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Systems of Quasi-Linear PDEs Arising in the Modelling of Biofilms and Related Dynamical Questions

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List of Frequently Used Notation

$\mathbb{N}$	positive integers
$\mathbb{Z}$	integers
$\mathbb{N}_0, \mathbb{Z}_+$	non-negative integers
$\mathbb{R}$	real numbers
$\mathbb{R}_+$	non-negative real numbers
$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathbb{R}^n_+$	componentwise non-negative vectors in $\mathbb{R}^n$
$\mathbb{T}$	$\mathbb{Z}$ or $\mathbb{R}$
$\mathbb{R}^{n\times n}$	real $n \times n$ matrices
$\leq, \precsim$	order relations in $\mathbb{R}^n$
$x \cdot y$	inner product in $\mathbb{R}^n$
$\lvert \cdot \rvert$	absolute value
$\lVert \cdot \rVert$	norm in $\mathbb{R}^n$
$\lfloor x \rfloor$	largest integer less than or equal to $x \in \mathbb{R}$
$\subset$	subset
$A \setminus B$	A without B
$\overline{A}$	closure of A
$\sharp A$	cardinality of A
$\sup$	supremum
$\inf$	infimum
(X, d_X)	metric space X with metric $d_X(\cdot,\cdot)$
$B_r^X(x)$	ball in the metric space X of radius $r > 0$ and centre $x \in X$
$(V, \lVert \cdot \rVert_V)$	normed space with norm $\lVert \cdot \rVert_V$
$\overline{A}^{\lVert\cdot\rVert_V}$	closure of the set $A \subset V$ in the norm topology
$(H, \langle \cdot,\cdot \rangle_H)$	Hilbert space H with inner product $\langle \cdot,\cdot \rangle_H$
$X \hookrightarrow Y$	continuous embedding of X in Y
$X \hookrightarrow\hookrightarrow Y$	compact embedding of X in Y

$f\|_A$	f restricted to the set A
$supp$	support
f_+, f_-	positive and negative part of f
$a.e.$	almost everywhere
$\lim, \to$	limit
$\rightharpoonup$	weak limit
Δ	Laplace operator
∇	gradient operator
$\partial_{x_i}, \partial_t$	partial derivatives
∂_ν	outward unit normal derivative
Ω	spatial domain
$\partial\Omega$	boundary of Ω
Q_T	parabolic cylinder $\Omega \times (0,T)$
$C^k(\Omega)$	k-times continuously differentiable functions on $\Omega, k \in \mathbb{N}_0$
$C_0^k(\Omega)$	functions in $C^k(\Omega)$ with compact support
$C^{k,l}(Q_T)$	functions that are k-times continuously differentiable with respect to x and l-times continuously differentiable with respect to t
$C^\alpha(\Omega)$	Hölder continuous functions on $\Omega, 0 < \alpha \leq 1$
$C^{\alpha,\beta}(Q_T)$	functions that are α-Hölder continuous with repect to x and β-Hölder continuous with respect to t
$L^p(\Omega)$	Lebesgue spaces, $1 \leq p \leq \infty$
$L^p_{loc}(\Omega)$	local L^p-spaces
$L^p(\Omega;\mathbb{R}^n)$	vector-valued functions with components in $L^p(\Omega)$
$\preccurlyeq, \precsim$	order relations in $L^2(\Omega;\mathbb{R}^n)$
K^+	positive cone in $L^2(\Omega;\mathbb{R}^n)$
$W^{k,p}(\Omega)$	Sobolev spaces, $k \in \mathbb{N}, 1 \leq p \leq \infty$
$H^k(\Omega)$	Sobolev spaces $W^{k,2}(\Omega)$
$H^s(\Omega)$	fractional Sobolev spaces, $s \in \mathbb{R}$
$H^s_0(\Omega)$	functions in $H^s(\Omega)$ vanishing on the boundary
$C([0,T];V)$	continuous functions on $[0,T]$ taking values in the Banach space V
$L^p((0,T);V)$	Bochner spaces

Id	identity operator
$\circ$	composition of operators
$\{T(t) \mid t \in \mathbb{T}_+\}$	semigroup
$\{U(t,s) \mid t \geq s\}$	evolution process
$\omega(A)$	ω-limit set of A
$\omega(A,t)$	pullback ω-limit set of A at time t
$\mathcal{A}$	global attractor
$\mathcal{M}$	exponential attractor
$\{\mathcal{A}(t) \mid t \in \mathbb{T}\}$	global pullback attractor
$\{\mathcal{M}(t) \mid t \in \mathbb{T}\}$	pullback exponential attractor
$\mathrm{dist}_{\mathrm{H}}^X(\,\cdot\,,\,\cdot\,)$	Hausdorff semi-distance in X
$N_\epsilon^X(A)$	minimal number of ϵ-balls in X needed to cover $A \subset X$
$\dim_{\mathrm{f}}^X(\,\cdot\,)$	fractal dimension in X
$\dim_{\mathrm{H}}^X(\,\cdot\,)$	Hausdorff dimension in X
$\mathcal{L}(X;Y)$	Banach space of bounded linear operators from X to Y
$\lVert \cdot \rVert_{\mathcal{L}(X;Y)}$	operator norm in $\mathcal{L}(X;Y)$
$\mathcal{D}(A)$	domain of the operator A
X^α	fractional power spaces
$(\Omega, \mathcal{F}, \mathbb{P})$	probability space
$\{W_t,\, t \in \mathbb{R}_+\}$	scalar Wiener process
dW_t	Itô differential
$\circ\, dW_t$	Stratonovich differential
E	expectation value

Introduction

Systems in biology, physics and other sciences are frequently modelled by evolutionary partial differential equations (PDEs). Many equations arising from mathematical physics were extensively studied and are yet well-understood (see [69] or [42]). Within the past decades, great interest arose in the modelling of biological systems and in new classes of PDEs emerging from this field. Analysing the behaviour of these models often requires new mathematical tools and ideas since the standard theory does not apply. A famous example is the chemotaxis system (see [44]), which describes the dynamics of a bacterial population in a spatial domain. The population follows the gradient of a chemotactic agent that is produced by the population itself and moves towards regions where substrate concentrations are higher. This problem attracted many mathematicians and deep analytical results were established.

Our focus lies on mathematical models that are formulated as systems of non-linear parabolic PDEs. We aim at studying the qualitative behaviour of solutions by using methods from the theory of dynamical systems. Each particular problem requires to choose appropriate function spaces and to prove the existence, uniqueness and continuous dependence of solutions on initial data. Once the well-posedness of the model is established, the time evolution of the system can be described in terms of semigroups acting in infinite dimensional spaces, or by evolution processes in the non-autonomous context. The central motivation for our analysis are systems of quasi-linear parabolic PDEs arising in the mathematical modelling of biofilms. The models describe the growth of spatially heterogeneous bacterial biofilm communities and are formulated as highly irregular density-dependent reaction-diffusion equations. The governing equations for the biomass density exhibit two degenerate diffusion effects simultaneously, which lead to difficulties in the analysis. Many interesting mathematical questions arise since standard methods are not applicable and new tools have to be applied to establish the well-posedness of the models.

Apart from proving the well-posedness of concrete mathematical models we are interested in the qualitative behaviour of solutions. As in the models for the growth of biofilms in most biological applications the solutions describe non-negative quantities. It is therefore essential for the mathematical model that solutions emanating from non-negative initial data remain non-negative as long as they exists. Models that do not guarantee the positivity of solutions are not valid or break down for small values of the solution. Motivated by the models describing the growth of bacterial biofilm communities we are particularly interested in systems of quasi-linear parabolic PDEs.

Another important qualitative aspect is the longtime behaviour of solutions. When we consider a system of competing species it is an interesting problem if and which species will persist, will become extinct or whether multiple species will coexist after transient states of the system have passed. Since we are dealing with models that are formulated as systems of PDEs the phase space of the generated dynamical system is infinite dimensional. However, the longtime dynamics can often be reduced to the dynamics on the global attractor. The global attractor is an invariant, compact subset of the initially infinite dimensional space which attracts all solutions. The theory of attractors is well established in the context of autonomous systems (see [5], [69] or [42]). However, time-dependent coefficients in the equations or random effects are significant in various cases. In biological applications the model parameters frequently depend on the life cycle of the involved species or daily or seasonal changes in its behaviour. In other cases random fluctuations of the environment should be taken into account. This leads to non-autonomous or random dynamical systems. The longtime behaviour and notion of attractors in the non-autonomous setting is far more complex, not yet very well understood and currently an active field of research.

Overview

The thesis consists of three major parts. Mathematical models for the growth of bacterial biofilms are addressed in the first chapter. The main result is the well-posedness of a mathematical model which describes a communication mechanism used by cells in growing biofilms to coordinate behaviour in groups. In Chapter 2 we formulate necessary and sufficient conditions for the positivity of solutions of systems of parabolic PDEs. Our results yield criteria for the positivity of solutions, which are easy to verify and allow to validate mathematical models. First, deterministic systems are considered and then stochastic perturbations of semi-linear parabolic systems. Chapter 3 is devoted to exponential attractors of infinite dimensional dynamical systems placing emphasis on non-autonomous problems. The central result is the construction of pullback exponential attractors for time continuous evolution processes in Banach spaces. Parts of the thesis are contained in the articles [10], [18], [31], [34] and [68].

Chapter 1

Mathematical Modelling of Biofilms

Biofilms are dense aggregations of microbial cells encased in a slimy extracellular matrix forming on biotic or abiotic surfaces in aqueous surroundings and play an important role in various fields. They are beneficially used in environmental engineering technologies, if they form on implants and natural surfaces in the human body they can provoke bacterial infections, and biofouling of industrial equipment can cause severe economic defects for the industry. Mathematical models of biofilms have been studied for several decades. They range from traditional one-dimensional models describing biofilms as homogeneous flat layers, to more recent two- and three-dimensional biofilm models that account for the spatial heterogeneity of biofilm communities. We study deterministic continuum models on the meso-scale ($50\mu m - 1mm$), the actual biofilm length scale. The biofilm as well as the aqueous surroundings are assumed to be continua.

The prototype of the models we address is a deterministic multidimensional biofilm growth model, which was first proposed in [24]. The model describes the growth of a bacterial biofilm community consisting of only one species, and is formulated as a highly non-linear reaction-diffusion system for the volume fraction occupied by biomass M and the concentration of the growth-controlling substrate S,

$$\begin{aligned} \partial_t S &= d_S \Delta S - k_1 \frac{SM}{k_2 + S}, \\ \partial_t M &= d \nabla \cdot (D_M(M) \nabla M) + k_3 \frac{MS}{k_2 + S} - k_4 M. \end{aligned} \tag{0.1}$$

The main difficulty is to model the spatial spreading mechanism of biomass: Expansion occurs locally only where and when the biomass density approaches values close to the maximal possible cell density, and biofilm and liquid surroundings are separated by a sharp interface. While the substrate concentration satisfies a standard semi-linear parabolic equation, the spatial spreading of biomass is described by the density-dependent diffusion operator

$$\nabla \cdot (D_M(M) \nabla M) = \nabla \cdot (\frac{M^a}{(1-M)^b} \nabla M),$$

where $a, b \geq 1$. The diffusion coefficient exhibits a polynomial degeneracy which is well-known from the porous medium equation and shows super diffusion. Both non-linear diffusion effects are necessary to reflect the experimentally observed characteristic growth behaviour of biofilms, and the highly irregular structure causes difficulties in the mathematical analysis. The single-species single-substrate model was mathematically analysed in [30], and the well-posedness of the model was established. Moreover, it was shown that the generated semigroup possesses a global attractor.

Various applications require to take further biofilm processes into account and to distinguish between multiple biomass components. The prototype biofilm model was therefore extended to reaction-diffusion systems involving several types of biomass and multiple dissolved substrates. The model introduced in [21] describes the diffusive resistance of biofilms against the penetration by antibiotics. In [45] an amensalistic biofilm control system was modelled, where a beneficial biofilm controls the growth of a pathogenic biofilm. The structure of the governing equations of the multi-species models differs essentially from the mono-species model, and the analytical results for the prototype model could not be carried over to the more involved multi-species case. In both articles, the model behaviour was studied numerically and the existence of solutions was established. The question of uniqueness of solutions, however, remained unanswered in both cases (see [21] and [45]).

Recently, another multi-component biofilm model was proposed, that describes quorum-sensing in growing biofilm communities (see [40]). Quorum-sensing is a cell-cell communication mechanism used by bacteria to coordinate behaviour in groups. The model was studied by numerical experiments, but analytical aspects were not addressed. It comprises a similar structure as the previous multi-component models [21] and [45], and is formulated as a non-linear reaction-diffusion system for the volume fractions occupied by up-regulated and down-regulated biomass, X and Y, the concentration of the growth limiting substrate S and the concentration of the signalling molecule A, which regulates the process of quorum-sensing,

$$\begin{aligned}
\partial_t S &= d_S \Delta S - k_1 \frac{SM}{k_2 + S}, \\
\partial_t A &= d_A \Delta A - \gamma A + \alpha X + (\alpha + \beta) Y, \\
\partial_t X &= d \nabla \cdot (D_M(M) \nabla X) + k_3 \frac{XS}{k_2 + S} - k_4 X - k_5 A^m X + k_5 Y, \\
\partial_t Y &= d \nabla \cdot (D_M(M) \nabla Y) + k_3 \frac{YS}{k_2 + S} - k_4 Y + k_5 A^m X - k_5 Y,
\end{aligned} \tag{0.2}$$

where $M = X + Y$ denotes the volume fraction of the total biomass. Compared to the previous multicomponent models the particularity of the quorum-sensing model is that adding the governing equations for the involved biomass components we recover exactly the mono-species biofilm model (0.1). Taking advantage of the known results for the prototype model we are able to prove the existence and uniqueness of solutions and its continuous dependence on initial data. The main result in Chapter 1 is the well-posedness of the quorum-sensing model and formulated in Theorem 1.4.

Theorem. *There exists a unique weak solution of the quorum-sensing model* (0.2)*, and the solution depends continuously on the initial data.*

In particular, it is the first time the uniqueness of solutions is established for multi-species reaction-diffusion models that extend the single-species biofilm model (0.1). Moreover, we improve previous regularity results for the solutions.

Chapter 2

Verifying Mathematical Models

The models for the growth of bacterial biofilm populations in Chapter 1 are formulated as systems of quasi-linear parabolic PDEs. The solutions describe the densities of biomass components and the concentrations of dissolved substrates and consequently, non-negative quantities. This is indeed the case in various applications modelled by convection-diffusion-reaction equations since the solutions of biological, physical or chemical models typically represent population densities, pressure, temperature or concentrations of nutrients and chemicals. Thus, it is an important property of the mathematical model that solutions emanating from non-negative initial data remain non-negative as long as they exist. Models that do not preserve the positivity of solutions are not valid. For scalar parabolic equations the non-negativity of solutions emanating from non-negative initial data is a direct consequence of the maximum principle. However, for systems of equations the maximum principle is not valid.

A general criterion for the positivity of solutions of semi-linear systems of reaction-diffusion-convection equations was formulated in [23]. Explicit necessary and sufficient conditions were obtained, that are easy to verify, and allow to validate mathematical models. Motivated by the class of PDEs arising in the modelling of biofilms we aim at generalizing the previous result for systems of quasi-linear reaction-diffusion-convection equations. For semi-linear systems, the diffusion and convection matrices are necessarily diagonal, while the quasi-linear case is essentially different. Here, cross-diffusion and cross-convection terms are allowed, however, the matrices are of a very particular form. For quasi-linear systems of the form

$$\begin{pmatrix}\partial_t u_1\\ \vdots\\ \partial_t u_k\end{pmatrix} = \begin{pmatrix}a_{11}(u) & \cdots & a_{1k}(u)\\ \vdots & & \vdots\\ a_{k1}(u) & \cdots & a_{kk}(u)\end{pmatrix}\begin{pmatrix}\Delta u_1\\ \vdots\\ \Delta u_k\end{pmatrix} + \sum_{l=1}^{n}\begin{pmatrix}\gamma_{11}^l(u) & \cdots & \gamma_{1k}^l(u)\\ \vdots & & \vdots\\ \gamma_{k1}^l(u) & \cdots & \gamma_{kk}^l(u)\end{pmatrix}\begin{pmatrix}\partial_{x_l} u_1\\ \vdots\\ \partial_{x_l} u_k\end{pmatrix} + \begin{pmatrix}f_1(u)\\ \vdots\\ f_k(u)\end{pmatrix}$$

we obtain the following positivity criterion (see Theorem 2.3 in Chapter 2).

Theorem. *The system of quasi-linear parabolic equations preserves positivity if and only if the interaction term f satisfies*

$$f_i(y_1,\dots,\underbrace{0}_{i},\dots,y_k) \geq 0 \qquad \text{for } y\in\mathbb{R}^k, y\geq 0, \tag{0.3}$$

and the diffusion and convection matrices fulfil

$$a_{ij}(y_1,\dots,\underbrace{0}_{i},\dots,y_k) = \gamma_{ij}^l(y_1,\dots,\underbrace{0}_{i},\dots,y_k) = 0 \qquad \text{for } y\in\mathbb{R}^k, y\geq 0$$

for all $i\neq j$, $1\leq i,j\leq k$ and $1\leq l\leq n$.

The theorem characterizes the class of quasi-linear parabolic systems that preserve the positivity of solutions and yields explicit necessary and sufficient conditions that are easy to verify in applications. In particular, the conditions on the matrices a and γ enforce a particular form of the matrices and we observe that if one component of the solution approaches zero, all cross-diffusion and cross-convection terms in the corresponding equation need to vanish. From the positivity criteria for semi-linear and quasi-linear parabolic systems we derive necessary and sufficient conditions for the validity of comparison theorems. For quasi-linear systems it is remarkable that the conditions for the validity of comparison principles are significantly stronger than the conditions for the positivity of solutions. In fact, all diffusion and convection matrices are necessarily diagonal and no cross-diffusion or cross-convection terms can appear.

The second part of Chapter 2 addresses stochastic perturbations of deterministic systems. Our aim is to characterize the class of stochastic perturbations that preserve the positivity property of deterministic systems of parabolic PDEs. Even for scalar ordinary differential equations (ODEs) it is well-known that additive noise destroys the positivity of solutions while perturbations by a linear multiplicative noise preserve the positivity property of the unperturbed deterministic problem.

We interpret the stochastic differential equations in the sense of Itô and consider stochastic perturbations of semi-linear parabolic systems of the form

$$\begin{pmatrix} du_1 \\ \vdots \\ du_k \end{pmatrix} = \left\{ \begin{pmatrix} -A^1 u_1 \\ \vdots \\ -A^k u_k \end{pmatrix} + \begin{pmatrix} f_1(u) \\ \vdots \\ f_k(u) \end{pmatrix} \right\} dt + \sum_{j=1}^{\infty} q_j \begin{pmatrix} g_j^1(u) \\ \vdots \\ g_j^k(u) \end{pmatrix} dW_t^i, \tag{0.4}$$

where A^i, $i = 1, \dots, k$, are linear elliptic differential operators of second order. We denote the system of stochastic PDEs by (f, g), and the corresponding unperturbed deterministic system by $(f, 0)$. By the deterministic positivity criterion for semi-linear systems we conclude that the unperturbed system $(f, 0)$ preserves the positivity of solutions if and only if the interaction function f satisfies Property (0.3).

To study the systems of stochastic partial differential equations (SPDEs) we consider smooth random approximations, since random equations can be interpreted pathwise and allow to apply deterministic methods. The approximations lead to a family of non-autonomous PDEs, and the solutions of the random approximations converge in expectation to the solution of a modified stochastic system. However, the original and the modified stochastic system are related through an explicit transformation. This allows to construct an auxiliary stochastic system (F, g) such that the solutions of the associated random approximations $(F_{\epsilon,\omega}, 0)$ converge to the solution of the original stochastic system (f, g). Using the deterministic result we derive necessary and sufficient conditions for the positivity of solutions of the random approximations. These conditions are explicit and preserved when passing to the limit. Moreover, they are invariant under the transformation relating the original and the modified stochastic system, which implies that the solutions of the stochastic system (f, g) preserve positivity. Finally, we observe that the solutions of the random approximations $(f_{\epsilon,\omega}, 0)$ associated to the stochastic system (f, g) converge to the solution of the original stochastic system if it is interpreted in the sense of Stratonovich. Our results are therefore valid independent of the choice of interpretation, and we obtain the following positivity criterion for stochastic systems (see Theorem 2.10 in Chapter 2).

Theorem. *Let (f, g) be a system of stochastic PDEs and $(F_{\epsilon,\omega}, 0)$ be the family of random approximations such that its solutions converge to the solution of the stochastic system (f, g). The solutions of the family of random approximations $(F_{\epsilon,\omega}, 0)$ preserve positivity if and only if the interaction function f satisfies Condition (0.3), and the stochastic perturbation fulfils*

$$g_j^i(y_1, \dots, \underbrace{0}_{i}, \dots, y_m) = 0 \qquad \text{for } y \in \mathbb{R}^k, y \geq 0,$$

for all $j \in \mathbb{N}$, $i = 1, \dots, k$.

These conditions imply that the stochastic system (f, g) preserves positivity for both Itô's and Stratonovich's interpretation.

If one component of the solution approaches zero, the stochastic perturbations in the corresponding equation need to vanish. Otherwise, the positivity of solutions cannot be guaranteed. In the particular case of scalar equations we recover the fact that positivity is preserved under perturbations by multiplicative noise while additive noise destroys the positivity of solutions.

Chapter 3

Exponential Attractors of Infinite Dimensional Dynamical Systems

Systems of parabolic PDEs generate infinite dimensional dynamical systems, and the time evolution of autonomous systems can be described in terms of semigroups. A semigroup in a metric space X is a family of continuous operators $S(t) : X \to X$, $t \geq 0$, that satisfies the properties

$$\begin{aligned} S(t)S(s) &= S(t+s) \qquad t, s \geq 0, \\ S(0) &= \mathrm{Id}. \end{aligned}$$

An important qualitative aspect is the behaviour of the system after transient states have passed. In many cases the longtime dynamics of semigroups is reduced to the dynamics on the global attractor. The global attractor is a compact, invariant subset of the phase space that attracts all solutions, and for large times the states of the system are well-approximated by the states within the global attractor. The global attractor is unique and the minimal closed subset that attracts all bounded sets of the phase space. Moreover, for various equations it was shown that the fractal dimension of the global attractor is finite (see [69] or [12]). When time tends to infinity the initially infinite dimensional dynamics is then in a certain sense reduced to finite dimensions.

The rate of convergence however can be arbitrarily slow, and the global attractor is generally not stable under perturbations. To overcome these drawbacks, the notion of an exponential attractor was introduced in [26] proposing to consider a larger set, which contains the global attractor, is still finite dimensional and attracts all bounded subsets at an exponential rate. Exponential attractors are only semi-invariant under the action of the semigroup and consequently, not unique. The construction of exponential attractors in [26] was developed for semigroups acting in Hilbert spaces. In [33] an alternative method and explicit algorithm for the construction of exponential attractors was proposed for discrete semigroups in Banach spaces. The construction essentially uses a smoothing or regularizing property of the semigroup and is the basis of our results. In the first part of Chapter 3 we recall the construction of exponential attractors for semigroups and generalize previous results.

While the theory of global and exponential attractors of autonomous dynamical systems is well-established, its counterpart in the non-autonomous setting is less developed and less understood. The solutions of non-autonomous problems do not only depend on the elapsed time, but also on the starting time. The rule of time evolution of non-autonomous systems is therefore described by a two-parameter family of operators. An evolution process is a family of continuous operators $U(t,s) : X \to X$, $t \geq s$, such that

$$
\begin{aligned}
U(t,s)U(s,r) &= U(t,r) && t \geq s \geq r,\\
U(t,t) &= \mathrm{Id} && t \in \mathbb{R}.
\end{aligned}
$$

Different concepts were proposed to extend the definition of global attractors of semigroups to the non-autonomous setting. We focus on the notion of pullback attractors which proved to be a useful tool to study the longtime dynamics of evolution processes.

Global non-autonomous attractors have the same drawbacks as global attractors of semigroups, which motivates to consider non-autonomous exponential attractors. The construction of autonomous exponential attractors was extended in [32] to discrete non-autonomous problems by using the concept of forwards attractors. Recently, the method was modified considering the pullback approach and the construction was generalized for time continuous evolution processes in [19] and [49]. The methods in [19] and [49] are similar, require strong regularity assumptions for the process and restrictive assumptions with respect to the pullback attraction. We modify the construction, generalize it for asymptotically compact processes and consider, instead of a fixed bounded pullback absorbing set, a family of time-dependent absorbing sets. This leads to exponential pullback attractors that are not necessarily uniformly bounded in the past, which is important when considering random attractors or unbounded non-autonomous terms in the equation. Theorem 3.10 contains the central result of Chapter 3.

Theorem. *Let $\{U(t,s) |\ t \geq s\}$ be a Lipschitz continuous evolution process in the Banach space V, and W be a normed space such that the embedding $V \hookrightarrow\hookrightarrow W$ is compact and dense. We assume $U = C + S$, where the family of operators C is a strict contraction in V, and S satisfies the smoothing property with respect to the spaces V and W. If there exists a semi-invariant family of bounded pullback absorbing sets for the evolution process U, the absorbing times are bounded in the past and the diameter of the absorbing sets grows at most sub-exponentially in the past, then there exists a pullback exponential attractor, and the fractal dimension of its sections is uniformly bounded.*

We also discuss the consequences of our construction when applied to autonomous evolution processes. For time continuous semigroups the method does not yield an exponential attractor in the strict sense but leads to a slightly weaker concept.

The existence of pullback exponential attractors implies the existence of the global pullback attractor and its finite dimensionality. We remark that the finite fractal dimension of pullback attractors that are unbounded in the past was an open problem (see [49] and [50]). In the final section of Chapter 3 we consider applications for our theoretical results and obtain an example for an unbounded pullback attractor of finite fractal dimension.

1. Mathematical Modelling of Biofilms

The dominant mode of microbial life in aquatic ecosystems are biofilm communities rather than planktonic cultures ([6]). Biofilms are dense aggregations of microbial cells encased in a slimy extracellular matrix forming on biotic or abiotic surfaces (called *substrata*) in aqueous surroundings. Such multicellular communities are a very successful life form and able to tolerate harmful environmental impacts that would eradicate free floating individual cells ([16], [57]). Whenever environmental conditions allow for bacterial growth, microbial cells can attach to a substratum and switch to a sessile life form. They start to grow and divide and produce a gel-like layer of *extracellular polymeric substances* (EPS) often forming complex spatial structures (see Figure 1.1). The self-produced EPS yields protection and allows survival in hostile environments. For example, the mechanisms of antibiotic resistance in biofilm cultures are essentially different from those of free swimming cells, which makes it difficult to eradicate bacterial biofilm infections. The EPS retards diffusion of antibiotics and the antibiotic agents fail to penetrate into the inner cores of the biofilm ([16], [57], [21]).

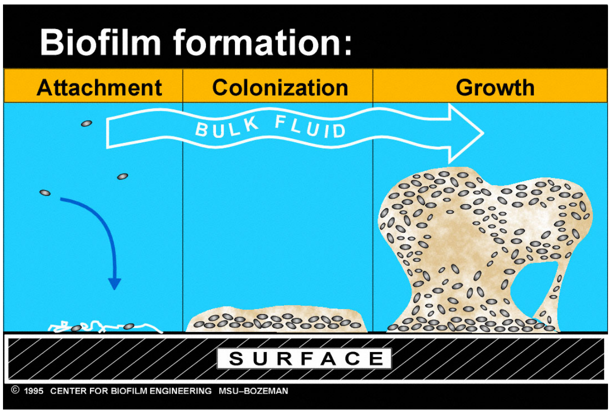


Figure 1.1.: The Formation of Biofilms (Montana State University, Center for Biofilm Engineering, 1995.)

Biofilms play an important role in various fields. They are beneficially used in environmental engineering technologies for groundwater protection and wastewater treatment. However, in most occurrences biofilm formations have negative effects. If they form on implants and natural surfaces in the human body they can provoke bacterial infections such as dental caries and otitis media ([16]). Biofilm contamination can lead to health risks in food processing environments, and biofouling of industrial equipment or ships can cause severe economic defects for the industry ([21], [68]).

Mathematical models of biofilms have been studied for several decades. They range from traditional one-dimensional models that describe biofilms as homogeneous flat layers, to more recent two- and three-dimensional biofilm models that account for the spatial heterogeneity of biofilm communities. A variety of mathematical modelling concepts has been suggested, including discrete stochastic particle based models and deterministic continuum models, that are based on the description of the mechanical properties of biofilms ([25], [68]). We are concerned with the latter, where biofilm and liquid surroundings are assumed to be continua, and its time evolution is governed by deterministic partial differential equations. The first continuum model [72] was a one-dimensional biofilm growth model and essentially based on the assumption of the biofilm as a homogeneous flat layer. Such models serve well for engineering applications on the macro-scale (larger than $1cm$) are however not capable to predict the often highly irregular spatial structure of microbial populations and the behaviour of biofilms on the meso-scale (between $50\mu m$ and $1mm$), the actual length scale of mature biofilms ([25]). Biofilms can form mushroom-like caps and contain clusters and channels, where substrates can circulate. Cells in different regions of the biofilm live in diverse micro-environments and exhibit differing behaviour ([16]).

To capture the spatial heterogeneity of biofilms a higher dimensional biofilm growth model was proposed in [24], which is based on the interpretation of a biofilm as a continuous, spatially structured microbial population. The essential difficulty is the modelling of the spatial spreading mechanism of biomass. The following characteristics of biofilms have been observed in experiments ([24]):

(i) The biomass density is bounded by a known maximum value.

(ii) Spatial spreading only takes place where the local biomass density approaches values close to its maximum possible value. In regions where the biomass density is low spatial spreading does not occur.

(iii) Biofilm and aqueous surroundings are separated by a sharp interface.

The mathematical model is formulated as a system of highly non-linear reaction-diffusion equations for the biomass density and concentration of a growth limiting nutrient and is the prototype of the biofilm models we discuss in this chapter. While the substrate concentration satisfies a standard semi-linear reaction-diffusion equation the governing equation for the biomass density exhibits two non-linear diffusion effects. The biomass diffusion coefficient degenerates like the porous medium equation and shows super diffusion, which causes difficulties in the mathematical analysis of the model. It was shown by numerical experiments that the model is capable to predict the heterogeneous spatial structure of biofilms and is in good agreement with experimental findings ([24]). In [30] and [28] the model was studied analytically. In particular, the existence and uniqueness of solutions could be established.

The prototype single-species single-substrate model was extended to model biofilms which consist of several types of biomass and account for multiple dissolved substrates. The model introduced in [21] describes the diffusive resistance of biofilms against the penetration by antibiotics. In [45] an amensalistic biofilm control system was modelled, where a beneficial biofilm controls the growth of a pathogenic biofilm. In both articles, existence proofs for the solutions were given, and numerical studies were presented. The structure of the governing equations of the multi-species models is similar, however, it differs essentially from the mono-species model. The analytical results for the prototype model could not all be carried over to the more involved multi-species case. For example, the question of uniqueness of solutions remained unanswered in [21] and [45]. Recently, in [40] another multi-component biofilm model was proposed, which combines the prototype model [24] with the mathematical model of quorum-sensing for suspended populations [55]. Quorum-sensing is a cell-cell communication mechanism used by bacteria to coordinate behaviour in groups. The model behaviour was studied by numerical experiments in [40], but analytical questions were not addressed. Compared to the previous multicomponent biofilm models, the particularity of the quorum-sensing model is, that adding the governing equations for the involved biomass components we recover exactly the mono-species biofilm model. Taking advantage of the known results for the prototype model we are able to prove the existence and uniqueness of solutions of the quorum-sensing model and the continuous dependence of solutions on initial data. It is the first time that a uniqueness result is obtained for multi-species reaction-diffusion models of biofilms that extend the single-species model [24].

In Section 1.1 we introduce the prototype biofilm growth model and summarize known analytical results. Multi-component biofilm models are addressed in Section 1.2. We first mention multi-species models that were studied analytically and recall previous existence results for the solutions. In Section 1.2.2 we present the quorum-sensing model, which is the central subject of this chapter. The main result is the proof of the well-posedness of the model, that we establish in Section 1.3. The existence proof is based on ideas and concepts that were applied for the models [21] and [45], but we obtain stronger regularity results for the solutions. The new approach allows us to show the uniqueness of solutions, which remained open for all previous multi-component models. In Section 1.3.4 we present numerical simulations to illustrate the model behaviour.

1.1. Prototype Biofilm Growth Model

1.1.1. Mathematical Model

The multi-dimensional biofilm growth model (see [24] and [30]) is formulated as a nonlinear reaction-diffusion system for the biomass density and the concentration of the growth controlling nutrient in a bounded domain $\Omega \subset \mathbb{R}^n$, where $n \in \{1, 2, 3\}$. The boundary of the domain $\partial\Omega$ is piecewise smooth. In dimensionless form the substrate concentration S is scaled with respect to the bulk concentration, and the biomass density is normalized with

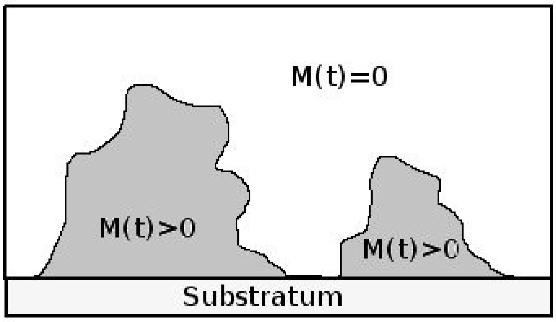


Figure 1.2.: Biofilm Domain

respect to the maximal bound for the cell density. Consequently, the dependent model variable M represents the volume fraction occupied by biomass. The EPS is implicitly taken into account, in the sense that the biomass volume fraction M describes the sum of biomass and EPS assuming that their volume ratio is constant. Both unknown functions depend on the spatial variable $x \in \Omega$ and time $t \geq 0$, and satisfy the parabolic system

$$\begin{aligned}
\partial_t S &= d_S \Delta S - k_1 \frac{SM}{k_2 + S} && \text{in } Q_T, \\
\partial_t M &= d\nabla \cdot (D_M(M)\nabla M) + k_3 \frac{SM}{k_2 + S} - k_4 M && \text{in } Q_T, \\
M|_{\partial\Omega} &= 0, \qquad S|_{\partial\Omega} = 1 && \text{on } \partial\Omega \times [0,T], \\
M|_{t=0} &= M_0, \qquad S|_{t=0} = S_0 && \text{in } \Omega \times \{0\},
\end{aligned} \tag{1.1}$$

where $T > 0$ and $Q_T := \Omega \times]0,T[$ is the parabolic cylinder. Furthermore, Δ denotes the Laplace operator and ∇ the gradient operator with respect to the spatial variable x. The constants d, d_S and k_2 are positive, and k_1, k_3 and k_4 are non-negative.

The solid region occupied by the biofilm as well as the liquid surroundings are assumed to be continua. The actual biofilm is described by the region $\Omega_2(t) := \{x \in \Omega \mid M(x,t) > 0\}$, and the liquid area by $\Omega_1(t) := \{x \in \Omega \mid M(x,t) = 0\}$. The substratum, on which the biofilm grows, is part of the boundary $\partial\Omega$ as illustrated in Figure 1.2.

Biomass is produced due to the consumption of nutrients. This process is described by the Monod interaction functions

$$k_3 \frac{SM}{k_2 + S} \qquad \text{and} \qquad -k_1 \frac{SM}{k_2 + S},$$

where k_3 denotes the maximum specific growth rate, and k_2 is the Monod half saturation constant. The constant k_1 is the maximum specific consumption rate. Natural cell death is also included in the model and described by the lysis rate k_4 in the equation for the biomass fraction.

While the nutrient is dissolved in the domain and the substrate concentration S satisfies a standard semi-linear reaction-diffusion equation, the spatial spreading of biomass is determined by the density-dependent diffusion coefficient

$$D_M(M) = \frac{M^a}{(1-M)^b} \qquad a, b \geq 1.$$

The biomass motility constant $d > 0$ is small compared to the diffusion coefficient d_S of the dissolved substrate, which reflects that the cells are to some extent immobilized in the EPS matrix. Accumulation of biomass leads to spatial expansion of the biofilm. We observe that the biomass diffusion coefficient vanishes when the total biomass approaches zero and blows up when the biomass density tends to its maximum value. The polynomial degeneracy M^a is well-known from the porous medium equation and guarantees that spatial spreading is negligible for low values of M. Moreover, it yields the separation of biofilm and liquid phase, that is, a finite speed of interface propagation. Spreading of biomass only takes place when and where the biomass fraction takes values close to its maximal possible value. For $M = 1$ instantaneous spreading occurs, which is known as the effect of super diffusion. The singularity at $M = 1$ ensures the maximal bound for the biomass density. Since biomass is produced as long as nutrients are available, this upper bound cannot be guaranteed by the growth terms alone. In fact, both non-linear diffusion effects are required to describe spatial expansion of biofilms. The degeneracy M^a alone does not yield the maximum bound for the cell density, while the singularity $(1 - M)^{-b}$ does not guarantee the separation of biofilm and liquid region by a sharp interface.

1.1.2. Analytical Results

A solution theory for System (1.1) was developed in [30]. Owing to the normalization we require that the initial data fulfil $S_0, M_0 \in L^\infty(\Omega)$ and

$$0 \leq S_0 \leq 1, \quad 0 \leq M_0 \leq 1 \qquad a.e. \text{ in } \Omega. \tag{1.2}$$

The corresponding solutions $S(t) := S(\cdot, t; S_0)$ and $M(t) := M(\cdot, t; M_0)$ should certainly satisfy the same bounds for $t > 0$. We summarize all relevant properties of the solutions of the mono-species model, which will later be needed to prove the well-posedness of the quorum-sensing model. The following theorem states the existence and regularity results for the solutions (see Theorem 3.1 in [30]).

Theorem 1.1. *We assume the initial data satisfies*

$$\begin{cases} S_0 \in L^\infty(\Omega) \cap H^1(\Omega), \quad S_0|_{\partial\Omega} = 1, \\ M_0 \in L^\infty(\Omega), \quad F(M_0) \in H_0^1(\Omega), \quad \|M_0\|_{L^\infty(\Omega)} < 1, \\ 0 \leq S_0 \leq 1, \quad 0 \leq M_0 \qquad a.e. \text{ in } \Omega, \end{cases} \tag{1.3}$$

where the function $F(v) := \int_0^v \frac{z^a}{(1-z)^b} dz$, *for* $0 \leq v < 1$. *Then, there exists a unique solution* (S, M) *satisfying System (1.1) in the sense of distributions, and the solution belongs to the* *class*

$$\begin{cases} M, S \in L^\infty(\Omega \times \mathbb{R}_+) \cap C(\mathbb{R}_+; L^2(\Omega)), \\ F(M), S \in L^\infty(\mathbb{R}_+; H^1(\Omega)) \cap C(\mathbb{R}_+; L^2(\Omega)), \\ \|M\|_{L^\infty(\Omega\times\mathbb{R}_+)} < 1, \\ 0 \le S, M \le 1 \quad a.e.\ in\ \Omega \times \mathbb{R}_+. \end{cases} \tag{1.4}$$

Furthermore, the following estimates hold

$$\begin{aligned} &\|S(t)\|^2_{H^1(\Omega)} + \|F(M(t))\|^2_{H^1(\Omega)} \ \le\ C(\|S_0\|^2_{H^1(\Omega)} + \|F(M_0)\|^2_{H^1(\Omega)} + 1), \\ &\|S(t)\|^2_{H^1(\Omega)} + \|\partial_t S(t)\|^2_{H^{-1}(\Omega)} + \|F(M(t))\|^2_{H^1(\Omega)} + \|M(t)\|^2_{H^s(\Omega)} + \|\partial_t M(t)\|^2_{H^{-1}(\Omega)} \\ &\le\ C(1 + \frac{1}{t^\kappa}), \end{aligned}$$

for $t > 0$ and some constants $C \ge 0$, $0 < s < \frac{1}{a+1}$ and $\kappa \ge 1$. The constants are independent of the initial data (S_0, M_0).

Moreover, it was shown that the solutions of System (1.1) are $L^1(\Omega)$-Lipschitz continuous with respect to initial data. The following result recalls Theorem 3.2 in [30].

