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Approximability of Scheduling with Fixed Jobs

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Approximability of Scheduling with Fixed Jobs *

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Abstract

The scheduling problem of minimizing the makespan is among the most well studied problems — especially in the field of approximation. In modern industrial software however, it has become standard to work on a variant of this problem, where some of the jobs are already fixed in the schedule. The remaining jobs are to be assigned to the machines in such a way that they do not overlap with fixed jobs. This problem variant is the root of many real world scheduling problems where pre–assignments on the machines are considered, such as free shifts, cleaning times or jobs that have already started. In our paper we first focus on simple algorithms for our problem which have a reasonable performance guarantee and are easy to implement in practical settings. This is followed by a detailed analysis on the approximability of the scheduling problem with fixed jobs. We present a polynomial time approximation scheme ($PTAS$) for the case that the number m of machines is constant. For our $PTAS$ we propose a new technique by partitioning an underlying packing problem into a reasonable unrelated family of restricted bin packing problems. The computation time is $\mathcal{O}(n \log n)$. We also generalize the $PTAS$ to the case that the machines are independent and run at different speeds. Moreover, we will demonstrate that, assuming $P \neq NP$, there is no arbitrarily close approximation in the general case when the number of machines is part of the input. This will be extended by showing that there is no asymptotic $PTAS$ in the general machine case. We finally show that there exists no $FPTAS$ in the constant machine case, unless $P = NP$. These results clearly mark the approximability gap to the classical problem of minimizing makespan since the latter does not assume for a $PTAS$ that the number m of machines is fixed. Moreover, the standard problem also has an $FPTAS$ for the fixed machine case.

key words: polynomial approximation scheme, scheduling, makespan

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1 Introduction

In the standard scheduling problem of minimizing the makespan one is given a set of n independent jobs with processing times p_j to be scheduled on m identical machines. The optimization problem is to assign all jobs to the machines, minimizing the latest completion time (the makespan) C_{max}. Even though this problem is NP–hard, it is usually considered as a relatively easy problem, in the sense that quite good and efficient approximation algorithms exist.
The investigation of such approximation algorithms with good (worst case) performance guarantee (the ratio of the makespan achieved by the algorithm over the makespan of the best schedule, maximized over all possible inputs) was pioneered by Graham. He showed that a simple list schedule achieves a performance ratio of $2 - \frac{1}{m}$. In subsequent papers this has been improved to a 1.2–approximation [Gr69], [CGJ78] and [Fr84]. In [HS87], Hochbaum and Shmoys proposed a linear time approximation scheme introducing the technique of *dual approximation*. Note that, as the minimum makespan problem is strongly NP–hard (see [GJ79]), no fully polynomial time approximation scheme can exist for this problem, unless $P = NP$. If the number of machines is not part of the input but fixed in advance (the so-called constant machine case), the situation is slightly better. Here already Graham [Gr69] had provided a polynomial approximation scheme. In 1976 Sahni [S76] then presented a fully polynomial time approximation scheme for the minimum makespan problem.
Unfortunately, these good news for the approximability for the standard scheduling problem is confronted by the fact that most real world scheduling problems come with additional side-constraints violating the simple setup of the standard scheduling problem considered above. Constraints like release dates, classes of precedence constraints or set up problems give an example, but in fact, the magnitude of variants and special models for scheduling is almost Babylonian. We refer to [LLRS93] and [Pi95] for a survey and additional pointers to the literature.
One aspect common to most variants of the scheduling problems considered in the literature is that they assume that the machines are freely available until all jobs are processed. This, however, again conflicts with many real world applications. Consider for example applications in production planning. Here one usually has to consider black out times on the machines due to maintenance services or unavailability of staff (evenings, weekends). Modeling such a kind of scheduling problem is easily achieved by considering some of the jobs, say k of the n jobs, as *fixed*. That is, they are a priori assigned to be performed on a given machine at a given time. The objective is to schedule the remaining $n-k$ jobs in such a way that C_{max} is minimized. Using a straightforward reduction from 3–*PARTITION* one immediately checks that the scheduling problem with fixed jobs is already strongly NP–hard in the single machine case. Scheduling with fixed jobs has appeared sporadically under various names in the literature, but only very few rigorous results are known, e.g. [Sch84]. In particular, to our knowledge the approximability of the scheduling problem with fixed jobs has not been considered yet.
In this paper we investigate the approximability of the scheduling problem with

fixed jobs. We start with some simple approaches which are adapted from the classical algorithms for bin packing. Secondly, we will present a polynomial time approximation scheme for the constant machine case. The approximation scheme will be further generalized to the case that the machines process the jobs at different speeds. This result is essentially best possible, as we also show that on the one hand in the case that the number of machines is specified as part of the input instance a polynomial time approximation scheme does not exist, unless $P = NP$, and that on the other hand in the constant machine case a fully polynomial time approximation scheme does not exist, unless $P = NP$. The remainder of the paper is organized as follows: In Section 2 we discuss some fast algorithms which have a constant performance guarantee. Then, in Section 3, we outline the crucial steps of the polynomial time approximation scheme for the scheduling problem with fixed jobs. Section 4 then presents the final algorithm. This is followed by a detailed analysis of the algorithm in Section 5. In Section 6 we deal with the uniform parallel machine case and Section 7 finally presents non–approximability results for the scheduling problem with fixed jobs.

2 LIST–based approximation algorithms

All algorithms discussed in this paper attack the scheduling problem as a bin packing problem with bins of variable sizes. The bin packing problem is obtained by defining the gaps between two subsequent fixed jobs on the same machine and before the first fixed job on every machine to be "closed" bins which must not be overloaded and the remaining extra bin on every machine that starts after the last fixed job to be an "open" bin which allows any load. The items to be packed are the $n - k$ jobs that are not fixed. For the design of a fast approximation algorithm it is natural to combine techniques for bin packing with a list scheduling rule on the open bins. The general framework of such an algorithm is as follows:

- Pack as many jobs as possible into the closed bins.
- Schedule the remaining jobs under a LIST–rule into the open bins.

