with $t' + x' = \widetilde{t}$, or $(t', x') \in \Theta^g_{1,1} \cup \Theta^g_{1,0}$ and $t' + x' = \widetilde{t} = \widetilde{t}_{n_0}$ for some $n_0 \in \{1, ..., N^f_{0,1}\}$. This yields $I^h_0 \le I^g_0 + I^g_1$. Next, we observe that the lower semicontinuity of f and g implies that $(\widetilde{\Theta}^h_{1,1} \cup \widetilde{\Theta}^h_{1,0}) \setminus (\Theta^g_{1,1} \cup \Theta^g_{1,0}) \subset \bigcup_{n_0=1}^{N^f_0} \{(t', x') \in \widetilde{\Theta} : t' + x' = \widetilde{t}_{n_0}\}$. Hence, if $\big|\{(t', x') \in S' : t' + x' = \widetilde{t}\}\big| = 1$ for some open line segment $S' \subset \widetilde{\Theta}^h_{1,1} \cup \widetilde{\Theta}^h_{1,0}$, then the thereby defined intersection point $\{(t'_0, x'_0)\} = \{(t', x') \in S' : t' + x' = \widetilde{t}\}$ satisfies either $(t'_0, x'_0) \in \Theta^g_{0,1}$ with $t'_0 + x'_0 = \widetilde{t} = \widetilde{t}_{n_0}$ for some $n_0 \in \{1, ..., N^f_{0,1}\}$, or $(t'_0, x'_0) \in \Theta^g_{1,1} \cup \Theta^g_{1,0}$. This implies $I^h_1 \le I^g_0 + I^g_1$.

In order to compute h from f, g, we use `GetDecomposition` and `GetPartialFunctions`.

Since we can solve linear equations in constant time, we can modify the elements in one object of the object lists L_l in constant time, $l \in \{0,1,2\}$. Since there are $\mathcal{O}(N_0^h J_0^h + N_1^h J_1^h)$ objects in all lists L_l, $l \in \{0,1,2\}$, we can modify all objects in $\mathcal{O}(N_0^h J_0^h + N_1^h J_1^h)$ time. We use the function `GetDecomposition` in order to determine which objects must be changed (resp., generated or deleted) in which manner. Since we can insert (resp., remove) elements into a linked list (resp., from a linked list) in constant time, we can perform the necessary modifications of the functions `GetDecomposition`, `GetPartialFunctions`, in $\mathcal{O}(N_0^h J_0^h + N_1^h J_1^h)$ arithmetic operations. (Observe that there exists a pointer from each object of `GetDecomposition` to the objects in the object lists L_l, and a pointer from each object in the object lists L_l to the objects of `GetPartialFunctions`.) Using Lemma 5.3.5 and the above upper bounds for $N_0^h, J_0^h, N_1^h, J_1^h$, the assertion follows. □

The next lemma is a special case of the preceding results.

Lemma 5.3.14 *Let $T' \subset \mathbb{R}$ be a closed interval with $\operatorname{int}(T') \neq \emptyset$, $\underline{\theta}, \overline{\theta} \in \mathbb{R}$ with $\underline{\theta} < \overline{\theta}$ and $\Theta = T' \times [\underline{\theta}, \overline{\theta}]$. Let $\widetilde{T} \subset \mathbb{R}$ be a closed interval with $\operatorname{int}(\widetilde{T}) \neq \emptyset$ and denote $\widetilde{\Theta} = \{(t', x') \in \Theta : t' + x' \in \widetilde{T}\}$. Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ denote the orthogonal projection to the first coordinate axis and denote $T = \pi(\widetilde{\Theta})$. Let $g \in \mathcal{PL}_c^2(\Theta)$ be a linear function with $\#g \leq (0, N_{0,1}^g, 0, 2, 1, 2, 2)$, let $f \in \mathcal{PL}_c^1(\widetilde{T})$ with $\#f = (0, N_{0,1}^f, N_1^f)$ and let the function $f^* : T \to \mathbb{R}$ be defined by*

$$f^*(t) = \min_{x' \in [\underline{\theta}, \overline{\theta}]} g(t, x') + f(t + x').$$

Then $f^* \in \mathcal{PL}_c^1(T)$ with $\#f^* \leq (0, 6N_{0,1}^f + 4, 6N_{0,1}^f + 3)$ and f^* can be computed from g, f in $\mathcal{O}\big(N_{0,1}^f \log(N_{0,1}^f)\big)$ arithmetic operations.

Proof From [65, Theorem 2.2.8] it immediately follows that f^* is continuous. In a similar manner as in Lemma 5.3.12 we see that $h : \widetilde{\Theta} \to \mathbb{R}$, $h(t', x') = g(t', x') + f(t' + x')$, satisfies $h \in \mathcal{PL}_c^2(\widetilde{\Theta})$. Let $\widetilde{\Theta}_{0,0}, \widetilde{\Theta}_{0,1}, \widetilde{\Theta}_{1,0}, \widetilde{\Theta}_{1,1} \subset \widetilde{\Theta}$ be the characterization of the breakpoints of h as in (5.43), (5.44) and define

$$\Lambda : T \to \mathcal{P}(\mathbb{R}), \qquad \Lambda(t) = \pi^{-1}(t) \cap (\widetilde{\Theta}_{0,0} \cup \widetilde{\Theta}_{0,1} \cup \widetilde{\Theta}_{1,1}).$$

In a similar manner as in Lemma 5.3.12 we see that the parametric optimization can be restricted to $x' \in \Lambda(t)$ for each $t \in T$. graph(Λ) consists of $N_{0,1}^f$ line segments

$$S'_{n_1} = \{(t', x') \in \widetilde{\Theta} : t' + x' = \widetilde{t}_{n_1}\}, \quad n_1 = 1, ..., N_{0,1}^f,$$

determined by the breakpoints $\widetilde{t}_1, ..., \widetilde{t}_{N_{0,1}^f}$ of f, and the two line segments $S'_{\underline{\theta}} = \{(t', x') \in \widetilde{\Theta} : x' = \underline{\theta}\}$, $S'_{\overline{\theta}} = \{(t', x') \in \widetilde{\Theta} : x' = \overline{\theta}\}$. Moreover, there are at most $2N_{0,1}^f$ mutual intersections of the line segments $S'_{\underline{\theta}}, S'_{\overline{\theta}}, S'_1, ..., S'_{N_{0,1}^f}$, whose projections to the first coordinate axis decompose T into at most $K = 2N_{0,1}^f + 1$ open intervals $T_1, ..., T_K$, cp. Lemma 5.3.5. Since g is linear, there holds $g(t', x') = g_0 + g_{t'} t' + g_{x'} x'$ for some $g_0, g_{t'}, g_{x'} \in \mathbb{R}$. Moreover, for each $n_1 \in \{1, ..., N_{0,1}^f\}$, $(t', x') \mapsto f(t' + x') = f(\widetilde{t}_{n_1})$ is constant on S'_{n_1}, which implies that for each $n_1 \in \{1, ..., N_{0,1}^f\}$ there exists a $h_{n_1} \in \mathbb{R}$, such that

$$h(t', \widetilde{t}_{n_1} - t') = (h_{n_1} + g_0 + \widetilde{t}_{n_1} g_{x'}) + (g_{t'} - g_{x'}) t', \qquad \forall t' \in \pi(S'_{n_1}).$$

Consequently, for any $n_1, n_1' \in \{1, ..., N_{0,1}^f\}$, the functions $t \mapsto h(t, \widetilde{t}_{n_1} - t)$ and $t \mapsto h(t, \widetilde{t}_{n_1'} - t)$, defined on $\pi(S'_{n_1})$ and $\pi(S'_{n_1'})$, respectively, cannot properly intersect each other. Hence, on each T_k, $k = 1, ..., K$, it is sufficient to consider the line segments $S'_{\underline{\theta}}, S'_{\overline{\theta}}, S'_{n_1^*(k)}$ in order to compute f^*, where

$$n_1^*(k) = \operatorname*{arg\,min}_{\substack{n_1 \in \{1,...,N_{0,1}^f\}: \\ T_k \subset \pi(S'_{n_1})}} h(t, \widetilde{t}_{n_1} - t), \qquad t \in T_k.$$

The continuity of f^* and Lemma 5.3.7 yield that $\# f^*|_{\mathrm{cl}(T_k)} \leq (0, 4, 3)$. As $T = \bigcup_{k=1}^K \mathrm{cl}(T_k)$ is a closed interval and $K \leq 2N_{0,1}^f + 1$, we obtain that $\# f^* \leq (0, 6N_{0,1}^f + 4, 6N_{0,1}^f + 3)$.
In order to compute $n_1^*(k)$ for all $k \in \{1, ..., K\}$, we first determine which line segments are relevant for $k = 1$ in $\mathcal{O}(N_{0,1}^f)$ arithmetic operations. We then sort the cost functions by building a binary heap in $\mathcal{O}\big(N_{0,1}^f \log(N_{0,1}^f)\big)$ arithmetic operations. Observe that, while k is increasing, only the line segment with minimal index is removed and the line segment with index one greater than the actually maximal index is inserted. Since the insertion (resp., removal) of an element into a binary heap (resp., from a binary heap) can be implemented in logarithmic time, we can compute $n_1^*(k)$ for all $k \in \{1, ..., K\}$ in $\mathcal{O}\big(N_{0,1}^f \log(N_{0,1}^f) + K \log\log(N_{0,1}^f)\big)$ arithmetic operations. As $K \leq 2N_{0,1}^f + 1$, f^* can be computed from g, f in $\mathcal{O}\big(N_{0,1}^f \log(N_{0,1}^f)\big)$ arithmetic operations. □

The next two lemmas are concerned with the relation between piecewise linear functions and point-to-set mappings.

Lemma 5.3.15 *Let $T \subset \mathbb{R}$ be the finite union of closed intervals and points and let the point-to-set mapping $\Gamma : T \to \mathcal{P}(\mathbb{R})$ be given by $\Gamma(t) = [\underline{\Gamma}(t), \overline{\Gamma}(t)]$ with $\underline{\Gamma} \in \mathcal{PL}^1_{lsc}(T)$, $\overline{\Gamma} \in \mathcal{PL}^1_{usc}(T)$ and $\underline{\Gamma}(t) \leq \overline{\Gamma}(t)$ for all $t \in T$. Then Γ is an upper-semicontinuous point-to-set mapping and* graph(Γ) *is the finite union of compact polygons and line segments. If $\underline{\Gamma}, \overline{\Gamma} \in \mathcal{PL}^1_c(T)$, then Γ is a continuous point-to-set mapping.*

Proof First we show that Γ is an upper semicontinuous point-to-set mapping. For each compact $K \subset T$, since $\underline{\Gamma}$ is lower semicontinuous, we have $\underline{\Gamma}(K) \subset [\underline{g}, \infty)$ for some $\underline{g} \in \mathbb{R}$. Similarly, for each compact $K \subset T$, since $\overline{\Gamma}$ is upper semicontinuous, we have $\overline{\Gamma}(K) \subset (-\infty, \overline{g}]$ for some $\overline{g} \in \mathbb{R}$. Consequently $\Gamma(t) \subset [\underline{g}, \overline{g}]$ for all $t \in K$. Since $\Gamma(t)$ is closed for all $t \in T$ this implies that $\Gamma(t)$ is compact for all $t \in T$. Now, let $t \in T$ and $U_\Gamma \subset \mathbb{R}$ be an open set such that $\Gamma(t) \subset U_\Gamma$. Since $\underline{\Gamma}(t)$ is lower semicontinuous and $\overline{\Gamma}$ is upper semicontinuous we obtain the existence of an open neighborhood $U_T \subset T$ of $t \in T$ such that $\Gamma(t) = [\underline{\Gamma}(t), \overline{\Gamma}(t)] \subset U_\Gamma$ for all $t \in U_T$. The result for $\underline{\Gamma}, \overline{\Gamma} \in \mathcal{PL}^1_c(T)$ follows in a similar manner. Next we show that graph(Γ) is the finite union of compact polygons, line segments and points: Let T' be a maximal open subset of T on which both $\underline{\Gamma}$ and $\overline{\Gamma}$ are linear. The number of such sets is finite, $\mathrm{cl}\big(\mathrm{graph}(\Gamma|_{T'})\big)$ is either a closed polygon or a closed line segment, and as Γ is upper semicontinuous there also holds $\mathrm{cl}\big(\mathrm{graph}(\Gamma|_{T'})\big) \subset \mathrm{graph}(\Gamma)$. Moreover, the set of all points $t \in T$ for which there exists no open neighborhood on which both $\underline{\Gamma}$ and $\overline{\Gamma}$ are linear is finite, and for each such t, $\mathrm{graph}(\Gamma|_{\{t\}})$ is either a closed line segment or a point. □

Lemma 5.3.16 *Let $T' \subset \mathbb{R}$ be a closed interval with* $\operatorname{int}(T') \neq \emptyset$ *and let the point-to-set mapping $\Gamma : T' \to \mathcal{P}(\mathbb{R})$ be given by $\Gamma(t') = [\underline{\Gamma}(t'), \overline{\Gamma}(t')]$ with $\underline{\Gamma}, \overline{\Gamma} \in \mathcal{PL}_c^1(T')$, $\#\underline{\Gamma} \leq (0, N_{0,1}, N_1)$, $\#\overline{\Gamma} \leq (0, N_{0,1}, N_1)$ and $\underline{\Gamma}(t') \leq \overline{\Gamma}(t')$ for all $t' \in T'$. Let $\widetilde{T} \subset \mathbb{R}$ be a closed interval or a point and let $\pi : \mathbb{R}^2 \to \mathbb{R}$ denote the orthogonal projection to the first coordinate axis. Then the set $T = \pi\big(\{(t', x') \in \operatorname{graph}(\Gamma) : t' + x' \in \widetilde{T}\}\big)$ consists of at most $N_{0,1} + 1$ connected components.*

Proof Let us first assume that $\widetilde{T}$ is a point, $\widetilde{T} = \{\widetilde{t}\}$. Denote $\gamma : T' \to \mathbb{R}$, $\gamma(t') = \widetilde{t} - t'$, then $T = \pi\big(\operatorname{graph}(\Gamma) \cap \operatorname{graph}(\gamma)\big)$. Note that $\underline{\Gamma} - \gamma, \overline{\Gamma} - \gamma \in \mathcal{PL}_c^1(T')$ with $\#\underline{\Gamma} - \gamma = \#\underline{\Gamma}$ and $\#\overline{\Gamma} - \gamma = \#\overline{\Gamma}$. Lemma 5.3.6 *(ii)* and *(iii)* imply that the zero set of $(\underline{\Gamma} - \gamma)(\overline{\Gamma} - \gamma)$ contains at most $2N_{0,1} + 1$ boundary points. Each boundary point of each connected component of T is either a boundary point of T' or a boundary point of a connected component of the zero set of $(\underline{\Gamma} - \gamma)(\overline{\Gamma} - \gamma)$. Let N_b denote the number of bounded connected components of T and let $N_u \leq 2$ denote the number of unbounded connected components of T. If $T = \mathbb{R}$ then the assertion is obviously true. Henceforth, we assume that $T \neq \mathbb{R}$. Since each bounded connected component of T has 2 boundary points, each unbounded connected component of T has 1 boundary point and T' has at most $2 - N_u$ boundary points, the maximal number of connected components of T is bounded from above by the objective function value of the following linear program:

$$
\begin{aligned}
\max N_b + N_u, &\qquad \text{subject to} \\
2N_b + N_u &\leq (2N_{0,1} + 1) + (2 - N_u), \\
N_u &\leq 2, \\
N_b, N_u &\geq 0.
\end{aligned}
$$

It is easily seen that the maximal value of the objective function is $N_{0,1} + 1$.
In a similar manner we establish the assertion if $\widetilde{T}$ is a closed interval with $\operatorname{int}(\widetilde{T}) \neq \emptyset$, by considering $\operatorname{bd}(\widetilde{T})$. □

Lemma 5.3.17 *Let $T' \subset \mathbb{R}$ be a closed interval with* $\operatorname{int}(T') \neq \emptyset$ *and let the point-to-set mapping $\Gamma : T' \to \mathcal{P}(\mathbb{R})$ be given by $\Gamma(t') = [\underline{\Gamma}(t'), \overline{\Gamma}(t')]$ with $\underline{\Gamma}, \overline{\Gamma} \in \mathcal{PL}_c^1(T')$ and $\underline{\Gamma}(t') \leq \overline{\Gamma}(t')$ for all $t' \in T'$. Let $\widetilde{T} \subset \mathbb{R}$ be a closed interval or a point and let $\pi : \mathbb{R}^2 \to \mathbb{R}$ denote the orthogonal projection to the first coordinate axis. If $\underline{\Gamma}, \overline{\Gamma}$ satisfy (3.4), then the set $T = \pi\big(\{(t', x') \in \operatorname{graph}(\Gamma) : t' + x' \in \widetilde{T}\}\big)$ consists of at most one connected component.*

Proof Let $\widetilde{\Theta} = \{(t', x') \in \operatorname{graph}(\Gamma) : t' + x' \in \widetilde{T}\}$ and let $(t_1', x_1'), (t_2', x_2') \in \widetilde{\Theta}$ be arbitrary but fixed. We now show that there exists a continuous curve from (t_1', x_1') to (t_2', x_2') which is contained in $\widetilde{\Theta}$. Without loss of generality we assume that $t_1' + x_1' \leq t_2' + x_2'$ and define the curve $\gamma_1 : T' \to \mathbb{R}^2$ by $\gamma_1(s) = (t_1' + s, x_1' - s)$. Let $s_1 \in \mathbb{R}_0^+$ be maximal such that $t_1' + s_1 \leq t_2'$ and $\gamma_1(s) \geq \underline{\Gamma}(t_1' + s)$ for all $s \in [0, s_1]$. Next, we define $\gamma_2 : (s_1, \infty) \to \mathbb{R}^2$, $\gamma_2(s) = (t_1' + s, \underline{\Gamma}(t_1' + s))$, denote $s_2 = t_2' - t_1'$, define $\gamma_3 : [s_2, \infty) \to \mathbb{R}^2$, $\gamma_3(s) = (t_2', \underline{\Gamma}(t_1' + s_2) + s - s_2)$ and set $s_3 = x_2' - \underline{\Gamma}(t_1' + s_2) + s_2$. Since $t' \mapsto \overline{\Gamma}(t')$ satisfies (3.4) and $s \mapsto (t_1' + s) + (x_1' - s)$ is constant, $\gamma_1(s) \in \widetilde{\Theta}$ for all $s \in [0, s_1]$. Since $t' \mapsto \underline{\Gamma}(t')$ satisfies (3.4), $\gamma_2(s) \in \widetilde{\Theta}$ for all $s \in (s_1, s_2)$. Finally, since $\gamma_3(s_2), \gamma_3(s_3) \in \Gamma(t_2')$ and $s_2 + \gamma_3(s_2), s_3 + \gamma_3(s_3) \in \widetilde{T}$, there holds

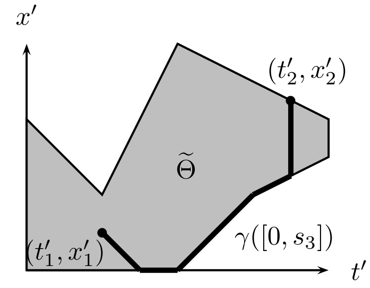


Figure 5.4.: The curve γ in (5.47).

$\gamma_3(s) \in \widetilde{\Theta}$ for all $s \in [s_2, s_3]$. Now, since $\underline{\Gamma}$ is continuous, the curve $\gamma : [0, s_3] \to \widetilde{\Theta}$,

$$\gamma(s) = \begin{cases} \gamma_1(s), & 0 \le s \le s_1 \\ \gamma_2(s), & s_1 < s < s_2 \\ \gamma_3(s), & s_2 \le s \le s_3 \end{cases} \tag{5.47}$$

is a continuous curve which connects (t'_1, x'_1) and (t'_2, x'_2), see also Figure 5.4. □

Before applying the above methodology to the time-dependent optimal path problem we prove the following recursion formulas.

Lemma 5.3.18 *(i) Let $a, b \in \mathbb{N}$ with $b \le 2 \le a$, then $a^{k+1} \ge a^k + ba^{k-1}$.*

(ii) Let $a_0, c, d, e \in \mathbb{N}$, $b \in \mathbb{R}_0^+$ and $a_n \in \mathbb{N}$ be defined by

$$a_n = c\Big(a_{n-1} + b + ed^{n-1}\Big)^2, \qquad \forall n \in \mathbb{N}.$$

Then

$$a_n = \mathcal{O}\Big(c^{2^n-1}(a_0 + b + ed)^{2^n}\Big), \quad \forall n \in \mathbb{N}.$$

(iii) Let $a_0, c_0, c_1, c \in \mathbb{N}$ and $a_n \in \mathbb{N}$ be defined by

$$a_n = c_1(a_{n-1} + c) + c_0, \qquad \forall n \in \mathbb{N}.$$

Then

$$a_n = c_1^n(a_0 + c) + \sum_{k=1}^{n-1} c_1^k(c + c_0) + c_0, \quad \forall n \in \mathbb{N}.$$

Proof *(i)* The assertion holds if and only if $(a - 1/2)^2 \ge b + 1/4$. This is true if $b \le 2 \le a$.
(ii) We first compute

$$a_1 = c(a_0 + b + ed^0)^2 = \mathcal{O}\Big(c^{2^1-1}(a_0 + b + ed)^{2^1}\Big).$$

Hence, the assertion is true for $n = 1$. Now assume that the assertion holds for a_n. Then

$$\begin{aligned} a_{n+1} &= \mathcal{O}\Big(c\Big(c^{2^n-1}(a_0+b+ed)^{2^n}+b+ed^n\Big)^2\Big) \\ &= \mathcal{O}\Big(c^{2^{n+1}-1}(a_0+b+ed)^{2^{n+1}}+c(b+ed^n)^2+2c^{2^n}(a_0+b+ed)^{2^n}(b+ed^n)\Big) \\ &= \mathcal{O}\Big(c^{2^{n+1}-1}(a_0+b+ed)^{2^{n+1}}\Big), \end{aligned}$$

since $2^n \geq 2n$ for all $n \in \mathbb{N}$.
(iii) Obviously, the assertion is correct for $n = 1$. Suppose that the assertion holds for $k = 1, ..., n$. Then

$$\begin{aligned} a_{n+1} &= c_1\Big(c_1^n(a_0+c)+\sum_{k=1}^{n-1}c_1^k(c+c_0)+c_0+c\Big)+c_0 \\ &= c_1^{n+1}(a_0+c)+\sum_{k=1}^{n-1}c_1^{k+1}(c+c_0)+c_1c_0+c_1c+c_0 \\ &= c_1^{n+1}(a_0+c)+\sum_{k=1}^{n}c_1^k(c+c_0)+c_0. \end{aligned}$$

□

5.3.2. Complexity Analysis of the Piecewise Linear Optimal Path Problem

After having introduced an appropriate concept of piecewise linear functions and an appropriate concept of the complexity of piecewise linear functions, we have proved some basic results on the summation, concatenation, pointwise minimum and parametric minimization of piecewise linear functions. We are now ready to apply the methodology of Subsection 5.3.1 to the time-dependent optimal path problem.
At this, we assume that the partial functions $\tau_e : \mathbb{R} \to \mathbb{R}_0^+$, $\tau_e(t) = \tau(e,t)$, and $\beta_e : \mathbb{R} \to \mathbb{R}$, $\beta_e(t) = \beta(e,t)$ are in $\mathcal{PL}_c^1(\mathbb{R})$ for each $e \in E$ with an identical decomposition of $\mathbb{R}$ according to the discussion preliminary to Definition 5.3.3. Moreover, we assume that $T(v)$ is a closed interval for all $v \in V$. We further assume that $\Delta T(v,t) = [\,\underline{\Delta T}(v,t), \overline{\Delta T}(v,t)]$ for all $(v,t) \in X$, at which the partial functions $t \mapsto \underline{\Delta T}_v(t) = \underline{\Delta T}(v,t)$ and $t \mapsto \overline{\Delta T}_v(t) = \overline{\Delta T}(v,t)$ are in $\mathcal{PL}_c^1\big(T(v)\big)$ for all $v \in V$. For each $v \in V$, we denote the graph of the partial point-to-set mapping $t \mapsto \Delta T(v,t)$ by Θ_v. Finally, we assume that, for each $v \in V$, the partial functions $\delta_v : \Theta_v \to \mathbb{R}$, $\delta_v(t,\Delta t) = \delta(v,t,\Delta t)$, are in $\mathcal{PL}_c^2(\Theta_v)$.
Since τ, β are derived from historical traffic data [41], [119], see also Section 2.3, which is usually stored according to a certain fixed discretization in the digital map, the above assumptions on $\{\tau_e\}_{e\in E}, \{\beta_e\}_{e\in E}$ are no restriction in most practical applications, see, e.g., [171], [75] and the references therein. Since waiting time constraints in the road network are usually given by earliest and latest departure times with respect to a given arrival time, neither the assumption on the structure of ΔT is a restriction in this particular application. Furthermore, the results which we prove below can easily be generalized to the general case by using Lemma 5.3.6, Lemma 5.3.15, Lemma 5.3.12 and Lemma 5.3.13.

In the remainder of this section, we always assume the above continuous and piecewise linear structure of the partial network functions. We also assume that there exists a $C \in \mathbb{N}$, $C \geq 2$, such that

$$\#\tau_e \leq (0, C-1, C), \qquad \#\beta_e \leq (0, C-1, C), \qquad \forall e \in E, \tag{5.48}$$

$$\#\underline{\Delta T}_v \leq (1, C, C), \qquad \#\overline{\Delta T}_v \leq (1, C, C), \qquad \forall v \in V, \tag{5.49}$$

$$\#\delta_v \leq (1, C, C, C, C, C, C), \qquad \forall v \in V. \tag{5.50}$$

Note that, if $C = 1$, then the partial network functions $\{\tau_e\}_{e \in E}$, $\{\beta_e\}_{e \in E}$ would be constant. The above assumptions on the (finite) complexity are no restriction in the case of a compact state space X. However, if X is unbounded, they imply that there exist $\underline{t}, \overline{t} \in \mathbb{R}$ such that all partial functions are linear for $t \in (-\infty, \underline{t}\,]$ and $t \in [\,\overline{t}, \infty)$. Such an assumption has also been imposed in [47] in order to prove that the time-dependent optimal path problem is solvable in finite time. We will show in Corollary 5.4.3 that, under weak assumptions, it is sufficient to consider only a compact time interval in order to compute the optimal value function in a periodical time-dependent network.
We now prove that the time-dependent optimal path problem with piecewise linear problem data is solvable in finite time. We also derive the order of the number of arithmetic operations which are necessary in order to compute the partial function $t_0 \mapsto b^*(v_0, t_0)$. The results in Theorem 5.3.21 imply that both the manner in which the waiting times are constrained, the specific form of the waiting cost function and the FIFO-property of the travel time function have a crucial impact on the complexity of the computation of the optimal value function.

Lemma 5.3.19 *Let $v_0, v' \in V$ be given and suppose that Assumption 4.2.3 holds. Let $(e_1, ..., e_n) \in E^n$ be a connected edge sequence with $\alpha(e_1) = v_0$ and $\omega(e_n) = v'$, let*

$$\widetilde{T}_0 = \big\{t_0 \in T(v_0) : \exists u = \big((\Delta t_k, e_k)\big)_{k=1,\dots,n} \in U(v_0, t_0)\big\},$$

$N_{\mathrm{bd}} = \big|\mathrm{bd}\big(T(v')\big)\big|$ and let $\widetilde{b}_0^* : \widetilde{T}_0 \mapsto \mathbb{R}$,

$$\widetilde{b}_0^*(t_0) = \inf \Big\{\mathcal{B}\big((v_0, t_0), u\big) : u = \big((\Delta t_k, e_k)\big)_{k=1,\dots,n} \in U(v_0, t_0)\Big\}$$

denote the optimal cost function along the edge sequence $(e_1, ..., e_n)$. Then $\widetilde{b}_0^* \in \mathcal{PL}_c^1(\widetilde{T}_0)$ and, denoting $\#\widetilde{b}_0^* = (N_{0,0}^0, N_{0,1}^0, N_1^0)$, we obtain:

(i) There holds $N_{0,0}^0 \leq C^{2n}$, and $N_{0,1}^0, N_1^0$ are of order

$$\mathcal{O}\bigg(C^{3 \cdot 2^n - 3}\Big(N_{\mathrm{bd}} + 2 + C^2\Big)^{2^n}\bigg).$$

Moreover, $\widetilde{b}_0^*$ can be computed in

$$\mathcal{O}\bigg(C^{9 \cdot 2^{n-1} - 5}\Big(N_{\mathrm{bd}} + 2 + C^2\Big)^{3 \cdot 2^{n-1}}\bigg)$$

arithmetic operations.

(ii) *If* $X = V \times \mathbb{R}$*, then* $N^0_{0,0} = 0$*, and* $N^0_{0,1}, N^0_1$ *are of order*

$$\mathcal{O}\Big(C^{3\cdot 2^n - 3} 2^{2^n}\Big).$$

Moreover, $\widetilde{b}^*_0$ *can be computed in*

$$\mathcal{O}\Big(C^{9\cdot 2^{n-1} - 5} 2^{3\cdot 2^{n-1}}\Big)$$

arithmetic operations.

(iii) *If* G *is a FIFO-network and the functions* $\underline{\Delta T}_v, \overline{\Delta T}_v$ *satisfy (3.4) for each* $v \in V$*, then* $N^0_{0,0} \leq 1$*, and* $N^0_{0,1}, N^0_1$ *are of order*

$$\mathcal{O}\Big(C^{2^n - 1}\Big(N_{\text{bd}} + \frac{7}{4}C\Big)^{2^n}\Big).$$

Moreover, $\widetilde{b}^*_0$ *can be computed in*

$$\mathcal{O}\Big(C^{3\cdot 2^{n-1} - 2}\Big(N_{\text{bd}} + \frac{7}{4}C\Big)^{3\cdot 2^{n-1}}\Big)$$

arithmetic operations.

(iv) *If* G *is a FIFO-network, the functions* $\underline{\Delta T}_v, \overline{\Delta T}_v$ *are constant for each* $v \in V$ *and* δ_v *is linear for each* $v \in V$*, then* $N^0_{0,0} \leq 1$*, and* $N^0_{0,1}, N^0_1$ *are of order*

$$\mathcal{O}\big((6C)^n (N_{\text{bd}} + C)\big).$$

Moreover, $\widetilde{b}^*_0$ *can be computed in*

$$\mathcal{O}\big(n(6C)^{n-1}(N_{\text{bd}} + C)\log(N_{\text{bd}} + C)\big)$$

arithmetic operations.

(v) *If* G *is a FIFO-network,* $\Delta T(v,t) = \{0\}$ *and* $\delta(v,t,0) = 0$ *for each* $(v,t) \in X$*, then* $N^0_{0,0} \leq 1$*, and* $N^0_{0,1}, N^0_1$ *are of order*

$$\mathcal{O}\big(Cn + N_{\text{bd}}\big).$$

Moreover, $\widetilde{b}^*_0$ *can be computed in*

$$\mathcal{O}\big((Cn^2 + nN_{\text{bd}})\log(C)\big)$$

arithmetic operations.

Proof Denote $v_k = \omega(e_k)$, $k = 1, ..., n$, and $\widetilde{\Delta T}_k, \widetilde{T}_k$ as in (4.3), (4.4), $k = 0, ..., n-1$. Since the result of Lemma 5.3.19 is trivial if $\widetilde{T}_0 = \emptyset$, we assume in the following that $\widetilde{T}_k \neq \emptyset$ for all $k = 0, ..., n$. Next, we denote $\theta_k, \beta_k : \mathbb{R} \to \mathbb{R}$, $\theta_k(t) = t + \tau(e_{k+1}, t)$, $\beta_k(t) = \beta(e_{k+1}, t)$,

$\delta_k : \Theta_{v_k} \to \mathbb{R}$, $\delta_k(t,\Delta t) = \delta(v_k, t, \Delta t)$. As $t \mapsto \underline{\Delta T}(v_k, t)$ and $t \mapsto \overline{\Delta T}(v_k, t)$ are $\mathcal{PL}_c^1\big(T(v_k)\big)$-functions, Lemma 5.3.15 yields that $\widetilde{\Delta T}_k$ is a continuous point-to-set mapping and that $\text{graph}(\widetilde{\Delta T}_k)$ is the finite union of closed polygons, closed line segments and points for each $k = 0, ..., n-1$. This also implies that $\widetilde{T}_k$ is the finite union of closed intervals and points for each $k = 0, ..., n-1$.
Next, we denote $\widetilde{b}_n^* : \widetilde{T}_n \to \mathbb{R}$, $\widetilde{b}_n^*(t) = 0$. Since $\widetilde{T}_n = T(v_n)$ is a closed interval, we either have $\#\widetilde{b}_n^* = (1,0,0)$ or $\#\widetilde{b}_n^* = (0, N_{\text{bd}}, 1)$. Moreover, for $k = 0, ..., n-1$, we denote $\widetilde{b}_k^* : \widetilde{T}_k \to \mathbb{R}$,

$$\widetilde{b}_k^*(t) = \inf_{\Delta t \in \widetilde{\Delta T}_k(t)} b_k(t, \Delta t),$$

where $b_k : \text{graph}(\widetilde{\Delta T}_k) \to \mathbb{R}$,

$$b_k(t,\Delta t) = \beta_k(t+\Delta t) + \widetilde{b}_{k+1}^*\big(\theta_k(t+\Delta t)\big) + \delta_k(t,\Delta t).$$

Furthermore, we define the auxiliary functions $h_k : \theta_k^{-1}(\widetilde{T}_{k+1}) \to \mathbb{R}$, $h_k(t) = \beta_k(t) + \widetilde{b}_{k+1}^*\big(\theta_k(t)\big)$, $k = 0, ..., n-1$. By backwards induction, we immediately obtain from [65, Theorem 2.2.8], Lemma 5.3.13 and Lemma 5.3.12 that $\widetilde{b}_k^* \in \mathcal{PL}_c^1(\widetilde{T}_k)$ with $\#\widetilde{b}_k^* = (N_{0,0}^k, N_{0,1}^k, N_1^k)$ for some $N_{0,0}^k, N_{0,1}^k, N_1^k \in \mathbb{N}_0$, and that h_k, b_k are continuous. We denote $N_0^k = N_{0,0}^k + N_{0,1}^k$ for $k = 0, ..., n$, and we denote the number of connected components of $\widetilde{T}_k$ by $\widetilde{N}_k$.
Let us consider the assertion *(i)*:
Since $\#\theta_k = \#\tau_{e_{k+1}}$, using Lemma 5.3.6 *(iii)*, we deduce that $\theta_k^{-1}(\widetilde{T}_{k+1})$ consists of at most $C\widetilde{N}_{k+1}$ connected components. Now, Lemma 5.3.6 *(iv)* and *(v)* yield that $h_k \in \mathcal{PL}_c^1\big(\theta_k^{-1}(\widetilde{T}_k)\big)$ with $\#h_k = (N_{0,0}^{h_k}, N_{0,1}^{h_k}, N_1^{h_k})$, satisfying $N_{0,1}^{h_k} \leq CN_{0,1}^{k+1} + C$. As h is continuous and $\theta_k^{-1}(\widetilde{T}_{k+1})$ consists of at most $C\widetilde{N}_{k+1}$ connected components we also have $N_{0,0}^{h_k} \leq C\widetilde{N}_{k+1}$. Moreover, as in the proof of Lemma 5.3.6 *(iv)*, we see that $N_{0,0}^{h_k} + N_{0,1}^{h_k} \leq CN_{0,1}^{k+1} + C$. Using Lemma 5.3.13, we obtain that $b_k \in \mathcal{PL}_c^2\big(\text{graph}(\widetilde{\Delta T}_k)\big)$ with $\#b_k = (N_{0,0}^{b_k}, N_{0,1}^{b_k}, J_0^{b_k}, I_0^{b_k}, N_1^{b_k}, J_1^{b_k}, I_1^{b_k})$, satisfying

$$\begin{aligned} N_{0,1}^{b_k} &\leq C + C(CN_{0,1}^{k+1} + C), \\ N_1^{b_k} &\leq C + C(CN_0^{k+1} + C), \\ J_1^{b_k} &\leq 2C + CN_0^{k+1}, \end{aligned}$$

Now, Lemma 5.3.12 yields

$$N_{0,1}^k \leq C + C(CN_{0,1}^{k+1} + C) + \Big[C + C(CN_0^{k+1} + C)\Big] \cdot \Big[2C + CN_0^{k+1} - 1\Big]. \tag{5.51}$$

Since $\theta_k^{-1}(\widetilde{T}_{k+1})$ consists of at most $C\widetilde{N}_{k+1}$ connected components, Lemma 5.3.16 yields $\widetilde{N}_k \leq C^2\widetilde{N}_{k+1}$. By induction, since $\widetilde{N}_n = 1$, we obtain that $\widetilde{N}_k \leq (C^2)^{n-k}$. Now, the continuity of $\widetilde{b}_k^*$ immediately yields $N_{0,0}^k \leq \widetilde{N}_k = \mathcal{O}\big((C^2)^{n-k}\big)$. According to Lemma 5.3.5 it is sufficient to consider the recursion of the $N_{0,1}^k$, $k = 0, ..., n$, in order to establish the assertion *(i)*. From (5.51), using $N_0^{k+1} \leq N_{0,1}^{k+1} + \widetilde{N}_{k+1}$ and $C \geq 2$, we obtain

$$N_{0,1}^k \leq C^3\Big(N_{0,1}^{k+1} + 2 + C^{2n-2k-2}\Big)^2.$$

From Lemma 5.3.18 *(ii)* we obtain

$$N_{0,1}^k = \mathcal{O}\bigg(C^{3\cdot 2^{n-k}-3}\Big(N_{\text{bd}} + 2 + C^2\Big)^{2^{n-k}}\bigg). \tag{5.52}$$

From the estimates for the necessary numbers of arithmetic operations in Lemma 5.3.6, Lemma 5.3.12 and Lemma 5.3.13 it is easily seen that the dominating order is given by $\mathcal{O}\big(N_1^{b_k}(J_1^{b_k})^2\big)$. Hence, $\widetilde{b}_{n-1}^*$ can be computed from $\widetilde{b}_n^*, \beta_{n-1}, \tau_{n-1}, \delta_{n-1}$ in $\mathcal{O}(C^4)$ arithmetic operations, since $N_{\text{bd}} = \mathcal{O}(1)$. Using (5.52), we further obtain that $\widetilde{b}_k^*$ can be computed from $\widetilde{b}_{k+1}^*, \beta_k, \tau_k, \delta_k$ in

$$\mathcal{O}\bigg(C^{9\cdot 2^{n-k-1}-5}\Big(N_{\text{bd}} + 2 + C^2\Big)^{3\cdot 2^{n-k-1}}\bigg)$$

arithmetic operations for $k = 0, ..., n-2$. Using Lemma 5.3.18 *(i)*, we establish that

$$\begin{aligned}\mathcal{O}(C^4) + \sum_{k=0}^{n-2} \mathcal{O}\bigg(C^{9\cdot 2^{n-k-1}-5}\Big(N_{\text{bd}} + 2 + C^2\Big)^{3\cdot 2^{n-k-1}}\bigg)\\ = \mathcal{O}\bigg(C^{9\cdot 2^{n-1}-5}\Big(N_{\text{bd}} + 2 + C^2\Big)^{3\cdot 2^{n-1}}\bigg).\end{aligned}$$

Let us now consider the assertion *(ii)*:
From Lemma 3.5.2 it follows that $\widetilde{T}_k = \mathbb{R}$ for all $k = 0, ..., n$. As $h_k, b_k, \widetilde{b}_k^*$ are continuous, there hold $N_{0,0}^{h_k} = N_{0,0}^{b_k} = J_0^{b_k} = N_{0,0}^k = 0$. Using Lemma 5.3.13, we obtain that

$$\begin{aligned}N_{0,1}^{b_k} &\leq C + C(CN_{0,1}^{k+1} + C),\\ N_1^{b_k} &\leq C + C(CN_{0,1}^{k+1} + C),\\ J_1^{b_k} &\leq 2C + CN_{0,1}^{k+1},\end{aligned}$$

Now, Lemma 5.3.12 yields

$$N_{0,1}^k \leq C + C(CN_{0,1}^{k+1} + C) + \Big[C + C(CN_{0,1}^{k+1} + C)\Big] \cdot \Big[2C + CN_{0,1}^{k+1} - 1\Big]. \tag{5.53}$$

According to Lemma 5.3.5 it is sufficient to consider the recursion of the $N_{0,1}^k$, $k = 0, ..., n$, in order to establish the assertion *(ii)*. Since $C \geq 2$ we obtain from (5.53) that

$$N_{0,1}^k \leq C^3\Big(N_{0,1}^{k+1} + 2\Big)^2.$$

Using Lemma 5.3.18 *(ii)* we establish, since $N_{\text{bd}} = 0$,

$$N_{0,1}^k = \mathcal{O}\bigg(C^{3\cdot 2^{n-k}-3} 2^{2^{n-k}}\bigg). \tag{5.54}$$

From the estimates for the necessary numbers of arithmetic operations in Lemma 5.3.6, Lemma 5.3.12 and Lemma 5.3.13 it is easily seen that the dominating order is given by $\mathcal{O}\big(N_1^{b_k}(J_1^{b_k})^2\big)$. Hence, $\widetilde{b}_{n-1}^*$ can be computed from $\widetilde{b}_n^*, \beta_{n-1}, \tau_{n-1}, \delta_{n-1}$ in $\mathcal{O}(C^4)$ arithmetic operations. Using (5.54), we further obtain that $\widetilde{b}_k^*$ can be computed from $\widetilde{b}_{k+1}^*, \beta_k, \tau_k, \delta_k$ in

$$\mathcal{O}\Big(C^{9\cdot 2^{n-k-1}-5}2^{3\cdot 2^{n-k-1}}\Big)$$

arithmetic operations for $k = 0, ..., n-2$. Using Lemma 5.3.18 *(i)*, we establish that

$$\mathcal{O}(C^4) + \sum_{k=0}^{n-2} \mathcal{O}\Big(C^{9\cdot 2^{n-k-1}-5}2^{3\cdot 2^{n-k-1}}\Big) = \mathcal{O}\Big(C^{9\cdot 2^{n-1}-5}2^{3\cdot 2^{n-1}}\Big).$$

Let us now assume that G is a FIFO-network and that for each $v \in V$ the partial functions $t \mapsto \underline{\Delta T}(v,t)$, $t \mapsto \overline{\Delta T}(v,t)$ satisfy (3.4). (Note, that this assumption holds in the cases *(iii)*, *(iv)*, *(v)* of Lemma 5.3.19.) Lemma 5.3.6 *(iii)* yields that $\theta_k^{-1}(\widetilde{T}_{k+1})$ consists of at most as many connected components as $\widetilde{T}_{k+1}$. Using Lemma 5.3.17 and an inductive argument, we obtain that $\widetilde{N}_k = 1$, for all $k = 0, ..., n$. Now, Lemma 5.3.6 *(iv)* and *(v)* yield that $h_k \in \mathcal{PL}_c^1\big(\theta_k^{-1}(\widetilde{T}_k)\big)$ with $\#h_k = (N_{0,0}^{h_k}, N_{0,1}^{h_k}, N_1^{h_k})$, satisfying $N_{0,1}^{h_k} \leq C + N_{0,1}^{k+1}$. As h is continuous and $\theta_k^{-1}(\widetilde{T}_{k+1})$ consists of at most 1 connected component we also have $N_{0,0}^{h_k} \leq 1$ and $N_{0,0}^{h_k} + N_{0,1}^{h_k} \leq C + N_{0,1}^{k+1}$. Using Lemma 5.3.13, we obtain that

$$\begin{aligned} N_{0,1}^{b_k} &\leq C + C(C + N_{0,1}^{k+1}), \\ N_1^{b_k} &\leq C + C(C + N_{0,1}^{k+1}), \\ J_1^{b_k} &\leq 2C + N_{0,1}^{k+1}, \end{aligned}$$

Let us now consider the assertion *(iii)*:
Lemma 5.3.12 yields

$$N_{0,1}^k \leq C + C(C + N_{0,1}^{k+1}) + \Big[C + C(C + N_{0,1}^{k+1})\Big] \cdot \Big[2C + N_{0,1}^{k+1} - 1\Big]. \tag{5.55}$$

According to Lemma 5.3.5 it is sufficient to consider the recursion of the $N_{0,1}^k$, $k = 0, ..., n$, in order to establish the assertion *(ii)*. Since $C \geq 2$ we obtain from (5.55) that

$$N_{0,1}^k \leq C\Big(N_{0,1}^{k+1} + \frac{7}{4}C\Big)^2.$$

Using Lemma 5.3.18 *(ii)* we establish

$$N_{0,1}^k = \mathcal{O}\bigg(C^{2^{n-k}-1}\Big(N_{\mathrm{bd}} + \frac{7}{4}C\Big)^{2^{n-k}}\bigg). \tag{5.56}$$

From the estimates for the necessary numbers of arithmetic operations in Lemma 5.3.6, Lemma 5.3.12 and Lemma 5.3.13 it is easily seen that the dominating order is given by $\mathcal{O}\big(N_1^{b_k}(J_1^{b_k})^2\big)$. Hence, $\widetilde{b}_{n-1}^*$ can be computed from $\widetilde{b}_n^*, \beta_{n-1}, \tau_{n-1}, \delta_{n-1}$ in $\mathcal{O}(C^4)$ arithmetic operations, since $N_{\mathrm{bd}} = \mathcal{O}(1)$. Using (5.56), we further obtain that $\widetilde{b}_k^*$ can be computed from $\widetilde{b}^*_{k+1}, \beta_k, \tau_k, \delta_k$ in

$$\mathcal{O}\Big(C^{3\cdot 2^{n-k-1}-2}\Big(N_{\mathrm{bd}}+\frac{7}{4}C\Big)^{3\cdot 2^{n-k-1}}\Big)$$

arithmetic operations for $k=0,...,n-2$. Using Lemma 5.3.18 *(i)*, we establish that

$$\mathcal{O}(C^4)+\sum_{k=0}^{n-1}\mathcal{O}\Big(C^{3\cdot 2^{n-k-1}-2}\Big(N_{\mathrm{bd}}+\frac{7}{4}C\Big)^{3\cdot 2^{n-k-1}}\Big)=\mathcal{O}\Big(C^{3\cdot 2^{n-1}-2}\Big(N_{\mathrm{bd}}+\frac{7}{4}C\Big)^{3\cdot 2^{n-1}}\Big).$$

Let us now consider the assertion *(iv)*:
Lemma 5.3.14 yields

$$N^k_{0,1}\leq 6C(N^{k+1}_{0,1}+C)+6C+4.$$

Using Lemma 5.3.18 *(iii)* we establish

$$N^{n-k}_{0,1}=\mathcal{O}\big((6C)^{n-k}(N_{\mathrm{bd}}+C)\big). \tag{5.57}$$

From the estimates for the necessary numbers of arithmetic operations in Lemma 5.3.6, and Lemma 5.3.14 it is easily seen that the dominating order is given by $\mathcal{O}\big(N^{h_k}_{0,1}\log(N^{h_k}_{0,1})\big)$. Hence, $\widetilde{b}^*_{n-1}$ can be computed from $\widetilde{b}^*_n, \beta_{n-1}, \tau_{n-1}, \delta_{n-1}$ in $\mathcal{O}\big((C+N_{\mathrm{bd}})\log(C+N_{\mathrm{bd}})\big)$. Using (5.57), we further obtain that $\widetilde{b}^*_k$ can be computed from $\widetilde{b}^*_{k+1}, \beta_k, \tau_k, \delta_k$ in

$$\mathcal{O}\big((n-k)(6C)^{n-k-1}(N_{\mathrm{bd}}+C)\log(N_{\mathrm{bd}}+C)\big)$$

arithmetic operations for $k=0,...,n-2$. Integrating $\sum_{k=0}^n kq^{k-1}$ with respect to q and using the formula for the geometric series [68, p.8], we establish that

$$\begin{aligned}\mathcal{O}\big((C+N_{\mathrm{bd}})\log(C+N_{\mathrm{bd}})\big)+(N_{\mathrm{bd}}+C)\log(N_{\mathrm{bd}}+C)\sum_{k=0}^{n-2}\mathcal{O}\big((n-k)(6C)^{n-k-1}\big)\\ =\mathcal{O}\big(n(6C)^{n-1}(N_{\mathrm{bd}}+C)\log(N_{\mathrm{bd}}+C)\big).\end{aligned}$$

Finally, we consider the assertion *(v)*:
In this case we obtain $\widetilde{b}^*_k\equiv h_k$ for each $k=0,...,n$. A simple inductive argument yields

$$N^k_{0,1}\leq N_{\mathrm{bd}}+(n-k)C=\mathcal{O}\big(C(n-k)+N_{\mathrm{bd}}\big). \tag{5.58}$$

From the estimates for the necessary numbers of arithmetic operations in Lemma 5.3.6 *(iv)*, *(v)*, we obtain that $\widetilde{b}^*_k$ can be computed from $\widetilde{b}^*_{k+1}, \tau, \beta$ in $\mathcal{O}\big((C(n-k)+N_{\mathrm{bd}})\log(C)\big)$ arithmetic operations. Hence, $\widetilde{b}^*_0$ can be computed in

$$\sum_{k=0}^{n-1}\mathcal{O}\big((C(n-k)+N_{\mathrm{bd}})\log(C)\big)=\mathcal{O}\big((Cn^2+nN_{\mathrm{bd}})\log(C)\big)$$

arithmetic operations. □

Remark 5.3.20 *The estimates of the complexity of the optimal cost function in Lemma 5.3.19 are not optimal. Better estimates can be proved by explicitly solving the recursions* *in the proof of Lemma 5.3.19 for the first few steps and by then deriving recursion formulas with smaller constants. However, since this approach yields the same qualitative (quadratic, exponential, double-exponential) results, we have restricted ourselves to the estimates in Lemma 5.3.19.*

Theorem 5.3.21 *Let $v_0, v' \in V$ be given, suppose that Assumption 4.2.3 holds and denote $N_{\mathrm{bd}} = \big|\mathrm{bd}\big(T(v')\big)\big|$. Suppose further that, for each $t_0 \in T_R(v_0)$, there exists an optimal control sequence $u^*(t_0) \in U(v_0, t_0)$ of topological length $n(t_0) = |u^*(t_0)|$, such that*

$$\sup_{t_0 \in T_R(v_0)} n(t_0) \leq N, \tag{5.59}$$

$$\Bigg| \bigcup_{t_0 \in T_R(v_0)} \Big\{ (e_1, ..., e_{n(t_0)}) \in E^{n(t_0)} : u^*(t_0) = \big((\Delta t_k, e_k)\big)_{k=1,\dots,n(t_0)} \Big\} \Bigg| \leq M, \tag{5.60}$$

for some $N, M \in \mathbb{N}$. Then the function $b_0^* : T_R(v_0) \mapsto \mathbb{R}$, $b_0^*(t_0) = b^*(v_0, t_0)$, satisfies $b_0^* \in \mathcal{PL}_{lsc}\big(T_R(v_0)\big)$, and denoting $\#b_0^* = (N_{0,0}, N_{0,1}, N_1)$, there hold:

(i) *$N_{0,0} = \mathcal{O}(MC^{2n})$ and $N_{0,1}, N_1$ are of order*

$$\mathcal{O}\bigg(M^2 C^{3\cdot 2^N - 3}\Big(N_{\mathrm{bd}} + 2 + C^2\Big)^{2^N}\bigg).$$

Moreover, b_0^* can be computed in

$$\mathcal{O}\bigg(MC^{9\cdot 2^{N-1}-5}\Big(N_{\mathrm{bd}} + 2 + C^2\Big)^{3\cdot 2^{N-1}} + M^3 C^{3\cdot 2^N - 3}\Big(N_{\mathrm{bd}} + 2 + C^2\Big)^{2^N}\bigg)$$

arithmetic operations.

(ii) *If $X = V \times \mathbb{R}$, then $b_0^* \in \mathcal{PL}_c\big(T_R(v_0)\big)$, $N_{0,0} = 0$, and $N_{0,1}, N_1$ are of order*

$$\mathcal{O}\bigg(M^2 C^{3\cdot 2^N - 3} 2^{2^N}\bigg).$$

Moreover, $\widetilde{b}_0^*$ can be computed in

$$\mathcal{O}\bigg(MC^{9\cdot 2^{N-1}-5} 2^{3\cdot 2^{N-1}} + M^3 C^{3\cdot 2^N - 3} 2^{2^N}\bigg)$$

arithmetic operations.

