A General Method to Speed Up Fixed-Parameter-Tractable Algorithms

Rolf Niedermeier, Peter Rossmanith

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Rolf Niedermeier
Wilhelm-Schickard-Institut für Informatik,
Universität Tübingen,
Sand 13, D-72076 Tübingen, Fed. Rep. of Germany

Peter Rossmanith
Institut für Informatik, Technische Universität München,
Arcisstr. 21, D-80290 München, Fed. Rep. of Germany

Abstract

A fixed-parameter-tractable algorithm, or *FPT* algorithm for short, gets an instance (I,k) as its input and has to decide whether $(I,k) \in \mathcal{L}$ for some parameterized problem $\mathcal{L}$. Many parameterized algorithms work in two stages: reduction to a problem kernel and bounded search tree. Their time complexity is then of the form $O(p(|I|) + q(k)\xi^k)$, where $q(k)$ is the size of the problem kernel. We show how to modify these algorithms to obtain time complexity $O(p(|I|)+\xi^k)$, if $q(k)$ is polynomial.

1 Introduction

A parameterized problem usually consists of two components—the input and aspects of the input that constitute a parameter. For example, the NP-complete Vertex Cover problem has an undirected graph G as its input and a positive integer k as its parameter; the question is whether there is a set of at most k vertices that cover all edges in G. The central question of parameterized complexity theory [4] is as follows: Given a parameterized problem $\mathcal{L}$ with input size n and parameter k, is there an algorithm solving $\mathcal{L}$ in time $f(k)n^{\alpha}$, where α is a constant independent of k and n and f is an arbitrary function depending only on k. A problem with such an algorithm is called *fixed parameter tractable* and the corresponding complexity class of problems is called FPT. Vertex Cover is in FPT [1, 3, 4], the currently best known FPT algorithm running in time faster than $O(kn + 1.3^k k^2)$ [2, 9, 10].

There is, however, a problem concerning the definition of FPT— the function f may grow arbitrarily fast. Thus, there are currently only a few parameterized problems known that have an (exponential) function f that grows as "slowly" as in the case of Vertex Cover. The development of efficient FPT algorithms hence is an active field of research [4, **?**, 7]. To the authors' best knowledge, at least the majority of *efficient* FPT algorithms known so far (e.g., [1, **?**, 5, 8, 9]) are based on combination of two standard methods: *bounded search trees* and *reductions to problem kernel* [4]. Here, we show how to significantly improve all FPT algorithms based on the combination of these two techniques. Hence, we contribute to the positive toolkit for designing *FPT* algorithms, which according to Downey and Fellows [4, page 20] belongs to the current research horizons in parameterized complexity: "The positive toolkit for designing *FPT* algorithms contains several key methods that are very deep and general—but for which practicality is still not yet clearly established." In the following, we provide a simple, practical, and generally applicable method to speed up *FPT* algorithms.

The basic idea of improvement is in a sense to *interleave* reduction to problem kernel and bounded search tree method. More specifically, assume that we have an *FPT* algorithm running in time $O(p(n) + q(k)\xi^k)$, where ξ is a (small) constant and p and q are polynomials. Moreover, ξ^k shall be the size of the bounded search tree and $q(k)$ the size of the problem kernel. Then our new technique shows how to get rid of the factor $q(k)$, thus transforming the above algorithm into a time $O(p(n) + \xi^k)$ one. It is important here to note that this improvement is not due to asymptotic tricks, but that $q(k)$ can be replaced by a *small* constant. We shall just swiftly mention that our technique leads to a significant improvement of the so-called klam values (cf. [4, pages 13–14]) for many problems. For example, consider Vertex Cover again. The first nontrivial parameterized algorithm for this problem had running time $O(kn + 1.32472^k k^2)$ [1], recently further developed to $O(kn + 1.29175^k k^2)$ and recently even further [2, 10]. Compare the growth of the two functions $1.32472^k k^2$ and $1.29175^k k^2$. For instance, for $k = 100$ the first one is bounded by 1.64×10^{16} and the second by 1.32×10^{15}, hence where $k = 100$, it improves by a factor of roughly 12. By way of contrast, the improvement from

$1.29175^k k^2$ to 1.29175^k is a factor of about 10000. In this context, observe that the second improvement involves, due to our technique, a small constant factor, which, however, is by magnitudes smaller than 10000. Summarized, this shows that for practical parameter sizes ($k \approx 100$ is very natural in the case of VERTEX COVER and many other parameterized problems) our improvement has a potentially much higher benefit than small (but in no way trivial) improvements in the exponential base ξ of the search tree size may have.