Possible strategies for packing the closed bins are given by the classical bin packing rules NEXT FIT, FIRST FIT, BEST FIT (see e. g. [CGJ84]) and a rule which we call EARLIEST FIT. EARLIEST FIT follows the idea of LIST–scheduling and packs an item into a bin, such that it can start as early as possible. We denote the makespan achieved under the above algorithms by C_L and the optimal makespan by C_{opt}. By a straightforward argument (similar to the proof of Graham's $2 - \frac{1}{m}$ bound for the standard scheduling problem of minimizing makespan [Gr66]) one can prove that these LIST–type algorithms have a performance guarantee not worse than 3, but, in contrast to the bin packing problem and the standard scheduling problem, one can indeed construct problem instances where the performance guarantee is worse than 2. Consider for instance a problem on 15 machines given by

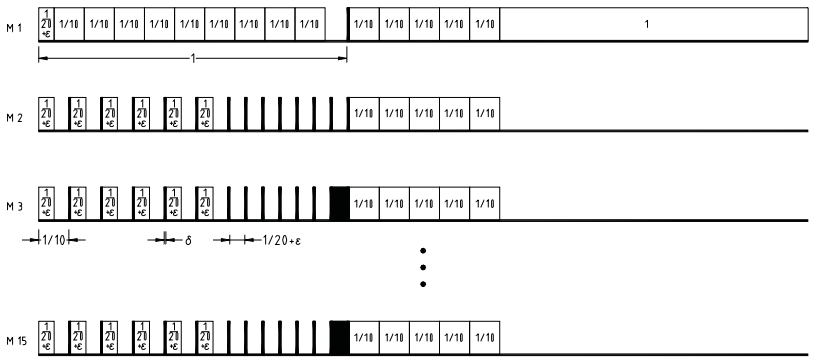

Figure 1: Solution for NEXT FIT, FIRST FIT or EARLIEST FIT

- 1 job with processing time 1,
- 84 jobs with processing time $\frac{1}{10}$,
- 85 jobs with processing time $\frac{1}{20} + \epsilon$ for a suitable small value $\epsilon > 0$.

The instance has also a series of fixed jobs, all of infinitesimal length δ such that machine one has a single closed bin of length 1 and machine two has six closed bins of length $\frac{1}{10}$ followed by seven closed bins of length $\frac{1}{20} + \epsilon$ and a fixed job which ends at $1 + \delta$. Similarly, the remaining machines have six closed bins of length $\frac{1}{10}$ followed by six closed bins of length $\frac{1}{20} + \epsilon$ and a fixed job that ends at $1 + \delta$. Since all jobs fit into the closed bins the value of an optimal solution equals $1+\delta$. The solutions delivered under NEXT FIT, FIRST FIT and EARLIEST FIT all may yield a makespan of $2+\frac{5}{10}+\delta$ (see Figure 1) depending on the sequence of jobs. For small δ we thus have $\frac{C_L}{C_{opt}} \approx 2.5$. An extended instance with a higher number of machines and one job with processing time one, but sufficiently small processing times for all remaining jobs even yields a performance ratio of 2.6.
In the given instance BEST FIT clearly outperforms the other packing rules since it produces an optimal solution. It is easy however to construct an instance were the ratio of the solution produced under BEST FIT towards the optimal solution gets arbitrarily close to 2. Moreover, the $\mathcal{O}(n^2)$ computation time is considerably higher than for the other, linear time, packing rules. We conclude this section with the observation that these simple algorithms provided here also work as on–line algorithms where items are packed/scheduled on their arrival. More sophisticated on–line algorithms however would try to better exploit the given bin sizes by keeping some free space for large jobs which might possibly arrive lateron. The development of such algorithms, however, is left for future work. In the remaining section of this paper we concentrate on developing approximation schemes.

3 The polynomial time approximation scheme

In what follows we will present the polynomial time approximation scheme for the scheduling problem with fixed jobs. Again the scheduling problem will be considered as a problem of packing variable size bins. The corresponding problem will be shortly called VBP. For the VBP, "closed" bins are given as described in Section 2, but the "open" bins are defined in a slightly different way. We use candidates C for the makespan and now define each remaining extra bin on every machine that starts after the last fixed job and ends at C to be an "open" bin. Here, "open" refers to the fact that the constraint on the bin sizes may be relaxed. Obviously, C is an upper bound for the makespan if and only if all items can be packed into the bins. Summarizing, our algorithm has the following framework:

- *Propose a candidate C for the makespan within a binary search procedure:*
  Check whether the underlying packing problem VBP has a feasible solution that does not overload the "closed" bins, but may overload the "open" bins slightly.
- *Output the schedule corresponding to the packing found for the smallest C.*

Let us first consider the special case of the standard scheduling problem of minimizing makespan. Here, VBP agrees with the standard bin packing problem: all bins have the same capacity and are "open" bins. The polynomial time approximation scheme proposed by Hochbaum and Shmoys [HS87] approximates the *feasibility* of the bin packing problem in that each bin is allowed to be overloaded by up to a factor of $1+\epsilon$. This yields — speaking in terms of the underlying scheduling problem — a $(1+\epsilon)$–approximation of the makespan. The resulting relaxed bin packing problem is fairly easy to solve as an integer linear program on a constant number of variables where items are rounded in their sizes. Small items are temporarily disregarded and later inserted by a first–fit strategy.
For the scheduling problem with fixed jobs the situation is considerably harder. In particular, two problems appear. Firstly, the rounding of item sizes could yield infeasible schedules caused by overlappings with fixed jobs. Secondly, even the small items turn out to be hard to pack (consider for an example an instance which is dominated by small closed bins and items of the same size). We clear these hurdles that come along with fixed jobs by building our algorithm out of three major parts. The first building block is a partition of the problem VBP into a family of (reasonably) unrelated restricted packing problems. We then develop a procedure for solving a single instance of such a packing subproblem as an integer linear program (ILP). The final part consists of a series of greedy packings that packs the items/jobs left over by the solutions of the packing subproblems.
In the next sections we will consider these ingredients of our algorithm in turn and show how they can be tied together to build up the entire algorithm for approximately solving the VBP. We start with some definitions:

Definition 3.1 *For any set $I = \{s_1, \ldots, s_n\}$, consisting of bins or items, let $size(s_i)$ denote the size of an element s_i and let $SIZE(I)$ denote the total size*

of all elements in I. We further define $\max(I) = \max_{s \in I} size(s)$. *Similarly,* $\min(I) = \min_{s \in I} size(s)$.

As a notational convenience we will further assume that all bins and items in instances of VBP are already scaled by $\frac{1}{C}$. Their sizes are therefore bounded by one. We also assume that $\epsilon > 0$ is an arbitrary but fixed constant.

3.1 The partition procedure

We will create a family of packing problems by partitioning the bin set according to bin sizes. The partition is based on identifying classes of bins which will have no significant impact on the packing and can therefore be neglected. This will then allow us to partition the packing problem VBP into separate instances of packing problems in each of which the minimum and the maximum bin capacity will not differ "too much".