(iii) *If G is a FIFO-network and the functions $\underline{\Delta T}_v, \overline{\Delta T}_v$ satisfy (3.4) for each $v \in V$, then $N_{0,0} = \mathcal{O}(M)$ and $N_{0,1}, N_1$ are of order*

$$\mathcal{O}\bigg(M^2 C^{2^N - 1}\Big(N_{\mathrm{bd}} + \frac{7}{4}C\Big)^{2^N}\bigg).$$

Moreover, b_0^* can be computed in

$$\mathcal{O}\bigg(MC^{3\cdot 2^{N-1}-2}\Big(N_{\mathrm{bd}} + \frac{7}{4}C\Big)^{3\cdot 2^{N-1}} + M^3 C^{2^N - 1}\Big(N_{\mathrm{bd}} + \frac{7}{4}C\Big)^{2^N}\bigg)$$

arithmetic operations.

(iv) *If G is a FIFO-network, the functions $\underline{\Delta T}_v, \overline{\Delta T}_v$ are constant for each $v \in V$ and δ_v is linear for each $v \in V$, then $N_{0,0} = \mathcal{O}(M)$ and $N_{0,1}, N_1$ are of order*

$$\mathcal{O}\big(M^2(6C)^N(N_{\text{bd}} + C)\big).$$

Moreover, b_0^* can be computed in

$$\mathcal{O}\big(MN(6C)^{N-1}(N_{\text{bd}} + C)\log(N_{\text{bd}} + C) + M^3(6C)^N(N_{\text{bd}} + C)\big)$$

arithmetic operations.

(v) *If G is a FIFO-network, $\Delta T(v,t) = \{0\}$ and $\delta(v,t,0) = 0$ for each $(v,t) \in X$, then $N_{0,0} = \mathcal{O}(M)$ and $N_{0,1}, N_1$ are of order*

$$\mathcal{O}\big(M^2(CN + N_{\text{bd}})\big).$$

Moreover, b_0^* can be computed in

$$\mathcal{O}\big(M(CN^2 + NN_{\text{bd}})\log(C) + M^3(CN + N_{\text{bd}})\big)$$

arithmetic operations.

Proof We separately consider all connected edge sequences $(e_1, ..., e_n) \in E^n$, $n \in \mathbb{N}$, with $\alpha(e_1) = v_0$, $\omega(e_n) = v'$. The complexity of the optimal cost function along each such edge sequence is given by Lemma 5.3.19. b_0^* is the pointwise minimum of all such restricted optimal cost functions. The isolated points in the decomposition of $T_R(v_0)$ with respect to b_0^* are contained in the union of the isolated points of the restricted optimal cost functions along the edge sequence. This yields the result for $N_{0,0}$. On each maximal open interval on which all restricted optimal value functions are linear, the maximal number of breakpoints of b_0^* is M according to Lemma 5.3.7. From this observation, we obtain the asserted order of $N_{0,1}, N_1$. Note that, if $X = V \times \mathbb{R}$, then $N_{\text{bd}} = 0$ since $T(v') = \mathbb{R}$ contains no boundary points. Moreover, since Lemma 3.5.2 implies that each node v is reachable at any time along any edge sequence from v_0 via v to v', the continuity of the optimal value functions along the edge sequences from v_0 to v' yields $N_{0,0} = 0$.
In order to compute b_0^* for each isolated point of $\big(T_R(v_0), b_0^*\big)$, we must compute the minimum of at most M numbers (i.e., we need $\mathcal{O}(M)$ arithmetic operations). Moreover, in order to compute b_0^* on each maximal open interval on which all restricted optimal cost functions are linear, we need $\mathcal{O}(M^2)$ arithmetic operations according to Lemma 5.3.7. Summing this to the number of arithmetic operations which are necessary in order to compute the M restricted optimal cost functions, we obtain the desired result for the necessary number of arithmetic operations. □

Remark 5.3.22 *Theorem 5.3.21 shows that the complexity of the time-dependent optimal path problem can be determined by the complexity of the network functions in the considered time interval. Decreasing the number of breakpoints of τ, β, and using linear waiting cost functions δ lead to a significant decrease in the complexity of the time-dependent optimal path problem. Both efficient and optimal methods are known in the literature, which yield a* *piecewise linear approximation of the original historical traffic data within a given accuracy or by using a given number of linear segments, see, e.g., [171], [75] and the references therein.*

Remark 5.3.23 *The complexity bound in Theorem 5.3.21 (iv) is only pseudo-polynomial. Recall that we have established an upper bound on the maximal topological length N of optimal paths in Lemma 5.1.9. However, as can be seen from Theorem 7.3.3, the number M of edge sequences from v_0 to v' which have to be considered when computing the optimal value function is not polynomial in the network size.*

Remark 5.3.24 *The estimates of the number of arithmetic operations in Theorem 5.3.21 require that the partial optimal value functions $t \to b^*(v,t)$ are only computed for nodes $v \in V$ which are contained in an optimal path. However, these nodes are possibly considered more than once in the estimates in Theorem 5.3.21. In particular, if k edge sequences which are traversed by optimal paths pass through some $v \in V$, then the complexity of computing $t \to b^*(v,t)$ is counted k times.*

5.4. Periodical Time-Dependent Networks

Definition 5.4.1 *We call a time-dependent network $G = (V, E, \tau; \beta, \delta)$ periodical with period $t_\mathrm{p} \in \mathbb{R}^+$, if*

$$\tau(e,t) = \tau(e, t + t_\mathrm{p}), \qquad \forall e \in E, t \in \mathbb{R}, \tag{5.61}$$
$$\beta(e,t) = \beta(e, t + t_\mathrm{p}), \qquad \forall e \in E, t \in \mathbb{R}, \tag{5.62}$$
$$\delta(v, t, \Delta t) = \delta(v, t + t_\mathrm{p}, \Delta t), \qquad \forall v \in V, t \in \mathbb{R}, \Delta t \in \mathbb{R}_0^+. \tag{5.63}$$

We call the arrival time constraints T and the waiting time constraints ΔT periodical with period $t_\mathrm{p} \in \mathbb{R}^+$, if for all $v \in V$ there hold

$$t \in T(v) \Longleftrightarrow t + t_\mathrm{p} \in T(v), \tag{5.64}$$
$$\Delta t \in \Delta T(v,t) \Longleftrightarrow \Delta t \in \Delta T(v, t + t_\mathrm{p}), \qquad \forall t \in T(v). \tag{5.65}$$

Theorem 5.4.2 *Let Assumptions 4.2.3 hold and let a source node $v_0 \in V$ and a goal node $v' \in V$ be given. If G, T and ΔT are periodical with period $t_\mathrm{p} \in \mathbb{R}^+$, then the partial function $t_0 \mapsto b^*(v_0, t_0)$ is periodical with period t_p.*

Proof We first prove that $U(v_0, t_0) = U(v_0, t_0 + t_\mathrm{p})$ for each $t_0 \in T(v_0)$. Let $t_0 \in T(v_0)$ be arbitrary but fixed. According to (5.64) there holds $(v_0, t_0 + t_\mathrm{p}) \in X$. Let $u = \big((\Delta t_k, e_k)\big)_{k=1,\dots,n} \in U(v_0, t_0)$, $n \in \mathbb{N}$, and denote $\widetilde{u}_i = \big((\Delta t_k, e_k)\big)_{k=1,\dots,i}$ for $i = 1, \dots, n$. Clearly $\widetilde{u}_i \in U(v_0, t_0)$ for all $i = 1, \dots, n$. We now show by induction that $\widetilde{u}_i \in U(v_0, t_0 + t_\mathrm{p})$ for all $i = 1, \dots, n$ and that

$$\big((v_k, t_k)\big)_{k=0,\dots,i} = \Phi\big((v_0, t_0), \widetilde{u}_i\big) \Longrightarrow \big((v_k, t_k + t_\mathrm{p})\big)_{k=0,\dots,i} = \Phi\big((v_0, t_0 + t_\mathrm{p}), \widetilde{u}_i\big). \tag{5.66}$$

According to (5.65) there holds $\Delta t_1 \in \Delta T(v_0, t_0 + t_\mathrm{p})$. Using (5.61), we obtain

$$(t_0 + \Delta t_1) + \tau(e_1, t_0 + \Delta t_1) + t_\mathrm{p} = (t_0 + t_\mathrm{p} + \Delta t_1) + \tau(e_1, t_0 + t_\mathrm{p} + \Delta t_1),$$

which implies that $(v_1, t_1+t_\mathrm{p}) = \varphi\big((v_0,t_0),\widetilde{u}_1\big) \in X$, cf. (5.64). The inductive step and the converse inclusion follow analogously.
It remains to prove that $\mathcal{B}\big((v_0,t_0),u\big) = \mathcal{B}\big((v_0,t_0+t_\mathrm{p}),u\big)$. This follows directly from (5.66), (5.62) and (5.63). □

Corollary 5.4.3 *Let Assumption 4.2.3 hold, suppose that β,δ are bounded on compact subsets of their respective domain and let a source node $v_0 \in V$ and a goal node $v' \in V$ be given. Suppose further that there exists a $N \in \mathbb{N}$ such that, for each $t_0 \in T(v_0)$, there exists a control sequence $u \in U(v_0,t_0)$ with $\omega(u) = v'$ and $|u| \leq N$. If G, T and ΔT are periodical with period $t_\mathrm{p} \in \mathbb{R}^+$, then there exists a $\bar{t} \in \mathbb{R}$ such that the partial function $b^*(v_0,t_0)$ can be computed for all $t_0 \in T(v_0)$ by considering the compact subset $X \cap \big(V \times [0,\bar{t}\,]\big)$ of X.*

Proof Since $t_0 \mapsto b^*(v_0,t_0)$ is periodical with period t_p according to Theorem 5.4.2, it is sufficient to compute $b^*(v_0,t_0)$ for all $t_0 \in [0,t_\mathrm{p}] \cap T(v_0)$. As $[0,t_\mathrm{p}]$ is compact, E is a finite set and τ is continuous and periodical with period t_p, there exists a $\overline{\tau} \in \mathbb{R}_0^+$, such that $\tau(e,t) \leq \overline{\tau}$ for all $e \in E$, $t \in \mathbb{R}$. Since ΔT is upper semicontinuous, ΔT is uniformly compact. Moreover, as V is finite, $[0,t_\mathrm{p}]$ is compact and ΔT is periodical with period t_p, there exist $\underline{\Delta T}, \overline{\Delta T} \in \mathbb{R}_0^+$ such that $\Delta T(x) \subset [\,\underline{\Delta T}, \overline{\Delta T}\,]$ for all $x \in X$. Furthermore, as β,δ are periodical with period t_p and bounded on compact subsets of their respective domain, $[0,t_\mathrm{p}]$ and $[0,t_\mathrm{p}] \times [\,\underline{\Delta T}, \overline{\Delta T}\,]$ are compact and V, E are finite, there exist $\overline{\beta}, \overline{\delta} \in \mathbb{R}$ such that $\beta(e,t) \leq \overline{\beta}$ for all $e \in E$, $t \in \mathbb{R}$ and $\delta(v,t,\Delta t) \leq \overline{\delta}$ for all $(v,t) \in V \times \mathbb{R}$, $\Delta t \in \Delta T(v,t) \subset [\,\underline{\Delta T}, \overline{\Delta T}\,]$.
Since for each $t_0 \in T(v_0)$ there exists a control sequence $u \in U(v_0,t_0)$ with $\omega(u) = v'$ and $|u| \leq N$, we obtain that $b^*(v_0,t_0) \leq N(\overline{\beta}+\overline{\delta})$ for all $t_0 \in T(v_0)$. Now, (4.7) implies that, for each $t_0 \in T(v_0)$, each optimal control sequence $u^* \in U(v_0,t_0)$ satisfies

$$|u^*| \leq |V| - 1 + \frac{N(\overline{\beta}+\overline{\delta}) - (|V|-1)\underline{\mathcal{B}}}{\underline{\mathcal{B}}^\circ}|V|.$$

Furthermore, if $u \in U(v_0,t_0)$, then $\mathcal{T}\big((v_0,t_0),u\big) \leq |u|(\overline{\tau} + \overline{\Delta T})$. Consequently, the partial function $b^*(v_0,t_0)$ can be computed for all $t_0 \in T(v_0)$ by considering the compact subset $X \cap \big(V \times [0,\bar{t}\,]\big)$ of X, whereat

$$\bar{t} = t_\mathrm{p} + \left(|V| - 1 + \frac{N(\overline{\beta}+\overline{\delta}) - (|V|-1)\underline{\mathcal{B}}}{\underline{\mathcal{B}}^\circ}|V|\right)(\overline{\tau} + \overline{\Delta T}).$$

□

Remark 5.4.4 *Clearly, if $T(v)$ is as in Theorem 5.1.3 (i) or (ii), then there exists a simple admissible path from each $v_0 \in V$ to each $v' \in V$, and we may set $N = |V| - 1$ in Corollary 5.4.3.*

6. A Problem Formulation for Robust Optimal Paths

In this chapter we consider the time-dependent optimal path problem under the assumption that neither the travel times nor the travel costs are precisely known. We provide a framework for the formulation of the time-dependent robust optimal path problem and give some basic results for the optimization with respect to the absolute robustness criterion. At this, we only consider the (robust) forward optimal path problem.
After reviewing some of the literature in the field of optimal path problems with uncertainty in Section 6.1, we introduce the model on which our further analysis is based in Section 6.2. We also introduce a generalization of the concept of interval data in Section 6.2 (namely the DP-property, cf. Definition 6.2.1) and show that the (time-independent) absolute robust optimal path problem is polynomially solvable if the set of possible scenarios satisfies the DP-property. Again by using the DP-property we prove the existence of optimal paths in Section 6.3. In Section 6.4, we impose some additional assumptions on the time-dependent network and prove the continuity of the optimal value function. Moreover, assuming that the scenario set is independent (cf. Definition 6.2.4), we show that the optimal value function is piecewise analytic in the sense of Definition 5.2.3 if the network functions are piecewise analytic in the sense of Definition 5.2.1. Under similar assumptions we provide a complexity analysis for the case in which all network functions are piecewise linear.

6.1. Literature Overview

There are two principal methods which have been proposed over the years to deal with data uncertainty in optimization and optimal control problems: Stochastic programming [148] (resp. stochastic optimal control [175]) and robust optimization [117], [24] (resp. robust optimal control [178]). The most common manner in which model uncertainties are incorporated into network models, especially in the field of (time-dependent) transportation networks, is the notion of stochastic (time-dependent) networks. In [134] and [123], a time-independent network was assumed in which the cost values on the edges are random variables with continuous distribution functions. While the computation of the path minimizing the expected cost was the objective in [134], the more general framework of utility functions was considered in [123]. In [151], [139] and [104] the cost values on the edges were modeled as Markov chains. The application under consideration in [104] was, similar to our motivation in Chapter 2, the road network. In [128], [73] and [127], a discrete-time time-dependent stochastic network was considered in which the path with least expected travel time should be determined. Motivated by the fact that after certain realizations of the travel times on the first edges of the path it might by optimal to choose another path than the one determined a priori, an optimal policy (which is referred to as a set of hyperpaths in [128]) rather than an optimal path was computed. In the language of mathematical control theory

[159], [59], [88], an optimal path corresponds to an optimal open-loop control, whereat an optimal policy (resp., set of hyperpaths) corresponds to an optimal feedback control. It is well-known that the cost function associated with the feedback control outvalues the cost function associated with the open-loop control [159], [59], [88], [25, Appendix G]. In these papers [128], [73] and [127], it was assumed that all probability distributions are stationary except for some finite time interval, and in [73], [127], it was additionally assumed that all travel times are deterministic except for some finite time interval. Similar assumptions were imposed in [71], [72], which also provide a brief overview of the problem variants in discrete-time stochastic time-dependent networks.

In [167], the continuous-time time-dependent stochastic fastest path problem was considered. The objective of the optimization in this work was not the minimization of the least expected travel time of a path, but the determination of the set of pareto-optimal paths. At this, a path p^* is called pareto-optimal if there exists no other path p whose cost function stochastically dominates the cost function of p^*, cp. [25, Section 2.3.4, Appendix G]. A generalization of the FIFO-property to stochastic time-dependent networks, namely stochastic consistency, was then formulated and used to construct an optimal pruning criterion for the search algorithm presented in the paper [167]. At this, the stochastic time-dependent network is said to be stochastically consistent if, for each $e \in E$ and all $t, t' \in \mathbb{R}$, there holds

$$t' \geq t \quad \Longrightarrow \quad \mathbb{P}\big\{t' + \tau(e,t') \leq c\big\} \leq \mathbb{P}\big\{t + \tau(e,t) \leq c\big\}, \quad \forall c \geq 0. \tag{6.1}$$

Note that Lemma 2.4.2 implies that a time-dependent stochastic road network in which the travel times are derived from a physical model of the traffic flow is stochastically consistent. A stochastic time-dependent network with a continuous time variable and the computation of paths which minimize the expected travel time were considered in [70]. In this paper, the expected arrival times along a path [70, Eq. (14)] were recursively approximated by first- and second-order approximations in time of the first two moments of the probability distributions of the arrival times. Clearly, this approach may lead to an accumulation of errors and may therefore result in the computation of a suboptimal solution. However, in many applications the distribution of the travel times and costs is not completely known, cf. [70], [41], [119], [141], [173], [55], and the recursion of the probability distributions is a nontrivial task. Both aspects impede and may even inhibit the consideration of the (continuous-time) time-dependent optimal path problem in a stochastic framework. In order to illustrate the complication that occurs in the recursion of the probability distributions, let us consider the recursion of the travel times. In the time-independent case, the travel time along a path is given by the sum of the travel times on the edges. Consequently, if each edge travel time is, e.g., normally distributed, then the travel time of the path is normally distributed and its expected value and variance can easily determined from the expected values and variances of the edge travel times. However, in the time-dependent case, the situation is more intricate: Suppose that the arrival time θ at some node $v \in V$ is a random variable with continuous probability density function f_θ, and suppose that the travel time at time t along an edge $e \in E$ emanating from v, $\tau(e,t)$, is a random variable with continuous probability density function $f_\tau\big(\cdot,(e,t)\big)$ for each $t \in \mathbb{R}$, at which (e,t) is considered as a parameter of f_τ. Note that $\tau(e,\cdot)$ is a stochastic process and let us assume that each realization of $\big\{\tau(e,t)\big\}_{t\in\mathbb{R}}$ is continuous in t. We denote by $\theta' = \theta + \tau(e,\theta)$ the arrival time at $\omega(e)$ which is a random variable depending on θ and $\big\{\tau(e,t)\big\}_{t\in\mathbb{R}}$. If θ and

$\{\tau(e,t)\}_{t\in\mathbb{R}}$ are stochastically independent, then

$$\begin{aligned}\mathbb{P}\{\theta' \le t\} = \mathbb{P}\{\theta + \tau(e,\theta) \le t\} &= \int_{\mathbb{R}} f_\theta(s)\mathbb{P}\{s + \tau(e,s) \le t\}\mathrm{d}s \\ &= \int_{\mathbb{R}} f_\theta(s) \int_{-\infty}^{t-s} f_\tau\big(\sigma,(e,s)\big)\mathrm{d}\sigma\,\mathrm{d}s,\end{aligned}$$

which implies that the probability density function $f_{\theta'}$ of θ' is given by (cp. [116, p.274, Differentiationssatz])

$$\begin{aligned}f_{\theta'}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}\{\theta' \le t\} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} f_\theta(s) \int_{-\infty}^{t-s} f_\tau\big(\sigma,(e,s)\big)\mathrm{d}\sigma\,\mathrm{d}s \\ &= \int_{\mathbb{R}} f_\theta(s) f_\tau\big(t-s,(e,s)\big)\mathrm{d}s.\end{aligned}$$

The dependence of the parameter (e,s) of f_τ on the integration variable s in the above integral is the reason why the recursion of the probability distributions is a nontrivial task. Closed solutions can only be obtained in very few special cases. Hence the approximation of the first two moments of the probability distributions is not only founded in the lack of a precise description of these distributions, but also in the difficulties that occur in their recursion.

In order to develop a computationally efficient method for solving the time-dependent optimal path problem subject to uncertainty, which does not require a precise stochastic description of all uncertainties in the network, we will assume that we are given a range of values which the travel times and travel costs can assume. This leads to the notion of minmax optimization.

The two most common approaches for formulating the uncertainty in robust optimization are the concept of scenarios and the concept of interval data. A discrete set of possible realizations of the problem data is commonly considered in the first case. In the second case the uncertainty is captured by certain interval ranges in which the realizations of the problem data are known to be contained. It has been shown in [15], that there are combinatorial optimization problems which are NP-hard in the case of a finite set of scenarios, but which are polynomially solvable in the case of the interval representation of the uncertainty, thereby justifying a differentiation of both models. We will see below how the concept of interval data can be formulated as a special case of the concept of scenarios and we will propose a generalization of the former approach in Definition 6.2.1, which we believe to be more meaningful, cp. Corollary 6.2.3.

There are also two (most common; for other definitions of the objective function see, e.g., [152] and the references therein) manners in which the objective function is formulated in the literature [117]: In the absolute robustness criterion (minmax optimization), the objective consists of the determination of a solution which, among all possible solutions, leads to the best worst-case performance. In the robust deviation criterion (minmax regret optimization), the objective consists of the determination of a solution which, among all possible solutions and maximizing over all scenarios, leads to the minimum deviation of the objective function value from the optimal objective function value of the particular scenario. The first criterion is also commonly used in the field of optimal control [166], [25, Chapter 1.6], although it generally leads to more conservative solutions [99].

Most results on the robust optimal path problem are formulated for the time-independent case. The absolute robust optimal path problem with scenario data is NP-hard [117], [176], whereat the absolute robust optimal path problem with interval data is solvable in polynomial time [28], [99]. The relative robust optimal path problem (i.e., the determination of the optimal path with respect to the robust deviation criterion) is NP-hard both for scenario data [117], [176] and for interval data [179], [16]. It has also been shown that, given a finite set of scenarios, the absolute and the relative robust optimal path problem are fully polynomial-time approximable if the number of possible scenarios is constant, whereat non-approximability results have been proved for both problems if the number of possible scenarios is not constant [12]. A number of algorithms have been developed in order to solve the different variants of the problem [176], [28], [99], [100], [130], [131], and in [58] the robust optimal path problem has been considered in a time-dependent road network in the context of vehicle routing.

6.2. Problem formulation

Let us assume for the remainder of this chapter that the travel time and travel cost functions τ, β, δ depend on the state of the network w, which is known to assume values in a set W of possible states of the network, i.e.,

$$\tau : E \times \mathbb{R} \times W \to \mathbb{R}_0^+, \tag{6.2}$$

$$\beta : E \times \mathbb{R} \times W \to \mathbb{R}, \tag{6.3}$$

$$\delta : V \times \mathbb{R} \times \mathbb{R}_0^+ \times W \to \mathbb{R}. \tag{6.4}$$

Let us further assume that the set of possible states of the network is subject to certain restrictions which depend on the state of and the control action applied by a commodity traveling in the network. This model accounts for both a time-dependent evolution of the network and an impact the commodity exerts on the set of possible states of the network. Note that the history of the commodity (i.e., the manner in which a particular state has been reached) is not taken into account and only the current state and control action are supposed to have an impact on the state of the network. Given a state x and a control action u, we denote by $\Omega(x, u)$ the restricted set of states of the network, i.e.,

$$\Omega : \bigcup_{(v,t) \in V \times \mathbb{R}} \big\{ \{(v,t)\} \times \{\mathbb{R}_0^+ \times E^+(v)\} \big\} \to \mathcal{P}(W) \setminus \emptyset. \tag{6.5}$$

Note that we may just as well interpret $\Omega(x, u)$ as the influences of the possible network states on those cost and transition functions which are affected by the application of the control u in the state x. Let

$$\mathcal{W} = \big\{ \mathcal{w} : \mathrm{dom}(\Omega) \to W |\ \mathcal{w}(x,u) \in \Omega(x,u) \quad \forall (x,u) \in \mathrm{dom}(\Omega) \big\}. \tag{6.6}$$

$\mathcal{W}$ is the set of possible scenarios for the time-dependent optimal path problem and each $\mathcal{w} \in \mathcal{W}$ is a possible scenario.
It would be possible to assume that not only the travel time and cost but also the goal node resulting from a particular control action are uncertain. In this case, we would have to replace the edges in the graph by hyperedges, which would result in a (time-dependent

and uncertain) hypergraph. The approach of determining an optimal path in a hypergraph has been chosen in [77] in order to solve the space-discretized version of a time-independent perturbed optimal control problem. We conjecture that the results presented in the following can be generalized to this setting. However, since we are especially interested in the time-dependent road network in which the travel times and travel costs may be uncertain but the topology of the road network is fixed, we leave this generalization as a topic for future research.

In a similar manner as in Chapter 3 we introduce the functions $\varphi, \Phi, \mathcal{B}, \mathcal{T}, b^*$. The state transition resulting from the application of a control action $(\Delta t, e)$ in the state (v,t) of the commodity traveling in the network, given the state $w \in W$ of the network, is specified by the control-to-state mapping

$$\varphi : \left(\bigcup_{(v,t)\in V\times\mathbb{R}} \{\{(v,t)\} \times \{\mathbb{R}_0^+ \times E^+(v)\}\} \right) \times W \to V \times \mathbb{R},$$
$$\varphi\big((v,t),(\Delta t, e), w\big) = \big(\omega(e), t + \Delta t + \tau(e, t+\Delta t, w)\big).$$

Given arrival time constraints T, waiting time constraints ΔT and a restriction of the set of possible network states Ω, we define the state space X as in (3.2). A control action u is called admissible for a given state $x = (v,t) \in X$, if $\varphi(x,u,w) \in X$ for all network states $w \in \Omega(x,u)$.

Let $x_0 = (v_0, t_0) \in X$ and $u = \big((\Delta t_k, e_k)\big)_{k=1,2,\ldots}$ denote a sequence of controls which satisfies $\alpha(e_{k+1}) = \omega(e_k)$ for all $k = 1,2,\ldots$. Given a scenario $\mathcal{W} \in \mathcal{W}$, the path $p = \Phi(x_0,u,\mathcal{W}) = (x_k)_{k=0,1,\ldots}$ produced by the application of u in x_0 is determined by

$$x_k = \varphi\big(x_{k-1}, u_k, \mathcal{W}(x_{k-1},u_k)\big), \qquad k = 1,2,\ldots. \tag{6.7}$$

The control sequence u is called admissible for the given state x_0 if, for each $\mathcal{W} \in \mathcal{W}$, $x_k \in X$ for all $k = 0,1,\ldots$. We denote the set of control sequences which are admissible for $x_0 \in X$ by $U(x_0)$. In a similar manner as in Chapter 3, we denote

$$\mathcal{B}(x_0,u,\mathcal{W}) = \sum_{k=1}^{|u|} \Big[\delta\big(v_{k-1}, t_{k-1}, \Delta t_k, \mathcal{W}(x_{k-1},u_k)\big) + \beta\big(e_k, t_{k-1} + \Delta t_k, \mathcal{W}(x_{k-1},u_k)\big)\Big],$$
$$\mathcal{T}(x_0,u,\mathcal{W}) = \sum_{k=1}^{|u|} \Big[\Delta t_k + \tau\big(e_k, t_{k-1} + \Delta t_k, \mathcal{W}(x_{k-1},u_k)\big)\Big],$$

where $x_0 \in X$, $u = \big((\Delta t_k, e_k)\big)_{k=1,2,\ldots} \in U(x_0)$ is an admissible control sequence, $\mathcal{W} \in \mathcal{W}$ is a network scenario and $p = \Phi(x_0,u,\mathcal{W}) = \big((v_k,t_k)\big)_{k=0,1,\ldots}$. Finally, given a destination node $v' \in V$, we define the optimal value function b^* of the forward time-dependent absolute robust optimal path problem according to:

$$b^*(x) = \inf_{\substack{u\in U(x):\\ \omega(u)=v'}} \sup_{\mathcal{W}\in\mathcal{W}} \mathcal{B}(x,u,\mathcal{W}). \tag{6.8}$$

We have so far modeled the uncertainty in the time-dependent network by a very general formulation using scenario data. However, the analysis in the following sections of this chapter will be based on a more restricted model which is a generalization of the concept of modeling the uncertainty by interval data. In the following, we suppose that the commodity which travels in the network does not know which scenario the network is in and in which manner the scenarios depend on the state of and the control action applied by the commodity. This is similar to assuming that the set of possible scenarios has a specific structure which is defined as follows:

Definition 6.2.1 *Let $(V, E, \tau; \beta, \delta)$ be a time-dependent network in which the functions τ, β, δ depend on some internal network state according to (6.2)-(6.4). Let $\mathcal{W}$ denote the set of possible scenarios of the time-dependent network according to (6.5) and (6.6).*
The set of possible scenarios $\mathcal{W}$ has the DP-property, if for each finite collection of $(x_n, u_n) = \big((v_n, t_n), (\Delta t_n, e_n)\big) \in \mathrm{dom}(\Omega)$, $n = 1, ..., N$, $N \in \mathbb{N}$, with $(x_m, u_m) \neq (x_n, u_n)$ for $m, n \in \{1, ..., N\}$, $m \neq n$ and each collection of $\mathcal{W}_n \in \mathcal{W}$, $n = 1, ..., N$, there exists a $\mathcal{W} \in \mathcal{W}$ such that

$$\begin{aligned} \tau\big(e_n, t_n + \Delta t_n, \mathcal{W}(x_n, u_n)\big) &= \tau\big(e_n, t_n + \Delta t_n, \mathcal{W}_n(x_n, u_n)\big), && n = 1, ..., N, \\ \beta\big(e_n, t_n + \Delta t_n, \mathcal{W}(x_n, u_n)\big) &= \beta\big(e_n, t_n + \Delta t_n, \mathcal{W}_n(x_n, u_n)\big), && n = 1, ..., N, \\ \delta\big(v_n, t_n, \Delta t_n, \mathcal{W}(x_n, u_n)\big) &= \delta\big(v_n, t_n, \Delta t_n, \mathcal{W}_n(x_n, u_n)\big), && n = 1, ..., N. \end{aligned}$$

If the set of possible scenarios has the DP-property, then it is not possible to infer from certain realizations of the network functions τ, β, δ for a finite collection of distinct state-control pairs $\{(x_n, u_n)\}_{n=1,...,N}$, $N \in \mathbb{N}$, to the realization of the network functions τ, β, δ for any other state-control pair (x, u). Whether this property originates from the set of possible scenarios or from the lack of knowledge of the commodity which travels in the network is arbitrary for the following considerations. However, assuming that the set of possible scenarios has the DP-property has a strong impact on the applicability of the principle of dynamic programming and on the computational complexity of solving the (time-independent) absolute robust optimal path problem, cf. Corollary 6.2.3. The key argument, on which the derivation of the dynamic programming equations for the time-dependent absolute robust path problem is based, is the following:

Lemma 6.2.2 *Suppose that the set of possible scenarios $\mathcal{W}$ has the DP-property and that there exists a $\underline{\mathcal{T}}^\circ \in \mathbb{R}$, $\underline{\mathcal{T}}^\circ > 0$, such that*

$$\mathcal{T}\big((v, t), u, \mathcal{W}\big) \geq \underline{\mathcal{T}}^\circ, \quad \forall u \in U(v, t) \text{ with } \omega(u) = \alpha(u), \forall \mathcal{W} \in \mathcal{W}. \tag{6.9}$$

If $x_0 = (v_0, t_0) \in X$ and $u = \big((\Delta t_k, e_k)\big)_{k=1,...,n} \in U(x_0)$, then

$$\sup_{\mathcal{W} \in \mathcal{W}} \mathcal{B}(x_0, u, \mathcal{W}) = \sup_{w_1 \in \Omega(x_0, u_1)} \cdots \sup_{w_n \in \Omega(x_{n-1}, u_n)} \sum_{k=1}^{|u|} \mathcal{B}_k(x_{k-1}, u_k, w_k), \tag{6.10}$$

where

$$\begin{aligned} x_k = (v_k, t_k) &= \varphi(x_{k-1}, u_k, w_k), && k = 1, ..., n, \\ \mathcal{B}_k(x_{k-1}, u_k, w_k) &= \delta\big(v_{k-1}, t_{k-1}, \Delta t_k, w_k\big) + \beta\big(e_k, t_{k-1} + \Delta t_k, w_k\big), && k = 1, ..., n. \end{aligned}$$

Proof We prove the assertion by induction on the length n of the control sequence u. Clearly, (6.10) holds for $n = 1$. Now, suppose that (6.10) holds for all states $x \in X$ and all control sequences $u \in U(x)$ of length at most $n-1$. Let $x_0 \in X$ and $u = \big(u_k\big)_{k=1,\dots,n} \in U(x_0)$. Then

$$\begin{aligned}\sup_{\mathcal{W}\in\mathcal{W}} \mathcal{B}(x_0,u,\mathcal{W}) &= \sup_{w_1\in\Omega(x_0,u_1)} \sup_{\substack{\mathcal{W}\in\mathcal{W}:\\ \mathcal{W}(x_0,u_1)=w_1}} \Big[\mathcal{B}_1(x_0,u_1,w_1) + \mathcal{B}\big(\varphi(x_0,u_1,w_1),u_{2:n},\mathcal{W}\big)\Big]\\ &= \sup_{w_1\in\Omega(x_0,u_1)} \Big[\mathcal{B}_1(x_0,u_1,w_1) + \sup_{\substack{\mathcal{W}\in\mathcal{W}:\\ \mathcal{W}(x_0,u_1)=w_1}} \mathcal{B}\big(\varphi(x_0,u_1,w_1),u_{2:n},\mathcal{W}\big)\Big]\end{aligned}$$

Let $w_1 \in \Omega(x_0,u_1)$ be arbitrary but fixed, let $x_1 = \varphi(x_0,u_1,w_1)$. Obviously, there holds $\{\mathcal{W}\in\mathcal{W} : \mathcal{W}(x_0,u_1)=w_1\} \subset \mathcal{W}$ and hence

$$\sup_{\substack{\mathcal{W}\in\mathcal{W}:\\ \mathcal{W}(x_0,u_1)=w_1}} \mathcal{B}\big(x_1,u_{2:n},\mathcal{W}\big) \le \sup_{\mathcal{W}\in\mathcal{W}} \mathcal{B}\big(x_1,u_{2:n},\mathcal{W}\big). \tag{6.11}$$

Let $\mathcal{W}\in\mathcal{W}$ be arbitrary but fixed and $(x_k)_{k=1,\dots,n} = \Phi(x_1,u_{2:n},\mathcal{W})$. (6.9) implies that $x_k \neq x_l$ for all $k,l \in \{0,\dots,n\}$ with $k \neq l$. Hence, as $\mathcal{W}$ has the DP-property, there exists a $\widetilde{\mathcal{W}} \in \mathcal{W}$ with $\widetilde{\mathcal{W}}(x_0,u_1) = w_1$, such that

$$\begin{aligned}\Phi(x_1,u_{2:n},\mathcal{W}) &= \Phi(x_1,u_{2:n},\widetilde{\mathcal{W}}),\\ \mathcal{B}(x_1,u_{2:n},\mathcal{W}) &= \mathcal{B}(x_1,u_{2:n},\widetilde{\mathcal{W}}).\end{aligned}$$

Together with (6.11), this implies that

$$\sup_{\substack{\mathcal{W}\in\mathcal{W}:\\ \mathcal{W}(x_0,u_1)=w_1}} \mathcal{B}\big(x_1,u_{2:n},\mathcal{W}\big) = \sup_{\mathcal{W}\in\mathcal{W}} \mathcal{B}\big(x_1,u_{2:n},\mathcal{W}\big).$$

Consequently,

$$\sup_{\mathcal{W}\in\mathcal{W}} \mathcal{B}(x_0,u,\mathcal{W}) = \sup_{w_1\in\Omega(x_0,u_1)} \Big[\mathcal{B}_1(x_0,u_1,w_1) + \sup_{\mathcal{W}\in\mathcal{W}} \mathcal{B}\big(\varphi(x_0,u_1,w_1),u_{2:n},\mathcal{W}\big)\Big].$$

Now (6.10) follows from the induction hypothesis. □

We now briefly consider the time-independent absolute robust optimal path problem under the assumption that the set of possible scenarios has the DP-property. The DP-property is adapted to time-independent networks as follows:
Given a set W of possible network states and a network $(V,E;\beta)$ with $\beta : E\times W \mapsto \mathbb{R}$ and a control-dependent restriction $\Omega : E \mapsto \mathcal{P}(W)\setminus\emptyset$ of the set of possible network states, the set of possible scenarios

$$\mathcal{W} = \big\{\mathcal{W} : E \to W |\ \mathcal{W}(e) \in \Omega(e) \quad \forall e \in E\big\}$$

has the DP-property, if for each function

$$\hat{\beta} : E \mapsto \mathbb{R},\ \text{with } \hat{\beta}(e) \in \beta(e,\Omega(e)), \quad \forall e \in E,$$

there exists a $\mathcal{w} \in \mathcal{W}$ such that

$$\hat{\beta}(e) = \beta\big(e, \mathcal{w}(e)\big), \quad \forall e \in E.$$

It is easily seen that this adaption is straightforward from Definition 6.2.1, since V, E are finite sets and, in the time-independent case, the state $v \in V$ is uniquely determined by the choice of an admissible control action $e \in E^+(v)$, i.e., $v = \alpha(e)$.
We will now show that the complexity results for the time-independent absolute robust optimal path problem with interval data can be generalized to scenario sets which have the DP-property. Note that modeling the uncertainty by interval data corresponds to modeling the uncertainty by a scenario set which has the DP-property and by considering a cost function β, which has the property that $\beta(e, \Omega(e))$ is an interval for each $e \in E$.

Corollary 6.2.3 *The time-independent absolute robust optimal path problem with nonnegative edge weights is polynomially solvable if the set of possible scenarios has the DP-property.*

Proof Let $(V, E; \beta)$ be an instance of a network in which $\beta : E \times W \to \mathbb{R}_0^+$, where W denotes the set of possible network states. Let Ω denote the restriction mapping of the network states and let $\mathcal{W}$ denote the set of possible scenarios. Let a source node $v_0 \in V$ and a goal node $v' \in V$ be given. Finally, let $(e_1, ..., e_n)$ be a connected edge sequence with $\alpha(e_1) = v_0$ and $\omega(e_n) = v'$. Suppose that there exist $k, l \in \{1, ..., n\}$ with $k > l$ such that $(e_k, ..., e_l)$ is a circle. The non-negativity of β implies that

$$\sum_{i=1,\dots,n} \beta(e_i, \mathcal{w}(e_i)) \geq \sum_{i=1,\dots,k-1} \beta(e_i, \mathcal{w}(e_i)) + \sum_{i=l+1,\dots,n} \beta(e_i, \mathcal{w}(e_i)) \qquad \forall \mathcal{w} \in \mathcal{W},$$

and hence there always exists a simple optimal path. Consequently, the optimal value function $b^* : V \to \mathbb{R}$ of the time-independent absolute robust optimal path problem satisfies

$$b^*(v) = \min_{(e_1,\dots,e_n)\in\mathcal{E}} \sup_{\mathcal{w}\in\mathcal{W}} \sum_{i=1}^{n} \beta(e_i, \mathcal{w}(e_i)),$$

where $\mathcal{E}$ is the set of all simple connected edge sequences which connect v_0 and v'. Using the DP-property of $\mathcal{W}$ and proceeding in a similar manner as in the proof of Lemma 6.2.2, we obtain

$$b^*(v) = \min_{(e_1,\dots,e_n)\in\mathcal{E}} \sum_{i=1}^{n} \overline{\beta}(e_i),$$

where $\overline{\beta}(e) = \sup_{w\in\Omega(e)} \beta(e, w)$ for all $e \in E$. Consequently, the time-independent absolute robust optimal path problem in $(V, E; \beta)$ can be solved by solving the time-independent optimal path problem in $(V, E; \overline{\beta})$ with source node v_0 and goal node v'. The latter problem is polynomially solvable. □

The above result is similar to the observation in [15], where it is pointed out that the polynomial solvability of certain robust combinatorial optimization problems with interval data is due to the structure of the set of scenarios which is a rectangular box. We have seen that, at least in the case of the optimal path problem, the DP-property of the set of scenarios (in the sense of the above adaption of Definition 6.2.4) is sufficient in order to establish the polynomial solvability. We conjecture that a similar generalization holds for

the combinatorial optimization problems in [15].
Furthermore, we conjecture that, in the time-dependent case, there exist similar relationships between the computational complexities of the absolute robust optimal path problem with fixed departure time. However, we leave a detailed analysis of this matter as a topic for further research.
In order to conclude this section we introduce the following assumption on the set of possible scenarios which simplifies the solution of the dynamic programming equations associated with the time-dependent absolute robust optimal path problem.

Definition 6.2.4 *Let $(V, E, \tau; \beta, \delta)$ be a time-dependent network in which the functions τ, β, δ depend on some internal network state according to (6.2)-(6.4). Let $\mathcal{W}$ denote the set of possible scenarios of the time-dependent network according to (6.5) and (6.6).*
The set of possible scenarios $\mathcal{W}$ is independent, if

(i) for each finite collection of $(x_n, u_n) = \big((v_n, t_n), (\Delta t_n, e_n)\big) \in \mathrm{dom}(\Omega)$, $n = 1, ..., N$, $N \in \mathbb{N}$, with $(x_m, u_m) \neq (x_n, u_n)$ for $m, n \in \{1, ..., N\}$, $m \neq n$, and any collection of

$$\begin{aligned} \hat{\tau}_n &\in \tau\big(e_n, t_n + \Delta t_n, \Omega(x_n, u_n)\big), && n = 1, ..., N, \\ \hat{\beta}_n &\in \beta\big(e_n, t_n + \Delta t_n, \Omega(x_n, u_n)\big), && n = 1, ..., N, \\ \hat{\delta}_n &\in \delta\big(v_n, t_n, \Delta t_n, \Omega(x_n, u_n)\big), && n = 1, ..., N, \end{aligned}$$

there exists a $\mathcal{w} \in \mathcal{W}$ such that

$$\begin{aligned} \hat{\tau}_n &= \tau\big(e_n, t_n + \Delta t_n, \mathcal{w}(x_n, u_n)\big), && \forall n = 1, ..., N, \\ \hat{\beta}_n &= \beta\big(e_n, t_n + \Delta t_n, \mathcal{w}(x_n, u_n)\big), && \forall n = 1, ..., N, \\ \hat{\delta}_n &= \delta\big(v_n, t_n, \Delta t_n, \mathcal{w}(x_n, u_n)\big), && \forall n = 1, ..., N, \end{aligned}$$

for all $n \in \{1, ..., N\}$, and

(ii) for any $\big((v, t_1), (\Delta t_1, e)\big), \big((v, t_2), (\Delta t_2, e)\big) \in \mathrm{dom}(\Omega)$ with $t_1 + \Delta t_1 = t_2 + \Delta t_2$, there holds

$$\begin{aligned} \tau\Big(e, t_1 + \Delta t_1, \Omega\big((v, t_1), (\Delta t_1, e)\big)\Big) &= \tau\Big(e, t_2 + \Delta t_2, \Omega\big((v, t_2), (\Delta t_2, e)\big)\Big), \\ \beta\Big(e, t_1 + \Delta t_1, \Omega\big((v, t_1), (\Delta t_1, e)\big)\Big) &= \beta\Big(e, t_2 + \Delta t_2, \Omega\big((v, t_2), (\Delta t_2, e)\big)\Big). \end{aligned}$$

Remark 6.2.5 *It is readily seen that if $\mathcal{W}$ is independent, then it has the DP-property. Moreover, if $\mathcal{W}$ is independent, then the realizations of the network functions τ, β, δ are independent of one another, and the realizations of τ and β depend only on the departure time on the respective edge.*

In the following sections, we will establish the existence of optimal paths and derive some properties of the optimal value function.

6.3. Existence of Optimal Paths and Dynamic Programming

Before proving the existence of time-dependent absolute robust optimal paths we prove two preliminary lemmas.

Lemma 6.3.1 *Let T, Y, W, Z denote topological spaces, $f : T \times Y \times W \to Z$ a continuous function, $\Lambda : T \to \mathcal{P}(Y)$ an upper semicontinuous point-to-set mapping and $\Omega : T \times Y \to \mathcal{P}(W)$ a lower semicontinuous point-to-set mapping. Suppose that Y is locally compact and satisfies the second axiom of countability, and suppose that W satisfies the first axiom of countability. If $C \subset Z$ is a closed set, then the point-to-set mapping $\Gamma : T \to \mathcal{P}(Y)$, $\Gamma(t) = \{y \in \Lambda(t) : f(t, y, w) \in C \ \forall w \in \Omega(t, y)\}$ is upper semicontinuous.*

Proof Suppose that there is a $t_0 \in T$, such that Γ is not upper semicontinuous in t_0. Then there exists an open set $U_Y \subset Y$ containing $\Gamma(t_0)$, such that there is no open neighborhood U_T of t_0 with $\Gamma(t) \subset U_Y$ for all $t \in U_T$. Hence, there is a sequence $(t_n)_{n\in\mathbb{N}}$, $\lim_{n\to\infty} t_n = t_0$, such that for each $n \in \mathbb{N}$ there exists at least one $y_n \in \Gamma(t_n) \setminus U_Y$. In a similar manner as in Lemma 4.2.2, using the upper semicontinuity of Λ, we obtain the existence of a subsequence (denoted again by $(y_n)_{n\in\mathbb{N}}$) which converges to some $y_0 \in \Lambda(t_0)$. Next, since $y_n \in \Gamma(t_n)$ for all $n \in \mathbb{N}$, we obtain that $f(t_n, y_n, w) \in C$ for all $w \in \Omega(t_n, y_n)$ and all $n \in \mathbb{N}$. Moreover, since $y_n \in Y \setminus U_Y$ and $Y \setminus U_Y$ is closed, we observe that $y_0 \in Y \setminus U_Y \subset Y \setminus \Gamma(t_0)$. We now claim that, as we have supposed that Γ is not upper semicontinuous in t_0, there exists at least one $w_0 \in \Omega(t_0, y_0)$ with $f(t_0, y_0, w_0) \notin C$. Otherwise, as C is closed and f is continuous, $\Gamma(t_0)$ would be a closed set which is compact (recall that $\Gamma(t_0) \subset \Lambda(t_0)$ and $\Lambda(t_0)$ is compact) and which contains y_0 (recall that $y_0 \in \Lambda(t_0)$), contradicting the fact that $y_0 \in Y \setminus \Gamma(t_0)$ and thereby proving the upper semicontinuity of Γ in t_0. In order to complete the contradiction, we will now show that there exists no $w_0 \in \Omega(t_0, y_0)$ with $f(t_0, y_0, w_0) \notin C$.
Let $(t_n)_{n\in\mathbb{N}}, t_0, (y_n)_{n\in\mathbb{N}}, y_0$ as defined above and suppose that there exists a $w_0 \in \Omega(t_0, y_0)$ with $f(t_0, y_0, w_0) \notin C$. As W satisfies the first axiom of countability [93, Definition 10.20], there exists a countable and nested filter base $(U_W^k)_{k\in\mathbb{N}}$ of open sets $U_W^k \subset W$ which converge to w_0 [93, p.8]. Since Ω is lower semicontinuous, there exists an open neighborhood $U_{T\times Y}^k \subset T \times Y$ of (t_0, y_0), such that $U_W^k \cap \Omega(t, y) \neq \emptyset$ for all $(t, y) \in U_{T\times Y}^k$ and each $k \in \mathbb{N}$. As $\lim_{n\to\infty}(t_n, y_n) = (t_0, y_0)$, for each $k \in \mathbb{N}$, there exists a $N(k) \in \mathbb{N}$, such that $U_W^k \cap \Omega(t_n, y_n) \neq \emptyset$ for all $n \geq N(k)$. Consequently, there exist $w_n \in \Omega(t_n, y_n)$ with $\lim_{n\to\infty} w_n = w_0$. As f is continuous, C is closed and $f(t_n, y_n, w_n) \in C$ for all $n \in \mathbb{N}$, we obtain that $f(t_0, y_0, w_0) = \lim_{n\to\infty} f(t_n, y_n, w_n) \in C$, thereby completing the contradiction.
□

Lemma 6.3.2 *Let T, Y, W be topological spaces. If $\Omega : T \times Y \to \mathcal{P}(W) \setminus \emptyset$ is a lower semicontinuous point-to-set mapping, and $f : T \times Y \times W \to \mathbb{R}$ is a lower semicontinuous function, then the function $f^* : T \times Y \to \mathbb{R}$,*

$$f^*(t, y) = \sup_{w\in\Omega(t,y)} f(t, y, w),$$

is lower semicontinuous.

Proof We denote

$$f^*(t, y) = \sup_{w\in\Omega(t,y)} f(t, y, w) = -\Big[\inf_{w\in\Omega(t,y)} -f(t, y, w)\Big].$$

As the function $(t, y, w) \mapsto -f(t, y, w)$ is upper semicontinuous, [65, Theorem 2.2.1] implies that the function $t \mapsto \inf_{w\in\Omega(t,y,w)} -f(t, y, w)$ is upper semicontinuous, which yields the lower semicontinuity of f^*. □

In the formulation of the following theorem we use a generalization of the set of reachable points in time $T_R(v_0)$ (cp. Definition 3.5.1) to time-dependent networks which are subject to uncertainty:

$$T_R(v_0) = \big\{t_0 \in T(v_0) : \exists u \in U(v_0, t_0) \text{ with } \omega(u) = v'\big\}.$$

This set consists of all departure times for which we can guarantee to reach the goal node v' without violating any constraint in any scenario $\mathcal{w} \in \mathcal{W}$.