Proposition 1.1. *Let (S, M) and $(\widetilde{S}, \widetilde{M})$ be two solutions of System (1.1) corresponding to initial data (S_0, M_0), $(\widetilde{S}_0, \widetilde{M}_0)$ respectively, and the initial data satisfy the assumptions of the previous theorem. Then, the following estimate holds*

$$\|S(t) - \widetilde{S}(t)\|_{L^1(\Omega)} + \|M(t) - \widetilde{M}(t)\|_{L^1(\Omega)} \le e^{(k_1+k_2+k_3)t}\big(\|S_0 - \widetilde{S}_0\|_{L^1(\Omega)} + \|M_0 - \widetilde{M}_0\|_{L^1(\Omega)}\big)$$

for $t \ge 0$. In particular, the solution is unique within the class (1.4).

The solution of the original system is obtained as the limit of solutions of regular approximations. For small $\epsilon > 0$ we define the non-degenerate auxiliary system for the single-species model (1.1) by

$$\begin{aligned} \partial_t S &= d_S \Delta S - k_1 \frac{SM}{k_2 + S} && \text{in } Q_T, \\ \partial_t M &= d\nabla \cdot (D_{\epsilon,M}(M)\nabla M) + k_3 \frac{SM}{k_2 + S} - k_4 M && \text{in } Q_T, \\ M|_{\partial\Omega} &= 0, \qquad S|_{\partial\Omega} = 1 && \text{on } \partial\Omega \times [0,T], \\ M|_{t=0} &= M_0, \qquad S|_{t=0} = S_0 && \text{in } \Omega \times \{0\}, \end{aligned} \tag{1.5}$$

where the regularized diffusion coefficient is defined as

$$D_{\epsilon,M}(z) := \begin{cases} \epsilon^a & z < 0 \\ \frac{(z+\epsilon)^a}{(1-z)^b} & z \le 1 - \epsilon \\ \frac{1}{\epsilon^b} & z \ge 1 - \epsilon. \end{cases}$$

For every (sufficiently small) $\epsilon > 0$ the auxiliary system (1.5) is regular parabolic and possesses a unique smooth solution (S_ϵ, M_ϵ). The solutions are uniformly bounded with respect to the regularization parameter $\epsilon > 0$, and if the initial data satisfies the assumptions of Theorem 1.1, the approximate solutions M_ϵ are separated from the singularity (see Proposition 1 and Proposition 6 in [30]). We summarize the auxiliary results in the following proposition.

Proposition 1.2. *If the initial data* (S_0, M_0) *satisfies the assumptions of Theorem 1.1 and*

$$\|M_0\|_{L^\infty(\Omega)} = 1 - \delta \qquad \text{for some } 0 < \delta < 1,$$

then, there exists $0 < \eta < 1$ *such that for all sufficiently small* $\epsilon > 0$ *the solutions* (S_ϵ, M_ϵ) *of the non-degenerate approximations (1.5) satisfy*

$$\|M_\epsilon(t)\|_{L^\infty(\Omega)} \leq 1 - \eta \qquad \text{for } t \geq 0,$$

where the constant η *depends on* δ *and* Ω *only and is independent of* $\epsilon > 0$. *Furthermore, the substrate concentrations are uniformly bounded,*

$$0 \leq S_\epsilon \leq 1 \qquad \text{in } \Omega \times \mathbb{R}_+.$$

Proposition 1.2 remains valid for the solution (S, M) of the original system (1.1), which is the limit of the solutions of the regular approximations in $C_{loc}(\mathbb{R}_+; L^2(\Omega))$ when ϵ tends to zero,

$$S_\epsilon \to S, \quad M_\epsilon \to M \qquad \text{strongly in } C_{loc}(\mathbb{R}_+; L^2(\Omega)).$$

Consequently, the biomass density does not attain the singularity as long as the initial concentration does not take this value. For further details and all proofs we refer to [30].

1.2. Multicomponent Biofilm Models

1.2.1. Antibiotic Disinfection of Biofilms

The prototype biofilm growth model presented in the previous section was extended to incorporate further biofilm processes. This requires to distinguish different types of biomass and to include governing equations for multiple biomass fractions and several dissolved substrates in the model. We discuss in this section multi-species models, that were studied analytically.

The first multi-species multi-substrate generalization of the prototype model (1.1) was suggested in [23]. In [21] existence results for the solutions were established and numerical simulations were presented. The model describes a growing biofilm community and its disinfection by antimicrobial agents. The dependent model variables are the volume fraction occupied by active biomass X, the volume fraction occupied by inert biomass Y, the concentration of the dissolved oxygen S, which controls the growth of the biomass, and the concentration of the antimicrobial agent B, which regulates the disinfection process.

As previously, the EPS is implicitly taken into account and we denote the total biomass fraction by $M := X + Y$. In dimensionless form the model is represented by the parabolic system

$$
\begin{aligned}
\partial_t S &= d_S \Delta S - k_1 \frac{SX}{k_2 + S} && \text{in } Q_T, \\
\partial_t B &= d_B \Delta B - \zeta_1 BX && \text{in } Q_T, \\
\partial_t X &= d\nabla \cdot (D_M(M)\nabla X) + k_3 \frac{SX}{k_2 + S} - k_4 X - \zeta_2 BX && \text{in } Q_T, \\
\partial_t Y &= d\nabla \cdot (D_M(M)\nabla Y) + \zeta_2 BX && \text{in } Q_T,
\end{aligned}
\tag{1.6}
$$

where we use the same notations as in Section 1.1.1. The additional constants ζ_1 and ζ_2 are positive, and $d_B > 0$ denotes the diffusion coefficient of the antimicrobial agent. Apart from the diffusion of the dissolved substrates and the growth and spatial spreading of biomass the mechanism of disinfection is included in the model. During this process antibiotic agents are consumed and active biomass is directly converted into inert biomass, which is determined by the disinfection parameters ζ_1 and ζ_2. In the absence of antimicrobial agents and inert biomass, the model reduces to the single species biofilm growth model (1.1).

In the article [21] the following boundary and initial values were assumed for the dependent model variables

$$
\begin{aligned}
&X|_{\partial\Omega} = 0, && Y|_{\partial\Omega} = 0, && S|_{\partial\Omega} = S_r, && B|_{\partial\Omega} = B_r && \text{on } \partial\Omega \times [0, T], \\
&X|_{t=0} = X_0, && Y|_{t=0} = Y_0, && S|_{t=0} = S_0, && B|_{t=0} = B_0 && \text{in } \Omega \times \{0\}.
\end{aligned}
$$

The non-negative functions B_r and S_r belong to the class $L^\infty(\partial\Omega)$, and the initial data satisfy $X_0, Y_0, S_0, B_0 \in L^\infty(\Omega)$,

$$
\begin{aligned}
&0 \le X_0, \quad 0 \le Y_0, \quad 0 \le B_0, \quad 0 \le S_0 \le 1 \qquad a.e. \text{ in } \Omega, \\
&\|X_0 + Y_0\|_{L^\infty(\Omega)} < 1.
\end{aligned}
$$

Definition 1.1. *We call the vector of functions (S, B, X, Y) a solution of System (1.6), if*

$$
S(\cdot, t), B(\cdot, t), X(\cdot, t), Y(\cdot, t) \in L^\infty(\Omega) \qquad t \ge 0,
$$

and it satisfies System (1.6) in distributional sense.

The following theorem yields the existence of solutions (see Theorem 2.3 in [21]).

Theorem 1.2. *If the initial data satisfies the stated assumptions, System (1.6) possesses a global solution in the sense of Definition 1.1, and the solution belongs to the space*

$$
S, B, X, Y \in L^\infty(\Omega \times \mathbb{R}_+).
$$

The solution is obtained as the limit of solutions of non-degenerate approximations for System (1.6). The regular parabolic auxiliary systems are the systems, where the diffusion coefficient D_M in the equations for the biomass fractions is replaced by the regularized diffusion coefficient $D_{\epsilon,M}$, which was defined in Section 1.1.2.

The model of an amensalistic biofilm control system [45] possesses a very similar structure as the model of antibiotic disinfection. The existence of solutions in the sense of Definition 1.1 was established by similar methods (see Theorem 3.3 in [45]). Since the pattern of the multi-component biofilm models is essentially different from the prototype model, and the equations are strongly coupled through the diffusion operators, the known results for the single-species model could not be carried over. The behaviour of the solutions was studied by numerical simulations in [21] and [45], but further analytical results were not obtained. In particular, the question of uniqueness of solutions remained unanswered in both cases.

1.2.2. Quorum-Sensing in Patchy Biofilm Communities

In this section we introduce a multicomponent biofilm model, which takes the process of *quorum-sensing* into account. The mechanism and benefit of quorum-sensing is not yet very well-understood, and there exist different biological theories and interpretations (see [43], [59]). It is currently an active field of research in experimental microbiology as well as in mathematical and theoretical biology, primarily for planktonic bacterial populations but also in the context of biofilms. Quorum-sensing is a cell-cell communication mechanism used by bacteria to coordinate gene expression and behaviour in groups. Bacteria constantly produce low amounts of signalling molecules that are released into the environment. Accumulation of autoinducers triggers a response by the cells and since the producing cells respond to their own signals the molecules are also called *autoinducers* ([55], [43]). When the concentration of autoinducers locally passes a certain threshold, the cells are rapidly induced, and switch from a so-called *down-regulated* to an *up-regulated state*. In an up-regulated state they typically produce the signalling molecule at a highly increased rate ([40]).

The quorum-sensing model was originally proposed in [40], where in numerical simulations the contribution of environmental hydrodynamics to the transport of signalling molecules and its effect on inter-colony communication and up-regulation was studied. Analytical aspects of the model were not addressed. It extends the prototype biofilm growth model and combines it with a model for quorum-sensing in planktonic cultures, which was suggested in [55].

A mathematical description of quorum-sensing in biofilms requires to distinguish two types of bacteria, the up-regulated and the down-regulated cells, and to include a mechanism provoking cells to switch between these two states. The dependent model variable X denotes the volume fraction occupied by down-regulated biomass and Y the volume fraction occupied by up-regulated biomass, where the EPS is implicitly taken into account. The dependent variable A reflects the concentration of the signalling molecule, and S the concentration of the growth controlling substrate. In dimensionless form the model is represented by the parabolic system

$$
\begin{aligned}
\partial_t S &= d_S \Delta S - k_1 \frac{SM}{k_2 + S} && \text{in } Q_T, \\
\partial_t A &= d_A \Delta A - \gamma A + \alpha X + (\alpha + \beta) Y && \text{in } Q_T, \\
\partial_t X &= d \nabla \cdot (D_M(M) \nabla X) + k_3 \frac{XS}{k_2 + S} - k_4 X - k_5 A^m X + k_5 Y && \text{in } Q_T, \\
\partial_t Y &= d \nabla \cdot (D_M(M) \nabla Y) + k_3 \frac{YS}{k_2 + S} - k_4 Y + k_5 A^m X - k_5 Y && \text{in } Q_T,
\end{aligned}
\tag{1.7}
$$

where we use the same notations as in Section 1.1.1 (see [40] and [68]). The constants d_A and γ are positive, $m \geq 1$ and α, β and k_5 are non-negative. The total biomass fraction $M = X + Y$ denotes the volume fraction occupied by up-regulated or down-regulated cells.

Since the biomass components are normalized with respect to the physically maximal possible cell density, the total biomass fraction should satisfy $M = X + Y \leq 1$ in Q_T. The actual biofilm is described by the region $\Omega_2(t) := \{x \in \Omega | \ M(x,t) > 0\}$, and the liquid surroundings by the region $\Omega_1(t) := \{x \in \Omega | \ M(x,t) = 0\}$. The autoinducer concentration A is normalized with respect to the threshold concentration for induction, and consequently, induction occurs locally in the biofilm if A reaches approximately 1 from below. If the concentration A locally decreases from a value larger than 1 to a value below 1, down-regulation at constant rate k_5 will dominate. Finally, the substrate concentration S is normalized with respect to a characteristic value for the system, such as the nutrient concentration at the boundary of the domain.

Under the hypothesis that induction switches the cells between down- and up-regulated states without changing their growth behaviour we can assume that the spatial spreading of both biomass fractions is described by the same diffusion operator. The biomass motility constant $d > 0$ is small compared to the diffusion coefficients $d_S > 0$ and $d_A > 0$ of the dissolved substrates. Apart from the spatial spreading of biomass and the diffusive transport of signalling molecules and nutrients the following processes are included in the model:

- Up-regulated and down-regulated biomass is produced due to the consumption of nutrients. This mechanism is described by Monod reaction terms, where the constant k_3 reflects the maximum specific growth rate, and k_2 the Monod half saturation constant. The constant k_1 is the maximum specific consumption rate.

- Natural cell death is included in the model and described by the lysis rate k_4. This effect can be dominant compared to cell growth, if the substrate concentration becomes sufficiently low.

- The signalling molecules decay abiotically at rate γ.

- Due to an increase of the autoinducer concentration A down-regulated cells are converted into up-regulated cells at rate $k_5 A^m$. In applications for the degree of polymerization m we typically take values $2 < m < 3$ (see [40] and [68]). Up-regulated cells are converted back into down-regulated cells at constant rate k_5. If the molecule concentration $A < 1$ the latter effect dominates, if $A > 1$ up-regulation is super-linear.

- Finally, down-regulated cells produce the signalling molecule at rate α, while up-regulated cells produce it at the increased rate $\alpha + \beta$, where β is one order of magnitude larger than α. For technical reasons, we require in the analysis $\alpha + \beta > \gamma$; that is, the signalling molecule production rate of the up-regulated cells is higher than the abiotic decay rate. This is not a severe model restriction; if the opposite was true no noteworthy accumulation of signalling molecules could take place.

In the following section we specify initial and boundary values for the biomass fractions and substrate concentrations to complete the model (1.8) and prove the well-posedness of the mathematical model.

1.3. Well-Posedness of the Quorum-Sensing Model

Compared to previous multicomponent biofilm models, the particularity of the quorum sensing model is, that adding the governing equations for the biomass fractions of up- and down-regulated cells we recover exactly the mono-species biofilm model. Taking advantage of the results for the single-species model we are able to prove the existence and uniqueness of solutions of the quorum-sensing model and the continuous dependence of solutions on initial data. It is the first time that a uniqueness result is obtained for multi-species diffusion-reaction models of biofilms that extend the prototype model [24]. The proof of the existence of solutions is based on the non-degenerate approximations developed in [30] and the methods applied in [21] and [45]. However, the approach we present in the following is different and leads to a uniqueness result for the solutions.

1.3.1. Preliminaries

For technical reasons we study the model in the auxiliary form

$$
\begin{aligned}
\partial_t S &= d_S \Delta S - k_1 \frac{SM}{k_2 + S} && \text{in } Q_T,\\
\partial_t A &= d_A \Delta A - \gamma A + \alpha X + (\alpha + \beta) Y && \text{in } Q_T,\\
\partial_t X &= d \nabla \cdot (D_M(M) \nabla X) + k_3 \frac{XS}{k_2 + S} - k_4 X - k_5 |A|^m X + k_5 |Y| && \text{in } Q_T,\\
\partial_t Y &= d \nabla \cdot (D_M(M) \nabla Y) + k_3 \frac{YS}{k_2 + S} - k_4 Y + k_5 |A|^m X - k_5 |Y| && \text{in } Q_T.
\end{aligned}
\tag{1.8}
$$

If the solutions of System (1.8) are non-negative, they are also solutions of System (1.7). On the other hand, non-negative solutions of System (1.7) solve System (1.8). After non-negativity of the solutions of System (1.8) is shown we can therefore remove the absolute value $|\cdot|$ from the first and second equation of System (1.8) and obtain the original model (1.7).

For the biomass components X and Y and the concentration of the signalling molecule A we assume homogeneous Dirichlet boundary conditions, and constant Dirichlet conditions for the nutrient concentration S,

$$\begin{aligned} &X|_{\partial\Omega} = Y|_{\partial\Omega} = A|_{\partial\Omega} = 0 && \text{on } \partial\Omega \times [0,T], \\ &S|_{\partial\Omega} = 1 && \text{on } \partial\Omega \times [0,T]. \end{aligned} \tag{1.9}$$

If the biofilm is contained in the inner region of the domain, away from the boundary $\partial\Omega$, this situation describes a growing biofilm in the absence of a substratum. Such biofilms are often called *microbial flocs*. The boundary conditions imposed on the concentration of nutrients reflect a constant unlimited nutrient supply at the boundary of the considered domain. Similarly, keeping A equal to zero at the boundary enforces a removal of autoinducers from the domain. These are specific boundary conditions, primarily chosen for convenience. The solution theory we develop in the following sections carries over to more general boundary values, which are relevant and often more appropriate for applications.

The initial data for the model variables are given by

$$X|_{t=0} = X_0, \quad Y|_{t=0} = Y_0, \quad S|_{t=0} = S_0, \quad A|_{t=0} = A_0 \qquad \text{in } \Omega, \tag{1.10}$$

where $S_0, X_0, Y_0, A_0 \in L^\infty(\Omega)$ satisfy the compatibility conditions and

$$\begin{aligned} &\|X_0 + Y_0\|_{L^\infty(\Omega)} < 1, \\ &0 \le S_0 \le 1, \qquad 0 \le A_0 \le 1, \qquad 0 \le X_0, \qquad 0 \le Y_0 \qquad \textit{a.e.} \text{ in } \Omega. \end{aligned} \tag{1.11}$$

In fact, in most relevant applications the initial autoinducer concentration A_0 is identically zero.

Definition 1.2. *We call the vector-valued function (S, A, X, Y) a* ***solution of System*** *(1.8) corresponding to the boundary and initial data (1.9) and (1.10), if its components belong to the class*

$$\begin{aligned} &X, Y, A, S \in C([0,T]; L^2(\Omega)) \cap L^\infty(Q_T), \\ &A, S \in L^2((0,T); H^1(\Omega)), \\ &D_M(M)\nabla X,\ D_M(M)\nabla Y \in L^2((0,T); L^2(\Omega; \mathbb{R}^n)) \end{aligned}$$

for any $T > 0$, and satisfy System (1.8) in distributional sense.

To be more precise, if (S, A, X, Y) is a solution according to Definition 1.2, then the equality

$$\begin{aligned} \int_\Omega X(x,T)\varphi(x)dx - \int_\Omega X_0(x)\varphi(x)dx &= -d\int_{Q_T} D_M(M(x,t))\nabla X(x,t) \cdot \nabla\varphi(x)dtdx \\ + \int_{Q_T} \left(k_3 \frac{X(x,t)S(x,t)}{k_2 + S(x,t)} - k_4 X(x,t) - k_5 |A(x,t)|^m X(x,t) + k_5 |Y(x,t)| \right) \varphi(x)dtdx \end{aligned}$$

holds for all test-functions $\varphi \in C_0^\infty(\Omega)$ and almost every $T > 0$. The determining equations for the other components of the solution are defined analogously.

Compared to other multi-component biofilm models such as [21] and [45], the particularity of the quorum-sensing model (1.8) is, that we recover the single-species biofilm growth model for the total biomass fraction M and the nutrient concentration S. Indeed, adding the equations for the biomass fractions X and Y in System (1.8) leads to

$$\begin{aligned}
\partial_t S &= d_S \Delta S - k_1 \frac{SM}{k_2 + S} && \text{in } Q_T, \\
\partial_t M &= d\nabla \cdot (D_M(M)\nabla M) + k_3 \frac{SM}{k_2 + S} - k_4 M && \text{in } Q_T,
\end{aligned} \tag{1.12}$$

with initial and boundary values

$$\begin{aligned}
M|_{\partial\Omega} &= 0, & S|_{\partial\Omega} &= 1 && \text{on } \partial\Omega \times [0, T], \\
M|_{t=0} &= M_0 = X_0 + Y_0, & S|_{t=0} &= S_0 && \text{in } \Omega \times \{0\},
\end{aligned}$$

which is exactly the prototype biofilm growth model discussed in Section 1.1. Consequently, the substrate concentration S and the total biomass density M can be regarded as known functions, and the original system (1.8) reduces to a system of equations for the biomass fraction X and the concentration of the quorum-sensing signaling molecule A,

$$\begin{aligned}
\partial_t X &= d\nabla \cdot (\mathcal{D}\nabla X) + k_3 \frac{XS}{k_2 + S} - k_4 X - k_5 |A|^m X + k_5 (M - X) && \text{in } Q_T, \\
\partial_t A &= d_A \Delta A - \gamma A + \alpha X + (\alpha + \beta)(M - X) && \text{in } Q_T,
\end{aligned}$$

where the diffusion coefficient of the biomass fraction is defined by

$$\mathcal{D}(x,t) := \frac{(M(x,t))^a}{(1 - M(x,t))^b} \qquad (x,t) \in Q_T.$$

In the reduction we used the positivity of the biomass component Y, which will be proved in Section 1.3.3. We rewrite this non-autonomous semi-linear system with bounded coefficients as

$$\begin{aligned}
\partial_t X &= d\nabla \cdot (\mathcal{D}\nabla X) + gX - k_5 |A|^m X + h && \text{in } Q_T, \\
\partial_t A &= d_A \Delta A - \gamma A - \beta X + l && \text{in } Q_T,
\end{aligned} \tag{1.13}$$

where the interaction terms are given by the known functions

$$\begin{aligned}
g(x,t) &:= k_3 \frac{S(x,t)}{k_2 + S(x,t)} - k_4 - k_5, \\
h(x,t) &:= k_5 M(x,t) \quad \geq 0, \\
l(x,t) &:= (\alpha + \beta) M(x,t) \quad \geq 0.
\end{aligned}$$

All coefficient functions are bounded, $g, h, l \in L^\infty(\Omega \times \mathbb{R}_+)$, and the diffusion coefficient $\mathcal{D}$ is non-negative and bounded by Theorem 1.1. Indeed, if the initial density of the total biomass satisfies $\|M_0\|_{L^\infty(\Omega)} < 1 - \delta$ for some $0 < \delta < 1$, then there exists a constant $0 < \eta < 1$ such that

$$0 \leq M(x,t) \leq 1 - \eta \qquad \text{for almost every } (x,t) \in \Omega \times \mathbb{R}_+.$$

Consequently, it follows the estimate

$$0 \leq \mathcal{D}(x,t) = \frac{(M(x,t))^a}{(1 - M(x,t))^b} \leq \frac{1}{(1 - M(x,t))^b} \leq \frac{1}{\eta^b},$$

which shows that the diffusion coefficient $\mathcal{D}$ is non-negative and satisfies $\mathcal{D} \in L^\infty(\Omega \times \mathbb{R}_+)$.

1.3.2. Uniqueness

In this paragraph we prove the uniqueness and $L^2(\Omega)$-Lipschitz-continuity of solutions with respect to initial data of the semi-linear parabolic system (1.13), which degenerates when the total biomass density M approaches zero.

Theorem 1.3. *Let the initial data (S_0, A_0, X_0, Y_0) satisfy $X_0, Y_0, A_0 \in H_0^1(\Omega)$, $S_0 \in H^1(\Omega)$ such that $S_0|_{\partial\Omega} = 1$, and*

$$\begin{aligned} &0 \leq S_0, X_0, Y_0, A_0 \leq 1 \qquad \textit{a.e. in } \Omega, \\ &\|X_0 + Y_0\|_{L^\infty(\Omega)} < 1. \end{aligned}$$

Then, there exists at most one non-negative solution (X, A) of the reduced System (1.13) within the class of solutions considered in Definition 1.2.

Proof. We assume that (X, A) and $(\widetilde{X}, \widetilde{A})$ are two such solutions corresponding to initial data (X_0, A_0), and define the differences $u := X - \widetilde{X}$ and $v := A - \widetilde{A}$. Then, v belongs to the space $L^2((0,T); H_0^1(\Omega))$, u satisfies $D_M(M(\cdot,t))\nabla u(\cdot,t) \in L^2(\Omega;\mathbb{R}^n)$ for almost every $t \in (0,T]$ and $\partial_t u, \partial_t v \in L^2((0,T); H^{-1}(\Omega))$ for every $T > 0$. Moreover, the functions u and v satisfy the system

$$\begin{aligned} \partial_t u &= d\nabla \cdot (\mathcal{D}\nabla u) + gu - k_5(A^m X - \widetilde{A}^m \widetilde{X}) && \text{in } Q_T, \\ \partial_t v &= d_A \Delta v - \gamma v - \beta u && \text{in } Q_T, \end{aligned}$$

with zero initial and boundary conditions

$$\begin{aligned} v|_{t=0} = u|_{t=0} &= 0 && \text{in } \Omega \times \{0\}, \\ u|_{\partial\Omega} = v|_{\partial\Omega} &= 0 && \text{on } \partial\Omega \times [0,T]. \end{aligned}$$

If we formally multiply the second equation by v and integrate over Ω, we obtain the estimate

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|v(\cdot,t)\|^2_{L^2(\Omega)} &= -d_A\|\nabla v(\cdot,t)\|^2_{L^2(\Omega;\mathbb{R}^n)} - \gamma\|v(\cdot,t)\|^2_{L^2(\Omega)} - \beta\langle u(\cdot,t), v(\cdot,t)\rangle_{L^2(\Omega)} \\ &\leq -\gamma\|v(\cdot,t)\|^2_{L^2(\Omega)} - \beta\langle u(\cdot,t), v(\cdot,t)\rangle_{L^2(\Omega)}, \end{aligned}$$

where we used the positivity of d_A. Moreover, multiplying the first equation by u and integrating over Ω yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(\cdot,t)\|^2_{L^2(\Omega)} &= -d\big\langle \mathcal{D}(x,t)\nabla u(\cdot,t), \nabla u(\cdot,t)\big\rangle_{L^2(\Omega;\mathbb{R}^n)} + \int_\Omega g(x,t)|u(x,t)|^2 dx \\ &\quad -k_5\int_\Omega [A^m(x,t)X(x,t) - \widetilde{A}^m(x,t)\widetilde{X}(x,t)]u(x,t)dx.\end{aligned}$$

In order to estimate the last integral we observe

$$A^m X - \widetilde{A}^m\widetilde{X} = A^m u + \widetilde{X}(A^m - \widetilde{A}^m) = A^m u + v\widetilde{X}m\int_0^1 (sA + (1-s)\widetilde{A})^{m-1}ds.$$

Since $\mathcal{D}$, A and $\widetilde{X}$ are non-negative we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(\cdot,t)\|^2_{L^2(\Omega)} &\le \int_\Omega g(x,t)|u(t,x)|^2 dx + k_5\int_\Omega A^m(x,t)u^2(x,t)dx \\ &+ k_5\int_\Omega \widetilde{X}(x,t)v(x,t)u(x,t)m\int_0^1 (sA(x,t)+(1-s)\widetilde{A}(x,t))^{m-1}dsdx \\ &\le C_1\|u(\cdot,t)\|^2_{L^2(\Omega)} + C_2\big\langle u(\cdot,t), v(\cdot,t)\big\rangle_{L^2(\Omega)},\end{aligned}$$

for some constants $C_1, C_2 \ge 0$. Here, we used that the functions $A, \widetilde{A}, \widetilde{X}$ and g belong to the class $L^\infty(Q_T)$. Adding both inequalities and using the Cauchy-Schwarz inequality yields

$$\frac{d}{dt}\big(\|u(.,t)\|^2_{L^2(\Omega)} + \|v(.,t)\|^2_{L^2(\Omega)}\big) \le C_3\big(\|u(.,t)\|^2_{L^2(\Omega)} + \|v(.,t)\|^2_{L^2(\Omega)}\big), \tag{1.14}$$

for some constant $C_3 \ge 0$. Invoking Gronwall's Lemma and using the initial conditions $u|_{t=0} = v|_{t=0} = 0$, we conclude $\|u(\cdot,t)\|_{L^2(\Omega)} = \|v(\cdot,t)\|_{L^2(\Omega)} = 0$ for all $t \in [0,T]$. □

We remark that the proof of Theorem 1.3 implies the Lipschitz-continuity of the solutions of System (1.13) with respect to initial data in the norm of $L^2(\Omega)\times L^2(\Omega)$.

Corollary 1.1. *Let (X, A) and $(\widetilde{X}, \widetilde{A})$ be two solutions of System (1.13) within the class of the previous theorem that correspond to initial data (X_0, A_0) and $(\widetilde{X}_0, \widetilde{A}_0)$ respectively. Then, the following estimate holds*

$$\|X(\cdot,t) - \widetilde{X}(\cdot,t)\|^2_{L^2(\Omega)} + \|A(\cdot,t) - \widetilde{A}(\cdot,t)\|^2_{L^2(\Omega)} \le e^{Ct}\big(\|X_0 - \widetilde{X}_0\|^2_{L^2(\Omega)} + \|A_0 - \widetilde{A}_0\|^2_{L^2(\Omega)}\big),$$

for some constant $C \ge 0$.

Proof. The estimate follows immediately from Inequality (1.14) in the proof of Theorem 1.3 and Gronwall's Lemma. □

The proof of the well-posedness of the original system (1.8) reduces to show the well-posedness of the semi-linear system (1.13). We formally obtained the uniqueness of solutions of the quorum-sensing model, the existence of solutions within the class of Definition 1.2 will be addressed in the following paragraph.

1.3.3. Existence

To prove the existence of solutions of the original system we consider non-degenerate auxiliary systems, and show that the solutions of the auxiliary systems converge to the solution of the degenerate problem when the regularization parameter tends to zero. The ideas are based on the method developed in [30] for the mono-species model and the strategy applied in [21] and [45] to prove the existence of solutions of multi-species biofilm models. For small $\epsilon > 0$ we define the non-degenerate approximation of System (1.8) by

$$
\begin{aligned}
\partial_t S &= d_S \Delta S - k_1 \frac{SM}{k_2 + S} && \text{in } Q_T,\\
\partial_t A &= d_A \Delta A - \gamma A + \alpha X + (\alpha + \beta) Y && \text{in } Q_T,\\
\partial_t X &= d \nabla \cdot (D_{\epsilon,M}(M) \nabla X) + k_3 \frac{XS}{k_2 + S} - k_4 X - k_5 |A|^m X + k_5 |Y| && \text{in } Q_T,\\
\partial_t Y &= d \nabla \cdot (D_{\epsilon,M}(M) \nabla Y) + k_3 \frac{YS}{k_2 + S} - k_4 Y + k_5 |A|^m X - k_5 |Y| && \text{in } Q_T,
\end{aligned}
\tag{1.15}
$$

where the regularized diffusion coefficient is defined by

$$
D_{\epsilon,M}(z) := \begin{cases} \epsilon^a & z < 0 \\ \frac{(z+\epsilon)^a}{(1-z)^b} & z \le 1 - \epsilon \\ \frac{1}{\epsilon^b} & z \ge 1 - \epsilon \end{cases}
$$

(see Section 1.1.2). Furthermore, we assume the initial data is regular and smooth; namely, that it belongs to the class

$$
\begin{aligned}
&S_0 \in L^\infty(\Omega) \cap H^1(\Omega), && S_0|_{\partial\Omega} = 1, && A_0 \in L^\infty(\Omega) \cap H_0^1(\Omega),\\
&M_0 = X_0 + Y_0 \in L^\infty(\Omega), && X_0, Y_0, F(M_0) \in H_0^1(\Omega), && \|M_0\|_{L^\infty(\Omega)} < 1,\\
&0 \le X_0, \quad 0 \le Y_0, && 0 \le S_0 \le 1, && 0 \le A_0 \le 1 \qquad \textit{a.e. } \text{in } \Omega,
\end{aligned}
\tag{1.16}
$$

where the function

$$
F(z) := \int_0^z \frac{z^a}{(1-z)^b} dz \qquad \text{for } 0 \le z < 1.
$$

Adding the equations for the biomass components X and Y of System (1.15) we recover the non-degenerate auxiliary system for the single-species model

$$
\begin{aligned}
\partial_t S &= d_S \Delta S - k_1 \frac{SM}{k_2 + S} && \text{in } Q_T,\\
\partial_t M &= d \nabla \cdot (D_{\epsilon,M}(M) \nabla M) + k_3 \frac{MS}{k_2 + S} - k_4 M && \text{in } Q_T.
\end{aligned}
\tag{1.17}
$$

We recall that for every (sufficiently small) $\epsilon > 0$ there exists a unique solution (S_ϵ, M_ϵ) of System (1.17), and the solutions are uniformly bounded with respect to the regularization parameter $\epsilon > 0$. Moreover, if the initial data belong to the class (1.16), the solution M_ϵ

is separated from the singularity. To be more precise, there exists a constant $\eta \in (0,1)$, which is independent of $\epsilon > 0$, such that $M_\epsilon < 1-\eta$ holds in Q_T (see Proposition 1.2 in Section 1.1.2). Hence, we may regard $M_\epsilon = X_\epsilon + Y_\epsilon$ and S_ϵ as known functions, and it suffices to prove the existence of solutions of the semi-linear parabolic problem

$$\begin{aligned} \partial_t X &= d\nabla \cdot (\mathcal{D}_\epsilon \nabla X) + g_\epsilon X - k_5|A|^m X + h_\epsilon && \text{in } Q_T, \\ \partial_t A &= d_A \Delta A - \gamma A - \beta X + l_\epsilon && \text{in } Q_T, \\ X|_{\partial\Omega} &= 0, \qquad A|_{\partial\Omega} = 0 && \text{on } \partial\Omega \times [0,T], \\ X|_{t=0} &= X_0, \quad\ A|_{t=0} = A_0 && \text{in } \Omega \times \{0\}. \end{aligned} \tag{1.18}$$

The diffusion coefficient for the biomass fraction is defined as $\mathcal{D}_\epsilon(x,t) := D_{\epsilon,M}(M_\epsilon(x,t))$, and the interaction functions are given by

$$\begin{aligned} g_\epsilon(x,t) &:= k_3 \frac{S_\epsilon(x,t)}{k_2 + S_\epsilon(x,t)} - k_4 - k_5, \\ h_\epsilon(x,t) &:= k_5 M_\epsilon(x,t) \ \geq 0, \\ l_\epsilon(x,t) &:= (\alpha+\beta) M_\epsilon(x,t) \ \geq 0, \end{aligned}$$

for $(x,t) \in Q_T$, where (M_ϵ, S_ϵ) denotes the solution of the non-degenerate approximation (1.17). In the reduction to the semi-linear system (1.18) we have already used the positivity of the biomass component Y_ϵ, which will be proved in the following lemma. To abbreviate notations we introduce the reaction terms f_1^ϵ and f_2^ϵ,

$$\begin{aligned} f_1^\epsilon(x,t,X(x,t),A(x,t)) &:= g_\epsilon(x,t)X(x,t) - k_5|A(x,t)|^m X(x,t) + h_\epsilon(x,t), \\ f_2^\epsilon(x,t,X(x,t),A(x,t)) &:= -\gamma A(x,t) - \beta X(x,t) + l_\epsilon(x,t). \end{aligned}$$

First, we show that all components of the solutions of the non-degenerate approximations are non-negative and bounded.

Lemma 1.1. *The components of the solution $(S_\epsilon, A_\epsilon, X_\epsilon, Y_\epsilon)$ of the auxiliary system (1.15) are non-negative and belong to the class $L^\infty(Q_T)$.*

Proof. The substrate concentration S_ϵ and the total biomass density $M_\epsilon = X_\epsilon + Y_\epsilon$ are non-negative and bounded by 1 according to Proposition 1.2. We will show that the components X_ϵ, Y_ϵ and A_ϵ are non-negative. Since $X_\epsilon + Y_\epsilon = M_\epsilon \leq 1$ in Q_T this immediately implies the boundedness of the biomass fractions X_ϵ and Y_ϵ. The boundedness of the molecule concentration A_ϵ then follows by a comparison theorem for scalar parabolic equations (see Theorem 10.1 in [67]). Indeed, by the hypothesis on the constants α, β and γ the constant $A_{max} := \frac{\alpha+\beta}{\gamma} > 1$ is a supersolution for A_ϵ. It satisfies $A_{max}|_{\partial\Omega} \geq 0 = A_\epsilon|_{\partial\Omega}$, $A_{max}|_{t=0} \geq A_0 = A_\epsilon|_{t=0}$ and

$$\begin{aligned} \partial_t A_{max} - d_A \Delta A_{max} + \gamma A_{max} - \alpha X_\epsilon - (\alpha+\beta) Y_\epsilon &= \gamma A_{max} - \alpha X_\epsilon - (\alpha+\beta) Y_\epsilon \\ &\geq \gamma A_{max} - \alpha - \beta = 0, \end{aligned}$$

where we used the assumption $\alpha + \beta > \gamma$ in Section 1.2.2.

Consequently, it remains to prove that the biomass fractions X_ϵ, Y_ϵ and the autoinducer concentration A_ϵ are non-negative. To show the non-negativity of the biomass fraction of down-regulated cells we again apply a comparison theorem for parabolic equations. The constant $\widetilde{X} = 0$ is a subsolution for the component X_ϵ. Indeed, it satisfies $X_\epsilon|_{\partial\Omega} \geq 0 = \widetilde{X}|_{\partial\Omega}$, $X_0 = X_\epsilon|_{t=0} \geq 0 = \widetilde{X}|_{t=0}$ and

$$\partial_t \widetilde{X} - d\nabla \cdot (D_{\epsilon,M}(M_\epsilon)\nabla \widetilde{X}) - k_3 \frac{\widetilde{X} S_\epsilon}{k_2 + S_\epsilon} + k_4 \widetilde{X} + k_5 |A_\epsilon|^m \widetilde{X} - k_5 |Y_\epsilon| = -k_5 |Y_\epsilon| \leq 0.$$

By the same arguments and owing to the positivity of X_ϵ, the constant solution $\tilde{Y} = 0$ is a subsolution for Y_ϵ, so we conclude $Y_\epsilon \geq 0$. Finally follows $A_\epsilon \geq 0$, by comparing with the subsolution $\tilde{A} = 0$ for the molecule concentration A_ϵ, and using the fact that the components X_ϵ and Y_ϵ are non-negative. □

Having established the positivity and uniform boundedness of the solutions we are in a position to prove the existence of solutions of the reduced system (1.18). To this end we treat the region, where the total biomass density becomes small, and its complement in Q_T separately. The solution (S, M) of the single-species model is obtained as the limit of the solutions (S_ϵ, M_ϵ) of the non-degenerate approximations

$$S = \lim_{\epsilon \to 0} S_\epsilon, \quad M = \lim_{\epsilon \to 0} M_\epsilon \qquad \text{in } C([0,T]; L^2(\Omega)),$$

where $T > 0$ is arbitrary (see Section 1.1.2). For some $\delta \in (0,1)$ we define the domains

$$Q_{\delta,T} := \{(x,t) \in Q_T \mid M(t,x) < \delta\}$$

and $Q^c_{\delta,T} := Q_T \setminus \overline{Q_{\delta,T}}$. We note that both sets are open due to the Hölder-continuity of the solution M (see [21]).

Lemma 1.2. *We assume the initial data belongs to the class (1.16). Then, for all sufficiently small $\epsilon > 0$ there exists a unique solution (A_ϵ, X_ϵ) of the auxiliary system (1.18) satisfying*

$$X_\epsilon, A_\epsilon \in L^2((0,T); H^1_0(\Omega)) \cap C([0,T]; L^2(\Omega)) \cap L^\infty(Q_T),$$
$$\partial_t X_\epsilon, \partial_t A_\epsilon \in L^2((0,T); H^{-1}(\Omega)).$$

Moreover, the solutions are uniformly bounded with respect to the regularization parameter $\epsilon > 0$, and satisfy the estimates

$$\max_{t\in[0,T]} \|X_\epsilon(\cdot,t)\|_{L^2(\Omega)} + \|X_\epsilon\|_{L^2((0,T);H^1_0(\Omega))} + \|\partial_t X_\epsilon\|_{L^2((0,T);H^{-1}(\Omega))} \leq C_{4,\epsilon}\left(1 + \|X_0\|_{L^2(\Omega)}\right),$$
$$\max_{t\in[0,T]} \|A_\epsilon(\cdot,t)\|_{L^2(\Omega)} + \|A_\epsilon\|_{L^2((0,T);H^1_0(\Omega))} + \|\partial_t A_\epsilon\|_{L^2((0,T);H^{-1}(\Omega))} \leq C_5(1 + \|A_0\|_{L^2(\Omega)}),$$

for $t > 0$ and some constants $C_{4,\epsilon}, C_5 \geq 0$, where the constant C_5 is independent of $\epsilon > 0$. The solutions are Hölder-continuous

$$X_\epsilon \in C^{\alpha_\epsilon, \frac{\alpha_\epsilon}{2}}(Q_T), \;\; A_\epsilon \in C^{\alpha, \frac{\alpha}{2}}(Q_T),$$

where constants α_ϵ and α are positive. The Hölder exponent α_ϵ depends on the parameter ϵ, the data and uniform bound of the approximate solutions only, the constant α is independent of $\epsilon > 0$.