Lemma 3.2 *Given an instance I for the packing subproblem VBP defined via C where all bins and items are scaled by $\frac{1}{C}$ and where the bins are sorted according to non–increasing sizes we can, for $t = \lceil \frac{4}{\epsilon} \rceil + 3$ and $\epsilon < 1$, identify in linear time a partition $B = \hat{B} \cup B_1 \cup \ldots \cup B_r$ of the bin set B such that*

$$\max(\hat{B}) \leq \epsilon \cdot \max(B), \tag{1}$$

$$SIZE(\hat{B}) \leq \epsilon \cdot SIZE(B), \tag{2}$$

and such that for all $i = 1, \ldots, r$:

$$\begin{aligned} \min(B_i) &\geq \epsilon^t \max(B_i), \quad \textit{and} \\ \max(B_{i+1}) &< \epsilon \min(B_i). \end{aligned} \tag{3}$$

Proof. We first partition the bin set B in linear time into a number $u \leq |B|$ of *buckets* $\bar{B}_i$, $1 \leq i \leq u$. We do that by placing the bins successively into buckets, opening a new one as soon as the size of the bin currently under consideration is smaller than ϵ times the size of the maximum bin in the current bucket. As we assumed that the bins are sorted by non–increasing sizes this can easily be done in linear time. The buckets $\bar{B}_i$ therefore have the following properties:

$$\min(\bar{B}_i) \geq \epsilon \max(\bar{B}_i) \qquad \text{and} \qquad \min(\bar{B}_i) > \max(\bar{B}_{i+1}) \qquad \text{for all } i \geq 1.$$

The following procedure will then identify the desired partition in linear time.

procedure partition

$s \leftarrow 1$; $j \leftarrow 2$; $\hat{B} \leftarrow \emptyset$; $i \leftarrow 1$; $B_1 \leftarrow \emptyset$;
while $j \leq u$ **do begin**
  if $\max(\bar{B}_j) \geq \epsilon \cdot \min(\bar{B}_{j-1})$ **then do begin**
    if $j - s = \lfloor \frac{t}{4} \rfloor - 1$ **then do begin**
      Choose $s \leq i^* \leq j$ such that $SIZE(\bar{B}_{i^*}) = \min_{s \leq i \leq j} SIZE(\bar{B}_i)$;
      if $i^* > s$ **then** $B_i \leftarrow B_i \cup \bar{B}_s \cup \ldots \cup \bar{B}_{i^*-1}$;
      if $B_i \neq \emptyset$ **then** $i{+}{+}$; $B_i \leftarrow \emptyset$;
      if $i^* = 1$ **then** $B_i \leftarrow \bar{B}_{i^*}$ **else** $\hat{B} \leftarrow \hat{B} \cup \bar{B}_{i^*}$;

if $i^* < j$ **then** $B_i \leftarrow B_i \cup \bar{B}_{i^*+1} \cup \ldots \cup \bar{B}_j$;
$s \leftarrow j+1$;
end
end
else do begin
if $s < j$ **then** $B_i \leftarrow B_i \cup \bar{B}_s \cup \ldots \cup \bar{B}_{j-1}$;
if $B_i \neq \emptyset$ **then** $i++$; $B_i \leftarrow \emptyset$;
$s \leftarrow j$;
end
$j++$;
end
$B_i \leftarrow B_i \cup \bar{B}_s \cup \ldots \cup \bar{B}_u$;

Informally speaking, the algorithm lineray travers the buckets starting with $\bar{B}_1$ and ending with $\bar{B}_u$. Whenever it detects a sequence of $\lfloor \frac{t}{4} \rfloor$ successive buckets it chooses the smallest of those buckets for $\hat{B}$. To see that the computed partition $\hat{B} \cup B_1 \cup \ldots \cup B_r$ has the desired properties, observe first that a class B_j consists of at most $2\lfloor \frac{t}{4} \rfloor \leq \frac{t}{2}$ successive buckets $\bar{B}_i$ and that, by construction, the minimum bin sizes of any two of these successive buckets are within a factor of ϵ^2. This implies (3). Property (1) follows from the special treatment of the case $i^* = 1$. To see that also (2) holds observe that whenever an index i^* is chosen it clearly satisfies the following property:

$$SIZE(\bar{B}_{i^*}) \leq \frac{1}{j-s+1} \sum_{i=s}^{j} SIZE(\bar{B}_i) = \frac{1}{\lfloor \frac{t}{4} \rfloor} \sum_{i=s}^{j} SIZE(\bar{B}_i).$$

Taking the sum over all buckets $\bar{B}_{i^*}$, we obtain (observe that the buckets involved on the right hand side are different for different buckets $\bar{B}_{i^*}$)

$$\sum_{s \in \hat{B}} size(s) = \sum_{i=1}^{r} SIZE(\bar{B}_{i^*}) \leq \frac{1}{\lfloor \frac{t}{4} \rfloor} SIZE(B) \leq \epsilon \cdot SIZE(B).$$

□

Equation (2) of Lemma 3.2 indicates that we can disregard all bins in $\hat{B}$ without affecting the optimal solution very much. This allows us to use the partition given in Lemma 3.2 in order to define a family of restricted packing problems, which we will call $RVBP_i$, $i = 1, \ldots, r$.
Problem $RVBP_i$ consists of
all bins contained in B_i,
job set $J_i := \{p \in J \mid \epsilon \min(B_i) \leq size(p) \leq \max(B_i)\}$.
It is an important observation that all packing problems $RVBP_i$ have the property that the size of the minimum item resp. bin and that of the maximum item resp. bin differ by at most a factor of ϵ^{t+1} resp. ϵ^t. In the next section we will show how such packing problems can be solved.

3.2 Restricted variable sized bin packing problems

In this section we present two versions of a restricted packing problem ($RVBP$). The first version of $RVBP$ is formulated in the form $RVBP[\delta, \rho, u, v]$, meaning that all item resp. bin sizes have to belong to the interval $[\delta, 1]$ resp. the interval $[\rho, 1]$ and u and v determine the maximum number of *distinct* item and bin sizes that are allowed. The second version, written as $RVBP[\delta, \rho]$, relaxes the constraints on the input instances and allows the bins and items to take on arbitrarily many distinct sizes. Their minimal sizes however remain bounded by δ and ρ. Thus, all item sizes are in the interval $[\delta, 1]$ and all bin sizes are in the interval $[\rho, 1]$. We consider the problems $RVBP$ as *optimization problems*. That is, we aim at finding a solution which minimizes the sum of the sizes of all items that remain unpacked.
In the next two subsections we show that $RVBP[\delta, \rho]$ can be (approximately) solved in linear time by solving a related $RVBP[\delta, \rho, u, v]$ problem exactly.
Before we do that we note that the problems $RVBP_i$ which were introduced at the end of the last section can all be viewed as instances of $RVBP[\epsilon^{t+1}, \epsilon^t]$ by simply scaling all bin and item sizes by $1/\max(B_i)$.