Assumption 6.3.3 *Let $G = (V, E, \tau; \beta, \delta)$ denote a time-dependent network in which the functions τ, β, δ depend on the network state $w \in W$. Suppose that the travel time function τ is continuous and the cost functions β, δ are lower semicontinuous. Suppose that $T(v)$ is a closed set for all $v \in V$, that the point-to-set mapping ΔT is upper semicontinuous, that the point-to-set mapping Ω is lower semicontinuous and that W satisfies the first axiom of countability. Further, suppose that the set of possible scenarios $\mathcal{W}$ has the DP-property and that there exist $\underline{\mathcal{B}}, \underline{\mathcal{B}}^\circ, \underline{\mathcal{T}}^\circ \in \mathbb{R}$, $\underline{\mathcal{B}}^\circ, \underline{\mathcal{T}}^\circ > 0$, such that there hold (6.9) and*

$$\mathcal{B}\big((v,t), u, \mathcal{w}\big) \geq \underline{\mathcal{B}}, \quad \forall u \in U(v,t) \text{ with } |u| = 1, \forall \mathcal{w} \in \mathcal{W}, \tag{6.12}$$

$$\mathcal{B}\big((v,t), u, \mathcal{w}\big) \geq \underline{\mathcal{B}}^\circ, \quad \forall u \in U(v,t) \text{ with } \omega(u) = \alpha(u), \forall \mathcal{w} \in \mathcal{W}. \tag{6.13}$$

Theorem 6.3.4 *Let $G = (V, E, \tau; \beta, \delta)$ denote a time-dependent network in which Assumption 6.3.3 holds and let $v_0, v' \in V$. Then, for any $t_0 \in T_R(v_0)$, there exists a (finite) optimal path from (v_0, t_0) to v' and the partial function $t_0 \mapsto b^*(v_0, t_0)$ is lower semicontinuous on $T_R(v_0)$.*

Proof We proceed in a similar manner as in the proof of Theorem 4.2.4. First, for any finite, connected edge sequence from v_0 to v', we define

$$\widetilde{T}_{(e_1,\ldots,e_n)} = \big\{t \in T(v_0) : \exists (\Delta t_k)_{k=1,\ldots,n} \text{ such that } \big((\Delta t_k, e_k)\big)_{k=1,\ldots,n} \in U(v_0, t)\big\}.$$

We denote the set of all finite, connected edge sequence from v_0 to v' by $\mathcal{E}$ and observe that

$$T_R(v_0) = \bigcup_{(e_1,\ldots,e_n)\in\mathcal{E}} \widetilde{T}_{(e_1,\ldots,e_n)}.$$

Next, we assume that a finite, connected edge sequence $(e_1, ..., e_n)$ from v_0 to v' has been fixed. We denote $v_{k-1} = \alpha(e_k)$ for $k = 1, ..., n$, and $v_n = \omega(e_n) = v'$. In order to ease the notation, we further introduce $\theta_k : \mathbb{R} \times W \to \mathbb{R}$, $\theta_k(t, w) = t + \tau(e_k, t, w)$, $k = 1, ..., n$. Note that θ_k is continuous for all $k = 1, ..., n$. We denote $\widetilde{T}_n = T(v_n)$, and for $k = 0, ..., n-1$, we define $\widetilde{\Delta T}_k : T(v_k) \to \mathcal{P}(\mathbb{R}_0^+)$,

$$\widetilde{\Delta T}_k(t) = \Big\{\Delta t \in \Delta T(v_k, t) : \theta_k\big(t + \Delta t, w\big) \in \widetilde{T}_{k+1} \ \forall w \in \Omega\big((v_k, t), (\Delta t, e_{k+1})\big)\Big\}, \tag{6.14}$$

$$\widetilde{T}_k = \operatorname{supp}(\widetilde{\Delta T}_k). \tag{6.15}$$

The DP-property of $\mathcal{W}$ and (6.9) imply that $\widetilde{T}_0 = \widetilde{T}_{(e_1,\ldots,e_n)}$, cp. Lemma 6.2.2. By backwards induction, as τ is continuous, ΔT is upper semicontinuous and $\widetilde{T}_n$ is closed, Lemma 6.3.1 implies that $\widetilde{\Delta T}_k$ is an upper semicontinuous point-to-set mapping and $\widetilde{T}_k$ is closed for all $k = 0, ..., n-1$. Note, that the set of admissible control actions at $(v_k, t) \in X$ along

$(e_1, ..., e_n)$ is given by $\widetilde{\Delta T}_k(t) \times \{e_k\}$.
We now analyze the optimal-cost function $\widetilde{b}^*$ along this edge sequence by backwards induction. Since the cost of each circle is strictly positive, we have

$$\widetilde{b}^*(v_n, t) = \widetilde{b}^*(v', t) = b^*(v', t) = 0$$

for all $t \in \widetilde{T}_n = T(v')$. Clearly, $\widetilde{b}^*(v_n, \cdot)$ is lower semicontinuous. Moreover, Lemma 6.2.2 implies that, for each $t_0 \in \widetilde{T}_0$, there holds

$$\begin{aligned}&\inf_{\Delta t_1 \in \widetilde{\Delta T}_1(t_0)} \cdots \inf_{\Delta t_n \in \widetilde{\Delta T}_n(t_{n-1})} \sup_{\mathcal{W} \in \mathcal{W}} \mathcal{B}(x_0, u, \mathcal{W}) \\ &= \inf_{\Delta t_1 \in \widetilde{\Delta T}_1(t_0)} \cdots \inf_{\Delta t_n \in \widetilde{\Delta T}_n(t_{n-1})} \sup_{w_1 \in \Omega(x_0, u_1)} \cdots \sup_{w_n \in \Omega(x_{n-1}, u_n)} \sum_{k=1}^{|u|} \mathcal{B}_k(x_{k-1}, u_k, w_k),\end{aligned}$$

where we have denoted, for $k = 1, ..., n$, $u_k = (\Delta t_k, e_k)$, $x_k = (v_k, t_k) = \varphi(x_{k-1}, u_k, w_k)$, and

$$\mathcal{B}_k(x_{k-1}, u_k, w_k) = \delta\big(v_{k-1}, t_{k-1}, \Delta t_k, w_k\big) + \beta\big(e_k, t_{k-1} + \Delta t_k, w_k\big).$$

As (6.12) implies that $\widetilde{b}_k^*(t) > -\infty$ for all $t \in \widetilde{T}_k$ and all $k = 0, ..., n-1$, [25, Proposition 1.6.1 et seq.] yields for all $t \in \widetilde{T}_k$ and all $k = 0, ..., n-1$:

$$\widetilde{b}^*(v_k, t) = \inf_{\Delta t \in \widetilde{\Delta T}_k(t)} \sup_{w \in \Omega((v_k, t), (\Delta t, e_{k+1}))} b_k(t, \Delta t, w), \tag{6.16}$$

where we used the function $b_k : \mathrm{graph}(\widetilde{\Delta T}_k) \times W \to \mathbb{R}$,

$$b_k(t, \Delta t, w) = \delta(v_k, t, \Delta t, w) + \beta(e_{k+1}, t + \Delta t, w) + \widetilde{b}^*\big(v_{k+1}, \theta_k(t + \Delta t, w)\big).$$

Since b_k is a real-valued lower semicontinuous function, $t \mapsto \widetilde{\Delta T}_k(t)$ is an upper semicontinuous point-to-set mapping and $(t, \Delta t) \mapsto \Omega\big((v_k, t), (\Delta t, e_{k+1})\big)$ is a lower semicontinuous point-to-set mapping, Lemma 6.3.2 and [65, Theorem 2.2.1] imply that $t \mapsto \widetilde{b}^*(v_k, t)$ is lower semicontinuous on $\widetilde{T}_k$. Moreover, as $\widetilde{\Delta T}_k(t)$ is compact and nonempty for each $t \in \widetilde{T}_k$, the minimum in (6.16) is attained by some $\Delta t_k^*(t)$, $k = 0, ..., n-1$.
Next, if $N \in \mathbb{N}$, we observe that any control sequence u with $|u| \geq N|V| + |V| - 1$ contains at least N circles, which implies

$$\mathcal{B}\big((v_0, t_0), u, \mathcal{W}\big) \geq (|V| - 1)\underline{\mathcal{B}} + N\underline{\mathcal{B}}^\circ, \qquad \forall \mathcal{W} \in \mathcal{W}. \tag{6.17}$$

The rest of the proof follows as in the proof of Theorem 4.2.4 by using (6.17) instead of (4.6). □

For the remainder of this chapter, we assume that $\Omega(x, u)$ is compact for each $x \in X$ and each $u \in U(x)$. This allows us to replace the supremum in (6.16) by a maximum. In the following propositions we establish the principle of dynamic programming for the time-dependent robust optimal path problem.

Proposition 6.3.5 *Suppose that Assumptions 6.3.3 holds and that a goal node $v' \in V$ is given. The optimal value function b^* defined in (6.8) satisfies the following dynamic* *programming equations:*

$$
\begin{aligned}
b^*(v',t) &= 0, && \forall t \in T(v'),\\
b^*(v,t) &= \min_{\substack{u\in U(v,t)\\ u=(\Delta t,e)}} \max_{w\in\Omega((v,t),u)} \Big[\delta(v,t,\Delta t,w) + \beta(e,t+\Delta t,w) + b^*\big(\varphi((v,t),u,w)\big)\Big],\\
& && \forall v \in V\setminus\{v'\}, t\in T(v).
\end{aligned}
$$

Proof As a consequence of (6.13) and since $\underline{\mathcal{B}}^\circ > 0$, we observe that $b^*(v',t) \geq 0$ for all $t \in T(v')$, and that the termination of the path from v' to v' in the initial state leads to the optimal cost $b^*(v',t) = 0$ for all $t \in T(v')$. Since we have already proved the existence of optimal paths in Theorem 6.3.4, the result follows from standard arguments (see, e.g., [26, Proposition 3.1.1]) by replacing expectation by maximization and using Lemma 6.2.2. □

We next consider the dynamic programming equations under the assumption that the set of possible scenarios is independent. We denote

$$
\begin{aligned}
\Theta &: E\times\mathbb{R} \to \mathcal{P}(\mathbb{R}), & \Theta(e,t) &= \bigcup_{w\in\Omega((v,t),(0,e))} \{t+\tau(e,t,w)\},\\
\overline{\beta} &: E\times\mathbb{R}\to\mathbb{R}, & \overline{\beta}(e,t) &= \max_{w\in\Omega((v,t),(0,e))} \beta(e,t,w),\\
\overline{\delta} &: V\times\mathbb{R}\times\mathbb{R}_0^+\to\mathbb{R}, & \overline{\delta}(v,t,\Delta t) &= \max_{w\in\Omega((v,t),(\Delta t,e))} \delta(v,t,\Delta t,w).
\end{aligned}
$$

Proposition 6.3.6 *Suppose that Assumption 6.3.3 holds, that the set of possible scenarios $\mathcal{W}$ is independent and that a goal node $v' \in V$ is given. The optimal value function b^* defined in (6.8) satisfies the following dynamic programming equations:*

$$
\begin{aligned}
b^*(v',t) &= 0, && \forall t\in T(v'),\\
b^*(v,t) &= \min_{\substack{u\in U(v,t)\\ u=(\Delta t,e)}} \Big[\overline{\delta}(v,t,\Delta t) + \overline{\beta}(e,t+\Delta t) + \max_{\theta\in\Theta(e,t+\Delta t)} b^*\big(\omega(e),\theta\big)\Big],\\
& && \forall v\in V\setminus\{v'\}, t\in T(v).
\end{aligned}
$$

Proof Recall that, if $\mathcal{W}$ is independent, then $\mathcal{W}$ has the DP-property, cp. Remark 6.2.5. Using Proposition 6.3.5 and Definition 6.2.4 *(i)*, we obtain

$$
\begin{aligned}
b^*(v,t) = \min_{\substack{u\in U(v,t)\\ u=(\Delta t,e)}} \Big[&\max_{w\in\Omega((v,t),u)} \delta(v,t,\Delta t,w) + \max_{w\in\Omega((v,t),u)} \beta(e,t+\Delta t,w)\\
&+ \max_{w\in\Omega((v,t),u)} b^*\big(\varphi((v,t),u,w)\big)\Big]. \qquad \forall v\in V\setminus\{v'\}, t\in T(v).
\end{aligned}
$$

Now, Definition 6.2.4 *(ii)* implies

$$
b^*(v,t) = \min_{\substack{u\in U(v,t)\\ u=(\Delta t,e)}} \Big[\overline{\delta}(v,t,\Delta t) + \overline{\beta}(e,t+\Delta t) + \max_{\theta\in\Theta(e,t+\Delta t)} b^*\big(\omega(e),\theta\big)\Big],
$$

for all $v \in V\setminus\{v'\}$ and all $t\in T(v)$. □

6.4. Properties of the Optimal Value Function

We will now apply some of the concepts developed in Chapter 5 to the problem setting of Section 6.2. First, the continuity of the optimal value function is established under similar assumptions to those in Section 5.1 by using the following result.

Assumption 6.4.1 *Let $v_0 \in V$ be a fixed source node and suppose that $T(v_0) = [\underline{t}, \infty)$ for some $\underline{t} \in \mathbb{R}$. Suppose further that $\tau, \Delta T, \Omega$ are continuous, that there exists a $\underline{\mathcal{T}}^\circ > 0$ such that (6.9) holds and that the set of possible scenarios $\mathcal{W}$ has the DP-property. Denote $\widetilde{T}(v_0) = T(v_0)$ and $\widetilde{T}(v) = \mathbb{R}$ for all $v \in V \setminus \{v_0\}$ and let $\widetilde{U}(v,t)$ denote the set of control sequences which are admissible for (v,t) in $(G, \widetilde{T}, \Delta T)$. For each $v \in V$, let*

$$\widetilde{\underline{t}}_R(v) = \inf_{t_0 \in T(v_0)} \inf_{\substack{u \in \widetilde{U}(v_0,t_0):\\ \omega(u)=v}} \inf_{\mathcal{W} \in \mathcal{W}} t_0 + \mathcal{T}\big((v_0,t_0), u, \mathcal{W}\big)$$

denote the earliest arrival time at v in the time-dependent network $(G, \widetilde{T}, \Delta T)$. Suppose that $T(v) \supset [\widetilde{\underline{t}}_R(v), \infty)$ for all $v \in V$.

Lemma 6.4.2 *Let v_0 be a fixed source node and suppose that Assumption 6.4.1 holds. Then, for all $v \in V$ and all $t \geq \widetilde{\underline{t}}_R(v)$, there holds $U(v,t) = \widetilde{U}(v,t)$. In particular, if $(v', t') = \varphi\big((v,t), u\big)$ for some $u \in U(v,t)$, $t \geq \widetilde{\underline{t}}_R(v)$, then $t' \geq \widetilde{\underline{t}}_R(v')$.*

Proof In a similar manner as in the proof of Lemma 6.2.2, we obtain that, for each $t_0 \in T(v_0)$ and each $u = \big((\Delta t_k, e_k)\big)_{k=1,\dots,n} \in \widetilde{U}(v_0, t_0)$, there holds

$$\inf_{\mathcal{W} \in \mathcal{W}} \mathcal{T}(x_0, u, \mathcal{W}) = \inf_{w_1 \in \Omega(x_0,u_1)} \cdots \inf_{w_n \in \Omega(x_{n-1},u_n)} \sum_{k=1}^{n} \mathcal{T}_k(x_{k-1}, u_k, w_k), \tag{6.18}$$

where

$$\begin{aligned} x_k = (v_k, t_k) &= \varphi(x_{k-1}, u_k, w_k), && k = 1, ..., n, \\ \mathcal{T}_k(x_{k-1}, u_k, w_k) &= \Delta t_k + \tau\big(e_k, t_{k-1} + \Delta t_k, w_k\big), && k = 1, ..., n. \end{aligned}$$

Using (6.18) and the continuity of Ω, the result follows in a similar manner as in the proof of Lemma 5.1.2. □

In the remainder of this section, we assume that a goal node $v' \in V$ is given.

Theorem 6.4.3 *Suppose that ΔT and Ω are continuous point-to-set mappings, τ, β, δ are continuous and there exist $\underline{\mathcal{B}}, \underline{\mathcal{B}}^\circ, \underline{\mathcal{T}}^\circ \in \mathbb{R}$, $\underline{\mathcal{B}}^\circ, \underline{\mathcal{T}}^\circ > 0$, such that (6.9), (6.12) and (6.13) hold. Suppose that the set of possible scenarios $\mathcal{W}$ has the DP-property.*

- *(i) Let a source node $v_0 \in V$ be given and let Assumption 6.4.1 hold, then the partial mapping $t_0 \mapsto b^*(v_0, t_0)$ is continuous on $T_R(v_0)$.*
- *(ii) If $X = V \times \mathbb{R}$ and (V, E) is strongly connected, then b^* is continuous.*

Proof The result follows in a similar manner as in Theorem 5.1.3 by using Lemma 6.4.2 instead of Lemma 5.1.2 and by applying [65, Theorem 2.2.2] to both the maximization with respect to $w \in \Omega\big((v,t), (\Delta t, e)\big)$ and the minimization with respect to $\Delta t \in \Delta T(v,t)$, cp. (6.16). □

In the preceding results of this chapter, the set of possible states of the network W was assumed to be an arbitrary topological space. We will now consider the computation of the optimal value function in the cases in which $W \subset \mathbb{R}^n$ for some $n \in \mathbb{N}$ and in which the network and constraint functions are piecewise analytic and piecewise linear, respectively. In general, we must then solve a problem of the form (6.16). In particular, we must maximize with respect to a variable $w \in \mathbb{R}^n$, at which the constraint function Ω depends on the pair $(t, \Delta t) \in \mathbb{R}^2$. Hence, the resulting problem is necessarily of a higher dimension than the problems which we have analyzed in Chapter 5. Although we believe that, at least in the case of piecewise linear network functions, a similar analysis as in Chapter 5 can be carried out for the time-dependent absolute robust optimal path problem, a number of notational and technical difficulties must be solved first. For this reason, we will not further consider this general problem setting in this thesis and assume that $W \subset \mathbb{R}$ and that the set of possible scenarios $\mathcal{W}$ is independent. Using the simplifications resulting from Proposition 6.3.6, we will see that an analysis of the time-dependent absolute robust optimal path problem can be carried out with the notation and the techniques of Chapter 5. The general problem is left as a topic for future research.
In the remainder of this chapter, we assume that $\Theta(e,t) = [\,\underline{\theta}(e,t), \overline{\theta}(e,t)\,]$ for some $\underline{\theta}, \overline{\theta}$: $E \times \mathbb{R} \to \mathbb{R}$ with $\underline{\theta}(e,t) \le \overline{\theta}(e,t)$ for all $(e,t) \in E \times \mathbb{R}$. Moreover, we assume that $\Delta T(x) = [\,\underline{\Delta T}(x), \overline{\Delta T}(x)\,]$ for some $\underline{\Delta T}, \overline{\Delta T} : X \to \mathbb{R}_0^+$ with $\underline{\Delta T}(x) \le \overline{\Delta T}(x)$ for all $x \in X$. Note that these assumptions are due to keeping the notation as simple as possible, and that the following results would also hold if Θ and ΔT were a finite union of such interval-valued point-to-set mappings. Using the techniques of the previous chapter, we obtain:

Theorem 6.4.4 *Suppose that the set of possible scenarios $\mathcal{W}$ is independent, $\underline{\theta}, \overline{\theta}, \overline{\beta} \in \mathcal{PC}^\omega(E \times \mathbb{R})$, $\overline{\delta} \in \mathcal{PC}^\omega(V \times \mathbb{R} \times \mathbb{R}_0^+)$, $\underline{\Delta T}, \overline{\Delta T} \in \mathcal{PC}^\omega(X)$ and there exist $\underline{\mathcal{B}}, \underline{\mathcal{B}}^\circ, \underline{\mathcal{T}}^\circ \in \mathbb{R}$, $\underline{\mathcal{B}}^\circ, \underline{\mathcal{T}}^\circ > 0$, such that (6.9), (6.12) and (6.13) hold.*

(i) Let a source node $v_0 \in V$ be given and let Assumption 6.4.1 hold, then the partial mapping $t_0 \mapsto b^*(v_0, t_0)$ is in $\mathcal{PC}^{1,\omega}\big(T_R(v_0)\big)$.

(ii) If $X = V \times \mathbb{R}$ and (V, E) is strongly connected, then $b^* \in \mathcal{PC}^{1,\omega}(X)$.

Proof We first assume that $\underline{\theta}, \overline{\theta}, \overline{\beta} \in \mathcal{C}^\omega(E \times \mathbb{R})$, $\overline{\delta} \in \mathcal{C}^\omega(V \times \mathbb{R} \times \mathbb{R}_0^+)$, $\underline{\Delta T}, \overline{\Delta T} \in \mathcal{C}^\omega(X)$. We proceed in a similar manner as in the proof of Theorem 4.2.4. Let $(e_1, ..., e_n)$ denote a finite, connected edge sequence from v_0 to v'. Denote $v_k = \omega(e_k)$, $k = 1, ..., n$, and $\widetilde{T}_k, \widetilde{\Delta T}_k$ as in (6.14), (6.15), $k = 0, ..., n-1$. Lemma 6.4.2 implies that $\widetilde{\Delta T}_k(t) = \Delta T(v_k, t)$ for all $t \in \widetilde{T}_k$ and hence $\widetilde{T}_k = \operatorname{supp}(\widetilde{\Delta T}_k) = T(v_k)$, $k = 0, ..., n-1$. Along the edge sequence $(e_1, ..., e_n)$ we must solve the following parametric minmax problem for $k = 0, ..., n-1$, cp. (6.16) and Proposition 6.3.6:

$$\widetilde{b}^*(v_k, t) = \inf_{\Delta t \in \widetilde{\Delta T}_k(t)} \Big[\overline{\delta}(v, t, \Delta t) + \overline{\beta}(e, t + \Delta t) + \max_{\theta \in \Theta(t+\Delta t)} \widetilde{b}^*(v_{k+1}, \theta)\Big], \quad \forall t \in \widetilde{T}_k, \quad (6.19)$$

where $\widetilde{b}^*(v_n, t) \equiv 0$ is in $\mathcal{C}^\omega(\widetilde{T}_n) \subset \mathcal{PC}^{1,\omega}(\widetilde{T}_n)$.
We will now prove by backwards induction that $t \mapsto \widetilde{b}^*(v_k, t)$ is in $\mathcal{PC}^{1,\omega}(\widetilde{T}_k)$. Let $k \in \{0, ..., n-1\}$ and assume that $t \mapsto \widetilde{b}^*(v_{k+1}, t)$ is in $\mathcal{PC}^{1,\omega}(\widetilde{T}_{k+1})$. According to Remark 5.2.4, it is sufficient to prove that, for an arbitrary compact interval $K \subset \widetilde{T}_k$, $t \mapsto \widetilde{b}^*(v_k, t)$ is in

$\mathcal{PC}^{1,\omega}(K)$. Since ΔT is continuous, $\operatorname{graph}(\Delta T|_{\{v_k\}\times K})$ is compact and the set

$$T'_k = \{t + \Delta t \in \mathbb{R} : (v_k, t, \Delta t) \in \operatorname{graph}(\Delta T|_{\{v_k\}\times K})\}$$

is a compact interval. Similarly, since Θ is continuous, $\operatorname{graph}(\Theta|_{\{e_{k+1}\}\times T'_k})$ is compact and the set

$$T''_{k+1} = \{\theta \in \mathbb{R} : (e_{k+1}, t', \theta) \in \operatorname{graph}(\Theta|_{\{e_{k+1}\}\times T'_k})\}$$

is a compact interval. (Recall that Lemma 6.4.2 implies that $T''_{k+1} \subset T(v_{k+1})$.) Using the fact that $t \mapsto \widetilde{b}^*(v_{k+1}, t)$ is in $\mathcal{PC}^{1,\omega}(\widetilde{T}_{k+1})$, there exists a decomposition of T''_{k+1} into $I_k \in \mathbb{N}$ compact intervals $T''_{k+1,i}$, $i = 1, ..., I_k$, such that $t \mapsto \widetilde{b}^*(v_{k+1}, t)$ is in $\mathcal{C}^{1,\omega}(T''_{k+1,i})$ for each $i \in \{1, ..., I_k\}$. For $i \in \{1, ..., I_k\}$ we denote $\Theta_{k,i} : T'_k \mapsto \mathcal{P}(\mathbb{R})$,

$$\Theta_{k,i}(t') = [\,\underline{\theta}(e_{k+1}, t'), \overline{\theta}(e_{k+1}, t')\,] \cap T''_{k+1,i}.$$

For all $i \in \{1, ..., I_k\}$, as $\underline{\theta}, \overline{\theta} \in \mathcal{C}^\omega(E \times \mathbb{R})$ and T'_k is compact, the set $\operatorname{supp}(\Theta_{k,i})$ is the finite union of $J_{k,i} \in \mathbb{N}$ compact intervals $T'_{k,ij}$, $j = 1, ..., J_{k,i}$, for each of which there exist $\underline{\theta}_{k,ij}, \overline{\theta}_{k,ij} \in \mathcal{C}^\omega(T'_{k,ij})$, such that $\Theta_{k,i}(t') = [\underline{\theta}_{k,ij}(t'), \overline{\theta}_{k,ij}(t')]$ for all $t' \in T'_{k,ij}$ (cf. [118, Corollary 1.2.6]). According to the construction, we obtain

$$T'_k = \bigcup_{i=1}^{I_k} \bigcup_{j=1}^{J_{k,i}} T'_{k,ij}, \qquad \Theta(e_k, t') = \bigcup_{i=1}^{I_k} \bigcup_{\substack{j\in\{1,...,J_{k,i}\}:\\ t'\in T'_{k,ij}}} [\underline{\theta}_{k,ij}(t'), \overline{\theta}_{k,ij}(t')], \quad \forall t' \in T'_k.$$

Lemma 5.2.12 implies that the function $f_{k,ij} : T'_{k,ij} \to \mathbb{R}$,

$$f_{k,ij}(t') = \max_{\theta\in[\,\underline{\theta}_{k,ij}(t'),\overline{\theta}_{k,ij}(t')]} \widetilde{b}^*(v_{k+1}, \theta) = -\min_{\theta\in[\,\underline{\theta}_{k,ij}(t'),\overline{\theta}_{k,ij}(t')]} -\widetilde{b}^*(v_{k+1}, \theta)$$

satisfies $f_{k,ij} \in \mathcal{PC}^{1,\omega}(T'_{k,ij})$. Next, we define $f_k : T'_k \to \mathbb{R}$,

$$f_k(t') = \min_{\substack{i\in\{1,...,I_k\}, j\in\{1,...,J_{k,i}\}:\\ t'\in T'_{k,ij}}} f_{k,ij}(t').$$

Using Lemma 5.2.9 on every nonempty set of the form $\bigcap_{i\in\mathcal{I}} \bigcap_{j\in\mathcal{J}_i} T'_{k,ij}$ with $\mathcal{I} \subset \{1, ..., I_k\}$ and $\mathcal{J}_i \subset \{1, ..., J_{k,i}\}$, we obtain that f_k is in $\mathcal{PC}^{1,\omega}(T'_k)$. Finally, we observe that (6.19) can be written as

$$\widetilde{b}^*(v_k, t) = \inf_{\Delta t\in\widetilde{\Delta T}_k(t)} \Big[\overline{\delta}(v, t, \Delta t) + \overline{\beta}(e, t + \Delta t) + f_k(t + \Delta t)\Big], \quad t \in \widetilde{T}_k.$$

Now the remaining part of the proof follows in a similar manner as in the proof of Theorem 5.2.14. □

We next consider the piecewise linear case. As in Subsection 5.3.2, and without loss of generality, we assume that the network functions are continuous. Before proving the main results, we establish a few preliminary lemmas:

Lemma 6.4.5 *Let $T, T' \subset \mathbb{R}$ be closed intervals or points, let $\underline{\theta}, \overline{\theta} \in \mathcal{PL}_c^1(T)$ be such that $\underline{\theta}(t) \le \overline{\theta}(t)$ for all $t \in T$ and such that the decomposition of T with respect to $\underline{\theta}$ and $\overline{\theta}$ is identical. Denote $\#\underline{\theta} = \#\overline{\theta} = (N_{0,0}, N_{0,1}, N_1)$ and define*

$$\widetilde{T} = \{t \in T : [\,\underline{\theta}(t), \overline{\theta}(t)] \subset T'\}.$$

If either T is a point or $\underline{\theta}, \overline{\theta}$ are monotonically increasing, then $\widetilde{T}$ consists of at most one connected component. Otherwise $\widetilde{T}$ consists of at most $2N_{0,1} + 1$ connected components.

Proof We first show that

$$\widetilde{T} = \underline{\theta}^{-1}(T') \cap \overline{\theta}^{-1}(T'). \tag{6.20}$$

Suppose that $\widetilde{t} \in \underline{\theta}^{-1}(T') \cap \overline{\theta}^{-1}(T')$. Then $\underline{\theta}(\widetilde{t}), \overline{\theta}(\widetilde{t}) \in T'$, and since T' is connected, there holds $[\,\underline{\theta}(\widetilde{t}), \overline{\theta}(\widetilde{t})] \subset T'$. Conversely, suppose that $\widetilde{t} \notin \underline{\theta}^{-1}(T') \cap \overline{\theta}^{-1}(T')$. Then $\{\underline{\theta}(\widetilde{t}), \overline{\theta}(\widetilde{t})\} \not\subset T'$, which implies that $[\,\underline{\theta}(\widetilde{t}), \overline{\theta}(\widetilde{t})] \not\subset T'$. Consequently, (6.20) holds.
If either T is a point (which implies $N_{0,0} = 1$ and $N_1 = 0$) or $\underline{\theta}, \overline{\theta}$ are monotonically increasing, then Lemma 5.3.6 (iii) implies that both $\underline{\theta}^{-1}(T')$ and $\overline{\theta}^{-1}(T')$ consist of at most one connected component. The intersection of two connected sets is a connected set.
Let us assume that $\operatorname{int}(T) \ne \emptyset$ and that $\underline{\theta}, \overline{\theta}$ are not monotonically increasing. Now Lemma 5.3.6 *(iii)* implies that both $\underline{\theta}^{-1}(T')$ and $\overline{\theta}^{-1}(T')$ consist of at most $N_{0,1} + 1$ connected components. Each left (resp., right) boundary point of a connected component of $\widetilde{T}$ must be a left (resp., right) boundary point of either $\underline{\theta}^{-1}(T')$ or $\overline{\theta}^{-1}(T')$, and hence $\widetilde{T}$ consists of at most $2N_{0,1} + 2$ connected components. However $\min \underline{\theta}^{-1}(T')$ and $\min \overline{\theta}^{-1}(T')$ (resp., $\max \underline{\theta}^{-1}(T')$ and $\max \overline{\theta}^{-1}(T')$) can only both be left (resp., right) boundary points of $\widetilde{T}$ if they coincide. Hence, $\widetilde{T}$ consists of at most $2N_{0,1} + 1$ connected components. □

Lemma 6.4.6 *Let $T \subset \mathbb{R}$ be a closed interval with $\operatorname{int}(T) \ne \emptyset$, let $T' \subset \mathbb{R}$ be the finite union of closed intervals and points, let $\underline{\theta}, \overline{\theta}, g \in \mathcal{PL}_c^1(T)$ be such that $\underline{\theta}(t) \le \overline{\theta}(t)$ for all $t \in T$ and such that the decomposition of T with respect to $\underline{\theta}, \overline{\theta}, g$ is identical. Let $f \in \mathcal{PL}_c^1(T')$. Denote $\#\underline{\theta} = \#\overline{\theta} = \#g = (0, N_{0,1}^\theta, N_1^\theta)$, $\#f = (N_{0,0}^f, N_{0,1}^f, N_1^f)$, $N_0^f = N_{0,0}^f + N_{0,1}^f$ and define*

$$\widetilde{T} = \{t \in T : [\,\underline{\theta}(t), \overline{\theta}(t)] \subset T'\}.$$

Then the function $f^* : \widetilde{T} \to \mathbb{R}$,

$$f^*(t) = g(t) + \max_{\theta \in [\,\underline{\theta}(t), \overline{\theta}(t)]} f(\theta),$$

satisfies $f^* \in \mathcal{PL}_c^1(\widetilde{T})$ and there holds

$$\#f^* \le (\, N_{0,1}^\theta + 2(N_{0,1}^\theta + 1)N_0^f,\ 3N_{0,1}^\theta + 6(N_{0,1}^\theta + 1)N_0^f + 2,\ 3N_{0,1}^\theta + 6(N_{0,1}^\theta + 1)N_0^f + 3\,),$$

and f^* can be computed from $f, g, \underline{\theta}, \overline{\theta}$ in $\mathcal{O}\big(N_{0,1}^\theta (N_0^f)^2\big)$ arithmetic operations.
If $\underline{\theta}, \overline{\theta}$ are monotonically increasing, then

$$\#f^* \le (\, N_{0,1}^\theta + 4N_0^f,\ 3N_{0,1}^\theta + 12N_0^f + 2,\ 3N_{0,1}^\theta + 12N_0^f + 3\,),$$

and f^* can be computed from $f, g, \underline{\theta}, \overline{\theta}$ in $\mathcal{O}\big(N_{0,1}^\theta N_0^f + (N_0^f)^2\big)$ arithmetic operations.

Proof Let $\#f^* = (N_{0,0}^{f^*}, N_{0,1}^{f^*}, N_1^{f^*})$, let $\{t'_{n'_0}\}_{n'_0=1,...,N_0^f}$, $\{T'_{n'_1}\}_{n'_1=1,...,N_1^f}$ be the decomposition of T' with respect to f and let the decomposition of T with respect to $\underline{\theta}$ and $\overline{\theta}$ be given by $\{t_{n_0}\}_{n_0=1,...,N_{0,1}^\theta}$, $\{T_{n_1}\}_{n_1=1,...,N_1^\theta}$. Assume that g is the zero function.
Let $T'_{\arg\max} \subset T'$ denote the set of local maxima of f and let

$$T^* = \mathrm{cl}(T'_{\arg\max}) \cap \bigcup_{n'_0=1}^{N_0^f} \{t'_{n'_0}\}.$$

We claim that, for each $t \in \widetilde{T}$, there holds

$$f^*(t) = \max\Big\{ f\big(\underline{\theta}(t)\big), f\big(\overline{\theta}(t)\big), \max_{t' \in T^* \cap [\underline{\theta}(t), \overline{\theta}(t)]} f(t') \Big\}. \tag{6.21}$$

We obviously have

$$f^*(t) = \max\Big\{ f\big(\underline{\theta}(t)\big), f\big(\overline{\theta}(t)\big), \max_{t' \in T'_{\arg\max} \cap [\underline{\theta}(t), \overline{\theta}(t)]} f(t') \Big\}.$$

Suppose that $t^* \in (T'_{\arg\max} \setminus T^*) \cap [\underline{\theta}(t), \overline{\theta}(t)]$ for some $t \in \widetilde{T}$. Then $t^* \in T'_{n'_1}$ for some $n'_1 \in \{1, ..., N_1^f\}$, f is constant on $T'_{n'_1}$, and consequently $f(t^*) = f(t'_{\underline{n}'_0}) = f(t'_{\overline{n}'_0})$ for $\underline{n}'_0, \overline{n}'_0 \in \{1, ..., N_{0,1}^f\}$ with $t'_{\underline{n}'_0}, t'_{\overline{n}'_0} \in \mathrm{bd}(T'_{n'_1})$ according to the continuity of f. Next, we observe that we either have $\{\underline{\theta}(t), \overline{\theta}(t)\} \cap T'_{n'_1} \neq \emptyset$, or $t'_{\underline{n}'_0}, t'_{\overline{n}'_0} \in [\underline{\theta}(t), \overline{\theta}(t)]$. In any case, there holds

$$f(t^*) = \max_{t' \in T^* \cap [\underline{\theta}(t), \overline{\theta}(t)]} f(t'),$$

which implies (6.21).
In a similar manner as in the proof of Lemma 5.3.6 *(iv)*, it follows that the set of breakpoints of $f \circ \underline{\theta}, f \circ \overline{\theta} : \widetilde{T} \to \mathbb{R}$ is contained in

$$\underline{T} = \bigcup_{n_0=1}^{N_{0,1}^\theta} \{t_{n_0}\} \cup \bigcup_{n'_0=1}^{N_0^f} \mathrm{bd}\big(\underline{\theta}^{-1}(\{t'_{n'_0}\})\big), \qquad \overline{T} = \bigcup_{n_0=1}^{N_{0,1}^\theta} \{t_{n_0}\} \cup \bigcup_{n'_0=1}^{N_0^f} \mathrm{bd}\big(\overline{\theta}^{-1}(\{t'_{n'_0}\})\big),$$

respectively. According to the construction of T^*, the set of breakpoints of the upper semicontinuous and piecewise constant function $f_{\max} : \widetilde{T} \to \mathbb{R}$,

$$f_{\max}(t) = \max_{t' \in T^* \cap [\underline{\theta}(t), \overline{\theta}(t)]} f(t')$$

is also contained in $\underline{T} \cup \overline{T}$. Finally, we observe that, if $\widetilde{t}$ is a boundary point of $\widetilde{T}$, then $\widetilde{t} \in \underline{\theta}^{-1}\big(\mathrm{bd}(T')\big) \cup \overline{\theta}^{-1}\big(\mathrm{bd}(T')\big) \subset \underline{T} \cup \overline{T}$.
Since each connected component of $\underline{\theta}^{-1}(\{t'_{n'_0}\})$ and $\overline{\theta}^{-1}(\{t'_{n'_0}\})$ contains at most two boundary points, $n'_0 = 1, ..., N_0^f$, Lemma 5.3.6 *(iii)* yields that the number of breakpoints N_0 in the common decomposition of $f \circ \underline{\theta}, f \circ \overline{\theta}, f_{\max}$ satisfies

$$N_0 \leq |\underline{T} \cup \overline{T}| \leq N_{0,1}^\theta + 2(N_{0,1}^\theta + 1)N_0^f,$$

and if $\underline{\theta}, \overline{\theta}$ are monotonically increasing, then

$$N_0 \leq |\underline{T} \cup \overline{T}| \leq N_{0,1}^{\theta} + 4N_0^f.$$

Obviously, there holds $N_{0,0}^{f^*} \leq N_0$. From Lemma 5.3.5 we obtain that the number of intervals N_1 in the common decomposition of $f \circ \underline{\theta}, f \circ \overline{\theta}, f_{\max}$ satisfies

$$N_1 \leq N_{0,1}^{\theta} + 2(N_{0,1}^{\theta} + 1)N_0^f + 1,$$

and if $\underline{\theta}, \overline{\theta}$ are monotonically increasing, then

$$N_1 \leq N_{0,1}^{\theta} + 4N_0^f + 1.$$

Using the continuity of f^* (cf. [65, Theorem 2.2.8]) and Lemma 5.3.7 on each of these intervals (by adding only the additional number of breakpoints on each interval) we obtain that

$$N_{0,1}^{f^*} \leq 3N_{0,1}^{\theta} + 6(N_{0,1}^{\theta} + 1)N_0^f + 2, \qquad N_1^{f^*} \leq 3N_{0,1}^{\theta} + 6(N_{0,1}^{\theta} + 1)N_0^f + 3,$$

and if $\underline{\theta}, \overline{\theta}$ are monotonically increasing, then

$$N_{0,1}^{f^*} \leq 3N_{0,1}^{\theta} + 12N_0^f + 2, \qquad N_1^{f^*} \leq 3N_{0,1}^{\theta} + 12N_0^f + 3.$$

If g is not the zero function, then the set of breakpoints of g is also contained in $\underline{T} \cup \overline{T}$. Hence, the result also holds if $g \neq 0$.
Finally, in order to compute f^*, we first compute the functions $f \circ \underline{\theta}, f \circ \overline{\theta}, f_{\max}$. According to Lemma 5.3.6 *(iv)*, *(v)*, we can generally compute $f \circ \underline{\theta}, f \circ \overline{\theta}$ in $\mathcal{O}(N_{0,1}^{\theta} N_0^f)$ arithmetic operations, and we can compute $f \circ \underline{\theta}, f \circ \overline{\theta}$ in $\mathcal{O}\big(\log(N_{0,1}^{\theta})N_0^f\big)$ arithmetic operations if $\underline{\theta}, \overline{\theta}$ are monotone increasing. In order to compute $f_{\max}$ at some $t \in \underline{T} \cup \overline{T}$, we must first determine $T^* \cap [\,\underline{\theta}(t), \overline{\theta}(t)]$. Using the array representation of f, we can determine the indices (i.e., the minimal and the maximal index) of the the breakpoints of f in $T^* \cap [\,\underline{\theta}(t), \overline{\theta}(t)]$ in $\mathcal{O}\big(\log(N_0^f)\big)$ by bisection. The maximizing argument of f in $T^* \cap [\,\underline{\theta}(t), \overline{\theta}(t)]$ can then be determined in $\mathcal{O}(N_0^f)$ arithmetic operations. We must repeat this procedure for each of the N_0 points in $\underline{T} \cup \overline{T}$, which generally yields a total of $\mathcal{O}\big(N_{0,1}^{\theta}(N_0^f)^2\big)$ arithmetic operations, and which yields a total of $\mathcal{O}\big(N_{0,1}^{\theta}N_0^f + (N_0^f)^2\big)$ arithmetic operations if $\underline{\theta}, \overline{\theta}$ are monotone increasing. The computation of f^* on each interval in the common decomposition of $\widetilde{T}$ with respect to $f \circ \underline{\theta}, f \circ \overline{\theta}, f_{\max}$ can be carried out in $\mathcal{O}(3^2) = \mathcal{O}(1)$ arithmetic operations according to Lemma 5.3.7. This yields the desired result. □

For the remainder of this section, we assume that, for each $e \in E$, the partial mappings $\underline{\theta}_e : \mathbb{R} \to \mathbb{R}$, $\underline{\theta}_e(t) = \underline{\theta}(e,t)$, $\overline{\theta}_e : \mathbb{R} \to \mathbb{R}$, $\overline{\theta}_e(t) = \overline{\theta}(e,t)$ and $\overline{\beta}_e : \mathbb{R} \to \mathbb{R}$, $\overline{\beta}_e(t) = \overline{\beta}(e,t)$ are in $\mathcal{PL}_c^1(\mathbb{R})$ with an identical decomposition of $\mathbb{R}$ according to the discussion preliminary to Definition 5.3.3. Furthermore, we assume that $T(v)$ is a closed interval for all $v \in V$. We also assume that the partial functions $t \mapsto \underline{\Delta T}_v(t) = \underline{\Delta T}(v,t)$ and $t \mapsto \overline{\Delta T}_v(t) = \overline{\Delta T}(v,t)$ are in $\mathcal{PL}_c^1\big(T(v)\big)$ for all $v \in V$. For each $v \in V$, we denote the graph of the partial point-to-set mapping $t \mapsto \Delta T(v,t)$ by Θ_v. Moreover, we assume that, for each $v \in V$, the partial functions $\overline{\delta}_v : \Theta_v \to \mathbb{R}$, $\overline{\delta}_v(t, \Delta t) = \overline{\delta}(v, t, \Delta t)$, are in $\mathcal{PL}_c^2(\Theta_v)$. Finally, we assume that there exists a $C \in \mathbb{N}$, $C \geq 2$, such that

$$\begin{aligned} &\#\underline{\theta}_e \leq (0, C-1, C), && \#\overline{\theta}_e \leq (0, C-1, C), \quad \#\overline{\beta}_e \leq (0, C-1, C), && \forall e \in E, \\ &\#\underline{\Delta T}_v \leq (1, C, C), && \#\overline{\Delta T}_v \leq (1, C, C), && \forall v \in V, \\ &\#\overline{\delta}_v \leq (1, C, C, C, C, C, C), && && \forall v \in V. \end{aligned}$$

Note that, if $C = 1$, then the partial network functions $\{\underline{\theta}_e\}_{e\in E}$, $\{\overline{\theta}_e\}_{e\in E}$, $\{\overline{\beta}_e\}_{e\in E}$ would be constant.

Lemma 6.4.7 *Let $v_0, v' \in V$ be given and suppose that Assumption 6.3.3 holds. Let $(e_1, ..., e_n) \in E^n$ be a connected edge sequence with $\alpha(e_1) = v_0$ and $\omega(e_n) = v'$, let*

$$\widetilde{T}_0 = \{t_0 \in T(v_0) : \exists u = \big((\Delta t_k, e_k)\big)_{k=1,\dots,n} \in U(v_0, t_0)\},$$

$N_{\mathrm{bd}} = \big|\mathrm{bd}\big(T(v')\big)\big|$ *and let* $\widetilde{b}_0^* : \widetilde{T}_0 \mapsto \mathbb{R}$,

$$\widetilde{b}_0^*(t_0) = \inf\Big\{\mathcal{B}\big((v_0, t_0), u\big) : u = \big((\Delta t_k, e_k)\big)_{k=1,\dots,|u|} \in U(v_0, t_0)\Big\}$$

denote the optimal cost function along the edge sequence $(e_1, ..., e_n)$. Then $\widetilde{b}_0^* \in \mathcal{PL}_c^1(\widetilde{T}_0)$ and, denoting $\#\widetilde{b}_0^* = (N_{0,0}^0, N_{0,1}^0, N_1^0)$, we obtain:

(i) There holds $N_{0,0}^0 \leq (2C^2)^n$ and $N_{0,1}^0, N_1^0$ are of order

$$\mathcal{O}\Big(C^{3\cdot 2^n - 3}\big(8N_{\mathrm{bd}} + 6 + 16C^2\big)^{2^n}\Big).$$

Moreover, $\widetilde{b}_0^*$ can be computed in

$$\mathcal{O}\bigg(C^{9\cdot 2^{n-1} - 5}\Big(8N_{\mathrm{bd}} + 6 + 16C^2\Big)^{3\cdot 2^{n-1}}\bigg)$$

arithmetic operations.

(ii) If $X = V \times \mathbb{R}$, then $N_{0,0}^0 = 0$ and $N_{0,1}^0, N_1^0$ are of order

$$\mathcal{O}\big(C^{3\cdot 2^n - 3}5^{2^n}\big).$$

Moreover, $\widetilde{b}_0^*$ can be computed in

$$\mathcal{O}\big(C^{9\cdot 2^{n-1} - 5}5^{3\cdot 2^{n-1}}\big)$$

arithmetic operations.

(iii) If $t \to \underline{\theta}(e, t), t \to \overline{\theta}(e, t)$ are monotonically increasing for each $e \in E$ and the functions $\underline{\Delta T}_v, \overline{\Delta T}_v$ satisfy (3.4) for each $v \in V$, then $N_{0,0}^0 \leq 1$ and $N_{0,1}^0, N_1^0$ are of order

$$\mathcal{O}\Big(C^{2^n - 1}\big(12N_{\mathrm{bd}} + 5C\big)^{2^n}\Big).$$

Moreover, $\widetilde{b}_0^*$ *can be computed in*

$$\mathcal{O}\Big(C^{3\cdot 2^{n-1}-2}\big(12N_{\text{bd}}+5C\big)^{3\cdot 2^{n-1}}\Big)$$

arithmetic operations.

(iv) *If* $t \to \underline{\theta}(e,t), t \to \overline{\theta}(e,t)$ *are monotonically increasing for each* $e \in E$*, the functions* $\underline{\Delta T}_v, \overline{\Delta T}_v$ *are constant for each* $v \in V$ *and* $\overline{\delta}_v$ *is linear for each* $v \in V$*, then* $N_{0,0}^0 \leq 1$ *and* $N_{0,1}^0, N_1^0$ *are of order*

$$\mathcal{O}\big(72^n(N_{\text{bd}}+C)\big).$$

Moreover, $\widetilde{b}_0^*$ *can be computed in*

$$\mathcal{O}\Big(72^{2n}(N_{\text{bd}}+C)^2\Big).$$

arithmetic operations.

(v) *If* $t \to \underline{\theta}(e,t), t \to \overline{\theta}(e,t)$ *are monotonically increasing for each* $e \in E$*,* $\Delta T(v,t) = \{0\}$ *and* $\overline{\delta}(v,t,0) = 0$ *for each* $(v,t) \in X$*, then* $N_{0,0}^0 \leq 1$ *and* $N_{0,1}^0, N_1^0$ *are of order*

$$\mathcal{O}\big(12^n(N_{\text{bd}}+C)\big).$$

Moreover, $\widetilde{b}_0^*$ *can be computed in*

$$\mathcal{O}\Big(12^{2n}(N_{\text{bd}}+C)^2\Big).$$

arithmetic operations.