Finally, restricted to the domain $Q^c_{\delta,T}$ the solutions X_ϵ satisfy all estimates uniformly. To be more precise, the constant $C_{4,\epsilon}$ in the inequality above and the Hölder exponent α_ϵ are independent of $\epsilon > 0$ for the family of approximate solutions $\{\widetilde{X}_\epsilon\}$, where $\widetilde{X}_\epsilon := X_\epsilon|_{Q^c_{\delta,T}}$.

Proof. If the initial data M_0 and S_0 belong to the class (1.16) the total biomass density M_ϵ satisfies $M_\epsilon < 1 - \eta$ in Q_T for some $\eta \in (0, 1)$, and the constant η is independent of $\epsilon > 0$. This implies that the diffusion coefficient $\mathcal{D}_\epsilon$ is positive and uniformly bounded from above by a constant independent of ϵ. Indeed, for all $\epsilon < \eta$ we obtain

$$\epsilon^a \leq \mathcal{D}_\epsilon(M_\epsilon(x,t)) = \frac{(M_\epsilon(x,t) + \epsilon)^a}{(1 - M_\epsilon(x,t))^b} \leq \frac{(1 - \eta + \epsilon)^a}{(1 - (1 - \eta))^b} \leq \frac{1}{\eta^b} \qquad \text{in } Q_T,$$

which shows that $\mathcal{D}_\epsilon \in L^\infty(Q_T)$ and $\mathcal{D}_\epsilon$ is strictly positive. Hence, for all sufficiently small $\epsilon > 0$ the semi-linear auxiliary system (1.18) is regular and uniformly parabolic. The functions $g_\epsilon, h_\epsilon, l_\epsilon, A_\epsilon$ and X_ϵ are uniformly bounded with respect to the regularization parameter $\epsilon > 0$ by Lemma 1.1, which implies that the interaction functions f_1^ϵ and f_2^ϵ are uniformly bounded in Q_T. By standard arguments (Galerkin approximations) follows the existence and uniqueness of the approximate solutions (X_ϵ, A_ϵ), the solutions belong to the class stated in the lemma and satisfy the given estimates (see Section 11.1 in [63]). Moreover, the Hölder-continuity of solutions follows from Theorem 10.1, Chapter III in [48].

Due to the uniform boundedness of the approximate solutions the component A_ϵ satisfies the parabolic equation

$$\partial_t A_\epsilon - d_A \Delta A_\epsilon = -\gamma A_\epsilon + H_\epsilon,$$

where the function H_ϵ is uniformly bounded, $\|H_\epsilon\|_{L^\infty(Q_T)} \leq c$ for some constant $c \geq 0$ which is independent of $\epsilon > 0$. Hence, the constants in the estimates for the component A_ϵ can be chosen independently of the regularization parameter $\epsilon > 0$.

Finally, if $\epsilon > 0$ is sufficiently small, then $M_\epsilon \geq \frac{\delta}{2}$ holds in the region $Q^c_{\delta,T}$. Consequently, the diffusion coefficient restricted to the domain $Q^c_{\delta,T}$ is uniformly bounded from above and below by a positive constant which is independent of $\epsilon > 0$,

$$\left(\frac{\delta}{2}\right)^a \leq (\frac{\delta}{2} + \epsilon)^a \leq \mathcal{D}_\epsilon(x,t) = \frac{(M_\epsilon(x,t) + \epsilon)^a}{(1 - M_\epsilon(x,t))^b} \leq \frac{1}{\eta^b} \qquad \text{in } Q^c_{\delta,T}.$$

Solutions of non-degenerate parabolic equations of second order with coefficients in $L^\infty(\Omega)$ satisfy the estimates stated in the lemma, and the bounds are determined in terms of the coefficients of the equation (see [48], Chapter V). Consequently, the estimates in the region $Q^c_{\delta,T}$ are uniform and do not dependent on $\epsilon > 0$. □

We will use this lemma to pass to the limit in the region $Q^c_{\delta,T}$. To pass to the limit in the region $Q_{\delta,T}$ requires further uniform estimates for the family of approximate solutions.

Lemma 1.3. *If $\epsilon > 0$ is sufficiently small, the product $\sqrt{\mathcal{D}_\epsilon}\nabla X_\epsilon$ is uniformly bounded in $L^2(Q_T;\mathbb{R}^n)$, and the approximate solutions satisfy $X_\epsilon(\cdot,t) \in H^s(\Omega)$ for some $s > 0$ and almost every $t \in [0,T]$. Moreover, there exists $\epsilon_0 > 0$ such that*

$$\|X_\epsilon\|_{L^2((0,T);H^s(\Omega))} \leq C \qquad \text{for all } 0 < \epsilon < \epsilon_0,$$

where the constant $C \geq 0$ is independent of the regularization parameter $\epsilon > 0$.

Proof. Multiplying the first equation of System (1.18) by X_ϵ and integrating over Ω we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|X_\epsilon(\cdot,t)\|^2_{L^2(\Omega)} + d\big\langle \mathcal{D}_\epsilon(\cdot,t)\nabla X_\epsilon(\cdot,t), \nabla X_\epsilon(\cdot,t)\big\rangle_{L^2(\Omega;\mathbb{R}^n)}\\ &= \int_\Omega X_\epsilon(x,t) f_1^\epsilon(x,t,X_\epsilon(x,t),A_\epsilon(x,t))\,dx \;\leq\; C_6,\end{aligned}$$

for some constant $C_6 \geq 0$. Due to Lemma 1.1 the constant C_6 is independent of $\epsilon > 0$. If we integrate this inequality from 0 to $T > 0$ it follows the first statement of the lemma.

Furthermore, for sufficiently small $\epsilon > 0$ we observe $X_\epsilon \leq M_\epsilon \leq 1 - \eta$ in Q_T and consequently,

$$X^a_\epsilon(x,t) \leq D_{\epsilon,M}(X_\epsilon(x,t)) = \frac{(X_\epsilon(x,t)+\epsilon)^a}{(1-X_\epsilon(x,t))^b} \leq \frac{(M_\epsilon(x,t)+\epsilon)^a}{(1-(M_\epsilon(x,t)))^b} = D_{\epsilon,M}(M_\epsilon(x,t)) \quad \text{in } Q_T.$$

This implies the estimate

$$\int_\Omega X^a_\epsilon(x,t)\|\nabla X_\epsilon(x,t)\|^2 dx \leq \int_\Omega D_{\epsilon,M}(M_\epsilon(x,t))\|\nabla X_\epsilon(x,t)\|^2 dx \leq C_7,$$

for some constant $C_7 \geq 0$, which is independent of the regularization parameter $\epsilon > 0$. This shows that $X_\epsilon^{\frac{a}{2}}(\cdot,t)\nabla X_\epsilon(\cdot,t) \in L^2(\Omega;\mathbb{R}^n)$ or equivalently, $X_\epsilon^{\frac{a}{2}+1}(t) \in H^1(\Omega)$ for almost every $t \in]0,T]$. Finally, if a function satisfies $\varphi^\beta \in H^1(\Omega)$ for some $\beta > 1$, then $\varphi \in W^{s,2\beta}(\Omega)$ holds for all $s \leq \frac{1}{\beta}$ (see Appendix B). This implies that $X_\epsilon(\cdot,t) \in W^{s,2(\frac{a}{2}+1)}(\Omega)$ for $s \leq \frac{1}{\frac{a}{2}+1}$. Since the domain Ω is bounded and $a \geq 1$ the embedding $W^{s,2+a}(\Omega) \hookrightarrow H^s(\Omega)$ is continuous and we obtain $X_\epsilon(\cdot,t) \in H^s(\Omega)$ for some positive $s > 0$. In particular, the family of approximate solutions $\{X_\epsilon\}_{\epsilon>0}$ is uniformly bounded in the Hilbert space $L^2((0,T);H^s(\Omega))$. □

Lemma 1.4. *There exist functions*

$$\begin{aligned}X^* &\in L^\infty(Q_T) \cap L^2((0,T);H^s(\Omega))\\ A^* &\in L^\infty(Q_T) \cap L^2((0,T);H^1_0(\Omega))\end{aligned}$$

and a sequence $(\epsilon_k)_{k\in\mathbb{N}}$ tending to zero for $k \to \infty$, such that the solutions of the auxiliary systems (1.18) converge weakly

$$X_{\epsilon_k} \rightharpoonup X^*, \ A_{\epsilon_k} \rightharpoonup A^*$$

in $L^2((0,T);H^s(\Omega))$, and $L^2((0,T);H^1_0(\Omega))$ respectively, and strongly

$$X_{\epsilon_k} \to X^*, \ A_{\epsilon_k} \to A^*$$

in $C([0,T];L^2(\Omega))$ when k tends to infinity.

Proof. We prove the convergence and existence of the limit for the biomass fraction X^*, the arguments are similar for the molecule concentration A^*. By Lemma 1.3 and for sufficiently small $\epsilon > 0$ the family of approximate solutions $\{X_\epsilon\}_{\epsilon>0}$ is uniformly bounded in the Hilbert space $L^2((0,T);H^s(\Omega))$ for some $s > 0$. Consequently, there exists an element $X^* \in L^2((0,T);H^s(\Omega))$ and a sequence $(\epsilon_k)_{k\in\mathbb{N}}$ tending to zero for $k \to \infty$ such that the sequence $(X_{\epsilon_k})_{k\in\mathbb{N}}$ converges weakly to X^* in $L^2((0,T);H^s(\Omega))$.

Furthermore, Lemma 1.3 implies that the product $\sqrt{\mathcal{D}_\epsilon}\nabla X_\epsilon$ is uniformly bounded in $L^2(Q_T;\mathbb{R}^n)$, and the diffusion coefficient satisfies $\mathcal{D}_\epsilon \in L^\infty(Q_T)$. Consequently, we obtain

$$\|\mathcal{D}_\epsilon \nabla X_\epsilon\|^2_{L^2(Q_T;\mathbb{R}^n)} \leq \|\mathcal{D}_\epsilon\|_{L^\infty(Q_T)} \|\sqrt{\mathcal{D}_\epsilon}\nabla X_\epsilon\|^2_{L^2(Q_T;\mathbb{R}^n)} \leq c,$$

for some constant $c \geq 0$ which is independent of $\epsilon > 0$. This proves the uniform boundedness of the derivative $\partial_t X_\epsilon$ in $L^2((0,T);H^{-1}(\Omega))$.

By Theorem 1.5, Chapter II in [12] now follows the strong convergence of the sequence of approximate solutions in the space $C([0,T];L^2(\Omega))$. □

It remains to show that the limits of the approximate solutions yield the solution of the degenerate problem.

Theorem 1.4. *The limits X^* and A^* of the solutions of the non-degenerate approximations in Lemma 1.4 are the unique weak solutions of the reduced system (1.13). In particular, there exists a unique solution of the quorum-sensing model (1.8) in the sense of Definition 1.2.*

Proof. We show that we can pass to the limit $\epsilon \to 0$ in the distributional formulation of the non-degenerate auxiliary system (1.18). We only prove the convergence for the biomass fraction X^* since the arguments are the same or simplify for the molecule concentration A^*. The functions X_ϵ are weak solutions of the auxiliary systems (1.18). Consequently, the equality

$$\begin{aligned} &\int_\Omega X_\epsilon(x,T)\varphi(x)dx - \int_\Omega X_0(x)\varphi(x)dx \\ = \quad & -d\int_{Q_T} \mathcal{D}_\epsilon(x,t)\nabla X_\epsilon(x,t)\cdot\nabla\varphi(x)dtdx + \int_{Q_T} f_1^\epsilon(x,t,A_\epsilon(x,t),X_\epsilon(x,t))\varphi(x)dtdx \end{aligned}$$

is satisfied for all test-functions $\varphi \in C_0^\infty(\Omega)$ and almost every $T > 0$. By Lemma 1.1 the family of approximate solutions is uniformly bounded in $L^\infty(Q_T)$ and we can immediately pass to the limit in all integrals, except for the diffusion term. Hence, it remains to show the convergence of the term

$$\begin{aligned} \int_{Q_T} \mathcal{D}_\epsilon(x,t)\nabla X_\epsilon(x,t)\cdot\nabla\varphi(x)dtdx &= \int_{Q_T} D_{\epsilon,M}(M_\epsilon(x,t))\nabla X_\epsilon(x,t)\cdot\nabla\varphi(x)dtdx \\ &\to \int_{Q_T} D_M(M(x,t))\nabla X(x,t)\cdot\nabla\varphi(x)dtdx, \end{aligned}$$

when the regularization parameter ϵ tends to zero. Note that the integrals are well-defined by Lemma 1.3. We split the difference and treat the domains $Q_{\delta,T}$ and $Q^c_{\delta,T}$ separately. To this end we define

$$\begin{aligned} R_\epsilon := I_\epsilon + J_\epsilon &:= \int_{Q_{\delta,T}} \big(\mathcal{D}_\epsilon(x,t)\nabla X_\epsilon(x,t) - D_M(M(x,t))\nabla X(x,t)\big)\cdot\nabla\varphi(x)dtdx \\ &+ \int_{Q^c_{\delta,T}} \big(\mathcal{D}_\epsilon(x,t)\nabla X_\epsilon(x,t) - D_M(M(x,t))\nabla X(x,t)\big)\cdot\nabla\varphi(x)dtdx, \end{aligned}$$

which does not depend on the parameter $\delta > 0$, and show that the term R_ϵ vanishes when ϵ tends to zero. To estimate the integral J_ϵ we express the difference in the following way

$$\mathcal{D}_\epsilon\nabla X_\epsilon - D_M(M)\nabla X = \left(D_{\epsilon,M}(M_\epsilon) - D_M(M)\right)\nabla X_\epsilon + D_M(M)\left(\nabla X_\epsilon - \nabla X\right).$$

For the first term in the integral we obtain

$$\begin{aligned} &\big|\big\langle (D_{\epsilon,M}(M_\epsilon) - D_M(M))\nabla X_\epsilon, \nabla\varphi\big\rangle_{L^2(Q^c_{\delta,T};\mathbb{R}^n)}\big| \\ \le\ & \|D_{\epsilon,M}(M_\epsilon) - D_M(M)\|_{L^\infty(Q^c_{\delta,T})}\big|\langle\nabla X_\epsilon,\nabla\varphi\rangle_{L^2(Q^c_{\delta,T};\mathbb{R}^n)}\big| \\ \le\ & \|D_{\epsilon,M}(M_\epsilon) - D_M(M)\|_{L^\infty(Q^c_{\delta,T})}\|\nabla\varphi\|_{L^2(Q^c_{\delta,T};\mathbb{R}^n)}\|\nabla X_\epsilon\|_{L^2(Q^c_{\delta,T};\mathbb{R}^n)} \\ \le\ & C_8\|D_{\epsilon,M}(M_\epsilon) - D_M(M)\|_{L^\infty(Q^c_{\delta,T})} \end{aligned}$$

for some constant $C_8 \ge 0$. Here, we used the Cauchy-Schwarz inequality and the uniform boundedness of the family of approximate solutions $\{X_\epsilon\}_{\epsilon>0}$, when restricted to the domain $Q^c_{\delta,T}$ in the norm induced by $L^2((0,T);H^1_0(\Omega))$ (see Lemma 1.2). The family of solutions M_ϵ of the non-degenerate approximations of the single-species model is uniformly bounded in the Hölder space $C^{\tilde\alpha,\frac{\tilde\alpha}{2}}(Q_T)$ for some $\tilde\alpha > 0$ (see [21]), which implies the strong convergence in the space $C(Q_T)$. Furthermore, the solutions of the auxiliary systems satisfy the uniform estimate $M_\epsilon \le 1-\eta$ in Q_T, and we conclude that $M \le 1-\eta$ in Q_T. On the interval $[0,1-\eta]$ the truncated function $D_{\epsilon,M} : [0,1-\eta] \to \mathbb{R}$ converges uniformly to the function D_M when ϵ tends to zero. Therefore, splitting the remaining term

$$\begin{aligned} &\|D_{\epsilon,M}(M_\epsilon) - D_M(M)\|_{L^\infty(Q^c_{\delta,T})} \\ \le&\|D_{\epsilon,M}(M_\epsilon) - D_{\epsilon,M}(M)\|_{L^\infty(Q^c_{\delta,T})} + \|D_{\epsilon,M}(M) - D_M(M)\|_{L^\infty(Q^c_{\delta,T})} \end{aligned}$$

we see that it vanishes when ϵ tends to zero.

Finally, the convergence of the second integral in J_ϵ

$$\left\langle D_M(M)\nabla\varphi, \nabla X_\epsilon - \nabla X^*\right\rangle_{L^2(Q^c_{\delta,T};\mathbb{R}^n)}$$

is an immediate consequence of Lemma 1.2. Indeed, restricted to the domain $Q^c_{\delta,T}$ the family of approximate solutions is uniformly bounded in the norm induced by $L^2((0,T);H^1_0(\Omega))$, which implies weak convergence in this space. Since the diffusion coefficient $D_M(M)$ belongs to $L^\infty(Q_T)$ by Proposition 1.5, the product $D_M(M)\nabla\varphi$ defines an element in the dual space and implies the convergence of the integral. Summarizing the above estimates we conclude that for every $\mu > 0$ there exists an $\epsilon_0 > 0$, which is independent of δ, such that the term $|J_\epsilon| < \mu$ for all $\epsilon < \epsilon_0$.

It remains to estimate the integral I_ϵ. We recall that the domain $Q_{\delta,T}$ was defined as the subset of Q_T where the total biomass density $M < \delta$. As M_ϵ converges strongly to M in $C(Q_T)$ there exists $\epsilon_1 > 0$ such that the approximate solutions $M_\epsilon < 2\delta$ in $Q_{\delta,T}$ for all $\epsilon < \epsilon_1$. For sufficiently small $\epsilon > 0$ we conclude

$$\mathcal{D}_\epsilon(x,t) = D_{\epsilon,M}(M_\epsilon(x,t)) = \frac{(M_\epsilon(x,t)+\epsilon)^a}{(1-M_\epsilon(x,t))^b} \le \frac{(3\delta)^a}{(1-2\delta)^b}$$

for all $(x,t) \in Q_{\delta,T}$.

Furthermore, the product $\sqrt{\mathcal{D}_\epsilon}\nabla X_\epsilon$ is uniformly bounded in $L^2(Q_T;\mathbb{R}^n)$ by Lemma 1.3, which allows us to use Hölder's inequality to estimate the integral

$$\begin{aligned}\Big|\int_{Q_{\delta,T}} \mathcal{D}_\epsilon(x,t)\nabla X_\epsilon(x,t)\cdot\nabla\varphi(x)dtdx\Big| &\le \big\|\sqrt{\mathcal{D}_\epsilon}\nabla X_\epsilon\big\|_{L^2(Q_T;\mathbb{R}^n)}\big\|\sqrt{\mathcal{D}_\epsilon}\nabla\varphi\big\|_{L^2(Q_{\delta,T};\mathbb{R}^n)} \\ \le\ C_9\left(\int_{Q_{\delta,T}} \mathcal{D}_\epsilon(x,t)\|\nabla\varphi(x)\|^2 dtdx\right)^{\frac{1}{2}} &\le C_9\frac{(3\delta)^{\frac{a}{2}}}{(1-2\delta)^{\frac{b}{2}}}\|\varphi\|^2_{L^2((0,T);H^1(\Omega))},\end{aligned}$$

where the constant $C_9 \ge 0$. Estimating the second integral of I_ϵ in the same way we obtain

$$\begin{aligned}|I_\epsilon| &\le \int_{Q_{\delta,T}} \big|\mathcal{D}_\epsilon(x,t)\nabla X_\epsilon(x,t)\cdot\nabla\varphi(x)\big|dtdx + \int_{Q_{\delta,T}} \big|D_M(M(x,t))\nabla X(x,t)\cdot\nabla\varphi(x)\big|dtdx \\ &\le C_{10}\frac{(3\delta)^{\frac{a}{2}}}{(1-2\delta)^{\frac{b}{2}}},\end{aligned}$$

for some constant $C_{10} \ge 0$.

To conclude the proof of the theorem let $\mu > 0$ be arbitrary. We first choose $\delta > 0$ and a corresponding $\epsilon_1 > 0$ such that

$$|I_\epsilon| < \frac{\mu}{2}$$

for all $\epsilon < \epsilon_1$. According to the first part of the proof there exists $\epsilon_0 > 0$, which does not dependent on $\delta > 0$, such that

$$|J_\epsilon| < \frac{\mu}{2}$$

for all $\epsilon < \epsilon_0$. Consequently, we obtain

$$|R_\epsilon| \leq |I_\epsilon| + |J_\epsilon| < \mu$$

for all $\epsilon < \min\{\epsilon_0, \epsilon_1\}$. This proves that the limit (X, A) is a solution of the reduced system (1.13), and the uniqueness of the solution follows by Theorem 1.3. The existence and uniqueness of solutions of the original system (1.8) now follows from the existence and uniqueness of the solution (S, M) of the single species model. □

Similar as in [30], the proof of the well-posedness of the quorum-sensing model can be extended to less regular initial data and other boundary conditions for the solutions. The boundary conditions for the dissolved substrates S and A, which describe mechanisms of substrate replenishment and autoinducer removal are thereby rather uncritical. For the biomass volume fractions X and Y the results carry over as long as the values remain below the threshold singularity. This is the case if $X + Y < 1$ is specified on some part of the boundary (see [30]).

1.3.4. Numerical Simulations

In this section we present numerical simulations by H. Eberl to illustrate the model behaviour. The model parameters correspond to a biofilm colony of *Pseudomonas putida*, the formation of the biofilm is controlled by carbon as the growth limiting substrate and the signalling molecules are *Acyl Homoserine Lactones* (AHL). For a detailed description of the data and the numerical experiments we refer to [68].

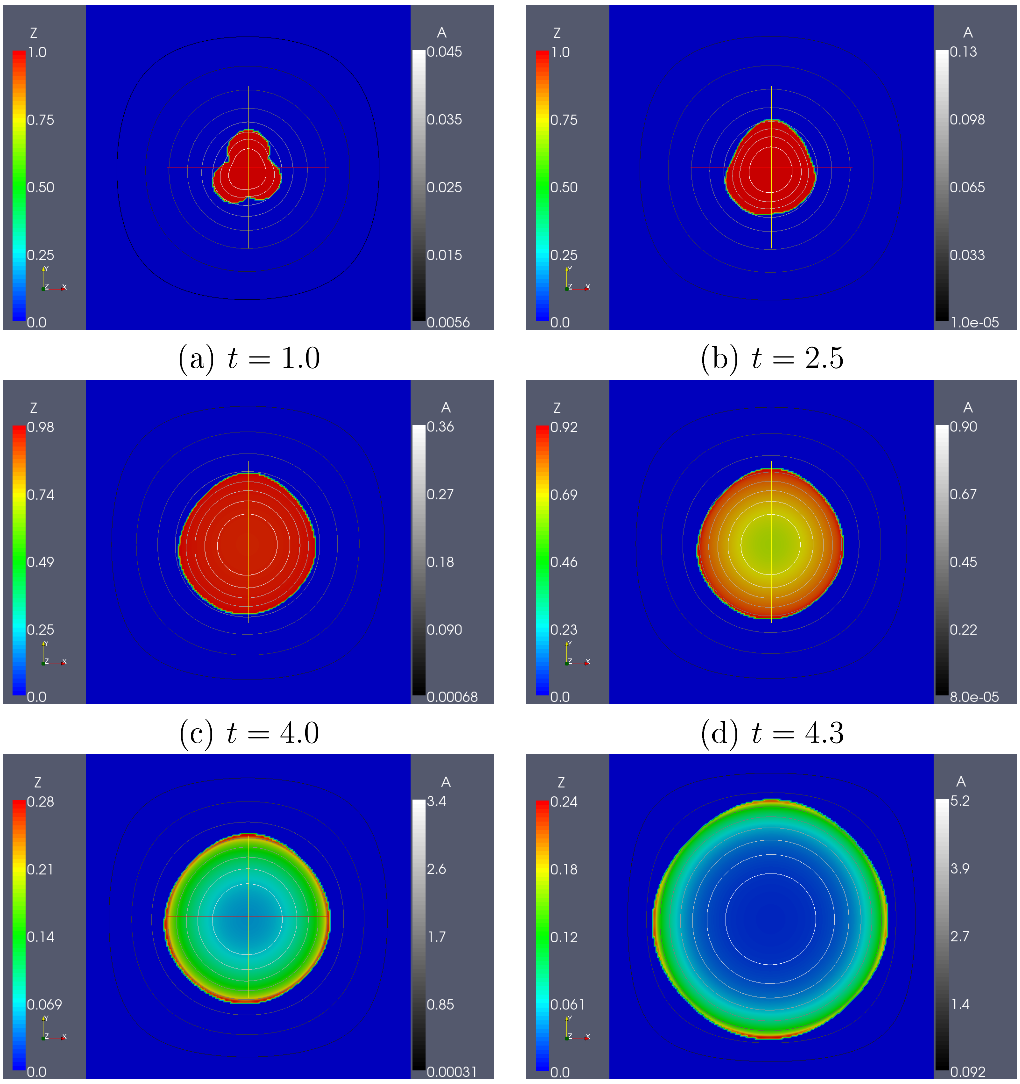
Microbial flocs

Biofilms in the absence of a substratum are aften called *microbial flocs*. Such bacterial aggregates enclosed by an EPS matrix are used in the industry for waste water treatment and also observed in natural settings ([59]). The first simulation reflects the Dirichlet boundary conditions (1.9). Initially, down-regulated biomass is only located in a heterogeneous region $\Omega_2(0)$ in the center of the domain, no up-regulated biomass and no AHL is assumed to be in the system. The substrate concentration is everywhere in Ω at the same level as on the boundary,

$$\begin{aligned} &A_0 = Y_0 \equiv 0, && && S_0 \equiv 1 && \text{in } \Omega, \\ &X_0 > 0 && \text{in } \Omega_2(0), && X_0 = 0 && \text{in } \Omega_1(0). \end{aligned}$$

This situation describes a heterogeneous microbial floc of down-regulated cells in the middle of the domain. In Figure 1.3 the development and process of up-regulation of the microbial floc is shown. The biofilm is represented by the ratio of down-regulated biomass to overall biomass, $Z = X/(X + Y)$ in the biofilm region $\Omega_2(t)$, while $Z = 0$ in the aqueous phase $\Omega_1(t)$. Moreover, the iso-concentration lines for the autoinducer A are coded in greyscale.

When the simulation starts the floc is formed by three overlapping circles in the center of the domain. Nutrients are available everywhere, the biomass in the system increases and



(a) $t = 1.0$ (b) $t = 2.5$

(c) $t = 4.0$ (d) $t = 4.3$

(e) $t = 4.8$ (f) $t = 6.5$

Figure 1.3.: Development and Up-Regulation of a Microbial Floc under Homogeneous Dirichlet Conditions for the Autoinducers: Shown are for selected times the fraction of down-regulated biomass, $Z := X/(X+Y)$, and isolines of the AHL concentration ([68]).

starts to expand where the biomass density locally reaches values close to 1. At time $t = 4$ the shape of the floc is almost spherical and we observe a small amount of up-regulated cells in the core of the floc . On the boundary the autoninducer concentration is kept at the constant level $A|_{\partial\Omega} = 0$. The highest concentrations are always found in the center of the floc, from where the molecules diffuse towards the boundary of the domain. At time $t = 4.3$ we note the onset of major up-regulation, and in the later snapshots the floc is everywhere dominated by up-regulated biomass. The highest fractions of down-regulated cells can be found in the outer-most layers.

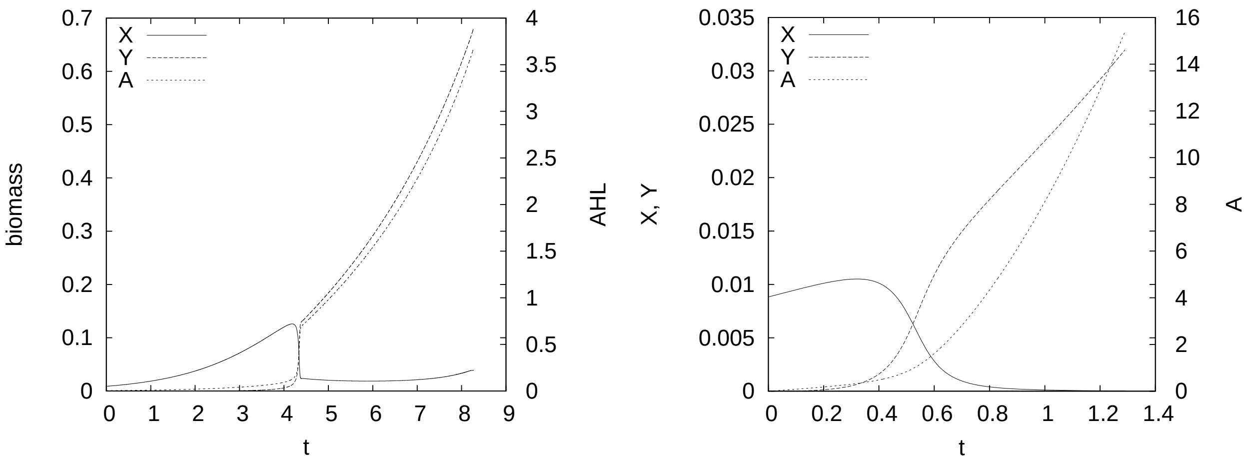


Figure 1.4.: Simulation of Quorum Sensing in a Microbial Floc: Plotted is the time evolution of X_{total}, Y_{total} and A_{total} for homogeneous Dirichlet conditions (left) and for homogeneous Neumann conditions (right) for the autoinducer concentration ([68]).

The the total amount of biomass fractions and autoinducers relative to the size of the domain $|\Omega|$ are plotted in the left panel of Figure 1.4,

$$X_{total}(t) = \frac{1}{|\Omega|}\int_\Omega X(x,t)dx, \quad Y_{total}(t) = \frac{1}{|\Omega|}\int_\Omega Y(x,t)dx, \quad A_{total}(t) = \frac{1}{|\Omega|}\int_\Omega A(x,t)dx.$$

The switch from a down- to an up-regulated system happens instantaneously, afterwards the biofilm develops at an unchanged rate and is now dominated by up-regulated cells. The corresponding results of a simulation, where homogeneous Neumann conditions for the autoinducer concentration are assumed, $\partial_\nu A|_{\partial\Omega} = 0$, are plotted in the second panel. Here, ∂_ν denotes the outward unit normal vector on the boundary of the domain. In this setting autoinducers cannot leave the domain, accumulate faster and very high autoinducer concentrations are attained. The onset of quorum-sensing occurs significantly earlier than under Dirichlet conditions, and soon after induction occurs, all biomass in the system is up-regulated. This illustrates that not only the number of cells in the system affects the process of up-regulation but also external mass transfer; namely, the removal of autoinducers from the system.

Under homogeneous Dirichlet boundary conditions for the autoinducer concentration significantly more biomass is produced before we observe the onset of up-regulation, and autoinducers accumulate slower. The slow increase of down-regulated biomass at approximately $t \approx 7$ is a boundary effect. The biomass in the system grows, which leads to lower substrate concentrations and higher AHL concentrations. Since the floc expands the biofilm/water interface approaches the boundary of the domain. The boundary conditions enforce that the flux of AHL out of the system and the flux of substrates into the system increases. Consequently, the up-regulation process of the floc is slower and the unlimited nutrient supply promotes the growth of down-regulated biomass in the outer layers.

Biofilms

The second simulation illustrates the process of quorum-sensing in a growing biofilm community in a rectangular domain.

The substratum is the bottom boundary of the domain. It is impermeable to biomass, substrate and AHL, which is reflected by homogeneous Neumann boundary conditions. Also at the lateral boundaries homogeneous Neumann conditions are assumed for all dependent variables. Through the top boundary Γ the growth limiting substrate S is added to the system and the autoinducer AHL removed, which is described by the Robin boundary conditions

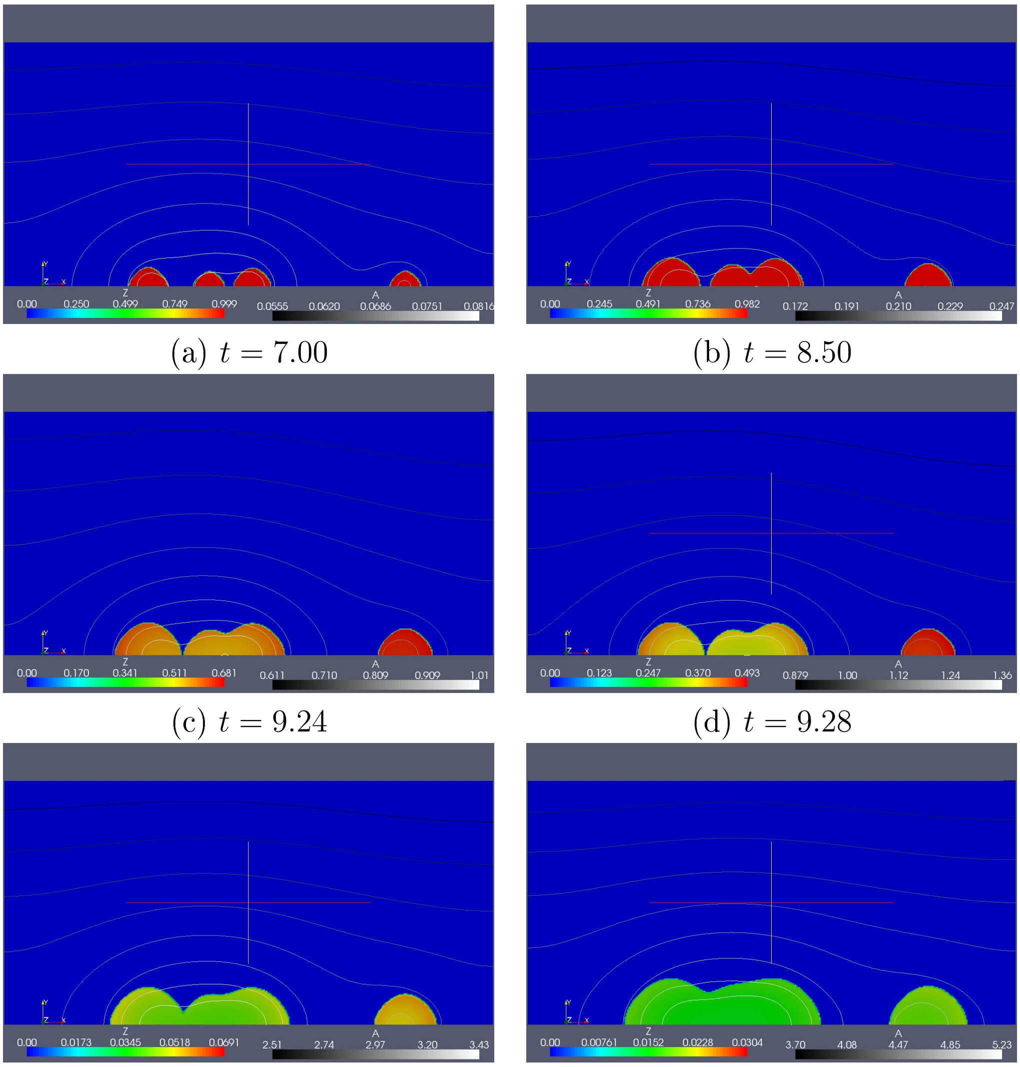
$$\big(S + \lambda \partial_\nu S\big)\big|_\Gamma = 1, \quad \big(A + \lambda \partial_\nu A\big)\big|_\Gamma = 0,$$

where the constant λ is positive. For both biomass fractions homogeneous Dirichlet conditions are assumed at the top boundary. Down-regulated biomass is placed initially in small pockets on the substratum. No up-regulated cells and no AHL are in the system, and the substrate concentration takes the bulk concentration value everywhere,

$$\begin{array}{llll} A_0 \equiv 0, & Y_0 \equiv 0, & S_0 \equiv 1 & \text{in } \Omega, \\ X_0 > 0 & \text{in } \Omega_2(0), & X_0 = 0 & \text{in } \Omega_1(0). \end{array}$$

Figure 1.5 shows the development of the biofilm and the process of up-regulation. As in the previous simulation, the biofilm is represented by the ratio of down-regulated to total biomass. When the simulation starts nutrients are available everywhere, the biomass starts growing, and expansion occurs locally when and where the biomass density approaches values close to 1. At time $t = 8.50$ the two middle colonies merged. The AHL concentrations are largest in the inner layers of the biofilm colonies and the signalling molecules diffuse from the biofilm colonies into to the aqueous phase.

Induction starts at approximately $t = 9.24$ in the clustered region, where more bacteria are concentrated, and AHL concentrations are higher. First, bacteria in the inner layers become up-regulated. The fraction of up-regulated cells, and the concentration of autoinducers in the smaller isolated colony on the right are lower. This causes a flux of AHL towards the single colony and consequently, the up-regulation pattern in this nearly hemispherical colony is not symmetric. At time $t = 9.28$ the average AHL concentration in



(a) $t = 7.00$

(b) $t = 8.50$

(c) $t = 9.24$

(d) $t = 9.28$

(e) $t = 9.78$

(f) $t = 11.28$

Figure 1.5.: Development and Up-Regulation of a Biofilm Colony: Shown are for selected times the fraction of down-regulated biomass, $Z := X/(X+Y)$, and isolines of the AHL concentration ([68]).

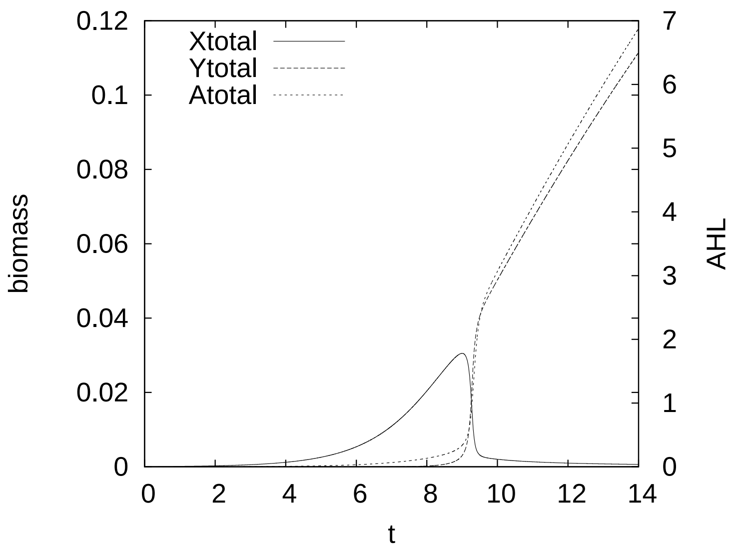


Figure 1.6.: Simulation of Quorum Sensing in a Biofilm Colony: Plotted is the time evolution of X_{total}, Y_{total} and A_{total} ([68]).

the domain reaches the threshold value, but the difference in the process of up-regulation between the clustered neighbouring colonies and the isolated colony is still clearly observable. In the next snapshot the AHL concentration is everywhere in the domain above the switching threshold, and the center colonies merged with the colony on the left. Only a small fraction of cells in the biofilm colonies is still down-regulated. Finally, at time $t = 11.28$ the colonies consist almost entirely of up-regulated cells.

The overall time evolution of the biofilm is summarized in Figure 1.6, where the lumped quantities X_{total}, Y_{total} and A_{total} are plotted. Initially, the biofilm shows exponential growth. Approximately at time $t \approx 9$ sufficient AHL has accumulated to induce up-regulation, and the switch from a mainly down-regulated biofilm to a biofilm dominated by up-regulated cells is almost immediate. This results in a drastic jump in the AHL accumulation. Afterwards the population continues to grow and consists of an almost entirely up-regulated biofilm.