3.2.1 $RVBP[\delta, \rho, u, v]$

The input instance $I = I(J, B)$ for $RVBP[\delta, \rho, u, v]$ can be written by two multisets $J = \{n_1 : p_1, n_2 : p_2, \ldots, n_u : p_u\}$ and $B = \{k_1 : s_1, k_2 : s_2, \ldots, k_v : s_v\}$, such that $1 \geq s_1 > s_2 > \cdots > s_v \geq \rho$, $1 \geq s_1 \geq p_1 > p_2 > \cdots > p_u \geq \delta$, $n = \sum_{i=1}^{u} n_i$ and $k = \sum_{i=1}^{v} k_v$, where n is the total number of items, k the total number of bins and where n_i and k_i give the number of items resp. bins of size p_i and s_i. Define a *configuration* to be a packing of a bin of a specific size with a specific set of items such that the sum of item sizes does not exceed the bin size. A configuration for a bin can be denoted by a u–vector $(l_1, \ldots, l_u)$ of non–negative integers, such that l_i is the number of items of size p_i that are packed into that bin. Since we are only packing items that are greater or equal than δ, we know that at most $\lceil \frac{1}{\delta} \rceil$ items can fit into a bin, thereby, limiting the number of possible configurations to a constant $q = q(\delta, u) \leq (u+1)^{\lceil \frac{1}{\delta} \rceil}$ for each of the v possible bin sizes.
Consider now a feasible solution x to an instance I of $RVBP[\delta, \rho, u, v]$. Clearly, x can be specified by a vector $x = (x_1, \ldots, x_{q \cdot v})$ where x_j denotes the number of bins that are packed according to configuration j.
Define now

- $\mathcal{T}_i$ the set of legal configurations for bins of size s_i
- a_{lj} as the number of items of size p_l that are used in configuration j

$RVBP[\delta, \rho, u, v]$ can then be formulated as the following integer program:

$$\begin{aligned} &minimize && \sum_{l=1}^{u} (n_l - \sum_{j} a_{lj} x_j) p_l && (4) \\ &subject\ to \end{aligned}$$

$$\forall 1 \leq l \leq u \qquad \sum_j a_{lj} x_j \leq n_l$$

$$\forall 1 \leq i \leq v \qquad \sum_{j \in \mathcal{T}_i} x_j \leq k_i$$

We call $OPT(I)$ the optimal value for the packing problem, i. e., the minimal possible total size of items that remain unpacked. Since we assumed that δ, ρ, and u are constants, we can obtain the ILP in time linear in v. Lenstra [Le83] has shown that the ILP can be solved in time polynomial in the number of constraints provided that the number of variables is fixed. That is, we can compute $OPT(I)$ in constant time for u and v fixed.

3.2.2 $RVBP[\delta, \rho]$

The second problem we consider is $RVBP[\delta, \rho]$ which is defined similar to $RVBP[\delta, \rho, u, v]$ except that the number of distinct bin and item sizes need no longer be constants. The bin and item sizes, however, are still restricted to the intervals $[\delta, 1]$ for items and $[\rho, 1]$ for the bins. Again, we assume that the items are sorted according to non–increasing item sizes.
We will show in the sequel how $RVBP[\delta, \rho, u, v]$ can be used for approximately solving $RVBP[\delta, \rho]$. The underlying idea of our reduction of an instance for the $RVBP[\delta, \rho]$ to an instance for $RVBP[\delta, \rho, u, v]$ is the so called linear grouping approach introduced in [DL81]. Here we will apply the grouping technique simultaneously to items *and* bins.
An instance for the $RVBP[\delta, \rho, u, v]$ is obtained from an instance $I = (J, B)$ of $RVBP[\delta, \rho]$ by the following construction: First, we group the item set J into groups $G_j = p_{(j-1)K_1+1} \ldots p_{jK_1}$ for $j = 1, \ldots, u$ and $G_{u+1} = p_{uK_1+1} \ldots p_{|J|}$, each of which, except the last group, consists of K_1 items where K_1 is a non–negative integer to be specified later. We define two functions INCREASING and DECREASING. The first function rounds the items of each group to its largest element. DECREASING rounds the item sizes of group G_j down to the size of the largest item in group G_{j+1} (the items in the last group are rounded down to the size of the smallest item). We will call the item groups obtained by INCREASING by $H = H_1 H_2 \ldots H_{u+1}$ and the groups obtained by DECREASING by $F = F_1 F_2 \ldots F_{u+1}$. The point of this definition is that the lists F_j and H_j, $(j = 1, \ldots u+1)$ are almost the same, except for the first resp. last item group. More specifically, as $F_j = H_{j+1}$, a packing for $\bigcup_{j \leq u} F_j$ *is* a packing for $\bigcup_{j=2}^{u+1} H_j$. This suggests the following algorithm: Find a packing for the list F, except for group F_{u+1}, and obtain a solution that is as good as the optimal solution of the original input (recall that items are rounded down in size). Consider the solution as a packing for the list H except of a group H_1 of items that are not packed. Clearly, the total size of items that belong to H_1 is at most K_1. Construct finally a feasible packing for the original input instance of the $RVBP[\delta, \rho]$ from the packing of H by decreasing the item sizes down to their original sizes. It clearly holds:

$$OPT(I_F) \leq OPT(I_H) \leq OPT(I_F) + K_1 \leq OPT(I) + K_1,$$

where I_H and I_F are the instances with item lists H resp. F.
Similarly, we will deal with the bins. We will group the bins into bin groups, each (except the last group) of which consisting of a number K_2 of bins, and define INCREASING to increase bins of the j^{th} bin group to the smallest bin in group $j-1$ and the bins of the first group to the largest bin. DECREASING on the other hand rounds all bins in one group down to the size of the smallest bin in the group. Solving the $RVBP$ on the modified input list where INCREASING is applied and rounding back bin sizes to their original sizes will again yield a feasible packing where the sum of the item sizes that are not packed is K_2. Putting together the grouping of bins and the grouping of the items in a single step will yield the desired reduction for I to an instance for the $RVBP[\delta, \rho, u, v]$ where we set $u = \lfloor \frac{|J|}{K_1} \rfloor$ and $v = \lfloor \frac{|B|}{K_2} \rfloor$. If K_1 and K_2 are defined by

$$K_1 := \lceil |J|\mu \rceil \qquad \text{and} \qquad K_2 := \lceil |B|\mu \rceil$$

(for an arbitrary but fixed constant $1 \geq \mu > 0$) then u and v are constants, and the corresponding problem $RVBP[\delta, \rho, u, v]$ can thus be solved in linear time, as shown in the previous section. The following lemma is therefore evident.

Lemma 3.3 *Let $1 \geq \mu > 0$ be an arbitrary, but fixed constant and let $K_1 := \lceil |J|\mu \rceil$ and $K_2 := \lceil |B|\mu \rceil$. Then we can find for any instance $I = I(J, B)$ for $RVBP[\delta, \rho]$ in linear time a solution such that the total size of all items that remain unpacked is at most*

$$OPT(I) + K_1 + K_2 \leq OPT(I) + \mu(|J| + |B|)) + 2.$$

(Here $OPT(I)$ denotes the total size of all items that remain unpacked in an optimal solution of the instance $I = I(J, B)$.)