Proof We denote $v_k = \omega(e_k)$, $k = 1,...,n$, $\widetilde{T}_n = T(v_n)$, and for $k = 0,...,n-1$, we define

$$\widetilde{\Delta T}_k : T(v_k) \to \mathcal{P}(\mathbb{R}_0^+), \qquad \begin{aligned} \widetilde{T}_{k,k+1} &= \{t \in \mathbb{R} : [\underline{\theta}(t), \overline{\theta}(t)] \subset \widetilde{T}_{k+1}\}, \\ \widetilde{\Delta T}_k(t) &= \big\{\Delta t \in \Delta T(v_k,t) : t + \Delta t \in \widetilde{T}_{k,k+1}\big\}, \\ \widetilde{T}_k &= \operatorname{supp}(\widetilde{\Delta T}_k). \end{aligned}$$

Next, we denote $\underline{\theta}_k, \overline{\theta}_k, \overline{\beta}_k : \mathbb{R} \to \mathbb{R}$, $\underline{\theta}_k(t) = \underline{\theta}(e_{k+1},t)$, $\overline{\theta}_k(t) = \overline{\theta}(e_{k+1},t)$, $\overline{\beta}_k(t) = \overline{\beta}(e_{k+1},t)$, $\overline{\delta}_k : \Theta_{v_k} \to \mathbb{R}$, $\overline{\delta}_k(t,\Delta t) = \overline{\delta}(v_k,t,\Delta t)$. Since the result of Lemma 5.3.19 is trivial if $\widetilde{T}_0 = \emptyset$, we assume in the following that $\widetilde{T}_k \neq \emptyset$ for all $k = 0,...,n$. Let us suppose that $\widetilde{T}_k$ is the finite union of closed intervals and points for some $k \in \{0,...,n-1\}$. Then Lemma 6.4.5 implies that $\widetilde{T}_{k-1,k}$ is the finite union of closed intervals. Furthermore, as $t \mapsto \underline{\Delta T}(v_{k-1},t), t \mapsto \overline{\Delta T}(v_{k-1},t)$ are $\mathcal{PL}_c^1\big(T(v_{k-1})\big)$-functions, Lemma 5.3.15 yields that $\widetilde{\Delta T}_{k-1}$ is a continuous point-to-set mapping and that $\operatorname{graph}(\widetilde{\Delta T}_{k-1})$ is the finite union of closed polygons, closed line segments and points. This also implies that $\widetilde{T}_{k-1}$ is the finite union of closed intervals and points. By induction, we obtain that $\widetilde{T}_{k+1,k}, \widetilde{T}_k$ are the finite union of closed intervals and points for each $k = 0,...,n-1$.
Next, we denote $\widetilde{b}_n^* : \widetilde{T}_n \to \mathbb{R}$, $\widetilde{b}_n^*(t) = 0$. Since $\widetilde{T}_n = T(v_n)$ is a closed interval, we either have $\#\widetilde{b}_n^* = (1,0,0)$ or $\#\widetilde{b}_n^* = (0, N_{\text{bd}}, 1)$. Moreover, for $k = 0,...,n-1$, we introduce the

functions $\widetilde{b}^*_{k,k+1} : \widetilde{T}_{k,k+1} \to \mathbb{R}$ and $\widetilde{b}^*_k : \widetilde{T}_k \to \mathbb{R}$,

$$\widetilde{b}^*_{k,k+1}(t) = \overline{\beta}_k(t) + \max_{\theta \in [\underline{\theta}_k(t), \overline{\theta}_k(t)]} \widetilde{b}^*_k(\theta), \qquad \widetilde{b}^*_k(t) = \min_{\Delta t \in \widetilde{\Delta T}_k(t)} f_k(t, \Delta t),$$

where $f_k : \text{graph}(\widetilde{\Delta T}_k) \to \mathbb{R}$, $f_k(t, \Delta t) = \widetilde{b}^*_{k,k+1}(t + \Delta t) + \overline{\delta}_k(t, \Delta t)$. By backwards induction, we immediately obtain from [65, Theorem 2.2.8], Lemma 6.4.6, Lemma 5.3.13 and Lemma 5.3.12 that $\widetilde{b}^*_{k,k+1} \in \mathcal{PL}^1_c(\widetilde{T}_{k,k+1})$, f_k is continuous and $\widetilde{b}^*_k \in \mathcal{PL}^1_c(\widetilde{T}_k)$ with $\#\widetilde{b}^*_k = (N^k_{0,0}, N^k_{0,1}, N^k_1)$ for some $N^k_{0,0}, N^k_{0,1}, N^k_1 \in \mathbb{N}_0$. We denote $N^k_0 = N^k_{0,0} + N^k_{0,1}$ for $k = 0, ..., n$, and we denote the number of connected components of $\widetilde{T}_k$ by $\widetilde{N}_k$.
Let us consider the assertion *(i)*:
Using Lemma 6.4.5, we deduce that $\widetilde{T}_{k,k+1}$ consists of at most $2C\widetilde{N}_{k+1}$ connected components. Moreover, Lemma 6.4.6 yields that $\widetilde{b}^*_{k,k+1} \in \mathcal{PL}^1_c(\widetilde{T}_{k,k+1})$ with $\#\widetilde{b}^*_{k,k+1} = (N^{k,k+1}_{0,0}, N^{k,k+1}_{0,1}, N^{k,k+1}_1)$, satisfying

$$N^{k,k+1}_{0,0} \leq C + 2CN^{k+1}_0, \qquad N^{k,k+1}_{0,1} \leq 3C + 6CN^{k+1}_0 + 2.$$

Using Lemma 5.3.13, we obtain that $f_k \in \mathcal{PL}^2_c\big(\text{graph}(\widetilde{\Delta T}_k)\big)$ with $\#f_k = (N^{f_k}_{0,0}, N^{f_k}_{0,1}, J^{f_k}_0, I^{f_k}_0, N^{f_k}_1, J^{f_k}_1, I^{f_k}_1)$, satisfying

$$\begin{aligned} N^{f_k}_{0,1} &\leq C + C(3C + 6CN^{k+1}_0 + 2), \\ N^{f_k}_1 &\leq C + C(4C + 8CN^{k+1}_0 + 2), \\ J^{f_k}_1 &\leq 5C + 8CN^{k+1}_0 + 2, \end{aligned}$$

Now, using $C \geq 2$ and Lemma 5.3.12, the above estimates yield

$$N^k_{0,1} \leq C + C(4C + 6CN^{k+1}_0) + \Big[C + C(5C + 8CN^{k+1}_0)\Big] \cdot \Big[5C + 8CN^{k+1}_0 + 1\Big]. \tag{6.22}$$

Since $\widetilde{T}_{k,k+1}$ consists of at most $2C\widetilde{N}_{k+1}$ connected components, Lemma 5.3.16 yields $\widetilde{N}_k \leq (2C^2)\widetilde{N}_{k+1}$. By induction, since $\widetilde{N}_n = 1$, we obtain that $\widetilde{N}_k \leq (2C^2)^{n-k}$. Now, the continuity of $\widetilde{b}^*_k$ immediately yields $N^k_{0,0} \leq \widetilde{N}_k \leq (2C^2)^{n-k}$. According to Lemma 5.3.5 it is sufficient to consider the recursion of the $N^k_{0,1}$, $k = 0, ..., n$, in order to establish the assertion *(i)*. From (6.22), using $N^{k+1}_0 \leq N^{k+1}_{0,1} + \widetilde{N}_{k+1}$ and $C \geq 2$, we obtain

$$N^k_{0,1} \leq C^3\Big(8N^{k+1}_{0,1} + 6 + 8(2C^2)^{n-k-1}\Big)^2,$$

and from Lemma 5.3.18 *(ii)* we obtain

$$N^k_{0,1} = \mathcal{O}\bigg(C^{3\cdot 2^{n-k}-3}\big(8N_{\text{bd}} + 6 + 16C^2\big)^{2^{n-k}}\bigg). \tag{6.23}$$

From the estimates for the necessary numbers of arithmetic operations in Lemma 6.4.6, Lemma 5.3.12 and Lemma 5.3.13 it is easily seen that the dominating order is given by $\mathcal{O}\big(N^{f_k}_1 (J^{f_k}_1)^2\big)$. Hence, $\widetilde{b}^*_{n-1}$ can be computed from $\widetilde{b}^*_n, \overline{\beta}_{n-1}, \underline{\theta}_{n-1}, \overline{\theta}_{n-1}, \overline{\delta}_{n-1}$ in $\mathcal{O}(C^4)$ arithmetic operations, since $N_{\text{bd}} = \mathcal{O}(1)$. Using (6.23), we further obtain that $\widetilde{b}^*_k$ can be computed from $\widetilde{b}^*_{k+1}, \underline{\theta}_k, \overline{\theta}_k, \overline{\beta}_k, \overline{\delta}_k$ in

$$\mathcal{O}\Big(C^{9\cdot 2^{n-k-1}-5}\big(8N_{\mathrm{bd}}+6+16C^2\big)^{3\cdot 2^{n-k-1}}\Big)$$

arithmetic operations for $k = 0, ..., n-2$. Using Lemma 5.3.18 *(i)*, we establish that

$$\mathcal{O}(C^4) + \sum_{k=0}^{n-2} \mathcal{O}\Big(C^{9\cdot 2^{n-k-1}-5}\big(8N_{\mathrm{bd}}+6+16C^2\big)^{3\cdot 2^{n-k-1}}\Big)$$
$$= \mathcal{O}\Big(C^{9\cdot 2^{n-1}-5}\big(8N_{\mathrm{bd}}+6+16C^2\big)^{3\cdot 2^{n-1}}\Big).$$

Let us now consider the assertion *(ii)*:
From Lemma 6.4.2 it follows that $\widetilde{T}_k = \widetilde{T}_{k,k+1} = \mathbb{R}$ for all $k = 0, ..., n-1$. As $\widetilde{b}^*_{k,k+1}, f_k, \widetilde{b}^*_k$ are continuous, there hold $N^{k,k+1}_{0,0} = N^{f_k}_{0,0} = J^{f_k}_0 = N^k_{0,0} = 0$. Moreover, Lemma 6.4.6 yields that

$$N^{k,k+1}_{0,1} \le 3C + 6CN^{k+1}_{0,1} + 2.$$

Using Lemma 5.3.13, we obtain that

$$N^{f_k}_{0,1} \le C + C(3C + 6CN^{k+1}_{0,1} + 2),$$
$$N^{f_k}_{1} \le C + C(3C + 6CN^{k+1}_{0,1} + 2),$$
$$J^{f_k}_{1} \le 4C + 6CN^{k+1}_{0,1} + 2,$$

Now, Lemma 5.3.12 yields

$$N^k_{0,1} \le C + C(3C + 6CN^{k+1}_{0,1} + 2) + \Big[C + C(3C + 6CN^{k+1}_{0,1} + 2)\Big] \cdot \Big[4C + 6CN^{k+1}_{0,1} + 1\Big] \tag{6.24}$$

According to Lemma 5.3.5 it is sufficient to consider the recursion of the $N^k_{0,1}$, $k = 0, ..., n$, in order to establish the assertion *(ii)*. Since $C \ge 2$ we obtain from (5.53) that

$$N^k_{0,1} \le C^3\big(6N^{k+1}_{0,1} + 5\big)^2,$$

Using Lemma 5.3.18 *(ii)* we establish, since $N_{\mathrm{bd}} = 0$,

$$N^k_{0,1} = \mathcal{O}\big(C^{3\cdot 2^{n-k}-3}5^{2^{n-k}}\big). \tag{6.25}$$

From the estimates for the necessary numbers of arithmetic operations in Lemma 6.4.6, Lemma 5.3.6, Lemma 5.3.12 and Lemma 5.3.13 it is easily seen that the dominating order is given by $\mathcal{O}\big(N^{f_k}_1 (J^{f_k}_1)^2\big)$. Hence, $\widetilde{b}^*_{n-1}$ can be computed from $\widetilde{b}^*_n, \overline{\beta}_{n-1}, \underline{\theta}_{n-1}, \overline{\theta}_{n-1}, \overline{\delta}_{n-1}$ in $\mathcal{O}(C^4)$ arithmetic operations, since $N_{\mathrm{bd}} = 0$. Using (5.54), we further obtain that $\widetilde{b}^*_k$ can be computed from $\widetilde{b}^*_{k+1}, \underline{\theta}_k, \overline{\theta}_k, \overline{\beta}_k, \overline{\delta}_k$ in

$$\mathcal{O}\big(C^{9\cdot 2^{n-k-1}-5}5^{3\cdot 2^{n-k-1}}\big)$$

arithmetic operations for $k = 0, ..., n-2$. Using Lemma 5.3.18 *(i)*, we establish that

$$\mathcal{O}(C^4) + \sum_{k=0}^{n-2} \mathcal{O}\big(C^{9\cdot 2^{n-k-1}-5} 5^{3\cdot 2^{n-k-1}}\big) = \mathcal{O}\big(C^{9\cdot 2^{n-1}-5} 5^{3\cdot 2^{n-1}}\big).$$

Let us now assume that, for each $e \in E$, the partial functions $t \mapsto \underline{\theta}(e,t)$, $t \mapsto \overline{\theta}(e,t)$ are monotonically increasing and that, for each $v \in V$, the partial functions $t \mapsto \underline{\Delta T}(v,t)$, $t \mapsto \overline{\Delta T}(v,t)$ satisfy (3.4). (Note, that this assumption holds in the cases *(iii)*, *(iv)*, *(v)* of Lemma 6.4.7.) Lemma 6.4.5 yields that $\widetilde{T}_{k,k+1}$ consists of at most as many connected components as $\widetilde{T}_{k+1}$. Using Lemma 5.3.17 and an inductive argument, we obtain that $\widetilde{N}_k = 1$, for all $k = 0, ..., n$. The continuity of $\widetilde{b}_k^*$ implies that either $\max\{N_1^{k+1}, N_{0,1}^{k+1}\} > 0$ or $N_{0,0}^{k+1} > 0$. Lemma 6.4.6 yields that $\widetilde{b}_{k,k+1}^* \in \mathcal{PL}_c^1(\widetilde{T}_{k,k+1})$ with $\#\widetilde{b}_{k,k+1}^* = (N_{0,0}^{k,k+1}, N_{0,1}^{k,k+1}, N_1^{k,k+1})$, satisfying $N_{0,0}^{k,k+1} \leq 1$ and

$$N_{0,1}^{k,k+1} \leq 3C + 12N_{0,1}^{k+1} + 2.$$

Since $\widetilde{b}_{k,k+1}^*$ is continuous and $\widetilde{N}_k = 1$, we either have $\#\widetilde{b}_{k,k+1}^* = (1,0,0)$ or $\#\widetilde{b}_{k,k+1}^* = (0, N_{0,1}^{k,k+1}, N_1^{k,k+1})$ for each $k = 0, ..., n-1$. Using Lemma 5.3.13, we obtain that $f_k \in \mathcal{PL}_c^2\big(\text{graph}(\widetilde{\Delta T}_k)\big)$ with $\#f_k = (N_{0,0}^{f_k}, N_{0,1}^{f_k}, J_0^{f_k}, I_0^{f_k}, N_1^{f_k}, J_1^{f_k}, I_1^{f_k})$, satisfying

$$\begin{aligned} N_{0,1}^{f_k} &\leq C + C(3C + 12N_{0,1}^{k+1} + 2), \\ N_1^{f_k} &\leq C + C(3C + 12N_{0,1}^{k+1} + 2), \\ J_1^{f_k} &\leq 4C + 12N_{0,1}^{k+1} + 2. \end{aligned}$$

Let us now consider the assertion *(iii)*:
Using $C \geq 2$, Lemma 5.3.12 yields

$$N_{0,1}^{k} \leq C + C(4C + 12N_{0,1}^{k+1}) + \Big[C + C(4C + 12N_{0,1}^{k+1})\Big] \cdot \Big[4C + 12N_{0,1}^{k+1} + 1\Big]. \tag{6.26}$$

According to Lemma 5.3.5 it is sufficient to consider the recursion of the $N_{0,1}^k$, $k = 0, ..., n$, in order to establish the assertion *(iii)*. Since $C \geq 2$ we obtain from (6.26) that

$$N_{0,1}^{k} \leq C\big(12N_{0,1}^{k+1} + 5C\big)^2.$$

Using Lemma 5.3.18 *(ii)* we establish

$$N_{0,1}^{k} = \mathcal{O}\bigg(C^{2^{n-k}-1}\big(12N_{\text{bd}} + 5C\big)^{2^{n-k}}\bigg). \tag{6.27}$$

From the estimates for the necessary numbers of arithmetic operations in Lemma 6.4.6, Lemma 5.3.6, Lemma 5.3.12 and Lemma 5.3.13 it is easily seen that the dominating order is given by $\mathcal{O}\big(N_1^{f_k}(J_1^{f_k})^2\big)$. Hence, $\widetilde{b}_{n-1}^*$ can be computed from $\widetilde{b}_n^*, \overline{\beta}_{n-1}, \underline{\theta}_{n-1}, \overline{\theta}_{n-1}, \overline{\delta}_{n-1}$ in $\mathcal{O}(C^4)$ arithmetic operations, since $N_{\text{bd}} = \mathcal{O}(1)$. Using (6.27), we further obtain that $\widetilde{b}_k^*$

can be computed from $\widetilde{b}^*_{k+1}, \underline{\theta}_k, \overline{\theta}_k, \overline{\beta}_k, \overline{\delta}_k$ in

$$\mathcal{O}\Big(C^{3\cdot 2^{n-k-1}-2}\big(12N_{\mathrm{bd}}+5C\big)^{3\cdot 2^{n-k-1}}\Big)$$

arithmetic operations. Using Lemma 5.3.18 *(i)*, we establish that

$$\mathcal{O}(C^4)+\sum_{k=0}^{n-2}\mathcal{O}\Big(C^{3\cdot 2^{n-k-1}-2}\big(12N_{\mathrm{bd}}+5C\big)^{3\cdot 2^{n-k-1}}\Big)=\mathcal{O}\Big(C^{3\cdot 2^{n-1}-2}\big(12N_{\mathrm{bd}}+5C\big)^{3\cdot 2^{n-1}}\Big).$$

Let us now consider the assertion *(iv)*:
Lemma 5.3.14 yields

$$N^k_{0,1}\le 6\big(3C+12N^{k+1}_{0,1}+2\big)+4=72\Big(N^{k+1}_{0,1}+\frac{1}{4}C\Big)+16.$$

Using Lemma 5.3.18 *(iv)* we establish

$$N^{n-k}_{0,1}=\mathcal{O}\big(72^{n-k}(N_{\mathrm{bd}}+C)\big). \tag{6.28}$$

From the estimates for the necessary numbers of arithmetic operations in Lemma 6.4.6, Lemma 5.3.6, and Lemma 5.3.14 it is easily seen that the dominating order is given by $\mathcal{O}\big(N^{n-1,n}_{0,1}\log(N^{n-1,n}_{0,1})\big)=\mathcal{O}\big(C\log(C)\big)$ and $\mathcal{O}\big((N^{k+1}_{0,1})^2\big)$ for $k=0,...,n-2$. Using (6.28), we obtain that $\widetilde{b}^*_k$ can be computed from $\widetilde{b}^*_{k+1}, \underline{\theta}_k, \overline{\theta}_k, \overline{\beta}_k, \overline{\delta}_k$ in

$$\mathcal{O}\Big(72^{2n-2k}(N_{\mathrm{bd}}+C)^2\Big)$$

arithmetic operations for $k=0,...,n-2$. Using the formula for the geometric series [68, p.8], we establish that

$$\mathcal{O}\big(C\log(C)\big)+(N_{\mathrm{bd}}+C)^2\sum_{k=0}^{n-2}\mathcal{O}\Big(72^{2n-2k}\Big)=\mathcal{O}\Big(72^{2n}(N_{\mathrm{bd}}+C)^2\Big).$$

Finally, we consider the assertion *(v)*:
In this case we obtain $\widetilde{b}^*_k\equiv\widetilde{b}^*_{k,k+1}$ for each $k=0,...,n-1$, i.e.,

$$N^k_{0,1}\le 3C+12N^{k+1}_{0,1}+2=12(N^{k+1}_{0,1}+C/4)+2.$$

Using Lemma 5.3.18 *(iv)* we establish

$$N^{n-k}_{0,1}=\mathcal{O}\big(12^{n-k}(N_{\mathrm{bd}}+C)\big). \tag{6.29}$$

From the estimates for the necessary numbers of arithmetic operations in Lemma 6.4.6, Lemma 5.3.6, and Lemma 5.3.14 it is easily seen that the dominating order is given by $\mathcal{O}\big(N^{n-1,n}_{0,1}\log(N^{n-1,n}_{0,1})\big)=\mathcal{O}\big(C\log(C)\big)$ and $\mathcal{O}\big((N^{k+1}_{0,1})^2\big)$ for $k=0,...,n-2$. Using (6.29), we obtain that $\widetilde{b}^*_k$ can be computed from $\widetilde{b}^*_{k+1}, \underline{\theta}_k, \overline{\theta}_k, \overline{\beta}_k, \overline{\delta}_k$ in

$$\mathcal{O}\Big(12^{2n-2k}(N_{\mathrm{bd}}+C)^2\Big)$$

arithmetic operations for $k = 0, ..., n-2$. Using the formula for the geometric series [68, p.8], we establish that

$$\mathcal{O}\big(C\log(C)\big) + (N_{\text{bd}} + C)^2 \sum_{k=0}^{n-2} \mathcal{O}\Big(12^{2n-2k}\Big) = \mathcal{O}\Big(12^{2n}(N_{\text{bd}} + C)^2\Big).$$

□

Theorem 6.4.8 *Let $v_0, v' \in V$ be given, let Assumptions 6.3.3 hold, suppose that $T(v)$ is a closed interval for all $v \in V$ and denote $N_{\text{bd}} = \big|\text{bd}\big(T(v')\big)\big|$. Suppose further that, for each $t_0 \in T_R(v_0)$, there exists an optimal control sequence $u^*(t_0) \in U(v_0, t_0)$ of topological length $n(t_0) = |u^*(t_0)|$, such that*

$$\sup_{t_0 \in T_R(v_0)} n(t_0) \leq N,$$

$$\Bigg| \bigcup_{t_0 \in T_R(v_0)} \Big\{(e_1, ..., e_{n(t_0)}) \in E^{n(t_0)} : u^*(t_0) = \big((\Delta t_k, e_k)\big)_{k=1,...,n(t_0)}\Big\} \Bigg| \leq M,$$

for some $N, M \in \mathbb{N}$. Then the function $b_0^* : T_R(v_0) \to \mathbb{R}$, $b_0^*(t_0) = b^*(v_0, t_0)$, satisfies $b_0^* \in \mathcal{PL}_{lsc}\big(T_R(v_0)\big)$, and denoting $\#b_0^* = (N_{0,0}, N_{0,1}, N_1)$, there hold:

(i) *$N_{0,0} = \mathcal{O}\big(M(2C^2)^N\big)$ and $N_{0,1}, N_1$ are of order*

$$\mathcal{O}\bigg(M^2 C^{3\cdot 2^N - 3}\big(8N_{\text{bd}} + 6 + 16C^2\big)^{2^N}\bigg).$$

Moreover, b_0^* can be computed in

$$\mathcal{O}\bigg(MC^{9\cdot 2^{N-1} - 5}\Big(8N_{\text{bd}} + 6 + 16C^2\Big)^{3\cdot 2^{N-1}} + M^3 C^{3\cdot 2^N - 3}\big(8N_{\text{bd}} + 6 + 16C^2\big)^{2^N}\bigg)$$

arithmetic operations.

(i) *If $X = V \times \mathbb{R}$, then $N_{0,0} = 0$ and $N_{0,1}, N_1$ are of order*

$$\mathcal{O}\bigg(M^2 C^{3\cdot 2^N - 3} 5^{2^N}\bigg).$$

Moreover, b_0^* can be computed in

$$\mathcal{O}\bigg(MC^{9\cdot 2^{N-1} - 5} 5^{3\cdot 2^{N-1}} + M^3 C^{3\cdot 2^N - 3} 5^{2^N}\bigg)$$

arithmetic operations.

(iii) *If $t \to \underline{\theta}(e, t), t \to \overline{\theta}(e, t)$ are monotonically increasing for each $e \in E$ and the functions $\underline{\Delta T}_v, \overline{\Delta T}_v$ satisfy (3.4) for each $v \in V$, then $N_{0,0} = \mathcal{O}(M)$ and $N_{0,1}, N_1$ are of order*

$$\mathcal{O}\bigg(M^2 C^{2^N - 1}\big(12N_{\text{bd}} + 5C\big)^{2^N}\bigg).$$

Moreover, b_0^* can be computed in

$$\mathcal{O}\Big(MC^{3\cdot 2^{N-1}-2}\big(12N_{\mathrm{bd}}+5C\big)^{3\cdot 2^{N-1}}+M^2C^{2^N-1}\big(12N_{\mathrm{bd}}+5C\big)^{2^N}\Big)$$

arithmetic operations.

(iv) *If $t \to \underline{\theta}(e,t), t \to \overline{\theta}(e,t)$ are monotonically increasing for each $e \in E$, the functions $\underline{\Delta T}_v, \overline{\Delta T}_v$ are constant for each $v \in V$ and $\overline{\delta}_v$ is linear for each $v \in V$, then $N_{0,0} = \mathcal{O}(M)$ and $N_{0,1}, N_1$ are of order*

$$\mathcal{O}\big(M^2 72^N(N_{\mathrm{bd}}+C)\big).$$

Moreover, b_0^* can be computed in

$$\mathcal{O}\Big(M72^{2N}(N_{\mathrm{bd}}+C)^2)+M^3 72^N(N_{\mathrm{bd}}+C)\Big)$$

arithmetic operations.

(v) *If $t \to \underline{\theta}(e,t), t \to \overline{\theta}(e,t)$ are monotonically increasing for each $e \in E$, $\Delta T(v,t) = \{0\}$ and $\overline{\delta}(v,t,0) = 0$ for each $(v,t) \in X$, then $N_{0,0} = \mathcal{O}(M)$ and $N_{0,1}, N_1$ are of order*

$$\mathcal{O}\big(M^2 12^N(N_{\mathrm{bd}}+C)\big).$$

Moreover, b_0^* can be computed in

$$\mathcal{O}\Big(M12^{2N}(N_{\mathrm{bd}}+C)^2)+M^3 12^N(N_{\mathrm{bd}}+C)\Big)$$

arithmetic operations.

Proof The results follows in a similar manner as in the proof of Theorem 5.3.21. □

7. Optimal Paths Without Waiting and with Fixed Departure Time

In the preceding chapters we have considered the problem of computing the optimal value function and optimal paths for varying departure times. In this case, the reachable part of the state space X_R had a hybrid structure. In particular, it consisted of pairs $\{v\} \times T_R(v)$, $v \in V$, at which $T_R(v) \subset \mathbb{R}$ generally was a time interval. Hence, X_R was innumerable in general.

In this chapter, we consider the problem of computing (the cost of) an optimal path for a fixed departure time in a FIFO-network in which waiting is prohibited. In particular, we assume that $\Delta T(v,t) = \{0\}$ and $\delta(v,t,0) = 0$ for all $(v,t) \in X$. If we consider the case of a fixed departure time and assume that only a finite number $M \in \mathbb{N}$ of paths of maximal length $N \in \mathbb{N}$ are admissible, then the reachable part of the state space consists of a finite number of states, i.e., the reachable part of the state space is discrete. For such types of state spaces the concept of the time-expanded network has been introduced in a number of discrete-time time-dependent network problems, see, e.g., [10], [39], [48], [158]. The time-expanded network is constructed from a time-dependent network as follows:

A node of the time-expanded network is a pair (v,t), where $v \in V$ is a node of the time-dependent network and $t \in \mathbb{R}$ is a (reachable) point in time. The edges of the time-expanded network are given by all state transitions $(v,t) \to (v',t')$ in the original network, i.e., by all pairs of time-expanded nodes $\big((v,t),(v',t')\big)$ for which there exists an edge e of the time-dependent network such that $v = \alpha(e)$, $v' = \omega(e)$ and $t' = t + \tau(e,t)$. (Recall that we are assuming that $\Delta T(v,t) = \{0\}$ for all $(v,t) \in X$.) The cost of traversing this edge of the time-expanded network is given by $\beta(e,t)$. If the (reachable part of the) state space of the time-dependent network is finite, the time-expanded network is a time-independent network which has the structure of the time-dependent network. The advantage of this approach is that all theoretical results and algorithms for the time-independent optimal path problem directly carry over to the formulation of the time-dependent optimal path problem in the time-expanded network. We will also use this notion in the following sections of this chapter.

If waiting is prohibited everywhere and there are no multiple edges in the network, then there is a one-to-one correspondence of paths p emanating from some initial state $x_0 \in X$ and control sequences $u \in U(x_0)$. This is not necessarily the case if waiting is allowed at certain nodes of the network, since $\{\Delta t \in \Delta T(v,t) : t + \Delta t + \tau(e, t + \Delta t) = t'\}$ may contain more than one possible waiting time for a given triple $(t,e,t') \in \mathbb{R} \times E \times \mathbb{R}$. Hence, assuming that there are no multiple edges in the network, we may use the notion of paths and control sequences interchangeably in this chapter. Moreover, we may relax Assumption 4.2.3 in order to guarantee the existence of optimal paths. The necessity of lower semicontinuous cost functions and continuous transition functions was due to the hybrid structure of the time-dependent optimal path problem with waiting times. If waiting is forbidden everywhere in the network then we are considering an essentially discrete problem (even if the time variable is continuous) which is specified by the topological structure (V,E) of the time-

dependent network. In this case, given that (4.1) and (4.2) hold, optimal paths exist even if the network functions are not (lower semi-) continuous.
In Section 7.1, we define two types of constraints which are of particular importance for the time-dependent optimal path problem in automotive navigation systems. For both types of constraints we derive pruning criteria in Section 7.2, which lead to a significant decrease of the number of reachable node-time pairs which must be considered during the computation of the optimal path. We then carry out a complexity analysis of the forward optimal path problem with fixed departure time in Section 7.3.

7.1. Problem Setting

In time-independent networks with nonnegative edge costs, optimal paths are always simple. They can be computed, e.g., by applying the principle of dynamic programming (e.g., by applying the Bellman-Ford algorithm [23]) or the algorithm of Dijkstra. Time-dependent optimal paths can contain circles, cp. Section 3.3. This is because, in a time-dependent network, the principle of dynamic programming is only generally valid in the time-expanded network [10], [48], cp. Proposition 4.3.1. This explains the difficulty in deriving computationally efficient algorithms for the time-dependent optimal path problem: The time-expanded network is usually very large in the case of a discrete time variable, and the set of reachable node-time pairs may even be innumerable in the case of a continuous time variable and fixed departure time [138]. Several pseudo-polynomial algorithms have been developed for discrete-time time-expanded networks, exploiting the fact that the time-expanded network is acyclic if all travel times are positive [10], [38], [35].
For some applications, like automotive navigation systems, it might be desirable to exclude circles in the topological structure of paths. This is on one hand motivated by the smaller number of feasible paths, which must be considered during search, and which will in almost all cases suffice for the computation of optimal paths. On the other hand, it is unlikely that an optimal path which contains a circle will be accepted by the driver.
Recall that the time-dependent optimal path problem can be formulated as a linear program in the space of positive Borel measures [142], [112], [133], [158], cp. Section 4.1. It can be shown that, if all cost functions are measurable, each extremal solution is a sum of Dirac-measures [112]. By inserting an additional constraint into [112, (LPM)], i.e., by requiring the Borel-measures x_e associated with the edges $e \in E$ of the network to be bounded from above by $x_e(\mathbb{R}) \leq 1$, $e \in E$, we conjecture that each feasible extremal solution of the resulting linear program can be shown to be a simple path. However, the travel times in the linear programming formulation of the time-dependent optimal path problem are restricted to be constant functions and few efficient algorithms have been developed to solve the resulting linear program [14], [133, Chapter 6]. Hence, the applicability of this approach is limited in applications in which the problem size is very large or the assumption of constant travel times is too restrictive. For this reason, we consider the problem of computing simple optimal paths in the framework of dynamic programming.
We have seen in Lemma 3.5.4 that certain fastest paths are easy to compute in FIFO-networks in which Assumption 3.5.3 holds. This motivates the introduction of the second constraint, which requires any feasible path to remain in some sense close to a fastest path. Such time constraints may occur in applications such as automotive navigations systems, in which it may be prohibitive to compute a route which requires more than 110% of the

optimal travel time. In order to ease the notation, we assume that the fixed departure time $t_0 \in \mathbb{R}$ always equals $t_0 = 0$. We now formulate the three problem settings which we consider in this chapter.

Problem 7.1.1 *Let $G = (V, E, \tau; \beta, \delta)$ be a time-dependent FIFO-network, let $T_1(v) = \mathbb{R}$ for all $v \in V$, let $\Delta T(v,t) = \{0\}$ and $\delta(v,t,0) = 0$ for all $(v,t) \in X$. Suppose that (V,E) is strongly connected and there exist $\underline{\mathcal{B}}, \underline{\mathcal{B}}^\circ \in \mathbb{R}$, $\underline{\mathcal{B}}^\circ > 0$, such that (4.1) and (4.2) hold. Let a source node $v_0 \in V$, the departure time $t_0 = 0$ and a goal node $v' \in V$ be given.*

(i) Compute an optimal path from $(v_0, 0)$ to v' in $(G, T_1, \Delta T)$.

Let $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ denote a monotonically increasing function with $\gamma(0) = 0$, and let $\Gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, $\Gamma(t) = t + \gamma(t)$. Denote by $\underline{t}_0(v)$ the earliest arrival time of a path from $(v_0, 0)$ to v and by $\bar{t}_{\Gamma(\underline{t}_0(v'))}(v)$ the latest departure time of a path from v to $\big(v', \Gamma(\underline{t}_0(v'))\big)$ in $(G, T_1, \Delta T)$, cp. Section 3.5. Let

$$T_2(v) = \big[\underline{t}_0(v), \infty\big) \cap \big(-\infty, \min\big\{\Gamma(\underline{t}_0(v)), \bar{t}_{\Gamma(\underline{t}_0(v'))}(v)\big\}\big], \qquad \forall v \in V.$$

(ii) Compute an optimal path from $(v_0, 0)$ to v' in $(G, T_2, \Delta T)$.

(iii) Compute an optimal simple path from $(v_0, 0)$ to v' in $(G, T_2, \Delta T)$.

Observe that $(G, T_1, \Delta T)$ satisfies Assumption 3.5.3. Hence, Lemma 3.5.4 implies that we can compute T_2 from T_1 in $\mathcal{O}\big(|E| + |V| \log |V|\big)$ time.

Lemma 7.1.2 *Consider Problem 7.1.1 (ii) and let $v \in V$ with $T_2(v) \neq \emptyset$. Then there exists a simple fastest path without waiting from $(v_0, 0)$ to $\big(v, \underline{t}_0(v)\big)$.*

Proof The assertion follows in a similar manner as in the proof of Lemma 3.5.4, since $\Gamma(\underline{t}_0(v)) \geq \underline{t}_0(v)$ for all $v \in V$. □

In the following, we consider two classes of time constraints, i.e., we choose γ as a linear function or as a logarithmic function. For the sake of simplicity, we denote

$$\gamma_{\text{lin}} : \mathbb{R}_0^+ \to \mathbb{R}_0^+, \qquad \gamma_{\text{lin}}(t) = t, \tag{7.1}$$

$$\gamma_{\text{log}} : \mathbb{R}_0^+ \to \mathbb{R}_0^+, \qquad \gamma_{\text{log}}(t) = \log(1 + t), \tag{7.2}$$

where log denotes the natural logarithm. Note that the introduction of constants or the choice of a logarithmic function to a different basis would not result in different orders of complexity in Section 7.3. Hence, the functions $\gamma_{\text{lin}}, \gamma_{\text{log}}$ can be viewed as representants for a whole class of functions. These classes have been chosen in analogy to the literature [140], [147], which investigates the effect of the accuracy of a given heuristic on the complexity of heuristic search (cf. Section 7.3). Yet, this choice is somewhat arbitrary, and results similar to those of Lemma 7.2.6, Theorem 7.3.1 and Corollary 7.3.9 can also be achieved for other function classes.

In an unconstrained optimal path problem (without waiting and with fixed departure time), none of the two constraints (i.e., the simple path constraint and the feasible arrival time constraint) is imposed. As each constraint has a different impact on the complexity of computing optimal paths, we will separately discuss the effects of the constraints both in continuous and in discrete time.

7.2. Pruning Techniques

In this section we consider the Problems 7.1.1 *(i)-(iii)*. Our first goal is to prune the search tree of an arbitrary optimal-path algorithm, based on the principle of branch and bound. We then consider Problem 7.1.1 *(iii)* and derive a bound on the number of predecessors of the head (resp., tail) of a path, which are relevant in order to maintain the simple path property. A key argument in the first part of this section is the bounded length of optimal paths, which is used to construct a Lipschitz constant for the optimal value function. Under weak assumptions we have already derived such bounds in Lemma 5.1.9. In practical applications, there are usually a plurality of more sophisticated techniques for the derivation of an upper bound of the length of each optimal path from v to v', such as, e.g., using landmarks [54]: Suppose that upper bounds $\overline{b}^*(v), \overline{b}_*(v')$ of the forward and reverse optimal value functions with respect to a landmark v^* are given for two nodes $v, v' \in V$. In other words, suppose that we know that we can reach v^* from (v, t) with a cost less or equal $\overline{b}^*(v)$ and we can reach v' from (v^*, t) with a cost less or equal $\overline{b}_*(v')$ for each $t \in \mathbb{R}$. Then, as a consequence of the triangle inequality, we also obtain that $b^*(v, t) \leq \overline{b}_*(v') + \overline{b}^*(v) =: \overline{\mathcal{B}}$, cp. Theorem 5.1.10. Note, that a smaller Lipschitz constant results in a stronger pruning criterion.
As we have pointed out at the beginning of this chapter, the computation of a solution to the time-dependent optimal path problem with fixed departure time and without waiting can be carried out in the time-expanded network. Generally, if the edge travel times are functions of a continuous time variable, most paths from v_0 to a node $v \in V$ with departure time t_0 result in different arrival times. As the time-expanded network may contain a large number of node-time-pairs in a small time interval, it is of high practical interest to prune any node-time-pair (and hence the search tree rooted in this node-time-pair), which cannot be contained in an optimal path. Although this is particularly important in the case of a continuous time variable, the following results hold also in the case of a discrete time variable.
In the next lemma we derive a pruning criterion for Problem 7.1.1 *(i)* which we then extend to Problem 7.1.1 *(ii)* in Lemma 7.2.5. Both results are formulated for forward search algorithms. Similar results can be proved for backward search algorithms.

Lemma 7.2.1 *Consider Problem 7.1.1 (i). Suppose that τ, β are Lipschitz-continuous with constants $L_\tau, L_\beta > 0$ and there exists $\overline{\mathcal{B}} \in \mathbb{R}$, $\overline{\mathcal{B}} > 0$, such that (5.15) holds. Denote*

$$L = \frac{L_\beta (1 + L_\tau)^N - 1}{L_\tau}, \tag{7.3}$$

with

$$N = |V| - 1 + \frac{|V|\overline{\mathcal{B}} - (|V| - 1)\underline{\mathcal{B}}}{\underline{\mathcal{B}}^\circ}|V|.$$

If $u, u' \in U(v_0, 0)$ with $\omega(u) = \omega(u')$, then u' cannot be extended to an optimal control sequence if

$$\mathcal{B}\big((v_0, 0), u'\big) > \mathcal{B}\big((v_0, 0), u\big) + L\Big|\mathcal{T}\big((v_0, 0), u\big) - \mathcal{T}\big((v_0, 0), u'\big)\Big|. \tag{7.4}$$

Proof Theorem 5.1.10 implies that b^* is Lipschitz-continuous with the Lipschitz-constant L given by (7.3). The minimum-cost extension of a control sequence $u \in U(v_0, 0)$ which leads to the goal node v' is the extension by an optimal control sequence $u^* \in U\big(v, \mathcal{T}\big((v_0, 0), u\big)\big)$

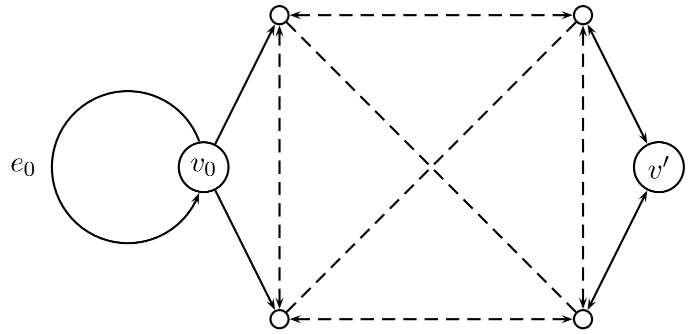


Figure 7.1.: Topological structure of the example network. The dashed center part of the graph may be arbitrary, but such that there exists at least one topological path from v_0 to v'. It might be, e.g., a symmetric grid graph of arbitrary size.

with $\omega(u^*) = v'$. Consequently, using the Lipschitz-continuity of b^*, (7.4) implies that

$$\begin{aligned}\mathcal{B}\big((v_0,0),u\big) + b^*\Big(v, \mathcal{T}\big((v_0,0),u\big)\Big) &\leq \mathcal{B}\big((v_0,0),u\big) + b^*\Big(v, \mathcal{T}\big((v_0,0),u'\big)\Big) \\ &\quad + L\Big|\mathcal{T}\big((v_0,0),u\big) - \mathcal{T}\big((v_0,0),u'\big)\Big| \\ &< \mathcal{B}\big((v_0,0),u'\big) + b^*\Big(v, \mathcal{T}\big((v_0,0),u'\big)\Big).\end{aligned}$$

Therefore, u' cannot be extended to an optimal control sequence. □

Remark 7.2.2 *Assuming that $\underline{\mathcal{B}} > 0$, the upper bound N of the length of an optimal path from v to v', can be significantly strengthened. If n is the minimum-hop distance from v to v', then we obtain $b^*(v,t) \leq n\overline{\mathcal{B}}$ which implies that $N \leq n\overline{\mathcal{B}}/\underline{\mathcal{B}}$ (cp. [107, Lemma 3]). Here, for $v, v' \in V$, the minimum hop distance from v to v' is defined as the minimum (topological) length of any (topological) path from v to v' in (V,E).*

The following simple example illustrates the use of the path pruning criterion: Consider the time-dependent network given by the graph in Figure 7.1, with

$$\tau(e_0,t) = 0.1\,, \qquad \beta(e_0,t) = 0.5\,, \qquad \forall t \in \mathbb{R}.$$

Suppose that τ, β are Lipschitz-continuous in the second argument with constants $L_\tau = L_\beta = 0.15$, and $\beta(e,t) \geq \underline{\beta} = 0.5$. Consider Problem 7.1.1 *(i)*. We assume that (e.g., from a time-independent preprocessing step) we know that $b^*(v_0,t) \leq 5$ for all $t \in \mathbb{R}$. This implies that the topological length of an optimal path is bounded from above by $N = \sup_{t\in\mathbb{R}} b^*(v_0,t)/\underline{\beta} \leq 10$. Consequently, the partial mapping $t \mapsto b^*(v_0,t)$ is Lipschitz-continuous with Lipschitz-constant

$$L = L_\beta \frac{(1+L_\tau)^N - 1}{L_\tau} \leq 3.1\,. \tag{7.5}$$

As the optimal path may contain circles, we must generally consider all copies of the source node v_0 in the time-expanded network. Since $b^*(v_0,t) \leq 5$ we must eventually consider 11 copies of v_0 if the node-time pairs are expanded by some forward search algorithm in an increasing order of cost. Let $u_{1:n} = \big((0,e_k)\big)_{k=1,\ldots,n}$ with $e_k = e_0$ for all $k = 1,\ldots,n$ and let $p_{0:n} = \Phi\big((v_0, 0), u_n\big) = \big((v_0, 0), ..., (v_0, 0.1 \cdot n)\big)$ denote the path n times cycling e_0. In addition to $(v_0, 0)$ (which may be considered as reached by the path p_0 of length 0 emanating from v_0), the node-time pairs $(v_0, 0.1 \cdot n)$ are reached by p_n, $n = 1, ..., 10$, respectively. The travel times and costs associated with p_n, $n = 0, ..., 10$, are

$$\mathcal{T}\big((v_0, 0), u_n\big) = 0.1 \cdot n\,, \qquad \mathcal{B}\big((v_0, 0), u_n\big) = 0.5 \cdot n\,.$$

Now, since

$$\mathcal{B}\big((v_0, 0), u_n\big) = 0.5 \cdot n > L \cdot 0.1 \cdot n = L\Big|\mathcal{T}\big((v_0, 0), u_n\big) - 0\Big|,$$

Lemma 7.2.1 implies that p_n cannot be extended to an optimal path, if $n = 1, ..., 10$. Hence, only by considering the source node, the application of the path pruning criterion has significantly reduced the size of the search space. Instead of 11 possible copies of v_0 in the time-expanded network, only $(v_0, 0)$ needs to be considered for the computation of the optimal time-dependent path. Of course, the same procedure can be repeated in any subsequent node, resulting in a further reduction of the search space. Although this is only an illustrative example, and the performance of the pruning criterion depends on the underlying network and the particular application, it shows the potential of the simple test given by equation (7.4).

In order to extend the result of Lemma 7.2.1 to the time-constrained case we first prove the following property of the sets of admissible control sequences.

Lemma 7.2.3 *Consider Problem 7.1.1 (ii). Then, for all $t, t' \in T(v)$, there holds*

$$t' \geq t \quad \Longrightarrow \quad U(v, t') \subset U(v, t),$$

and if there exists a $\underline{\mathcal{T}} \in \mathbb{R}^+$ such that

$$\underline{\mathcal{T}} \leq \mathcal{T}\big((v, t), u\big), \qquad \forall (v, t) \in X, u \in U(v, t), \tag{7.6}$$

then $U\big(v, \underline{t}_0(v)\big)$ is a finite set for each $v \in V$.

Proof According to Lemma 7.1.2, there exists a simple fastest path from v_0 to each $v \in V$ with $T_2(v) \neq \emptyset$. Let $u^* \in U(v_0, 0)$ be such that $\underline{t}_0(v) = \mathcal{T}\big((v_0, 0), u^*\big)$. Let $t, t' \in T(v)$ with $t' \geq t$ and let $u \in U(v, t')$, $n = |u|$, $v_k = \omega(u_k)$, $k = 1, ..., n$. The FIFO-property implies that

$$t + \mathcal{T}\big((v, t), u_{1:i}\big) \leq t' + \mathcal{T}\big((v, t'), u_{1:i}\big) \leq \min\Big\{\Gamma\big(\underline{t}_0(v_i)\big), \bar{t}_{\Gamma(\underline{t}_0(v'))}\Big\}, \qquad i = 1, ..., n.$$

If we had $t + \mathcal{T}\big((v, t), u_{1:i}\big) < \underline{t}_0(v_i)$, then the concatenation $(u^*, u_{1:i})$ of u^* and $u_{1:i}$ would satisfy

$$\mathcal{T}\big((v_0, 0), (u^*, u_{1:i})\big) \leq t + \mathcal{T}\big((v, t), u_{1:i}\big) < \underline{t}_0(v_i)$$

according to the FIFO-property, thereby contradicting the definition of $\underline{t}_0(v_i)$, $i = 1, ..., n$. Consequently, $u \in U(v, t)$.

Now assume that there exists a $\underline{\mathcal{T}} \in \mathbb{R}^+$ such that (7.6) holds. In a similar manner as in the proof of Lemma 5.1.9 *(i)* we obtain that, for any $v \in V$, the length of any admissible control sequence $u \in U(v, t)$ is bounded from above by $N = \big[\Gamma\big(\underline{t}_0(v')\big) - \underline{t}_0(v)\big]/\underline{\mathcal{T}}$. Hence, $|U(v, t)| \leq |E|^N$ for all $t \in T(v)$. □

The optimal value function is not necessarily continuous in the case of time-constrained optimal paths. This is due to the fact that a control sequence u may produce very low values of the cost function $\mathcal{B}$ but become infeasible at a certain time t, due to the constraints on the arrival times, cp. Remark 5.1.4. In such a case the optimal value function would jump to the value defined by the next-best feasible path (see Figure 7.2). In particular, we obtain the following corollary of Lemma 7.2.3:

Corollary 7.2.4 *Consider Problem 7.1.1 (ii) and assume that τ, β are continuous. Then the partial function $t \mapsto b^*(v,t)$ is continuous from the left for each $v \in V$. If there exists a $\underline{\mathcal{T}} \in \mathbb{R}^+$ such that (7.6) holds, then the set of discontinuities of the partial function $t \mapsto b^*(v,t)$ on $T_R(v)$ is finite.*

Proof Let $t \in T_R(v)$. Lemma 7.2.3 implies that

$$\lim_{s\uparrow t} b^*(v,s) = \lim_{s\uparrow t} \min_{\substack{u\in U(v,s):\\ \omega(u)=v'}} \mathcal{B}\big((v,s),u\big) \leq \min_{\substack{u\in U(v,t):\\ \omega(u)=v'}} \mathcal{B}\big((v,t),u\big) = b^*(v,t),$$

since $U(v,t) \subset U(v,s)$ for $t \geq s$ and the partial mapping $t \mapsto \mathcal{B}\big((v,t),u\big)$ is continuous for each $u \in U\big(v, \underline{t}_0(v)\big)$.
According to the FIFO-property and the continuity of τ the set $\widetilde{T}(u) = \big\{t \in T(v) : u \in U(v,t)\big\}$ is a closed connected set for each $u \in U\big(v, \underline{t}_0(v)\big)$. The set of discontinuities of the partial function $t \mapsto b^*(v,t)$ on $T_R(v)$ is contained in $\{\max \widetilde{T}(u)\}_{u\in U(v,\underline{t}_0(v))}$. If there exists a $\underline{\mathcal{T}} \in \mathbb{R}^+$ such that (7.6) holds, Lemma 7.2.3 implies that $U\big(v, \underline{t}_0(v)\big)$ is finite, and hence the set of discontinuities of the partial function $t \mapsto b^*(v,t)$ on $T_R(v)$ is finite. □

This leads to the following extension of Lemma 7.2.1 to the time-constrained case.

Corollary 7.2.5 *Consider Problem 7.1.1 (ii). Assume that τ, β are Lipschitz-continuous in the second argument with constants $L_\tau, L_\beta > 0$, and that there exists a $\underline{\mathcal{T}} \in \mathbb{R}^+$ such that (7.6) holds. Let $v \in V$ with $T_2(v) \neq \emptyset$, let $N = \big[\Gamma\big(\underline{t}_0(v')\big) - \underline{t}_0(v)\big]/\underline{\mathcal{T}}$, and let L be defined by (7.3). If $u, u' \in U(v_0, 0)$ with $\omega(u) = \omega(u')$, then u' cannot be extended to an optimal control sequence if $\mathcal{T}\big((v_0,0),u'\big) \geq \mathcal{T}\big((v_0,0),u\big)$ and*

$$\mathcal{B}\big((v_0,0),u'\big) > \mathcal{B}\big((v_0,0),u\big) + L\Big(\mathcal{T}\big((v_0,0),u'\big) - \mathcal{T}\big((v_0,0),u\big)\Big). \tag{7.7}$$

Proof From Lemma 5.1.9 *(i)* we obtain that the length of any optimal path is bounded from above by $N = \big[\Gamma\big(\underline{t}_0(v')\big) - \underline{t}_0(v)\big]/\underline{\mathcal{T}}$. Proceeding as in the proof of Lemma 5.1.8, the partial function $t \mapsto b^*(v,t)$ is Lipschitz-continuous with the Lipschitz-constant L given by (7.3) on every time interval $T' \subset T(v)$ which contains no discontinuity. Let $t_1, ..., t_j$, $j \in \mathbb{N}$, denote the time instants at which the partial function $t \mapsto b^*(v,t)$ is discontinuous, and let $b_i = \lim_{t\downarrow t_i} b^*(v,t) - \lim_{t\uparrow t_i} b^*(v,t)$, $i = 1, ..., j$, denote the height of the i-th jump. According to Corollary 7.2.4, $b_i > 0$ for all $i = 1, ..., j$. Consequently, if $t' \geq t$, there holds

$$b^*(v,t') \geq b^*(v,t) - L(t'-t) + \sum_{\substack{i\in\{1,...,j\}:\\ t\leq t_i<t'}} b_i \geq b^*(v,t) - L(t'-t). \tag{7.8}$$

The minimum-cost extension of a control sequence $u \in U(v_0, 0)$ which leads to the goal node v' is the extension by an optimal control sequence from v to v'. For $u, u' \in U(v_0, 0)$

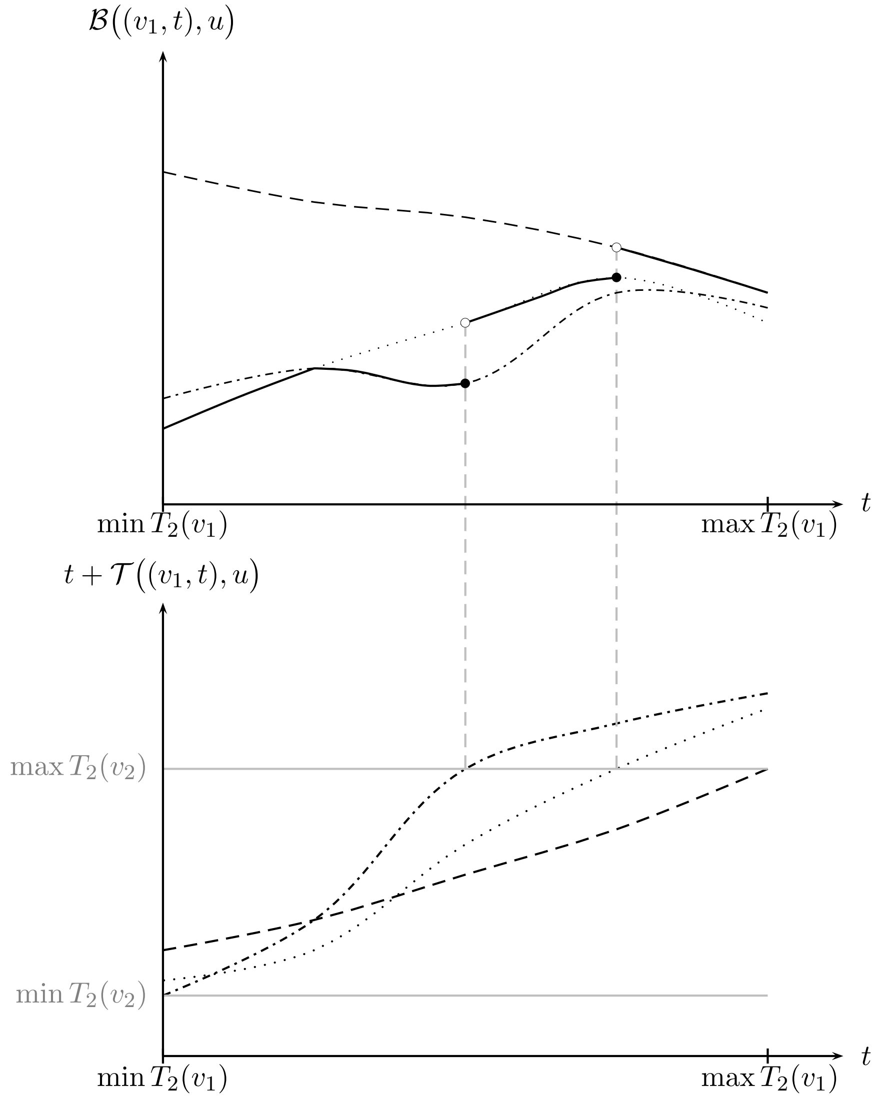


Figure 7.2.: Cost functions and arrival time functions of time-dependent paths, corresponding to three control sequences u with $\alpha(u) = v_1$, $\omega(u) = v_2$ and varying departure times t (dashed, chain-dotted, dotted black curves). The gray line in the lower drawing constitutes the time constraint in v_2, the solid black curve in the upper drawing illustrates the resulting optimal cost function in the time-constrained network.

with $v = \omega(u) = \omega(u')$ let $t = \mathcal{T}\big((v_0, 0), u\big)$, $t' = \mathcal{T}\big((v_0, 0), u'\big)$ and assume that $t' \geq t$. Now, (7.7) and (7.8) imply that

$$\begin{aligned}\mathcal{B}\big((v_0,0),u\big) + b^*(v,t) &\leq \mathcal{B}\big((v_0,0),u\big) + b^*(v,t') + L(t'-t)\\ &< \mathcal{B}\big((v_0,0),u'\big) + b^*(v,t').\end{aligned}$$

Therefore, u' cannot be extended to an optimal control sequence. □

Let us now consider Problem 7.1.1 *(iii)*. In this case, any solution algorithm has to remember the history of each path during the expansion process. Hence, a solution algorithm must expand paths rather than nodes. In contrast to the algorithms of Dijkstra or Bellman-Ford, which only need to remember the direct predecessor of each node, this must be considered as a severe drawback. The following result shows that the number of predecessors which are relevant for a further expansion of a path by some forward search algorithm is bounded.

Lemma 7.2.6 *Consider Problem 7.1.1 (iii). Suppose that γ is either linear or logarithmic, and suppose that there exist a constants $\underline{\mathcal{T}}, \overline{\mathcal{T}} \in \mathbb{R}^+$ such that there hold (7.6) and*

$$\mathcal{T}\big((v,t),u\big) \leq \overline{\mathcal{T}}, \qquad \forall (v,t) \in X, u \in U(v,t). \tag{7.9}$$

Let n denote the minimum-hop distance from v_0 to v'. Then the number N of predecessors which are relevant for the expansion of any path rooted in v_0, is bounded from above by $N \leq \gamma\big(n\overline{\mathcal{T}}\big)/\underline{\mathcal{T}} - 1$.

Proof As in the proof of Lemma 7.2.3 we see that the length of any feasible path is bounded from above by $\Gamma\big(\underline{t}_0(v')\big)/\underline{\mathcal{T}}$. Clearly $\underline{t}_0(v') \leq n\overline{\mathcal{T}}$. Let $u \in U(v_0, 0)$ be an admissible control sequence of maximum length $K \in \mathbb{N}$, and $p = \big((v_k, t_k)\big)_{k=0,\dots,K} = \Phi\big((v_0, 0), u\big)$. For each v_j, $j = 1, \dots, K$, the set of admissible arrival times satisfies $T(v_j) \subset \big[0, \Gamma\big(\mathcal{T}\big((v_0,0),u_{1:j}\big)\big)\big]$, as Γ is monotone increasing and $\mathcal{T}\big((v_0,0),u_{1:j}\big) \geq \underline{t}_0(v_j) \geq 0$. A necessary condition for the relevance of v_k for the further extension of $p_{0:j}$, $k \leq j < K$, is therefore

$$\mathcal{T}\big((v_0,0),u_{1:j}\big) + \underline{\mathcal{T}} \leq \Gamma\Big(\mathcal{T}\big((v_0,0),u_{1:k}\big)\Big), \tag{7.10}$$

because v_k must still be reachable and $\mathcal{T}\big((v_0,0),u_{1:j+1}\big) \geq \mathcal{T}\big((v_0,0),u_{1:j}\big) + \underline{\mathcal{T}}$.
Since $\mathcal{T}\big((v_0,0),u_{1:j}\big) = \mathcal{T}\big((v_0,0),u_{1:k}\big) + \mathcal{T}\big((v_k,t_k),u_{k+1:j}\big)$ as well as $\Gamma\big(\mathcal{T}\big((v_0,0),u_{1:k}\big)\big) = \mathcal{T}\big((v_0,0),u_{1:k}\big) + \gamma\big(\mathcal{T}\big((v_0,0),u_{1:k}\big)\big)$, (7.10) implies that

$$\mathcal{T}\big((v_k,t_k),u_{k+1:j}\big) + \underline{\mathcal{T}} \leq \gamma\Big(\mathcal{T}\big((v_0,0),u_{1:k}\big)\Big) \tag{7.11}$$

is necessary for the relevance of v_k for the expansion of $p_{0:j}$. As v' must always be reachable, another necessary condition for the further extension of $p_{0:j}$ is given by

$$\mathcal{T}\big((v_0,0),u_{1:j}\big) + \underline{\mathcal{T}} \leq \Gamma(n\overline{\mathcal{T}}). \tag{7.12}$$

Let $r = j - i$ denote the number of relevant predecessors of a path of length $j < K$, $\tau_i = \mathcal{T}\big((v_0,0),u_{1:i}\big)/i$ the average edge travel time on $p_{0:i}$, $1 \leq i < j$ and $\tau_r = \mathcal{T}\big((v_i,t_i),u_{i+1:j}\big)/r$ the average edge travel time on $p_{i:j}$. (7.6) and (7.9) imply that $\underline{\mathcal{T}} \leq \tau_i \leq \overline{\mathcal{T}}$ and $\underline{\mathcal{T}} \leq \tau_r \leq \overline{\mathcal{T}}$.