2 *FPT* algorithms

Many *FPT* algorithms work in two stages [4]: Firstly, the instance is transformed into an equivalent one that is smaller in size. To be specific, its size is bounded by a function that depends *on the parameter only.* This stage is called *reduction to problem kernel.* Secondly, the new small instance is solved recursively by solving several derived instances with *smaller parameters.* Since the parameters in the recursive calls are smaller, the recursion eventually terminates (either by finding a solution or by realizing that no solution exists because of $k \leq 0$). That stage is called *bounded search tree.*

In the following we describe each stage in more detail and introduce all necessary notation that is needed to improve the algorithm. We also illustrate each concept within the example of VERTEX COVER. The instance of VERTEX COVER is an undirected graph and a parameter k. The question the algorithm must answer is whether or not a vertex cover of size at most k

exists. (A vertex cover is a subset of vertices C such that every edge in the graph is incident to at least one vertex in C.)

2.1 Reduction to problem kernel

Let $\mathcal{L}$ be a parameterized problem, i.e., $\mathcal{L}$ consists of pairs (I, k), where I has a solution of size k. In the case of Vertex Cover $\mathcal{L}$ consists of all (G, k), where G is an undirected graph that has a vertex cover of size k. Reduction to problem kernel consists of replacing the original instance (I, k) with a new one (I', k') so that $k' \leq k$, $|I'| \leq q(k')$, and $(I, k) \in \mathcal{L} \Leftrightarrow (I', k') \in \mathcal{L}$. What is particularly important is that the size of the new instance is bounded by a function of the parameter alone. We call this function q. In general, q might be arbitrary, but in this paper we restrict q to being polynomial as is usually the case for efficient FPT algorithms.[1]

In the case of Vertex Cover, reduction to problem kernel is carried out as follows: If the degree of some vertex is bigger than k then delete this vertex from the graph and decrease k by 1. This leads to an equivalent instance as this vertex has to part of every vertex cover of size k (otherwise all its incident edges would have to be covered by other vertices, which is not possible since there are too many of them). If no vertex with such a high degree remains, the number of vertices of the graph can be at most $k(k+1)$, if there is a vertex cover of size k: The vertex cover itself contains k vertices and all other vertices

[1] In general, $k' \leq k$ is not necessary. All results also hold if k' is bounded by some polynomial in k.

must be adjacent to one of them. Since each member of the vertex cover has at most k neighbors there can be at most k^2 additional vertices. To finish the reduction to problem kernel it needs merely be observed as to whether there are more than $k(k+1)$ vertices and, if yes, replace the instance with some small instance that has no solution.[2] In that way, we replaced the original instance (G, k) with (G', k'). In particular $q(k) = k(k+1)$.

Let $\mathcal{R}$ denote the function that performs the reduction to problem kernel, i.e., $\mathcal{R}(I, k) = (I', k')$ and let $P(|I|)$ be the number of steps required to perform the reduction. We demand P be bounded by some polynomial. For Vertex Cover $P(|G|) = O(|G|)$ if the graph is represented by an adjacency list.

2.2 Bounded search trees

Let (I', k') be an instance after reduction to problem kernel. Many algorithms solve the problem by constructing a search tree that looks exhaustively for solutions. In order to gain efficiency, branches will be pruned. Pruning of branches is subject chiefly to two conditions: Either we can be sure that the branch contains no solution or, if there are two branches A and B we can prune B if we can be sure that a solution in B implies a solution in A of the same size at most. The main objective to find ever more efficient *FPT* algorithms involved decreasing the size of the search tree. In the following we analyze first the *size*

[2]This is only done for technical reasons here. The algorithm could already stop, since in this case it is clear that no vertex cover of size at most k exists.

of the tree as well, but then take a look at the *time* taken for processing the tree. The next section building on this analysis improves the overall *time* to traverse the search tree, but not its *size*, which will not be affected at all.