3.2.3 Solving the subproblems $RVBP_i$

As already outlined above, the problems $RVBP_i$ defined at the end of Section 3.1 can all be viewed as instances of $RVBP[\epsilon^{t+1}, \epsilon^t]$ by simply scaling all bin and item sizes by $1/\max(B_i)$. Clearly, Lemma 3.3 also applies for these scaled instances. That is, the sum of the sizes of unpacked items for problem $RVBP_i$ can thus be bounded by $OPT(RVBP_i) + (\mu(|J_i| + |B_i|) + 2)\max(B_i)$. We will see later that an appropriate choice for μ will guarantee that the sum of sizes of unpacked items is small enough for an $(1+\epsilon)$–approximation of the makespan.
For the $RVBP_1$, we have to consider two cases. In the case that all bins are smaller than ϵ we solve the $RVBP_1$ approximately as for all other problems $RVBP_i$. Otherwise, we proceed in a slightly different way. (The reason for this is that items that remain unpacked may be as large as C_{opt}, and contrary to all other subproblems $RVBP_i$ their effect on the final schedule is therfore so significant that they have to be packed more carefully.) For solving $RVBP_1$ in that case, we first observe that if the number of items with size at least ϵ^{t+2} exceeds $\frac{m}{\epsilon^{t+2}}$ there exists no feasible packing, as the total bin capacity of all bins in VBP is bounded by m (recall that we scaled all bins and items by $1/C$). For

$RVBP_1$ we consider the bin set $\bar{B}_1 := B_1 \cup \{b \in \hat{B} \mid size(b) \geq \min(J_1)\}$, that is we may view $RVBP_1$ directly as a problem of type $RVBP[\epsilon^{t+2}, \epsilon^{t+2}, \frac{m}{\epsilon^{t+2}}, |\bar{B}_1|]$ which can be solved optimally in constant time, as $|\bar{B}_1|$ is also, trivially, bounded by $|\bar{B}_1| \leq \frac{m}{\epsilon^{t+2}}$. If some items remain unpacked we can safely infer that there exists no feasible packing for VBP.

3.3 Greedy Packing

We will now come to the final ingredient of our algorithm, the greedy packing. It will be used to pack the two classes of items that have not been packed into any bin yet. Those are the sets

$\hat{J}_i \subset J_i$, and

$J^* = J - \bigcup_{i=1}^{r} J_i$,

where $\hat{J}_i$ denotes the set of items that remain unpacked in the solution of the problem $RVBP_i$. The second set J^* is the union of all items that have not been considered in any packing subproblem $RVBP_i$, as simply their sizes are "between" the item sizes of two subsequent packing problems. It is natural, to use the partition $J^* := \bigcup_{i=1}^{r} J_i^*$, where $J_i^* := \{p \in J \mid \epsilon \min(B_i) > size(p) > \max(B_{i+1})\}$ (with the convention that $\max(B_{r+1}) = 0$).
We proceed as follows: After having solved a problem $RVBP_i$ $(i = 1, \ldots, r)$ we pack the items $p \in \hat{J}_i$ that remain unpacked in the solution of the $RVBP_i$ into the bins with size greater than $\max(B_i)$ (including bins from $\hat{B}$). For short, we will call those bins $B_i^{\leq}$, with $B_1^{\leq}$ being the empty set. In this step we allow each bin, even the "closed" bins, to be overloaded by one item $p \in \hat{J}_i$. Secondly, we greedy pack for each i the items in J_i^*. Again, we allow the bins to be overloaded by an item p – if its size is not larger than that of the bin.
At the end of the algorithm we remove all items that overload a closed bin and pack them with a greedy algorithm into the open bins. This final step will guarantee that the corresponding schedule will be feasible, as all overlappings with fixed jobs are cleared.

4 The complete algorithm

We are now able to present the complete algorithm. Recall that an instance I for the scheduling problem with fixed jobs is given by

- A set of n jobs $p_1, \ldots, p_n$ with integer processing times.
- A subset $p_{i_1}, \ldots, p_{i_k}$ of k fixed jobs. Each fixing consists of a starting time and an assigned machine.

The number m of machines and the desired approximation ratio are not part of the input. They are considered to be fixed. The completion time of the last fixed job is denoted by C_F. According to the context, jobs p_j are viewed as items with size $size(p_j)$ corresponding to their processing times.

As already outlined in the beginning of Section 3, the basic framework of our algorithm is a binary search for the optimum makespan C_{opt}. A useful lower bound for C_{opt} is $\max\{\sum_{j=1}^{n} \frac{size(p_j)}{m}, \max_j size(p_j), C_F\}$. Similar to Section 2 it can be easily shown that any list schedule has a makespan at most $3\max\{\sum_{j=1}^{n} \frac{size(p_j)}{m}, \max_j size(p_j), C_F\}$.
One easily checks that there are a few simple criteria which indicate that a proposed candidate C for the makespan is smaller than C_{opt} resp. that the packing problem VBP corresponding to C is infeasible. We defer the precise arguments for these facts to the proof of Lemma 5.3.
The algorithm can then be stated as follows:

[Algorithm A_ϵ for the makespan problem with fixed jobs]

Input: Instance I consisting of fixed and free jobs.
Output: A feasible schedule for I.

1. Sort jobs by non-increasing sizes.
2. $ub \leftarrow 3\max\{\sum_{j=1}^{n} \frac{size(p_j)}{m}, \max_j size(p_j), C_F\}$;
   $lb \leftarrow \max\{\sum_{j=1}^{n} \frac{size(p_j)}{m}, \max_j size(p_j), C_F\}$;

   while $(ub - lb \geq 1)$ **do begin**

   $C \leftarrow lb + \frac{ub-lb}{2}$;

   Obtain an instance of a packing problem VBP with bins B and jobs J by linearly traversing the machines and scaling all bins and items by $\frac{1}{C}$.

   Call the function for solving VBP defined via the scaled bins and items. If a solution for the packing problem is obtained, set $ub \leftarrow C$, else set $lb \leftarrow C$.

   end
3. If no packing has been detected so far, set $C \leftarrow ub$ and again solve the packing problem VBP on the bins and items scaled by $\frac{1}{C}$.
4. Interpret the obtained packing as a schedule and return it.

We will now present the function for packing the items into the bins, which is the essential part of the algorithm.

[Function for solving VBP]

Input: Set J of items and set B of bins
Output: A packing of items into bins (which may overload the "open" but not the "closed" bins) or a proof of the infeasibility of the packing problem

1. Identify the bin sets $B_1, \ldots, B_r$ according to Lemma 3.2 and construct the problems $RVBP_i$ as indicated at the end of Section 3.1.

For $i = 1, \ldots, r$ let

$$\begin{aligned} B_i^* &:= \{s \in B \mid \min(B_i) > size(s) > \max(B_{i+1})\} \text{ and} \\ J_i^* &:= \{p \in J \mid \epsilon \min(B_i) > size(p) > \max(B_{i+1})\} \end{aligned}$$

(with the convention that $\max(B_{r+1}) = 0$);

2. **if** $\max(B_1) \geq \epsilon$ **do begin**

if items $p \in J$ exist with $size(p) > \max(B_1)$, exit (infeasibility detected)

if $|J_1| \geq \frac{m}{\epsilon^{t+1}}$, exit (infeasibility detected) **else** solve $RVBP_1$ optimally (cf.Section 3.2.1), hereby we consider all bins from $B_1 \cup \{b \in \hat{B} \mid size(b) \geq \min(J_1)\}$. Exit if some items remain unpacked (infeasibility detected);

else

Solve $RVBP_1$ approximately (cf. Section 3.2.2) with $\mu := \epsilon^{t+2}$.