We now consider the following nonlinear optimization problem:

$$\min_{(i,r,\tau_i,\tau_r)} \; -r, \qquad \text{subject to} \tag{7.13}$$

$$-i \leq 0, \tag{7.14}$$

$$-r \leq 0, \tag{7.15}$$

$$\underline{\mathcal{T}} - \tau_i \leq 0, \tag{7.16}$$

$$\tau_i - \overline{\mathcal{T}} \leq 0, \tag{7.17}$$

$$\underline{\mathcal{T}} - \tau_r \leq 0, \tag{7.18}$$

$$\tau_r - \overline{\mathcal{T}} \leq 0, \tag{7.19}$$

$$-\gamma(\tau_i i) + \tau_r r + \underline{\mathcal{T}} \leq 0, \tag{7.20}$$

$$-\Gamma(\overline{\mathcal{T}}n) + \tau_r r + \tau_i i + \underline{\mathcal{T}} \leq 0. \tag{7.21}$$

The constraints (7.14), (7.15) ensure that only paths of nonnegative length are considered. (7.16)-(7.19) denote the edge travel time constraints and (7.20), (7.21) coincide with (7.11), (7.12). If $x^* = (i^*, r^*, \tau_i^*, \tau_r^*)$ is an optimal solution of (7.13)-(7.21), then the number of relevant predecessors is bounded from above by r^*. Let $f : \mathbb{R}^4 \to \mathbb{R}$ denote the objective function of (7.13), and let $q : \mathbb{R}^4 \to \mathbb{R}^8$, with the components q_l, $l = 1, ..., 8$, be defined by (7.14)-(7.21). According to [37, Theorem 3.3.5], a necessary condition for the optimality of x^* is the existence of $\mu_l \in \mathbb{R}$, $\mu_l \leq 0$, $l = 1, ..., 8$, such that

$$-\nabla f(x^*) + \sum_{l=1}^{8} \mu_l \nabla q_l(x^*) = 0, \tag{7.22}$$

$$\mu_l q_l(x^*) = 0, \quad l = 1, ..., 8, \tag{7.23}$$

if the set $\Omega = \{x \in \mathbb{R}^4 : q(x) \leq 0\}$ satisfies the constraint qualification [37, Definition 3.3.1] in x^*. This is guaranteed by the existence of $\delta x \in \mathbb{R}^4$ with

$$\langle \nabla q_l(x^*), \delta x \rangle < 0, \quad \forall l \in \{1, ..., 8\} \text{ with } q_l(x^*) = 0. \tag{7.24}$$

according to [37, Theorem 3.3.21].
If $\gamma \equiv \gamma_{\text{lin}}$, an analysis of (7.22) and (7.23) yields the admissible solutions $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_6 = 0$, $\mu_5 = -(n\overline{\mathcal{T}} - \underline{\mathcal{T}})/\underline{\mathcal{T}}^2$, $\mu_7 = \mu_8 = -1/2\underline{\mathcal{T}}$, $i^* = n\overline{\mathcal{T}}/\tau_i^*$, $r^* = (n\overline{\mathcal{T}} - \underline{\mathcal{T}})/\underline{\mathcal{T}}$, $\tau_r^* = \underline{\mathcal{T}}$ and $\tau_i^* \in [\underline{\mathcal{T}}, \overline{\mathcal{T}}]$ arbitrary. Obviously, the choice of τ_i^* does not affect the value of the objective function. We therefore choose $\tau_i^* = \overline{\mathcal{T}}$ and $i^* = n$ as candidates for an optimal solution. The constraint qualification is satisfied in the thereby defined point $x^* = (i^*, r^*, \tau_i^*, \tau_r^*)$, as $\delta x = (0, -3n/\underline{\mathcal{T}}, -1, n\underline{\mathcal{T}}/(n\overline{\mathcal{T}} - \underline{\mathcal{T}}))$ satisfies (7.24). Hence the number of relevant predecessors is bounded from above by $r^* = \gamma_{\text{lin}}(n\overline{\mathcal{T}})/\underline{\mathcal{T}} - 1$, if $\gamma \equiv \gamma_{\text{lin}}$.
If $\gamma \equiv \gamma_{\log}$, an analysis of (7.22) and (7.23) yields the (unique) admissible solution $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_6 = 0$, $\mu_5 = [\log(1+n\overline{\mathcal{T}})/\underline{\mathcal{T}} - 1]/\underline{\mathcal{T}}$, $\mu_7 = -(1+n\overline{\mathcal{T}})/[\underline{\mathcal{T}}(2+n\overline{\mathcal{T}})]$, $\mu_8 = -1/[\underline{\mathcal{T}}(2+n\overline{\mathcal{T}})]$, $i^* = n$, $r^* = \log(1 + n\overline{\mathcal{T}})/\underline{\mathcal{T}} - 1$, $\tau_i^* = \overline{\mathcal{T}}$, $\tau_r^* = \underline{\mathcal{T}}$. The constraint qualification is satisfied in the thereby defined point $x^* = (i^*, r^*, \tau_i^*, \tau_r^*)$, as $\delta x = (0, -2n^2\overline{\mathcal{T}}/\underline{\mathcal{T}}, -1, 1)$ satisfies (7.24). Hence the number of relevant predecessors is bounded from above by $r^* = \gamma_{\log}(n\overline{\mathcal{T}})/\underline{\mathcal{T}} - 1$, if $\gamma \equiv \gamma_{\log}$. □

Remark 7.2.7 *Note that the upper bound on the number of predecessors given by Lemma*

7.2.6 is valid for any feasible path in the time-dependent network, given a source node and a goal node of minimum-hop distance n. In the same manner in which this bound was derived in the proof of Lemma 7.2.6, replacing n by k, a bound for any feasible path of length $k \in \mathbb{N}$ can be derived.

7.3. Complexity Analysis

We have derived two pruning techniques in the last section, which allow a significant reduction of the cost of computing time-dependent optimal paths. Nevertheless, the computation of such paths is still in general NP-hard. In this section, we prove some complexity results for the Problems 7.1.1 *(ii)* and *(iii)*.
There has been considerable effort in bounding the number of nodes expanded by heuristic search algorithms, such as the A* algorithm [80], in terms of the accuracy of the heuristic. Assuming that the graph is a tree, it has been shown that the number of nodes expanded by the A* algorithm is polynomial in the length of the optimal solution (in the worst case), if the accuracy of the heuristic is constant [146] or logarithmic [140]. By contrast, the number of nodes expanded by the A* algorithm is exponential (in the worst case) if the accuracy of the heuristic is linear [147]. Although the setting considered in these works does not carry over to the time-dependent case, a similar result holds if the time variable is discrete and time constraints of varying order are considered.
As we have argued in Section 7.2, the constraint of allowing only simple paths for expansion leads to a different notion of expansion. In contrast to the usual optimal path algorithms (such as Dijkstra or Bellman-Ford), it is necessary to expand paths rather than nodes. As the number of simple paths grows exponentially with the number of feasible nodes, we cannot expect a polynomial bound on the number of paths. Hence, as long as we consider a discrete time variable, we only consider Problem 7.1.1 *(ii)* and we do not require paths to be simple.
The following results are formulated for a continuous time variable and a discrete-valued travel time function. The set of reachable nodes in the time-expanded network is then a discrete set.

Theorem 7.3.1 *Consider Problem 7.1.1 (ii) and assume that $\tau(E \times \mathbb{R}) \subset \{\underline{\tau}, ..., \overline{\tau}\}$ with $\underline{\tau}, \overline{\tau} \in \mathbb{N}$. Let n denote the minimum-hop distance from v_0 to v'. If (V, E) is a symmetric directed r-ary tree, then the number N of reachable nodes in the time-expanded network is*

$$N = \mathcal{O}\left(n^3 r^{n\overline{\tau}/(2\underline{\tau})}\right), \qquad \text{if } \gamma \equiv \gamma_{lin}, \tag{7.25}$$

$$N = \mathcal{O}\left(n^{1+1/(2\underline{\tau})} \log(n) r^{1/(2\underline{\tau})}\right), \qquad \text{if } \gamma \equiv \gamma_{\log}. \tag{7.26}$$

Proof The fastest path subtree S of (V, E) is a directed tree rooted in v_0. As any admissible path must visit v_0 at an admissible time $t \in T_2(v_0) = \{0\}$, the only edge emanating from v_0 must be an edge on a fastest path from v_0 to v'. Due to the FIFO-property, the fastest path from v_0 to v' is simple and therefore uniquely determined. We denote the nodes which are passed by this path by $v_0, v_1, ..., v_{n-1}, v_n$, with $v_n = v'$. Let S_k, $k = 1, ..., n$, denote the subtree of S rooted in v_k and containing (except for v_k) only nodes not passed by the fastest path from v_0 to v' (see Figure 7.3). The number of reachable nodes in the time-expanded network is given by the set of all node-time pairs in the time-expansions of the subtrees S_k, $k = 1, ..., n$. As γ is monotonically increasing, the maximum number of feasible copies of v_k is given by $\lfloor \gamma(k\overline{\tau}) \rfloor$. The maximum depth of S_k is therefore bounded from above by $\lfloor \gamma(k\overline{\tau})/(2\underline{\tau}) \rfloor$, because v_k must be reachable at an admissible arrival time from any node $v \in S_k$. Moreover, if we consider a node v_{kj} at depth $j \in \mathbb{N}$ in S_k (see Figure 7.3), then $T_2(v_{kj})$ contains no more than $\gamma(k\overline{\tau}) - 2j\underline{\tau}$ reachable and admissible arrival times. The number N_k of reachable and admissible node-time pairs in the time-expansion of S_k is therefore bounded from above by

$$N_k \leq \gamma(k\overline{\tau}) + (r-1) \sum_{j=1}^{\lfloor \gamma(k\overline{\tau})/(2\underline{\tau}) \rfloor} r^{j-1}(\gamma(k\overline{\tau}) - 2j\underline{\tau}). \tag{7.27}$$

From our reasoning above we have $N \leq \sum_{k=1}^{n} N_k$.
If $\gamma \equiv \gamma_{\text{lin}}$, then (7.27) becomes

$$N_k \leq k\overline{\tau} + (r-1) \sum_{j=1}^{\lfloor (k\overline{\tau})/(2\underline{\tau}) \rfloor} r^{j-1}(k\overline{\tau} - 2j\underline{\tau}) = \mathcal{O}\left(k^2 r^{k\overline{\tau}/(2\underline{\tau})}\right),$$

which results in (7.25).
If $\gamma \equiv \gamma_{\log}$, using the formula for the geometric series, (7.27) becomes

$$\begin{aligned} N_k &\leq \log(1+k\overline{\tau}) + (r-1) \sum_{j=1}^{\lfloor \log(1+k\overline{\tau})/(2\underline{\tau}) \rfloor} r^{j-1}(\log(1+k\overline{\tau}) - 2j\underline{\tau}) \\ &\leq \log(1+k\overline{\tau}) \left(1 + (r-1) \sum_{j=0}^{\lfloor \log(1+k\overline{\tau})/(2\underline{\tau}) \rfloor - 1} r^j \right) \\ &= \log(1+k\overline{\tau}) \left(1 + (r-1) \frac{r^{\lfloor \log(1+k\overline{\tau})/(2\underline{\tau}) \rfloor} - 1}{r-1} \right) \\ &= \mathcal{O}\left(\log(k)(kr)^{1/(2\underline{\tau})}\right), \end{aligned}$$

which results in (7.26). □

Remark 7.3.2 *In the situation of Theorem 7.3.1, since the set of reachable states is finite, an optimal path exists even if (4.1) and (4.2) do not hold.*

A major difficulty when adapting this methodology to general graphs is the fact that there exists more than one simple solution path. In a grid graph, which may be considered as an appropriate model for the road network of an urban area, neither the complexity results concerning the accuracy of a heuristic, nor the results derived in Theorem 7.3.1 apply. Considering a continuous variable, independent of the simple path constraint, even the following negative result holds.

Theorem 7.3.3 *Consider Problem 7.1.1 (ii) or Problem 7.1.1 (iii), suppose that there exists a* $\underline{T} > 0$ *such that (7.6) holds and that* (V, E) *is a grid graph. Let* n *denote the minimum-hop distance from* v_0 *to* v'. *If* $\gamma \not\equiv 0$, *then in the worst case there exist* $\Omega(2^{n/2})$ *optimal paths from* v_0 *to* v' *and* $\Omega(n 2^{n/2})$ *reachable node-time pairs.*

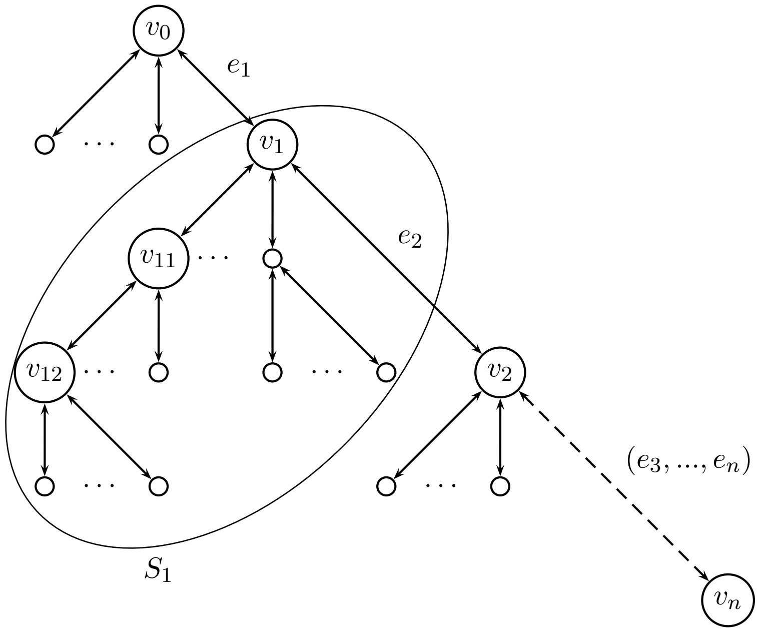


Figure 7.3.: Labeling of the symmetric directed r-ary tree used in the proof of Theorem 7.3.1. The edge sequence $(e_1, ..., e_n)$ constitutes the topological structure of the optimal path.

Proof In order to localize a node in the grid graph, we use a coordinate system and choose v_0 as the origin. The coordinates $(x, y) \in \mathbb{Z}^2$ of any node $v \in V$ in the grid graph are then given by the (directed) number of hops x in the horizontal direction and the (directed) number of hops y in the vertical direction, which are required to reach v from v_0. Without loss of generality, we assume that v' is located at $(x', y') \in \mathbb{Z}^2$, with $0 \leq x' \leq y'$, $n = x' + y'$. We will now consider the set $V_\Box$ of nodes v with coordinates $(x, y) \in \mathbb{Z}^2$, $0 \leq x \leq x'$, $0 \leq y \leq y'$, i.e., those nodes which are contained in minimum-hop paths from v_0 to v'. As $\underline{t}_0(v) \geq \underline{\mathcal{T}} > 0$ and $\gamma \not\equiv 0$, $T_2(v)$ contains an infinite number of points in time for all $v \in V_\Box$, $v \neq v_0$. We may therefore choose the edge travel times τ in such a way that each minimum-hop path from v_0 to $v \in V_\Box$ is admissible, and such that the node-time pairs passed by all minimum-hop path are distinct. Furthermore, we may choose the edge cost function β in such a way that $\beta(e, t) = \underline{\beta}$ for some $\underline{\beta} \in \mathbb{R}^+$, all $t \in T_2(\alpha(e))$ and all $e \in E$ with $e = (v_1, v_2)$ for some $v_1, v_2 \in V_\Box$, and $\beta(e, t) > \underline{\beta}$ otherwise. With this choice, each minimum-hop path from v_0 to v' is admissible and optimal. Each of these paths can be represented by a sequence of x' horizontal and y' vertical hops. Hence the number of all minimum-hop paths from v_0 to v' is given by the number of permutations of a set containing x' indistinguishable elements of one type (horizontal hops) and y' indistinguishable elements of another type (vertical hops). Therefore, there are

$$\frac{(x' + y')!}{x'!y'!} \tag{7.28}$$

minimum-hop paths. Choosing, without loss of generality, $x' = y' = n/2$, we obtain $\Omega(2^{n/2})$ optimal paths from v to v'. Since each such path is of topological length n, we obtain $\Omega(n2^{n/2})$ reachable node-time pairs. $\Box$

Remark 7.3.4 *Note, that the exponential number of reachable node-time pairs results from the fact that the terminal node-time pair of each time-dependent path possibly defines a new reachable node-time pair. In this case, it might be beneficial to introduce the simple path constraint, as the number of simple paths of length l in a grid graph is of the order μ^l, $2.62002 \leq \mu \leq 2.67919$ [122], whereas the number of paths of length l is of the order 4^l. Although this may lead to a considerable decrease in the number of reachable node-time pairs, exponential worst-case complexity can only be avoided by choosing $\gamma \equiv 0$.*

Despite the negative result given by Theorem 7.3.3, the number of reachable node-time pairs in a time-dependent grid graph remains polynomial in the minimum-hop distance of the source and goal node if the time variable is discrete. In order to establish this result we need the following Lemma:

Lemma 7.3.5 *Let (V, E) be a grid graph. The number of nodes $v \in V$ of minimum-hop distance k from a given node v_0 is bounded from above by $4k$.*

Proof Associating the same coordinate system with the grid graph as in the proof of Theorem 7.3.3, the number of nodes of distance k is given by the number of solutions $(i, j) \in \mathbb{Z}^2$ of $|i| + |j| = k$. These solutions form a $\pi/4$-rotated square in $\mathbb{Z}^2$, with each edge of the square containing $k + 1$ grid points. As each corner of the square is contained in two edges, there are $4(k + 1) - 4 = 4k$ nodes of minimum-hop distance k from v_0. □

We now derive an upper bound of the number of reachable node-time pairs which implies the desired complexity result for discrete-time time-expanded grid graphs.

Theorem 7.3.6 *Consider Problem 7.1.1 (iii) and assume that $\tau(E \times \mathbb{R}) = \{\underline{\tau}, ..., \overline{\tau}\}$ with $\underline{\tau}, \overline{\tau} \in \mathbb{N}$. Let n denote the minimum-hop distance from v_0 to v'. Suppose that the number of nodes of minimum-hop distance k from v_0 is bounded by $\nu(k)$. Then the number N of reachable nodes in the time-expanded network is bounded from above by*

$$N \leq \sum_{k=1}^{\lfloor \Gamma(n\overline{\tau})/\underline{\tau} \rfloor} \nu(k)\gamma(k\overline{\tau}). \tag{7.29}$$

Proof Let v_k be a node of minimum-hop distance k from the source node v_0, and let $t_k = \underline{t}_0(v_k)$. Clearly, there hold $k\underline{\tau} \leq t_k \leq k\overline{\tau}$, and $\underline{t}_0(v') \leq n\overline{\tau}$. Relaxing the time constraint which ensures that v' can be reached at an admissible time from each $t \in T_2(v_k)$, $\{v_k\} \times T_2(v_k)$ contains at most $\lfloor \gamma(t_k) \rfloor$ reachable node-time pairs and t_k is bounded from above by $\overline{t} = \Gamma(n\overline{\tau})$. An upper bound for the number of reachable node-time pairs of minimum-hop distance at most K from v_0 is therefore given by the following optimization problem:

$$\max_{(t_1,...,t_K)} \sum_{k=1}^{K} \nu(k)\gamma(t_k), \qquad \text{subject to} \tag{7.30}$$

$$\underline{\tau} \leq t_1 \leq \overline{\tau}, \tag{7.31}$$

$$\underline{\tau} \leq t_{k+1} - t_k \leq \overline{\tau}, \qquad k = 1, ..., K - 1, \tag{7.32}$$

$$t_k \leq \overline{t}, \qquad k = 1, ..., K. \tag{7.33}$$

The constraints (7.31) and (7.32) take the range of τ into account. Obviously, all t_k, $k = 1, ..., K$, are bounded from above by $K\overline{\tau}$, hence $\sum_{k=1}^{K} \nu(k)\gamma(t_k)$ is bounded from above, and if there exists a solution, there also exists an optimal solution with a finite value N_K of the objective function (7.30). As we have required $t_k \leq \overline{t}$ for all $k = 1, ..., K$, a solution can only exist if $K \leq \overline{t}/\underline{\tau}$. Hence, the number of reachable node-time pairs is bounded by

$$N \leq \max_{K \in \{1,...,\lfloor \overline{t}/\underline{\tau} \rfloor\}} N_K. \tag{7.34}$$

Since γ is monotone increasing, for any $K \in \{1, ..., \lfloor \overline{t}/\underline{\tau} \rfloor\}$, $\sum_{k=1}^{K} \nu(k)\gamma(t_k)$ is maximized if the variables t_k are maximized simultaneously, i.e., if for some $k^* \in \{1, ..., K\}$

$$t_k = \overline{t} - (K - k)\underline{\tau}, \qquad k^* + 1 \leq k \leq K, \tag{7.35}$$

$$t_{k^*} = \overline{t} - (K - k^* + 1)\underline{\tau} - (k^* - 1)\overline{\tau}, \tag{7.36}$$

$$t_k = k\overline{\tau}, \qquad 1 \leq k \leq k^* - 1. \tag{7.37}$$

From (7.35)-(7.37) we see that $t_k \leq k\overline{\tau}$ for all $k = 1, ..., K$. Consequently, because γ is monotone increasing, we obtain $\gamma(t_k) \leq \gamma(k\overline{\tau})$ and

$$\sum_{k=1}^{K} \nu(k)\gamma(t_k) \leq \sum_{k=1}^{K} \nu(k)\gamma(k\overline{\tau}).$$

Finally, as $K \leq \overline{t}/\underline{\tau} = \Gamma(n\overline{\tau})/\underline{\tau}$, we obtain (7.29). □

Remark 7.3.7 *In the proof of Theorem 7.3.6, the optimization problem (7.30)-(7.32) defines an upper bound for the number of reachable node-time pairs which only accounts for the distance to the source node v_0. Considering, in addition to (7.31)-(7.33), the constraint that v' must be reachable at an admissible arrival time from any admissible node-time pair, a more sophisticated and more accurate upper bound for the number of reachable node-time pairs can be defined as follows: Associate with any $v \in V$ the minimum-hop distance i from v_0 to v and the minimum-hop distance j from v to v'. (Note, that we must assume that the number of predecessors of minimum-hop distance j from v' is bounded by $\nu(j)$.) Then, for any $K \in \{1, ..., \lfloor \Gamma(n\overline{\tau})/\underline{\tau} \rfloor\}$, solve the following maximization problem:*

$$\max_{\nu_{ij}, t_{ij}} \sum_{i+j \leq K,\, i,j \geq 0} \nu_{ij}\gamma(t_{ij}), \qquad \textit{subject to} \tag{7.38}$$

$$i\underline{\tau} \leq t_{ij} \leq i\overline{\tau}, \qquad i + j \leq K,\ i, j \geq 0, \tag{7.39}$$

$$\Gamma(n\overline{\tau}) - j\overline{\tau} \leq t_{ij} \leq \Gamma(n\overline{\tau}) - j\underline{\tau}, \qquad i + j \leq K,\ i, j \geq 0, \tag{7.40}$$

$$\nu_{ij} \leq \nu(i), \qquad i + j \leq K,\ i, j \geq 0, \tag{7.41}$$

$$\nu_{ij} \leq \nu(j), \qquad i + j \leq K,\ i, j \geq 0. \tag{7.42}$$

In this formulation, (7.39) and (7.40) take into account the time constraints at v, whereas (7.41) and (7.42) take into account the topological structure of the time-dependent network. The maximum value of the objective function in (7.38) defines an upper bound for the maximum number of reachable node-time pairs. As long as neither γ nor ν are exponential functions, this procedure only yields a more accurate upper bound, but does not improve the result of Theorem 7.3.6 in the order of complexity. For this reason, we have not further *followed this approach.*

Remark 7.3.8 *Note, that the application of Theorem 7.3.6 to a symmetrical r-ary tree results in different orders of complexity than Theorem 7.3.1, i.e., $N = \mathcal{O}\left(n^2 r^{2n\overline{\tau}/\underline{\tau}}\right)$ if $\gamma \equiv \gamma_{lin}$ and $N = \mathcal{O}\left(n^{1+1/\underline{\tau}} \log(n) r^{n\overline{\tau}/\underline{\tau}+1/\underline{\tau}}\right)$ if $\gamma \equiv \gamma_{\log}$. The fact, that N grows exponentially with n even if $\gamma \equiv \gamma_{\log}$ is due to the weaker structural assumptions in Theorem 7.3.6.*

Corollary 7.3.9 *Consider Problem 7.1.1 (iii) and assume that $\tau(E \times \mathbb{R}) = \{\underline{\tau}, ..., \overline{\tau}\}$ with $\underline{\tau}, \overline{\tau} \in \mathbb{N}$. Let n denote the minimum-hop distance from v_0 to v'. If (V, E) is a grid graph, then the number N of reachable node-time pairs in the time-expanded network is*

$$N = \mathcal{O}(n^3), \qquad \textit{if } \gamma \equiv \gamma_{lin}, \tag{7.43}$$

$$N = \mathcal{O}(n^2 \log(n)), \qquad \textit{if } \gamma \equiv \gamma_{\log}. \tag{7.44}$$

Proof The assertion follows directly from Lemma 7.3.5 and Theorem 7.3.6, since for $\gamma \equiv \gamma_{\text{lin}}$ and $\gamma \equiv \gamma_{\log}$ we have $\gamma(k\overline{\tau}) = \mathcal{O}(\gamma(k))$ and $\Gamma(n\overline{\tau}/\underline{\tau}) = \mathcal{O}(n)$. □

Part III.

Algorithmic Solutions

8. An Exact Method for the Computation of Optimal Paths

In this chapter, we introduce a solution technique which, similar to the previously published decreasing order of time (DOT) algorithms [38], [47, Chapter 6], computes the forward optimal value function and the corresponding optimal paths by scanning backwards in time. By using a different interpretation of chronological scan algorithms, i.e., by extending Dijkstra's idea of sorting cost values to sorting cost functions, we generalize the concept of DOT methods to an heuristic search algorithm. As in large graphs, such as the road network, heuristic search is often the only possibility to obtain acceptable query times in real-time applications [132], [107], this generalization must be considered as a great improvement with respect to the algorithms published in the past [138], [143], [144], [38], [47], [49].
In Section 8.1 we introduce the algorithm and prove its correctness. The progression of the algorithm is illustrated with a simple numerical example in Section 8.2. A more detailed study of the algorithm, including a comparison with an approximative method (cf. Chapter 9) is then carried out in Appendix A.

8.1. The DOT* Algorithm

In the remainder of this chapter we suppose that a source node $v_0 \in V$, an earliest departure time $\underline{t}$ at v_0, a goal node $v' \in V$ and a latest arrival time $\overline{t}$ at v' are given. For simplicity, we also assume that $T(v) = [\,\underline{t}, \overline{t}\,]$ for all $v \in V$. Note, that results similar to the ones derived in the following hold for any compact state space.
In view of reachability, the time constraints at v_0 and v' may result in even stronger time constraints at intermediate nodes if $\tau \neq 0$ (cf. the definition of $T_R(v)$ for $v \in V$ in Definition 3.5.1). If Assumption 3.5.3 holds, then $T_R(v)$ can be computed in polynomial time for all $v \in V$ (cp. Corollary 3.5.7), and hence the question whether there exists a feasible finite path from v_0 to v' can be answered in polynomial time. Note that, by restricting all computations to the reachable points of the state space, any knowledge about $\{T_R(v)\}_{v \in V}$ can be used to reduce the computational overhead. In the DOT*-algorithm (Algorithm 8.1.1), we take this into account by considering lower bounds of the travel times. We will get back to the idea more rigorously in Chapter 9.
In the following, we present a new decision rule which determines a node $\widehat{v}$ and a time interval $\widehat{I}$ for which the optimal value function b^* can be determined in one iteration of a chronological scan algorithm. This decision rule can be understood as a generalization of the decision rule in Dijkstra's shortest path algorithm [56] or in the A* algorithm [80] to time-dependent networks. Indeed, the DOT* algorithm simplifies to Dijkstra's shortest path algorithm if $\underline{t} = \overline{t}$, $\tau = 0$, $\delta = 0$, β is constant and nonnegative and no heuristic is used. Recall that in each iteration of (a backwards search implementation of) Dijkstra's algorithm, the open node with the minimum cost value is identified and declared as closed. Then, each of its non-closed predecessors is declared as open and its cost value is updated

according to the dynamic programming equation. The proceeding of the A* algorithm is similar, with the exception of choosing the node $\widehat{v}$ with the minimum sum of its actual cost value and a lower bound of the cost to reach $\widehat{v}$ from the source node (in backwards search). The lower bound is also referred to as the heuristic utilized by the A* algorithm. The sum of the actual cost value and the heuristic underestimates the true cost of the optimal path from v_0 to v' constrained to pass through $\widehat{v}$, and thus, roughly speaking, prefers nodes close to the optimal path. It has been shown in [80], that the more accurate the utilized heuristic is, the less nodes are expanded by the A* algorithm.
Hence, in each iteration of both algorithms, the optimal cost value of one node is identified. In a time-dependent network, we cannot expect to be able to compute the optimal value function of one node in one iteration. However, the computation of one cost value in each iteration is only sufficient in discrete-time time-dependent networks [38], and decision rules for continuous-time time-dependent networks have only been developed for the piecewise linear case so far [47, Chapter 6].
Similar to [38], [47, Chapter 6], we use the fact that it is only possible to travel forward in time in order to determine an appropriate node and an appropriate time interval for which the optimal value function can be computed. The main idea behind this solution strategy is to avoid computational overhead: Label-correcting methods, such as the algorithms proposed in [138],[47, Chapter 7], repeatedly evaluate the dynamic programming equation at all nodes and all times. In comparison to [47, Chapter 6], besides the fact that our method is also applicable if the network functions are not piecewise linear, the main advantage of our algorithm is the incorporation of heuristic search.
Considering fastest paths with a fixed departure time in a discrete-time context, the A* algorithm has been adapted to time-dependent networks in [39]. In the following, we generalize this idea to the computation of the optimal value function in a continuous-time time-dependent network. In the next definition we recall some properties of (time-independent) heuristics [50] which we will use in the following. Note that, in contrast to [50], we formulate the properties of a heuristic for backwards search.

Definition 8.1.1 *Let $G = (V, E; \gamma)$ be a (time-independent) network in which $\gamma : E \to \mathbb{R}^+$, and let $c^* : V \times V \to \mathbb{R}_0^+$ be such that, for all $v_1, v_2 \in V$, $c^*(v_1, v_2)$ denotes the cost of an optimal path from v_1 to v_2 in G. Suppose that a source node $v_0 \in V$ has been fixed.*
A heuristic $\pi : V \to \mathbb{R}$ is called admissible, if $\pi(v) \leq c^*(v_0, v)$ for all $v \in V$. A heuristic $\pi : V \to \mathbb{R}$ is called consistent, if $\pi(v_2) \leq \pi(v_1) + c^*(v_1, v_2)$ for all $v_1, v_2 \in V$.

We use the split network to iteratively compute the optimal value function at nodes of increasing distance from the goal node v'. As in Section 3.2, we denote the nodes which result from the splitting of $v \in V$ by v_{w} and v_{nw}. Furthermore, we use lower bounds $\pi_t, \pi_b : V \to \mathbb{R}$ of the cost and travel time to reach the goal node, as well as lower bounds $\widetilde{\pi}_t, \widetilde{\pi}_b : V \times V \to \mathbb{R}$ of the cost and travel time of any path between a pair of nodes.
In particular, we suppose that for all $v \in V$, we have already computed a lower bound $\pi_t(v)$ of the optimal travel time function $t_0 \mapsto t^*(v_{0,\mathrm{nw}}, t_0)$ with respect to the goal node $v_{\mathrm{nw}} \in V$ and a lower bound $\pi_b(v)$ of the optimal value function $t_0 \mapsto b^*(v_{0,\mathrm{nw}}, t_0)$ with respect to the goal node $v_{\mathrm{nw}} \in V$.
Moreover, we suppose that for all $v_1, v_2 \in V$, $v_1 \neq v_2$, we have computed a lower bound $\widetilde{\pi}_t(v_1, v_2)$ of the optimal travel time function $t \mapsto t^*(v_{1,\mathrm{nw}}, t)$ with respect to the goal node $v_{2,\mathrm{nw}}$ and a lower bound $\widetilde{\pi}_b(v_1, v_2)$ of the optimal value function $t \mapsto b^*(v_{1,\mathrm{nw}}, t)$ with respect to the goal node $v_{2,\mathrm{nw}}$. For $v_1 = v_2 = v$, we suppose that we have computed $\widetilde{\pi}_t(v, v)$ and

$\widetilde{\pi}_b(v,v)$ as lower bounds of the travel time and cost, respectively, of any circle containing v_{nw}.
Such lower bounds $\pi.,\widetilde{\pi}.$ (the subscript $\cdot$ stands for t or b in the following) can be computed, e.g., by solving optimal path problems in the static networks $G_{\underline{\tau}} = (V,E;\underline{\tau})$ and $G_{\underline{\beta}} = (V,E;\underline{\beta})$, weighted by

$$\underline{\tau}(e) = \min_{t\in[\underline{t},\overline{t}]} \Big[\tau(e,t) + \min_{\Delta t\in\Delta T(\omega(e),t+\tau(e,t))} \Delta t\Big], \qquad \forall e\in E,$$

$$\underline{\beta}(e) = \min_{t\in[\underline{t},\overline{t}]} \Big[\beta(e,t) + \min_{\Delta t\in\Delta T(\omega(e),t+\tau(e,t))} \delta\big(\omega(e),t+\tau(e,t),\Delta t\big)\Big], \qquad \forall e\in E,$$

respectively. Note, that it is not necessary to solve an all-to-all optimal path problem in order to compute the lower bounds $\widetilde{\pi}.$: They can be determined using an admissible and consistent heuristic $\pi. : V \to \mathbb{R}_0^+$, which underestimates the time and cost, respectively, to reach any $v\in V$ from *one* (previously determined) source node $v_0\in V$. If such a heuristic $\pi.$ is known, the lower bound can be set to $\widetilde{\pi}.(v_1,v_2) = \pi.(v_2)-\pi.(v_1)$. Since $\pi.$ is admissible, $\widetilde{\pi}.(v_0,v_1) = \pi.(v_1)-\pi.(v_0) = \pi.(v_1)$ underestimates the cost to reach v_1 from v_0 and since $\pi.$ is consistent, $\widetilde{\pi}.(v_1,v_2) = \pi.(v_2)-\pi.(v_1)$ underestimates the cost to reach v_2 from v_1. Using landmarks and the triangle inequality (see, e.g., [54] for details), the lower bounds can be improved by defining them as the maximum over a set of heuristics.
In [160], the idea of the A* algorithm has been generalized to multiobjective search. This is by some means similar to the solution strategy in the DOT* algorithm, in which we compute the optimal value function at each $v\in V$ in decreasing order of time and by minimizing the cost of the respective control sequences. In particular, the decisions in the DOT* algorithm are based on maximal time and minimal cost.
We proceed by describing the notation used in the DOT* algorithm: Cost values are denoted by the function $\widehat{b} : (V\cup E)\times[\underline{t},\overline{t}] \to \mathbb{R}_0^+\cup\{\infty\}$, where $\widehat{b}(v,t)$ denotes the best cost value computed so far for reaching v' from (v,t). Furthermore, $\widehat{b}(e,t)$ denotes the best cost value computed so far for reaching v' from $(\alpha(e)_{\mathrm{nw}},t)$, constrained to depart on e at time t. (Here, $\alpha(e)_{\mathrm{nw}}$ denotes the virtual node associated with $\alpha(e)$, at which waiting is prohibited, cf. Section 3.2.) The corresponding control policies are denoted by the functions $\mu_{\mathrm{nw}} : V\times[\underline{t},\overline{t}] \to E\cup\{0,\infty\}$ and $\mu_{\mathrm{w}} : V\times[\underline{t},\overline{t}] \to \mathbb{R}_0^+\cup\{\infty\}$. Here, $\mu_{\mathrm{w}}(v,t)\in\mathbb{R}_0^+$ denotes the waiting time at (v,t), and $\mu_{\mathrm{nw}}(v,t)\in E$ denotes the edge to be traversed first on the path to v', when leaving v at time t. The cost functions and the control policies are initialized with the value ∞ at $v\in V$, $v\neq v'$, and with the value 0 at v' (termination).
As we have mentioned before, in each iteration of the DOT* algorithm, the optimal value function at a certain node and for a certain time interval is computed. (See Figure 8.1 for the illustration of one iteration of the algorithm.) In order to distinguish which points in time are relevant for the current iteration, we use the following notation: By $t^+(v)$, $v\in V$, we denote the latest point in time for which $\widehat{b}(v,\cdot)$ has not yet been computed. $t^+(v)$ is initialized with the value ∞, and is assigned the value $-\infty$ as soon as the computation of $\widehat{b}(v,\cdot)$ has been completed. Similarly, for $e\in E$, $t^+(e)$ denotes the latest departure time on $e\in E$ which has not yet been considered in the computation of the cost function $\widehat{b}(\alpha(e),\cdot)$. $t^+(e)$ is initialized with the value $-\infty$, increased the first time a path from $\alpha(e)$ to v' has been determined, then decreased whenever the cost function $\widehat{b}(\alpha(e),\cdot)$ is computed for some time interval, and finally assigned the value $-\infty$ as soon as the computation of $\widehat{b}(\alpha(e),\cdot)$ has been completed.

In each iteration of the DOT* algorithm, one node $\widehat{v} \in V$ is chosen in such a way that there exists an edge $e^* \in E^+(\widehat{v})$ with

$$e^* \in E^* = \arg\max_{e \in E} \big(t^+(e) - \pi_t(\alpha(e))\big),$$
$$\widehat{b}(e^*, \widehat{t}\,) = \min_{e \in E^*} \Big(\widehat{b}\big(e, t^+(e)\big) + \pi_b(\alpha(e))\Big).$$

At this node $\widehat{v}$, and for times $t \leq \widehat{t} = t^+(e^*)$, cost values are to be computed in the current iteration. By t^-, we denote the supremum of all points in time, for which we cannot guarantee the optimality of the cost values which are being determined at $\widehat{v}$. The time interval for which (the candidate for) the optimal value function is being computed at $\widehat{v}$ is denoted by $\widehat{I} := (t^-, \widehat{t}\,] \cap [\underline{t}, \overline{t}\,]$. (Note, that we may possibly have $t^- = -\infty$.) Hence, there are two values to be chosen in each iteration of the algorithm: the choice of $\widehat{v}$ (cf. lines 14-16) and the choice of $\widehat{I}$ (cf. lines 17-18).
In order to determine an appropriate value of t^-, we use the following observation: Since $\widetilde{\pi}_t, \widetilde{\pi}_b$ are lower bounds of the travel time and cost, respectively, at least one path departing on some $e^+ \in E^+(\widehat{v})$ at time t is optimal, if

$$\min_{e^+ \in E^+(\widehat{v})} \widehat{b}(e^+, t) \leq \min_{e \in E, \theta \geq t} \Big(b^*\big(\alpha(e)_{\text{nw}}, \theta + \widetilde{\pi}_t(\widehat{v}, \alpha(e))\big) + \widetilde{\pi}_b(\widehat{v}, \alpha(e))\Big). \tag{8.1}$$

In the DOT* algorithm, a similar criterion is used, which is only based on the cost functions determined by the algorithm (cf. line 17). Its validity is proved in Theorem 8.1.8 and illustrated in Figure 8.2.
Different approaches can be applied to solve an equation of the form (8.1), cp. line 17. One possibility is to sort the edge cost functions, i.e., to sort the values $\{\widehat{b}(e,t)\}_{e \in E}$ and $\big\{\widehat{b}\big(e, t + \widetilde{\pi}_t(\widehat{v}, \alpha(e)\big) + \widetilde{\pi}_b(\widehat{v}, \alpha(e))\big\}_{e \in E}$ for all $t \in [\underline{t}, \overline{t}\,]$. However, a sorting of the latter cost functions is generally costly, since the functions depend on the iteration node. Yet, if $\widetilde{\pi}_t, \widetilde{\pi}_b$ are constant or have been defined using a small number of landmarks, a small number of sorted lists (each list sorted with respect to one particular landmark) can be used to determine the right-hand side in (8.1). Another possibility is the organization of the edge cost functions in a priority queue. This approach has led to a very efficient implementation of Dijkstra's algorithm using Fibonacci heap [69]. We do not want to go into the details of potential sorting procedures here, and leave this as a topic for future work. Nevertheless, we explicitly note that it is never necessary to resort all edge cost functions for all times: It is always sufficient to use the sorting of a small number of sorted edges to compute the left-hand side of (8.1). In order to compute the right-hand side of (8.1), it is sufficient to remove a small number of edges for a given time interval (cf. line 25), and to add a small number of edges for a given time interval (cf. line 23). In any case, only those points in time t have to be considered, for which $t \leq t^+(e)$, cf. lines 16, 17, 19.
Alternatively, depending on the space-time trade-off chosen in the particular application, it is also possible to compute both the left-hand side and the right-hand side in (8.1) whenever line 17 is executed.
Once the expansion node, the expansion time interval and the optimal edge policy at $\widehat{v}_{\text{nw}}$ have been determined, the optimal waiting policy must be computed (cf. line 20) by solving a parametric optimization problem of the form (4.19). Recall that this task is equivalent to the determination of the optimal edge policy in the piecewise linear case, cf. Remark

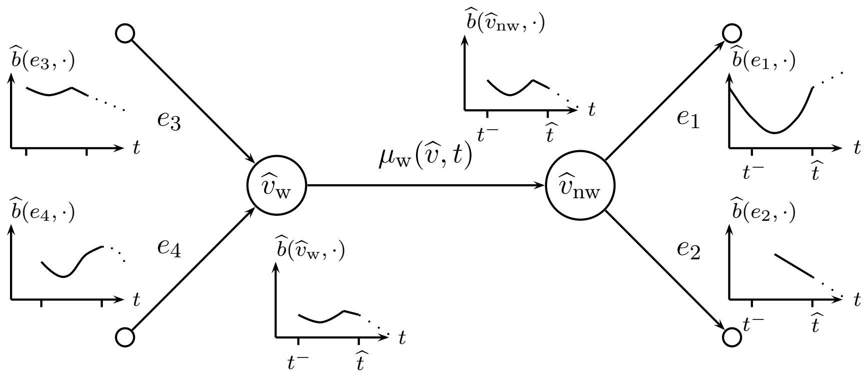


Figure 8.1.: Information flow in one iteration of the DOT* algorithm: The cost functions associated with the edges e_1, e_2 are used for the determination of the cost function at $\widehat{v}_{\text{nw}}$, which is again decreased by the application of the optimal waiting policy at $\widehat{v}_{\text{w}}$. Finally, this cost function is used for the computation of the cost functions associated with e_3, e_4.

4.3.2. At the end of each iteration, the edges in $E^-(\widehat{v})$ are updated using the information just computed at $\widehat{v}$ (cf. lines 22-23). Finally, $t^+(\widehat{v})$ and $\{t^+(e)\}_{e\in E^+(\widehat{v})}$ are decreased to t^- (cf. lines 24-25), since the computation of $\widehat{b}(\widehat{v},\cdot)$ has been completed on $(t^-,\bar{t}\,]\cap[\,\underline{t},\bar{t}\,]$ (cf. Lemma 8.1.3 and Lemma 8.1.5).

Remark 8.1.2 *If the condition $t^+(v_0) > -\infty$ is omitted in the while-loop (cf. line 13), then the DOT* algorithm becomes an all-to-one solution method, i.e., the optimal value function with respect to the goal node v' and the given time constraints is computed at all nodes.*

In the following, we derive some properties of the algorithm and prove its correctness. For this purpose, we assume that there exists a $\underline{\pi}_t \geq 0$, such that the lower bounds $\pi_t, \widetilde{\pi}_t$ satisfy

$$\pi_t(v_2) - \pi_t(v_1) \leq \widetilde{\pi}_t(v_1, v_2) - \underline{\pi}_t, \qquad \forall v_1, v_2 \in V. \tag{8.2}$$

Note, that (8.2) may be understood as a sharpened consistency assumption, which ensures that the nodes v on the fastest path from v_0 to v' are expanded in increasing distance from the goal node v'. This is reasonable, since we need to know (at least the relevant part of) the optimal value function of the successors of $\widehat{v}$ when $\widehat{v}$ is expanded. The first two lemmas follow easily from the lines of the algorithm.

Lemma 8.1.3 (Decreasing order of time) *Suppose that $\underline{\pi}_t > 0$, let $\widehat{v} \in V$ and denote by $\widehat{I}_i(\widehat{v})$ the time interval for which $\widehat{b}(\widehat{v},\cdot)$ is computed the i-th time $\widehat{v}$ is chosen for expansion in line 16 of the DOT* algorithm. Then* $\max \widehat{I}_1(\widehat{v}) = \bar{t}_R(\widehat{v})$, *and for all $i \geq 2$ there holds* $\max \widehat{I}_i(\widehat{v}) = \inf \widehat{I}_{i-1}(\widehat{v})$.

Proof Clearly, the assertion holds for v' in the initialization (cf. lines 1-4). It then follows by induction that $\max \widehat{I}_1(\widehat{v}) = \bar{t}_R(\widehat{v})$, cp. lines 9, 22, and $\max \widehat{I}_i(\widehat{v}) = \inf \widehat{I}_{i-1}(\widehat{v})$ for each $\widehat{v} \in V$, since $t^+(\widehat{v})$ is only decreased throughout the course of the algorithm, cp. the proof of Theorem 8.1.6. $\square$

Algorithm 8.1.1 DOT* algorithm

Require: time-dependent network $G = (V, E, \tau; \beta, \delta)$, source node v_0, goal node v', time interval $[\,\underline{t}, \overline{t}\,]$, waiting time constraints ΔT

Ensure: optimal value function $\widehat{b}$, optimal waiting policy μ_{w}, optimal edge policy μ_{nw}

% Initialize v'
$\widehat{b}(v', t) \leftarrow 0$, for all $t \in [\,\underline{t}, \overline{t}\,]$
$\mu_{\mathrm{nw}}(v', t) \leftarrow 0$, for all $t \in [\,\underline{t}, \overline{t}\,]$
$\mu_{\mathrm{w}}(v', t) \leftarrow 0$, for all $t \in [\,\underline{t}, \overline{t}\,]$
$t^+(v') \leftarrow -\infty$
% Initialize $v \in V \setminus \{v'\}$
$\widehat{b}(v, t) \leftarrow \infty$, for all $v \in V \setminus \{v'\}$, $t \in [\,\underline{t}, \overline{t}\,]$
$\mu_{\mathrm{nw}}(v, t) \leftarrow \infty$, for all $v \in V \setminus \{v'\}$, $t \in [\,\underline{t}, \overline{t}\,]$
$\mu_{\mathrm{w}}(v, t) \leftarrow \infty$, for all $v \in V \setminus \{v'\}$, $t \in [\,\underline{t}, \overline{t}\,]$
$t^+(v) \leftarrow \infty$, for all $v \in V \setminus \{v'\}$
% Initialize $e \in E^-(v')$
$t^+(e) \leftarrow \min\big\{t^+(\alpha(e)), \max\{t \in [\,\underline{t}, \overline{t}\,] : t + \tau(e,t) \le \overline{t}\}\big\}$, for all $e \in E^-(v')$
$\widehat{b}(e, t) \leftarrow \beta(e, t)$, for all $e \in E^-(v')$, $t \in [\,\underline{t} + \pi_t(\alpha(e)), t^+(e)]$
% Initialize $e \in E \setminus E^-(v')$
$t^+(e) \leftarrow -\infty$, for all $e \in E \setminus E^-(v')$
$\widehat{b}(e, t) \leftarrow \infty$, for all $e \in E \setminus E^-(v')$, $t \in [\,\underline{t}, \overline{t}\,]$
while $t^+(v_0) > -\infty$ and $\max_{e \in E} t^+(e) \neq -\infty$ **do**
  % Choose node for expansion
  $E^* \leftarrow \arg\max_{e \in E} \big[t^+(e) - \pi_t(\alpha(e))\big]$
  Choose $e^* \in \arg\min_{e \in E^*} \Big(\widehat{b}\big(e, t^+(e)\big) + \pi_b(\alpha(e))\Big)$
  $(\widehat{v}, \widehat{t}\,) \leftarrow (\alpha(e^*), t^+(e^*))$
  % Determine time interval for expansion
  $t^- \leftarrow \sup\Big\{t \in [\,\underline{t} + \pi_t(\widehat{v}), \widehat{t}\,] : \min_{e^+ \in E^+(\widehat{v})} \widehat{b}(e^+, t)$
    $> \min_{\theta \in [t, \widehat{t}\,]} \min_{e \in E : t^+(e) \ge \theta + \widetilde{\pi}_t(\widehat{v}, \alpha(e))} \Big(\widehat{b}\big(\, e,\, \theta + \widetilde{\pi}_t(\widehat{v}, \alpha(e))\,\big) + \widetilde{\pi}_b(\widehat{v}, \alpha(e))\Big)\Big\}$
  $\widehat{I} \leftarrow (t^-, \widehat{t}\,] \cap [\,\underline{t} + \pi_t(\widehat{v}), \overline{t}\,]$
  % Expansion
  Choose $\mu_{\mathrm{nw}}(\widehat{v}, t) \in \arg\min_{e^+ \in E^+(\widehat{v})} \widehat{b}(e^+, t)$, for all $t \in \widehat{I}$
  $\mu_{\mathrm{w}}(\widehat{v}, t) \leftarrow \arg\min_{\Delta t \in \Delta T(\widehat{v}, t)} \big\{\delta(\widehat{v}, t, \Delta t) + \widehat{b}\big(\, \mu_{\mathrm{nw}}(\widehat{v}, t + \Delta t),\, t + \Delta t\,\big)\big\}$, for all $t \in \widehat{I}$
  $\widehat{b}(\widehat{v}, t) \leftarrow \delta(\widehat{v}, t, \mu_{\mathrm{w}}(\widehat{v}, t)) + \widehat{b}\big(\, \mu_{\mathrm{nw}}(\widehat{v}, t + \mu_{\mathrm{w}}(\widehat{v}, t)),\, t + \mu_{\mathrm{w}}(\widehat{v}, t)\,\big)$, for all $t \in \widehat{I}$
  % Prepare edges terminating in $\widehat{v}$ for future expansion
  $t^+(e) \leftarrow \min\big\{t^+(\alpha(e)), \max\{t \in [\,\underline{t}, \overline{t}\,] : t + \tau(e, t) \le \widehat{t}\,\}\big\}$,
    for all $e \in E^-(\widehat{v})$ for which $t^+(e) = -\infty$
  $\widehat{b}(e, t) \leftarrow \widehat{b}(\widehat{v}, t + \tau(e, t)) + \beta(e, t)$,
    for all $e \in E^-(\widehat{v})$, $t \in [\,\underline{t} + \pi_t(\alpha(e)), t^+(e)]$ with $t + \tau(e, t) \in \widehat{I}$
  % Update expanded time intervals
  $t^+(\widehat{v}) \leftarrow t^-$
  $t^+(e) \leftarrow \min\{t^+(e), t^+(\widehat{v})\}$, for all $e \in E^+(\widehat{v})$
end while

Remark 8.1.4 *Let $(\widehat{v}_j, \widehat{t}_j)$ denote the values of $\widehat{v}$ and $\widehat{t}$ which are determined in line 16 of the j-th iteration of the DOT* algorithm. If π_t is admissible and consistent, then the sequence $\big(\widehat{t}_j - \pi_t(\widehat{v}_j)\big)_{j=1,2,\dots}$ is monotone decreasing and satisfies $\widehat{t}_j - \pi_t(\widehat{v}_j) \geq t^+(v_0)$.*

Lemma 8.1.5 (Label setting) *Suppose that $\underline{\pi}_t > 0$. Then the DOT* algorithm is a label-setting algorithm, in the sense that once a value $b(v,t), \mu_w(v,t), \mu_{nw}(v,t)$ has been computed for $(v,t) \in V \times [\,\underline{t}, \overline{t}\,]$, or a value $\widehat{b}(e,t)$ has been computed for $(e,t) \in E \times [\,\underline{t}, \overline{t}\,]$, it is never changed again.*

Proof The result is a direct consequence of Lemma 8.1.3. □

Theorem 8.1.6 (Termination) *Suppose that $\underline{\pi}_t > 0$. Then the DOT* algorithm terminates after at most*

$$\sum_{v \in V \setminus \{v'\}} \max\left\{0, 1 + \left\lfloor \frac{(\,\overline{t} - \widetilde{\pi}_t(v,v')) - (\,\underline{t} + \pi_t(v))}{\underline{\pi}_t} \right\rfloor \right\} \tag{8.3}$$

iterations.