In general, let (I, k) be a node of the search tree. To solve (I, k), it is replaced by several instances $(I_1, k - d_1), (I_2, k - d_2), \ldots, (I_m, k - d_m)$ so that $d_i > 0$ and $|I_i| \leq |I|$ for all $i \in \{1, \ldots, m\}$ and $(I, k) \in \mathcal{L}$ iff $(I_i, k - d_i) \in \mathcal{L}$ for some $i \in \{1, \ldots, m\}$. The leaves consist commonly of those instances with $k \leq 0$. Since all $d_i > 0$, the children's parameters are strictly smaller and the tree has a finite size. An upper bound on the size of the tree is easy to obtain by solving the corresponding recurrence for the number of leaves:

$$S_k = S_{k-d_1} + S_{k-d_2} + \cdots + S_{k-d_m}.$$

The solution has the general form $S_k = \Theta(p(k)\xi^k)$, where $1/\xi$ is the smallest positive, real root of the reflected characteristic polynomial

$$1 - z^{d_1} - z^{d_2} - \cdots - z^{d_m}$$

and $p(k)$ is a polynomial [6]. If ξ is a unique root, as is almost always the case, p is simply a constant and therefore $S_k = \Theta(\xi^k)$. In the following we assume that ξ is a unique root. If that were not the case, then p is not a constant, but some polynomial of degree > 0. In that case ξ^k should be replaced by $p(k)\xi^k$ in the next section.

Finally, let $R(|I|)$ be the time needed to compute $(I_1, k - d_1), (I_2, k - d_2), \ldots, (I_m, k - d_m)$ from (I, k). Again we demand that $R(|I|)$ be bounded polynomially.

For Vertex Cover several complicated methods for the construction of small bounded search trees exist. We present a comparatively simple one, which yields a relatively big search tree: Let (G, k) be an instance of Vertex Cover. Pick any one edge $\{x, y\}$. Each vertex cover has to contain at least one of x and y in order to cover $\{x, y\}$. So let $(G_1, k-1)$ and $(G_2, k-1)$ be the children of (G, k) in the search tree, where we get G_1 by deleting x and all its incident edges from G and G_2 analogously by deleting y. The size of the search tree is at most 2^k, i.e., $\xi = 2$ and $R(|G|) = O(|G|)$ if G is represented by an adjacency list [3].

The overall time complexity for the second stage *bounded search tree* is $O(R(q(k))\xi^k)$. For Vertex Cover we thus get $O(k^2 2^k)$.

3 Accelerating *FPT* algorithms

In the following, we will deal with a large class of fixed-parameter-tractable algorithms. Let us summarize the conditions that these algorithms have to undergo: They have to be *FPT* algorithms that work in two stages, *reduction to problem kernel* and *bounded search tree.* Reduction to problem kernel takes $P(|I|)$ steps and results in an instance of size at most $q(k)$, where both P and q are polynomially bounded. The expansion of a node in the search tree takes $R(|I|)$ steps, which must also be bounded by some polynomial, the search tree size being $O(\xi^k)$. The overall time complexity of the algorithm is then

$$O(P(|I|) + R(q(k))\xi^k),$$

where (I,k) is the instance to be solved. In the following we show how to modify the second stage of the algorithm in order to improve the time complexity to