Let $\bar{J}_1$ denote the set of all items which remain unpacked.

if $SIZE(\bar{J}_1) > (\mu \cdot (|J_1| + |B_1|) + 2) \max(B_1) + SIZE(B_1^*)$ **then** exit (infeasibility detected);

end

Pack all items $p \in J_1^*$ with a greedy algorithm into the bins $s \in B_1 \cup B_1^*$. The greedy algorithm is allowed to put an item into any bin that is not yet completely filled (even if – after the item is inserted – the capacity of the bin is exceeded), provided the size of the item does not exceed the (total) size of the bin. Hereby, the bins should be filled up according to non–increasing sizes and the items should also be considered in sorted order according to non–increasing sizes. Exit if unpacked items remain (infeasibility detected).

3. **for** $i = 2$ **to** r **do begin**

Solve $RVBP_i$ approximately (cf. Section 3.2.3) with $\mu := \epsilon^{t+2}$.

Let $\hat{J}_i$ denote the set of items which remain unpacked and let

$$B_i^< := B_1 \cup B_1^* \cup \ldots \cup B_{i-1} \cup B_{i-1}^*.$$

Pack items $p \in \hat{J}_i$ with a greedy algorithm into the bins of $B_i^<$. We allow that an item is put into a bin which is not yet completely filled even if the capacity of the bin is exceeded. Let $\bar{J}_i$ denote the set of all items which remain unpacked.

if $SIZE(\bar{J}_i) > (\mu(|J_i| + |B_i|) + 2)\max(B_i) + SIZE(B_i^*)$, exit (infeasibility detected);

Pack all items $p \in J_i^*$ with a greedy algorithm into the bins of $B_i^< \cup B_i \cup B_i^*$. The greedy algorithm is allowed to put an item into any bin that is not yet completely filled (even if – after the item is inserted – the capacity of the bin is exceeded), provided the size of the item does not exceed the (total) size of the bin. Hereby, the bins should be filled up according to non–increasing sizes and the items should also be considered in sorted order according to non–increasing sizes. Exit if unpacked items remain (infeasibility detected).

end

4. For every overloaded bin remove the *last* item from this bin. Let J_R denote the set of all removed items.

   Let $\bar{J} := \bar{J}_1 \cup \cdots \cup \bar{J}_r \cup J_R$.

   Pack all items in $\bar{J}$ with a greedy algorithm into the "open" bins, putting each item into that bin which has currently the least overload.

   end

5 Analysis

We are now ready to tie the results of the previous sections together in order to obtain the following theorem:

Theorem 5.1 *Algorithm A_ϵ finds in time $\mathcal{O}(n \log n)$ a feasible schedule for an instance I of the scheduling problem with fixed jobs and a constant number of machines. The performance ratio of Algorithm A_ϵ is $1 + 10\epsilon$.*

We prove the theorem in two steps. We first show that the algorithm will always construct a feasible schedule with the desired performance ratio. Then we prove the bound $\mathcal{O}(n \log n)$ on the computation time.

Lemma 5.2 *For all $C \geq C_{opt}$ the function for solving VBP constructs a packing that does not overload the "closed" bins.*

Proof. We first consider the items in J_i. Items to be packed by the greedy algorithm are all items of the set J_i that remain unpacked by the solution of $RVBP_i$. Note that these items can only be packed into bins $B_i^< \cup B_i \cup B_i^*$ but not into any smaller bin. So, all bins $\bigcup_{j=i+1}^{r}(B_j \cup B_j^*)$ need not be considered for items in J_i. Within the solution of $RVBP_i$ items in J_i are packed in the bins B_i by solving $RVBP_i$ (almost) optimally (resp. optimally, if $i = 1$ and $\max(B_1) \geq \epsilon$). The value of the optimal solution for the $RVBP_i$ is at most the size of the items of J_i that are not packed into B_i in the optimal schedule. That is, the total size $SIZE(\hat{J}_i)$ of all items in $\hat{J}_i$ exceeds the total size of all items

from J_i that are not packed in an optimal solution into bins of B_i by at most the error of the obtained solution for $RVBP_i$, that is by at most

$$(\mu(|J_i| + |B_i|) + 2) \cdot \max(B_i)$$

(resp. zero for $RVBP_1$, if $\max(B_1) \geq \epsilon$.)
Observe finally, that all items that are packed into $B_i^<$ before applying the algorithm for $RVBP_i$ on J_i, are also packed into $B_i^<$ in the optimal solution. Since we assumed $C \geq C_{opt}$, it therefore holds that $SIZE(\hat{J}_i) \leq fcap_i + SIZE(B_i^*) + (\mu(|J_i|+|B_i|)+2) \cdot \max(B_i)$, where $fcap_i$ denotes the remaining total free capacity of all bins in $B_i^<$ before the start of the greedy algorithm. As the greedy algorithm may overload the bins, it will only leave items unpacked iff all bins in $B_i^<$ are completely full. That is, the total size of all items that remain unpacked will be at most $(\mu(|J_i|+|B_i|)+2) \cdot \max(B_i) + SIZE(B_i^*)$. This guarantees that the algorithm does not exit at the corresponding **if**-condition whenever $C \geq C_{opt}$.
Now consider the items in J_i^*. Observe that they, as well, fit only into bins from $B_i^< \cup B_i \cup B_i^*$. Again it holds that all items that are packed into $B_i^< \cup B_i \cup B_i^*$ before applying the greedy algorithm to J_i^*, are also packed into these bins in the optimal solution. Furthermore, as we do pack the items in sorted order and do fill the bins according to non–increasing size, a similar property holds throughout the greedy packing for all bins from B_i^*: whenever we consider an item p and the largest bin that is not yet completely filled is b then all bins with size at least $size(b)$ contain only items of size at least $size(p)$. That is, either p can be packed into b (if $size(p) \leq size(b)$) or we have detected that the packing is infeasible, which cannot be the case for $C \geq C_{opt}$. That is, the greedy algorithm has to succeed with packing all items from J_i^*.
This shows that for all $C \geq C_{opt}$ the algorithm will never exit without a packing. As one also easily checks that the algorithm does indeed pack all items and, by construction, no "closed" bin is overloaded, this completes the proof of the lemma. □