Proof From the choice of $(\widehat{v}, \widehat{t})$ in lines 14-16 and (8.2) we see that there exists a $e^+ \in E^+(\widehat{v})$, such that

$$\begin{aligned} \widehat{t} = t^+(e^+) &\geq t^+(e) - \pi_t(\alpha(e)) + \pi_t(\widehat{v}) \\ &\geq t^+(e) - \widetilde{\pi}_t(\widehat{v}, \alpha(e)) + \underline{\pi}_t, \quad \forall e \in E. \end{aligned}$$

This implies that there is no $e \in E$, such that $t^+(e) - \widetilde{\pi}_t(\widehat{v}, \alpha(e)) \geq \theta$ for $\theta > \widehat{t} - \underline{\pi}_t$. Hence,

$$\inf_{\theta \in [t, \widehat{t}\,]} \min_{\substack{e \in E, \\ t^+(e) \geq \theta + \widetilde{\pi}_t(\widehat{v}, \alpha(e))}} \widehat{b}\big(e, \theta + \widetilde{\pi}_t(\widehat{v}, \alpha(e))\big) + \widetilde{\pi}_b(\widehat{v}, \alpha(e)) = \infty$$

for all $t > \widehat{t} - \underline{\pi}_t$. Consequently, $t^- \leq \widehat{t} - \underline{\pi}_t$ (cf. line 17). Moreover, since $t^+(e) \leq \overline{t}$ (cf. lines 9, 11, 22, 25), there holds $t^- \leq \min\{\,\overline{t}, \widehat{t}\,\} - \underline{\pi}_t$. As $t^+(\widehat{v}) \geq \max_{e \in E^+(\widehat{v})} t^+(e) = \widehat{t} \leq \overline{t}$ (cf. lines 8, 24 and recall that $t^- \leq \max_{e \in E} t^+(e)$), we have $t^- \leq \min\{\,\overline{t}, t^+(\widehat{v})\} - \underline{\pi}_t$ (cf. line 17), and $\min\{\,\overline{t}, t^+(\widehat{v})\}$ is decreased at least by $\underline{\pi}_t$ (cf. line 24).
Since, for each $v \in V$, $\widetilde{\pi}_t(v, v')$ is a lower bound for the travel time from v to v', the latest departure time $\overline{t}_R(v)$ at v satisfies $\overline{t}_R(v) \leq \overline{t} - \widetilde{\pi}_t(v, v')$. Moreover, the choice of t^- and $\widehat{I}$ in lines 17, 18, implies that the optimal value function at v is only computed for $t \geq \underline{t} + \pi_t(v)$. Since, in each iteration, an interval of the form $(t^-, \widehat{t}\,] \cap [\,\underline{t}, \overline{t}\,]$ is processed, there is possibly one extra iteration for the left endpoint of $[\,\underline{t} + \pi_t(v), \overline{t} - \widetilde{\pi}_t(v, v')\,]$. Therefore, according to Lemma 8.1.5, a node $v \in V$ can be iterated at most $1 + \big\lfloor \big((\overline{t} - \widetilde{\pi}_t(v, v')) - (\underline{t} + \pi_t(v))\big)/\underline{\pi}_t \big\rfloor$ times. Summing over all nodes in the network and taking into account that the computation at v' is completed in the initialization, we obtain (8.3). □

Remark 8.1.7 *The result of Theorem 8.1.6 suggests that optimal paths can be computed in polynomial time. This is certainly not true in general, since the complexity of the algorithm highly depends on the complexity of summing, sorting and concatenating the cost functions [111], cp. Section 5.3 for the piecewise linear case.*

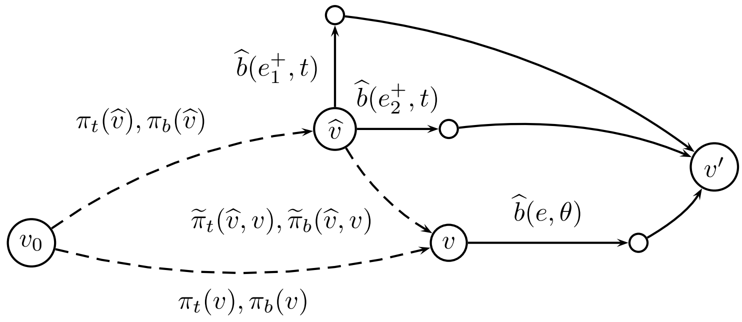


Figure 8.2.: Determination of $\widehat{v}$ (cf. lines 14-16) and t^- (cf. line 17) in the DOT* algorithm, using the lower bounds $\pi_t, \widetilde{\pi}_t, \pi_b, \widetilde{\pi}_b$. All optimal value functions along the solid lines have been computed, the paths along the dashed lines have not yet been expanded.

Theorem 8.1.8 (Correctness) *Once the DOT* algorithm has terminated,* $b^*(v_0, t_0) = \widehat{b}(v_0, t_0)$ *for all* $t_0 \in [\underline{t}, \bar{t}\,]$*. Moreover, for all* $t_0 \in [\underline{t}, \bar{t}\,]$ *for which* $\widehat{b}(v_0, t_0) \neq \infty$*, the optimal path* $p^* = \big((v_k, t_k)\big)_{k=0,1,\dots}$ *from* (v_0, t_0) *to* v' *and the corresponding optimal control sequence* $u^* = \big((\Delta t_k, e_k)\big)_{k=1,2,\dots}$ *can be constructed recursively by*

$$(\Delta t_k, e_k) = \Big(\, \mu_w(v_{k-1}, t_{k-1}),\, \mu_{nw}\big(v_{k-1}, t_{k-1} + \mu_w(v_{k-1}, t_{k-1})\big) \Big), \quad k = 1, 2, \dots, \tag{8.4}$$

$$(v_k, t_k) = \varphi\big((v_{k-1}, t_{k-1}), (\Delta t_k, e_k)\big), \quad k = 1, 2, \dots, \tag{8.5}$$

terminating as soon as $v_k = v'$ *for some* $k \in \mathbb{N}$*.*

Proof Since $\mu_{\mathrm{w}}(v, t) \in \Delta T(v, t)$ for all $(v, t) \in V \times [\underline{t}, \bar{t}\,]$ (cf. line 20), and $\widehat{b}(e, t) \neq \infty$ only if $t \in [\underline{t}, \bar{t}\,]$ and $t + \tau(e, t) \in [\underline{t}, \bar{t}\,]$ (cf. lines 9, 10, 22, 23), we have

$$\big(\, \mu_{\mathrm{nw}}(v, t + \mu_{\mathrm{w}}(v, t)),\, \mu_{\mathrm{w}}(v, t) \,\big) \in U(v, t), \quad \forall (v, t) \in X \text{ with } \widehat{b}(v, t) \neq \infty.$$

Moreover, if $\widehat{b}(v, t)$ is computed, then the cost value in the successor state has been computed in a previous iteration of the DOT* algorithm, because $\widehat{b}(e, t)$ is only computed if $\widehat{b}\big(\omega(e), t + \tau(e, t)\big)$ is computed (cf. lines 1, 10 and 21, 23). Consequently, the recursion given by (8.5) can only terminate in the goal node v'. From line 21, it follows that $\widehat{b}(v, t)$ is the cost value produced by the application of $\big(\mu_{\mathrm{nw}}(v, t + \mu_{\mathrm{w}}(v, t)), \mu_{\mathrm{w}}(v, t)\big)$ in (v, t). Since Theorem 4.2.4 implies that infinite paths generate infinite cost, (8.5) must define a finite path, and hence the recursion terminates in v'. Consequently, the proof is complete, if we show that $b^*(v, t) = \widehat{b}(v, t)$ for all $(v, t) \in X$ for which $t > t^+(v)$. We now prove this assertion by induction over the number of iterations.

Obviously, the assertion is true for $(v, t) \in \{v'\} \times [\underline{t}, \bar{t}\,]$ in the initialization phase of the DOT* algorithm. Now, suppose that in some iteration $\widehat{v} \in V$ has been chosen, and that $\widehat{b}(\widehat{v}, t)$ is being computed for $t \in \widehat{I}$. We show that $\widehat{b}(\widehat{v}, t)$ satisfies the Bellman equation (4.14). We split $\widehat{v}$ into the nodes $\widehat{v}_{\mathrm{w}}, \widehat{v}_{\mathrm{nw}}$ (see Section 4.2) and set $\widehat{b}(\widehat{v}_{\mathrm{nw}}, t) = \widehat{b}\big(\mu_{\mathrm{nw}}(\widehat{v}, t), t\big)$ and $\widehat{b}(\widehat{v}_{\mathrm{w}}, t) = \widehat{b}\big(\widehat{v}_{\mathrm{nw}}, t + \mu_{\mathrm{w}}(\widehat{v}, t)\big) + \delta\big(\widehat{v}, t, \mu_{\mathrm{w}}(\widehat{v}, t)\big)$. As a consequence of Lemma 8.1.3 and the induction hypothesis, and in view of line 21 and equations (4.19) and (4.20), it is sufficient to show that

$$\widehat{b}(\widehat{v}_{\mathrm{nw}},t) = \min_{(0,e)\in U(\widehat{v}_{\mathrm{nw}},t)} \Big[b^*\big(\omega(e),t+\tau(e,t)\big)+\beta(e,t)\Big], \tag{8.6}$$

$$\widehat{b}(\widehat{v}_{\mathrm{w}},t) = \min_{\Delta t\in\Delta T(v,t)} \Big[\widehat{b}(\widehat{v}_{\mathrm{nw}},t+\Delta t)+\delta(\widehat{v},t,\Delta t)\Big], \tag{8.7}$$

for all $t \in \widehat{I}$. From line 20 it is obvious that (8.7) holds if (8.6) holds. As a consequence of the induction hypothesis and the definition of $\widehat{b}(e,t)$ (cf. lines 10, 23), (8.6) would be equivalent to

$$\widehat{b}(\widehat{v}_{\mathrm{nw}},t) = \min_{(0,e)\in U(\widehat{v}_{\mathrm{nw}},t)} \widehat{b}(e,t), \tag{8.8}$$

if $\widehat{b}(e,t)$ had already been computed for all $(e,t)\in E^+(\widehat{v})\times\widehat{I}$, for which $(0,e)\in U(\widehat{v}_{\mathrm{nw}},t)$. We now show that (8.6) and (8.8) are equivalent although we might have $\widehat{b}(e,t)=\infty$ for some $(e,t)\in E^+(\widehat{v})\times\widehat{I}$.
Let us assume the contrary: Denote $\widetilde{v}_0=\widehat{v}_{\mathrm{nw}}$ and suppose that there is a $\widetilde{t}_0\in\widehat{I}$ and an optimal control sequence in the split network $u=\big((\Delta t_k,e_k)\big)_{k=1,\dots,n}\in U(\widetilde{v}_0,\widetilde{t}_0)$ such that

$$\mathcal{B}\big((\widetilde{v}_0,\widetilde{t}_0),u\big) < \min_{e^+\in E^+(\widetilde{v}_0):t^+(e^+)\geq\widetilde{t}_0} \widehat{b}(e^+,\widetilde{t}_0).$$

Let $p=\big((\widetilde{v}_k,\widetilde{t}_k)\big)_{k=0,\dots,n}$ denote the path (in the split network) generated by u at $(\widetilde{v}_0,\widetilde{t}_0)$ and let $u_{j:n}=\big((\Delta t_k,e_k)\big)_{k=j,\dots,n}$ for $j=1,...,n$. Let further $J\in\{1,...,n\}$ such that $(\widetilde{v}_J,\widetilde{t}_J)$ is the first state on p at which waiting is allowed and for which $\widehat{b}(\widetilde{v}_J,\widetilde{t}_J)$ has already been computed. As a consequence of the induction hypothesis and the principle of optimality, we have

$$\mathcal{B}\big((\widetilde{v}_J,\widetilde{t}_J),u_{J+1:n}\big)=\widehat{b}(\widetilde{v}_J,\widetilde{t}_J)=b^*(\widetilde{v}_J,\widetilde{t}_J).$$

Hence, $\widehat{b}(e_J,\widetilde{t}_{J-1}+\Delta t_J)$ has already been computed (cf. line 23), which implies that $J\geq 2$. (If $J=1$, then $\Delta t_1=0$ since $\widetilde{v}_0=\widehat{v}_{\mathrm{nw}}$, and we would have

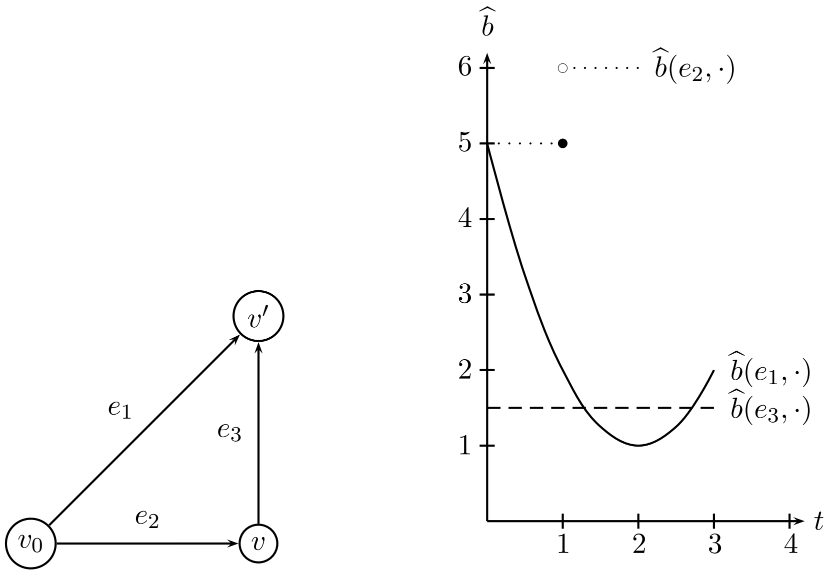
$$\widehat{b}(e_1,\widetilde{t}_0)=\mathcal{B}\big((\widetilde{v}_0,\widetilde{t}_0),u\big) < \min_{e^+\in E^+(\widehat{v}),t^+(e^+)\geq\widetilde{t}_0} \widehat{b}(e^+,\widetilde{t}_0)$$

for $e_1\in E^+(\widehat{v})$ and $t^+(e_1)\geq\widetilde{t}_0$. This is a contradiction.) Consequently, $t^+(e_J)\geq\widetilde{t}_{J-1}+\Delta t_J$. Note also that $\widetilde{t}_0\geq\underline{t}+\pi_t(\widehat{v})$. Now, since $\widetilde{\pi}_t(\widehat{v},\alpha(e_J))$ is a lower bound for the travel time from $\widehat{v}_{\mathrm{nw}}=\widetilde{v}_0$ to $\widetilde{v}_{J-1}$, (8.2) implies (cp. Figure 8.2)

$$\begin{aligned}\underline{t}+\pi_t(\alpha(e_J)) &\leq \widetilde{t}_0+\pi_t(\alpha(e_J))-\pi_t(\widehat{v})\leq\widetilde{t}_0+\widetilde{\pi}_t(\widehat{v},\alpha(e_J))\\ &\leq \widetilde{t}_{J-1}+\Delta t_J.\end{aligned} \tag{8.9}$$

Moreover, as $\widetilde{\pi}_b(\widehat{v},\alpha(e_J))$ is a lower bound for the travel cost from $\widehat{v}$ to $\widetilde{v}_{J-1}$, we establish (cp. Figure 8.2)

$$\begin{aligned}\min_{e^+\in E^+(\widehat{v}),t^+(e^+)\geq\widetilde{t}_0} \widehat{b}(e^+,\widetilde{t}_0) &> \mathcal{B}\big((\widetilde{v}_0,\widetilde{t}_0),u\big)\geq\mathcal{B}\big((\widetilde{v}_{J-1},\widetilde{t}_{J-1}),u_{J:n}\big)+\widetilde{\pi}_b(\widehat{v},\alpha(e_J))\\ &=\widehat{b}(e_J,\widetilde{t}_{J-1}+\Delta t_J)+\widetilde{\pi}_b(\widehat{v},\alpha(e_J)).\end{aligned} \tag{8.10}$$



(a) Topology of the time-dependent network.

(b) Plot of the edge cost functions determined by the DOT* algorithm.

Figure 8.3.: Topology and edge cost functions of the example network.

Now, (8.9) and (8.10) contradict the choice of t^- (cf. line 17). □

8.2. A numerical example

In this section, we briefly illustrate the progression of the DOT* algorithm. For this purpose we consider the time-dependent network $G = (V, E, \tau; \beta, \delta)$ with V, E as shown in Figure 8.3(a), with travel times

$$\tau(e_1, t) = \begin{cases} 3 - 2t, & 0 \leq t \leq 1 \\ 1, & 1 < t \leq 4 \end{cases}, \qquad \tau(e_2, t) = 1, \qquad \tau(e_3, t) = 1,$$

travel cost

$$\beta(e_1, t) = 1 + (t-2)^2, \qquad \beta(e_2, t) = \begin{cases} 7/2, & 0 \leq t \leq 1 \\ 9/2, & 1 < t \leq 4 \end{cases}, \qquad \beta(e_3, t) = 3/2,$$

and waiting cost

$$\delta(v_0, t, \Delta t) = \Delta t, \qquad \delta(v, t, \Delta t) = \Delta t^2, \qquad \delta(v', t) = 0.$$

We consider $[\underline{t}, \overline{t}\,] = [0, 4]$ and the waiting time constraints $\Delta T(v_0, t) = [0, 4-t]$, $\Delta T(v, t) = [0, 1]$, $\Delta T(v', t) = \{0\}$. Note that the time-dependent network G is neither continuous, nor FIFO, nor piecewise linear.
We first set $\pi_t(v) = \pi_b(v) = 0$ for all $v \in V$, and $\widetilde{\pi}_t(v_1, v_2) = \widetilde{\pi}_b(v_1, v_2) = 1$ for all $v_1, v_2 \in V$. With this choice we obtain a Dijkstra-like behavior of the DOT* algorithm:

In the initialization phase, we set $\widehat{b}(v',t) = 0$ for all $t \in [0,4]$ and $t^+(v') = -\infty$. Further, we set $\widehat{b}(e_1,t) = 1 + (t-2)^2$ for $0 \le t \le 3$, $t^+(e_1) = 3$, and $\widehat{b}(e_3,t) = 3/2$ for $0 \le t \le 3$, $t^+(e_3) = 3$.

Iteration 1:

Since $\widehat{b}(e_1,3) > \widehat{b}(e_3,3)$, we clearly have $\widehat{t} = 3$, $\widehat{v} = v$. Now

$$\widetilde{\pi}_b(v,v_0) + \inf_{\theta \in [t+\widetilde{\pi}_t(v,v_0),\widehat{t}\,]} \widehat{b}(e_1,\theta) = \begin{cases} 2, & 0 \le t \le 1 \\ 2 + (t-1)^2, & 1 < t \le 2 \end{cases},$$

$$\widetilde{\pi}_b(v,v) + \inf_{\theta \in [t+\widetilde{\pi}_t(v,v),\widehat{t}\,]} \widehat{b}(e_3,\theta) = 5/2, \quad 0 \le t \le 2,$$

and $\widehat{b}(e_3,t) = 3/2$, which results in $t^- = -\infty$. Hence, $\mu_{\text{nw}}(v,t) = e_3$ for $0 \le t \le 3$ and

$$\mu_{\text{w}}(v,t) = \operatorname*{arg\,min}_{\Delta t \in [0,\min\{3-t,1\}]} \Delta t^2 + 3/2 = 0.$$

(Note, that we have to fulfill $0 \le \Delta t \le \min\{3-t,1\}$, since $\widehat{b}(e_3,t) = \infty$ for $t > 3$.) This results in $\widehat{b}(v,t) = 3/2$ for $0 \le t \le 3$, $t^+(v) = -\infty$, $t^+(e_3) = -\infty$, and finally, we set $t^+(e_2) = 2$ and

$$\widehat{b}(e_2,t) = \begin{cases} 5, & 0 \le t \le 1 \\ 6, & 1 < t \le 2 \end{cases}.$$

Iteration 2:

Again, we have $\widehat{t} = 3$, which results in $\widehat{v} = v_0$. Since

$$\widetilde{\pi}_b(v_0,v_0) + \inf_{\theta \in [t+\widetilde{\pi}_t(v_0,v_0),\widehat{t}\,]} \widehat{b}(e_1,\theta) = \begin{cases} 2, & 0 \le t \le 1 \\ 2 + (t-1)^2, & 1 < t \le 2 \end{cases},$$

$$\widetilde{\pi}_b(v_0,v_0) + \inf_{\theta \in [t+\widetilde{\pi}_t(v_0,v_0),\widehat{t}\,]} \widehat{b}(e_2,\theta) = \begin{cases} 6, & t = 0 \\ 7, & 0 < t \le 1 \end{cases},$$

and $\widehat{b}(e_1,t) = 1 + (t-2)^2$, we obtain $t^- = 1$. Hence, we set $\mu_{\text{nw}}(v_0,t) = e_1$ for $1 < t \le 3$. We now compute

$$\mu_{\text{w}}(v_0,t) = \operatorname*{arg\,min}_{\Delta t \in [0,3-t]} \Delta t + 1 + (t + \Delta t - 2)^2 = \begin{cases} 3/2 - t, & 1 < t < 3/2 \\ 0, & 3/2 \le t \le 3 \end{cases},$$

$$\widehat{b}(v_0,t) = \begin{cases} 11/4 - t, & 1 < t < 3/2 \\ 1 + (t-2)^2, & 3/2 < t \le 3 \end{cases},$$

and $t^+(v_0) = 1$, $t^+(e_1) = 1$, $t^+(e_2) = 1$.

Iteration 3:

It is obvious that $\widehat{v} = v_0$, and for $t \in [0,1]$ we compute $\mu_{\text{nw}}(v_0,t) = e_1$, $\mu_{\text{w}}(v_0,t) = 3/2 - t$ and $\widehat{b}(v_0,t) = 11/4 - t$. Since we now set $t^+(v_0) = t^- = -\infty$, the algorithm terminates. Note, that $\widehat{b}(v_0,\cdot)$ cannot be computed for all $t \in [0,3]$ in iteration 2. Suppose that there was an additional edge $e_4 \sim (v_0,v_0)$ with $\tau(e_4,t) = 1$ and $\beta(e_4,t) = 1$. In this case, it would be necessary to compute $\widehat{b}(e_4,\cdot)$ before computing $\widehat{b}(v_0,\cdot)$ for $t \in [0,1]$ in order to guarantee

the correctness of the algorithm.
Let us now use more informed heuristics, which leads to a A*-like behavior of the DOT* algorithm. For this purpose, we assume that $\widetilde{\pi}_t(v_1, v_2) = 1$ for all $v_1, v_2 \in V$ and $\widetilde{\pi}_b(v_1, v_2) = 1$ for all $v_1, v_2 \in V$ except for $\widetilde{\pi}_b(v_0, v) = \widetilde{\pi}_b(v, v_0) = 7/2$, $\widetilde{\pi}_b(v_0, v_0) = 4$. Moreover, we assume that $\pi_t(v) = \pi_b(v) = 1/2$ for $v \in V$, $v \neq v_0$, and $\pi_t(v_0) = \pi_b(v_0) = 0$. The initialization phase remains unchanged.
<u>Iteration 1:</u>
As $t^+(e_1) - 0 > t^+(e_3) - 1/2$, we have $\widehat{t} = 3$, $\widehat{v} = v_0$. Since

$$\widetilde{\pi}_b(v_0, v_0) + \inf_{\theta \in [t + \widetilde{\pi}_t(v_0, v_0), \widehat{t}\,]} \widehat{b}(e_1, \theta) = \begin{cases} 5, & 0 \leq t \leq 1 \\ 5 + (t-1)^2, & 1 < t \leq 2 \end{cases},$$

$$\widetilde{\pi}_b(v_0, v) + \inf_{\theta \in [t + \widetilde{\pi}_t(v_0, v), \widehat{t}\,]} \widehat{b}(e_3, \theta) = 5, \quad 0 \leq t \leq 2$$

and $\widehat{b}(e_1, t) \leq 5$ for all $0 \leq t \leq 3$ we obtain $t^- = -\infty$ and the algorithm terminates after one iteration. The optimal value function at v_0 and the corresponding optimal paths coincide in both runs of the algorithm.

9. An Approximative Method for the Computation of Optimal Paths

In the preceding chapter, we have introduced an exact solution technique for the time-dependent forward optimal path problem which allows the incorporation of heuristic search and which reduces the computational overhead with respect to the methods published in the past. Since the computational complexity of determining an exact solution may still be infeasible in many practical applications, we now present an algorithm which approximates the forward optimal value function and the corresponding optimal paths. Assuming that the network satisfies the FIFO-property, we generate two initial solutions in polynomial time, which we use to iteratively approximate the optimal solution. We show that all iterates fulfill a certain monotonicity property, which provides an explicit estimate of the accuracy of the found solutions. Furthermore, we prove that the algorithm converges under mild assumptions after a finite number of iterations.
In Section 9.1, we introduce the algorithm and prove its properties. We illustrate its progression with a simple numerical example in Section 9.2. A more detailed study of the algorithm, including a comparison with an exact method (cf. Chapter 8) is then carried out in Appendix A.

9.1. The TD-APX Algorithm

In this section, we assume that an earliest departure time $\underline{t}$ at v_0 and a latest arrival time $\bar{t}$ at v' are given, that (in addition to Assumption 4.2.3) Assumption 3.5.3 holds and that $\tau > 0$. Moreover, we assume that a norm $\|\cdot\|$ for piecewise continuous real-valued functions $f : D \to \mathbb{R}$ with compact domain $D \subset \mathbb{R}$ is given, such as, e.g.,

$$\|f\| = \sup_{t \in D} |f(t)| + \int_D |f(t)| \, \mathrm{d}t,$$

We now present a time-dependent approximative algorithm (TD-APX, Algorithm 9.1.1) which solves the problem of computing the optimal value function and the optimal paths in a time-dependent network with compact state space. The method is based on the principles of branch and bound and policy iteration. After computing a lower bound and an upper bound for the optimal value function, we use branch and bound to reduce the set of states which are possibly contained in an optimal path. Since the set of states in the time-dependent network is innumerable in general, we only branch by splitting the state space into $\big\{\{v\} \times [\underline{t}, \bar{t}\,]\big\}$ for $v \in V$. In order to tighten the bounds associated with each node, we iterate the control policy at a given node by solving the associated dynamic programming equations (4.13), (4.14). This corresponds to the principle of policy iteration, a well known solution technique in the field of Markov decision processes.
The TD-APX algorithm proceeds in three steps. In the first step, fastest paths from $(v_0, \underline{t}\,)$

to each $v \in V$ and from each $v \in V$ to $(v', \bar{t})$ are computed. The respective travel times are used to determine the set of times $T_R(v) \subset [\underline{t}, \bar{t}\,]$ for which a node $v \in V$ is reachable, cp. Lemma 3.5.6. The set of all reachable nodes is denoted by V_R, the reachable part of the state space is denoted by $X_R = \bigcup_{v \in V_R} \big\{\{v\} \times T_R(v)\big\} \subset X$, cp. Definition 3.5.1, and the set of all control sequences which are admissible in $(v,t) \in X_R$ is denoted by $U_R(v,t)$. Moreover, the control sequences corresponding to (simple) fastest paths (without waiting) from $(v_0, \underline{t}\,)$ to v and from v to $(v', \bar{t}\,)$ are stored as initial iterates and used to define upper bounds $\bar{b}_0(v), \bar{b}'(v)$ of the optimal value function, respectively. The admissibility of this approach is proved in Lemma 9.1.5.
In the second step, we compute optimal paths from v_0 to each $v \in V$ and from each $v \in V$ to v' in the time-independent network $\underline{G} = (V_R, E_R; \underline{\beta}, \underline{\delta})$ which is defined by

$$\begin{aligned} E_R &= \{e \in E : \alpha(e) \in V_R \setminus \{v'\}, \omega(e) \in V_R\}, \\ \underline{\beta}(e) &= \min_{t \in T_R(\alpha(e))} \beta(e,t), && e \in E_R, \\ \underline{\delta}(v) &= \min_{t \in T_R(v)} \Big[\min_{\Delta t \in \Delta T(v,t)} \delta(v,t,\Delta t) \Big], && v \in V_R. \end{aligned}$$

The corresponding control sequences are used to define lower bounds $\underline{b}_0(v), \underline{b}'(v)$ of the optimal value function. Since there may exist circles with negative costs in $\underline{G}$, we do not use these control sequences as initial iterates. However, we use the upper bounds $\bar{b}_0(v), \bar{b}'(v)$ and the lower bounds $\underline{b}_0(v), \underline{b}'(v)$ in order to eliminate all nodes from the considered node set V_R which cannot be contained in an optimal path (cf. lines 12, 26, cp. Remark 9.1.2).
In the third step, we denote by $i(v)$ the number of times a node $v \in V_R$ has been iterated. Moreover, we denote by $t \mapsto \bar{b}_{i(v)|i(v)}(v,t)$ and $t \mapsto \underline{b}_{i(v)|i(v)}(v,t)$ the upper bounds and lower bounds of the partial function $t \mapsto b^*(v,t)$ which have been computed in the $i(v)$-th iteration of a node $v \in V_R$. Finally, we denote by $t \mapsto \bar{b}_{i(v)+1|i(v)}(v,t)$ and $t \mapsto \underline{b}_{i(v)+1|i(v)}(v,t)$ the current upper bounds and lower bounds of the partial function $t \mapsto b^*(v_{\text{nw}},t)$. (Here, v_{nw} denotes the copy of v in the split network, at which waiting is prohibited, cp. Section 4.2.) The upper and lower bounds of the cost functions are then iterated in order to approximate the optimal value function. One iteration consists of the update of the functions at v_{nw} (and the corresponding optimal edge policies), the computation of the potential improvement $\pi(v)$ (which is the $\|\cdot\|$-change of the cost functions at v_{nw} since the last computation of the optimal waiting policy), the choice of an appropriate candidate node $\hat{v} \in V_R$ for the computation of a new waiting policy (cf. line 21 and Remark 9.1.3) and the computation of the new cost and waiting policy at $\hat{v}$. The iteration loop is terminated as soon as the optimal value function at v_0 is approximated within a given accuracy $\epsilon \geq 0$ in $\|\cdot\|$.
We further partition the nodes under consideration V_R into three sets. $V_F \subset V_R$ denotes the set of all nodes v_F for which we have already finished the computation of the partial function $t \mapsto b^*(v_F,t)$. $V_O \subset V_R$ consists of all nodes v_O for which an improvement of the upper and lower bounds is possible by means of computing a new waiting policy. We may understands these nodes as labeled "open", the set of remaining nodes $V_C = V_R \setminus (V_F \cup V_O)$ may be understood as labeled "closed".
In order to simplify the notation of the algorithm we cite the following update equations,

cp. (4.20) and (4.19):

$$\underline{b}_{i(v)+1|i(v)}(v,t) \leftarrow \min_{(0,e)\in U_R(v_{\mathrm{nw}},t)} \Big[\underline{b}_{i(\omega(e))|i(\omega(e))}\big(\omega(e), t+\tau(e,t)\big) + \beta(e,t)\Big], \tag{9.1}$$

$$\underline{\mu}_{\mathrm{nw},i(v)+1}(v,t) \leftarrow \operatorname*{arg\,min}_{(0,e)\in U_R(v_{\mathrm{nw}},t)} \Big[\underline{b}_{i(\omega(e))|i(\omega(e))}\big(\omega(e), t+\tau(e,t)\big) + \beta(e,t)\Big], \tag{9.2}$$

$$\overline{b}_{i(v)+1|i(v)}(v,t) \leftarrow \min_{(0,e)\in U_R(v_{\mathrm{nw}},t)} \Big[\overline{b}_{i(\omega(e))|i(\omega(e))}\big(\omega(e), t+\tau(e,t)\big) + \beta(e,t)\Big], \tag{9.3}$$

$$\overline{\mu}_{\mathrm{nw},i(v)+1}(v,t) \leftarrow \operatorname*{arg\,min}_{(0,e)\in U_R(v_{\mathrm{nw}},t)} \Big[\overline{b}_{i(\omega(e))|i(\omega(e))}\big(\omega(e), t+\tau(e,t)\big) + \beta(e,t)\Big], \tag{9.4}$$

$$\underline{b}_{i(v)|i(v)}(v,t) \leftarrow \min_{\substack{\Delta t\in\Delta T(v,t):\\ t+\Delta t\in T_R(v)}} \Big[\underline{b}_{i(v)|i(v)-1}(v,t+\Delta t) + \delta(v,t,\Delta t)\Big], \tag{9.5}$$

$$\underline{\mu}_{\mathrm{w},i(v)}(v,t) \leftarrow \operatorname*{arg\,min}_{\substack{\Delta t\in\Delta T(v,t):\\ t+\Delta t\in T_R(v)}} \Big[\underline{b}_{i(v)|i(v)-1}(v,t+\Delta t) + \delta(v,t,\Delta t)\Big], \tag{9.6}$$

$$\overline{b}_{i(v)|i(v)}(v,t) \leftarrow \min_{\substack{\Delta t\in\Delta T(v,t):\\ t+\Delta t\in T_R(v)}} \Big[\overline{b}_{i(v)|i(v)-1}(v,t+\Delta t) + \delta(v,t,\Delta t)\Big], \tag{9.7}$$

$$\overline{\mu}_{\mathrm{w},i(v)}(v,t) \leftarrow \operatorname*{arg\,min}_{\substack{\Delta t\in\Delta T(v,t):\\ t+\Delta t\in T_R(v)}} \Big[\overline{b}_{i(v)|i(v)-1}(v,t+\Delta t) + \delta(v,t,\Delta t)\Big]. \tag{9.8}$$

(Note that Assumption 4.2.3 implies that the minima in the above equations are attained.) The resulting procedure is summarized in Algorithm 6.1.

Remark 9.1.1 *The TD-APX algorithm can be aborted if it is determined in line 4 that $T_R(v_0) = \emptyset$. In this case we have $T_R(v) = \emptyset$ for all $v \in V$ and the goal node is not reachable from the source node, cf. Lemma 3.5.6. Note that this also implies that $V_O = \emptyset$.*

Remark 9.1.2 *If a node v_R is removed from V_R in line 12 or 26, then there exists no optimal path from v_0 to v' which passes through v_R. By construction of the upper and lower bounds $\overline{b}_0(v), \overline{b}'(v), \underline{b}_0(v), \underline{b}'(v)$, Theorem 4.4.2 yields that*

$$\underline{b}_0(v) + \underline{b}'(v) \le b_*(v,t) + b^*(v,t) \le \overline{b}_0(v) + \overline{b}'(v), \quad \forall t \in T_R(v).$$

As a consequence of the principle of optimality, there also holds

$$b^*(v_0,t_0) \le \sup_{t\in T_R(v)} \big[b_*(v,t) + b^*(v,t)\big], \quad \forall t_0 \in T_R(v_0).$$

Consequently, v_R cannot be contained in an optimal path from v_0 to v' if

$$\underline{b}_0(v_R) + \underline{b}'(v_R) > \min_{v\in V_R} \big[\overline{b}_0(v) + \overline{b}'(v)\big].$$

Remark 9.1.3 *The manner in which the iteration node is chosen in line 21 is not the only way to achieve the properties of the algorithm which we prove below. It constitutes a trade-off between the (heuristic) probability a node v is contained in an optimal path (i.e., $\underline{b}_0(v)+\underline{b}'(v)$*

Algorithm 9.1.1 TD-APX algorithm

Require: time-dependent network $G = (V, E, \tau; \beta, \delta)$, source node v_0, goal node v', time interval $[\underline{t}, \overline{t}\,]$, waiting time constraints ΔT, desired accuracy ϵ

Ensure: approximation of the optimal value function $\overline{b}$, corresponding waiting policy $\overline{\mu}_{\mathrm{w}}$, corresponding edge policy $\overline{\mu}_{\mathrm{nw}}$

% STEP 1:

Compute $\underline{t}_R(v)$ and the corresponding optimal control sequences $\overline{u}(v_0, v)$ for all $v \in V$, such that the path generated by $\overline{u}(v_0, v)$ is simple and without waiting.

Compute $\overline{t}_R(v)$ and the corresponding optimal control sequences $\overline{u}(v, v')$ for all $v \in V$, such that the path generated by $\overline{u}(v, v')$ is simple and without waiting.

$V_R \leftarrow \{v \in V : \underline{t}_R(v) \leq \overline{t}_R(v)\}$

$T_R(v) = [\,\underline{t}_R(v), \overline{t}_R(v)]$ for all $v \in V_R$

$\overline{b}_{0|-1}(v, t) \leftarrow \mathcal{B}\big((v, t), \overline{u}(v, v')\big) - \delta(v, t, 0)$ for all $(v, t) \in X_R$

$\overline{b}_{0|0}(v, t) \leftarrow \mathcal{B}\big((v, t), \overline{u}(v, v')\big)$ for all $(v, t) \in X_R$

$(\overline{\mu}_{\mathrm{nw},0}, \overline{\mu}_{\mathrm{w},0})(v, t) = \overline{u}_1(v, v')$ for all $(v, t) \in X_R$

$\overline{b}_0(v) \leftarrow \sup_{t_0 \in T_R(v_0) :\, \overline{u}(v_0, v) \in U_R(v_0, t_0)} \mathcal{B}\big((v_0, t_0), \overline{u}(v_0, v)\big)$ for all $v \in V_R$

$\overline{b}'(v) \leftarrow \sup_{t \in T_R(v)} \mathcal{B}\big((v, t), \overline{u}(v, v')\big)$ for all $v \in V_R$

% STEP 2:

Compute in $\underline{G}$ the optimal cost $\underline{b}_0(v)$ from v_0 to v for all $v \in V_R$

Compute in $\underline{G}$ the optimal cost $\underline{b}'(v)$ from v to v' for all $v \in V_R$

$V_R \leftarrow \big\{v_R \in V_R : \underline{b}_0(v_R) + \underline{b}'(v_R) \leq \min_{v \in V_R} \big[\,\overline{b}_0(v) + \overline{b}'(v)\big]\big\}$

$\underline{b}_{0|-1}(v, t) \leftarrow \underline{b}'(v) - \underline{\delta}(v)$ for all $(v, t) \in X_R$

$\underline{b}_{0|0}(v, t) \leftarrow \underline{b}'(v)$ for all $(v, t) \in X_R$

% STEP 3:

$i(v) \leftarrow 0$ for all $v \in V_R$

$V_F \leftarrow \big\{v \in V_R : \big\|\overline{b}_{i(v)|i(v)}(v, \cdot) - \underline{b}_{i(v)|i(v)}(v, \cdot)\big\| = 0\big\}$

Compute (9.1)-(9.4) for all $v \in V_R \setminus V_F$ and all $t \in T_R(v)$

$\pi(v) \leftarrow \sum_{b \in \{\underline{b}, \overline{b}\}} \big\|b_{i(v)+1|i(v)}(v, \cdot) - b_{i(v)|i(v)-1}(v, \cdot)\big\|$ for all $v \in V_R \setminus V_F$

$V_O \leftarrow \{v \in V_R \setminus V_F : \pi(v) > 0\}$

while $V_O \neq \emptyset$ and $\big\|\overline{b}_{i(v_0)|i(v_0)}(v_0, \cdot) - \underline{b}_{i(v_0)|i(v_0)}(v_0, \cdot)\big\| > \epsilon$ **do**

    Choose $\widehat{v} \in \arg\min_{v \in V_O} \big[\,\underline{b}_0(v) + \underline{b}'(v) - \pi(v)\big]$

    $i(\widehat{v}) \leftarrow i(\widehat{v}) + 1$

    Compute (9.5)-(9.8) for all $\widehat{v}$ and all $t \in T_R(\widehat{v})$

    $\underline{b}'(\widehat{v}) \leftarrow \min_{t \in T_R(\widehat{v})} \underline{b}_{i(\widehat{v})|i(\widehat{v})}(\widehat{v}, t)$

    $\overline{b}'(\widehat{v}) \leftarrow \sup_{t \in T_R(\widehat{v})} \overline{b}_{i(\widehat{v})|i(\widehat{v})}(\widehat{v}, t)$

    $V_R \leftarrow \big\{v_R \in V_R : \underline{b}_0(v_R) + \underline{b}'(v_R) \leq \min_{v \in V_R} \big[\,\overline{b}_0(v) + \overline{b}'(v)\big]\big\}$

    Compute (9.1)-(9.4) for all $v \in V_R^-(\widehat{v}) = V^-(\widehat{v}) \cap (V_R \setminus V_F)$ and all $t \in T_R(v)$

    $\pi(v) \leftarrow \sum_{b \in \{\underline{b}, \overline{b}\}} \big\|b_{i(v)+1|i(v)}(v, \cdot) - b_{i(v)|i(v)-1}(v, \cdot)\big\|$ for all $v \in V_R^-(\widehat{v})$

    $V_F \leftarrow \big\{v \in V_R : \big\|\overline{b}_{i(v)|i(v)}(v, \cdot) - \underline{b}_{i(v)|i(v)}(v, \cdot)\big\| = 0\big\}$

    $V_O \leftarrow \{v \in V_R \setminus V_F : \pi(v) > 0\}$

end while

should be small) and the room for improvement at v (i.e., the potential improvement $\pi(v)$ should be big).

Remark 9.1.4 *The iteration number $i(v)$ associated with each node $v \in V_R$ has been introduced in order to formulate the monotonicity of the iteration process (cf. Lemma 9.1.7). For algorithmic purposes, it would be sufficient to formulate a two-step recursion, where the first step (update potential improvement) is defined by (9.1)-(9.4) and the second step (update computation) by (9.5)-(9.8).*

In general, the complexity of the TD-APX algorithm depends highly on the complexity of the functional operations in (9.1)-(9.8) and lines 18, 28, [111]. In the piecewise linear case, the time complexity of STEP 1 and STEP 2 is polynomial in the size of the network topology. Hence, an admissible initial solution is computed in polynomial time, cf. Lemma 3.5.4 and Lemma 9.1.5. Since the number of linear pieces of the optimal value function grows exponentially with the size of the network topology in the worst case, the iterations in STEP 3 become more costly as the algorithm proceeds. If the optimal value function is approximated exactly by the lower and upper bounds, the worst-case time complexity of STEP 3 is exponential in the size of the network topology, cp. Section 5.3.
Let us now prove some properties of the TD-APX algorithm.

Lemma 9.1.5 (Admissibility) *At each iteration j of the algorithm, and for each $t_0 \in T_R(v_0)$, the control sequence $u = \big((\Delta t_k, e_k)\big)_{k=1,2,\dots}$, defined recursively by*

$$\Delta t_k = \overline{\mu}_{w,i(v_{k-1})}(v_{k-1}, t_{k-1}), \qquad k = 1, 2, \dots, \tag{9.9}$$

$$e_k = \overline{\mu}_{nw,i(v_{k-1})}(v_{k-1}, t_{k-1} + \Delta t_k), \qquad k = 1, 2, \dots, \tag{9.10}$$

$$(v_k, t_k) = \varphi\big((v_{k-1}, t_{k-1}), (\Delta t_k, e_k)\big), \qquad k = 1, 2, \dots \tag{9.11}$$

is admissible and terminates in $v' \times T_R(v')$ after a finite number $n \in \mathbb{N}$ of steps, thereby defining an admissible path $\big((v_k, t_k)\big)_{k=0,\dots,n}$ from v_0 to v'.

Proof We prove the assertion by induction over the number of iterations j of the algorithm. The admissibility of the initial iterates is a consequence of the FIFO-property and follows from the same arguments as in Lemma 3.5.4. Since $\tau > 0$ and X is compact, there exists a $\underline{\tau} > 0$, such that $\tau(e, t) \geq \underline{\tau}$ for all $e \in E$, $t \in T_R(\alpha(e))$. Clearly, the iterative steps (9.1)-(9.4) and (9.5)-(9.8) preserve these properties, since all constraints are explicitly taken into account. Consequently, the recursion (9.9)-(9.11) terminates in v' after a finite number of steps. (Otherwise the corresponding path would need infinitely long to reach v'.) □

Remark 9.1.6 *Recall that the control policy $\underline{\mu}_{nw}, \underline{\mu}_w$ is not initialized in STEP 2. Hence, a recursive construction of the control sequences and paths as in (9.9)-(9.11) is not possible in general. However, if all nodes have been iterated at least once, the control policy $\underline{\mu}_{nw}, \underline{\mu}_w$ can be used to construct admissible control sequences terminating in v' in a similar manner as in Lemma 9.1.5. However, as a consequence of the computation of the initial cost values in $\underline{G}$, the cost associated with these control sequences is generally unknown.*

Lemma 9.1.7 (Monotonicity) *At each iteration j of the algorithm, for all $(v, t) \in X_R$ with $i := i(v) \geq 1$, there holds*

$$\underline{b}_{i-1|i-1}(v, t) \leq \underline{b}_{i|i}(v, t) \leq b^*(v, t) \leq \overline{b}_{i|i}(v, t) \leq \overline{b}_{i-1|i-1}(v, t). \tag{9.12}$$

Proof The assertion follows by induction from Theorem 4.4.2 and [26, Lemma 1.1.1]. □

Theorem 9.1.8 (Optimality) *If the TD-APX algorithm terminates, then*

$$\left\| b^*(v_0,\cdot) - \underline{b}_{i(v_0)|i(v_0)}(v_0,\cdot)\right\| \le \epsilon, \qquad \left\| b^*(v_0,\cdot) - \overline{b}_{i(v_0)|i(v_0)}(v_0,\cdot)\right\| \le \epsilon, \tag{9.13}$$

and the corresponding ϵ-optimal paths and control sequences are characterized in Lemma 9.1.5.

Proof In view of Lemma 9.1.7, Remark 9.1.1 and Remark 9.1.2 we only need to prove that $V_O \neq \emptyset$ throughout the algorithm unless $T_R(v_0) = \emptyset$. If $T_R(v_0) \neq \emptyset$ and $V_O = \emptyset$, then for all $v \in V_R$ we must have (cf. lines 19, 30, 18, 28)

$$\left\| b_{i(v)+1|i(v)}(v,\cdot) - b_{i(v)|i(v)-1}(v,\cdot)\right\| = 0, \qquad b \in \{\underline{b}, \overline{b}\},$$

which implies that reapplying the iteration procedure (9.5)-(9.8) at any $v \in V_R$ would result in the same cost functions. Hence

$$\begin{aligned} b_{i(v)|i(v)}(v,t) = \min_{(\Delta t,e)\in U_R(v,t)} \Big[&\delta(v,t,\Delta t) + \beta(e,t+\Delta t) \\ &+ b_{i(\omega(e))|i(\omega(e))}\big(\omega(e), t+\Delta t + \tau(e,t+\Delta t)\big)\Big], \qquad b \in \{\underline{b}, \overline{b}\}. \end{aligned}$$

Since we have $\underline{b}_{i(v')|i(v')}(v',t') = \overline{b}_{i(v')|i(v')}(v',t') = 0$ for all $t \in T(v')$ (cf. lines 6, 14), and v' is reachable from each $(v,t) \in X_R$, the current iterates $\underline{b}_{i(\cdot)|i(\cdot)}(\cdot,\cdot)$, $\overline{b}_{i(\cdot)|i(\cdot)}(\cdot,\cdot)$, are fixed points of the dynamic programming equations and must therefore be optimal [47][Proposition 4.1]. □

Theorem 9.1.9 (Termination) *The TD-APX algorithm terminates after a finite number of iterations.*

Proof As in the proof of Theorem 9.1.8, we see that, if the algorithm terminates due to $V_O = \emptyset$, then we also have $\left\| \overline{b}_{i(v_0)|i(v_0)}(v_0,\cdot) - \underline{b}_{i(v_0)|i(v_0)}(v_0,\cdot)\right\| \le \epsilon$. Hence, it is sufficient to prove the assertion for the termination criterion $V_O = \emptyset$. We suppose that the algorithm does not terminate after a finite number of steps. Then, there must be a subset V_∞ of nodes, $V_\infty \subset V_R$, which are iterated infinitely often. If a node v is chosen to be iterated, we must have $v \in V_O$, i.e., cf. lines 19, 30, 18, 28,

$$\sum_{b\in\{\underline{b},\overline{b}\}} \left\| b_{i(v)+1|i(v)}(v,\cdot) - b_{i(v)|i(v)-1}(v,\cdot)\right\| > 0.$$

Hence, there must be a point in time at which one of the predecessors of v has been changed in one of the preceding iterations. For $v \in V_\infty$, let $T_j(v) \subset T_R(v)$ denote the set of those points in time $t \in T_R(v)$, for which at least one of the cost values $\overline{b}_{i(v)|i(v)}(v,t), \underline{b}_{i(v)|i(v)}(v,t)$ is changed in an iteration $j' \ge j$. We obviously have $T_j(v) \subset [\underline{t}_R(v), \overline{t}_R(v)]$. Now, let $J_k \in \mathbb{N}$, $k \in \mathbb{N}$ denote the number of iterations, such that each $v \in V_\infty$ has been iterated at least k times after J_k iterations of the algorithm. Since $\tau > 0$ and X_R is compact, there exists a $\underline{\tau} > 0$, such that $\tau \ge \underline{\tau}$. As each change in a cost function is transported backwards in time to the predecessors of the respective node, we have $T_{J_k}(v) \subset [\underline{t}_R(v), \overline{t}_R(v) - k\underline{\tau}]$.