Interpretation

Many features and processes in bacterial cells are regulated by autoinducer signalling, but the mechanisms and its ecological rule are still not yet very well-understood. Autoinducer signalling is used to regulate the expression of specific sets of genes. It is often related with the switching from one life-strategy to another, affects virulence factors and therefore the pathogenic potential of biofilms ([43], [6], [59]). Moreover, experimental findings support the hypothesis that autoinducers are required for the formation of biofilms, cause cell aggregation and affect the structure of a developing biofilm community ([59]). A better understanding of the underlying mechanisms is desirable to develop methods that allow to manipulate the behaviour of bacterial biofilms or to eradicate them.

Quorum-sensing in the strict sense is commonly characterized as a mechanism by which the cells measure the local density of the population to react accordingly for group benefits. The related concept of *diffusion-sensing* supports the hypothesis that single cells explore the local environmental conditions. Namely, if mass transfer is sufficiently limited for the secretion of molecules ([43]).

The simulations in Figure 1.5 illustrate that the spatial arrangement of cell colonies has a significant impact on the process of up-regulation in a growing biofilm. The switching behavior in one colony can be affected by the size and location of the other colonies. Comparing the development and up-regulation process of the microbial floc in Figure 1.4 under different boundary conditions we observe that also environmental conditions play an important role. The purely diffusive transport of autoinducers can affect the onset of switching greatly. Therefore, the numerical simulations indicate that spatial effects are crucial in the process of quorum-sensing and support the recent hypothesis of *efficiency-sensing* ([43], [68]). It aims that cells measure a combination of cell-densities, mass-transfer properties and the spatial distribution of cells.

1.4. Concluding Remarks

Only few analytical results were obtained for the mathematical models describing the growth of spatially heterogeneous biofilm communities. A solution theory for the prototype model was developed in [30], and the existence of the global attractor of the generated semigroup was shown. The global attractor was further studied in [28]. Not all results could be carried over to the more involved models that account for multiple biomass components and several dissolved substrates. In particular, the question of uniqueness of solutions remained open for the models [21] and [45]. The quorum-sensing model is the first of the multi-species biofilm models for which a uniqueness result could be established (see Theorem 1.4). Our approach to show the well-posedness is different from the approach applied in [30] for the single-species model. We expect that the solution theory developed in Section 1.3 extends to other multi-component biofilm models, and that the uniqueness of solutions can be proved for the models [21] and [45] by similar arguments.

The longtime behaviour of solutions and the existence of attractors has not yet been analysed for multi-species biofilm models and is an interesting problem. The setting and the phase space of the generated semigroup is different from the single-species model.

Another important and biologically relevant aspect is the extension of the models to allow for time-dependent interaction functions. For particular applications it can be important to take daily changes or changes in the life cycle of the bacteria into account, which leads to time-dependent coefficients in the equations. Under appropriate assumptions on the non-autonomous functions the solution theory carries over to such models. However, non-autonomous reaction terms can lead to interesting effects in the longtime dynamics, and the attractors can be essentially more complex (see Chapter 3).

2. Verifying Mathematical Models Including Diffusion, Transport and Interaction

The solutions of many systems of convection-diffusion-reaction equations arising in biology, physics or engineering describe quantities such as population densities, pressure or concentrations of nutrients and chemicals. Consequently, a natural property to require is positivity of the solutions. Models that do not guarantee positivity are not valid or break down for small values of the solution. Moreover, showing that a particular model does not preserve positivity often leads to a better understanding of the model and its limitations ([29], [34]). In this chapter we address systems of parabolic PDEs and analyse whether solutions originating from non-negative initial data remain non-negative as long as they exist. In other words, we study the invariance of the positive cone for the model under consideration.

For scalar parabolic equations the non-negativity of solutions emanating from non-negative initial data is a direct consequence of the maximum principle (see [62] or [51]). However, for systems of equations the maximum principle is not valid. In the particular case of monotone systems the situation resembles the case of scalar equations. Sufficient conditions for preserving the positive cone can be found in [66] (Chapter 7). Further, a general result for the flow invariance of regions of the phase-space is known as the Nagumo-Brezis Theorem ([60], Theorem 4.2). It is formulated for abstract differential equations in Banach spaces and states that the tangential condition ([60], p. 70) is necessary and sufficient for the flow invariance of a certain region. One could apply this result to study the invariance of the positive cone but it provides abstract conditions that are difficult to verify in general, and does not yield an explicit characterization of the class of differential operators that satisfy the tangential condition. For systems of ordinary differential equations, in fact, the tangential condition allows to formulate explicit conditions for the flow invariance of the positive cone ([60], Corollary 4.2).

An explicit characterization of the class of parabolic systems that preserve the positivity of solutions is important since it provides the modeller with a tool, which is easy to verify, to approach the question of the positive invariance of the model. Necessary and sufficient conditions for the positivity of solutions of systems of semi-linear reaction-diffusion-convection equations were formulated in [29]. They are not obtained by applying the tangential condition for this particular class of operators, the proof is based on a direct approach to derive conditions for the positivity of solutions. Since an increasing number of mathematical models exhibit density-dependent diffusion terms our aim is to extend the previous result to quasi-linear parabolic systems. Moreover, we apply the positivity criteria to deduce necessary and sufficient conditions for the validity of comparison principles for semi-linear and for quasi-linear systems.

The second part of this chapter is devoted to stochastic perturbations of deterministic parabolic systems which play an important role in the modelling of a variety of phenomena in physics and biology. We seek an explicit characterization of the class of stochastic perturbations that preserve the invariance of the positive cone of the unperturbed deterministic model. For stochastic scalar ODEs it is well-known that additive noise destroys the positivity of solutions while the positivity property is preserved under perturbations by a linear multiplicative noise. Our main result for systems of stochastic PDEs resembles this observation. To study the positivity property of stochastic systems we construct a family of random PDEs such that its solutions converge in expectation to the solution of the stochastic system. We formulate necessary and sufficient conditions for the positivity of the solutions of the family of random approximations. The positivity of the random approximations then implies the positivity of the solutions of the stochastic system. Moreover, we show that the positivity is preserved for both, Itô's and Stratonovich's interpretation of stochastic differential equations.

For stochastic perturbations of systems of ODEs the classical Nagumo-Brezis Theorem was generalized in [53]. The tangential condition was formulated in the stochastic setting and shown that it is necessary and sufficient for the invariance of regions of the phase space. The result is valid for Itô's and for Stratonovich's interpretation (see [53], Theorem 1). As its deterministic counterpart the tangential condition is formulated in an abstract form and has to be verified for each particular problem. We cannot apply this criterion to analyse the invariance of the positive cone for systems of stochastic PDEs but it allows to deduce explicit necessary and sufficient conditions for the positivity of solutions of systems of stochastic ODEs. Sufficient conditions for the validity of comparison principles for systems of stochastic ODEs can be found in [14] (Theorem 6.4.1), which imply sufficient conditions for the positivity of solutions. The proof uses a conjugacy between stochastic and random differential equations, but cannot be applied for systems of stochastic PDEs. For stochastic perturbations of a single scalar parabolic PDE explicit necessary and sufficient conditions for the positivity of solutions of the stochastic system were proved in [47] (Corollary 2.6 and Theorem 2.9). The proof is not based on random approximations. We apply results from the deterministic theory and formulate necessary and sufficient conditions for the invariance of the positive cone for the random approximations, which yield sufficient conditions for the positivity of the solutions of the stochastic system. To show that these conditions are also necessary presumably requires different techniques.

The outline of this chapter is as follows. In Section 2.1 we recall the positivity criterion obtained in [29] for systems of semi-linear parabolic PDEs before we derive necessary and sufficient conditions for the positivity of solutions of systems of quasi-linear convection-diffusion-reaction-equations. It turns out that for semi-linear systems, the diffusion and convection matrices are necessarily diagonal, while the quasi-linear case is essentially different. Here, cross-diffusion and -convection terms are allowed, however, the matrices are of a very particular form. As a consequence of the positivity criteria we deduce necessary and sufficient conditions for the validity of comparison principles for solutions of semi-linear and quasi-linear systems in Section 2.2. In Section 2.3 we present several applications and consider quasi-linear systems arising in the modelling of biological systems.

The second part of the chapter is devoted to stochastic perturbations of deterministic systems. In Section 2.4 we motivate our results and consider simple examples where a direct transformation relates the stochastic system with a family of random equations. In the general case, where such a simple transformation is not applicable, we study the stochastic problem by considering smooth random approximations since random equations can be interpreted pathwise and allow to apply deterministic methods. We recall an approximation theorem for stochastic perturbations of semi-linear parabolic systems in Section 2.5.1. The solutions of the random approximations do not converge to the solution of the original system but to the solution of a modified stochastic system. However, the relation is explicit and it is possible to construct a family of random approximations such that its solutions converge to the solution of the original stochastic system. The main result is formulated in Section 2.5.2 and yields necessary and sufficient conditions for the positivity of solutions of the random approximations. The conditions ensure that the stochastic system preserves positivity. Moreover, the conditions are invariant under the transformation relating the original system and the auxiliary system, and the transformation coincides with the relation connecting Itô's and Stratonovich's interpretation of stochastic differential equations. Consequently, the positivity of solutions is guaranteed, independent of the choice of interpretation. As a consequence of the positivity criterion we formulate conditions for the validity of comparison principles for stochastic systems in Section 2.5.3. In Section 2.5.4 we consider an application and verify the positivity property of a stochastic model.

2.1. Positivity Criteria for Deterministic Systems

2.1.1. Semi-Linear Systems

In this section we recall the positivity criterion obtained in [29] for systems of semi-linear parabolic equations. It yields explicit necessary and sufficient conditions for the positivity of solutions of semi-linear convection-diffusion-reaction equations of the form

$$\begin{aligned} \partial_t u &= a \cdot \Delta u - \gamma \cdot Du + f(u) && \Omega \times (0,T), \\ u|_{\partial\Omega} &= 0 && \partial\Omega \times [0,T], \\ u|_{t=0} &= u_0 && \Omega \times \{0\}, \end{aligned} \tag{2.1}$$

where $u = (u_1, \dots, u_k) : \Omega \times [0,T] \to \mathbb{R}^k$, $k \in \mathbb{N}$, is a vector-valued function of the spatial variable $x \in \Omega$ and time $t \in [0,T]$. Here, $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, denotes a bounded domain with boundary $\partial\Omega$ and $T > 0$.

The diffusion matrix $a = (a_{ij})_{1 \le i,j, \le k}$ has constant coefficients $a_{ij} \in \mathbb{R}$ and

$$a \text{ is positive definite.} \tag{2.2}$$

The convection term is defined by

$$\gamma \cdot \mathrm{D}u := \sum_{l=1}^{n} \gamma^l \cdot \partial_{x_l} u,$$

where $\gamma^l = (\gamma^l_{ij})_{1\le i,j\le k}$, $1 \le l \le n$, are matrices with constant coefficients $\gamma^l_{ij} \in \mathbb{R}$. The partial derivatives ∂_t and ∂_{x_l}, $1 \le l \le n$, as well as the Laplace operator $\Delta = \Delta_x$ are applied componentwise to the vector-valued function u. Moreover, we assume that the interaction function $f = (f_1, \ldots, f_k)$ is continuously differentiable,

$$f \in C^1(\mathbb{R}^k; \mathbb{R}^k). \tag{2.3}$$

We will formulate explicit conditions on the matrices a and γ^l, $1 \le l \le n$, and the interaction function f such that the solutions of System (2.1) preserve positivity.

Let $L^p(\Omega; \mathbb{R}^k)$, where $1 \le p \le \infty$, be the space of vector-valued functions $u : \Omega \to \mathbb{R}^k$ such that the components $u_i \in L^p(\Omega)$, $1 \le i \le k$. The scalar product in the Hilbert space $L^2(\Omega; \mathbb{R}^k)$ is defined by

$$\langle u, v\rangle_{L^2(\Omega;\mathbb{R}^k)} := \sum_{i=1}^{k} \langle u_i, v_i\rangle_{L^2(\Omega)} \qquad u, v \in L^2(\Omega; \mathbb{R}^k).$$

For vectors $y \in \mathbb{R}^k$ we write $y \ge 0$ if the inequality is satisfied componentwise,

$$y_i \ge 0 \qquad \text{for all } \; 1 \le i \le k,$$

and denote all non-negative vectors by $\mathbb{R}^k_+ := \{y \in \mathbb{R}^k |\; y \ge 0\}$.

Definition 2.1. *The* ***positive cone*** *in $L^2(\Omega; \mathbb{R}^k)$ is the set*

$$K^+ := \big\{u \in L^2(\Omega; \mathbb{R}^k) \;\big|\; u \ge 0 \text{ a.e. in } \Omega\big\}.$$

Moreover, we say that System (2.1) fulfils the ***positivity property*** *if for every initial data $u_0 \in K^+$ the corresponding solution $u(\,\cdot\,, \cdot\,; u_0) : \Omega \times [0, t_{max}] \to \mathbb{R}^k$ satisfies*

$$u(\cdot, t; u_0) \in K^+ \qquad \textit{for } \; t \in [0, t_{max}],$$

where $t_{max} > 0$ and $[0, t_{max}]$ denotes the maximal existence interval of the solution.

Our aim is not to study the well-posedness of the initial-/boundary value problem (2.1), we are interested in the qualitative behaviour of solutions. Therefore, in the sequel we assume that for every initial data $u_0 \in K^+$ there exists a unique solution of System (2.1), and the solution satisfies L^∞-estimates,

$$u(\cdot, t; u_0) \in L^\infty(\Omega; \mathbb{R}^k) \qquad \text{for } \; t \in [0, t_{max}]. \tag{2.4}$$

Sufficient conditions on the data and the coefficients of the equations that justify this assumption can be found in [48]. The following theorem characterizes the class of semi-linear systems (2.1) that satisfy the positivity property.

Theorem 2.1. *Let the assumptions (2.2) - (2.4) be fulfilled and the initial data* $u_0 \in K^+$ *satisfy the compatibility conditions. Then, System (2.1) possesses the positivity property, if and only if the matrices* a *and* γ *are diagonal, and the interaction function satisfies*

$$f_i(y) \geq 0 \qquad \text{for all } y \in \mathbb{R}^k_+ \text{ such that } y_i = 0, \tag{2.5}$$

where $1 \leq i \leq k$.

For the proof of Theorem 2.1 we refer to [29] and [34].

Definition 2.2. *We say that the function* $f : \mathbb{R}^k \to \mathbb{R}^k$ *fulfils the* ***positivity condition*** *if its components satisfy the inequalities (2.5) in Theorem 2.1.*

In the spatially homogeneous case, for systems of ODEs Theorem 2.1 is equivalent to the tangential condition for the invariance of the positive cone. In this case, explicit conditions for the positivity of solutions can be derived from the Nagumo-Brezis Theorem (see [60] or [71]). For the proof of the following criterion we refer to [60], Corollary 4.2.

Theorem 2.2. *Let* $f : \mathbb{R}^k \to \mathbb{R}^k$ *fulfil the hypothesis (2.3) and* $u_0 \in \mathbb{R}^k$*. Then, the system of ODEs*

$$\begin{aligned} \frac{d}{dt}u &= f(u), \\ u|_{t=0} &= u_0, \end{aligned} \tag{2.6}$$

where $u = (u_1, \ldots, u_k) : \mathbb{R}_+ \to \mathbb{R}^k$*, satisfies the positivity property if and only if the function* f *satisfies the positivity condition.*

Theorem 2.1 states that a given system of ODEs which satisfies the positivity property will preserve this property when diffusion and convection effects are taken into account if and only if no cross-diffusion and no cross-convection terms are present.

2.1.2. Quasi-Linear Systems

An increasing number of models exhibits density-dependent diffusion and convection terms. To study the positivity property of these models we generalize Theorem 2.1 for systems of quasi-linear parabolic equations of the form

$$\begin{aligned} \partial_t u &= a(u) \cdot \Delta u - \gamma(u) \cdot Du + f(u) & \Omega \times (0,T), \\ u|_{\partial\Omega} &= 0 & \partial\Omega \times [0,T], \\ u|_{t=0} &= u_0 & \Omega \times \{0\}, \end{aligned} \tag{2.7}$$

where we use the notations of the previous section.

We assume the diffusion matrix $a(u) = (a_{ij}(u))_{1 \leq i,j, \leq k}$ is density-dependent with continuously differentiable coefficient functions $a_{ij} : \mathbb{R}^k \to \mathbb{R}$ and $a(u)$ is positive definite,

$$y^T a(u) y \geq \mu \qquad \text{for all } u, y \in \mathbb{R}^k, y \neq 0, \tag{2.8}$$

where the constant $\mu > 0$ and y^T denotes the transposed vector. The convection term is given by

$$\gamma(u) \cdot \mathrm{D}u := \sum_{l=1}^{n} \gamma^l(u) \cdot \partial_{x_l} u,$$

where the coefficient functions $\gamma_{ij}^l : \mathbb{R}^k \to \mathbb{R}$ of the matrices $\gamma^l(u) = (\gamma_{ij}^l(u))_{1 \le i,j \le k}$ are continuously differentiable, $1 \le l \le n$. Moreover, we suppose the interaction function $f = (f_1, \ldots, f_k)$ is continuously differentiable,

$$f \in C^1(\mathbb{R}^k; \mathbb{R}^k). \tag{2.9}$$

Since we are interested in the qualitative behaviour of solutions we assume that for any non-negative initial data $u_0 \in K^+$ there exists a unique solution of System (2.7), and the solution and its derivatives with respect to x satisfy L^∞-estimates,

$$u(\cdot, t; u_0),\ \partial_{x_l} u(\cdot, t; u_0) \in L^\infty(\Omega; \mathbb{R}^k) \qquad t \in [0, t_{max}], \tag{2.10}$$

for all $1 \le l \le n$, where $[0, t_{max}]$ denotes the maximal existence interval of the solution.

The following theorem yields explicit conditions on the matrix functions a and γ^l and the interaction term f that are necessary and sufficient for the positivity property of System (2.7).

Theorem 2.3. *Let the conditions (2.8) - (2.10) be fulfilled, and the initial data $u_0 \in K^+$ satisfy the compatibility assumptions. Moreover, we assume that the second partial derivatives of the functions a_{ij} for $i \neq j, 1 \le i,j \le k$, exist and belong to the space $L^\infty_{loc}(\mathbb{R}^k)$. Then, System (2.7) satisfies the positivity property, if and only if the interaction term f satisfies the positivity condition and the matrices a and γ^l fulfil*

$$a_{ij}(y) = \gamma_{ij}^l(y) = 0 \qquad \text{for all } y \in \mathbb{R}^k_+ \text{ such that } y_i = 0, \tag{2.11}$$

where $i \neq j$, $1 \le i,j \le k$ and $1 \le l \le n$.

The conditions (2.11) on the diffusion and convection matrices in Theorem 2.3 imply that the matrices can be represented in the form

$$a(u) = \begin{pmatrix} a_{11}(u) & u_1 A_{12}(u) & u_1 A_{13}(u) & \cdots & u_1 A_{1k}(u) \\ u_2 A_{21}(u) & a_{22}(u) & u_2 A_{23}(u) & \cdots & u_2 A_{2k}(u) \\ \vdots & \vdots & \vdots & & \vdots \\ u_k A_{k1}(u) & u_k A_{k2}(u) & u_k A_{k3}(u) & \cdots & a_{kk}(u) \end{pmatrix}$$

$$\gamma^l(u) = \begin{pmatrix} \gamma_{11}^l(u) & u_1 \Gamma_{12}^l(u) & u_1 \Gamma_{13}^l(u) & \cdots & u_1 \Gamma_{1k}^l(u) \\ u_2 \Gamma_{21}^l(u) & \gamma_{22}^l(u) & u_2 \Gamma_{23}^l(u) & \cdots & u_2 \Gamma_{2k}^l(u) \\ \vdots & \vdots & \vdots & & \vdots \\ u_k \Gamma_{k1}^l(u) & u_k \Gamma_{k2}^l(u) & u_k \Gamma_{k3}^l(u) & \cdots & \gamma_{kk}^l(u) \end{pmatrix}$$

with bounded functions $A_{ij}(u)$ and $\Gamma^l_{ij}(u)$, $i \neq j, 1 \leq l \leq n$.

Proof. ***Necessity:*** We assume the solution $u = u(\,\cdot\,, \,\cdot\,; u_0) : \Omega \times [0, t_{max}] \to \mathbb{R}^k$ corresponding to initial data $u_0 \in K^+$ remains non-negative for $t > 0$ and prove the necessity of the stated conditions. In the following we make formal calculations, for its validity we refer to [48]. Taking smooth initial data u_0 and an arbitrary function $v \in K^+$, which is orthogonal to u_0 in $L^2(\Omega; \mathbb{R}^k)$, we obtain

$$\begin{aligned}
\big\langle \partial_t u|_{t=0}, v \big\rangle_{L^2(\Omega;\mathbb{R}^k)} &= \big\langle \lim_{t\to 0_+} \frac{u(\,\cdot\,, t; u_0) - u_0}{t}, v \big\rangle_{L^2(\Omega;\mathbb{R}^k)} \\
&= \lim_{t\to 0_+} \big\langle \frac{u(\,\cdot\,, t; u_0)}{t}, v \big\rangle_{L^2(\Omega;\mathbb{R}^k)} - \lim_{t\to 0_+} \big\langle \frac{u_0}{t}, v \big\rangle_{L^2(\Omega;\mathbb{R}^k)} \\
&= \lim_{t\to 0_+} \big\langle \frac{u(\,\cdot\,, t; u_0)}{t}, v \big\rangle_{L^2(\Omega;\mathbb{R}^k)} \geq 0,
\end{aligned}$$

where we used the orthogonality of u_0 and v as well as the hypothesis $u(\,\cdot\,, t; u_0) \in K^+$ for $t > 0$, and $t \to 0_+$ denotes the derivative from the right. We remark that for the particular initial data u_0 that we will choose in the sequel there always exists an orthogonal element $v \in K^+$. On the other hand, since u is the solution of System (2.7) corresponding to initial data u_0, we observe

$$\big\langle \partial_t u|_{t=0}, v \big\rangle_{L^2(\Omega;\mathbb{R}^k)} = \big\langle a(u_0) \cdot \Delta u_0 - \gamma(u_0) \cdot \mathrm{D} u_0 + f(u_0), v \big\rangle_{L^2(\Omega;\mathbb{R}^k)} \geq 0. \tag{2.12}$$

In particular, for fixed $i \in \{1, \ldots, k\}$ choosing the functions $u_0 = (\tilde{u}_1, \ldots, \underbrace{0}_{i}, \ldots, \tilde{u}_k)$ and $v = (0, \ldots, \underbrace{\tilde{v}}_{i}, \ldots, 0)$ with $u_0, v \in K^+$ leads to the scalar inequality

$$\Big\langle \sum_{j=1, j\neq i}^{k} a_{ij}(u_0) \Delta \tilde{u}_j - \sum_{l=1}^{n} \sum_{j=1, j\neq i}^{k} \gamma^l_{ij}(u_0) \partial_{x_l} \tilde{u}_j + f_i(u_0),\ \tilde{v} \Big\rangle_{L^2(\Omega)} \geq 0.$$

Since this inequality holds for arbitrary non-negative $\tilde{v} \in L^2(\Omega)$, we obtain the pointwise estimate

$$\sum_{j=1, j\neq i}^{k} a_{ij}(u_0) \Delta \tilde{u}_j - \sum_{l=1}^{n} \sum_{j=1, j\neq i}^{k} \gamma^l_{ij}(u_0) \partial_{x_l} \tilde{u}_j + f_i(u_0) \geq 0 \qquad \text{a.e. in } \Omega. \tag{2.13}$$

This implies the conditions on the diffusion and convection matrices,

$$a_{ij}(\tilde{u}_1, \ldots, \underbrace{0}_{i}, \ldots, \tilde{u}_k) = \gamma^l_{ij}(\tilde{u}_1, \ldots, \underbrace{0}_{i}, \ldots, \tilde{u}_k) = 0 \qquad \tilde{u}_j \geq 0,\ j \neq i,$$

for all $1 \leq j \leq k$, and $1 \leq l \leq n$ (see Lemma 2.1 below).

From Inequality (2.13) now follows that the components of the interaction term satisfy

$$f_i(\tilde{u}_1, \ldots, \underbrace{0}_{i}, \ldots, \tilde{u}_k) \geq 0 \qquad \tilde{u}_j \geq 0,\ j \neq i,$$

for all $1 \leq i, j \leq k$.
Sufficiency: We show that the stated conditions on a, γ and f ensure that the solution $u = u(\cdot, \cdot; u_0)$ corresponding to initial data $u_0 \in K^+$ remains non-negative. First, we assume that the properties (2.11) and the positivity condition (2.5) are satisfied for all $y \in \mathbb{R}^k$ such that $y_i = 0$. The system of equations then takes the form

$$\partial_t u_i = a_{ii}(u)\Delta u_i + \sum_{j=1, j\neq i}^{k} u_i A_{ij}(u)\Delta u_j - \sum_{l=1}^{n} \gamma_{ii}^l(u)\partial_{x_l} u_i - \sum_{l=1}^{n}\sum_{j=1, j\neq i}^{k} u_i \Gamma_{ij}^l(u)\partial_{x_l} u_j + f_i(u),$$

for $1 \leq i \leq k$, where the functions $A_{ij}, \Gamma_{ij}^l : \mathbb{R}^k \to \mathbb{R}$ are defined by

$$\begin{aligned} A_{ij}(y) &:= \int_0^1 \partial_i a_{ij}(y_1, \ldots, s y_i, \ldots, y_k) ds \qquad && y \in \mathbb{R}^k, \\ \Gamma_{ij}^l(y) &:= \int_0^1 \partial_i \gamma_{ij}^l(y_1, \ldots, s y_i, \ldots, y_k) ds \qquad && y \in \mathbb{R}^k. \end{aligned}$$

For a function $u \in L^2(\Omega)$ we denote its positive and negative part by $u_+ := \max\{u, 0\}$ and $u_- := \max\{-u, 0\}$, respectively, and obtain the representation $u = u_+ - u_-$. Its absolute value is given by $|u| = u_+ + u_-$. By the definition immediately follows $u_-\, u_+ = 0$. Furthermore, if $u \in \mathrm{H}^1(\Omega)$, then also its positive and negative part, $u_+, u_- \in \mathrm{H}^1(\Omega)$, and

$$\partial_{x_l} u_- = \begin{cases} -\partial_{x_l} u & u < 0 \\ 0 & u \geq 0 \end{cases} \qquad \partial_{x_l} u_+ = \begin{cases} \partial_{x_l} u & u > 0 \\ 0 & u \leq 0 \end{cases}$$

for all $1 \leq l \leq n$ (cf. [41]). This implies

$$(\partial_{x_l} u_+)\, u_- = u_+\, \partial_{x_l} u_- = (\partial_{x_l} u_+)\, \partial_{x_m} u_- = 0 \qquad 1 \leq l, m \leq n.$$

In order to prove the positivity of the solution u corresponding to initial data $u_0 \in K^+$ we show that $(u_0)_{i-} = 0$ implies $u_{i-} := (u_i(\cdot, t; u_0))_- = 0$ for $t > 0$ and all $1 \leq i \leq k$. Multiplying the i-th equation by the negative part u_{i-} and integrating over Ω yields

$$\begin{aligned} \langle \partial_t u_i, u_{i-} \rangle_{L^2(\Omega)} \;=\; & \langle a_{ii}(u)\Delta u_i, u_{i-} \rangle_{L^2(\Omega)} + \sum_{j=1, j\neq i}^{k} \langle u_i A_{ij}(u)\Delta u_j, u_{i-} \rangle_{L^2(\Omega)} \\ & - \sum_{l=1}^{n} \langle \gamma_{ii}^l(u)\partial_{x_l} u_i, u_{i-} \rangle_{L^2(\Omega)} - \sum_{l=1}^{n}\sum_{j=1, j\neq i}^{k} \langle u_i \Gamma_{ij}^l(u)\partial_{x_l} u_j, u_{i-} \rangle_{L^2(\Omega)} \\ & + \langle f_i(u), u_{i-} \rangle_{L^2(\Omega)}. \end{aligned}$$

We observe that the left-hand side of the equation can be written as

$$\langle \partial_t u_i, u_{i-} \rangle_{L^2(\Omega)} = -\langle \partial_t u_{i-}, u_{i-} \rangle_{L^2(\Omega)} = -\frac{1}{2}\partial_t \|u_{i-}\|_{L^2(\Omega)}^2.$$

Taking into account the homogeneous Dirichlet boundary conditions we obtain for the first term on the right-hand side of the equation

$$\begin{aligned}\langle a_{ii}(u)\Delta u_i, u_{i-}\rangle_{L^2(\Omega)} &= -\langle a_{ii}(u)\Delta u_{i-}, u_{i-}\rangle_{L^2(\Omega)} = \langle \nabla(a_{ii}(u)u_{i-}), \nabla u_{i-}\rangle_{L^2(\Omega;\mathbb{R}^n)} \\ &= \langle a_{ii}(u)\nabla u_{i-}, \nabla u_{i-}\rangle_{L^2(\Omega;\mathbb{R}^n)} + \sum_{j=1}^{k}\langle \partial_j a_{ii}(u)u_{i-}\nabla u_j, \nabla u_{i-}\rangle_{L^2(\Omega;\mathbb{R}^n)}.\end{aligned}$$

We further estimate the second integral by

$$\Big|\sum_{j=1}^{k}\langle \partial_j a_{ii}(u)u_{i-}\nabla u_j, \nabla u_{i-}\rangle_{L^2(\Omega;\mathbb{R}^n)}\Big| \leq C_1\sum_{l=1}^{n}\langle |\partial_{x_l}u_{i-}|, u_{i-}\rangle_{L^2(\Omega)},$$

for some constant $C_1 \geq 0$. Here, we used the hypothesis (2.10) and the regularity assumption $a_{ii} \in C^1(\mathbb{R}^k;\mathbb{R})$. For the second diffusion term we obtain

$$\begin{aligned}&\Big|\Big\langle \sum_{j=1,j\neq i}^{k} u_i A_{ij}(u)\Delta u_j, u_{i-}\Big\rangle_{L^2(\Omega)}\Big| \\ =& \Big| -\sum_{j=1,j\neq i}^{k}\big\langle u_{i-}A_{ij}(u)\Delta u_j, u_{i-}\big\rangle_{L^2(\Omega)}\Big| \leq \sum_{j=1,j\neq i}^{k}\Big|\Big\langle \nabla\big(A_{ij}(u)(u_{i-})^2\big), \nabla u_j\Big\rangle_{L^2(\Omega;\mathbb{R}^n)}\Big| \\ \leq& \sum_{j=1,j\neq i}^{k}\Big(\Big|\Big\langle 2A_{ij}(u)u_{i-}\nabla u_{i-}, \nabla u_j\Big\rangle_{L^2(\Omega;\mathbb{R}^n)}\Big| + \sum_{m=1}^{k}\Big|\Big\langle \partial_m A_{ij}(u)(u_{i-})^2\nabla u_m, \nabla u_j\Big\rangle_{L^2(\Omega;\mathbb{R}^n)}\Big|\Big) \\ \leq& C_2\sum_{l=1}^{n}\langle |\partial_{x_l}u_{i-}|, u_{i-}\rangle_{L^2(\Omega)} + C_3\|u_{i-}\|^2_{L^2(\Omega)},\end{aligned}$$

for some constants $C_2, C_3 \geq 0$. As before, we used the assumption (2.10) and that the second partial derivatives of the functions a_{ij} belong to $L^\infty_{loc}(\Omega)$. Similarly, we derive an estimate for the convection terms

$$\begin{aligned}&\Big| -\sum_{l=1}^{n}\langle \gamma^l_{ii}(u)\partial_{x_l}u_i, u_{i-}\rangle_{L^2(\Omega)} - \sum_{l=1}^{n}\sum_{j=1,j\neq i}^{k}\langle u_i\Gamma^l_{ij}(u)\partial_{x_l}u_j, u_{i-}\rangle_{L^2(\Omega)}\Big| \\ \leq& \sum_{l=1}^{n}\Big(\langle |\gamma^l_{ii}(u)\partial_{x_l}u_{i-}|, u_{i-}\rangle_{L^2(\Omega)} + \sum_{j=1,j\neq i}^{k}\langle |\Gamma^l_{ij}(u)\partial_{x_l}u_j|u_{i-}, u_{i-}\rangle_{L^2(\Omega)}\Big) \\ \leq& C_4\sum_{l=1}^{n}\langle |\partial_{x_l}u_{i-}|, u_{i-}\rangle_{L^2(\Omega)} + C_5\|u_{i-}\|^2_{L^2(\Omega)},\end{aligned}$$

for some constants $C_4, C_5 \geq 0$. Here, we used that the coefficient functions $\gamma^l_{ij} \in C^1(\mathbb{R}^k;\mathbb{R})$, $1 \leq i, j \leq k, 1 \leq l \leq n$, and the hypothesis (2.10). To estimate the interaction term we use that $f \in C^1(\mathbb{R}^k;\mathbb{R}^k)$, which leads to

$$\begin{aligned} f_i(u_1,\dots,u_k) &= f_i(u_1,\dots,\underbrace{0}_{i},\dots,u_k) + u_i \int_0^1 \partial_i f_i(u_1,\dots,su_i,\dots,u_k)ds \\ &= f_i(u_1,\dots,\underbrace{0}_{i},\dots,u_k) + u_i\, F_i(u_1,\dots,u_k), \end{aligned}$$

where the function $F_i : \mathbb{R}^k \to \mathbb{R}$ is bounded. This representation yields

$$\begin{aligned} \langle f_i(u), u_{i-}\rangle_{L^2(\Omega)} &= \langle f_i(u_1,\dots,\underbrace{0}_{i},\dots,u_k), u_{i-}\rangle_{L^2(\Omega)} + \langle u_i F_i(u_1,\dots,u_k), u_{i-}\rangle_{L^2(\Omega)} \\ &= \langle f_i(u_1,\dots,\underbrace{0}_{i},\dots,u_k), u_{i-}\rangle_{L^2(\Omega)} - \langle F_i(u_1,\dots,u_k)u_{i-}, u_{i-}\rangle_{L^2(\Omega)}. \end{aligned}$$

Summing up all terms we obtain

$$\begin{aligned} \frac{1}{2}\partial_t \|u_{i-}\|^2_{L^2(\Omega)} + \langle a_{ii}(u)\nabla u_{i-}, \nabla u_{i-}\rangle_{L^2(\Omega;\mathbb{R}^n)} \;\le\; & C_6 \sum_{l=1}^{n} \langle |\partial_{x_l} u_{i-}|, u_{i-}\rangle_{L^2(\Omega)} + C_7 \|u_{i-}\|^2_{L^2(\Omega)} \\ & - \langle f_i(u_1,\dots,\underbrace{0}_{i},\dots,u_k), u_{i-}\rangle_{L^2(\Omega)}, \end{aligned}$$

for some constants $C_6, C_7 \ge 0$.

To estimate the mixed terms we use Young's inequality. Namely, for every $\epsilon > 0$ there exists a constant $C_\epsilon \ge 0$ such that

$$\sum_{l=1}^{n} \langle |\partial_{x_l} u_{i-}|, u_{i-}\rangle_{L^2(\Omega)} \le \epsilon \|\nabla u_{i-}\|^2_{L^2(\Omega;\mathbb{R}^n)} + C_\epsilon \|u_{i-}\|^2_{L^2(\Omega)}.$$

If we choose $\epsilon > 0$ sufficiently small and take Hypothesis (2.8) into account, it follows

$$\partial_t \|u_{i-}\|^2_{L^2(\Omega)} \le C_8 \|u_{i-}\|^2_{L^2(\Omega)} - 2\langle f_i(u_1,\dots,\underbrace{0}_{i},\dots,u_k), u_{i-}\rangle_{L^2(\Omega)},$$

for some constant $C_8 \ge 0$. Since in the beginning we assumed that $f_i(y) \ge 0$ for all $y \in \mathbb{R}^k$ such that $y_i = 0$, $1 \le i \le k$, we obtain the estimate

$$\partial_t \|u_{i-}\|^2_{L^2(\Omega)} \le C_8 \|u_{i-}\|^2_{L^2(\Omega)}.$$

By Gronwall's Lemma and the initial condition $(u_0)_{i-} = 0$ follows $\|u_{i-}\|_{L^2(\Omega)} = 0$.

It remains to justify our initial assumptions. To this end we consider the modified system

$$\begin{aligned} &\partial_t \hat{u} = \hat{a}(\hat{u}) \cdot \Delta\hat{u} - \hat{\gamma}(\hat{u}) \cdot D\hat{u} + \hat{f}(\hat{u}) && \Omega \times (0,T), \\ &\hat{u}|_{\partial\Omega} = 0 && \partial\Omega \times [0,T], \\ &\hat{u}|_{t=0} = u_0 && \Omega \times \{0\}, \end{aligned}$$

where the function $\hat{f} : \mathbb{R}^k \to \mathbb{R}^k$ is given by

$$\hat{f}_i(y) = f_i(|y_1|, \dots, \underbrace{0}_{i}, \dots, |y_k|) + y_i F_i(y) \qquad y \in \mathbb{R}^k,$$

and the function F_i was defined as

$$F_i(y_1, \dots, y_k) \;:=\; \int_0^1 \partial_i f_i(y_1, \dots, s y_i, \dots, y_k) ds \qquad y \in \mathbb{R}^k.$$

The modified diffusion and convection matrices are given by

$$\begin{aligned} \hat{\gamma}^l_{ij}(y_1, \dots, y_k) &:= \gamma^l_{ij}(|y_1|, \dots, \underbrace{0}_{i} \dots, |y_k|) + y_i \Gamma^l_{ij}(y) \qquad && y \in \mathbb{R}^k, \\ \hat{a}_{ij}(y_1, \dots, y_k) &:= a_{ij}(|y_1|, \dots, \underbrace{0}_{i}, \dots, |y_k|) + y_i A_{ij}(y) \qquad && y \in \mathbb{R}^k, \end{aligned}$$

for $1 \le i, j \le k$, $1 \le l \le n$. Following the same arguments we conclude that the solution $\hat{u}$ of the modified system remains non-negative. However, if the function $\hat{u}$ is non-negative we can remove the absolute values, and $\hat{u}$ is a solution of the original system

$$\begin{aligned} \partial_t u &= a(u) \cdot \Delta u - \gamma(u) \cdot Du + f(u) \qquad && \Omega \times (0, T), \\ u|_{\partial\Omega} &= 0 && \partial\Omega \times [0, T], \\ u|_{t=0} &= u_0 && \Omega \times \{0\}. \end{aligned}$$

By the uniqueness of solutions corresponding to initial data u_0 follows that $u = \hat{u}$, which implies $u(\,\cdot\,, t; u_0) \in K^+$ for $t > 0$, and concludes the proof of the theorem. □

Lemma 2.1. *Let $j \neq i, 1 \le i, j \le k$, and $1 \le l \le n$. We assume the hypothesis of Theorem 2.3 are satisfied. If the pointwise inequality*

$$\sum_{j=1, j\neq i}^{k} a_{ij}(\tilde{u}) \Delta \tilde{u}_j - \sum_{l=1}^{n} \sum_{j=1, j\neq i}^{k} \gamma^l_{ij}(\tilde{u}) \partial_{x_l} \tilde{u}_j + f_i(\tilde{u}) \ge 0$$

is valid for every initial data $\tilde{u} = (\tilde{u}_1, \dots, \underbrace{0}_{i}, \dots, \tilde{u}_k) \in K^+$, then

$$a_{ij}(y) = \gamma^l_{ij}(y) = 0 \qquad \textit{for all } y \in \mathbb{R}^k_+ \textit{ such that } y_i = 0.$$

Proof. We argue by contradiction and suppose that there exists $y \in \mathbb{R}^k_+$ such that $y_i = 0$ and $a_{ij}(y) \neq 0$. First, we assume that $y_j > 0$. Let $x_0 \in \Omega$ and $\mathcal{U}_{x_0}$ be an open neighbourhood of x_0 that is compactly contained in Ω.