Lemma 5.3 *For $C \geq C_{opt}$ the total size of all elements in J_R is bounded by*

$$SIZE(J_R) \leq 2\epsilon \cdot SIZE(B). \tag{5}$$

Proof. The set J_R consists of all elements which had to be removed from an overfull bin. By construction the maximum size of an item which created an overload in a bin from B_i is $\epsilon \min(B_i)$ and the maximum size of an item p that created an overload in a bin b from B_i^* is $size(b)$. Moreover, the way B_i^* is defined, we have $\sum_{b \in \bigcup_{i=1}^r B_i^*} size(b) = SIZE(\hat{B})$ with $\hat{B}$ being defined according to Lemma 3.2. Using Lemma 3.2, we obtain

$$\begin{aligned}
\sum_{p \in J_R} size(p) &\leq \sum_{i=1}^{r} \epsilon \cdot min(B_i)|B_i| + \sum_{i=1}^{r} \sum_{b \in B_i^*} size(b) \\
&\leq \sum_{i=1}^{r} (\epsilon \cdot SIZE(B_i)) + SIZE(\hat{B}) \\
&\leq 2\epsilon \cdot SIZE(B).
\end{aligned}$$

□

Together with the trivial bounds,

$$\begin{aligned} SIZE(J) &\leq m, \text{ and} \\ SIZE(B) &\leq m \end{aligned}$$

(which hold as we assumed that all items and bins were scaled by $1/C$) we obtain the following corollary:

Corollary 5.4 *For $C_{opt} \leq C$ and $\epsilon \leq \frac{1}{2}$ the total size of all elements in $\bar{J}$ is bounded by*

$$SIZE(\bar{J}) \leq \epsilon \cdot (5m + 4). \tag{6}$$

Proof. The minimum size of an item resp. bin given in an instance of problem $RVBP_i$ is not smaller than $\max(B_i)\epsilon^{t+1}$. This tells us that $|B_i| \leq \frac{SIZE(B_i)}{\epsilon^{t+1}\max(B_i)}$ and that $|J_i| \leq \frac{SIZE(J_i)}{\epsilon^{t+1}\max(B_i)}$. Thus we have

$$\sum_{i=1}^{r} |B_i| \max(B_i) \leq \sum_{i=1}^{r} \frac{SIZE(B_i)}{\epsilon^{t+1}} \leq \frac{m}{\epsilon^{t+1}} \tag{7}$$

and

$$\sum_{i=1}^{r} |J_i| \max(B_i) \leq \sum_{i=1}^{r} \frac{SIZE(J_i)}{\epsilon^{t+1}} \leq \frac{m}{\epsilon^{t+1}}. \tag{8}$$

Observe that $\bar{J_1}$ is empty whenever we solved $RVBP_1$ optimally, that is, whenever $\max(B_1) \geq \epsilon$. That is, using the notation $i_0 := 1$ if $\max(B_1) < \epsilon$ and $i_0 := 2$ otherwise, we obtain (recall that we chose $\mu := \epsilon^{t+2}$)

$$\begin{aligned} \sum_{p \in \bar{J}} size(p) &= \sum_{i=i_0}^{r} \sum_{p \in \bar{J_i}} size(p) + \sum_{p \in J_R} size(p) \\ &\leq \sum_{i=i_0}^{r} [(\mu(|B_i| + |J_i|) + 2) \cdot \max(B_i) + SIZE(B_i^*)] + 2\epsilon \cdot SIZE(B) \\ &\leq 2\epsilon m + 2\epsilon \sum_{i=0}^{\infty} \epsilon^i + SIZE(\hat{B}) + 2\epsilon \cdot SIZE(B) \\ &\leq \epsilon \cdot (2m + 4 + 3m), \end{aligned}$$

again using that $SIZE(\hat{B}) \leq \epsilon \cdot SIZE(B)$ and $SIZE(B) \leq m$. □

Corollary 5.4 immediately allows us to bound the makespan computed by algorithm A_ϵ.

Corollary 5.5 *For $C \geq C_{opt}$ and $\epsilon \leq \frac{1}{2}$ the algorithm finds a feasible schedule with makespan C_{A_ϵ} such that*

$$C_{A_\epsilon} \leq (1 + 9\epsilon) \cdot C.$$

Proof. First observe that all jobs in $J \setminus \bar{J}$ are packed in such a way that the resulting makespan is at most C. We thus only have to consider the items in $\bar{J}$. Clearly, if we pack them into m (initially empty) bins in a greedy fashion choosing for each item the bin which has currently the least load, the maximum load of the m bins will be, for $m > 1$, bounded by

$$\frac{1}{m} \sum_{p_j \in \bar{J}} size(p_j) + \max_{p_j \in \bar{J}} size(p_j) \leq 5\epsilon + 4\frac{1}{m} \cdot \epsilon + \epsilon \leq 8\epsilon,$$

where the first inequality follows from Corollary 5.4 and the fact that – by construction – all items in $\bar{J}$ have size at most ϵ. Similary, for $m = 1$ we can directly apply Corollary 5.4 and obtain a load of the bin bounded by

$$\sum_{p_j \in \bar{J}} size(p_j) \leq 9\epsilon.$$

Clearly, by also using up the remaining free capacity of the "open" bins the resulting load can only decrease.
That is, the makespan of the resulting schedule satisfies

$$C_{A_\epsilon} \leq C + 9\epsilon \cdot C.$$

(Note that the additional factor of C is due to the fact that within the packing problem all item and bin sizes where scaled by $1/C$.) □

In what follows, we will proof the $\mathcal{O}(n \log n)$ computation time of algorithm A_ϵ. Clearly, items can be sorted by non–increasing sizes in time $\mathcal{O}(n \log n)$. Consider now the computation time needed for solving a single instance of VBP:

- We need time $\mathcal{O}(n \log n)$ for sorting bins by non–increasing size and then for each packing problem VBP. Computing the classes B_i can then be done in linear time according to Lemma 3.2.

- For each packing problem $RVBP_i$, $1 \leq i \leq r$ we need linear time for constructing the ILP and constant time for solving the ILP. In the case that problem $RVBP_1$ is solved by complete enumeration, then again constant time is needed. Cf. Section 3.2.1 resp. 3.2.3.

  Packing items with the greedy algorithm consumes linear time, since we simply fill up the bins successively.

- Identifying the set J_R and packing all items in $\bar{J}$ can again be carried out in linear time.

This shows that we can solve a VBP in time $\mathcal{O}(n \log n)$. It remains to count the number of iterations of VBP that are carried out within the binary search frame. In the original scheduling problem all processing times are assumed to be integers, so the binary search can be terminated, when $ub - lb < 1$. This yields already a polynomial time bound for our algorithm. We will use the following lemma, though, to improve the computation time.

Lemma 5.6 *If algorithm A_ϵ is executed with k iterations of binary search, the resulting schedule has a makespan of at most $(1+9\epsilon)(1+2^{-(k-1)})C_{opt}$.*

Thus, only $\mathcal{O}(\log(\frac{1}{\epsilon}))$ iterations are required to obtain a $(1+10\epsilon)$-approximation. The lemma is easily proven by observing that we have $ub - lb = 2^{-(k-2)} \max\{\sum_{j=1}^{n} \frac{size(p_j)}{m}, \max_j size(p_j), C_F\}$ in iteration k and that $C_{opt} \geq lb \geq \max\{\sum_{j=1}^{n} \frac{size(p_j)}{m}, \max_j size(p_j), C_F\}$.