$$O(P(|I|) + \xi^k).$$

Generally, we now use the following algorithm to expand a node (I,k) in the search tree:

```
**if** $|I| > c \cdot q(k)$ **then** replace $(I,k)$ with $\mathcal{R}(I,k)$ **fi**;
replace $(I,k)$ with $(I_1, k-d_1), (I_2, k-d_2), \ldots, (I_m, k-d_m)$
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Here $c \geq 1$ is a constant that can be chosen with the aim of further optimizing the running time. There is a tradeoff in choosing c: The optimal choice depends on the implementation of the algorithm, but in the end it affects only the constant factor in the overall time complexity. Therefore we neglect optimizing c in this paper.

A closer look shows that we in fact seem to *increase* the time needed to expand a node in the search tree. This is generally speaking true: Sometimes we apply reduction to problem kernel prior to splitting into recursive calls. However, these additional reductions to problem kernel also *decrease* the instance size in the middle of the search tree. Since the time for splitting is bounded polynomially in the *instance size*, this also helps to *decrease* the time to expand a node. It proves to be the case that while we waste time near the root of the search tree, we gain much more time near the leaves. Note that the technique of *interleavin* reduction to problem kernel and bounded search trees was already used for developing efficient *FPT* algorithms for Vertex Cover [**?**, 10]. There, however, it was used to reduce the number of case distinctions in the search tree; it was not considered with the aim of removing the factor $R(q(k))$ as we do.

In order to analyze the running time of the above mathematically, we describe the time to expand a node (I, k) and all its descendants by a recurrence. Let T_k denote an upper bound on the *time* to process (I, k). The following recurrence exists for T_k:

$$T_k = T_{k-d_1} + T_{k-d_2} + \cdots + T_{k-d_m} + O(P(q(k)) + R(q(k)))$$

The time to expand (I, k) itself is at most $O(P(q(k))+R(q(k)))$, since $|I| = O(q(k))$ since $|I| > c \cdot q(k)$ is constantly prevented. In order to solve this non-homogeneous linear recurrence we need a special solution. To get its general solution we add the general solution of the corresponding homogeneous recurrence $T_k = T_{k-d_1} + T_{k-d_2} + \cdots + T_{k-d_m}$. However, we already know that all solutions of this homogeneous recurrence are bounded by $O(\xi^k)$. Consequently we are only required to find a small special solution of the non-homogeneous recurrence. In our case the inhomogeneity is a polynomial. Therefore, there exists a special solution that is also a polynomial in k. It is easy to construct such a special solution explicitly. There is always a polynomial solution that has the same degree as the inhomogeneity p. (If r is a polynomial special solution then $r(k) - \sum_{i=1}^{m} r(k-d_i) = p(k)$ and the highest degree monomials on the left side cannot cancel each other.) All solutions of T_k are therefore bounded by $O(\xi^k)$.

In order to illustrate this, let us solve the recurrence for the simple algorithm that solves Vertex Cover (cf. Subsection 2.2). The recurrence reads

$$T_k = 2T_{k-1} + C \cdot k^2 + D \cdot k + E,$$

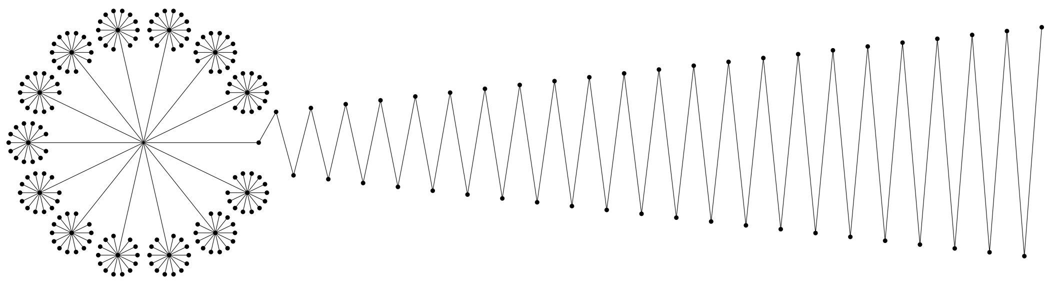
where C, D and E are constants that depend on the implementation of the algorithm. The initial conditions are simple, say, $T_0 = 0$. The reflected characteristic polynomial is $1 - 2z$ and its unique root is $1/2$. The general solution of the homogeneous recurrence is $\lambda 2^k$ for $\lambda \in \mathbf{R}$. Since it is a recurrence of first order, the dimension of its space of solutions is one, too.

A special solution is $T_k = -Ck^2 - (4C+D)k - (6C+2D+E)$. The general solution is then $\lambda 2^k - Ck^2 - (4C + D)k - (6C + 2D + E)$ and the solution for $T_0 = 0$ is $T_k = (6C + 2D + E) \cdot 2^k - Ck^2 - (4C + D)k - (6C + 2D + E)$.

4 The modification is necessary

In this section, we show that an improved analysis alone cannot achieve the speedup of the last section. That is, the interleaving of reduction to problem kernel and the bounded search tree really is necessary to get the claimed improvements. Without modification, the algorithms in general have a running time of $\Omega(P(|I|) + R(f(k))\xi^k)$. As an example, we can again use Vertex Cover. Look at Figure 1 for a definition of a family of instances of Vertex Cover defined for odd k. There is no solution of size $\leq k$, since the optimal vertex cover has size $\frac{5}{2}k - \frac{3}{2}$ (in the head $k - 2$ vertices and half the vertices of the tail).