If $a_{ij}(y) > 0$ we define the function $\tilde{u} : \Omega \to \mathbb{R}^k$ by

$$\tilde{u}_m(x) := \begin{cases} y_m & m \neq i, m \neq j \\ y_j e^{-\frac{1}{\epsilon}\|x - x_0\|^2} & m = j \\ 0 & m = i \end{cases} \qquad \text{for } x \in \mathcal{U}_{x_0},$$

where $1 \leq m \leq k$, and extend it to a smooth non-negative function on Ω that vanishes on the boundary. Computing the derivatives we observe

$$\begin{aligned}\nabla \tilde{u}_j(x) &= -\frac{2}{\epsilon} y_j (x - x_0) e^{-\frac{1}{\epsilon}\|x-x_0\|^2},\\ \Delta \tilde{u}_j(x) &= -\frac{2}{\epsilon} y_j e^{-\frac{1}{\epsilon}\|x-x_0\|^2} + \frac{4}{\epsilon^2} y_j \|x - x_0\|^2 e^{-\frac{1}{\epsilon}\|x-x_0\|^2},\end{aligned}$$

for $x \in \mathcal{U}_{x_0}$, and consequently,

$$\begin{aligned}\partial_{x_l} \tilde{u}_m(x_0) &= 0,\\ \Delta \tilde{u}_m(x_0) &= \begin{cases} -\frac{2}{\epsilon} y_j & m = j\\ 0 & m \neq j\end{cases}\end{aligned}$$

for all $1 \leq m \leq k, 1 \leq l \leq n$. Since $\epsilon > 0$ can be chosen arbitrarily small, the inequality (2.13) is violated in the point $x_0 \in \Omega$.

On the other hand, if $a_{ij}(y) < 0$, we define the function $\tilde{u} : \Omega \to \mathbb{R}^k$ by

$$\tilde{u}_m(x) := \begin{cases} y_m & m \neq i, m \neq j\\ y_j(e^{-\frac{1}{\epsilon}\|x-x_0\|^2} + \frac{1}{\epsilon^2}\|x - x_0\|^2) & m = j\\ 0 & m = i\end{cases} \qquad \text{for } x \in \mathcal{U}_{x_0},$$

where $1 \leq m \leq k$, and extend it to a smooth non-negative function on Ω that vanishes on the boundary. Computing the derivatives we observe

$$\begin{aligned}\nabla \tilde{u}_j(x) &= y_j\big(-\frac{2}{\epsilon}(x - x_0)e^{-\frac{1}{\epsilon}\|x-x_0\|^2} + \frac{2}{\epsilon^2}(x - x_0)\big),\\ \Delta \tilde{u}_j(x) &= y_j\big(-\frac{2}{\epsilon} e^{-\frac{1}{\epsilon}\|x-x_0\|^2} + \frac{4}{\epsilon^2}\|x - x_0\|^2 e^{-\frac{1}{\epsilon}\|x-x_0\|^2} + \frac{2}{\epsilon^2}\big),\end{aligned}$$

for all $x \in \mathcal{U}_{x_0}$, and consequently,

$$\begin{aligned}\partial_{x_l} \tilde{u}_m(x_0) &= 0,\\ \Delta \tilde{u}_m(x_0) &= \begin{cases} y_j \frac{2}{\epsilon}(\frac{1}{\epsilon} - 1) & m = j\\ 0 & m \neq j,\end{cases}\end{aligned}$$

for all $1 \leq m \leq k, 1 \leq l \leq n$. If we choose $\epsilon > 0$ sufficiently small the inequality (2.13) is violated in the point $x_0 \in \Omega$.

It remains to consider the case that the function $a_{ij} : \mathbb{R}^k \to \mathbb{R}$ is identically zero on the set $\{y \in \mathbb{R}^k_+ | \; y_i = 0, y_j > 0\}$. By the continuity of a_{ij} then follows $a_{ij}(y) = 0$ for all $y \in \mathbb{R}^k_+$ such that $y_i = y_j = 0$. This concludes the proof for the conditions on the diffusion matrix.

To derive the assumptions on the convection terms we again argue by contradiction and suppose that there exists $y \in \mathbb{R}^k_+$ such that $y_i = 0$ and $\gamma^l_{ij}(y) \neq 0$. Without loss of generality we assume that $y_j > 0$. Otherwise, the claim follows by the continuity of the function γ_{ij}^l in the same way as for the diffusion matrix. Let $x_0 \in \Omega$ and $\mathcal{U}_{x_0}$ be an open neighbourhood of x_0 which is compactly contained in Ω.

If $\gamma_{ij}^l(y) > 0$ we define the function $\tilde{u} : \Omega \to \mathbb{R}_+^k$ by

$$\tilde{u}_m(x) := \begin{cases} y_m & m \neq i, m \neq j \\ y_j(1 + \sin(\frac{x_l - (x_0)_l}{\epsilon})) & m = j \\ 0 & m = i \end{cases} \qquad \text{for } x \in \mathcal{U}_{x_0},$$

for $1 \leq m \leq k$, and extend it to a smooth non-negative function on Ω that vanishes on the boundary. Computing the derivatives we observe

$$\partial_{x_l} \tilde{u}_j(x) = \frac{1}{\epsilon} y_j \cos(\frac{x_l - (x_0)_l}{\epsilon}),$$
$$\partial_{x_l}^2 \tilde{u}_j(x) = -\frac{1}{\epsilon^2} y_j \sin(\frac{x_l - (x_0)_l}{\epsilon}),$$

for all $x \in \mathcal{U}_{x_0}$, and consequently,

$$\partial_{x_l} \tilde{u}_m(x_0) = \begin{cases} \frac{1}{\epsilon} y_j & m = j \\ 0 & m \neq j, \end{cases}$$
$$\Delta \tilde{u}_m(x_0) = 0,$$

for all $1 \leq m \leq k$. Choosing $\epsilon > 0$ sufficiently small the inequality (2.13) is violated in the point $x_0 \in \Omega$.

Otherwise, if $\gamma_{ij}^l(y) < 0$, we define the function $\tilde{u} : \Omega \to \mathbb{R}_+^k$ by

$$\tilde{u}_m(x) := \begin{cases} y_m & m \neq i, m \neq j \\ y_j(1 - \sin(\frac{x_l - (x_0)_l}{\epsilon}) & m = j \\ 0 & m = i \end{cases} \qquad \text{for } x \in \mathcal{U}_{x_0},$$

for all $1 \leq m \leq k$, and extend it to a smooth non-negative function on Ω that vanishes on the boundary. In this case we obtain

$$\partial_{x_l} \tilde{u}_m(x_0) = \begin{cases} -\frac{1}{\epsilon} y_j & m = j \\ 0 & m \neq j, \end{cases}$$
$$\Delta \tilde{u}_m(x_0) = 0,$$

for all $m \neq i, 1 \leq m \leq k$. Choosing $\epsilon > 0$ sufficiently small leads to a contradiction to Inequality (2.13) in the point $x_0 \in \Omega$. □

The conditions on the diffusion and convection matrices that are necessary and sufficient for the positivity of solutions of semi-linear and quasi-linear systems are essentially different. We illustrate the results considering a simple example.

Example 2.1. *For $k = 3$ and $\gamma \equiv 0$ the semi-linear system (2.1) takes the form*

$$\begin{aligned}\partial_t u_1 &= a_{11}\Delta u_1 + a_{12}\Delta u_2 + a_{13}\Delta u_3 + f_1(u),\\ \partial_t u_2 &= a_{21}\Delta u_1 + a_{22}\Delta u_2 + a_{23}\Delta u_3 + f_2(u),\\ \partial_t u_3 &= a_{31}\Delta u_1 + a_{32}\Delta u_2 + a_{33}\Delta u_3 + f_3(u).\end{aligned} \tag{2.14}$$

If the assumptions of Theorem 2.1 are satisfied, System (2.14) satisfies the positivity property if and only if

$$f_1(0,y,z) \geq 0, \quad f_2(y,0,z) \geq 0, \quad f_3(y,z,0) \geq 0 \qquad \text{for all } y \geq 0, z \geq 0, \tag{2.15}$$

and the matrix $a = (a_{ij})_{1 \leq i,j \leq 3}$ is diagonal. Consequently, all cross-diffusion terms are zero and the system is of the form

$$\begin{aligned}\partial_t u_1 &= a_{11}\Delta u_1 + f_1(u),\\ \partial_t u_2 &= a_{22}\Delta u_2 + f_2(u),\\ \partial_t u_3 &= a_{33}\Delta u_3 + f_3(u).\end{aligned}$$

The quasi-linear system (2.7) for $k = 3$ and $\gamma \equiv 0$ takes the form

$$\begin{aligned}\partial_t u_1 &= a_{11}(u)\Delta u_1 + a_{12}(u)\Delta u_2 + a_{13}(u)\Delta u_3 + f_1(u),\\ \partial_t u_2 &= a_{21}(u)\Delta u_1 + a_{22}(u)\Delta u_2 + a_{23}(u)\Delta u_3 + f_2(u),\\ \partial_t u_3 &= a_{31}(u)\Delta u_1 + a_{32}(u)\Delta u_2 + a_{33}(u)\Delta u_3 + f_3(u).\end{aligned} \tag{2.16}$$

If the assumptions of Theorem 2.3 are satisfied, System (2.16) satisfies the positivity property if and only if the interaction function possesses the property (2.15) and

$$a_{ij}(y) = 0 \qquad \text{for all } y \in \mathbb{R}^3_+ \text{ such that } y_i = 0,$$

for all $i \neq j, 1 \leq i,j \leq 3$. This implies that System (2.16) can be represented as

$$\begin{aligned}\partial_t u_1 &= \quad a_{11}(u)\,\Delta u_1 + u_1 A_{12}(u)\Delta u_2 + u_1 A_{13}(u)\Delta u_3 + f_1(u),\\ \partial_t u_2 &= u_2 A_{21}(u)\Delta u_1 + \quad a_{22}(u)\,\Delta u_2 + u_2 A_{23}(u)\Delta u_3 + f_2(u),\\ \partial_t u_3 &= u_3 A_{31}(u)\Delta u_1 + u_3 A_{32}(u)\Delta u_2 + \quad a_{33}(u)\,\Delta u_3 + f_3(u),\end{aligned}$$

where the functions A_{ij}, $i \neq j$, were defined in the proof of Theorem 2.3.

Summarizing we observe that cross-diffusion terms destroy the positivity property of semi-linear systems. They may appear in the quasi-linear case, but are necessarily of a very particular form. Namely, if one component of the solution approaches zero, the cross-diffusion terms in the corresponding equation need to vanish.

2.2. Comparison Principles for Deterministic Systems

We apply the positivity criteria of the previous section to derive necessary and sufficient conditions for the validity of comparison theorems for the solutions of semi-linear and quasi-linear parabolic systems.

2.2.1. Semi-Linear Systems

For vectors y and z in $\mathbb{R}^k$ we write $y \geq z$ if the inequality holds componentwise,

$$y_i \geq z_i \qquad \text{for all } 1 \leq i \leq k.$$

Definition 2.3. *We define the (partial) order relation $\preccurlyeq$ on the space of vector-valued functions $L^2(\Omega; \mathbb{R}^k)$ by*

$$u \preccurlyeq v \qquad \textit{if} \quad v - u \in K^+,$$

where $u, v \in L^2(\Omega; \mathbb{R}^k)$.

Furthermore, we call System (2.7) (or System (2.1)) ***order preserving with respect to the order relation*** $\preccurlyeq$ *if for every initial data $u_0, v_0 \in L^2(\Omega; \mathbb{R}^k)$ such that $u_0 \preccurlyeq v_0$ the corresponding solutions satisfy*

$$u(\,\cdot\,; t, u_0) \preccurlyeq v(\,\cdot\,; t, u_0) \qquad \textit{for } t > 0,$$

as long as both solutions exists.

Theorem 2.4. *Under the assumptions of Theorem 2.1, System (2.1) is order preserving with respect to $\preccurlyeq$ if and only if the matrices a and γ are diagonal, and the reaction term f satisfies*

$$f_i(y) \geq f_i(z) \qquad \textit{for all } y, z \in \mathbb{R}^k \textit{ such that } y \geq z,\ y_i = z_i, \tag{2.17}$$

for all $1 \leq i \leq k$.

Definition 2.4. *We call the function $f : \mathbb{R}^k \to \mathbb{R}^k$* ***quasi-monotone*** *if it satisfies Property (2.17) in Theorem 2.4.*

Sketch of the proof. Let u_0 and v_0 be given initial data such that $u_0 \succcurlyeq v_0$. We prove that the order relation is preserved by the corresponding solutions u and v, if and only if the matrices a and γ are diagonal, and the reaction term fulfills the stated monotonicity conditions. Defining the difference of the solutions $w := u - v$ it satisfies the system

$$\begin{aligned} \partial_t w &= a \cdot \Delta w - \gamma \cdot Dw + f(u) - f(v) && \Omega \times (0, T), \\ w|_{\partial\Omega} &= 0 && \partial\Omega \times [0, T], \\ w|_{t=0} &= w_0 && \Omega \times \{0\}, \end{aligned} \tag{2.18}$$

where $w_0 := u_0 - v_0 \in K^+$. Moreover, System (2.1) is order preserving with respect to $\preccurlyeq$ if and only if System (2.18) satisfies the positivity property.

Necessity: We only indicate the ideas and refer to the proof of Theorem 2.6 for details. Let the index $i \in \{1, \ldots, k\}$ be fixed. If the solutions preserve the order relation we follow similar arguments as in the proof of the positivity criterion and obtain the scalar inequality

$$\sum_{j=1, j\neq i}^{k} a_{ij} \Delta(\tilde{u}_j - \tilde{v}_j) - \sum_{l=1}^{n} \sum_{j=1, j\neq i}^{k} \gamma_{ij}^l \partial_{x_l}(\tilde{u}_j - \tilde{v}_j) + f_i(u_0) - f_i(v_0) \geq 0,$$

for every $u_0 = (\tilde{u}_1, \ldots, \tilde{u}_k)$ and $v_0 = (\tilde{v}_1, \ldots, \tilde{v}_k)$ such that $u_0 \succcurlyeq v_0$ and $\tilde{u}_i = \tilde{v}_i$. It follows that the off-diagonal coefficients of the matrices a and γ_l, $1 \leq l \leq n$, are identically zero (see Lemma 2.2) and the interaction function is quasi-monotone.
Sufficiency: The sufficiency of the stated conditions can be shown as in the proof of Theorem 2.6. The arguments for semi-linear systems simplify. □

Next, we analyse conditions on the interaction function ensuring that System (2.1) is order-preserving with respect to an arbitrary order relation.

Definition 2.5. *To define the order relation* $\gtrsim$ *on* $\mathbb{R}^k$ *let* σ_1 *and* σ_2 *be disjoint sets such that* $\sigma_1 \cup \sigma_2 = \{1, \ldots, k\}$. *For vectors* y *and* z *in* $\mathbb{R}^k$ *we write* $y \gtrsim z$ *if*

$$\begin{cases} y_j \geq z_j & \text{for } j \in \sigma_1 \\ y_j \leq z_j & \text{for } j \in \sigma_2. \end{cases}$$

For vector-valued functions u *and* v *in* $L^2(\Omega; \mathbb{R}^k)$ *we use the same notation and write* $u \gtrsim v$ *if the inequalities* $u \gtrsim v$ *hold pointwise a.e. in* Ω.

Theorem 2.5. *Under the hypothesis of Theorem 2.1 the semi-linear system (2.1) is order preserving with respect to* $\precsim$ *if and only if the matrices* a *and* γ *are diagonal, and the interaction term* f *satisfies*

$$\begin{aligned} f_i(y) &\leq f_i(z) \qquad \text{if } i \in \sigma_1, \\ f_i(y) &\geq f_i(z) \qquad \text{if } i \in \sigma_2, \end{aligned}$$

for all $y, z \in \mathbb{R}^k$ *such that* $y \precsim z$ *and* $y_i = z_i$, *where* $1 \leq i \leq k$.

Proof. Let u_0 and v_0 be given initial data and assume $u_0 \gtrsim v_0$. We prove that the order $\gtrsim$ is preserved by the corresponding solutions u and v, if and only if the matrices a and γ are diagonal, and the reaction term f fulfils the stated conditions. Defining the function w by

$$w_i := \begin{cases} u_i - v_i & \text{if } i \in \sigma_1 \\ -(u_i - v_i) & \text{if } i \in \sigma_2, \end{cases}$$

it satisfies the system

$$\begin{aligned} \partial_t w &= \tilde{a} \cdot \Delta w - \tilde{\gamma} \cdot Dw + F(u, v) && \Omega \times (0, T), \\ w|_{\partial\Omega} &= 0 && \partial\Omega \times [0, T], \\ w|_{t=0} &= w_0 && \Omega \times \{0\}, \end{aligned} \tag{2.19}$$

with initial data $w_0 \in K^+$. The function F is defined by

$$F_i(u, v) := \begin{cases} f_i(u) - f_i(v) & \text{if } i \in \sigma_1 \\ -(f_i(u) - f_i(v)) & \text{if } i \in \sigma_2, \end{cases}$$

the diffusion matrix $\tilde{a}$ is given by

$$\tilde{a}_{ij} := \begin{cases} a_{ij} & \text{if } i,j \in \sigma_1 \text{ or } i,j \in \sigma_2 \\ -a_{ij} & \text{otherwise,} \end{cases}$$

and the convection matrices $\tilde{\gamma}^l$ are defined by

$$\tilde{\gamma}^l_{ij} := \begin{cases} \gamma^l_{ij} & \text{if } i,j \in \sigma_1 \text{ or } i,j \in \sigma_2 \\ -\gamma^l_{ij} & \text{otherwise,} \end{cases}$$

for all $1 \le l \le n$ and $1 \le i,j \le k$. We observe that System (2.1) is order preserving with respect to $\succsim$ if and only if System (2.19) satisfies the positivity property. It follws as in Theorem 2.4 that the matrices a and γ^l are diagonal, $1 \le l \le n$, and the function F satisfies

$$F_i(y,z) \ge 0 \qquad \text{for all } y,z \in \mathbb{R}^k \text{ such that } w = y - z \in \mathbb{R}^k_+ \text{ and } w_i = 0,$$

for $1 \le i \le k$. Consequently, by the definition of the function F we obtain

$$\begin{cases} f_i(y) - f_i(z) \ge 0 & \text{if } i \in \sigma_1 \\ -\big(f_i(y) - f_i(z)\big) \ge 0 & \text{if } i \in \sigma_2 \end{cases} \qquad \text{for all } y,z \in \mathbb{R}^k \text{ such that } y_i = z_i,\ y \succsim z.$$

□

2.2.2. Quasi-Linear Systems

In this subsection we analyse the validity of comparison principles for quasi-linear systems. Owing to the stronger coupling of the equations we cannot deduce the results directly from the positivity criterion like in the semi-linear case. Indeed, allowing for comparison between arbitrary solutions, and not only with the zero solution, leads to essentially stronger conditions for the diffusion and convection matrices.

Theorem 2.6. *In addition to the hypothesis of Theorem 2.3 we assume that the partial derivatives of second order of the diagonal coefficient functions a_{ii} exist and belong to the space $L^\infty_{loc}(\mathbb{R}^k)$ for all $1 \le i \le k$. Then, the quasi-linear system (2.7) is order preserving with respect to $\preccurlyeq$ if and only if the matrices a and γ^l are diagonal, the coefficient functions a_{ii} and γ^l_{ii} depend on the component u_i of the solution only, for all $1 \le i \le k, 1 \le l \le n$, and the interaction term f is quasi-monotone.*

Proof. Let u_0 and v_0 be given initial data such that $u_0 \succcurlyeq v_0$. We show that the order $\succcurlyeq$ is preserved by the corresponding solutions u and v, if and only if a, γ and f fulfil the stated conditions. Defining the difference of the solutions $w := u - v$ it satisfies the system

$$\begin{aligned} \partial_t w &= a(u)\cdot\Delta u - a(v)\cdot\Delta v - \gamma(u)\cdot Du + \gamma(v)\cdot Dv + f(u) - f(v) && \Omega \times (0,T), \\ w|_{\partial\Omega} &= 0 && \partial\Omega \times [0,T], \quad (2.20) \\ w|_{t=0} &= w_0 && \Omega \times \{0\}, \end{aligned}$$

where the initial data $w_0 := u_0 - v_0 \in K^+$.
Necessity: We assume that the solutions u and v preserve the order relation $\succcurlyeq$, which is equivalent to the positivity property of System (2.20). Following the arguments in the first part of the proof of Theorem 2.3 leads to the scalar inequality

$$\Big\langle a(u_0) \cdot \Delta u_0 - a(v_0) \cdot \Delta v_0 - \gamma(u_0) \cdot Du_0 + \gamma(v_0) \cdot Dv_0 + f(u_0) - f(v_0), \varphi \Big\rangle_{L^2(\Omega;\mathbb{R}^k)} \geq 0,$$

where φ is an arbitrary function in K^+ which is orthogonal to w_0 in $L^2(\Omega;\mathbb{R}^k)$. Let the index $i \in \{1,\dots,k\}$ be fixed. Choosing smooth functions $u_0 = (\tilde{u}_1,\dots,\tilde{u}_k)$ and $v_0 = (\tilde{v}_1,\dots,\tilde{v}_k)$ such that $u_0 \succcurlyeq v_0$, $\tilde{u}_i = \tilde{v}_i$, and $\varphi = (0,\dots,\underbrace{\tilde{\varphi}}_{i},\dots,0)$, where $\tilde{\varphi} \in L^2(\Omega)$ is an arbitrary non-negative function, the functions w_0 and φ are orthogonal in $L^2(\Omega;\mathbb{R}^k)$. By the inequality above we obtain the pointwise estimate

$$\begin{aligned}&\sum_{j=1,j\neq i}^{k} (a_{ij}(u_0)\Delta\tilde{u}_j - a_{ij}(v_0)\Delta\tilde{v}_j) + (a_{ii}(u_0) - a_{ii}(v_0))\Delta\tilde{u}_i \\ &-\sum_{l=1}^{n}\sum_{j=1,j\neq i}^{k} (\gamma_{ij}^l(u_0)\partial_{x_l}\tilde{u}_j + \gamma_{ij}^l(v_0)\partial_{x_l}\tilde{v}_j) + \sum_{l=1}^{n}(\gamma_{ii}(u_0) - \gamma_{ii}(v_0))\partial_{x_l}\tilde{u}_i + f_i(u_0) - f_i(v_0) \geq 0\end{aligned} \tag{2.21}$$

in Ω. It follows that the coefficient functions a_{ij} and γ_{ij}^l are identically zero, for all $1 \leq l \leq n$, $1 \leq j \leq k$, $i \neq j$, and the diagonal coefficient functions satisfy

$$\begin{cases} a_{ii}(y) = a_{ii}(z) \\ \gamma_{ii}^l(y) = \gamma_{ii}^l(z) \end{cases} \qquad \text{for all } y, z \in \mathbb{R}^k \text{ such that } y \geq z, y_i = z_i,$$

where $1 \leq l \leq n$ (see Lemma 2.2 below). This implies that the functions a_{ii} and γ_{ii}^l depend on the component u_i of the solution only. Using these relations we conclude from Inequality (2.21) the monotonicity conditions for the interaction term,

$$f_i(y) \geq f_i(z) \qquad \text{for all } y, z \in \mathbb{R}^k \text{ such that } y \geq z, y_i = z_i,$$

where $1 \leq i \leq k$.
Sufficiency: Under the stated assumptions on a, γ and f, System (2.20) takes the form

$$\begin{aligned}\partial_t w_i &= a_{ii}(u)\Delta u_i - a_{ii}(v)\Delta v_i - \sum_{l=1}^{n}\big(\gamma_{ii}^l(u)\partial_{x_l}u_i - \gamma_{ii}^l(v)\partial_{x_l}v_i\big) + f_i(u) - f_i(v), \\ w|_{\partial\Omega} &= 0, \\ w|_{t=0} &= w_0,\end{aligned} \tag{2.22}$$

for $1 \leq i \leq k$, where the initial data $w_0 \in K^+$. To show the positivity property of this system we prove for the solution $w = w(\,\cdot\,, t; w_0)$ that the initial assumption $(w_0)_{i-} = 0$

implies $w_{i-} = 0$ for $t > 0$ and all $1 \leq i \leq k$. First, we assume that the quasi-monotonicity condition (2.17) is satisfied for all $y, z \in \mathbb{R}^k$ such that $y_i = z_i$. The assumptions on the functions a_{ii} lead to the equality

$$\begin{aligned} a_{ii}(v) &= \int_0^1 \frac{d}{ds} a_{ii}(v_1, \ldots, sv_i + (1-s)u_i, \ldots, v_k) ds + a_{ii}(v_1, \ldots, u_i, \ldots, v_k) \\ &= (v_i - u_i) \int_0^1 \partial_i a_{ii}(v_1, \ldots, sv_i + (1-s)u_i, \ldots, v_k) ds + a_{ii}(u_1, \ldots, u_i, \ldots, u_k) \\ &= (v_i - u_i) \tilde{A}_{ii}(u, v) + a_{ii}(u), \end{aligned}$$

for all $1 \leq i \leq k$, where the function $\tilde{A}_{ii}$ is bounded. Hence, we obtain

$$a_{ii}(u)\Delta u_i - a_{ii}(v)\Delta v_i = a_{ii}(u)\Delta w_i + w_i \tilde{A}_{ii}(u, v)\Delta v_i$$

and, using an analogous representation for the functions γ_{ii}^l follows

$$\begin{aligned} \gamma_{ii}^l(u)\partial_{x_l} u_i - \gamma_{ii}^l(v)\partial_{x_l} v_i &= \gamma_{ii}^l(u)\partial_{x_l} w_i + w_i \int_0^1 \partial_i \gamma_{ii}^l(v_1, \ldots, sv_i + (1-s)u_i, \ldots, v_k) ds \partial_{x_l} v_i \\ &= \gamma_{ii}^l(u)\partial_{x_l} w_i + w_i \tilde{\Gamma}_{ii}^l(u, v)\partial_{x_l} v_i, \end{aligned}$$

for all $1 \leq l \leq n$, $1 \leq i \leq k$, where the function $\tilde{\Gamma}_{ii}^l$ is bounded. Multiplying the i-th equation by the negative part w_{i-} and integrating over Ω yields

$$\begin{aligned} -\partial_t \|w_{i-}\|_{L^2(\Omega)}^2 &= -\langle a_{ii}(u)\Delta w_{i-}, w_{i-}\rangle_{L^2(\Omega)} - \langle w_{i-}\tilde{A}_{ii}(u, v)\Delta v_i, w_{i-}\rangle_{L^2(\Omega)} \\ &\quad + \sum_{l=1}^n \big(\langle \gamma_{ii}^l(u)\partial_{x_l} w_{i-}, w_{i-}\rangle_{L^2(\Omega)} + \langle w_{i-}\tilde{\Gamma}_{ii}^l(u, v)\partial_{x_l} v_i, w_{i-}\rangle_{L^2(\Omega)} \big) \\ &\quad + \langle f_i(u) - f_i(v), w_{i-}\rangle_{L^2(\Omega)}. \end{aligned}$$

Taking into account the homogeneous Dirichlet boundary conditions we derive for the first diffusion term

$$\begin{aligned} -\langle a_{ii}(u)\Delta w_{i-}, w_{i-}\rangle_{L^2(\Omega)} &= \langle \nabla \big(a_{ii}(u) w_{i-}\big), \nabla w_{i-}\rangle_{L^2(\Omega;\mathbb{R}^n)} \\ &= \langle a_{ii}(u)\nabla w_{i-}, \nabla w_{i-}\rangle_{L^2(\Omega;\mathbb{R}^n)} + \sum_{j=1}^k \langle w_{i-}\partial_j a_{ii}(u)\nabla u_j, \nabla w_{i-}\rangle_{L^2(\Omega;\mathbb{R}^n)}. \end{aligned}$$

We further estimate the second integral by

$$\Big| \sum_{j=1}^k \langle w_{i-}\partial_j a_{ii}(u)\nabla u_j, \nabla w_{i-}\rangle_{L^2(\Omega;\mathbb{R}^n)} \Big| \leq c_1 \sum_{l=1}^n \langle |\partial_{x_l} w_{i-}|, w_{i-}\rangle_{L^2(\Omega)},$$

for some constant $c_1 \geq 0$. Here, we used the hypothesis (2.10) and that the function $a_{ii} \in C^1(\mathbb{R}^k;\mathbb{R})$, $1 \leq i \leq k$. For the second diffusion term we obtain

$$\begin{aligned}\big| - \big\langle w_{i-}\tilde{A}_{ii}(u,v)\Delta v_i, w_{i-}\big\rangle_{L^2(\Omega)}\big| &= \big|2\big\langle w_{i-}\tilde{A}_{ii}(u,v)\nabla v_i, \nabla w_{i-}\big\rangle_{L^2(\Omega;\mathbb{R}^n)}\big| \\ &\quad + \big|\big\langle (w_{i-})^2\nabla(\tilde{A}_{ii}(u,v)), \nabla v_i\big\rangle_{L^2(\Omega;\mathbb{R}^n)}\big| \\ &\leq c_2\sum_{l=1}^{n}\langle|\partial_{x_l}w_{i-}|, w_{i-}\rangle_{L^2(\Omega)} + c_3\|w_{i-}\|^2_{L^2(\Omega)},\end{aligned}$$

for some constants $c_2, c_3 \geq 0$. Again, we used the hypothesis (2.10) and the regularity assumptions on the functions a_{ii}, $1 \leq i \leq k$. In a similar way we estimate the convection terms

$$\begin{aligned}&\big|\sum_{l=1}^{n}\big(\langle\gamma^l_{ii}(u)\partial_{x_l}w_{i-}, w_{i-}\rangle_{L^2(\Omega)} + \langle w_{i-}\tilde{\Gamma}^l_{ii}(u,v)\partial_{x_l}v_i, w_{i-}\rangle_{L^2(\Omega)}\big)\big| \\ &\leq c_4\sum_{l=1}^{n}\langle|\partial_{x_l}w_{i-}|, w_{i-}\rangle_{L^2(\Omega)} + c_5\|w_{i-}\|^2_{L^2(\Omega)},\end{aligned}$$

for some constants $c_4, c_5 \geq 0$. Finally, we observe

$$\begin{aligned}f_i(u) - f_i(v) =& f_i(u) - f_i(v_1,\dots,u_i,\dots,v_k) + w_i\int_0^1 \partial_i f_i(v_1,\dots,sv_i + (1-s)u_i,\dots,v_k)ds \\ =& f_i(u) - f_i(v_1,\dots,u_i,\dots,v_k) + w_iF_i(u,v).\end{aligned}$$

This representation yields an estimate for the remaining integral,

$$\begin{aligned}&-\langle f_i(u) - f_i(v), w_{i-}\rangle_{L^2(\Omega)} \\ =& -\langle f_i(u_1,\dots,u_k) - f_i(v_1,\dots,u_i,\dots,v_k), w_{i-}\rangle_{L^2(\Omega)} + \langle w_{i-}F_i(u,v), w_{i-}\rangle_{L^2(\Omega)} \\ \leq& \big|\langle w_{i-}F_i(u,v), w_{i-}\rangle_{L^2(\Omega)}\big| \leq c_6\ \|w_{i-}\|^2_{L^2(\Omega)},\end{aligned}$$

for some constant $c_6 \geq 0$. Here, we used our initial assumption that the quasi-monotonicity condition (2.17) is satisfies for all $y, z \in \mathbb{R}^k$ such that $y_i = z_i$. Summing up the terms and estimating all mixed integrals of the form $\sum_{l=1}^{n}\langle|\partial_{x_l}w_{i-}|, w_{i-}\rangle_{L^2(\Omega)}$ by Young's inequality we conclude

$$\partial_t\|w_{i-}\|^2_{L^2(\Omega)} \leq c_7\|w_{i-}\|^2_{L^2(\Omega)},$$

for some constant $c_7 \geq 0$. By Gronwall's lemma and the hypothesis $(w_0)_{i-} = 0$ follows $w_{i-} = 0$, which proves that System (2.22) satisfies the positivity property. Finally, System (2.22) satisfies the positivity property if and only if System (2.7) is order preserving with respect to the order relation $\preccurlyeq$.

It remains to justify our initial assumption on the interaction function. To this end we consider the modified system

$$\begin{aligned}\partial_t \hat{w}_i &= a_{ii}(u)\Delta u_i - a_{ii}(v)\Delta v_i - \sum_{l=1}^{n}\big(\gamma_{ii}^l(u)\partial_{x_l}u_i - \gamma_{ii}^l(v)\partial_{x_l}v_i\big) + \hat{F}_i(u,v),\\ \hat{w}|_{\partial\Omega} &= 0,\\ \hat{w}|_{t=0} &= w_0,\end{aligned}$$

where the function $\hat{F}:\mathbb{R}^k\times\mathbb{R}^k\to\mathbb{R}$ is given by

$$\hat{F}_i(y,z) := f_i(\tilde{y}_1,\dots,\tilde{y}_k) - f_i(\tilde{z}_1,\dots,y_i,\dots,\tilde{z}_k) + (y_i - z_i)F_i(y,z).$$

The function F_i was defined above and

$$\tilde{y}_j := \begin{cases} y_j & \text{if } y_j \geq z_j \\ -y_j & \text{if } y_j \leq z_j \end{cases} \qquad \tilde{z}_j := \begin{cases} z_j & \text{if } y_j \geq z_j \\ -z_j & \text{if } y_j \leq z_j \end{cases} \qquad \text{for all } 1\leq j\leq k,\ y,z\in\mathbb{R}^k.$$

Following the same arguments we conclude that the function $\hat{w}$ remains non-negative. However, if the solution $\hat{w}$ is non-negative it satisfies the original system (2.22) and, by the uniqueness of solutions follows $\hat{w} = w$. □

Lemma 2.2. *Let $j\neq i, 1\leq i,j\leq k$, and $1\leq l\leq n$. We assume the hypothesis of Theorem 2.6 are satisfied, and the pointwise inequality*

$$\begin{aligned}&\sum_{j=1,j\neq i}^{k}\big(a_{ij}(\tilde{u})\Delta\tilde{u}_j - a_{ij}(\tilde{v})\Delta\tilde{v}_j\big) + \big(a_{ii}(\tilde{u}) - a_{ii}(\tilde{v})\big)\Delta\tilde{u}_i\\ &-\sum_{l=1}^{n}\sum_{j=1,j\neq i}^{k}\big(\gamma_{ij}^l(\tilde{u})\partial_{x_l}\tilde{u}_j + \gamma_{ij}^l(\tilde{v})\partial_{x_l}\tilde{v}_j\big) + \sum_{l=1}^{n}\big(\gamma_{ii}(\tilde{u}) - \gamma_{ii}(\tilde{v})\big)\partial_{x_l}\tilde{u}_i + f_i(\tilde{u}) - f_i(\tilde{v}) \geq 0\end{aligned}$$

is valid in Ω for every initial data $\tilde{u} = (\tilde{u}_1,\dots,\tilde{u}_k)$ and $\tilde{v} = (\tilde{v}_1,\dots,\tilde{v}_k)$ such that $\tilde{u}\succcurlyeq\tilde{v}$ and $\tilde{u}_i = \tilde{v}_i$. Then, the coefficient functions a_{ij} and γ_{ij}^l are identically zero, and the diagonal coefficient functions a_{ii} and γ_{ii}^l depend on the component u_i of the solution only.

Proof. We argue by contradiction and suppose that the function a_{ij} is not identically zero. Then, there exists $y\in\mathbb{R}^k$ such that $a_{ij}(y)\neq 0$, and without loss of generality we can assume that $y_j > 0$. Let $x_0\in\Omega$ and $\mathcal{U}_{x_0}$ be an open neighbourhood of x_0 that is compactly contained in Ω.

If $a_{ij}(y) > 0$ we define the function $\tilde{u}:\Omega\to\mathbb{R}^k$ by

$$\tilde{u}_m(x) := \begin{cases} y_m & m\neq j \\ y_j e^{-\frac{1}{\epsilon}\|x-x_0\|^2} & m = j \end{cases} \qquad \text{for } x\in\mathcal{U}_{x_0},$$

where $1\leq m\leq k$, and extend it to a smooth function on Ω that vanishes on the boundary and such that the component $\tilde{u}_j$ is non-negative in Ω. Furthermore, we define

$$\tilde{v}_m(x) := \begin{cases} \tilde{u}_m(x) & m\neq j \\ 0 & m = j \end{cases} \qquad \text{for } x\in\Omega.$$

Certainly, the functions satisfy $\tilde{u} \succcurlyeq \tilde{v}$ and $\tilde{u}_i = \tilde{v}_i$. Computing the derivatives and evaluating the functions in the point $x_0 \in \Omega$ we observe

$$\begin{aligned}
\partial_{x_l}\tilde{u}_m(x_0) &= 0, \\
\Delta\tilde{u}_m(x_0) &= \begin{cases} -\frac{2}{\epsilon}y_j & m = j \\ 0 & m \neq j, \end{cases} \\
\partial_{x_l}\tilde{v}_m(x_0) &= \Delta\tilde{v}_m(x_0) = 0,
\end{aligned}$$

for all $1 \leq m \leq k, 1 \leq l \leq n$. Since $\epsilon > 0$ can be chosen arbitrarily small, the inequality (2.21) is violated in the point $x_0 \in \Omega$.

On the other hand, if $a_{ij}(y) < 0$, we define the function $\tilde{u} : \Omega \to \mathbb{R}^k$ by

$$\tilde{u}_m(x) := \begin{cases} y_m & m \neq j \\ y_j(e^{-\frac{1}{\epsilon}\|x-x_0\|^2} + \frac{1}{\epsilon^2}\|x - x_0\|^2) & m = j \end{cases} \qquad \text{for } x \in \mathcal{U}_{x_0},$$

where $1 \leq m \leq k$, and extend it to a smooth function on Ω that vanishes on the boundary and such that $\tilde{u}_j$ is non-negative in Ω. As before, if we define the function $\tilde{v}$ by

$$\tilde{v}_m(x) := \begin{cases} \tilde{u}_m(x) & m \neq j \\ 0 & m = j \end{cases} \qquad \text{for } x \in \Omega,$$

then the functions satisfy $\tilde{u} \succcurlyeq \tilde{v}$ and $\tilde{u}_i = \tilde{v}_i$. Evaluating the derivatives in the point x_0 we obtain

$$\begin{aligned}
\partial_{x_l}\tilde{u}_m(x_0) &= 0, \\
\Delta\tilde{u}_m(x_0) &= \begin{cases} y_j\frac{2}{\epsilon}(\frac{1}{\epsilon} - 1) & m = j \\ 0 & m \neq j, \end{cases} \\
\partial_{x_l}\tilde{v}_m(x_0) &= \Delta\tilde{v}_m(x_0) = 0,
\end{aligned}$$

for all $1 \leq m \leq k, 1 \leq l \leq n$. If we choose $\epsilon > 0$ sufficiently small the inequality (2.21) is violated in the point $x_0 \in \Omega$, which proves that the function a_{ij} is identically zero.

Next, we assume that there exist $y, z \in \mathbb{R}^k$ such that $y \geq z$, $y_i = z_i$, and $a_{ii}(y) \neq a_{ii}(z)$. Without loss of generality we assume that $y_i = z_i > 0$. If the difference $a_{ii}(y) - a_{ii}(z) > 0$ we define the function

$$\tilde{u}_m(x) := \begin{cases} y_m & m \neq i \\ y_i e^{-\frac{1}{\epsilon}\|x-x_0\|^2} & m = i \end{cases} \qquad \text{for } x \in \mathcal{U}_{x_0},$$

where $1 \leq m \leq k$, and extend it to a smooth function on Ω that vanishes on the boundary. Furthermore, we define the function $\tilde{v} : \Omega \to \mathbb{R}^k$ by

$$\tilde{v}_m(x) := \begin{cases} z_m & m \neq i \\ \tilde{u}_i(x) & m = i \end{cases} \qquad \text{for } x \in \mathcal{U}_{x_0},$$

where $1 \leq m \leq k$, and extended to a smooth function on Ω that vanishes on the boundary and such that the relations $\tilde{u} \succcurlyeq \tilde{v}$ and $\tilde{u}_i = \tilde{v}_i$ are valid. Computing the derivatives in the point x_0 we obtain

$$\partial_{x_l}\tilde{u}_m(x_0) = \partial_{x_l}\tilde{v}_m(x_0) = 0,$$
$$\Delta\tilde{u}_m(x_0) = \Delta\tilde{v}_m(x_0) = \begin{cases} -y_j\frac{2}{\epsilon} & m = j \\ 0 & m \neq j, \end{cases}$$

for all $1 \leq m \leq k, 1 \leq l \leq n$. Consequently, choosing $\epsilon > 0$ sufficiently small leads to a contradiction to Inequality (2.21) in the point x_0.