Consequently, there is a $k^* \in \mathbb{N}$, and a finite number of iterations J_{k^*}, such that none of the cost functions in V_∞ can be changed in any subsequent iteration of the algorithm. This is a contradiction. □

Remark 9.1.10 *Observe that, if* $p = \big((v_k, t_k)\big)_{k=0,\dots,n}$ *is an optimal path from* v_0 *to* $v_n = v'$*, then the TD-APX algorithm yields*

$$\underline{b}_{i(v_k)|i(v_k)}(v_k, t_k) = b^*(v_k, t_k) = \overline{b}_{i(v_k)|i(v_k)}(v_k, t_k) \tag{9.14}$$

if v_k *is iterated and we already have* $\underline{b}_{i(v_{k+1})|i(v_{k+1})}(v_{k+1}, t_{k+1}) = \overline{b}_{i(v_{k+1}),i(v_{k+1})}(v_{k+1}|t_{k+1}) = b^*(v_{k+1}, t_{k+1})$*. Hence, if* $\widehat{v}_j$ *denotes the node chosen in the* j*-th iteration of the TD-APX algorithm, and* $(v_k)_{k=n-1,\dots,0}$ *is a subsequence of* $(\widehat{v}_j)_{j=1,\dots,J}$ *for some* $J \in \mathbb{N}$*, then (9.14) holds. Consequently, if we modify the TD-APX algorithm in such a way that*

$$\max_{v \in V_O} i(v) / \min_{v \in V_O} i(v) < C$$

throughout the course of the algorithm for some $C > 1$*, then the TD-APX algorithm terminates also in the case of negative travel times. (This is due to the fact that both the length of each optimal path and the number of nodes in the network is bounded.)*

9.2. The numerical example revisited

Let us again consider the network depicted in Figure 8.3(a), with the same travel times, travel cost, waiting cost, arrival time restrictions, waiting time restrictions, source node v_0 and goal node v' as in Section 8.2, except for

$$\tau(e_1, t) = \begin{cases} 3 - t, & 0 \le t \le 2 \\ 1, & 2 < t \le 4 \end{cases} .$$

Note that now the network is a FIFO-network. Let us perform the operations of the TD-APX algorithm.
In STEP 1, we compute

$$\begin{aligned} \underline{t}_R(v_0) &= \underline{t} = 0, & \overline{t}_R(v_0) &= 3, & T_R(v_0) &= [0, 3], \\ \underline{t}_R(v) &= 1, & \overline{t}_R(v) &= 3, & T_R(v) &= [1, 3], \\ \underline{t}_R(v') &= 2, & \overline{t}_R(v') &= \overline{t} = 4, & T_R(v') &= [2, 4], \end{aligned}$$

and set

$$\begin{aligned} (\overline{\mu}_{\mathrm{nw},0}, \overline{\mu}_{\mathrm{w},0})(v_0, t) &= (e_1, 0), & \overline{b}_{-1|0}(v_0, t) &= \overline{b}_{0|0}(v_0, t) = 1 + (t-2)^2, \\ (\overline{\mu}_{\mathrm{nw},0}, \overline{\mu}_{\mathrm{w},0})(v, t) &= (e_3, 0), & \overline{b}_{-1|0}(v, t) &= \overline{b}_{0|0}(v, t) = 3/2, \end{aligned}$$

as well as $\overline{b}_{-1|0}(v',t) = \overline{b}_{0|0}(v',t) = 0$. Moreover, we compute

$$\begin{aligned} \overline{b}_0(v_0) &= 0, & \overline{b}'(v_0) &= 5, \\ \overline{b}_0(v) &= 9/2, & \overline{b}'(v) &= 3/2, \\ \overline{b}_0(v') &= 6, & \overline{b}'(v') &= 0. \end{aligned}$$

Hence we obtain $V_R = V$, $E_R = E$ and the cost functions $\underline{\beta}, \underline{\delta}$ of the network $\underline{G} = (V_R, E_R; \underline{\beta}, \underline{\delta})$:

$$\begin{aligned} \underline{\beta}(e_1) &= 1, & \underline{\beta}(e_2) &= 7/2, & \underline{\beta}(e_3) &= 3/2, \\ \underline{\delta}(v_0) &= 0, & \underline{\delta}(v) &= 0, & \underline{\delta}(v') &= 0. \end{aligned}$$

In STEP 2, we compute

$$\begin{aligned} \underline{b}_0(v_0) &= 0, & \underline{b}'(v_0) &= 1, & \underline{b}_{-1|0}(v_0,t) &= \underline{b}_{0|0}(v_0,t) = 1, \\ \underline{b}_0(v) &= 7/2, & \underline{b}'(v) &= 3/2, & \underline{b}_{-1|0}(v,t) &= \underline{b}_{0|0}(v,t) = 3/2, \\ \underline{b}_0(v') &= 1, & \underline{b}'(v') &= 0. & \underline{b}_{-1|0}(v',t) &= \underline{b}_{0|0}(v',t) = 0. \end{aligned}$$

Since $\min_{v \in V_R} \overline{b}_0(v) + \overline{b}'(v) = 5 \geq \underline{b}_0(v_R) + \underline{b}'(v_R)$ for all $v_R \in V_R$, none of the nodes in V_R can be excluded from further consideration in STEP 2.
In STEP 3, we first determine that $V_F = \{v, v'\}$ since the lower and upper bounds for the cost functions at both nodes coincide. (9.3) and (9.4) at v_0 lead to $\overline{b}_{1|0}(v_0,\cdot) = \overline{b}_{-1|0}(v_0,\cdot)$ since $\overline{b}_{-1|0}(v_0,\cdot)$ is already optimal (cp. Section 8.2). (9.1) and (9.2) lead to

$$\underline{b}_{1|0}(v_0,t) = \left\{ \begin{array}{ll} \min\{1+(t-2)^2, 5\}, & 0 \leq t \leq 1 \\ \min\{1+(t-2)^2, 6\}, & 1 < t \leq 2 \\ 1+(t-2)^2, & 2 < t \leq 3 \end{array} \right\} = 1+(t-2)^2$$

and $\underline{\mu}_{\mathrm{nw},1}(v_0,t) = e_1$. Hence, $V_O = \{v_0\}$, v_0 is iterated and, since $\overline{b}_{1|0}(v_0,\cdot) = \underline{b}_{1|0}(v_0,\cdot)$, we obtain $b^*(v_0,\cdot) = \overline{b}_{1|1}(v_0,\cdot) = \underline{b}_{1|1}(v_0,\cdot)$, which leads to the termination of the TD-APX algorithm.

10. Conclusion

We conclude this thesis by summarizing our results and indicating directions for further research.

10.1. Summary

In this thesis, we have considered the time-dependent optimal path problem with arrival time and waiting time constraints. This problem arises in many applications, among which we were particularly interested in the computation of fuel-optimal paths in the time-dependent road network. Since in real-world applications the problem data is usually subject to uncertainty, we have also considered a robust formulation of the time-dependent optimal path problem in which the travel time and cost functions are only known to assume values in a certain (time-varying) range.

We have derived necessary and sufficient conditions for the existence of optimal paths and the (lower semi-) continuity of the optimal value function. Considering piecewise analytic problem data, we were able to prove the directional differentiability of the optimal value function. Piecewise analytic functions also constitute the largest class of functions for which this result can be shown to hold since the (pointwise) minimum of two functions must generally be formed in the optimization process. For practical considerations, a piecewise linear description of the travel time, cost and constraint functions is the most appropriate. In any other case, the exact solution of the optimal path problem would involve the computation of the roots of general nonlinear functions. Motivated by the particular importance of the piecewise linear optimal path problem we have carried out a detailed complexity analysis for this kind of problem data. The results indicate that both the FIFO-property of the network and the form of the waiting time restrictions have a crucial impact on the space and time complexity of computing the optimal solution.

Traditionally, the computation of optimal paths for a fixed departure time is most common in applications such as navigation systems. For this problem setting, we have derived a pruning criterion which allows a significant reduction of the search space of any optimal path algorithm. A study of the impact of the arrival time constraints on the problem complexity has shown that the discrete-time time-dependent optimal path problem with fixed departure time is polynomially solvable in FIFO-networks if the arrival time constraints are tight enough, whereas no such result holds in continuous time.

We have proposed two algorithmic solutions of the time-dependent optimal path problem with varying departure time, which are likely to outperform the methods published in the past. The DOT* algorithm generalizes the decreasing order of time algorithms to heuristic search. Thereby, the manner in which a node and a time interval are chosen for expansion by the algorithm does not rely on the piecewise linear structure of the network. Furthermore, the heuristic is used to reduce the part of the state space which must be explored by the algorithm. The TD-APX algorithm, which is designed for FIFO-networks, generates an admissible solution in polynomial time. Maintaining an upper bound on the accuracy of the found solutions, the optimal value is then iteratively and monotonically approximated. This approach seems to be promising, especially when the computation time rather then the accuracy of the solution is critical, since the approximation process can be terminated with adjustable accuracy. Both methods have been experimentally evaluated with real-world data from the road network of Ingolstadt. Especially the results achieved with the TD-APX algorithm demonstrated the practical applicability of the proposed approaches.

10.2. Directions for future research

During the study of the time-dependent optimal path problem, a couple of interesting directions for future research have emerged. We now list the most promising among them.
From a modeling point of view, in order to obtain suitable input data for physical consumption models, there is a need for extending the known theories of urban traffic in such a way that, depending on a given traffic density, a common distribution of vehicle speeds and accelerations can be derived. These distributions can then be used as input data for the physical consumption models in order to compute the fuel consumption more accurately. It would also be interesting to simultaneously consider the optimization of the route and the optimization of the velocity plot on the route. Although such an optimization problem must respect the constraints which are imposed by the traffic, we believe the resulting solution to have high fuel-saving potential. However, we argue that an exact solution of this optimization problem will be computationally infeasible because of the large problem size.
Our results on the robust time-dependent optimal path problem should rather be seen as a basis for further work than a complete body of research. Both the assumption which allows the application of dynamic programming without extending the state space and the assumption which allows the separate consideration of the travel time and cost are quite restrictive and may not be fulfilled in practical applications. A complexity analysis of the absolute robust time-dependent optimal path problem with fixed departure time also remains as an open question.
In view of the applicability of our results for electric vehicles it would be interesting to incorporate further resource constraints into the optimal path problem, such as the finite capacity of the battery. If the remaining capacity of the battery is introduced as an additional state, the dimensionality of the resulting dynamic programming problem would increase and thereby necessitate the extension of the results of this thesis to two-dimensional parametric optimization problems.
From an algorithmic point of view, the verification of the optimality of the DOT* algorithm (in the sense in which the A* algorithm is optimal) remains as an open question. Further research on good choices of the iteration node in the TD-APX algorithm seems also promising. Moreover, it would be interesting to generalize the proposed methods in such a way that also the absolute robust optimal path problem can be solved. In order to use the algorithms for real-time applications, the parallelization of both methods will be a challenge for the future.

Appendix

A. A Case Study in the Road Network of Ingolstadt

In order to illustrate and road-test the theoretical and algorithmic results of this thesis we have carried out a case study in the road network of the German city of Ingolstadt. This case study comprises the computation of the energy consumption using traffic data and a physical consumption model as well as the experimental evaluation of the proposed algorithms in the resulting time-dependent road network of Ingolstadt.

As it has already been mentioned in Section 1.1 and Chapter 2, the computation of the energy consumption associated with a route in the road network is not only relevant for vehicles with combustion engines but also for electric vehicles. Indeed, one of the core problems of the development of electric vehicles is the conflict of objectives between the cruising range and the size and cost of the battery [125]. That is because on the one hand the energy capacity is proportional to the cruising range, and on the other hand the battery is the most expensive component of hybrid and electric vehicles, since it must meet the high demands on energy capacity and power demand in road traffic [126]. Moreover, since up to now the infrastructure which allows the charging of the battery of an electric vehicle is sparse, the prediction of the energy consumption associated with a route (or the prediction of the cruising range) is crucial, even if it is not the objective of optimization. These framework conditions were also taken into account in the MUTE project [4] (in which researchers of the Technische Universität München develop a prototype of a technically and economically convincing electric vehicle) and it was decided to incorporate the computation of energy-optimal routes into the advanced driver assistance systems of the MUTE-vehicle. The vehicle parameters used in the planning phase of the MUTE-vehicle were provided by the project team for an evaluation of the theoretical results derived in this thesis and are denoted in Table A.1.

In the following sections we describe the collection of the time-dependent traffic data (cf. Section A.1) and in which manner the vehicle parameters provided in Table A.1 and the average speed data were used in order to define the time-dependent network of Ingolstadt (cf. Sections A.2 and A.3). In Section A.4, we provide a detailed discussion of the computational experiments which were carried out in this road network.

A.1. Data Collection

The speed values which form the basis for the evaluation of the algorithms are derived from taxi floating car data, which was kindly provided by Taxi-Funk Ingolstadt GmbH & Co. KG. The time and position information which are repeatedly sent from each taxi are first received via a telnet stream. A map matching and a routing procedure are then used to determine which road segments the taxi has been traveling through between two consecutive information transmissions (for details, see [57]). In order to assign a speed value to each road segment which was partially or completely traveled through, it is assumed that the vehicle

Vehicle parameter	Planning value in the MUTE project
Vehicle mass (including battery)	500 kg
Friction coefficient	0.007
Frontal area	1.696 m^2
Air-drag coefficient	0.28
Transmission ratio [1]	9.11
Motor moment of inertia	0.03 kg m^2
Wheel perimeter	1.7813 m
Wheel moment of inertia	0.80 kg m^2
Basic power consumption	770 W
Engine power	15 kW
Energy content of the battery	9 kWh

Table A.1.: Vehicle parameters used in the planning phase of the MUTE project [4].

No Vacation	Vacation	Holiday
Monday	Monday	Monday-Sunday
Tuesday-Thursday	Tuesday-Thursday	
Friday	Friday	
Saturday	Saturday	
Sunday	Sunday	

Table A.2.: Class definition of calendar days with similar traffic conditions.

speed in the time interval between two transmissions is independent both of the time and the position. Based on all data which has been received for one particular road segment e during a time interval $[t_k - 15\,\text{min}, t_k]$, $t_k = k \cdot 5\,\text{min}$, $k = 0, ..., 287$, the current average speed $\bar{v}_k(e)$ on this road segment is determined. In particular, if $n_k(e)$ taxis have been traveling through some road segment e of length $L(e)$ during the time interval $[t_k - 15\,\text{min}, t_k]$ and $\tau_{k,n}(e)$ denotes the travel time of the n-th taxi, then $\bar{v}_k(e)$ is set to

$$\bar{v}_k(e) = \frac{1}{n_k(e)} \sum_{n=1}^{n_k(e)} \frac{L(e)}{\tau_{k,n}(e)}. \tag{A.1}$$

These average speeds have been stored for each road segment in a data base for the duration of about one year.
As many research works show [169], [94], [121], the traffic state strongly depends on the calendar day, since traffic demand on working days is very different to that on weekend or on holidays. In order to identify calendar days with similar traffic conditions a correlation analysis has been applied, which led to the class definition in Table A.2.

Since traffic may also change from season to season due to different weather conditions, the average speed values are aggregated for a time duration of three months and based on the classification in Table A.2. As the validity of speed information decreases with the age of the measurement, the aggregated average speeds (except for the class "Holiday") are weighted with half-life (for details, see [108]).

[1] The MUTE-vehicle has only one gear.

A.2. Data Postprocessing

As the topological structure of the time-dependent network of Ingolstadt we used the graph (V, E) contained in the digital map which was kindly provided by the PTV AG for research purposes. In the first step of the data postprocessing, the time-dependent speed data resulting from the data collection described in Section A.1 were matched to the road segments E. This results, for each road segment $e \in E$ in the road network, in a collection $\big(t_k, \bar{v}_k(e), n_k(e)\big)_{k=0,\dots,287}$ of times of day t_k, average speeds $\bar{v}_k(e)$ and number of observations $n_k(e)$. Since the measurements on a road segment may be sparse or unevenly distributed throughout the day (i.e., $n_k(e) = 0$ for many k), single discordant values may lead to an unlikely variation in time of the average speed. In order to cope with this situation we have implemented a weighted kernel regression using a Gaussian kernel [157]. If, at some point in time t_k an average speed $\bar{v}_k(e)$ had been determined from $n_k(e)$ measurements, then we associated the weight $w_k(e) = n_k(e)$ with $\bar{v}_k(e)$. Otherwise, we used the weight $w_k(e) = 0.1$ together with the average speed $\bar{v}_k(e) = \bar{v}_{\text{map}}(e)$ which was contained in the digital map as an input to the kernel regression. The standard deviation for the kernel function was set to $0.5\text{h}\cdot\sqrt{288/\sum_{k=0}^{287} w_k(e)}$ which led to a satisfactory tradeoff between the preservation of the local variations in and the smoothing of the average speed data. We denote the k-th smoothed average speed on a road segment e by $g\big(\bar{v}_k(e)\big)$.

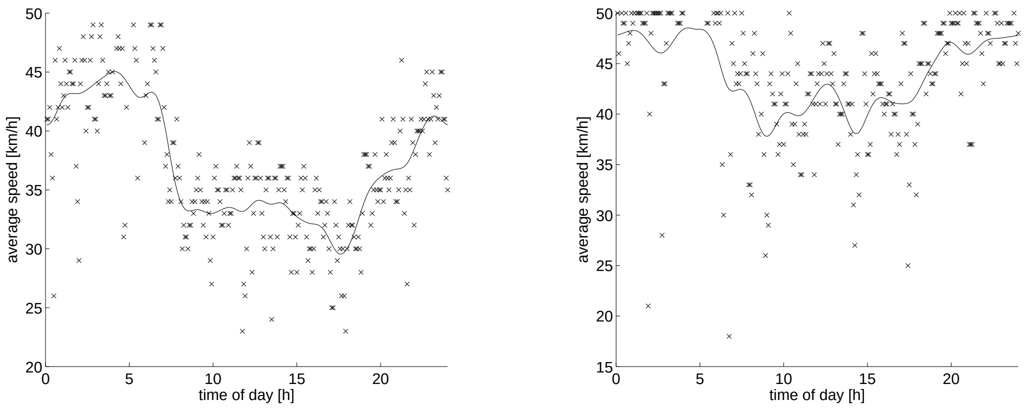
The output data of the postprocessing step are, for each road segment e in the road network, a collection $\big(t'_k, \bar{v}'_k(e), \tau'_k(e)\big)_{k=0,\dots,71}$ of times of day t'_k, average speeds $\bar{v}'_k(e)$ and travel times $\tau'_k(e)$. Time-dependent average speed (average travel time) data were only assigned to those road segments on which measured average speeds had been associated with at least 12.5 % of the points in time of the input data. For the remaining road segments, the time-independent speed data $\bar{v}_{\text{map}}(e)$ (resp., travel time $\tau(e) = L/\bar{v}_{\text{map}}(e)$) of the digital map was used. This led to a proportion of 30 % of time-dependent road segments in the network of Ingolstadt.

Since the average speed $\bar{v}_k(e)$ on a road segment e of length $L(e)$ at time t_k has been determined according to (A.1) from measurements during the time interval $[t_k - 15\,\text{min}, t_k]$, we set

$$\tau(e, t_k - 5\,\text{min}) = \frac{L(e)}{g\big(\bar{v}_k(e)\big)}. \tag{A.2}$$

In this manner, the point in time $t'_k - 5\,\text{min}$ with which the travel time is associated is chosen in such a way that it is both an inner point of the measurement interval and not contained in the $(k-1)$-th measurement interval. Note that (A.2) corresponds to the (smoothed) harmonic mean of the measured travel times.

In order to obtain an explicit network description, which is suitable (in view of a tradeoff between the accuracy of the time-dependent description and the computational complexity) for the computation of energy-optimal paths in the road network of Ingolstadt we only used 72 data points, corresponding to $t'_k = k\cdot 20\,\text{min}$, $\bar{v}'_k(e) = g\big(\bar{v}_{4(k-1)+1}(e)\big)$ and $\tau'_k(e) = \tau(e, t'_k)$, $k = 0, \dots, 71$, for each road segment e. The average travel times between two such points in time were then determined by linear interpolation, leading to a piecewise linear and continuous description of the travel time function τ. An examination of the derivatives of the linear pieces of the partial functions $t \mapsto \tau(e, t)$, $e \in E$, revealed the FIFO-property of τ and an examination of the breakpoints of the partial functions $t \mapsto \tau(e, t)$, $e \in E$, revealed that $\tau > 0$. An illustration of the postprocessing step is provided in Figure A.1.



(a) Road segment of category "main", $\bar{v}_{\text{map}} = 25\,\text{km/h}$.

(b) Road segment of category "freeway", $\bar{v}_{\text{map}} = 45\,\text{km/h}$.

Figure A.1.: Postprocessing of the average speed data associated with two road segment. The crosses in the plots correspond to the input data, the solid line corresponds to the smoothed output function. The data of the class "Tuesday-Thursday" was used in both plots.

A.3. Definition of the Time-Dependent Cost Functions

In order to estimate the speed distributions which serve as an input for the physical consumption model, we distinguish between the road categories "Local street", "Main street", "Arterial" and "Freeway". We do not distinguish between different driver types or between certain functional zones of the city of Ingolstadt. This coarse resolution is due to the lack of precise driving data (i.e., driving data including the speed and acceleration distributions) and due to the independence of our vehicle model from the gear changing behavior. (The MUTE-vehicle has only one gear, cp. Table A.1.)

In order to determine the speed distribution for a particular road segment and average speed, we use the data published in [60] which was also used to evaluate the impact of driving patterns on fuel-use in [62]. This data results from the recording of the driving data of 29 families for two weeks in the Swedish city of Västerås. The size of Västerås is comparable to the size of Ingolstadt, and hence we assume that (although none of the families drove an electric vehicle and there is a certain variation between the driving patterns in different cities [63]) the speed distributions are comparable to those of Ingolstadt.

Using this data for a calibration of the model, we adapt the methodology presented in Part I as follows: For simplicity, we approximate (2.13) by

$$V = (1 - d_{\text{s}})\big[d_{\text{f}} V_{\text{f}} + (1 - d_{\text{f}}) V_{\text{t}}\big], \tag{A.3}$$

where $d_{\text{s}} \in \{0, 1\}$ is a decision variable which models that the vehicle is in a stop with probability $p_{\text{s}} = \mathbb{P}(d_{\text{s}} = 1) \in [0, 1]$, $d_{\text{f}} \in \{0, 1\}$ is a decision variable which models that a moving vehicle is in free flow with probability $p_{\text{f}} = \mathbb{P}(d_{\text{f}} = 1) \in [0, 1]$, V_{f} is normally distributed,

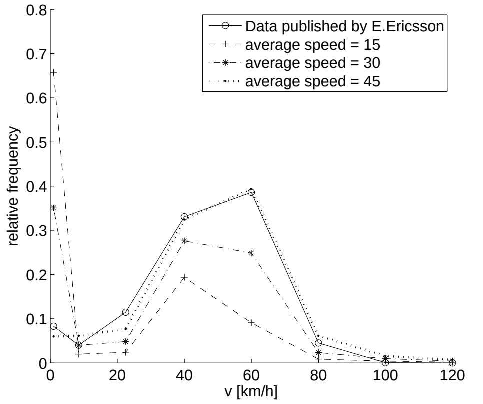
$V_{\mathrm{f}} \sim \mathcal{N}(\mu_{\mathrm{f}}, \sigma_{\mathrm{f}})$, and V_{t} is half-normally distributed, $V_{\mathrm{t}} \sim \mathcal{HN}(\sigma_{\mathrm{t}})$. Hence, in contrast to the model in [87], we do not assume that either $V = 0$ or $V = \bar{v}$, but we associate a normal distribution with the free flow speeds and we introduce a half-normal distribution in order to model the transition between $V = 0$ and the free flow speeds. Observe that, except for the assumption that V_{t} is half normally distributed, (A.3) is equivalent to (2.13). Moreover, the form of the probability density function during an acceleration (resp. deceleration) phase which has been derived in Section 2.2 resembles the probability density function of a half-normally distributed random variable, cp. Figure 2.4. Note that the data published in [60] allows the derivation of a speed distribution of the form (A.3) even if the parameters which were introduced in Section 2.2 are not completely known (as is usually the case in practical applications).
We choose σ_{t} such that $\mathbb{E}[v_{\mathrm{t}}] = \sqrt{2/\pi}\sigma_{\mathrm{t}} = \mu_{\mathrm{f}} - \sigma_{\mathrm{f}}$. Hence, we expect the transition to lead to the “center” of the free flow speed distribution. Indeed, $\mathbb{P}(v_{\mathrm{f}} < \mu_{\mathrm{f}} - \sigma_{\mathrm{f}}) < 16\,\%$. Furthermore, we expect the travel time incurred for a transition to be proportional to the target speed (i.e., to μ_{f}) and the travel time incurred during free flow to be inversely proportional to μ_{f} (since the length of the road segment is fixed). This implies that

$$p_{\mathrm{f}} = \frac{1}{\lambda\mu_{\mathrm{f}}^2 + 1}, \qquad 1 - p_{\mathrm{f}} = \frac{\lambda\mu_{\mathrm{f}}^2}{\lambda\mu_{\mathrm{f}}^2 + 1}$$

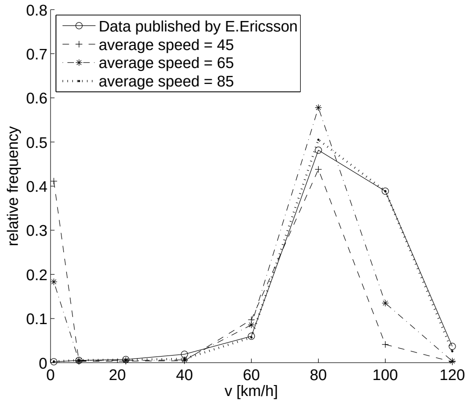
for some $\lambda \geq 0$. Finally, assuming that all random variables are mutually independent, we determine p_{s} from (A.3) according to

$$p_{\mathrm{s}} = 1 - \frac{\mathbb{E}[V]}{p_{\mathrm{f}}\mu_{\mathrm{f}} + (1 - p_{\mathrm{f}})(\mu_{\mathrm{f}} - \sigma_{\mathrm{f}})}.$$

Using this approach, we reconstruct the probability distribution in [60] with a cumulative error of $\approx 5\,\%$ per road category and speed limit.
Since the speed data for each road category in [60] has been recorded for approximately constant traffic conditions and does not allow the consideration of time-dependent variations of the average speed, we determine the distribution parameters for varying traffic conditions from (2.15)-(2.20). Recall that we have assumed that the traffic is undersaturated in (2.15), which is a simplification since there are time periods during which the traffic in Ingolstadt is saturated.
In (2.15) we use the fit parameters $T_{\mathrm{los}} = 6\,\mathrm{s}$ (corresponding to two time periods of length $3\,\mathrm{s}$ of amber light), the average length L of a road of the respective category in Västerås (we have assumed that every second road segment in the digital map leads to an intersection in order to obtain an average intersection density of $> 200\mathrm{m}$ between two intersections as indicated in [60]), the free speed v_0 contained in the digital map, the outflow capacity $\hat{Q} = \max\left\{1800, 1800 \cdot v_0/(50 \ \mathrm{km/h})\right\}$ veh./hour/lane and $\delta = 0.1$ (cp. [87, Fig. 6]). The parameter s is estimated for each road category and speed limit from the average traffic density $\bar{\rho}$ and average speed $\bar{v}$ contained in [60]. The range of the fitted parameter $s \in [1.3, 2]$ reflects the low density of signalized junctions in Västerås and the average priorities in traffic associated with the different road categories and speed limits. The average vehicle length in (2.16) has been set to $v_{\mathrm{veh}} = 5\,\mathrm{m}$.
In (2.18) we have fixed the parameters $\rho_{\max} = 160\,\mathrm{veh/km}$, $T_r = 0.8\,\mathrm{s}$ [84], [85] as well as $v_{\max} = v_0$ and determined ρ_0, v_2, v_3 in such a way that $\rho_{\mathrm{f}} \mapsto \mu_{\mathrm{f}}(\rho_{\mathrm{f}})$ is continuously



(a) Road segment of category "main", speed limit $v_0 = 50\,\text{km/h}$ and length $L = 145\,\text{m}$. The average speed in [60] is $\approx 44\,\text{km/h}$.



(b) Road segment of category "freeway", speed limit $v_0 = 90\,\text{km/h}$ and length $L = 320\,\text{m}$. The average speed in [60] is $\approx 85\,\text{km/h}$.

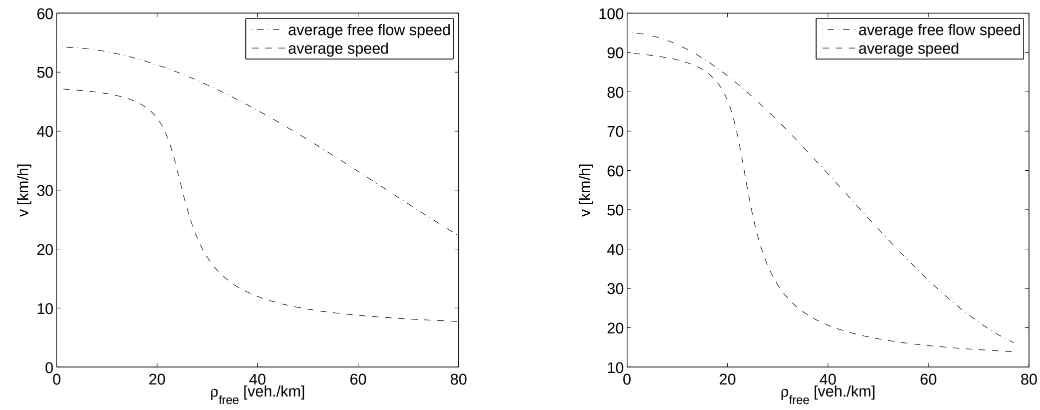
Figure A.2.: Relative frequency of the vehicle speed for different traffic densities on two road segments. The solid line corresponds to the data published in [60], the dotted, dash-dotted and dashed lines correspond to the model (A.3) and (2.15)-(2.20).

differentiable and matches the data published in [60]. The parameter a_0 in (2.20) has been chosen in such a way that (2.19) holds for the data published in [60]. An example of resulting speed distributions for varying traffic densities is depicted in Figure A.2, the corresponding relations between the free flow density and the average speed $\bar{v}$ as well as the average free flow speed μ_f are depicted in Figure A.3.

Given a pair $(t'_k, \bar{v}'_k(e))$ on some road segment e, we estimate the distribution parameters as described above and determine the associated speed distribution which results from (A.3). This speed distribution and the speed-dependent acceleration distributions published in [8] are used to determine a common speed-acceleration distribution associated with the pair $(t'_k, \bar{v}'_k(e))$. The energy consumption function β for the road segment e at time t'_k is then defined according to

$$\beta(e, t'_k) = \int_0^{\tau(e,t'_k)} \mathbb{E}_{v,a|e,k} \left[\widetilde{P}(v, a, \alpha) \right] \mathrm{d}t, \qquad k = 0, ..., l'. \tag{A.4}$$

Here, $\mathbb{E}_{v,a|e,k}$ denotes the expected value with respect to the common speed-acceleration distribution associated with the pair $(t'_k, \bar{v}'_k(e))$ and $\widetilde{P}(v, a, \alpha)$ denotes the power which must be provided by the battery of the vehicle in order to countervail the driving resistances, cp. Section 2.4.2. Since the longitudinal elevation α of the road segments is not contained in the digital map we used the value $\alpha = 0$ in our computations. The energy consumption between two points in time is then determined by linear interpolation, leading to a piecewise linear description of the travel cost function β.



(a) Road segment of category "main", speed limit $v_0 = 50\,\mathrm{km/h}$ and length $L = 145\,\mathrm{m}$.

(b) Road segment of category "freeway", speed limit $v_0 = 90\,\mathrm{km/h}$ and length $L = 320\,\mathrm{m}$.

Figure A.3.: Relations between the free flow density and the average speed $\bar{v}$ as well as the average free flow speed μ_{f}. The range of ρ_{f} corresponds to the range $\rho \in [0, \rho_{\max}]$.

A.4. Experimental Evaluation

In the following subsections we summarize the experimental evaluation of the DOT* algorithm and the TD-APX algorithm in the time-dependent network of Ingolstadt. We first describe the characteristics of the network (cf. Subsection A.4.1), the computer and the implementations (cf. Subsection A.4.2). We then discuss the experimental results in Subsection A.4.3.

A.4.1. Network Description

We consider the time-dependent optimal path problem with a fixed time frame $[\underline{t}, \bar{t}\,]$, $\underline{t} < \bar{t}$, as in Part III, i.e., $T(v) = [\underline{t}, \bar{t}\,]$ for all $v \in V$. We use the graph (V, E) contained in the digital map, the travel time function τ defined in (A.2) and the travel cost function defined in (A.4). Moreover, we set $\Delta T(v, t) = \{0\}$ for all $(v, t) \in X$ and $\delta(v, t, 0) = 0$. As we have mentioned in Section A.2, the travel time function is strictly positive and satisfies the FIFO-property. Hence, denoting the time-dependent network of Ingolstadt by $G = (V, E, \tau; \beta, \delta)$, Assumption 3.5.3 holds in $(G, T, \Delta T)$.
The time-dependent network of Ingolstadt contains 4447 nodes, 11808 (directed) edges, and the partial network functions $t \mapsto \tau_e(t) = \tau(e, t)$, $t \mapsto \beta_e(t) = \beta(e, t)$, $t \in [\underline{t}, \bar{t}\,]$, satisfy

$$\#\tau_e \leq \left(0, \left\lceil \frac{\bar{t} - \underline{t}}{20\,\mathrm{min}} \right\rceil + 2, \left\lceil \frac{\bar{t} - \underline{t}}{20\,\mathrm{min}} \right\rceil + 1\right),$$
$$\#\beta_e \leq \left(0, \left\lceil \frac{\bar{t} - \underline{t}}{20\,\mathrm{min}} \right\rceil + 2, \left\lceil \frac{\bar{t} - \underline{t}}{20\,\mathrm{min}} \right\rceil + 1\right),$$

for all $e \in E$ according to (A.2) and (A.4). For the time-dependent network functions associated with the class "Tuesday-Thursday", there hold $\tau(E \times \mathbb{R}) \subset [0.42\,\mathrm{s}, 375.8\,\mathrm{s}]$ and

$\beta(E \times \mathbb{R}) \subset [0.00065\,\mathrm{kWh}, 0.23\,\mathrm{kWh}]$, and non-constant functions are associated with 3959 edges. Let $\widetilde{E} \subset E$ denote the set of edges for which the partial functions τ_e, associated with the data of the class "Tuesday-Thursday", are non-constant. There holds

$$\max_{e \in \widetilde{E}} \frac{\max_{t \in [0\,\mathrm{h}, 24\,\mathrm{h}]} \tau(e,t)}{\min_{t \in [0\,\mathrm{h}, 24\,\mathrm{h}]} \tau(e,t)} \approx 8.93, \qquad \max_{e \in \widetilde{E}} \frac{\max_{t \in [0\,\mathrm{h}, 24\,\mathrm{h}]} \beta(e,t)}{\min_{t \in [0\,\mathrm{h}, 24\,\mathrm{h}]} \beta(e,t)} \approx 6.34,$$

$$\frac{1}{|\widetilde{E}|} \sum_{e \in \widetilde{E}} \frac{\max_{t \in [0\,\mathrm{h}, 24\,\mathrm{h}]} \tau(e,t)}{\min_{t \in [0\,\mathrm{h}, 24\,\mathrm{h}]} \tau(e,t)} \approx 1.85, \qquad \frac{1}{|\widetilde{E}|} \sum_{e \in \widetilde{E}} \frac{\max_{t \in [0\,\mathrm{h}, 24\,\mathrm{h}]} \beta(e,t)}{\min_{t \in [0\,\mathrm{h}, 24\,\mathrm{h}]} \beta(e,t)} \approx 1.53,$$

which is in accordance with the empirical observations in [103].
Moreover, for the time-dependent network functions associated with the class "Sunday", there hold $\tau(E \times \mathbb{R}) \subset [0.42\,\mathrm{s}, 375.8\,\mathrm{s}]$ and $\beta(E \times \mathbb{R}) \subset [0.00058\,\mathrm{kWh}, 0.23\,\mathrm{kWh}]$, and non-constant functions are associated with 3341 edges. Let $\widetilde{E} \subset E$ denote the set of edges for which the partial functions τ_e, associated with the data of the class "Sunday", are non-constant. There holds

$$\max_{e \in \widetilde{E}} \frac{\max_{t \in [0\,\mathrm{h}, 24\,\mathrm{h}]} \tau(e,t)}{\min_{t \in [0\,\mathrm{h}, 24\,\mathrm{h}]} \tau(e,t)} \approx 8.78, \qquad \max_{e \in \widetilde{E}} \frac{\max_{t \in [0\,\mathrm{h}, 24\,\mathrm{h}]} \beta(e,t)}{\min_{t \in [0\,\mathrm{h}, 24\,\mathrm{h}]} \beta(e,t)} \approx 7.08,$$

$$\frac{1}{|\widetilde{E}|} \sum_{e \in \widetilde{E}} \frac{\max_{t \in [0\,\mathrm{h}, 24\,\mathrm{h}]} \tau(e,t)}{\min_{t \in [0\,\mathrm{h}, 24\,\mathrm{h}]} \tau(e,t)} \approx 1.75, \qquad \frac{1}{|\widetilde{E}|} \sum_{e \in \widetilde{E}} \frac{\max_{t \in [0\,\mathrm{h}, 24\,\mathrm{h}]} \beta(e,t)}{\min_{t \in [0\,\mathrm{h}, 24\,\mathrm{h}]} \beta(e,t)} \approx 1.48,$$

The maxima and minima of τ in the classes "Tuesday-Thursday" and "Sunday" are both attained for $e \in E \setminus \widetilde{E}$.

A.4.2. Description of the Computer and the Implementation

The computational experiments are carried out on an 2×quad-core Xeon E3750 with each processor clocked at 2.33 GHz and provided with 8 MB of L2 cache. The machine has 16 GB of RAM, the operating system is Ubuntu Linux 2.6.32-25-generic, which was compiled with GCC 4.4.3.
Our implementation is written in Java and compiled with Java version 1.6.0_18. The results of Subsection 5.3.2 imply that we must expect the piecewise linear functions to contain a large number of linear pieces. In order to cope with the space complexity of storing the partial optimal value functions, we have set the initial and the maximal heap size of the Java Virtual Machine to 3 GB. The graph (V, E) is represented by saving $E^-(v), E^+(v)$ for each $v \in V$ and $\alpha(e), \omega(e)$ for each $e \in E$. We have used two implementations of piecewise linear functions: The partial network functions τ_e, β_e, $e \in E$, are implemented as arrays in order to rapidly access one particular linear piece. The partial optimal value functions are implemented as linked lists since they are repeatedly modified by the algorithms. This allows a space-efficient storage of the partial optimal value functions and the insertion and removal of linear pieces in $O(1)$ space and time complexity. Linked lists are also well-suited to scroll through the linear pieces, since, in most cases, either no linear piece or all linear pieces of a partial optimal value function must be accessed in one iteration of each of the algorithms.
In a preprocessing step, the all-to-all optimal path problems in the networks $(V, E; \underline{\tau})$ and

$(V,E;\underline{\beta})$ are solved with the Floyd-Warshall algorithm [9, p.147], where $\underline{\tau}: E \to \mathbb{R}^+$ and $\underline{\beta}: E \to \mathbb{R}^+$,

$$\underline{\tau}(e) = \min_{t\in[\underline{t},\bar{t}]} \tau(e,t), \qquad \underline{\beta}(e) = \min_{t\in[\underline{t},\bar{t}]} \beta(e,t), \qquad \forall e \in E,$$

respectively. By $c_t(v_1,v_2)$, we denote the cost of an optimal path from $v_1 \in V$ to $v_2 \in V$ in $(V,E,\underline{\tau})$, and by $c_b(v_1,v_2)$, we denote the cost of an optimal path from $v_1 \in V$ to $v_2 \in V$ in $(V,E,\underline{\beta})$. Finally, we denote

$$\underline{c}_t = \min_{e\in E} \underline{\tau}(e) = \min_{v_1,v_2\in V: v_1\neq v_2} c_t(v_1,v_2),$$
$$\underline{c}_b = \min_{e\in E} \underline{\beta}(e) = \min_{v_1,v_2\in V: v_1\neq v_2} c_b(v_1,v_2).$$

There holds $\underline{c}_t, \underline{c}_b > 0$ for the two time-dependent networks of Ingolstadt, which are generated from the traffic data associated with the classes "Tuesday-Thursday" and "Sunday". In the DOT* algorithm, a hash table [43, Chapter 11] is used in order to choose the iteration node. The hash table contains, for each edge $e \in E$, the value $t^+(e) - \pi_t\big(\alpha(e)\big)$, and the value $\max_{e\in E}\big[t^+(e) - \pi_t\big(\alpha(e)\big)\big]$ and an array of the edges $\arg\max_{e\in E}\big[t^+(e) - \pi_t\big(\alpha(e)\big)\big]$ are separately stored. In each iteration, the edges $\arg\max_{e\in E}\big[t^+(e) - \pi_t\big(\alpha(e)\big)\big]$ are accessed via the hash table and the iteration nodes is chosen according to line 16 of Algorithm 8.1.1. The lower bounds $\pi_\cdot$ (the subscript $\cdot$ stands for t or b in the following) are defined as

$$\pi_\cdot(v) = s \cdot c_\cdot(v_0, v), \qquad \forall v \in V, \tag{A.5}$$

where $s \in [0,1)$. The lower bounds $\widetilde{\pi}_\cdot$ are either defined as

$$\widetilde{\pi}_\cdot(v_1,v_2) = \underline{c}_\cdot, \qquad \forall v_1, v_2 \in V, \tag{A.6}$$

or as

$$\widetilde{\pi}_\cdot(v_1,v_2) = \begin{cases} \max\big\{\underline{c}_\cdot, c_\cdot(v_0,v_2) - c_\cdot(v_0,v_1)\big\}, & \text{if } v_1 \neq v_2 \\ 2\underline{c}_\cdot, & \text{if } v_1 = v_2 \end{cases}, \qquad \forall v_1, v_2 \in V, \tag{A.7}$$

or as

$$\widetilde{\pi}_\cdot(v_1,v_2) = \begin{cases} c_\cdot(v_1,v_2), & \text{if } v_1 \neq v_2 \\ \min_{v\in V: v\neq v_1} c_\cdot(v_1,v) + \min_{v\in V: v\neq v_2} c_\cdot(v,v_2), & \text{if } v_1 = v_2 \end{cases}, \qquad \forall v_1, v_2 \in V. \tag{A.8}$$

As (V,E) contains no loops, the values $\widetilde{\pi}_t(v,v), \widetilde{\pi}_b(v,v)$ in (A.7) and (A.8) are indeed lower bounds of the cost and travel time of any circle containing $v \in V$, respectively. The parameter s scales the length of the time interval for which the partial optimal value function can be computed in one iteration of the DOT* algorithm, cp. (8.2). Small values of s lead to longer time intervals, bigger values of s lead to the expansion of less nodes, cp. Figures A.4(a)-A.4(c). While the definition (A.7) only requires the solution of an one-to-all optimal path problem with the source node v_0, the definition (A.8) requires the solution of an all-to-all optimal path problem. However, the lower bounds $\widetilde{\pi}_\cdot$ defined in (A.8) enable the processing of longer time intervals in each iteration of the DOT* algorithm, since they are greater than or equal to the lower bounds $\widetilde{\pi}_\cdot$ defined in (A.7). In order to compute t^- in

line 17 of the DOT* algorithm, the pointwise minimum of the (shifted) edge cost functions is computed in each iteration. A sorting of the edge cost functions (resp. the sortings of several edge cost functions with respect to several landmarks) has not been implemented because of memory restrictions.
In the first step of the TD-APX algorithm, a bidirectional Dijkstra search is used to determine the sets $T_R(v)$ for all $v \in V$. A bidirectional A* search is then used to define the lower bounds of the cost functions in the second step. The bidirectional A* algorithm is used in order to compute both lower bounds $\underline{b}(v_R), \overline{b}(v_R)$ as soon as possible for some $v_R \in V_R \subset V$. Since nodes are expanded in an increasing order of cost, all nodes which have not yet been expanded by the bidirectional A* algorithm when $\underline{b}_0(v_R) + \underline{b}'(v_R) > \min_{v \in V_R} \left[\overline{b}_0(v) + \overline{b}'(v) \right]$ for the first time for some $v_R \in V_R$, can be removed from V_R, cp. Algorithm 9.1.1, line 12. In the third step of the TD-APX algorithm, a Fibonacci heap [43, Chapter 20] is used to choose the iteration node. The seminorm $\|f\| = \int_D |f(t)|\,\mathrm{d}t$ on the space of real-valued lower semicontinuous functions f with compact domain D is used in the implementation of the TD-APX algorithm. Thereby the repeated computation of the minimum of the partial optimal value functions is avoided. (Note that, as a consequence of Lemma 7.2.3 and Corollary 7.2.4, the above seminorm would be a norm if we considered the time interval $]\,\underline{t}, \overline{t}\,]$ instead of the time interval $[\,\underline{t}, \overline{t}\,]$.)
A more detailed analysis of the algorithms, including a detailed description of the implementations will be presented in [46], [170].

A.4.3. Experimental Results

We have observed in some preliminary experiments that the TD-APX algorithm solves the time-dependent optimal path problem in the road network of Ingolstadt much faster than the DOT* algorithm. This is due to the fact that $\min_{(e,t) \in E \times \mathbb{R}} \tau(e,t) = 0.42\,\mathrm{s}$. If a time frame of 42 minutes is considered and 1000 nodes have to be expanded by the DOT* algorithm, then we must expect $\mathcal{O}(6 \cdot 10^6)$ iterations in the worst case, cp. Theorem 8.1.6. Taking into account that (depending on the complexity of the partial optimal value functions computed so far) one iteration requires a computation time of approximately 0.02 s-0.2 s, we must expect a total computation time of approximately 33 h-333 h. Since this is computationally infeasible both for a statistical evaluation of the algorithm and an utilization in practical applications we are considering four kinds of test scenarios:
In the first scenario, we evaluate the benefits of using precise lower bounds in the DOT* algorithm. Due to the above estimates on the worst-case runtime, only a small number of very small test cases are evaluated. In the second scenario, we compare the average runtimes of the TD-APX algorithm (with $\epsilon = 0\,\mathrm{kWh}$) and the DOT* algorithm (with different definitions of the lower bounds) on a small number of test scenarios in the city center. We then evaluate the computational complexity of the time-dependent optimal path problem in more detail in the third scenario, in which we solve a large set of test cases in the complete network with the TD-APX algorithm. In the fourth scenario, we focus on the differences in the energy consumption and the number of edge sequences which are traversed by optimal paths at different times of day and days of the week.

Scenario 1

In order to evaluate the benefits of using precise lower bounds in the DOT* algorithm, we use a set of 10 test cases. Here, the source and goal node are chosen in a minimum-hop distance $d \in \{3, 4, 5, 6\}$, and the time frame [7:58 h, 8:02 h] is taken as a basis. Three different settings of the DOT* algorithm are used to solve these optimal path problems, in each of which π_t, π_b are defined according to (A.5). In the first setting, the scale parameter s is set to 0, and $\widetilde{\pi}_t, \widetilde{\pi}_b$ are defined according to (A.6). In the second and third setting, the scale parameter s is varied between 0.1 and 0.9, and $\widetilde{\pi}_t, \widetilde{\pi}_b$ are defined according to (A.7) and (A.8). The runtimes and numbers of iterations associated with these test cases are listed in Table A.3. As a reference, the results of the TD-APX algorithm are also provided.

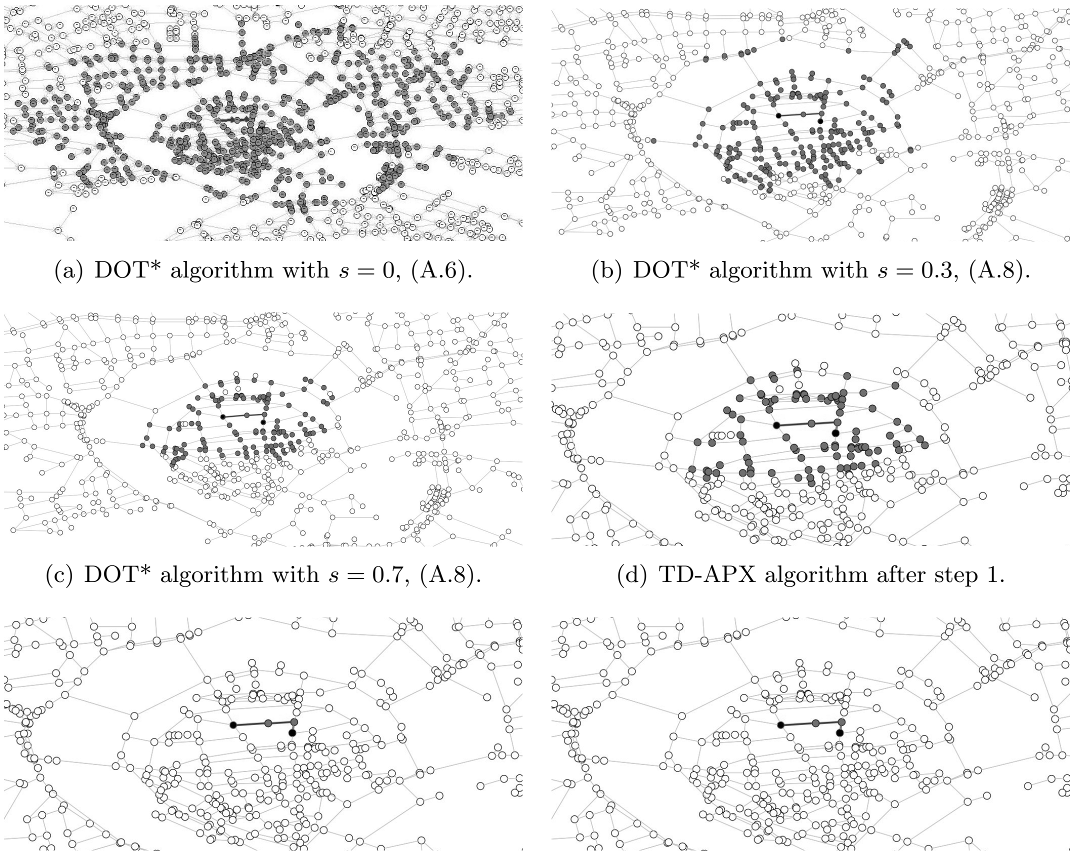
It can be seen that the utilization of precise lower bounds has a drastic impact on the

Algorithm	Setting	Av. comp. time	Min. comp. time	Max. comp. time	Av. no. of iterations
DOT*	lower bound	3751 s	0.001 s	12850 s	65423
	$s = 0.1$, (A.7)	2301 s	0.02 s	7331 s	47179
	$s = 0.1$, (A.8)	86 s	0.03 s	187 s	5551
	$s = 0.3$, (A.7)	993 s	0.02 s	3978 s	25564
	$s = 0.3$, (A.8)	64 s	0.01 s	149 s	4435
	$s = 0.5$, (A.7)	747 s	0.001 s	3293 s	21909
	$s = 0.5$, (A.8)	41 s	0.2 s	95 s	4143
	$s = 0.7$, (A.7)	756 s	0.001 s	3088 s	20848
	$s = 0.7$, (A.8)	38 s	0.01 s	119 s	2550
	$s = 0.9$, (A.7)	813 s	0.001 s	3867 s	20925
	$s = 0.9$, (A.8)	33 s	0.01 s	156 s	1960
TD-APX	$\epsilon = 0$ kWh	0.006 s	0.003 s	0.01 s	13

Table A.3.: Computation time and number of iterations for very short paths. The network functions were generated from the traffic data of the class "Tuesday-Thursday" (working week). 10 pairs of distinct source and goal nodes of minimum-hop distance $d \in \{3, 4, 5, 6\}$ were chosen in the city center.

average runtime and on the average number of iterations of the DOT* algorithm. With an appropriate choice of the scale parameter s, (A.7) improves the average runtime of the DOT* algorithm by one order of magnitude with respect to the setting in which $s = 0$ and $\widetilde{\pi}_t, \widetilde{\pi}_b$ are defined according to (A.6). (A.8) even improves the average runtime of the DOT* algorithm by two orders of magnitude with respect to the setting in which $s = 0$ and $\widetilde{\pi}_t, \widetilde{\pi}_b$ are defined according to (A.6). While this small number of test cases suggests that the optimal scale parameter s is between 0.5 and 0.7 for the lower bounds (A.7), an optimal scale parameter of approximately 0.9 can be observed for the lower bounds (A.8). Furthermore, we observe that the runtime of the DOT* algorithm is highly sensitive to the input data. The minimal and maximal runtimes of each setting of the algorithm differ by 6 orders of magnitude. While the algorithm terminates fast if the edge cost functions differ significantly from each other, a very long runtime is produced if the DOT* algorithm must expand nodes with approximately identical shifted cost functions.

In Figure A.4, we have depicted the node sets which are expanded by the algorithms in one of the above test cases. Despite of the small set of test cases which were evaluated in the first test scenario, the usefulness of precise lower bounds is underlined by the results in Table A.3 and Figure A.4.



(a) DOT* algorithm with $s = 0$, (A.6).

(b) DOT* algorithm with $s = 0.3$, (A.8).

(c) DOT* algorithm with $s = 0.7$, (A.8).

(d) TD-APX algorithm after step 1.

(e) TD-APX algorithm after step 2.

(f) TD-APX algorithm after step 3.

Figure A.4.: Nodes which are expanded by different settings of the DOT* algorithm and in different steps of the TD-APX algorithm ($\epsilon = 0\,\text{kWh}$). The same source and goal nodes and the time frame [7:58 h, 8:02 h] are taken as a basis of all plots.