6 The uniform parallel machine case

In the uniform parallel machine case we assume that the machines run at different speeds $s_1, \cdots, s_m$. More precisely, if job j is executed on machine i it takes p_j/s_i time units to be completed. Again the goal is to assign the jobs to the machines so that the last job finishes as early as possible. In [HS88], Hochbaum and Shmoys had presented a $PTAS$ for the standard version of the scheduling problem where no jobs are fixed. For the $PTAS$ they generalized their approach of dual approximation to the case where bins have different sizes. The computation time of their algorithm is a high polynomial in n.
The $PTAS$ presented in this paper for the scheduling problem with fixed jobs can also be generalized to the case where machines run at different speeds and the number of machines is a constant m. For simplicity we normalize the speed of the fastest machine to 1 and so we have $s_i \leq 1$, $(i = 1, \cdots, n)$. Let us further assume a binary search procedure proposes a candidate C for the makespan. Since every machine i can process s_i units of processing time in one time unit, it can process a total of $C \cdot s_i$ of processing time before the deadline C.
On the other hand, C defines a packing problem with open and closed bins of variable sizes. We scale those bins and all the jobs by $\frac{1}{C}$ and multiply the sizes of bins that correspond to machine i with s_i. That is, an item p that is placed into any bin on machine i occupies $\frac{size(p)}{s_i} \cdot C$ time units in the schedule on machine i. Again, the task is to decide whether there exists a packing of the scaled item set J into the resulting set of bins B. For our $PTAS$ we refine algorithm A_ϵ in that we place all items that are assigned to the set $\bar{J}$ now to the fastest machine. As $SIZE(J) \leq m$ and $SIZE(B) \leq m$, Corollary 5.4 still applies and so, for $C \geq C_{opt}$ and $\epsilon \leq \frac{1}{2}$, we obtain a total makespan of at most $C + \frac{(5m+4)\epsilon}{\max_i(s_i)} \cdot C = C + (5m+4)\epsilon \cdot C$. Calling this refined algorithm A'_ϵ we obtain the following theorem:

Theorem 6.1 *Algorithm A'_ϵ finds in time $\mathcal{O}(n \log n)$ a feasible schedule for an instance I of the scheduling problem with fixed jobs and a constant number m of machines with independent processing times. The performance ratio of Algorithm A_ϵ is $1 + (5m+5) \cdot \epsilon$.*

7 Non-approximability results

The following two theorems indicate the limits in the approximability of the scheduling problem with fixed jobs.

Theorem 7.1 *For a constant number m of machines, there is no fully polynomial time approximation scheme for the scheduling problem with fixed jobs, unless $P = NP$.*

Proof. It is well known (*cf.* [GJ79]) that no strongly NP-complete problem can be approximated by a fully polynomial time approximation scheme, unless $P = NP$. It thus suffices to reduce some strongly NP-complete problem to the scheduling problem with fixed jobs and a constant number, say 1, of machines. We will reduce from

3-$PARTITION$
INPUT: A finite set A of $3n$ elements of sizes $s_1, \ldots, s_{3n} \in \mathbb{Z}^+$ and a bound $B \in \mathbb{Z}^+$ such that $\frac{B}{4} < s_i < \frac{B}{2}$ for all $1 \leq i \leq 3n$ and that $\sum_{i=1}^{3n} s_i = n \cdot B$.
QUESTION: Does there exist a partition of A into n disjoint sets $A_1, \ldots, A_n$ such that for $1 \leq i \leq n$, $\sum_{s_j \in A_i} s_j = B$?

This problem is known to be strongly NP-complete, see [GJ79]. Given an instance of 3-$PARTITION$ we construct an instance of the 1-machine scheduling problem as follows. For every item we introduce a job whose processing time corresponds to the size of the item. Furthermore, we add a fixed job of size 1 at times $k \cdot B + (k-1)$ for all $1 \leq k \leq n$. Clearly, there exists a schedule with makespan $n \cdot (B+1)$ if and only if there exists the desired partition of the items into n sets. □

Theorem 7.2 *If the number m of the machines is part of the input, there exists no polynomial time algorithm that solves the scheduling problem with fixed jobs within a performance guarantee of $\frac{3}{2} - \epsilon$ for all $\epsilon > 0$, unless $P = NP$.*

Proof. This time we reduce from the following variant of 3-$PARTITION$, which is also known to be NP-complete, see [GJ79]:
INPUT: Disjoint sets A and B containing n resp. $2n$ elements of sizes $a_1, \ldots, a_n \in \mathbb{Z}^+$ resp. $b_1, \ldots, b_{2n} \in \mathbb{Z}^+$ and a bound $L \in \mathbb{Z}^+$ such that $\sum a_i + \sum b_j = nL$.
QUESTION: Does there exist a permutation $\pi \in \mathcal{S}_{2n}$ such that for all $1 \leq i \leq n$: $a_i + b_{\pi(2i-1)} + b_{\pi(2i)} = L$?

This time we provide n machines, choose an integer number K such that $K \geq (\frac{1}{2} - \epsilon)L/(2\epsilon)$, and fix for every machine M_i, $1 \leq i \leq n$, a job with processing time a_i which ends at time $2K + L$. Furthermore, we have $2n$ non-fixed jobs with processing times $K + b_i$, $1 \leq i \leq 2n$. Clearly, there exists a schedule with makespan $2K + L$ if and only if there exists the desired permutation π. In addition, if such a permutation does not exist every schedule will have a makespan of at least $3K + L$. As $(3K + L)/(2K + L) \geq 3/2 - \epsilon$ by choice of K, the claim of the theorem follows. □

The proof of Theorem 7.2 indicates that the lower bound $\frac{3}{2}$ for the approximability of the makespan is also valid for large C_{opt}. As a consequence, there exists no *asymptotic PTAS* for the scheduling problem with fixed jobs, if the number m of machines is specified as part of the input, unless $P = NP$.

8 Conclusion

We presented a polynomial time approximation scheme for the scheduling problem with fixed jobs, provided the number of machines is fixed. Our $PTAS$ is based on a new reduction to a series of restricted variable size bin packing problems ($RVBP$s) which are obtained by eliminating bins that are not important for the optimal solution. The algorithm then solves the $RVBP$s with the help of an integer linear programming approach. Items that cannot be packed are subsequently considered by a greedy algorithm. A final list scheduling algorithm packs only remaining items. The computation time is $\mathcal{O}(n \log n)$ for every fixed desired performance guarantee of $1 + \epsilon$. Also, the $PTAS$ can be extended to the case where machines run at different speeds. Our algorithm is best possible on the other hand, in that we also showed that neither a $PTAS$ for the scheduling problem with fixed jobs and an arbitrary number of machines nor an $FPTAS$ for the constant machine case exists, unless $P = NP$. In view of the practical relevance of our problem, we also discussed some fast, LIST–based algorithms for the scheduling problem with fixed jobs. These algorithms also work in on–line settings and have a constant wost case guarantee.

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