Figure 1: An instance of VERTEX COVER. The following graph is the $k = 15$ member of a family of instances (G_k, k) for VERTEX COVER. The graph G_k consists of a tree with degree $k-1$ and depth 2 to which a path with $3k+1$ vertices is attached (called the *tail*). It is easy to see that the smallest vertex cover for G_k has size $\frac{5}{2}k - \frac{3}{2}$ and therefore the whole family has no members in VERTEX COVER.



The graph contains exactly $(k-1)(k-2)+1$ vertices in the head and $3k+1$ vertices in the tail (altogether k^2+4). Reduction to problem kernel does not affect this graph since the degree of every vertex is at most k, although its size is very near the maximum possible $k(k+1)$. Now assume that the unmodified algorithm chooses edges from right to left. This leads to a search tree of size 2^k, the largest possible. While the algorithm examines this graph, it removes nodes and edges, but the *head* remains unchanged. Consequently, instances have size $\Omega(k^2)$ during *each* splitting step. The overall time complexity therefore is the worst possible — $\Omega(k^2 2^k)$. Of course, a better time complexity can also be achieved by changing the order of choosing edges. Nevertheless, the time bound is $\Theta(k^2 2^k)$ in the worst case.

After the modification the running time is decreased tremendously. After the *second* edge is removed and k decreased by two, the whole head is removed from the graph.

5 Applications: Improving klam values

To measure the goodness of an FPT algorithm, Downey and Fellows introduced the notion of *klam* values [4, pages 13–14]: Let U be some reasonable (depending on technology) speed limit, say $U = 10^{20}$. Since it is known that every problem in FPT is solvable in time $f(k) + n^c$ for a constant c, the question is how big k may be so that $f(k) \leq U$ remains. This is called the

Table 1: Comparing klam values for various FPT algorithms for Vertex Cover.

$f(k)$	klam	Reference	New $f(k)$	New klam
$k^2 2^k$	≈ 55	[3]	2^k	≈ 65
$k^2 1.325^k$	≈ 130	[1]	1.325^k	≈ 160
$k^2 1.320^k$	≈ 132	[?]	1.320^k	≈ 165
$k^2 1.292^k$	≈ 150	[9]	1.292^k	≈ 179
$\max\{k^2 1.256^k, k 1.291^k\}$	≈ 157	[10]	1.291^k	≈ 180
$k 1.271^k$	≈ 170	[2]	1.271^k	≈ 192

number of klams that the algorithm is worth.

Let us once more consider Vertex Cover. Table 1 lists some known results for the $f(k)$ function for Vertex Cover, also showing each of the improvements due to our new technique. Note that we omit taking constant factors into account, which would be only slightly increased due to our new technique. Compared to the asymptotic improvements we obtain, however, these may be neglected for the sake of ease.

Table 1 shows that, in principle, the improvements due to our new technique increase according to the size of the problem kernel. Here, we have kernel sizes of $O(k^2)$ and $O(k)$. It also shows, however, that the improvements concerning the klam values commonly are significantly larger through our technique than they are through improved bases of the exponential factors obtained in a series of papers. Moreover, observe that the very recent improvement of Stege and Fellows [10] to our previous result [9] becomes very small when our technique is applied.

Finally, note that it is fairly easy to give an algorithm running in time $O(n^{O(1)} + k^6 3^k)$ for the Hitting Set for Size Three Sets problem with applications in computational biology [4]. Clearly, it is conceivable that the exponential factor can be improved to some degree. By means of our new technique, however, the above result can automatically be improved, so that $O(n^{O(1)} + 3^k)$. Thus, the klam value improves from 24 to 41. This shows the great potential of our new technique especially for comparatively large problem kernel sizes (here $O(k^6)$). Downey and Fellows' monograph [4] contains dozens of further problems where our technique works out of the box.

6 Conclusion

We introduced a new, simple, and prospective technique for speeding up *FPT* algorithms based on reductions to problem kernel and bounded search trees. As a rule, the potential for improvement due to our method increases the larger the problem kernel in the underlying parameterized problem is. For example, associated candidate problems (see [4] for details) are k-Leaf Spanning Tree (problem kernel size $O(k^2)$) and Hitting Set for Size Three Sets (problem kernel size $O(k^6)$). Thus, our method belongs in the toolkit of every designer of efficient *FPT* algorithms.

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