Scenario 2

Based on the results of the first scenario, we evaluate the performance of the TD-APX algorithm with $\epsilon = 0\,\text{kWh}$ and the performance of the DOT* algorithm with π_t, π_b defined as in (A.5) and $\widetilde{\pi}_t, \widetilde{\pi}_b$ as in (A.8). For the parameter s in (A.5) we use the values $0.5, 0.7$ and 0.9, which led to relatively short average and maximal runtimes in the first test scenario. The source node v_0 and the goal node v' are chosen randomly in the city center (approximately $1\,\text{km}^2$, cp. Figure A.5), and the time frame [7:50 h, 8:00 h] is taken as a basis. 20 pairs $(v_0, v') \in V^2$, $v_0 \neq v'$, are chosen and used as input of each optimal path algorithm. The results of these tests are depicted in Table A.4. In order to compare these results with static optimal path algorithms, the average computation times of Dijkstra's algorithm and the A* algorithm (in the networks $(V, E; \underline{\tau})$ and $(V, E; \underline{\beta})$, respectively) are also provided.

As in the test cases of the first test scenario, the scale parameter $s = 0.9$ yields the best average runtimes of the DOT* algorithm. Furthermore, we again observe a very high spread in the computation times of the DOT* algorithm. For this reason we neither present test cases for longer time frames than the one underlying Table A.4. In some preliminary tests we have observed that the computation time of the DOT* algorithm repeatedly exceeds 12

Algorithm	Setting	[7:50 h, 8:00 h]		
		Av. comp. time	Min. comp. time	Max. comp. time
Dijkstra	-	0.002 s	0.001 s	0.2 s
A*	-	0.002 s	0.001 s	0.006 s
DOT*	$s = 0.5$, (A.8)	1523 s	0.04 s	6505 s
	$s = 0.7$, (A.8)	1243 s	0.001 s	5012 s
	$s = 0.9$, (A.8)	834 s	0.001 s	4264 s
TD-APX	$\epsilon = 0\,\mathrm{kWh}$	0.3 s	0.001 s	1.4 s

Table A.4.: Average computation time of different optimal path algorithms in the time-dependent road network of Ingolstadt. The network functions were generated from the traffic data of the class "Tuesday-Thursday" (working week), the source and goal node were both chosen randomly in the city center.

hours in the time frame [7:50 h, 8:10 h]. This is neither acceptable for practical applications nor suitable for a statistic evaluation.
Although the utilization of precise lower bounds speeds up the DOT* algorithm by a factor of approximately 100, it is obvious from the test results that, in its present implementation, the DOT* algorithm is not suited for practical applications. The average runtime of the TD-APX algorithm outvalues the average runtime of the DOT* algorithm by 3 orders of magnitude. In order to evaluate the DOT* algorithm more extensively in the future we suggest to take into account the following lessons learned from our first experimental results: One focus of future implementations should be on keeping down the number of linear pieces associated with the edge and node cost functions. In most iterations of the DOT* algorithm very short time intervals are expanded. However, when the same node or edge is expanded in a subsequent iteration, it is very likely that the rightmost linear piece computed in the latter iteration extends the leftmost piece of the previous iteration. An efficient implementation of checking such a compliance can be used to reduce the computational overhead. Note that all operations on piecewise linear functions (with the exception of appending a new linear piece to the head or tail of the linked list) require at least logarithmic time in the number of linear pieces. As can be seen from Figure A.6, the number of linear pieces of the partial optimal value function is not exceedingly large in the medium-sized time-dependent optimal path problems, which were considered in this case study. It might therefore be interesting to implement the choice of the iteration node of the DOT* algorithm using a sorting of the edge cost functions. If such an implementation is too memory-consuming (as must be anticipated for larger networks), the computation of the functions

$$t \mapsto \min_{e^+ \in E^+(\widehat{v})} \widehat{b}(e^+, t),$$

$$t \mapsto \min_{\theta \in [t, \widehat{t}\,]} \min_{e \in E : t^+(e) \geq \theta + \widetilde{\pi}_t(\widehat{v}, \alpha(e))} \Big(\widehat{b}\big(\, e,\, \theta + \widetilde{\pi}_t(\widehat{v}, \alpha(e))\,\big) + \widetilde{\pi}_b(\widehat{v}, \alpha(e)) \Big)$$

in line 17 of Algorithm 8.1.1 should be executed simultaneously and in decreasing order of time, and it should be aborted as soon as the value t^- is determined. However, altogether it appears that the optimization strategy of the TD-APX algorithm (local iteration) is superior to the optimization strategy of the DOT* algorithm (global optimization) in the

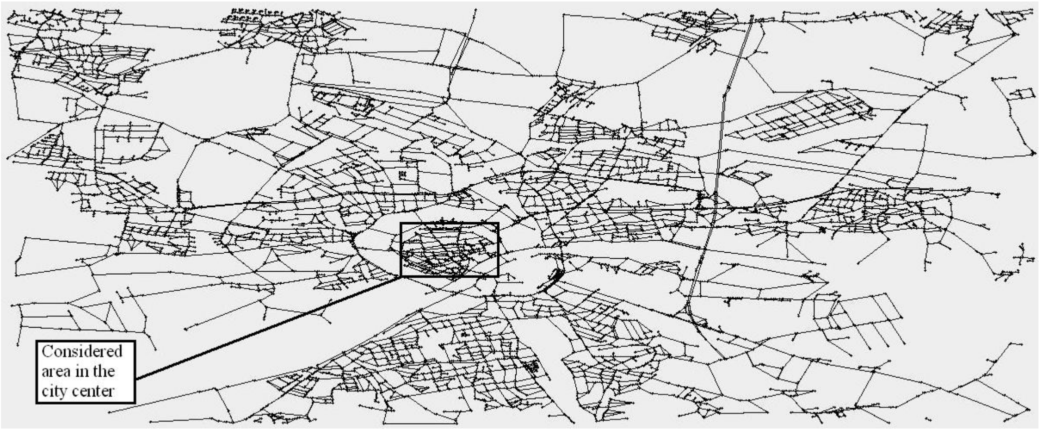


Figure A.5.: The road network of the German city of Ingolstadt.

time-dependent network of Ingolstadt.

Scenario 3

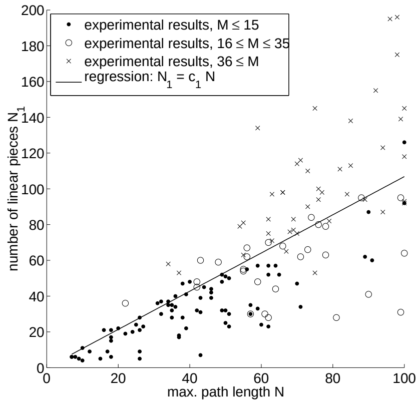
In the third scenario we evaluate the TD-APX algorithm in more detail. We choose the accuracy ϵ as 0%, 1% and 5% of the energy content of the battery, i.e., we set ϵ equal to 0 kWh, 0.09 kWh and 0.45 kWh. Moreover, we vary the time frame $[\underline{t}, \overline{t}]$ by setting $\underline{t}$ = 7:00 h and using the values 7:30 h, 8:00 h, 8:30 h, 9:00 h, 10:00 h, 11:00 h and 12:00 h for $\overline{t}$. For each time frame, a certain number of pairs $(v_0, v') \in V^2$, $v_0 \neq v'$, are chosen and used as input of the TD-APX algorithm. The results of this test are depicted in Table A.5. As a reference, the average computation times of Dijkstra's algorithm and the A* algorithm (in the networks $(V, E; \underline{\tau})$ and $(V, E; \underline{\beta})$, respectively) are also provided.

In order to illustrate the results of one test run of the TD-APX algorithm, the partial

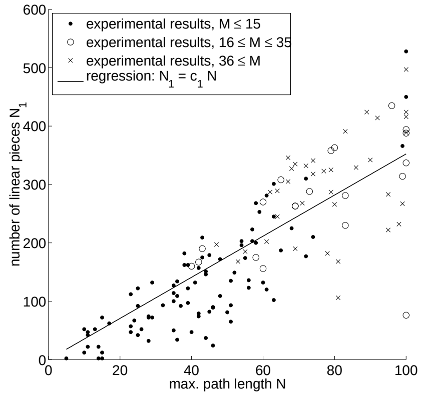
Algorithm	Time frame	Number of test cases	Average computation time		
			$\epsilon = 0$ kWh	$\epsilon = 0.09$ kWh	$\epsilon = 0.45$ kWh
Dijkstra	-	1269	0.0019 s	0.0019 s	0.0019 s
A*	-	1269	0.0015 s	0.0015 s	0.0015 s
TD-APX	[7:00 h, 7:30 h]	238	8.3 s	2.0 s	0.041 s
	[7:00 h, 8:00 h]	243	14 s	2.9 s	0.059 s
	[7:00 h, 8:30 h]	148	22 s	4.2 s	0.097 s
	[7:00 h, 9:00 h]	121	31 s	7.7 s	0.13 s
	[7:00 h, 10:00 h]	88	44 s	10 s	0.18 s
	[7:00 h, 11:00 h]	87	60 s	15 s	0.22 s
	[7:00 h, 12:00 h]	100	120 s	30 s	0.36 s

Table A.5.: Average computation time of the TD-APX algorithm in the time-dependent road network of Ingolstadt. The network functions were generated from the traffic data of the class "Tuesday-Thursday" (working week), the source and goal node were both chosen randomly in the city.

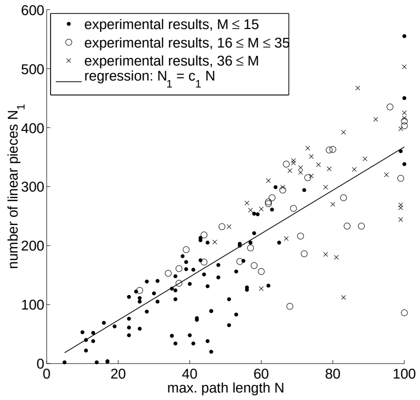
optimal value function $t_0 \mapsto b^*(v_0, t_0)$, the cost function associated with the initial iterate



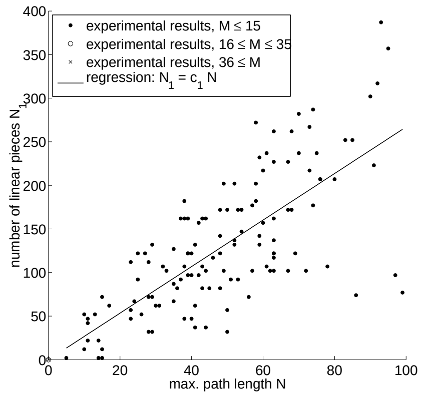
(a) $\underline{t} = 7{:}00\,\text{h}$, $\bar{t} = 7{:}30\,\text{h}$, $\epsilon = 0\,\text{kWh}$.



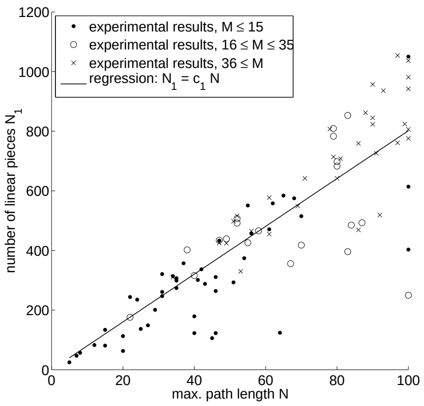
(b) $\underline{t} = 7{:}00\,\text{h}$, $\bar{t} = 9{:}00\,\text{h}$, $\epsilon = 0.09\,\text{kWh}$.



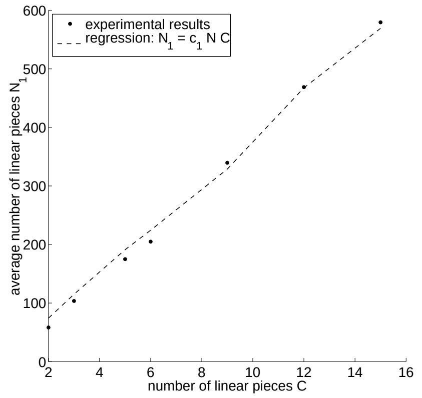
(c) $\underline{t} = 7{:}00\,\text{h}$, $\bar{t} = 9{:}00\,\text{h}$, $\epsilon = 0\,\text{kWh}$.



(d) $\underline{t} = 7{:}00\,\text{h}$, $\bar{t} = 9{:}00\,\text{h}$, $\epsilon = 0.45\,\text{kWh}$.

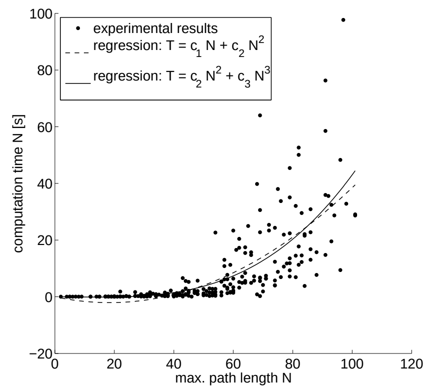


(e) $\underline{t} = 7{:}00\,\text{h}$, $\bar{t} = 11{:}00\,\text{h}$, $\epsilon = 0\,\text{kWh}$.

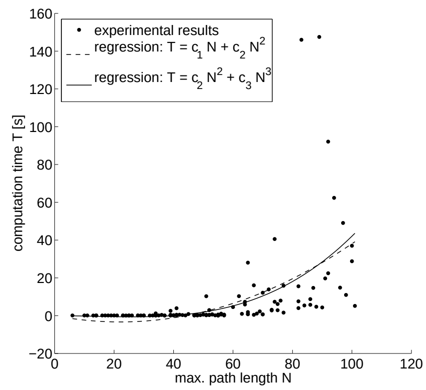


(f) $\epsilon = 0\,\text{kWh}$.

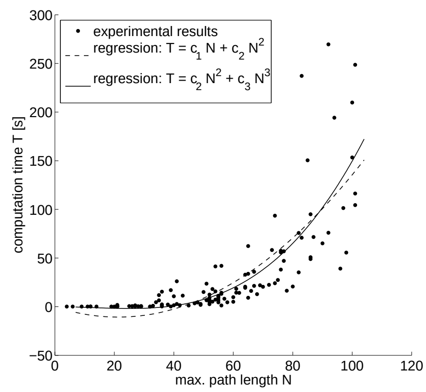
Figure A.6.: Dependence of the number of linear pieces of the partial optimal value function $t_0 \to b^*(v_0, t_0)$ on the maximal topological length N of all optimal paths and on the number C of linear pieces of the partial network functions τ_e, β_e, $e \in E$. In the plots, M denotes the number of mutually distinct edge sequences which are traversed by optimal paths.



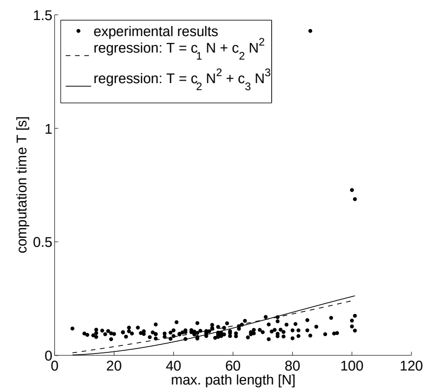
(a) $\underline{t} = 7{:}00\,\mathrm{h}$, $\bar{t} = 7{:}30\,\mathrm{h}$, $\epsilon = 0\,\mathrm{kWh}$.



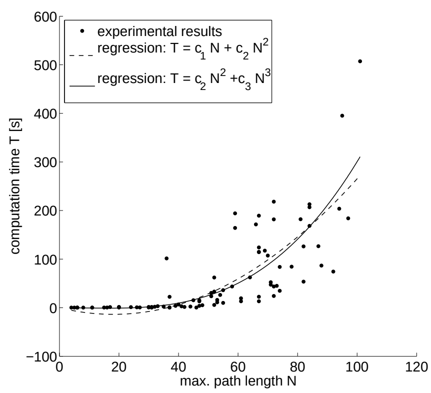
(b) $\underline{t} = 7{:}00\,\mathrm{h}$, $\bar{t} = 9{:}00\,\mathrm{h}$, $\epsilon = 0.09\,\mathrm{kWh}$.



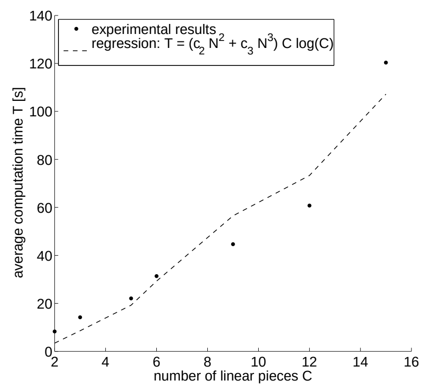
(c) $\underline{t} = 7{:}00\,\mathrm{h}$, $\bar{t} = 9{:}00\,\mathrm{h}$, $\epsilon = 0\,\mathrm{kWh}$.



(d) $\underline{t} = 7{:}00\,\mathrm{h}$, $\bar{t} = 9{:}00\,\mathrm{h}$, $\epsilon = 0.45\,\mathrm{kWh}$.



(e) $\underline{t} = 7{:}00\,\mathrm{h}$, $\bar{t} = 11{:}00\,\mathrm{h}$, $\epsilon = 0\,\mathrm{kWh}$.



(f) $\epsilon = 0\,\mathrm{kWh}$.

Figure A.7.: Dependence of the computation time of the TD-APX algorithm on the maximal topological length N of all optimal paths and on the number C of linear pieces of the partial network functions τ_e, β_e, $e \in E$.

of the control policy, and a subset of the optimal paths are depicted in Figure A.8.

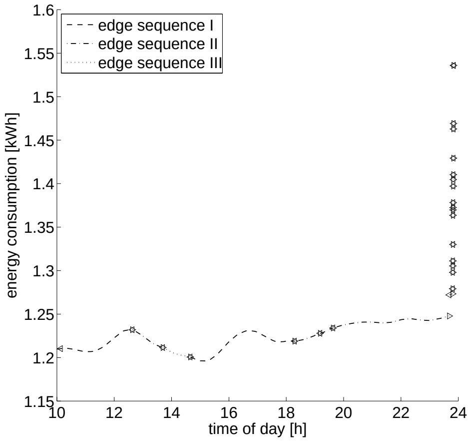
In order to evaluate the predictions of the complexity of the time-dependent optimal path problem with varying departure time, the number of linear pieces of the partial optimal value function $t_0 \mapsto b^*(v_0, t_0)$ is plotted in Figure A.6 and the average computation time of the TD-APX algorithm is depicted in Figure A.7. In these figures, regression analysis is used to analyze the dependence of the number of linear pieces and the average computation time on the maximal topological length N of all optimal paths and on the number C of linear pieces of the partial network functions τ_e, β_e, $e \in E$, cp. (5.59) and (5.48).

From Theorem 5.3.21 *(v)*, we expect the number of linear pieces of the partial optimal value function $t_0 \mapsto b^*(v_0, t_0)$ to be of the order $\mathcal{O}\big(M^2(CN + N_{\text{bd}})\big)$, where M denotes the number of mutually distinct edge sequences from v_0 to v' and $N_{\text{bd}} = 2$ denotes the number of boundary points of $[\underline{t}, \overline{t}\,]$. By contrast, in the evaluation of the experimental results we find the values of N and M to be correlated and the influence of M to be negligible. However, we have noticed that even if the number of mutually distinct edge sequences from v_0 to v' is large, the number of distinct edges is usually very small. This is consistent with the statement in Remark 5.3.24. By inspection of Figures A.6(a), A.6(c) and A.6(e), it appears that, for a fixed value of C, the number of linear pieces of the partial optimal value function grows approximately linearly with the topological length of the longest optimal path. This is consistent with the theoretical findings in Theorem 5.3.21 *(v)*. The large variation in the number of linear pieces is due to the topological structure of the road network of Ingolstadt (the road network is structured in neighborhoods and suburbs) and the distribution of the road segments with which non-constant data is associated.

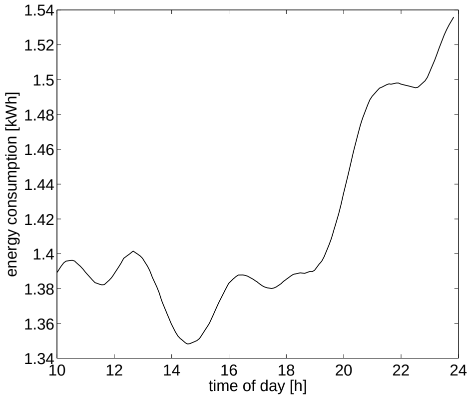
In Figure A.6(f), the average number of linear pieces of the partial optimal value function is plotted in dependence on the number of linear pieces of the network functions. Assuming that the number of linear pieces is approximately proportional to N for a fixed time frame (see above), the computational experiments confirm the result of Theorem 5.3.21 *(v)*, which predicts the average number of linear pieces of the partial optimal value function to be approximately proportional to C for a fixed maximal path length N. In Figure A.6(f), the average maximal path lengths N resulting from the test cases in the respective time frames are used for the regression function.

Based on the result of Theorem 5.3.21 *(v)*, we expect the computation time to be of the order $\mathcal{O}\big(M(CN^2 + NN_{\text{bd}})\log(C) + M^3(CN + N_{\text{bd}})\big)$. As in the evaluation of the number of linear pieces of the partial optimal value function, we find the influence of M on the computation time to be negligible. Moreover, by inspection of Figures A.7(a), A.7(c) and A.7(e), we find that, for a fixed value of C, $\mathcal{O}(N^3 + N^2)$ describes the average computation time better than $\mathcal{O}(N^2 + N)$. This is due to the fact that, in an algorithmic solution of the time-dependent optimal path problem, we need to compute the partial optimal value functions at $\mathcal{O}(N^2)$ nodes. Since the partial optimal value functions at the "inner nodes" of the set of all expanded nodes $V_R \subset V$ are needed to compute the partial optimal value functions at the $\mathcal{O}(N)$ "boundary nodes" of V_R, we must expect N times the computation time which is predicted by Theorem 5.3.21 *(v)*. This is indeed what we can see in Figures A.7(a), A.7(c) and A.7(e).

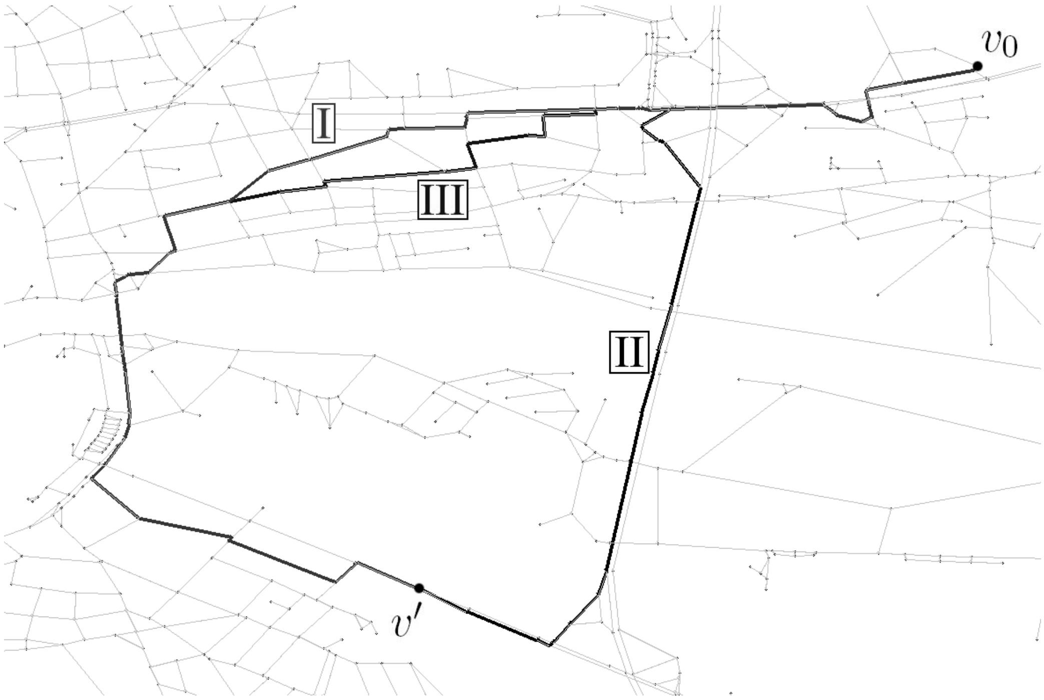
In Figure A.7(f), the average computation time of the TD-APX algorithm for $\epsilon = 0\,\text{kWh}$ is plotted in dependence on the number of linear pieces of the network functions. Assuming that the computation time is roughly proportional to $\mathcal{O}(N^3 + N^2)$ for a fixed time frame (see above), the computational experiments confirm the result of Theorem 5.3.21 *(v)*, which predicts the average computation time to be roughly proportional to $C\log(C)$ for a fixed



(a) Optimal value function. The time domains during which all optimal paths traverse the same edge sequence are delimited by triangles.



(b) Cost function associated with the initial iterate of the control policy.



(c) Subset of the optimal paths. The time domains associated with the edge sequences I, II and III are depicted in Figure A.8(a).

Figure A.8.: Results of one test run of the TD-APX algorithm corresponding to a fixed pair $(v_0, v') \in V^2$ and the time frame [10:00 h, 24:00 h].

maximal path path length N. In Figure A.7(f), the average maximal path lengths N resulting from the test cases in the respective time frames are used for the regression function. When varying the value of ϵ from 0 kWh to 0.45 kWh, the number of linear pieces is decreased by a factor of approximately 1.5, whereas the average computation time is decreased by two orders of magnitude. In fact, as can be seen in Figure A.7(d), the computation time hardly depends on the maximal length of the optimal path if $\epsilon = 0.45$ kWh. This is due to the fact that, if ϵ is large enough, then the initial iterates of the upper and lower bounds of the partial optimal value functions, $t_0 \mapsto \overline{b}_{0|0}(v_0, t_0)$, $t_0 \mapsto \underline{b}_{0|0}(v_0, t_0)$, already satisfy the termination condition $\|\overline{b}_{0|0}(v_0, t_0) - \underline{b}_{0|0}(v_0, t_0)\| \leq \epsilon$. (In this case the number M of edge sequences which are traversed by the ϵ-optimal paths equals 1, cp. Figure A.6(c).) Only the few test cases in which the termination condition is not satisfied in the initial iteration of step 3 of the TD-APX algorithm cause the variation of the average computation times which are listed in Table A.5. The factor 1.5 between the numbers of linear pieces corresponds to the additional breakpoints of the optimal value function which are caused by the switching of the edge sequences traversed by optimal paths. Although the impact of this switching behavior is not of the order of magnitude which we predicted in the worst-case analysis in Theorem 5.3.21 *(v)*, a significant difference of the number of linear pieces can be observed in Figures A.6(c) and A.6(d).

Scenario 4

In the fourth scenario, we evaluate the influence of the time of day and the influence of the day of the week on the solutions of the time-dependent optimal path problem. For this purpose, we compare the time frames [1:00 h, 2:00 h] and [7:30 h, 8:30 h] for the classes "Tuesday-Thursday" (working week) and "Sunday". 400 pairs $(v_0, v') \in V^2$, $v_0 \neq v'$, are chosen randomly and used as the input of the TD-APX algorithm for both time frames and both classes of the days of the week. The value $\epsilon = 0$ kWh is generally used for the TD-APX algorithm in this scenario. The solutions of the optimal path problems are analyzed in terms of the average energy consumption and in terms of the average number of routes. The average energy consumptions are computed as the average of the optimal value functions (i.e., the cost functions associated with the energy-optimal paths $p^*(t)$, $t \in T_R(v_0)$) and as the average of the cost functions associated with the paths $p_{\mathrm{LDT}}(t)$, which are generated by the initial iterate of the TD-APX algorithm, $t \in T_R(v_0)$. The latter paths correspond to the control sequence $\overline{u}(v_0, v')$ which allows the latest departure time (LDT) at v_0, cp. Algorithm 9.1.1, line 1. In particular, $p_{\mathrm{LDT}}(t) = \Phi\big((v_0, t), \overline{u}(v_0, v')\big)$ for $t \in T_R(v_0)$. The average energy consumptions associated with these paths are summarized in Table A.6. In Table A.7, we have listed the average number of (mutually distinct) edge sequences which are traversed by optimal paths from a fixed source node to a fixed goal node and the percentage of test runs for which this number is greater than 1.

We observe that, in the considered test cases, the average energy assumption associated with energy-optimal paths is about 10 % lower than the average energy consumption associated with the control sequence which allows the latest departure time at the source node. (Note that the path $p_{\mathrm{LDT}}(\overline{t}_R(v_0))$ is a fastest path.) We also find that the difference between these average consumption values is smaller at night than in the morning. Furthermore, with the exception of the value associated with energy-optimal paths in the time frame [7:30 h,8:30 h], "Tuesday-Thursday", the average energy consumption is lower

[3] Cost of the path which allows the latest departure time (LDT).

Day	Time	Criterion	Average energy consumption
Tue-Thu	[1:00 h,2:00 h]	Energy	0.7405 kWh
		LDT [3]	0.8196 kWh
	[7:30 h,8:30 h]	Energy	0.7396 kWh
		LDT [3]	0.8303 kWh
Sun	[1:00 h,2:00 h]	Energy	0.7358 kWh
		LDT [3]	0.8133 kWh
	[7:30 h,8:30 h]	Energy	0.7306 kWh
		LDT [3]	0.8159 kWh

Table A.6.: Average energy consumption associated with the paths $p^*(t), p_{\mathrm{LDT}}(t)$ for $t \in T_R(v_0)$.

Day	Time	Average number of edge sequences	Percentage of queries
Tue-Thu	1:00-2:00 h	36	94
	7:30-8:30 h	34	91
Sun	1:00-2:00 h	30	92
	7:30-8:30 h	34	91

Table A.7.: Alternation of the edge sequences traversed by optimal paths from a fixed source node to a fixed goal node.

on Sundays than on Tuesdays-Thursdays. Although the differences are small, they can be ascribed to the heavier traffic conditions during the working week and during the rush hour. We also observe that the average energy consumption associated with the control sequences which allows the latest departure time at the source node are particularly high in commuter traffic, whereas the average energy consumption associated with the optimal control sequences is particularly low in commuter traffic. This can be explained as follows: Driving at high speeds in commuter traffic necessitates a large number of strong acceleration and deceleration maneuvers, which increases the cost associated with the control sequences which allow the latest departure time at the source node. Since driving at lower speeds (≈ 40 km/h, cp. Figure A.10) is generally more economic for the MUTE vehicle than driving at higher speeds, the cost associated with the optimal control sequences is decreased if slow synchronized traffic is found. Since only approximately one third of the edges of the network are attributed with time-dependent data, a larger variation between the average energy consumptions must be expected in general.
From the results in Table A.7 we deduce that the edge sequences traversed by energy-optimal paths differ from the edge sequences traversed by fastest paths in more than 90 % of the considered test cases. As can be seen from Figure A.9, the partial optimal value function increases steeply in the time interval [23:40 h, 23:50 h], where 23:50 h$\approx \bar{t}_R(v_0)$, while the edge sequences associated with the respective cost values are rapidly alternating. This behavior constitutes the transition from the unconstrained to the constrained optimal (fastest) path. (At $\bar{t}_R(v_0)$ only fastest paths are admissible at v_0.) The number of edge sequences which are traversed by the respective optimal paths is surprisingly large and leads to an average of over 30 different edge sequences which are traversed by optimal paths associated with

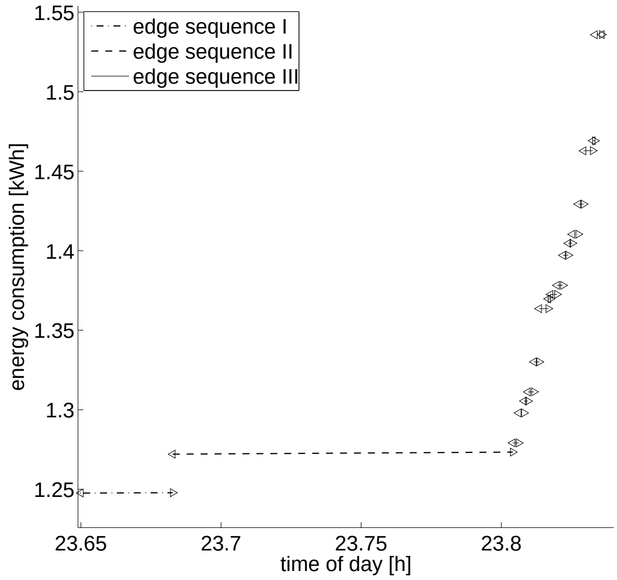


Figure A.9.: Partial optimal value function $t_0 \mapsto b^*(v_0, t_0)$ in a neighborhood of the latest departure time $\bar{t}_R(v_0) \approx$ 23:50 h. The time domains during which all optimal paths traverse the same edge sequence are delimited by triangles. The underlying test case is described and illustrated in Figure A.8.

the considered pairs $(v_0, v') \in V^2$, $v_0 \neq v'$.

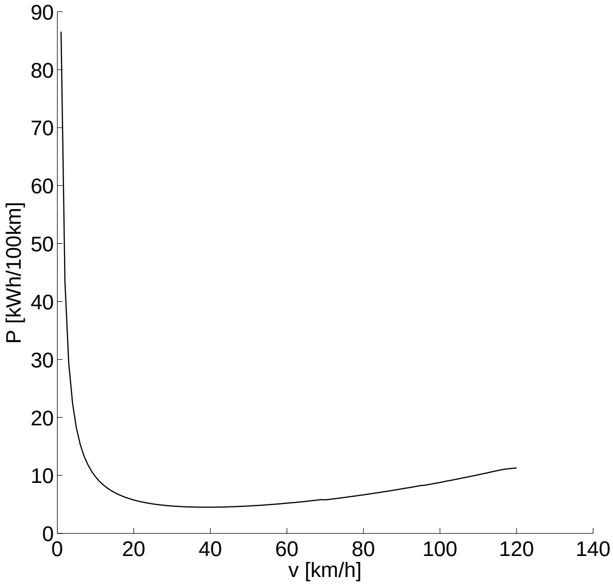


Figure A.10.: Distance-related energy consumption of the MUTE vehicle. The plotted consumption values correspond to the energy consumption associated with a drive at constant speed and no longitudinal elevation.

B. List of Publications

The following articles have been published before the submission of this thesis:

- S.Kluge, M.Brokate and K.Reif, “New complexity results for time-constrained dynamical optimal path problems”, Journal of Graph Algorithms and Applications, vol. 14, no. 2, January 2010, pp. 123-147.
- S.Kluge, K.Reif and M.Brokate, “Stochastic Stability of the Extended Kalman Filter With Intermittent Observations”, IEEE Transactions on Automatic Control, vol. 55, no. 2, February 2010, pp. 514-518.

A revised version of the following article has been submitted on November 19, 2010 for publication in the journal “Networks”:

- S.Kluge, K.Reif and M.Brokate, “On the computation of the optimal value function in time-dependent networks”.

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List of Symbols

Network description:

Symbol	*Description of the variable*
V	Set of nodes
v	Node
E	Set of directed edges, $E \subset V \times V$
e	Edge
α	Tail mapping, $\alpha : E \to V$
ω	Head mapping, $\omega : E \to V$
$E^-(v)$	Set of edges terminating in v
$E^+(v)$	Set of edges emanating from v
$V^-(v)$	Set of predecessors of v
$V^+(v)$	Set of successors of v
$\deg^+(v)$	Outdegree of v
$\deg^-(v)$	Indegree of v
t	Time variable, $t \in \mathbb{R}$
Δt	Waiting time variable $\Delta t \in \mathbb{R}_0^+$
τ	Edge travel time mapping, $\tau : E \times \mathbb{R} \to \mathbb{R}_0^+$
β	Edge cost mapping, $\beta : E \times \mathbb{R} \to \mathbb{R}$
δ	Waiting cost mapping, $\delta : V \times \mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}$
G	Time-dependent network, $G = (V, E, \tau; \beta, \delta)$

Problem Description:

Symbol	*Description of the variable*
$T(v)$	Set of admissible points in time at $v \in V$
X	State space, $X = \bigcup_{v \in V} \big\{\{v\} \times T(v)\big\}$
$\Delta T(v,t)$	Set of admissible waiting times at $(v,t) \in X$
P	Set of admissible paths (state sequences)
p	Path (state sequence), $p = \big((v_k, t_k)\big)_{k=0,1,\ldots}$
$U(v,t)$	Set of admissible control sequences at $(v,t) \in X$
u	Control sequence, $u = \big((\Delta t_k, e_k)\big)_{k=1,2,\ldots}$
$\lvert p\rvert, \lvert u\rvert$	Length of the path and control sequence, respectively
p_k, u_k	k-th component of the path and control sequence, respectively
$p_{i:j}, u_{i:j}$	components $i, ..., j$ of the path and control sequence, respectively
v_0	Source node
v'	Goal node
$\underline{t}_R(v)$	Earliest arrival time at $v \in V$
$\bar{t}_R(v)$	Latest departure time at $v \in V$
$T_R(v)$	Set of reachable points in time at $v \in V$, $T_R \subset \mathbb{R}$
X_R	Reachable part of the state space, $X_R = \bigcup_{v \in V} \big\{\{v\} \times T_R(v)\big\}$

Functions on the Network:

Symbol	*Description of the variable*
φ	Control-to-state mapping, $\varphi : \bigcup_{x \in X} \{(x, u) : u \in U(x), \lvert u \rvert = 1\} \to X$
Φ	Control-to-path mapping, $\Phi : \bigcup_{x \in X} \{(x, u) : u \in U(x)\} \to P$
$\mathcal{T}$	Path travel time function, $\mathcal{T} : \bigcup_{x \in X} \{(x, u) : u \in U(x)\} \to \mathbb{R}_0^+$
$\mathcal{B}$	Path cost function, $\mathcal{B} : \bigcup_{x \in X} \{(x, u) : u \in U(x)\} \to \mathbb{R}$
t^*	Optimal travel time function, $t^* : X \to \mathbb{R}_0^+ \cup \{\infty\}$
t_*	Reverse optimal travel time function, $t^* : X \to \mathbb{R}_0^+ \cup \{\infty\}$
b^*	Optimal value function, $b^* : X \to \mathbb{R} \cup \{\pm\infty\}$
b_*	Reverse optimal value function, $b^* : X \to \mathbb{R} \cup \{\pm\infty\}$
μ^*	Optimal control policy $\mu^* : X \to \mathbb{R}_0^+ \times E$

Notational Convention:

- Lower bounds of a quantity are denoted by underlining, upper bounds by overlining the quantity, respectively.
- Optimality is denoted by the subscript and superscript $*$, respectively.
- Duality is denoted by the superscript T.
- The interior of a set S is denoted by $\mathrm{int}(S)$, the closure of S is denoted by $\mathrm{cl}(S)$, the boundary of S is denoted by $\mathrm{bd}(S)$.
- The support and the graph of a mapping $f : S \to Y$ are denoted by $\mathrm{supp}(f)$ and $\mathrm{graph}(f)$, respectively.
- The set of k-times continuously differentiable real-valued functions on an open set $S \subset \mathbb{R}^n$ is denoted by $\mathcal{C}^k(S)$, $k \in \mathbb{N} \cup \{\infty\}$. The set of real analytic functions on an open set $S \subset \mathbb{R}^n$ is denoted by $\mathcal{C}^\omega(S)$.
- If $S \subset \mathbb{R}$ is an open set and $f \in \mathcal{C}^k(S)$ then the first derivative of f is denoted by f', the k-th derivative of f is denoted by $f^{(k)}$, $k \geq 2$.
  If $S \subset \mathbb{R}^n$ is an open set and $f : S \to \mathbb{R}^m$ is Fréchet-differentiable, then we denote the Fréchet-derivative of f at $x \in S$ by $Df(x)$.
- If $S \subset \mathbb{R}^n$ is an open set and $f \in \mathcal{C}^k(S)$ then the partial derivative of f with respect to the i-th coordinate is denoted by $\partial_i f$, $i = 1, ..., n$. If the variables of f have distinct names, e.g., $(t, x) \mapsto f(t, x)$ then the partial derivative of f with respect to t is denoted by $\partial_t f$ and the partial derivative with respect to x is denoted by $\partial_x f$.
- In the split network (cf. Section 3.2), the waiting node is denoted by the subscript w, the node at which no waiting is allowed will be denoted by the subscript nw.
- The probability of an event E is denoted by $\mathbb{P}\{E\}$, the expected value of a real-valued random variable X is denoted by $\mathbb{E}[X]$.
- The abbreviations "p.d.f.", "i.i.d." and "a.s." stand for "probability density function", "independent and identically distributed" and "almost surely", respectively.

Chapter-Specific Notation, Chapter 2:

Symbol	*SI unit*	*Description of the variable*
t	[s]	Time
d	[m]	Traveled distance
v	[m/s]	Velocity
a	[m/s^2]	Acceleration
α	[rad]	Angle of elevation
ω	[1/s]	Rotational speed
τ	[Nm]	Torque
p	[bar]	Break pressure
n	[1]	Gear
A_f	[m^2]	Frontal area of the vehicle
m_v	[kg]	Vehicle mass
m_w	[kg]	Wheel mass
c_d	[1]	Drag coefficient
r_w	[m]	Wheel radius
gr	[1]	Gear ratio
c_{rr}	[1]	Rolling resistance coefficient
H_l	[MJ/l]	Lower heating value
g	[N/kg]	Gravitational acceleration
ρ_a	[kg/m^3]	Air density
η_e	[1]	Engine efficiency
η_t	[1]	Transmission efficiency
λ	[1]	Molding body surcharge factor
I_w	[kg m^2]	Wheel moment of inertia
I_m	[kg m^2]	Motor moment of inertia
F_r	[N]	Rolling resistance
F_c	[N]	Climbing resistance
F_a	[N]	Aerodynamic resistance
F_i	[N]	Inertial resistance
F	[N]	Driving resistance
P	[W]	Power
P_0	[W]	Basic power consumption
Q	[l/s]	Fuel flow rate
B	[l]	Fuel consumption
L	[m]	Length of the road segment
ρ	[veh./km]	Traffic density
ρ_{f}	[veh./km]	Traffic density under free flow conditions
V_{f}	[m/s]	Vehicle speed under free flow conditions (random variable)
μ_v	[m/s]	Expected value of vehicle speed under free flow conditions
σ_v	[m/s]	Standard deviation of vehicle speed under free flow conditions
V_0^+	[m/s]	Terminal vehicle speed in an acceleration process (random variable)
V_0^-	[m/s]	Initial vehicle speed in a deceleration process (random variable)
A	[m/s^2]	Vehicle acceleration and deceleration (random variable)
σ_a	[m/s^2]	Distribution parameter of the vehicle acceleration and deceleration

Symbol	*SI unit*	*Description of the variable*
A^+	[m/s^2]	Vehicle acceleration (random variable)
A^-	[m/s^2]	Vehicle deceleration (random variable)
T_h	[s]	Waiting time at the junction (random variable)
$\overline{T}$	[s]	Maximal waiting time at the junction
p_t	[1]	Probability of stopping at the junction
p_s	[1]	Probability of finding the vehicle in a halt
V	[m/s]	Vehicle speed on the urban road segment (random variable)
T	[s]	Expected travel time on the urban road segment
$\mathcal{N}(\mu, \sigma)$		Normal distribution with parameters $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}_0^+$
$\mathcal{HN}(0, \sigma)$		Half-normal distribution with parameter $\sigma \in \mathbb{R}^+$
$\mathcal{U}(I)$		Uniform distribution over the interval $I \subset \mathbb{R}$

Chapter-Specific Notation, Chapter 3:

Symbol	*Description of the variable*
ρ	Turn restriction mapping, $\rho : \bigcup_{v \in V}\{E^-(v) \times E^+(v)\} \to \{0, 1\}$
σ	Turn travel time mapping, $\sigma : \bigcup_{v \in V}\{E^-(v) \times E^+(v)\} \times \mathbb{R} \to \mathbb{R}_0^+$
ι	Turn travel cost mapping, $\iota : \bigcup_{v \in V}\{E^-(v) \times E^+(v)\} \times \mathbb{R} \to \mathbb{R}$

Chapter-Specific Notation, Chapter 5:

Symbol	*Description of the variable*
$\mathcal{PC}^\omega(\Theta)$	Set of all piecewise analytic functions $f : \Theta \to \mathbb{R}$, $\Theta \subset \mathbb{R}^n$, in the sense of Definition 5.2.1
$\mathcal{C}^{1,\omega}(T)$	Set of all analytic functions $f : T \to \mathbb{R}$, $T \subset \mathbb{R}$, in the sense of Definition 5.2.3
$\mathcal{PC}^{1,\omega}(T)$	Set of all piecewise analytic functions $f : \Theta \to \mathbb{R}$, $T \subset \mathbb{R}$, in the sense of Definition 5.2.3
$\mathcal{PL}^n(\Theta)$	Set of all piecewise linear functions $f : \Theta \to \mathbb{R}$, $\Theta \subset \mathbb{R}^n$, $n \in \{1, 2\}$
$\mathcal{PL}_c^n(\Theta)$	Set of all continuous piecewise linear functions $f : \Theta \to \mathbb{R}$, $\Theta \subset \mathbb{R}^n$, $n \in \{1, 2\}$
$\mathcal{PL}_{lsc}^n(\Theta)$	Set of all lower semicontinuous piecewise linear functions $f : \Theta \to \mathbb{R}$, $\Theta \subset \mathbb{R}^n$, $n \in \{1, 2\}$
$\mathcal{PL}_{usc}^n(\Theta)$	Set of all upper semicontinuous piecewise linear functions $f : \Theta \to \mathbb{R}$, $\Theta \subset \mathbb{R}^n$, $n \in \{1, 2\}$
$\#f$	Complexity of the piecewise linear function $f : \Theta \to \mathbb{R}$, $\Theta \subset \mathbb{R}^n$, $n \in \{1, 2\}$, in the sense of Definition 5.3.3 and Definition 5.3.11.

Chapter-Specific Notation, Chapter 6:

Symbol	*Description of the variable*
W	Set of all possible states of the network
$\Omega(t)$	Restriction of the set of possible states of the network at time $t \in \mathbb{R}$
$w(t)$	State of the network at time $t \in \mathbb{R}$
$\mathcal{W}$	Set of all functions $w : \mathbb{R} \to W$ with $w(t) \in \Omega(t)$ for all $t \in \mathbb{R}$

Chapter-Specific Notation, Chapter 7:

Symbol	*Description of the variable*
γ	Travel time constraint function, $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$
Γ	Travel time constraint function, $\Gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, $\Gamma(t) = t + \gamma(t)$

Chapter-Specific Notation, Chapter 8:

Symbol	*Description of the variable*
$\underline{t}$	Earliest departure time at v_0, $\underline{t} \in \mathbb{R}$
$\bar{t}$	Latest arrival time at v', $\bar{t} \in \mathbb{R}$
$\widehat{v}$	Node to be expanded by the DOT* algorithm, $\widehat{v} \in V$
$\widehat{t}$	Point in time, at which $\widehat{v}$ is chosen by the DOT* algorithm, $\widehat{t} \in [\underline{t}, \bar{t}\,]$
$\widehat{b}$	Cost function computed by the DOT* algorithm, $\widehat{b} : X \to \mathbb{R}_0^+$
$\widehat{\Delta t}$	Waiting policy determined by the DOT* algorithm, $\widehat{\Delta t} : X \to \mathbb{R}_0^+$
$\widehat{e}$	Edge policy determined by the DOT* algorithm, $\widehat{e} : X \to E$
$t^+(v)$	Maximum point in time, for which $\widehat{b}(v, \cdot)$ has not yet been computed by the DOT* algorithm $t^+(v) \in [\underline{t}, \bar{t}\,] \cup \{-\infty, +\infty\}$
$t^+(e)$	Maximum point in time, for which $\widehat{b}(e, \cdot)$ is relevant for the DOT* algorithm, $t^+(e) \in [\underline{t}, \bar{t}\,] \cup \{-\infty\}$
t^-	Maximum point in time, for which optimality cannot be guaranteed in the current iteration of the DOT* algorithm
$\widehat{I}$	Time interval for which optimality can be guaranteed in the current iteration of the DOT* algorithm
$\pi_t(v)$	Lower bound of the travel time from v_0 to $v \in V$
$\pi_b(v)$	Lower bound of the travel cost from v_0 to $v \in V$
$\widetilde{\pi}_t(\widehat{v}, v)$	Lower bound of the travel time from $\widehat{v}_{\text{nw}}$ to v_{w} in the split network, $\widehat{v}, v \in V$.
$\widetilde{\pi}_b(\widehat{v}, v)$	Lower bound of the travel cost from $\widehat{v}_{\text{nw}}$ to v_{w} in the split network, $\widehat{v}, v \in V$.

Chapter-Specific Notation, Chapter 9:

Symbol	*Description of the variable*
$\\| \cdot \\|$	Norm on the space of piecewise continuous functions
π	Potential improvement, $\pi : V \to \mathbb{R}_0^+$
$i(v)$	Number of times the node $v \in V$ has been iterated
$\underline{b}_{i(v)\|i(v)}$	Lower bound of the optimal value function at v_{w} after $i(v)$ iterations of the node $v \in V$, $\underline{b}_{i(v)\|i(v)} : \{v\} \times T_R(v) \to \mathbb{R}$
$\underline{b}_{i(v)+1\|i(v)}$	Lower bound of the optimal value function at v_{nw} after $i(v)$ iterations of the node $v \in V$, $\underline{b}_{i(v)+1\|i(v)} : \{v\} \times T_R(v) \to \mathbb{R}$
$\bar{b}_{i(v)\|i(v)}$	Upper bound of the optimal value function at v_{w} after $i(v)$ iterations of the node $v \in V$, $\underline{b}_{i(v)\|i(v)} : \{v\} \times T_R(v) \to \mathbb{R}$
$\bar{b}_{i(v)+1\|i(v)}$	Upper bound of the optimal value function at v_{nw} after $i(v)$ iterations of the node $v \in V$, $\underline{b}_{i(v)+1\|i(v)} : \{v\} \times T_R(v) \to \mathbb{R}$
$\underline{e}_{i(v)}$	Edge policy associated with $\underline{b}_{i(v)\|i(v)}$, $v \in V$, $\underline{e}_{i(v)} : \{v\} \times T_R(v) \to E$
$\bar{e}_{i(v)}$	Edge policy associated with $\bar{b}_{i(v)\|i(v)}$, $v \in V$, $\bar{e}_{i(v)} : \{v\} \times T_R(v) \to E$

Symbol	*Description of the variable*
$\underline{\Delta t}_{i(v)}$	Waiting policy associated with $\underline{b}_{i(v)\|i(v)}$, $v \in V$, $\underline{\Delta t}_{i(v)} : \{v\} \times T_R(v) \to \mathbb{R}_0^+$
$\overline{\Delta t}_{i(v)}$	Waiting policy associated with $\overline{b}_{i(v)\|i(v)}$, $v \in V$, $\overline{\Delta t}_{i(v)} : \{v\} \times T_R(v) \to \mathbb{R}_0^+$
$\underline{b}_0(v)$	Lower bound of the reverse optimal value function $b_*(v,t)$ with respect to the source node v_0
$\underline{b}'(v)$	Lower bound of the optimal value function $b_*(v,t)$ with respect to the goal node v'
$\overline{b}_0(v)$	Upper bound of the reverse optimal value function $b_*(v,t)$ with respect to the source node v_0
$\overline{b}'(v)$	Upper bound of the optimal value function $b_*(v,t)$ with respect to the goal node v'
V_R	Set of all reachable nodes $v \in V$ for which it has not been excluded that there exists an optimal path from v_0 to v' which passes through v.
V_F	Set of all nodes $v \in V_R$ for which the computation of the optimal value function has already been finished.
V_O	Set of all nodes $v \in V_R$ for which an iteration may result in an improvement of the upper or lower bound.
V_C	Set of all nodes $v \in V_R \setminus V_F$ for which an iteration cannot result in an improvement of the upper or lower bound.

Chapter-Specific Notation, Chapter A:

Symbol	*SI unit*	*Description of the variable*
α	[rad]	Angle of elevation
$\widetilde{P}(v,a,\alpha)$	[W]	Power which must be which must be provided by the battery of the vehicle when driving at a speed v an acceleration a and an angle of elevation α
L	[m]	Length of the road segment
V_f	[m/s]	Vehicle speed under free flow conditions (random variable)
μ_f	[m/s]	Expected value of vehicle speed under free flow conditions
σ_f	[m/s]	Standard deviation of vehicle speed under free flow conditions
V_t	[m/s]	Vehicle speed under transition conditions (random variable)
σ_t	[m/s]	Distribution parameter of the vehicle speed under transition conditions
p_s	[1]	Probability of finding the vehicle in a halt
p_f	[1]	Probability of finding a moving vehicle in free flow conditions
$\mathcal{N}(\mu,\sigma)$		Normal distribution with parameters $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}_0^+$
$\mathcal{HN}(0,\sigma)$		Half-normal distribution with parameter $\sigma \in \mathbb{R}^+$

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