Similarly, if $a_{ii}(y) - a_{ii}(z) < 0$, we define

$$\tilde{u}_m(x) := \begin{cases} y_m & m \neq i \\ y_j(e^{-\frac{1}{\epsilon}\|x-x_0\|^2} + \frac{1}{\epsilon^2}\|x - x_0\|^2) & m = i \end{cases} \qquad \text{for } x \in \mathcal{U}_{x_0},$$

where $1 \leq m \leq k$, and extend it to a smooth function on Ω that vanishes on the boundary. We define the function $\tilde{v} : \Omega \to \mathbb{R}^k$ by

$$\tilde{v}_m(x) := \begin{cases} z_m & m \neq i \\ \tilde{u}_m(x) & m = i \end{cases} \qquad \text{for } x \in \mathcal{U}_{x_0},$$

where $1 \leq m \leq k$, and extend it to a smooth function on Ω that vanishes on the boundary and such that the relations $\tilde{u} \succcurlyeq \tilde{v}$ and $\tilde{v}_i = \tilde{u}_i$ are satisfied. If we compute the derivatives in the point x_0 we obtain

$$\partial_{x_l}\tilde{u}_m(x_0) = \partial_{x_l}\tilde{v}_m(x_0) = 0,$$
$$\Delta\tilde{u}_m(x_0) = \Delta\tilde{v}_m(x_0) = \begin{cases} y_j\frac{2}{\epsilon}(\frac{1}{\epsilon} - 1) & m = j \\ 0 & m \neq j, \end{cases}$$

for all $1 \leq m \leq k, 1 \leq l \leq n$. Choosing $\epsilon > 0$ sufficiently small this leads to a contradiction to the inequality (2.21) in the point x_0.

In a similar way follow the conditions for the convection matrices. Here, we may use the functions constructed in the second part of the proof of Lemma 2.1 to derive the conclusions. □

A direct consequence of Theorem 2.6 is a criterion for the validity of comparison principles with respect to an arbitrary order relation.

Theorem 2.7. *In addition to the hypothesis of Theorem 2.3 we assume that the partial derivatives of second order of the diagonal coefficient functions a_{ii} exist and belong to the space $L^\infty_{loc}(\mathbb{R}^k)$ for all $1 \leq i \leq k$. Then, System (2.7) is order preserving with respect to $\precsim$ if and only if the matrices a and γ^l are diagonal, the coefficient functions a_{ii} and γ^l_{ii} depend* *on the component u_i of the solution only for all $1 \le i \le k, 1 \le l \le n$, and the interaction term satisfies*

$$\begin{cases} f_i(y) \le f_i(z) & \text{if } i \in \sigma_1 \\ f_i(y) \ge f_i(z) & \text{if } i \in \sigma_2 \end{cases} \qquad \text{for all } y, z \in \mathbb{R}^k \text{ such that } y_i = z_i,\ y \precsim z,$$

for $1 \le i \le k$.

Proof. Let u_0 and v_0 be given initial data such that $u_0 \succsim v_0$. We show that the order is preserved by the corresponding solutions u and v, if and only if a, γ and f fulfill the stated conditions. Defining the function w by

$$w_i := \begin{cases} u_i - v_i & \text{if } i \in \sigma_1 \\ -(u_i - v_i) & \text{if } i \in \sigma_2 \end{cases}$$

it satisfies the system

$$\begin{aligned} \partial_t w &= \tilde{a}(u) \cdot \Delta u - \tilde{a}(v) \cdot \Delta v - \tilde{\gamma}(u) \cdot Du + \tilde{\gamma}(v) \cdot Dv + \tilde{F}(u, v), \\ w|_{\partial\Omega} &= 0, \\ w|_{t=0} &= w_0, \end{aligned} \tag{2.23}$$

where $w_0 \in K^+$, and the function $\tilde{F}$ is given by

$$\tilde{F}_i(u, v) := \begin{cases} f_i(u) - f_i(v) & i \in \sigma_1 \\ -(f_i(u) - f_i(v)) & i \in \sigma_2, \end{cases}$$

for $1 \le i \le k$. The coefficient functions of the diffusion matrix $\tilde{a}$ are given by

$$\tilde{a}_{ij}(u) := \begin{cases} a_{ij}(u) & \text{if } j \in \sigma_1 \\ -a_{ij}(u) & \text{if } j \in \sigma_2, \end{cases}$$

and the convection terms are defined by

$$\tilde{\gamma}^l_{ij}(u) := \begin{cases} \gamma^l_{ij}(u) & \text{if } j \in \sigma_1 \\ -\gamma^l_{ij}(u) & \text{if } j \in \sigma_2, \end{cases}$$

for all $1 \le i, j \le k$ and $1 \le l \le n$. In the proof of Theorem 2.6 we verified that System (2.23) satisfies the positivity property if and only if the matrices a and γ^l are diagonal, and the functions a_{ii} and γ^l_{ii} depend on the component u_i of the solution only, for $1 \le l \le n$, $1 \le i \le k$. Furthermore, the interaction term satisfies

$$\tilde{F}_i(y, z) \ge 0 \qquad \text{for all } y, z \in \mathbb{R}^k \text{ such that } y_i = z_i, y \ge z.$$

By the definition of the function $\tilde{F}$ follow the stated monotonicity conditions for the function f. Finally, the positivity property of System (2.23) is equivalent to the statement that System (2.7) is order-preserving with respect to $\precsim$, which concludes the proof of the Theorem. ☐

We recall Example 2.1 to illustrate the results and to compare the conditions that are necessary and sufficient for the positivity of solutions and the validity of comparison principles, respectively.

Example 2.2. *If the hypothesis of Theorem 2.4 are satisfied the semi-linear system (2.14) is order-preserving with respect to the order relation $\preccurlyeq$ if and only if the diffusion matrix is diagonal, and the interaction function is quasi-monotone,*

$$\begin{cases} f_1(w,y,z) \geq f_1(w,\tilde{y},\tilde{z}), \\ f_2(y,w,z) \geq f_2(\tilde{y},w,\tilde{z}), \\ f_3(y,z,w) \geq f_3(\tilde{y},\tilde{z},w) \end{cases} \qquad \text{for all } y \geq \tilde{y}, z \geq \tilde{z}, w \in \mathbb{R}. \tag{2.24}$$

Next, we assume the quasi-linear system (2.16) satisfies the assumptions of Theorem 2.6. Then, the system is order-preserving with respect to $\preccurlyeq$ if and only if the interaction function possesses the property (2.24), the diffusion matrix is diagonal and the coefficient functions a_{ii} depend on the component u_i of the solution only, for $1 \leq i \leq 3$. This implies that the quasi-linear system takes the form

$$\begin{aligned} \partial_t u_1 &= a_{11}(u_1)\,\Delta u_1 + f_1(u), \\ \partial_t u_2 &= a_{22}(u_2)\,\Delta u_2 + f_2(u), \\ \partial_t u_3 &= a_{33}(u_3)\,\Delta u_3 + f_3(u). \end{aligned}$$

We observe that in the semi-linear case the conditions on the diffusion terms are the same for the positivity property of the system and for the validity of comparison theorems. Quasi-linear systems that satisfy the positivity property may exhibit cross-diffusion terms of a particular form (see Example 2.1). However, if we allow for comparison between arbitrary solutions the diffusion matrix is necessarily diagonal, and the diagonal coefficient functions a_{ii} are functions of the component u_i of the solution only.

2.3. Generalizations and Applications

The proof of the positivity criteria can be generalized in various directions. For simplicity we formulated the results for quasi-linear and semi-linear systems of the form (2.1) and (2.7), respectively. We applied the method in [31] to an infinite system of semi-linear parabolic equations. Moreover, the results remain valid for equations in heterogeneous media, where the coefficient functions and the interaction function depend on the spatial variable, and for time-dependent interaction terms. The results can also be generalized for arbitrary elliptic differential operators of second order and for many degenerate parabolic systems. Before we apply the positivity criterion to verify the positivity property of mathematical models we extend Theorem 2.1 and Theorem 2.3 for different boundary conditions for the solution, which are often more relevant in applications. For further generalizations we refer to [31], [34] and Section 2.4.3.

2.3.1. Other Boundary Values

Inhomogeneous Dirichlet Boundary Conditions

In the sequel we assume the domain Ω is sufficiently regular such that the divergence theorem holds and the solutions are smooth solutions. Let $g : \partial\Omega \to \mathbb{R}^k$ be a continuous, componentwise non-negative function. We suppose the solutions of System (2.1) and System (2.7) satisfy the inhomogeneous Dirichlet boundary conditions

$$u\big|_{\partial\Omega} = g \qquad \text{on } \partial\Omega \times [0, T], \tag{2.25}$$

where $g = (g_1, \ldots, g_k)$. The non-negativity of the function g is a natural and necessary assumption if we require that the systems satisfy the positivity property. We show that the proof of the sufficiency of the stated conditions in Theoram 2.1 and Theorem 2.3 remains valid. The boundary conditions are used when we multiply the equations by the negative part of the solution and integrate the diffusion terms by parts. Assuming the Dirichlet boundary conditions (2.25) we obtain in the semi-linear case

$$\langle a_i \Delta u_{i-}, u_{i-} \rangle_{L^2(\Omega)} \;=\; \int_{\partial\Omega} a_i(\frac{\partial}{\partial\nu} u_{i-}) u_{i-} dS - a_i \|\nabla u_{i-}\|^2_{L^2(\Omega;\mathbb{R}^n)} = -a_i \|\nabla u_{i-}\|^2_{L^2(\Omega;\mathbb{R}^n)},$$

for $1 \le i \le k$, where $\frac{\partial}{\partial\nu}$ denotes the outward-pointing unit normal derivative on the boundary and $\int_{\partial\Omega} dS$ the boundary integral. Since the solution takes non-negative values on the boundary, $u_{i-}|_{\partial\Omega} = g_{i-} = 0$, the boundary integral is zero. Consequently, the proof of Theorem 2.1 continues as in the case of homogeneous Dirichlet conditions.

The same applies to quasi-linear systems, where we obtain two additional boundary integrals, one for the diagonal coefficient functions

$$\begin{aligned}\langle a_{ii}(u) \Delta u_{i-}, u_{i-} \rangle_{L^2(\Omega)} &= \int_{\partial\Omega} a_{ii}(u)(\frac{\partial}{\partial\nu} u_{i-}) u_{i-} dS - \langle a_{ii}(u) \nabla u_{i-}, \nabla u_{i-} \rangle_{L^2(\Omega;\mathbb{R}^n)} \\ &= -\langle a_{ii}(u) \nabla u_{i-}, \nabla u_{i-} \rangle_{L^2(\Omega;\mathbb{R}^n)},\end{aligned}$$

and one for the cross-diffusion terms

$$\begin{aligned}\sum_{j=1, j\neq i}^{k} \left\langle u_{i-} A_{ij}(u) \Delta u_j, u_{i-} \right\rangle_{L^2(\Omega)} &= \sum_{j=1, j\neq i}^{k} \int_{\partial\Omega} A_{ij}(u)(\frac{\partial}{\partial\nu} u_j)(u_{i-})^2 dS \\ &\quad - \sum_{j=1, j\neq i}^{k} \left\langle \nabla(A_{ij}(u)(u_{i-})^2), \nabla u_j \right\rangle_{L^2(\Omega;\mathbb{R}^n)} \\ &= - \sum_{j=1, j\neq i}^{k} \left\langle \nabla(A_{ij}(u)(u_{i-})^2), \nabla u_j \right\rangle_{L^2(\Omega;\mathbb{R}^n)},\end{aligned}$$

for $1 \le i \le k$ (see the proof of Theorem 2.3). Since the the function g is non-negative, $u_{i-}\mid_{\partial\Omega} = g_{i-} = 0$, both boundary integrals vanish, and the proof of Theorem 2.3 remains unchanged.

Homogeneous Neumann Boundary Conditions

Next, we assume the solutions of System (2.1) or System (2.7) satisfy homogeneous Neumann boundary conditions,

$$\frac{\partial}{\partial \nu} u\big|_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega \times [0,T]. \tag{2.26}$$

This reflects the situation that the boundary of the domain is impermeable and substrates cannot leave the system. Using the representation $\frac{\partial}{\partial \nu} u_i = \frac{\partial}{\partial \nu} u_{i+} - \frac{\partial}{\partial \nu} u_{i-}$ the boundary conditions (2.26) imply that

$$\frac{\partial}{\partial \nu} u_{i+}\big|_{\partial\Omega} = \frac{\partial}{\partial \nu} u_{i-}\big|_{\partial\Omega} \qquad \text{on } \partial\Omega \times [0,T],$$

for $1 \leq i \leq k$. For the boundary integral in the semi-linear case follows

$$\int_{\partial\Omega} a_i(\frac{\partial}{\partial \nu} u_{i-}) u_{i-} dS \;\; = \;\; \int_{\partial\Omega} a_i(\frac{\partial}{\partial \nu} u_{i+}) u_{i-} dS = 0,$$

since the supports of the positive part and the negative part of u_i are disjoint. The same applies to the first boundary integral that we obtain for quasi-linear systems,

$$\int_{\partial\Omega} a_{ii}(u)(\frac{\partial}{\partial \nu} u_{i-}) u_{i-} dS \;\; = \;\; \int_{\partial\Omega} a_{ii}(u)(\frac{\partial}{\partial \nu} u_{i+}) u_{i-} dS = 0.$$

Moreover, the boundary conditions (2.26) immediately imply that the boundary integrals for the cross-diffusion terms vanish,

$$\sum_{j=1, j\neq i}^{k} \int_{\partial\Omega} A_{ij}(u)(\frac{\partial}{\partial \nu} u_j)(u_{i-})^2 dS = 0,$$

for all $1 \leq i \leq k$.

Certainly, the boundary values for the components of the solutions need not necessarily be of the same type. We could impose homogeneous Neumann boundary conditions for some components of the solution and non-negative Dirichlet boundary conditions for the other components, which leaves the arguments unchanged.

2.3.2. Positivity Property of Deterministic Models

In this section we present examples of quasi-linear models that satisfy the positivity property. For applications formulated as semi-linear systems of reaction-diffusion equations we refer to [29], [66] and [69].

Chemotaxis

The Keller-Segel Model describes the dynamics of a population in a spatial domain Ω following the gradient of a chemotactic agent, which is produced by the population itself. The following system of parabolic PDEs is based on the Keller-Segel model and was analysed in [44],

$$\begin{aligned}
\partial_t u &= \Delta u - \chi \nabla \cdot (u \nabla v) && \Omega \times (0,\infty),\\
\partial_t v &= \Delta v - (u-1) && \Omega \times (0,\infty),\\
\frac{\partial u}{\partial \nu} &= \frac{\partial u}{\partial \nu} = 0 && \partial\Omega \times (0,\infty),\\
u(\cdot,0) &= u_0, \quad v(\cdot,0) = v_0 && \Omega \times \{0\},
\end{aligned} \tag{2.27}$$

where u denotes the population density and v the concentration of the chemotactic agent. Furthermore, $\chi > 0$ is a positive constant and $\Omega \subset \mathbb{R}^2$ is a bounded domain with C^1-boundary. The initial data $u_0, v_0 \in C^1(\Omega;\mathbb{R})$, are non-negative and satisfy the boundary conditions.

Rewriting the first equation in the form

$$\partial_t u = \Delta u - \chi(\nabla u \cdot \nabla v + u \Delta v)$$

we note that the cross-diffusion term is of the form required by Theorem 2.3. The proof of Theorem 2.3 extends to systems of the form (2.27). Indeed, assuming boundedness of the solutions and their derivatives we multiply the first equation by the negative part u_-, integrate over Ω and obtain

$$-\frac{d}{dt}\|u_-\|^2_{L^2(\Omega)} = \|\nabla u_-\|^2_{L^2(\Omega;\mathbb{R}^2)} - \chi \langle u_- \nabla v, \nabla u_- \rangle_{L^2(\Omega;\mathbb{R}^2)}.$$

Young's inequality implies that for every $\epsilon > 0$ there exists a constant $C_\epsilon \geq 0$ such that

$$\left|\chi \langle u_- \nabla v, \nabla u_- \rangle_{L^2(\Omega;\mathbb{R}^2)}\right| \leq \epsilon \|\nabla u_-\|^2_{L^2(\Omega;\mathbb{R}^2)} + C_\epsilon \|u_-\|^2_{L^2(\Omega)}.$$

Consequently, if we choose $\epsilon > 0$ sufficiently small follows

$$\frac{d}{dt}\|u_-\|^2_{L^2(\Omega)} \leq c\|u_-\|^2_{L^2(\Omega)},$$

for some constant $c \geq 0$, and the proof of Theorem 2.3 stays valid. Since in Section 2.3.1 we extended the proof for homogeneous Neumann boundary conditions we conclude that the density u remains non-negative.

Furthermore, if the population density u is bounded by 1 the interaction function $f(u,v) := -(u-1)$ in the second equation satisfies

$$f(u,0) = -(u-1) \geq 0 \qquad \text{for} \quad 0 \leq u \leq 1.$$

In this case, Theorem 2.3 implies that System (2.27) satisfies the positivity property.

Prototype Biofilm Growth Model

Next, we illustrate that the method applied in the proof Theorem 2.3 can also be used to verify the positivity property of degenerate parabolic equations such as the biofilm models discussed in Chapter 1.

We recall that the solution of the prototype biofilm growth model (1.1) is obtained as the limit of the solutions (S_ϵ, M_ϵ) of the non-degenerate approximations

$$\begin{aligned}
\partial_t S_\epsilon &= d_S \Delta S_\epsilon - k_1 \frac{S_\epsilon M_\epsilon}{k_2 + S_\epsilon} && \Omega \times (0,T), \\
\partial_t M_\epsilon &= d \nabla \cdot (D_{\epsilon,M}(M_\epsilon) \nabla M_\epsilon) + k_3 \frac{S_\epsilon M_\epsilon}{k_2 + S_\epsilon} - k_4 M_\epsilon && \Omega \times (0,T), \\
M_\epsilon|_{\partial\Omega} &= 0, \qquad S_\epsilon|_{\partial\Omega} = 1 && \partial\Omega \times [0,T], \\
M_\epsilon|_{t=0} &= M_0, \qquad S_\epsilon|_{t=0} = S_0 && \Omega \times \{0\},
\end{aligned} \tag{2.28}$$

where the regularized diffusion coefficient is given by

$$D_{\epsilon,M}(z) := \begin{cases} \epsilon^a & z < 0 \\ \frac{(z+\epsilon)^a}{(1-z)^b} & 0 \leq z \leq 1 - \epsilon \\ \frac{1}{\epsilon^b} & z \geq 1 - \epsilon. \end{cases}$$

We assume the initial data (S_0, M_0) are smooth, non-negative and satisfy the compatibility conditions. It was shown that for every sufficiently small $\epsilon > 0$ the auxiliary system (2.28) possesses a unique solution (S_ϵ, M_ϵ), and the solutions S_ϵ and M_ϵ are uniformly bounded by 1 (see Section 1.1.2).

The positivity of the substrate concentration follows from Theorem 2.1, since no cross-diffusion terms are present, and the interaction function satisfies $f_1(0, z) = 0$ for all $z \in \mathbb{R}$, where

$$f_1(y, z) := -k_1 \frac{yz}{k_2 + y} \qquad (y, z) \in \mathbb{R}^2.$$

Furthermore, the reaction function in the second equation fulfils the positivity condition since $f_2(y, 0) = 0$ for all $y \in \mathbb{R}$, where

$$f_2(y, z) := k_3 \frac{yz}{k_2 + y} - k_4 z \qquad (y, z) \in \mathbb{R}^2.$$

If we formally multiply the equation for the biomass density by the negative part $M_{\epsilon-}$ and integrate over Ω we obtain

$$-\partial_t \|M_{\epsilon-}\|^2_{L^2(\Omega)} = d \langle D_{\epsilon,M}(M) \nabla M_{\epsilon-}, \nabla M_{\epsilon-} \rangle_{L^2(\Omega;\mathbb{R}^n)} - \langle k_3 \frac{S_\epsilon M_{\epsilon-}}{k_2 + S_\epsilon}, M_{\epsilon-} \rangle_{L^2(\Omega)} + k_4 \|M_{\epsilon-}\|^2_{L^2(\Omega)}.$$

Since the regularized diffusion coefficient $D_{\epsilon,M}(M_\epsilon)$ is strictly positive in $\Omega \times (0,T)$ follows the estimate

$$\partial_t \|M_{\epsilon-}\|^2_{L^2(\Omega)} \leq C \|M_{\epsilon-}\|^2_{L^2(\Omega)},$$

for some constant $C \geq 0$, which implies the non-negativity of the biomass fraction M_ϵ.

The solutions (S_ϵ, M_ϵ) of the non-degenerate approximations converge to the solution (S, M) of the original system (1.1). It follows that the solution (S, M) remains non-negative and we conclude that the biofilm model satisfies the positivity property.

Quorum-Sensing in Biofilm Communities

Finally, we analyse the positivity property of the quorum-sensing model. The model was studied in Section 1.3 and is formulated as the system of quasi-linear reaction-diffusion equations

$$\begin{aligned}
\partial_t S &= d_S \Delta S - k_1 \frac{SM}{k_2 + S} && \Omega \times (0,T), \\
\partial_t A &= d_A \Delta A - \gamma A + \alpha X + (\alpha + \beta) Y && \Omega \times (0,T), \\
\partial_t X &= d \nabla \cdot (D_M(M) \nabla X) + k_3 \frac{XS}{k_2 + S} - k_4 X - k_5 A^m X + k_5 Y && \Omega \times (0,T), \\
\partial_t Y &= d \nabla \cdot (D_M(M) \nabla Y) + k_3 \frac{YS}{k_2 + S} - k_4 Y + k_5 A^m X - k_5 Y && \Omega \times (0,T),
\end{aligned} \tag{2.29}$$

where the biomass diffusion coefficient is defined by

$$D_M(M) = \frac{M^a}{(1-M)^b},$$

and $M := X + Y$ denotes the volume fraction of the total biomass. The solutions take the initial and boundary values

$$\begin{aligned}
&X|_{\partial\Omega} = 0, && Y|_{\partial\Omega} = 0, && A|_{\partial\Omega} = 0, && S|_{\partial\Omega} = 1 && \partial\Omega \times [0,T], \\
&X|_{t=0} = X_0, && Y|_{t=0} = Y_0, && S|_{t=0} = S_0, && A|_{t=0} = A_0 && \Omega \times \{0\}.
\end{aligned}$$

The solution of System (2.29) is obtained as the limit of the solutions of the non-degenerate approximations (see Section 1.3.3). To verify the positivity property of the model it suffices to check the positivity condition for the reaction terms. Indeed, the arguments applied in the previous example for the mono-species model justify that the method in the proof of Theorem 2.3 extends to the non-degenerate approximations for the quorum-sensing model. The interaction function $f : \mathbb{R}^4 \to \mathbb{R}^4$ is given by

$$\begin{aligned}
f_1(u,v,w,z) &= -k_1 \frac{u(w+z)}{k_2 + u}, \\
f_2(u,v,w,z) &= -\gamma v + \alpha w + (\alpha + \beta) z, \\
f_3(u,v,w,z) &= k_3 \frac{uw}{k_2 + u} - k_4 w - k_5 v^m w + k_5 z, \\
f_4(u,v,w,z) &= k_3 \frac{uz}{k_2 + u} - k_4 z + k_5 v^m w - k_5 z,
\end{aligned}$$

for $(u,v,w,z) \in \mathbb{R}^4$, and we easily verify the positivity condition,

$$\begin{aligned}
&f_1(0,v,w,z) = 0 && v \geq 0, w \geq 0, z \geq 0,\\
&f_2(u,0,w,z) = \alpha w + (\alpha+\beta) z \ \geq 0, && u \geq 0, w \geq 0, z \geq 0,\\
&f_3(u,v,0,z) = k_5 z \geq 0, && u \geq 0, v \ \geq 0, z \geq 0,\\
&f_4(u,v,w,0) = k_5 v^m w \geq 0, && u \geq 0, v \ \geq 0, w \geq 0.
\end{aligned}$$

2.4. Stochastic Perturbations of Deterministic Systems

In the following sections we analyse the positivity property of parabolic systems under stochastic perturbations. We are interested in an explicit characterization of the the class of stochastic perturbations that preserve the positivity property of deterministic systems since it allows to specify admissible models in applications where the solutions describe non-negative quantities.

In the context of stochastic differential equations we use the triple $(\Omega, \mathcal{F}, \mathbb{P})$ to denote the probability space. This should not lead to confusion with previous notations, where we used the symbol Ω to represent the spatial domain. Whenever we address stochastic PDEs we denote the spatial domain by $\mathcal{O}$ instead of Ω.

2.4.1. Motivation: Additive Versus Multiplicative Noise

To motivate our results we discuss the positivity of solutions in two simple examples of stochastic ODEs. Let $\{W_t,\, t \in \mathbb{R}_+\} = \{W_t(\omega), t \in \mathbb{R}_+\}_{\omega\in\Omega}$ be a scalar real-valued Wiener process, $(\Omega, \mathcal{F}, \mathbb{P})$ be the canonical Wiener space and dW_t denote the corresponding Itô differential. To indicate Stratonovich's interpretation of stochastic differential equations we use the notation $\circ\, dW_t$ (see [14] or [56]).

Let $u : \mathbb{R}_+ \to \mathbb{R}$ be the solution of the deterministic ODE

$$\begin{aligned}
\frac{du}{dt} &= 0,\\
u|_{t=0} &= u_0,
\end{aligned} \tag{2.30}$$

where $u_0 \in \mathbb{R}_+$. The initial value problem certainly satisfies the positivity property. Indeed, the solution of (2.30) is the constant function $u(t;u_0) = u_0$, which is non-negative for $t > 0$ if and only if the initial data u_0 is non-negative. However, if we perturb the system by *additive noise*,

$$\begin{aligned}
du\ &= 0\, dt + dW_t,\\
u|_{t=0} &= u_0,
\end{aligned} \tag{2.31}$$

the positivity is not preserved by the solutions of the perturbed stochastic system.

Proposition 2.1. *We assume the initial data* $u_0 \in \mathbb{R}_+$*. Then, there exists* $t^* > 0$ *such that the solution* u *of System (2.31) satisfies* $u(t^*, \omega; u_0) < 0$ *for almost every* $\omega \in \Omega$*. This is also valid for Stratonovic's interpretation of the stochastic differential equation (2.31).*

Proof. For additive noise Itô's and Stratonovich's interpretation of the stochastic differential equation (2.31) lead to the same integral equation (see [56], Section 5.1). The solution of the stochastic differential equation is the process

$$u(t, \omega; u_0) = u(0) + \int_0^t dW_s = u_0 + W_t(\omega) - W_0(\omega) = u_0 + W_t(\omega),$$

where we used that the Wiener process satisfies $W_0(\omega) = 0$, $\omega \in \Omega$. The law of iterated logarithm states that

$$\liminf_{t\to\infty} \frac{W_t}{\sqrt{2t \log\log t}} = -1 \qquad \mathbb{P}\text{-almost surely}$$

(see [56], Theorem 5.1.2). Consequently, there exists an increasing sequence $\{t_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}_+$, $\lim_{n\to\infty} t_n = \infty$, such that

$$\lim_{n\to\infty} \frac{W_{t_n}}{\sqrt{2t_n \log\log t_n}} = -1 \qquad \mathbb{P}\text{-almost surely.}$$

For sufficiently large $N_0 \in \mathbb{N}$ follows

$$W_{t_n}(\omega) < -\frac{1}{2}\sqrt{2t_n \log\log t_n} \qquad \text{for } \mathbb{P}\text{-almost every } \omega \in \Omega,$$

for all $n \geq N_0$, which proves that the sequence $W_{t_n} \to -\infty$ $\mathbb{P}$-almost surely when n tends to infinity. We conclude that the solution satisfies $u(t_n, \omega; u_0) < 0$ for $\mathbb{P}$-almost every $\omega \in \Omega$ if n is sufficiently large. □

Instead of additive noise we consider the perturbation of the initial value problem (2.30) by a linear, *multiplicative noise* of the form

$$\begin{aligned} du &= 0\,dt + \alpha u \circ dW_t, \\ u|_{t=0} &= u_0, \end{aligned} \tag{2.32}$$

where the constant $\alpha \in \mathbb{R}$. For convenience we use Stratonovich's interpretation of the stochastic differential equation since in this case ordinary chain rule formulas apply under a change of variables (see [56], Section 3.3). The solution $u : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ of the Statonovich differential equation (2.32) is the stochastic process

$$u(t, \omega; u_0) = u_0 e^{\alpha W_t(\omega)}.$$

We observe that, independent of the sign of $\alpha \in \mathbb{R}$, the stochastic initial value problem (2.32) satisfies the positivity property. The same is valid when we interpret the stochastic differential equation (2.32) in the sense of Itô.

Proposition 2.2. *The stochastic problem (2.32) satisfies the positivity property independent of the choice of Itô's or Stratonovich's interpretation.*

Proof. It remains to prove the statement for Itô's interpretation of the stochastic differential equation. There is an explicit formula relating the integral equations obtained through Itô's and Stratonovich's interpretation (see [56], Section 3.3). Namely, the Itô equation

$$du = 0\,dt + \alpha u\,dW_t,$$

is equivalent to the following Stratonovich equation

$$du = (0 - \frac{\alpha^2}{2}u)dt + \alpha u \circ dW_t,$$

which can be solved explicitly. The transformation $v(t,\omega) := e^{-\alpha W_t(\omega)}u(t,\omega)$ leads to the ordinary differential equation

$$\begin{aligned} dv &= (-\frac{\alpha^2}{2}v)dt, \\ v|_{t=0} &= u_0. \end{aligned}$$

Its solution is the function $v(t,\omega) = u_0 e^{-\frac{\alpha^2}{2}t}$, and we obtain as solution of the original problem

$$u(t,\omega;u_0) = u_0 e^{-(\frac{\alpha^2}{2}t - \alpha W_t(\omega))}.$$

If the initial data u_0 is non-negative, the solution remains non-negative for $t > 0$ independent of the sign of α. This shows that System (2.32) satisfies the positivity property for both Itô's and Stratonovich's interpretation. □

This first example illustrates that additive noise destroys the positivity property of deterministic equations while the positivity property is preserved under perturbations by a linear, multiplicative noise.

Next, we analyse systems of stochastic ODEs. Since additive noise destroys the positivity property we consider perturbations by a linear, multiplicative noise in each component. Let $T > 0$ and $(u,v,w) : [0,T] \times \Omega \to \mathbb{R}^3$ be the solution of the system of Stratonovic equations

$$\begin{aligned} du &= f_1(u,v,w)dt + \alpha_1 u \circ dW_t, \\ dv &= f_2(u,v,w)dt + \alpha_2 v \circ dW_t, \\ dw &= f_3(u,v,w)dt + \alpha_3 w \circ dW_t, \\ (u,v,w)|_{t=0} &= (u_0,v_0,w_0), \end{aligned} \tag{2.33}$$

where the constants $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, the initial data $(u_0,v_0,w_0) \in \mathbb{R}^3_+$, and the interaction function $f = (f_1,f_2,f_3) : \mathbb{R}^3 \to \mathbb{R}^3$ is continuously differentiable. We apply an analogous transformation as in the previous example. To be more precise, defining the functions

$$\begin{aligned} &\tilde{u}(t,\omega) := e^{-\alpha_1 W_t(\omega)}u(t,\omega), \quad \tilde{v}(t,\omega) := e^{-\alpha_2 W_t(\omega)}v(t,\omega), \\ &\tilde{w}(t,\omega) := e^{-\alpha_3 W_t(\omega)}w(t,\omega) \end{aligned}$$

leads to the family of random equations

$$\begin{aligned}\frac{d\tilde{u}}{dt} &= e^{-\alpha_1 W_t} f_1(e^{\alpha_1 W_t}\tilde{u}, e^{\alpha_2 W_t}\tilde{v}, e^{\alpha_3 W_t}\tilde{w}),\\ \frac{d\tilde{v}}{dt} &= e^{-\alpha_2 W_t} f_2(e^{\alpha_1 W_t}\tilde{u}, e^{\alpha_2 W_t}\tilde{v}, e^{\alpha_3 W_t}\tilde{w}),\\ \frac{d\tilde{w}}{dt} &= e^{-\alpha_3 W_t} f_3(e^{\alpha_1 W_t}\tilde{u}, e^{\alpha_2 W_t}\tilde{v}, e^{\alpha_3 W_t}\tilde{w}).\end{aligned} \tag{2.34}$$

Random equations can be interpreted pathwise and studied by deterministic methods. The deterministic positivity criteria can be generalized for non-autonomous equations (see Section 2.4.3), and we conclude that for fixed $\omega \in \Omega$ the solutions of System (2.34) preserve positivity if and only if the interaction terms

$$F_i^\omega(t,x,y,z) := e^{-\alpha_i W_t(\omega)} f_i(e^{\alpha_1 W_t(\omega)}x, e^{\alpha_2 W_t(\omega)}y, e^{\alpha_3 W_t(\omega)}z) \qquad i = 1,2,3,$$

satisfy

$$F_1^\omega(t,0,y,z) \geq 0, \quad F_2^\omega(t,x,0,z) \geq 0, \quad F_3^\omega(t,x,y,0) \geq 0$$

for all $t \in [0,T]$ and $x,y,z \geq 0$. We observe that this is the case if and only if the original reaction function satisfies the positivity condition,

$$f_1(0,y,z) \geq 0, \quad f_2(x,0,z) \geq 0, \quad f_3(x,y,0) \geq 0 \qquad \text{for all } x,y,z \geq 0.$$

Consequently, the positivity property of the unperturbed deterministic system is equivalent to the positivity property of the random system (2.34) and of the system of Stratonovic equations (2.33).

Finally, we discuss the positivity property of the stochastic system (2.33) when it is interpreted in the sense of Itô. The system of Itô equations is equivalent to the system of Stratonovich equations

$$\begin{aligned}du &= (f_1(u,v,w) - \frac{\alpha_1^2}{2}u)dt + \alpha_1 u \circ dW_t,\\ dv &= (f_2(u,v,w) - \frac{\alpha_2^2}{2}v)dt + \alpha_2 v \circ dW_t,\\ dw &= (f_3(u,v,w) - \frac{\alpha_3^2}{2}w)dt + \alpha_3 w \circ dW_t,\\ (u,v,w)|_{t=0} &= (u_0,v_0,w_0),\end{aligned} \tag{2.35}$$

and the previous transformations lead to the random system

$$\begin{aligned}\frac{d\tilde{u}}{dt} &= -\frac{\alpha_1^2}{2}\tilde{u} + e^{-\alpha_1 W_t} f_1(e^{\alpha_1 W_t}\tilde{u}, e^{\alpha_2 W_t}\tilde{v}, e^{\alpha_3 W_t}\tilde{w}),\\ \frac{d\tilde{v}}{dt} &= -\frac{\alpha_2^2}{2}\tilde{v} + e^{-\alpha_2 W_t} f_2(e^{\alpha_1 W_t}\tilde{u}, e^{\alpha_2 W_t}\tilde{v}, e^{\alpha_3 W_t}\tilde{w}),\\ \frac{d\tilde{u}}{dt} &= -\frac{\alpha_3^2}{2}\tilde{u} + e^{-\alpha_3 W_t} f_3(e^{\alpha_1 W_t}\tilde{u}, e^{\alpha_2 W_t}\tilde{v}, e^{\alpha_3 W_t}\tilde{w}).\end{aligned} \tag{2.36}$$

By the deterministic positivity criterion for non-autonomous equations we conclude that for fixed $\omega \in \Omega$ the solutions of System (2.36) preserve positivity if and only if the interaction term satisfies

$$\tilde{F}_1^\omega(t,0,y,z) \geq 0, \quad \tilde{F}_2^\omega(t,x,0,z) \geq 0, \quad \tilde{F}_3^\omega(t,x,y,0) \geq 0$$

for all $t \in [0,T]$ and $x,y,z \geq 0$. Here, the modified interaction function $\tilde{F}$ is defined by

$$\tilde{F}_1^\omega(t,x,y,z) := F_1^\omega(t,x,y,z) - \frac{\alpha_1^2}{2}x, \quad \tilde{F}_2^\omega(t,x,y,z) := F_2^\omega(t,x,y,z) - \frac{\alpha_2^2}{2}y,$$
$$\tilde{F}_3^\omega(t,x,y,z) := F_3^\omega(t,x,y,z) - \frac{\alpha_3^2}{2}z.$$

Owing to the particular form of the additional term we obtain when we apply Itô's interpretation the interaction function $\tilde{F}^\omega = (\tilde{F}_1^\omega, \tilde{F}_2^\omega, \tilde{F}_3^\omega)$ satisfies the positivity condition if and only if the function $F^\omega = (F_1^\omega, F_2^\omega, F_3^\omega)$ fulfils the positivity condition. This in turn is equivalent to the positivity condition for the interaction function $f = (f_1, f_2, f_3)$ of the unperturbed deterministic system. We summarize our discussion in the following proposition.

Proposition 2.3. *The stochastic system of Stratonovich equations (2.33) satisfies the positivity property if and only if the corresponding system of Itô equations fulfils the positivity property. Furthermore, this is valid if and only if the unperturbed deterministic system satisfies the positivity property.*

The positivity condition for the interaction function f, which is necessary and sufficient for the positivity property of the unperturbed deterministic system, is equivalent to the positivity condition for the functions F^ω and $\tilde{F}^\omega$. Consequently, stochastic perturbations by a linear, multiplicative noise do not affect the qualitative behaviour of solutions with respect to positivity, independent of the choice of interpretation. This is valid owing to the explicit relation between the equations corresponding to Itô's and Stratonovich's interpretation, and the particular transformation that leads to the family of random equations. The conditions for the positivity property of the unperturbed deterministic system are invariant under all these transformations.

In general, the qualitative behaviour of solutions of stochastic differential equations depends on the choice of interpretation. We refer to [56], Example 5.1.1, which illustrates that the asymptotic behaviour of solutions of stochastic differential equations can be essentially different for Itô's and for Stratonovic's interpretation.

Our aim is to study stochastic perturbations of systems of parabolic PDEs, that we interpret in the sense of Itô. To analyse the general case, where we cannot apply such a simple transformation which directly leads to systems of random PDEs, we consider smooth random approximations of the stochastic systems. An approximation theorem obtained in [15] for stochastic perturbations of semi-linear parabolic equations allows us to construct a family of random equations such that its solutions converge in expectation to the solution of the stochastic system. We formulate necessary and sufficient conditions for the positivity property and for the validity of comparison theorems for the family of random approximations and prove that the property is preserved by the stochastic system, independent of the choice of Itô's or Stratonovich's interpretation. In other words, for the class of stochastic systems we consider the qualitative behaviour of solutions regarding positivity and the validity of comparison principles is independent of the choice of interpretation.

2.4.2. Stochastic Perturbations of Semi-Linear Parabolic Systems

We consider systems of semi-linear parabolic equations under stochastic perturbations of the form

$$du_l(x,t) = \Big(- \sum_{i=1}^{m} A_i^l(x,D)u_i(x,t) + f^l(x,t,u(x,t))\Big)dt + \sum_{i=1}^{\infty} q_i g_i^l(x,t,u(x,t))dW_t^i, \tag{2.37}$$

where $1 \leq l \leq m$, $m \in \mathbb{N}$, and the solution $u = (u_1, \ldots, u_m)$ is a vector-valued process. Furthermore, $x \in \mathcal{O}$ denotes the spatial variable and $t \in [0,T]$ the time variable, where $\mathcal{O} \subset \mathbb{R}^n$, $n \in \mathbb{N}$, is a bounded domain and $T > 0$. The linear differential operators A_i^l are of second order and elliptic. Moreover, $\{W_t^i,\, t \in \mathbb{R}_+\}_{i\in\mathbb{N}}$ is a family of mutually independent standard scalar Wiener processes on the canonical Wiener space $(\Omega, \mathcal{F}, \mathbb{P})$, and dW_t^i denotes the corresponding Itô differential. The non-negative parameters q_i are normalization factors. We assume the solution satisfies the boundary conditions

$$\big(\delta_l u_l + (1-\delta_l)\frac{\partial}{\partial \nu} u_l\big)\big|_{\partial\mathcal{O}} = 0 \qquad \partial\mathcal{O} \times [0,T],$$

where $\frac{\partial}{\partial \nu}$ denotes the outward normal derivative on the boundary $\partial\mathcal{O}$ and $\delta_l \in \{0,1\}$, for $1 \leq l \leq m$. Finally, the initial values of the solution are given by

$$u|_{t=0} = u_0 \qquad \mathcal{O} \times \{0\},$$

where the deterministic function $u_0 : \mathcal{O} \to \mathbb{R}^m$.

We denote the system of Itô equations (2.37) by (A, f, g), and the corresponding unperturbed deterministic system by $(A, f, 0)$. We aim at deriving explicit conditions on the coefficient functions of the differential operator A and the functions f and g to ensure that System (2.37) satisfies the positivity property. To this end we consider smooth random approximations of the stochastic problem since random equations can be interpreted pathwise and allow to apply deterministic methods. The random approximations, however, lead to a family of non-autonomous parabolic equations. In the next section we therefore generalize the deterministic positivity criterion for semi-linear systems (Theorem 2.1) for non-autonomous parabolic systems of the form $(A, f, 0)$. We show that the deterministic system $(A, f, 0)$ satisfies the positivity property if and only if the differential operators are diagonal, and the interaction function satisfies the non-autonomous positivity property.

Consequently, it suffices to consider stochastic systems with *diagonal* differential operators of the form

$$du_l(x,t) = \big(-A^l(x,D)u_l(x,t) + f^l(x,t,u(x,t))\big)\,dt + \sum_{i=1}^{\infty} q_i g_i^l(x,t,u(x,t))dW_t^i, \quad (2.38)$$

where $1 \le l \le m$. We denote the system of stochastic PDEs (2.38) by (f,g), and the corresponding unperturbed deterministic system by $(f,0)$. To analyse the stochastic problem (f,g) with diagonal differential operators we apply a Wong-Zakaï type approximation theorem obtained in [15], which yields a family of random approximations $(f_{\epsilon,\omega},0)$ for the stochastic system. The solutions of the random approximations do not converge to the solution of the original system, but to the solution of a modified stochastic system. Therefore, we first construct an auxiliary stochastic system (F,g) such that the solutions of the corresponding random approximations $(F_{\epsilon,\omega},0)$ converge to the solution of our original problem (f,g). We apply the deterministic results to derive explicit necessary and sufficient conditions for the positivity property and for the validity of comparison theorems for the random systems $(F_{\epsilon,\omega},0)$. Moreover, the conditions are preserved when taking the limit and are invariant under the transformation relating the original and the modified system. This observation allows us to formulate explicit conditions on the stochastic perturbation g and interaction function f that ensure the positivity property or the validity of comparison theorems for the stochastic system (f,g). Furthermore, the solution of the modified stochastic system coincides with the solution of the original stochastic system when we interpret it in the sense of Stratonovich. Our criteria are therefore independent of the choice of interpretation.

2.4.3. A Positivity Criterion for Non-Autonomous Deterministic Systems

Since the Wong-Zakaï approximations lead to a family of non-autonomous parabolic systems we generalize the deterministic positivity criterion for non-autonomous interaction functions and moreover, we allow for arbitrary linear elliptic differential operators of second order. We consider semi-linear parabolic systems of the form

$$\partial_t u_l(x,t) = -\sum_{i=1}^{m} A_i^l(x,D)u_i(x,t) + f^l(x,t,u(x,t)) \qquad \mathcal{O}\times(0,T), \qquad (2.39)$$

where $1 \le l \le m$, $\mathcal{O} \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\mathcal{O}$ and the function $u = (u_1,\dots,u_m)$. The solution satisfies the boundary and initial conditions

$$\big(\delta_l u_l + (1-\delta_l)\frac{\partial}{\partial\nu}u_l\big)\big|_{\partial\mathcal{O}} = 0 \qquad \partial\mathcal{O}\times[0,T], \qquad (2.40)$$

$$u|_{t=0} = u_0 \qquad \mathcal{O}\times\{0\}, \qquad (2.41)$$

where $1 \le l \le m$, and the initial data $u_0 : \mathcal{O} \to \mathbb{R}^m$ satisfies the compatibility conditions. The differential operators $A_i^l(x, D)$ are defined by

$$A_i^l(x,D) = -\sum_{k,j=1}^{n} a_{kj}^{il}(x)\partial_{x_k}\partial_{x_j} + \sum_{k=1}^{n} a_k^{il}(x)\partial_{x_k} \qquad \text{for } x \in \mathcal{O},\ i,l = 1,\dots,m,$$

where we omit the zero-order terms. Analysing the positivity property of semi-linear parabolic systems it seems more natural to absorb these terms in the interaction function f.

We assume the coefficient functions satisfy $a_{kj}^{il} = a_{jk}^{il}$, and the operators are uniformly elliptic,

$$\mu|\zeta|^2 \le \sum_{k,j=1}^{n} a_{kj}^{il}(x)\zeta_k\zeta_j \qquad \text{for all } x \in \mathcal{O}, \zeta \in \mathbb{R}^n,\ i,l = 1,\dots,m. \tag{2.42}$$

Moreover, all coefficient functions of the operator A are continuously differentiable and bounded in the domain $\mathcal{O}$.

The interaction functions f^l are continuously differentiable with respect to u and we suppose that

$$f^l \text{ and } \partial_u f^l \text{ are bounded on } \mathcal{O} \times [0,T] \times \mathbb{R}^m \text{ for bounded values of } u, \tag{2.43}$$

where $1 \le l \le m$.

Finally, we assume that for every initial data $u_0 \in K^+$ there exists a unique solution of System (2.39), and for $t > 0$ the solution satisfies L^∞-estimates,

$$u(\cdot, t; u_0) \in L^\infty(\mathcal{O}; \mathbb{R}^m) \qquad \text{for } t \in [0, t_{max}], \tag{2.44}$$

where $[0, t_{max}]$ denotes the maximal existence interval of the solution.

The following theorem generalizes Theorem 2.1 for semi-linear parabolic systems of the form (2.39). The proof of the sufficiency of the stated conditions also follows from the results by H. Amann (see [2] and [15]), but the method we apply in our proof is different.

Theorem 2.8. *Let the hypothesis (2.42) -(2.44) be satisfied and the initial data $u_0 \in K^+$ be smooth and fulfil the compatibility conditions. Then, System (2.39) satisfies the positivity property if and only if for all $1 \le j,k \le n$ the matrices $\left(a_{kj}^{il}\right)_{1\le i,l\le m}$ and $\left(a_k^{il}\right)_{1\le i,l\le m}$ are diagonal, and the components of the reaction term satisfy*

$$f^l(x,t,y) \ge 0, \qquad \textit{for } x \in \mathcal{O}, t \in [0,t_{max}] \textit{ and } y \in \mathbb{R}_+^m \textit{ such that } y_l = 0, \tag{2.45}$$

for all $1 \le l \le m$.

Proof. We rewrite System (2.39) in the form

$$\partial_t u(x,t) \;=\; \sum_{k,j=1}^{n} a_{kj}(x)\partial_{x_k}\partial_{x_j}u(x,t) - \sum_{k=1}^{n} a_k(x)\partial_{x_k}u(x,t) + f(x,t,u(x,t)), \tag{2.46}$$

where the matrices a_{kj} and a_k are defined by

$$a_{kj}(x) = \begin{pmatrix} a_{kj}^{11}(x) & \cdots & a_{kj}^{1m}(x) \\ \vdots & \ddots & \vdots \\ a_{kj}^{m1}(x) & \cdots & a_{kj}^{mm}(x) \end{pmatrix}, \quad a_k(x) = \begin{pmatrix} a_k^{11}(x) & \cdots & a_k^{1m}(x) \\ \vdots & \ddots & \vdots \\ a_k^{m1}(x) & \cdots & a_k^{mm}(x) \end{pmatrix},$$

and all derivatives in System (2.46) are applied componentwise to the vector-valued function $u = (u_1, \ldots, u_m)$.

Necessity: We assume the solution $u(\cdot, \cdot; u_0) : \mathcal{O} \times [0, t_{max}] \to \mathbb{R}^m$ corresponding to initial data $u_0 \in K^+$ remains non-negative for $t > 0$ and prove the necessity of the stated conditions. To this end we follow the arguments in the proof of Theorem 2.3. Taking smooth initial data u_0 and an arbitrary function $v \in K^+$, that is orthogonal to u_0 in $L^2(\mathcal{O}; \mathbb{R}^m)$, we conclude

$$\begin{aligned} \big\langle \partial_t u|_{t=0}, v \big\rangle_{L^2(\mathcal{O};\mathbb{R}^m)} =& \big\langle \sum_{k,j=1}^{n} a_{kj}(\cdot)\partial_{x_k}\partial_{x_j} u_0 - \sum_{k=1}^{n} a_k(\cdot)\partial_{x_k} u_0, v \big\rangle_{L^2(\mathcal{O};\mathbb{R}^m)} \\ &+ \big\langle f(\cdot, 0, u_0), v \big\rangle_{L^2(\mathcal{O};\mathbb{R}^m)} \geq 0. \end{aligned} \tag{2.47}$$

Let $i, l \in \{1, \ldots, m\}$ such that $i \neq l$. If we choose the functions $u_0 = (0, \ldots, \underbrace{\tilde{u}}_{l}, \ldots, 0)$ and $v = (0, \ldots, \underbrace{\tilde{v}}_{i}, \ldots, 0)$ with $u_0, v \in K^+$ follows the scalar inequality

$$\int_{\mathcal{O}} \Big(\sum_{k,j=1}^{n} a_{kj}^{il}(x)\partial_{x_k}\partial_{x_j}\tilde{u}(x) - \sum_{k=1}^{n} a_k^{il}(x)\partial_{x_k}\tilde{u}(x) + f^i(x, 0, u_0(x)) \Big) \tilde{v}(x)\ dx \geq 0.$$

Since the inequality holds for an arbitrary non-negative function $\tilde{v} \in L^2(\mathcal{O})$, we obtain the pointwise estimate

$$\sum_{k,j=1}^{n} a_{kj}^{il}(x)\partial_{x_k}\partial_{x_j}\tilde{u}(x) - \sum_{k=1}^{n} a_k^{il}(x)\partial_{x_k}\tilde{u}(x) + f^i(x, 0, u_0(x)) \geq 0$$

almost everywhere in $\mathcal{O}$. This implies that the coefficient functions of the differential operator are zero,

$$a_{kj}^{il}(x) = a_k^{il}(x) = 0 \qquad x \in \mathcal{O},$$

for $1 \leq i, l \leq m$, $i \neq l$ (see the proof of Lemma 2.1), and shows that the matrices a_{kj} and a_k are necessarily diagonal for all $1 \leq j, k \leq n$. Next, we choose the functions $u_0 = (u_1, \ldots, \underbrace{0}_{i}, \ldots, u_m)$ and $v = (0, \ldots, \underbrace{\tilde{v}}_{i}, \ldots, 0)$ such that $u_0, v \in K^+$ and conclude from Inequality (2.47)

$$f^i(x, 0, \tilde{u}_1, \ldots, \underbrace{0}_{i}, \ldots \tilde{u}_m) \geq 0 \qquad \text{for } \tilde{u}_1, \ldots \tilde{u}_m \geq 0, x \in \mathcal{O},$$

for all $1 \le i \le m$.

It remains to show that this property is satisfied for $t > 0$. If the solution remains strictly positive for $t > 0$ we do not obtain an additional assumption. Otherwise, if for some time $t_0 > 0$ the solution approaches a boundary point of the positive cone K^+, then there exists an index $1 \le i \le m$ such that the component $u_i|_{t=t_0} = 0$. Choosing the function $v = (0, \dots, \underbrace{\tilde{v}}_{i}, \dots, 0)$ with arbitrary non-negative $\tilde{v}$, it is orthogonal to the solution $u(\,\cdot\,, t_0; u_0)$ in $L^2(\mathcal{O}; \mathbb{R}^m)$. Consequently, we obtain

$$\begin{aligned}\big\langle \partial_t u|_{t=t_0}, v\big\rangle_{L^2(\mathcal{O};\mathbb{R}^m)} &= \big\langle \lim_{t\to(t_0)_+} \frac{u_i(\,\cdot\,,t;u_0) - u_i(\,\cdot\,,t_0;u_0)}{t-t_0}, \tilde{v}\big\rangle_{L^2(\mathcal{O})} \\ &= \lim_{t\to(t_0)_+} \big\langle \frac{u_i(\,\cdot\,,t;u_0)}{t-t_0}, \tilde{v}\big\rangle_{L^2(\mathcal{O})} - \lim_{t\to(t_0)_+} \big\langle \frac{u_i(\,\cdot\,,t_0;u_0)}{t-t_0}, \tilde{v}\big\rangle_{L^2(\mathcal{O})} \\ &= \lim_{t\to(t_0)_+} \big\langle \frac{u_i(\,\cdot\,,t;u_0)}{t-t_0}, \tilde{v}\big\rangle_{L^2(\mathcal{O})} \ge 0,\end{aligned}$$

where $t \to (t_0)_+$ denotes the limit from the right. We used that at time $t = t_0$ the component $u_i|_{t=t_0} = 0$ and the positivity of the solution $u(\,\cdot\,, t; u_0) \in K^+$ for $t > 0$. On the other hand, u is a solution of the initial value problem, which implies

$$\begin{aligned}\big\langle \partial_t u|_{t=t_0}, v\big\rangle_{L^2(\mathcal{O};\mathbb{R}^m)} &= \big\langle \sum_{k,j=1}^{n} a_{kj}(\,\cdot\,)\partial_{x_k}\partial_{x_j} u|_{t=t_0} - \sum_{k=1}^{n} a_k(\,\cdot\,)\partial_{x_k} u|_{t=t_0}, v\big\rangle_{L^2(\mathcal{O};\mathbb{R}^m)} \\ &\quad + \big\langle f(\,\cdot\,, t_0, u|_{t=t_0}), v\big\rangle_{L^2(\mathcal{O};\mathbb{R}^m)} \ge 0.\end{aligned}$$

We argue as before and use the diagonality of the matrices a_{kj} and a_k to obtain the pointwise inequality

$$f^i(x, t_0, \tilde{u}_1|_{t=t_0}, \dots, \underbrace{0}_{i}, \dots, \tilde{u}_m|_{t=t_0}) \ge 0$$

almost everywhere in $\mathcal{O}$. This implies the positivity condition for the interaction function and concludes the proof of the necessity of the stated conditions.

Sufficiency: We assume the stated conditions are satisfied and denote the diagonal coefficient functions of the differential operators by $a^l_{kj} := a^{ll}_{kj}, a^l_k := a^{ll}_k$, for $1 \le k, j \le n$. The system of equations then takes the form

$$\partial_t u_l(x,t) = \sum_{k,j=1}^{n} a^l_{kj}(x)\partial_{x_k}\partial_{x_j} u_l(x,t) - \sum_{k=1}^{n} a^l_k(x)\partial_{x_k} u_l(x,t) + f^l(x,t,u(x,t)), \qquad (2.48)$$

where $1 \le l \le m$. We follow the strategy in the proof of Theorem 2.3 and multiply the l-th equation of System (2.48) by the negative part $u_{l-} = (u(\cdot,t))_{l-}$. Integrating over the domain $\mathcal{O}$ we obtain

$$-\frac{1}{2}\partial_t\|u_{l-}\|^2_{L^2(\mathcal{O})} = \big\langle \partial_t u_l, u_{l-}\big\rangle_{L^2(\mathcal{O})} = \big\langle \sum_{k,j=1}^n a^l_{kj}(\,\cdot\,)\partial_{x_k}\partial_{x_j}u_l, u_{l-}\big\rangle_{L^2(\mathcal{O})}$$
$$-\big\langle \sum_{k=1}^n a^l_k(\,\cdot\,)\partial_{x_k}u_l, u_{l-}\big\rangle_{L^2(\mathcal{O})} + \big\langle f^l(\,\cdot\,,t,u), u_{l-}\big\rangle_{L^2(\mathcal{O})}.$$

Without loss of generality we assume that all components of the solution satisfy homogeneous Dirichlet boundary conditions (for homogeneous Neumann boundary conditions we refer to Section 2.3.1). For the first term on the right hand side of the equation we obtain

$$\big\langle \sum_{k,j=1}^n a^l_{kj}(\,\cdot\,)\partial_{x_k}\partial_{x_j}u_l, u_{l-}\big\rangle_{L^2(\mathcal{O})} = -\big\langle \sum_{k,j=1}^n a^l_{kj}(\,\cdot\,)\partial_{x_k}\partial_{x_j}u_{l-}, u_{l-}\big\rangle_{L^2(\mathcal{O})}$$
$$= \big\langle \sum_{k,j=1}^n a^l_{kj}(\,\cdot\,)\partial_{x_j}u_{l-}, \partial_{x_k}u_{l-}\big\rangle_{L^2(\mathcal{O})} + \big\langle \sum_{k,j=1}^n \partial_{x_k}a^l_{kj}(\,\cdot\,)\partial_{x_j}u_{l-}, u_{l-}\big\rangle_{L^2(\mathcal{O})}.$$

By Young's inequality follow the estimates

$$\Big|\big\langle \sum_{k,j=1}^n \partial_{x_k}a^l_{kj}(\,\cdot\,)\partial_{x_j}u_{l-}, u_{l-}\big\rangle_{L^2(\mathcal{O})}\Big| \le \epsilon\|\nabla u_{l-}\|^2_{L^2(\mathcal{O};\mathbb{R}^m)} + C_{\epsilon,1}\|u_{l-}\|^2_{L^2(\mathcal{O})},$$

for some constant $C_{\epsilon,1} \ge 0$, and

$$\Big|\big\langle \sum_{k=1}^n a^l_k(\,\cdot\,)\partial_{x_k}u_{l-}, u_{l-}\big\rangle_{L^2(\mathcal{O})}\Big| \le \epsilon\|\nabla u_{l-}\|^2_{L^2(\mathcal{O};\mathbb{R}^m)} + C_{\epsilon,2}\|u_{l-}\|^2_{L^2(\mathcal{O})},$$

for some $C_{\epsilon,2} \ge 0$. Like in the proof of Theorem 2.3 we represent the interaction term by

$$f^l(x,t,u) = f^l(x,t,u_1,\dots,\underbrace{0}_{l},\dots,u_m) + u_lF^l(x,t,u),$$

for $l = 1,\dots,m$, where the functions F^l are bounded. Then, using the uniform parabolicity assumption (2.42) and collecting all terms we obtain

$$\frac{1}{2}\partial_t\|u_{l-}\|^2_{L^2(\mathcal{O})} + \mu\|\nabla u_{l-}\|^2_{L^2(\mathcal{O};\mathbb{R}^m)} \le \frac{1}{2}\partial_t\|u_{l-}\|^2_{L^2(\mathcal{O})} + \big\langle \sum_{k,j=1}^n a^l_{kj}(\cdot)\partial_{x_j}u_{l-}, \partial_{x_k}u_{l-}\big\rangle_{L^2(\mathcal{O})}$$
$$\le\Big|\big\langle \sum_{k,j=1}^n \partial_{x_k}a^l_{kj}(\cdot)\partial_{x_j}u_{l-}, u_{l-}\big\rangle_{L^2(\mathcal{O})} + \big\langle \sum_{k=1}^n a^l_k(\cdot)\partial_{x_k}u_{l-}, u_{l-}\big\rangle_{L^2(\mathcal{O})}\Big|$$
$$-\big\langle f^l(\cdot,t,u_1,\dots,\underbrace{0}_{l},\dots,u_m), u_{l-}\big\rangle_{L^2(\mathcal{O})} + \big\langle u_{l-}, F^l(\cdot,t,u)u_{l-}\big\rangle_{L^2(\mathcal{O})}$$
$$\le 2\epsilon\|\nabla u_{l-}\|^2_{L^2(\mathcal{O};\mathbb{R}^m)} + (C_{\epsilon,1} + C_{\epsilon,2} + C)\|u_{l-}\|^2_{L^2(\mathcal{O})} - \big\langle f^l(\cdot,t,u_1,\dots,0,\dots,u_m), u_{l-}\big\rangle_{L^2(\mathcal{O})},$$

for some constant $C \geq 0$. We assume the interaction function satisfies the positivity condition (2.5) for all $y \in \mathbb{R}^m$ such that $y_l = 0$. Choosing $\epsilon > 0$ sufficiently small then follows the estimate

$$\partial_t \|u_{l-}\|^2_{L^2(\mathcal{O})} \leq c\|u_{l-}\|^2_{L^2(\mathcal{O})},$$

for some constant $c \geq 0$. By Gronwall's Lemma and the initial assumption $(u_0)_{l-} = 0$ we conclude that $u_{l-}(\cdot, t; u_0) = 0$ almost everywhere in $\mathcal{O}$ for $t > 0$. The assumption on the interaction function can be justified as in the proof of Theorem 2.3. □

This generalizes the semi-linear positivity criterion for systems with arbitrary elliptic second order differential operators and non-autonomous interaction functions. Since the deterministic system (2.46) satisfies the positivity property if and only if the differential operators are diagonal it suffices to consider stochastic perturbations of semi-linear systems of the form (2.38).

2.5. Stochastic Systems: Positivity Property and Comparison Principles

To study the stochastic system (f, g) we construct a family of random equations such that its solutions converge in expectation to the solution of the stochastic problem. We use the deterministic results to formulate criteria for the positivity property and validity of comparison theorems for the family of random PDEs, which then imply the corresponding property of the stochastic system.

2.5.1. Random Approximations of Stochastic Systems

E. Wong and M. Zakaï ([73],[74]) studied the relation between ordinary and stochastic differential equations and introduced a smooth approximation of the Brownian motion to approximate stochastic integrals by ordinary integrals. In this way, they obtain an approximation of the stochastic differential equation by a family of random differential equations. However, when the smoothing parameter tends to zero the random solutions do not converge to the solution of the original stochastic problem, but to the solution of a modified one. The appearing correction term is called *Wong-Zakaï correction term.* The Wong-Zakaï approximation theorem was generalized in various directions. In this section, we briefly recall the main result in [15] about a Wong-Zakaï-type approximation theorem for stochastic systems of semi-linear parabolic PDEs, which is applicable for the class of systems we consider.

In the sequel we analyse stochastic systems (f, g) of the form (2.38) and assume that the functions $g^l_j : \mathcal{O} \times [0, T] \times \mathbb{R} \to \mathbb{R}$ are continuously differentiable and are bounded for bounded values of the solution, where $j \in \mathbb{N}, l = 1, \ldots, m$.

Predictable Approximation of the Wiener Process

A general notion of a smooth predictable approximation of the Wiener process is given in [15], Definition 4.1. In the following, we will take the main example in this article as a definition (see [15], Proposition 4.2).

Let $\{W_t, t \in \mathbb{R}_+\}$ be a standard scalar Wiener process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t, t \in \mathbb{R}_+\}$. The ***smooth predictable approximation*** of the Wiener process $\{W_t, t \in \mathbb{R}_+\}$ is the family of random processes $\{W_\epsilon(t), t \in \mathbb{R}_+\}_{\epsilon>0}$ defined by

$$W_\epsilon(t) = \int_0^\infty \phi_\epsilon(t-\tau) W_\tau d\tau,$$

where $\phi_\epsilon(t) = \frac{1}{\epsilon}\phi(\frac{t}{\epsilon})$, and $\phi : \mathbb{R} \to \mathbb{R}$ is a function with the properties

$$\phi \in C^1(\mathbb{R}), \quad \text{supp}\phi \subset [0,1], \quad \int_0^1 \phi(t)dt = 1.$$

We will need the following result (see [15], p.1442), which states that the derivative of the smooth predictable approximation W_ϵ, denoted by $\dot{W}_\epsilon$, can be written as a stochastic integral of the form

$$\dot{W}_\epsilon(t) \quad = \quad \int_{t-\epsilon}^t \phi_\epsilon(t-\tau) dW_\tau, \quad t \geq \epsilon. \tag{2.49}$$

As a consequence, the process $\dot{W}_\epsilon$ is Gaussian.

Smoothing of Itô's Problem and Random Systems

Using the family of smooth predictable approximations $\{W_\epsilon^j(t), t \in \mathbb{R}_+\}_{\epsilon>0, j\in\mathbb{N}}$ of the family of Wiener processes $\{W_t^j, t \in \mathbb{R}_+\}_{j\in\mathbb{N}}$ the predictable smoothing of Itô's problem (2.38) is the family of random equations

$$du_l(x,t) = \big(-A^l(x,D)u_l(x,t) + f^l(x,t,u(x,t))\big)dt + \big(\sum_{j=1}^\infty q_j g_j(x,t,u(x,t)) \dot{W}_\epsilon^j(t)\big)dt, \tag{2.50}$$

where $1 \leq l \leq m$. In our notation, this leads to the following definition.

Definition 2.6. *The* ***smooth random approximation of the stochastic system*** (f, g) *with respect to the smooth predictable approximation* $\{W_\epsilon(t), t \in \mathbb{R}_+\}_{\epsilon>0}$ *is the family of random PDEs* $(f_{\epsilon,\omega}, 0)$*, where*

$$f_{\epsilon,\omega}^l(x,t,u(x,t)) = f^l(x,t,u(x,t)) + \sum_{j=1}^\infty q_j g_j^l(x,t,u(x,t)) \dot{W}_\epsilon^j(t) \qquad \epsilon > 0.$$

Wong-Zakaï Approximation Theorem

Following the approach in [15] we consider mild solutions of the stochastic system of PDEs (f, g).

Definition 2.7. *A random function $u(x,t,\omega) = (u_1(x,t,\omega), \dots, u_m(x,t,\omega))$ is called a* ***mild solution*** *of the stochastic problem (f, g) in the space $H^1_B(\mathcal{O};\mathbb{R}^m)$ on the interval $[0,T]$, if $u \in C([0,T]; L^2(\mathcal{O}\times\Omega))$ is a predictable process such that*

$$\int_0^T E \parallel u(t) \parallel^2_{H^1(\mathcal{O};\mathbb{R}^m)} dt < \infty,$$

where $u(t) = u(\cdot, t, \cdot)$, and satisfies the integral equation

$$u(t) = S(t)u_0 + \int_0^t S(t-\tau)f(\tau, u(\tau))d\tau + \sum_{j=1}^{\infty} q_j \int_0^t S(t-\tau)g_j(\tau, u(\tau))dW^j_\tau, \qquad (2.51)$$

where we assume that all integrals in (2.51) exist.

In Definition 2.7 the operator E denotes the mean value operator on $(\Omega, \mathcal{F}, \mathbb{P})$ and the family $\{S(t),\ t \in \mathbb{R}_+\}$ the analytic semigroup in $L^2(\mathcal{O};\mathbb{R}^m)$ generated by the linear operator A with domain

$$H^2_B(\mathcal{O};\mathbb{R}^m) := \{u \in H^2(\mathcal{O};\mathbb{R}^m) \,|\, u \text{ satisfies the boundary conditions (2.40)}\}.$$

Here, B indicates the boundary operator and

$$H^k(\mathcal{O};\mathbb{R}^m) := \{u \in L^2(\mathcal{O}) \,|\, D^\alpha u_l \in L^2(\mathcal{O}) \text{ for all } |\alpha| \le k, l = 1, \dots, m\}.$$

For further details we refer to [15] and [2].

Definition 2.8. *Let (f, g) be a stochastic system of PDEs and u be its mild solution. We say that the mild solutions u_ϵ of a family of random PDEs $(F_{\epsilon,\omega}, 0)$* ***converge to the mild solution of the stochastic system*** *(f, g) if*

$$\lim_{\epsilon\to 0} \int_0^T E \parallel u(t) - u_\epsilon(t) \parallel^2_{H^1(\mathcal{O};\mathbb{R}^m)} dt = 0.$$

The main result in [15] is the following approximation theorem (Theorem 4.3, [15]).

Theorem 2.9. *Assume that the stated assumptions on the operator A and the functions f and g are satisfied. Moreover, let $\sum_{j=1}^\infty q_j < \infty$, the initial data $u_0 \in C^2(\mathcal{O};\mathbb{R}^m)$ satisfy the compatibility conditions, be $\mathcal{F}_0$-measurable and $E\|u_0\|^r_{C^2(\mathcal{O};\mathbb{R}^m)} < \infty$ for some $r > 8$. We assume the associated system of random PDEs $(f_{\epsilon,\omega}, 0)$ has a mild solution u_ϵ belonging to the class $C([0,T]; L^r(\Omega; X_{\alpha,p}))$ for all $0 \le \alpha < 1$ and $p > 1$, and for this solution there exists a constant $c \ge 0$ independent of $\epsilon > 0$ such that*

$$\sup_{t\in[0,T]} E \parallel u_\epsilon \parallel^r_{L^p(\mathcal{O};\mathbb{R}^m)} \le c \qquad \text{for all } p > 1.$$

Then, the mild solutions u_ϵ converge to a solution u_{cor} of the corrected stochastic system of PDEs (f_{cor}, g) when ϵ tends to zero, where

$$f^l_{\mathrm{cor}} = f^l + \frac{1}{2}\sum_{j=1}^{\infty} q_j^2 \sum_{i=1}^{m} g^i_j \frac{\partial g^l_j}{\partial u_i} \qquad \textit{for } l = 1, \dots m.$$

The spaces $X_{\alpha,p}$ denote the fractional power spaces associated to the operator A. For further details we refer to [15].

2.5.2. A Positivity Criterion for Stochastic Systems

We aim at analysing the qualitative behaviour of the solutions of the stochastic system (f, g). Hence, in the sequel we assume that a unique solution of the stochastic initial value problem (2.38) exists, and the solutions of the random approximations converge to the solution of the modified stochastic system (f_{cor}, g) (see Theorem 2.9). Sufficient conditions for the existence and uniqueness of solutions can be found in the article [15]. Since the solutions of the random approximations do not converge to the solution of the original system we construct an auxiliary stochastic system as follows:

- Let (F, g) be a given stochastic system. The corresponding family of random approximations $(F_{\epsilon,\omega}, 0)$, $\epsilon > 0$, $\omega \in \Omega$ is explicit, depends on the definition of the smooth approximation $\{W_\epsilon(t), t \in \mathbb{R}_+\}$ of the Wiener process $\{W_t, t \in \mathbb{R}_+\}$, and is given by

$$F^l_{\epsilon,\omega} = F^l + \sum_{j=1}^{\infty} q_j g^l_j \dot{W}^j_\epsilon \qquad l = 1, \dots, m.$$

- Theorem 2.9 states that the solutions of the random systems converge in expectation to the solution of the modified stochastic system (F_{cor}, g), where

$$F^l_{cor} = F^l + \frac{1}{2}\sum_{j=1}^{\infty} q_j^2 \sum_{i=1}^{m} g^i_j \frac{\partial g^l_j}{\partial u_i} \qquad l = 1, \dots, m.$$

- To analyse the stochastic system (f, g) we therefore construct an auxiliary system (F, g) such that the solutions of the associated system of random PDEs $(F_{\epsilon,\omega}, 0)$ converge to the solutions of our original system (f, g).

- We then use the deterministic positivity criterion to derive necessary and sufficient conditions for the positivity property of the family of random approximations $(F_{\epsilon,\omega}, 0)$. Finally, we show that this property is preserved by the transformation relating the original system and the modified system and by passing to the limit when the smoothing parameter ϵ goes to zero.

Let (f, g) be a system of stochastic PDEs. If we define the auxiliary stochastic system (F, g) by

$$F^l = f^l - \frac{1}{2}\sum_{j=1}^{\infty} q_j^2 \left(g_j^1 \frac{\partial g_j^l}{\partial u_1} + \cdots + g_j^m \frac{\partial g_j^l}{u_m} \right) \qquad l = 1, \ldots, m,$$

the solutions of the associated family of random PDEs $(F_{\epsilon,\omega}, 0)$ converge in expectation to the solution of the original stochastic system (f, g).

Motivated by Theorem 2.8 we extend the definition of the positivity condition for non-autonomous problems.

Definition 2.9. *We say that the function*

$$f : \mathcal{O} \times [0, T] \times \mathbb{R}^m \to \mathbb{R}^m, \ f(x, t, y) = (f^1(x, t, y), \ldots, f^m(x, t, y)),$$

satisfies the ***positivity condition*** *if it satisfies Property (2.5) in Theorem 2.8 for all* $t \in [0, T]$.

The following lemma will be essential for the proof of the stochastic positivity criterion.

Lemma 2.3. *Let* (f, g) *be a given stochastic system of PDEs. We assume that the functions* g_j^l *are twice continuously differentiable with respect to* u *and satisfy*

$$g_j^l(x, t, u_1, \ldots, \underbrace{0}_{l}, \ldots, u_m) = 0 \qquad x \in \mathcal{O}, t > 0, \ u_k \geq 0, \tag{2.52}$$

for all $j \in \mathbb{N}$ *and* $k, l = 1, \ldots, m$. *Then, the following statements are equivalent:*

(a) The function f *satisfies the positivity condition.*

(b) The modified function F *satisfies the positivity condition.*

(c) The associated random functions $F_{\epsilon,\omega}$ *satisfy the positivity condition for all* $\epsilon > 0$ *and* $\omega \in \Omega$.

Proof. The proof is a simple computation. Let $j \in \mathbb{N}$ and $1 \leq l \leq m$. Since the function g_j^l is continuously differentiable with respect to u_l and satisfies Property (2.52) we can represent it in the form $g_j^l(x, t, u) = u_l G_j^l(x, t, u)$, where the function G_j^l is continuously differentiable. For the sum appearing in the Wong-Zakaï correction term we obtain

$$\sum_{i=1}^{m} g_j^i \frac{\partial g_j^l}{\partial u_i} = \sum_{i=1}^{m} g_j^i \frac{\partial (u_l G_j^l)}{\partial u_i} = \sum_{i \neq l} g_j^i u_l \frac{\partial G_j^l}{\partial u_i} + g_j^l \frac{\partial (u_l G_j^l)}{\partial u_l},$$

which leads to an associated function F of the form

$$F^l = f^l - \frac{1}{2}\sum_{j=1}^{\infty} q_j^2 \sum_{i=1}^{m} g_j^i \frac{\partial g_j^l}{\partial u_i} = f^l - \frac{1}{2}\sum_{j=1}^{\infty} q_j^2 \left(\sum_{i \neq l} g_j^i u_l \frac{\partial G_j^l}{\partial u_i} + g_j^l \frac{\partial (u_l G_j^l)}{\partial u_l} \right).$$

Due to the hypothesis (2.52) we note that the modified function F satisfies the positivity condition if and only if the interaction function f satisfies the positivity condition since all correction terms vanish if the component u_l of the solution is zero. Finally, the associated system of random PDEs $(F_{\epsilon,\omega}, 0)$ is given by

$$F^l_{\epsilon,\omega} = F^l + \sum_{j=1}^{\infty} q_j g^l_j \dot{W}^j_\epsilon.$$

The assumption (2.52) therefore implies that

$$\begin{aligned} F^l_{\epsilon,\omega}(x, t, u_1, \ldots, \underbrace{0}_{l}, \ldots, u_m) &= F^l(x, t, u_1, \ldots, \underbrace{0}_{l}, \ldots, u_m) \\ &= f^l(x, t, u_1, \ldots, \underbrace{0}_{l}, \ldots, u_m), \end{aligned}$$

which concludes the proof of the lemma. □

Applying Lemma 2.3 we derive necessary and sufficient conditions for the positivity property of the random approximations.

Theorem 2.10. *Let (f, g) be a system of stochastic PDEs and $(F_{\epsilon,\omega}, 0)$ be the associated family of random approximations. We assume that the functions g^l_j are twice continuously differentiable with respect to u, for all $j \in \mathbb{N}$ and $l = 1, \ldots, m$. Then, the family of random approximations $(F_{\epsilon,\omega}, 0)$ satisfies the positivity property for all $\omega \in \Omega$ and (sufficiently small) $\epsilon > 0$ if and only if f satisfies the positivity condition and the stochastic perturbation g fulfils Condition (2.52). In this case, the stochastic system of Itô equations (f, g) satisfies the positivity property.*

Proof. ***Sufficiency:*** By assumption, the interaction function f satisfies the positivity condition. Moreover, since the stochastic perturbation fulfils Property (2.52), Lemma 2.3 implies the positivity condition for the family of random functions $F_{\epsilon,\omega}$, where $\omega \in \Omega$ and $\epsilon > 0$. We apply the deterministic positivity criterion (Theorem 2.8) to conclude that the solutions of the random approximations are non-negative. Finally, the Wong-Zakaï approximation theorem states that the solutions of the random approximations $(F_{\epsilon,\omega}, 0)$ converge in expectation to the solution of the stochastic system (f, g), which implies that the stochastic system (f, g) satisfies the positivity property.
Necessity: We assume the family of random PDEs $(F_{\epsilon,\omega}, 0)$ satisfies the positivity property. By Theorem 2.8 this is equivalent to the positivity condition for the random functions $F^l_{\epsilon,\omega}$,

$$F^l_{\epsilon,\omega}(x, t, \tilde{u}) \;=\; F^l(x, t, \tilde{u}) + \sum_{j=1}^{\infty} q_j g^l_j(x, t, \tilde{u}) \dot{W}^j_\epsilon(t) \;\geq 0 \qquad x \in \mathcal{O}, t > 0, \quad (2.53)$$

where $\tilde{u} \in \mathbb{R}^m_+$, such that $\tilde{u}_l = 0$, for $l = 1, \ldots, m$. The derivative of the smooth approximations $\{W_\epsilon(t), t \in \mathbb{R}_+\}$ of the Wiener process can be represented as the stochastic integral

(2.49) and takes arbitrary values. If we assume the function $g_j^l\big|_{u_l=0}$ is not identically zero, then for sufficiently small $\epsilon > 0$ Inequality (2.53) is violated almost surely. This proves the necessity of the condition on the stochastic perturbation. If Property (2.52) holds, the positivity condition for the family of random approximations is equivalent to the positivity condition for the interaction function f by Lemma 2.3. □

The same result is valid if we apply Stratonovich's interpretation of stochastic differential equations. In other words, the positivity property of solutions of the stochastic system is independent of the choice of interpretation.

Corollary 2.1. *Let (f, g) be a system of stochastic Itô PDEs. We assume the hypothesis of Theorem 2.10 are satisfied and the family of random approximations $(F_{\epsilon,\omega}, 0)$ satisfies the positivity property. Then, the stochastic system $(f, g)_{Strat}$ obtained when we use Stratonovich's interpretation of the stochastic differential equations satisfies the positivity property.*

Proof. The Wong-Zakaï correction term coincides with the transformation relating Ito's and Stratonovich's interpretation of the stochastic system (see [70], Section 6.1). Consequently, the solutions of the random approximations $(f_{\epsilon,\omega}, 0)$ converge to the solution of the given stochastic system, when interpreted in the sense of Stratonovich. The corollary is therefore an immediate consequence of Theorem 2.10 and Lemma 2.3. □

The intuitive interpretation of the condition on the stochastic perturbation is the following: In the critical case, when one component of the solution approaches zero, the stochastic perturbation needs to vanish. Otherwise, the positivity of the solution cannot be guaranteed. For scalar stochastic ODEs this resembles our observation in Section 2.4.1 that additive noise destroys the positivity property of the deterministic system while the positivity property is preserved under perturbations by a linear multiplicative noise.

2.5.3. Comparison Principles for Stochastic Systems

As a direct consequence of the positivity criterion we obtain necessary and sufficient conditions for the random approximations to satisfy comparison principles. We extend the definition of quasi-monotonicity for non-autonomous interaction functions.

Definition 2.10. *We call the function $f : \mathcal{O} \times [0, T] \times \mathbb{R}^m \to \mathbb{R}^m$* ***quasi-monotone****, if it satisfies*

$$f^l(x, t, y) \leq f^l(x, t, z)$$

for all $x \in \mathcal{O}$, $t \in \times[0, T]$ and all $y, z \in \mathbb{R}^m$ such that $y \leq z$ and $y_l = z_l$, where $1 \leq l \leq m$.

Theorem 2.11. *Let (f, g) be a system of stochastic Itô PDEs, $(F_{\epsilon,\omega}, 0)$ be the associated family of random approximations and the hypothesis of Theorem 2.10 be satisfied. Then, the family of random approximations $(F_{\epsilon,\omega}, 0)$ is order preserving with respect to the order* *relation $\preccurlyeq$ if and only if the interaction function f is quasi-monotone, and the functions g_j^l depend on the component u_l of the solution only,*

$$g_j^l(x,t,u_1,\ldots,u_m) = g_j^l(x,t,u_l) \qquad \text{for all } j \in \mathbb{N}, 1 \le l \le m.$$

In this case, the stochastic system (f,g) is order preserving with respect to $\preccurlyeq$.

Proof. Let u_0 and v_0 be given initial data such that $u_0 \succcurlyeq v_0$. Applying Theorem 2.10 we derive necessary and sufficient conditions to ensure that the order is preserved by the solutions of the associated random approximations. Since the differential operator A is linear, the difference $w := u - v$ is a solution of the stochastic system $(\tilde{f}, \tilde{g})$ where

$$\tilde{f}^l(x,t,w) := f^l(x,t,u) - f^l(x,t,v) \quad \text{and} \quad \tilde{g}_j^l(x,t,w) := g_j^l(x,t,u) - g_j^l(x,t,v)$$

for $j \in \mathbb{N}$, $1 \le l \le m$. Furthermore, by the definition of the function w the family of random approximations $(F_{\epsilon,\omega}, 0)$ corresponding to the original system (f,g) is order preserving with respect to $\preccurlyeq$ if and only if the random approximations $(\tilde{F}_{\epsilon,\omega}, 0)$ associated to the stochastic system $(\tilde{f}, \tilde{g})$ satisfy the positivity property.

Theorem 2.10 yields necessary and sufficient conditions for the latter. Namely, the random family $(\tilde{F}_{\epsilon,\omega}, 0)$ satisfies the positivity property if and only if the function $\tilde{f}$ satisfies the positivity condition and the stochastic perturbation fulfils

$$\tilde{g}_j^l(x,t,w_1,\ldots,w_{l-1},0,w_{l+1},\ldots,w_m) = 0 \qquad x \in \mathcal{O}, t > 0, w_k \ge 0,$$

for all $j \in \mathbb{N}$ and $1 \le k,l \le m$. This is equivalent to the condition

$$\tilde{g}_j^l(x,t,y) = g_j^l(x,t,z) \qquad \text{for all } y,z \in \mathbb{R}^m \text{ such that } y_l = z_l, y \ge z,$$

and $x \in \mathcal{O}, t > 0$. Consequently, the functions g_j^l depend on the component u_l of the solution only. The positivity condition for the function $\tilde{f}$ is equivalent to the quasi-monotonicity of the original interaction term f.

By Theorem 2.9 the solutions of the associated random family $(F_{\epsilon,\omega}, 0)$ converge in expectation to the solution of the original system (f,g), which proves that the order is preserved by the solutions of the stochastic system. □

It is well-known in the deterministic theory of PDEs that the quasi-monotonicity of the interaction function f ensures that the system $(f,0)$ is order preserving (see [66]). The conditions on the functions g_j^l in the previous theorem guarantee the persistence of this property under stochastic perturbations. Theorem 5.6 in [15] yields sufficient conditions for the validity of comparison principles for stochastic systems of the form (f,g). We show in Theorem 2.11 that these conditions are also necessary to ensure that the family of random approximations is order preserving.

Moreover, the conditions for the validity of comparison principles for the random approximations imply that the stochastic system is order preserving when it is interpreted in the sense of Stratonovich.

Corollary 2.2. *Let (f, g) be a system of stochastic Itô PDEs. We assume the hypothesis of Theorem 2.10 are satisfied and the associated family of random approximations is order preserving with respect to the order relation $\preccurlyeq$. Then, the stochastic system $(f, g)_{Strat}$ we obtain when we apply Stratonovich's interpretation of stochastic differential equations is order preserving with respect to $\preccurlyeq$.*

Proof. Theorem 2.11 implies that the stochastic perturbations g_j^l, $j \in \mathbb{N}, l = 1, \ldots, m$, depend on the component u_l of the solution only. In this case it is easy to verify that the following statements are equivalent:

(a) The function f is quasi-monotone.

(b) The associated random functions $f_{\epsilon,\omega}$ are quasi-monotone, where $\epsilon > 0$ and $\omega \in \Omega$.

(c) The modified function F is quasi-monotone.

(d) The associated random functions $F_{\epsilon,\omega}$ are quasi-monotone, where $\epsilon > 0$ and $\omega \in \Omega$.

The solutions of the random approximations $(f_{\epsilon,\omega}, 0)$ converge to the solution of the given stochastic system, when interpreted in the sense of Stratonovich. Hence, the statement of the corollary is an immediate consequence of Theorem 2.11 and the equivalence relations (a)-(d). □

Like in the deterministic case we immediately obtain necessary and sufficient conditions for the random family $(F_{\epsilon,\omega}, 0)$ to satisfy comparison principles with respect to an arbitrary order relation in $\mathbb{R}^m$.

Corollary 2.3. *Let (f, g) be a system of stochastic PDEs and the hypothesis of Theorem 2.10 be satisfied. Then, the associated family of random approximations $(F_{\epsilon,\omega}, 0)$ is order preserving with respect to the order relation $\gtrsim$ if and only if*

$$\begin{cases} f^l(x,t,y) \geq f^l(x,t,z) & l \in \sigma_1 \\ f^l(x,t,y) \leq f^l(x,t,z) & l \in \sigma_2, \end{cases}$$

for $x \in \mathcal{O}, t > 0$ and all $y, z \in \mathbb{R}^m$ such that $y \gtrsim z$ and $y_l = z_l$, and the functions g_j^l depend on the component u_l of the solution only,

$$g_j^l(x,t,u_1,\ldots,u_m) = g_j^l(x,t,u_l) \qquad \text{for all } j \in \mathbb{N}, 1 \leq l \leq m.$$

In this case, the stochastic system (f, g) is order preserving with respect to the order relation $\gtrsim$ for both Itô's and Stratonovich's interpretation.

Proof. We define the function

$$w_j := \begin{cases} u_j - v_j & \text{if } j \in \sigma_1 \\ v_j - u_j & \text{if } j \in \sigma_2. \end{cases}$$

Then, w is a solution of the stochastic system $(\tilde{f}, \tilde{g})$, where

$$\tilde{f}^l(x,t,w) := \begin{cases} f^l(x,t,u) - f^l(x,t,v) & \text{if } j \in \sigma_1 \\ f^l(x,t,v) - f^l(x,t,u) & \text{if } j \in \sigma_2, \end{cases}$$
$$\tilde{g}^l(x,t,w) := \begin{cases} g^l(x,t,u) - g^l(x,t,v) & \text{if } j \in \sigma_1 \\ g^l(x,t,v) - g^l(x,t,u) & \text{if } j \in \sigma_2. \end{cases}$$

The solutions of the random approximations $(F_{\epsilon,\omega}, 0)$ of the stochastic system (f, g) are order preserving with respect to the order relation $\succsim$ if and only if the family of random approximations $(\tilde{F}_{\epsilon,\omega}, 0)$ corresponding to the stochastic system $(\tilde{f}, \tilde{g})$ satisfies the positivity property. Like in the proof of Theorem 2.11 we conclude that the random family associated to the system $(\tilde{f}, \tilde{g})$ satisfies the positivity property if and only if the functions $\tilde{g}^l_j$ depend on the component u_l of the solution only and the interaction term $\tilde{f}$ fulfils the positivity condition. This is equivalent to the conditions on the functions g and f stated in the theorem.

The solutions of the random approximations $(F_{\epsilon,\omega}, 0)$ are order preserving with respect to the order relation $\succsim$ and converge to the solution of the stochastic system (f, g), which implies that the solutions of the system of Itô equations (f, g) preserve the order relation $\succsim$. The result for the solutions of the stochastic system $(f, g)_{Strat}$ when we apply Stratonovich's interpretation of stochastic differential equations follows from the proof of Corollary 2.2. □

For the validity of comparison principles, the critical situation occurs when one component of the solutions u and v approaches the same value. Then, the other components of the solution should have no influence on the intensity of the stochastic perturbation, and the stochastic perturbations in the corresponding equation necessarily coincide.

2.5.4. Verifying Stochastic Models

We apply our results to verify the positivity property of a deterministic predator-prey system under stochastic perturbations that was discussed as a sample application in [4] (Section 5). The deterministic model is formulated as reaction-diffusion system for the predator u and the prey v in a bounded spatial domain $\mathcal{O} \subset \mathbb{R}^3$ with smooth boundary $\partial\mathcal{O}$,

$$\begin{aligned} \partial_t u &= \Delta u - \beta_1\big(\big|\frac{v}{u}\big|\big)u + c\beta_2\big(\big|\frac{v}{u}\big|\big)v, \\ \partial_t v &= \Delta v + \big[\gamma - \beta_2\big(\big|\frac{v}{u}\big|\big)\big]v, \\ \frac{\partial}{\partial\nu}u|_{\partial\mathcal{O}} &= 0, \quad \frac{\partial}{\partial\nu}v|_{\partial\mathcal{O}} = 0, \\ (u,v)|_{t=0} &= (u_0, v_0), \end{aligned} \tag{2.54}$$

where the constants c and γ are positive and the functions $\beta_1, \beta_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are smooth and non-negative. We observe that the interaction function satisfies the positivity condition and no cross-diffusion terms are present. Consequently, the deterministic model (2.54) preserves the positivity of solutions by Theorem 2.1.

The model includes a certain uncertainty since it is impossible to determine the exact model parameters γ, β_1 and β_2 ([4]). One possibility to take this into account is to add noise, which leads to the following stochastic model

$$\begin{aligned} du &= \big\{\Delta u - \beta_1\big(\big|\frac{v}{u}\big|\big)u + c\beta_2\big(\big|\frac{v}{u}\big|\big)v\big\}dt + udW_t, \\ dv &= \big\{\Delta v + \big[\gamma - \beta_2\big(\big|\frac{v}{u}\big|\big)\big]v\big\}dt + vdW_t, \end{aligned} \tag{2.55}$$

where $\{W_t,\ t \in \mathbb{R}_+\}$ denotes a standard scalar Wiener process and dW_t the corresponding Itô differential (see [4]). If one component of the solution approaches zero the stochastic perturbation in the corresponding equation vanishes. Theorem 2.10, Theorem 2.1 and the positivity condition of the deterministic interaction function therefore imply that the stochastic system (2.55) satisfies the positivity property. Moreover, this is valid independent of the choice of Itô's or Stratonovich's interpretation of stochastic differential equations.

2.6. Concluding Remarks

We formulated general criteria for the positivity of solutions of semi-linear and quasi-linear parabolic systems and for stochastic perturbations of semi-linear systems.

The Wong-Zakaï approximation theorem proved in [15] allowed us to study the stochastic systems by considering smooth random approximations. Our results for non-autonomous deterministic systems yield necessary and sufficient conditions for the family of random approximations, which imply the positivity property of the original stochastic system. Initially, we were hoping to obtain a stronger result. Namely, that the conditions in Theorem 2.10 are also necessary for the positivity property of the stochastic system. The difficulty is that we cannot directly deduce the positivity of the random approximations from the positivity of the solution of the stochastic system. To show the necessity for the stochastic system presumably requires different methods or stronger assumptions on the solution. For scalar parabolic equations the necessity was shown in [47], but the proof is not based on random approximations.

For systems of stochastic ODEs we can derive explicit necessary and sufficient conditions for the positivity property from an abstract result obtained in [53], which generalizes the Nagumo-Brezis Theorem and the tangential condition for stochastic systems of ODEs. The conditions on the stochastic perturbations we obtain in this particular case coincide with the conditions formulated in Theorem 2.10.

Another interesting problem which is important in numerical simulations are criteria for the positivity of solutions of discrete systems. We expect that the method applied in the proof of the deterministic positivity criterion (Theorem 2.3) can be used to derive explicit necessary and sufficient conditions for the positivity of solutions of finite difference schemes.

3. Exponential Attractors of Infinite Dimensional Dynamical Systems

The longtime behaviour of solutions of various dissipative evolution equations arising in mathematical physics, biology and other sciences can be studied in terms of attractors of the generated semigroup, which acts in infinite dimensional phase spaces ([5], [42], [69]). To illustrate the ideas we consider the Cauchy problem for a semi-linear heat equation in a smooth bounded domain $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$,

$$\begin{aligned} \partial_t u(x,t) &= \Delta u(x,t) + f(u(x,t)) && \Omega \times (0,T), \\ u|_{\partial\Omega}(x,t) &= 0 && \partial\Omega \times [0,T], \\ u(x,0) &= u_0(x) && \Omega \times \{0\}, \end{aligned} \tag{3.1}$$

where $T > 0$, $\partial\Omega$ denotes the boundary of the domain Ω and u is a scalar function depending on the spatial variable $x \in \Omega$ and the time variable $t \in [0,T]$. Under appropriate conditions on the reaction function f and for a suitably chosen Banach space of functions V there exists for every initial data $u_0 \in V$ a unique solution u of the initial-/boundary-value problem (3.1) taking values in V; that is, the solution $u(\cdot,t) \in V$ for all $t \in [0,T]$. Moreover, if the solution exists globally and depends continuously on the initial data, the time evolution of the system can be described in terms of a semigroup acting in the Banach space V. For $t \geq 0$ we define the operator $T(t) : V \to V$ by

$$T(t)u_0 := u(\cdot,t),$$

where $u(\cdot,t) \in V$ is the unique global solution of (3.1) corresponding to initial data $u_0 \in V$. The operator $T(t)$, $t > 0$, maps a given initial state u_0 of the system to the state of the system at time t after starting, and the family of operators $\{T(t) | \; t \geq 0\}$ satisfies the properties of a semigroup in V.

An important mathematical question is the qualitative behaviour of the system when time tends to infinity. In the modelling of population dynamics for instance we are interested whether the involved species will persist or become extinct in the far future, after transient states of the system have passed. The longtime dynamics of dissipative systems can often be described by the dynamics on the global attractor. The global attractor is a compact, invariant subset of the phase space, which attracts all solutions and hence, captures all relevant limit dynamics of the system. For large times the dynamics in the initially infinite dimensional phase space is reduced to a small (compact) subset, and the states of the system are well-approximated by the states of the system within the attractor.

The global attractor is unique, minimal within the family of closed subsets that attract all bounded sets and the maximal bounded invariant subset of the phase space. Moreover, it was shown in many cases that the dimension of the global attractor is finite ([12], [26], [69]).

However, the rate of convergence to the attractor is generally unknown, it can be arbitrarily slow, and the global attractor is in general not stable under perturbations. To overcome these drawbacks the concept of an exponential attractor was introduced ([26]). Exponential attractors are larger subsets of the phase space, contain the global attractor, attract all bounded subsets at an exponential rate and are still finite dimensional. The main obstacle of exponential attractors is that they are only semi-invariant under the action of the semigroup and therefore not unique.

While the theory of attractors of semigroups is well-established and well-understood its counterpart in the non-autonomous setting is less understood and far more complex ([8], [50]). Let us again consider the Cauchy problem for a semi-linear heat equation, however with time-dependent reaction function f,

$$\begin{aligned} \partial_t u(x,t) &= \Delta u(x,t) + f(t,u(x,t)) && \Omega \times (s,T), \\ u|_{\partial\Omega}(x,t) &= 0 && \partial\Omega \times [s,T], \\ u(x,s) &= u_s(x) && \Omega \times \{s\}, \end{aligned} \tag{3.2}$$

where $s \in \mathbb{R}$ and $T > s$. Under appropriate conditions on the reaction function f and for a suitably chosen Banach space of functions V, there exists for every initial data $u_s \in V$ and initial time $s \in \mathbb{R}$ a unique solution u taking values in V; that is, $u(\cdot,t) \in V$ for all $t \in [s,T]$. Moreover, we assume the solution exists globally and depends continuously on the initial data. Different from autonomous problems, where the solution at time $t > s$ only depends on the elapsed time after starting $t - s$, the solution of non-autonomous problems also depends on the initial time $s \in \mathbb{R}$. The rule of time evolution of the system is then described in terms of a two-parameter family of operators acting in the Banach space V. For $s \in \mathbb{R}$ and $t > s$ we define the operator $U(t,s) : V \to V$ by

$$U(t,s)u_s := u(\cdot,t) \quad t \geq s,$$

where $u(\cdot,t) \in V$ is the unique global solution of (3.2) corresponding to initial data $u_s \in V$ and initial time $s \in \mathbb{R}$. The operator $U(t,s)$, $t > s$, maps a given initial state u_s at initial time $s \in \mathbb{R}$ to the state of the system at a later time $t > s$, and the family of operators $\{U(t,s)|\ t,s \in \mathbb{R},\ t \geq s\}$ satisfies the properties of an evolution process in V.

The first attempt to extend the notion of global attractors for evolution processes was the concept of uniform attractors ([12]). Uniform attractors are fixed compact subsets of the phase space that attract all solutions uniformly with respect to the initial time. It is a suitable concept for certain classes of non-autonomous terms or for small non-autonomous perturbations of autonomous problems. To capture more general non-autonomous functions, however, requires to weaken the notion of convergence. This leads to the definition of forwards and pullback global attractors, which comprise of families of time-dependent subsets of the phase space that attract all solutions in forwards or pullback sense, respectively ([11]). Since global non-autonomous attractors have the same favourable properties and drawbacks as global attractors of semigroups, it is of interest to extend the concept of exponential attractors for evolution processes ([19], [49]). Our aim is to analyse the existence of pullback exponential attractors.

The outline of this chapter is as follows. In Section 3.1.1 we introduce basic concepts and recall a general existence theorem for global attractors of semigroups. Section 3.1.2 is devoted to the dimension of attractors, and we summarize properties of the fractal dimension that we frequently use in the subsequent sections. We define exponential attractors of semigroups in Section 3.1.3 and give an overview of previous existence results. In Section 3.1.4 we present an algorithm for the construction of exponential attractors for asymptotically compact semigroups in Banach spaces. Properties of the exponential attractor are discussed in Section 3.1.5.

The second part of Chapter 3 is devoted to non-autonomous attractors. We introduce evolutions processes and recall different notions of non-autonomous attractors in Section 3.2.1. In the sequel we use the concept of pullback convergence. We recall an existence result for global pullback attractors and summarize previous results regarding pullback exponential attractors in Section 3.2.2. The main result of this chapter is a new construction of pullback exponential attractors for asymptotically compact evolution processes in Banach spaces and is formulated in Section 3.2.3. In Section 3.2.4 we analyse properties of the pullback exponential attractor. Finally, applications are addressed in Section 3.2.5, where we consider initial value problems for a non-autonomous Chafee-Infante equation and a non-autonomous damped wave equation.

3.1. Autonomous Evolution Equations

3.1.1. Semigroups and Global Attractors

We study the longtime behaviour of evolutionary PDEs by using concepts from the theory of dynamical systems. Namely, we analyse the existence of attractors for the generated semigroup (or evolution process) in infinite dimensional phase spaces.

In the sequel we use the letter $\mathbb{T}$ to denote $\mathbb{R}$ or $\mathbb{Z}$ and define $\mathbb{T}_+ := \{t \in \mathbb{T} | \ t \geq 0\}$.

Definition 3.1. *Let $T(t) : X \to X$, $t \in \mathbb{T}_+$, be operators in a metric space (X, d_X). We call the family $\{T(t) | \ t \in \mathbb{T}_+\}$ a* ***semigroup*** *in X if it satisfies the properties*

$$\begin{aligned} &T(t) \circ T(s) = T(t+s) \quad \text{for } t, s \in \mathbb{T}_+, \\ &T(0) = \mathrm{Id}, \\ &(t,x) \mapsto T(t)x \quad \text{is continuous from } \mathbb{T}_+ \times X \to X, \end{aligned}$$

where $\circ$ denotes the composition of operators and Id *the identity operator in X.*

If $\mathbb{T} = \mathbb{R}$ we call $\{T(t) | \ t \in \mathbb{R}_+\}$ a ***time continuous semigroup*** *and for $\mathbb{T} = \mathbb{Z}$ a* ***discrete semigroup*** *in X.*

We are interested in the behaviour of the system when time tends to infinity. The limiting dynamics is in many cases reduced to the dynamics on the global attractor, which is a compact invariant set that attracts all bounded subsets of the phase space.

Definition 3.2. *The set $\mathcal{A} \subset X$ is the* ***global attractor*** *for the semigroup $\{T(t)|\ t \in \mathbb{T}_+\}$ if $\mathcal{A}$ is a non-empty, compact subset of X and strictly invariant under the action of the semigroup, $T(t)\mathcal{A} = \mathcal{A}$ for all $t \in \mathbb{T}_+$. Moreover, $\mathcal{A}$ attracts every bounded subset $D \subset X$,*

$$\lim_{t\to\infty} \mathrm{dist_H}(T(t)D, \mathcal{A}) = 0.$$

Here, $\mathrm{dist_H}(\cdot,\cdot)$ denotes the Hausdorff semi-distance in X,

$$\mathrm{dist_H}(A, B) := \sup_{a\in A} d_X(a, B) = \sup_{a\in A} \inf_{b\in B} d_X(a, b) \qquad \textit{for subsets } A, B \subset X.$$

The global attractor is unique, the minimal closed set that attracts all bounded subsets and the maximal bounded invariant subset of the phase space. To show that semigroups generated by non-linear PDEs possess a global attractor, one generally derives a priori estimates to prove the existence of a bounded absorbing or attracting set for the semigroup.

Definition 3.3. *We call the subset $B \subset X$ an* ***absorbing set (attracting set)*** *for the semigroup $\{T(t)|\ t \in \mathbb{T}_+\}$, if all trajectories emanating from a bounded set eventually enter the set B (a neighbourhood of the set B) and remain within it for all later times. To be more precise, for every bounded set $D \subset X$ there exists $T_D \in \mathbb{T}_+$ such that*

$$T(t)D \subset B \qquad \textit{for all } t \geq T_D$$
$$\big(\lim_{t\to\infty} \mathrm{dist_H}(T(t)D, B) = 0 \qquad \textit{for every bounded subset } D \subset X\big).$$

If a semigroup possesses a compact attracting set follows the existence of the global attractor (see [12], Theorem II.3.1). Here and in the sequel, $\overline{A}$ denotes the closure of a subset $A \subset X$.

Theorem 3.1. *Let $\{T(t)|\ t \in \mathbb{T}_+\}$ be a semigroup in a complete metric space X, and $K \subset X$ be a compact attracting set. Then, the global attractor for the semigroup exists and coincides with the ω-limit set of K,*

$$\mathcal{A} = \omega(K),$$

where $\omega(K) := \bigcap_{s\in\mathbb{T}_+} \overline{\bigcup_{t\geq s} S(t)K}$.

The converse statement of Theorem 3.1 is certainly also true and we observe: *A semigroup $\{T(t)|\ t \in \mathbb{T}_+\}$ in a complete metric space X possesses a global attractor if and only if there exists a compact attracting set for the semigroup.*

3.1.2. On the Dimension of Attractors

The existence of global attractors was established for semigroups generated by many dissipative evolution equations, and in most cases it was shown that the attractor is finite dimensional (see [5], [12], [69]). In general, the global attractor is a complex object and possibly fractal. The most commonly used concepts of dimension in the theory of infinite dimensional dynamical systems are the Hausdorff dimension and the fractal (or upper box-counting) dimension.

Definition 3.4. *Let (X, d_X) be a complete metric space and $A \subset X$ be a precompact subset. For positive $\rho > 0$ and $\epsilon > 0$ we define*

$$\mu_H(A, \rho, \epsilon) := \inf\{\sum_{i \in I} r_i^\rho |\ I \text{ finite}\},$$

*where the infimum is taken over all finite coverings of the set A by balls with radii $r_i \leq \epsilon$, $i \in I$. The **Hausdorff dimension** $\dim_{\mathrm{H}}^X(A)$ of the set A in X is defined as the infimum over all $\rho > 0$ such that*

$$\mu_H(A, \rho) := \lim_{\epsilon \to 0} \mu_H(A, \rho, \epsilon) = 0.$$

*Moreover, the **fractal dimension** of the set A is defined as*

$$\dim_{\mathrm{f}}^X(A) := \lim_{\epsilon \to 0} \frac{\ln(N_\epsilon^X(A))}{\ln(\frac{1}{\epsilon})},$$

*where $N_\epsilon^X(A)$, $\epsilon > 0$, denotes the minimal number of balls in the metric space X with radius ϵ and centres in A needed to cover the set A. The number $N_\epsilon^X(A)$ is often called the (Kolmogorov) **ϵ-entropy** of the set A.*

If it is clear from the context in which space X we measure the dimension we will frequently omit the superscript X. The fractal dimension is an upper bound for the Hausdorff dimension of precompact sets, but these notions do not coincide in general (see for instance [26]). For reasons we explain in the sequel we use the fractal dimension as a measure for the size of exponential attractors. For some evolution equations it was shown that the dimension of the global attractor is infinite (see [12], [37]). In this case, the Kolmogorov ϵ-entropy turned out to be a useful concept to estimate the complexity of the attractor. It was first introduced in [46] and measures the massiveness of precompact subsets of metric spaces in terms of the order of growth of the minimal number of ϵ-balls needed to cover the set when $\epsilon > 0$ tends to zero.

In the following proposition we summarize properties of the fractal dimension that we frequently use in the next sections. For the proof we refer to [38], Section 3.2.

Proposition 3.1. *Let (X, d_X) be a complete metric space and $A, B \subset X$ be precompact subsets. The fractal dimension satisfies the following properties:*

(i) Monotonicity: If $A \subset B$, then

$$\dim_{\mathrm{f}}^X(A) \leq \dim_{\mathrm{f}}^X(B).$$

(ii) Finite stability:

$$\dim_{\mathrm{f}}^X(A \cup B) = \max\{\dim_{\mathrm{f}}^X(A), \dim_{\mathrm{f}}^X(B)\}.$$

(iii) Fractal dimension of the closure:

$$\dim_{\mathrm{f}}^X(A) = \dim_{\mathrm{f}}^X(\overline{A}).$$

(iv) If (Y, d_Y) is another complete metric space and the mapping $F : X \to Y$ is Hölder continuous in A,

$$d_Y(F(x), F(y)) \leq C(d_X(x, y))^\theta \qquad \text{for all} \quad x, y \in A,$$

where the constant $C \geq 0$ and $0 < \theta \leq 1$, then

$$\dim_{\mathrm{f}}^Y(F(A)) \leq \frac{1}{\theta}\dim_{\mathrm{f}}^X(A).$$

In particular, for Lipschitz continuous maps $F : X \to Y$ we obtain

$$\dim_{\mathrm{f}}^Y(F(A)) \leq \dim_{\mathrm{f}}^X(A).$$

Furthermore, the fractal dimension is an upper bound for the Hausdorff dimension,

$$\dim_{\mathrm{H}}^X(A) \leq \dim_{\mathrm{f}}^X(A).$$

We remark that the Hausdorff dimension of every countable set is zero, which is not valid for the fractal dimension. Moreover, the Hausdorff dimension is countably stable, while the fractal dimension is only finitely stable.

If the existence and finite dimensionality of the global attractor is known the longtime behaviour of the semigroup is reduced to a finite dimensional subset of the phase space. To study the dynamics on the attractor by known methods from the theory of finite dimensional dynamical systems it is necessary to project the attractor onto subsets of the Euclidean space. Almost every projection of a compact subset A of a Banach space with finite fractal dimension $\dim_{\mathrm{f}}(A) = d$ onto subspaces of dimension greater than $2d$ is injective. Mañé has stated this result in [52] for subsets of finite Hausdorff dimension. His proof is however not applicable for arbitrary subsets A of finite Hausdorff dimension, since he uses the fact that the Hausdorff dimension of the set of differences

$$A - A := \{x - y \mid x, y \in A\}$$

is finite, which is not valid in general (see [64]). Indeed, in the appendix of [65] a countable compact subset of $\mathbb{R}^m$ is constructed such that no projection onto the Euclidean space $\mathbb{R}^n$, where $n < m$, is injective. This counterexample was extended in [7] for infinite dimensional spaces, where a countable compact subset (of zero Hausdorff dimension) is constructed such that no injective linear mapping into an Euclidean space $\mathbb{R}^n, n \in \mathbb{N}$, exists. The fractal dimension however possesses the property that $\dim_{\mathrm{f}}(A) < \infty$ implies for the set of differences $\dim_{\mathrm{f}}(A - A) \leq 2\dim_{\mathrm{f}}(A)$ and consequently, Mané's proof of the embedding theorem is valid for subsets of Banach spaces with finite fractal dimension. His result was further generalized and the Hölder continuity of the inverse of Mañé's projection was shown. For Hilbert spaces the embedding theorem was proved in [26] (Appendix A):

Theorem 3.2. *Let A be a compact subset of a Hilbert space H with finite fractal dimension* $\dim_{\mathrm{f}}(A) = d$. *Then, for any integer $k > 2d$ the set of projections $L : V \to \mathbb{R}^k$ admits a G_δ dense subset consisting of projections that are injective on the set A.*

A generalization of the result for Banach spaces and the proof of the Hölder continuity of the inverse of Mañé's projection can be found in [64] (Theorem 5.1).

Different methods were developed to show the finite dimensionality of attractors of semigroups and to derive upper bounds for their dimension (see [69], [9]). The construction of exponential attractors which we present in the sequel is one way of proving the existence and finite dimensionality of global attractors, but provides only rough estimates for the fractal dimension. Essentially better bounds are obtained by using Lyaponov exponents (see [69], Section V.2). However, this method is restricted to semigroups acting in Hilbert spaces and requires the differentiability of the semigroup.

3.1.3. Exponential Attractors of Semigroups

Global attractors have all the mentioned favourable properties, in various applications however we encounter difficulties. We consider two simple examples which illustrate the drawbacks.

Example 3.1. *The solution of the scalar ODE*

$$\begin{aligned} \frac{d}{dt}x(t) &= -\big(x(t)\big)^2 && t \in \mathbb{R}_+, \\ x(0) &= x_0 && x_0 \in \mathbb{R}, \end{aligned}$$

is the function $x : \mathbb{R}_+ \to \mathbb{R},\ t \mapsto \frac{x_0}{1+tx_0}$. When time t tends to infinity all solutions converge to zero, and the global attractor $\mathcal{A}$ consists of the singleton set $\{0\}$. The rate of convergence to the attractor however is like $\frac{1}{t}$.

Example 3.2. *The scalar ODE*

$$\begin{aligned} \frac{d}{dt}x(t) &= -x(t)\big(x(t)-1\big)^2 && t \in \mathbb{R}_+, \\ x(0) &= x_0 && x_0 \in \mathbb{R}, \end{aligned}$$

possesses two equilibria, the stable equilibrium $\{0\}$ and the unstable equilibrium $\{1\}$. The global attractor of the generated semigroup is the closed interval connecting these points, $\mathcal{A} = \{[0,1]\}$.

However, if we perturb the equation by an arbitrarily small parameter $\epsilon > 0$, the perturbed problem

$$\begin{aligned} \frac{d}{dt}x(t) &= -x(t)\big(x(t)-1\big)^2 - \epsilon && t \in \mathbb{R}_+, \\ x(0) &= x_0 && x_0 \in \mathbb{R}, \end{aligned}$$

possesses only one equilibrium $\{y_\epsilon\} \in \mathbb{R}$. *Furthermore, all solutions converge to the equilibrium point* $\{y_\epsilon\}$ *when time* t *goes to infinity, and the global attractor of the perturbed system is reduced to a single point,* $\mathcal{A}_\epsilon = \{y_\epsilon\}$.

These examples indicate the major drawbacks of global attractors: The rate of convergence to the attractor is in general unknown and can be arbitrarily slow. Moreover, global attractors are generally not stable under perturbations and may completely change its structure under an arbitrarily small perturbation of the system. To overcome these drawbacks we may consider larger sets instead, which contain the global attractor, are still finite dimensional, attract all bounded sets at a fast rate and are therefore more robust under perturbations. Comparing with the concept of global attractors this requires to weaken the strict invariance property of the attracting set.

A first approach in that direction was to embed the global attractor into a finite-dimensional manifold.

Definition 3.5. *Let* $\{T(t)|\ t \in \mathbb{T}_+\}$ *be a semigroup in a separable Hilbert space* H. *The subset* $M \subset H$ *is an* ***inertial manifold*** *for the semigroup* $\{T(t)|\ t \in \mathbb{T}_+\}$ *if* M *is*

(i) *a finite dimensional Lipschitz manifold,*

(ii) *positively semi-invariant,* $T(t)M \subset M$ *for all* $t \in \mathbb{T}_+$, *and*

(iii) *attracts all bounded subsets exponentially; that is, there exists a constant* $\omega > 0$ *such that*

$$\lim_{t\to\infty} e^{\omega t} \mathrm{dist}_{\mathrm{H}}(T(t)D, M) = 0 \qquad \textit{for all bounded sets } D \subset H.$$

Inertial manifolds were introduced in [39], and are semi-invariant Lipschitz manifolds that exponentially attract all bounded subsets of the phase space. They are stable under perturbations and allow to describe the longtime dynamics of the semigroup in terms of a finite system of ODEs. Inertial manifolds are defined and constructed for semigroups acting in Hilbert spaces and all known methods are based on a so-called spectral gap condition. However, various counterexamples were presented illustrating that the spectral gap condition is a restrictive assumption (see [26]).

Owing to these obstacles exponential attractors were proposed in [26], which are more general and less regular objects. In particular, their construction is not based on the spectral gap condition.

Definition 3.6. *Let* $\{T(t)|\ t \in \mathbb{T}_+\}$ *be a semigroup in a metric space* (X, d_X). *We call the non-empty compact subset* $\mathcal{M} \subset X$ *an* **exponential attractor** *for the semigroup* $\{T(t)|\ t \in \mathbb{T}_+\}$ *if* $\mathcal{M}$ *is*

(i) *of finite fractal dimension,* $\dim_{\mathrm{f}}(\mathcal{M}) < \infty$,

(ii) *semi-invariant,* $T(t)\mathcal{M} \subset \mathcal{M}$ *for all* $t \in \mathbb{T}_+$, *and*

(iii) *attracts all bounded subsets exponentially; that is, there exists a constant* $\omega > 0$ *such that*
$$\lim_{t\to\infty} e^{\omega t}\mathrm{dist}_{\mathrm{H}}(T(t)D, \mathcal{M}) = 0 \qquad \textit{for every bounded subset } D \subset X.$$

Thanks to the exponential rate of attraction exponential attractors are more robust under perturbations ([26], [32], [35]). Furthermore, if a semigroup possesses an exponential attractor, Theorem 3.1 implies that the global attractor $\mathcal{A}$ is contained in the exponential attractor $\mathcal{M}$ and given by its ω-limit set, $\mathcal{A} = \omega(\mathcal{M})$. An immediate consequence of the existence of an exponential attractor is therefore the existence and finite-dimensionality of the global attractor. However, exponential attractors are only semi-invariant under the action of the semigroup and consequently not unique. Indeed, if $\mathcal{M}$ is an exponential attractor for the semigroup $\{T(t)|\ t \in \mathbb{T}_+\}$, then any iterate $T(t)\mathcal{M}$ is also an exponential attractor, for $t \in \mathbb{T}_+$.

The first existence proof and method for the construction of exponential attractors was developed for semigroups acting in Hilbert spaces (see [26]). It is based on the so-called squeezing property of the semigroup and essentially uses the Hilbert structure of the phase space. Since Zorn's Lemma is applied the proof is non-constructive. Moreover, the existence of a compact absorbing set for the semigroup is a priori assumed, which ensures the existence of the global attractor. The exponential attractor is constructed by adding to the global attractor an appropriate semi-invariant subset of the phase space such that all trajectories emanating from bounded sets are attracted exponentially fast. The main difficulty in the construction is to control the fractal dimension of the added set.

Later, the construction of exponential attractors was extended to semigroups acting in Banach spaces in [22]. The proof is based on the covering method developed in [52] to show the finite fractal dimension of negatively invariant sets under maps that are continuously differentiable and such that the derivative is the sum of a compact map and a contraction. This covering method was further developed and applied in several cases to prove the finite dimensionality of global attractors (see [9]). The construction of the exponential attractor in [22] is based on the method and ideas in [26]. It requires the differentiability of the semigroup, the existence of the global attractor is a priori known and the proof is non-constructive.

In [33] an alternative method and explicit algorithm for the construction of exponential attractors was proposed for discrete semigroups acting in Banach spaces. It is based on the compact embedding of the phase space into an auxiliary normed space and uses the regularizing or smoothing property of the semigroup with respect to these spaces. The rate of convergence and the bound on the fractal dimension of the exponential attractor can explicitly be estimated in terms of the entropy properties of this embedding. This approach is the basis of our results. The method for the construction of discrete exponential attractors for semigroups in [33] was further developed in [32] and also extended for discrete non-autonomous problems. Furthermore, in [13], based on the results in [33], exponential attractors for time-continuous semigroups were constructed and estimates for the fractal dimension of global and exponential attractors established. The construction of exponential attractors we present in the following section generalizes the results in [33], [32] and [13] for semigroups, and we improve the bounds on the fractal dimension of the attractors.

3.1.4. Existence Results for Exponential Attractors

We first construct exponential attractors for discrete asymptotically compact semigroups and derive bounds on their fractal dimension before we extend the construction for time continuous semigroups. Compared to former work (see [33], [32], [13]) we modify the setting and construction and consider semigroups that are asymptotically compact in the stronger space. Previous settings and results are discussed in Section 3.1.5. For continuous semigroups an additional regularity property in time is required to obtain finite dimensional exponential attractors. Exponential attractors for continuous semigroups were also obtained in [13], however under less general assumptions, and in [19] as a corollary of the non-autonomous construction. Our results in the time continuous case generalize the previous results and improve the estimates on the fractal dimension of the attractors in [13] and [19].

The construction of exponential attractors is based on the compact embedding of the phase space into an auxiliary normed space and a certain smoothing or regularizing property of the semigroup with respect to these spaces.

Let $\{T(t)|\ t \in \mathbb{T}_+\}$ be a semigroup in a Banach space $(V, \|\cdot\|_V)$.

(H_0) We assume $(W, \|\cdot\|_W)$ is another normed space such that the embedding $V \hookrightarrow\hookrightarrow W$ is dense, compact and

$$\|v\|_W \leq \mu \|v\|_V \qquad \text{for all} \quad v \in V,$$

where the constant $\mu > 0$.

Moreover, we suppose that the semigroup possesses a bounded absorbing set and satisfies the smoothing property asymptotically. Namely, the semigroup can eventually be represented as a sum $T = S + C$, where S satisfies the smoothing property and C is a contraction in V. To be more precise, let $\{T(t)|\ t \in \mathbb{T}_+\}$ be a semigroup in V such that $T(t) = S(t) + C(t)$, where $\{S(t)|\ t \in \mathbb{T}_+\}$ and $\{C(t)|\ t \in \mathbb{T}_+\}$ are families of operators that satisfy the properties:

$(S1)$ There exists a bounded absorbing set $B \subset V$ for the semigroup $\{T(t)|\ t \in \mathbb{T}_+\}$; that is, for all bounded subsets $D \subset V$ there exists $T_D \in \mathbb{T}_+$ such that

$$T(t)D \subset B \qquad \text{for all } t \geq T_D.$$

$(S2)$ The family $\{S(t)|\ t \in \mathbb{T}_+\}$ satisfies the smoothing property within the absorbing set: There exists $\tilde{t} \in \mathbb{T}_+ \backslash \{0\}$ such that

$$\|S(\tilde{t})u - S(\tilde{t})v\|_V \leq \kappa \|u - v\|_W \qquad u, v \in B,$$

for some constant $\kappa > 0$.

($S3$) The family $\{C(t)|\ t \in \mathbb{T}_+\}$ is a family of contractions within the absorbing set:

$$\|C(\tilde{t})u - C(\tilde{t})v\|_V \leq \lambda \|u - v\|_V \qquad u, v \in B,$$

where the constant $0 \leq \lambda < \frac{1}{2}$.

The smoothing property implies that the operator $S(\tilde{t}) : V \to V$ is compact. We do not require that the families of operators $\{S(t)|\ t \in \mathbb{T}_+\}$ and $\{C(t)|\ t \in \mathbb{T}_+\}$ are semigroups, but remark that in applications the family of contractions $\{C(t)|\ t \in \mathbb{T}_+\}$ often forms a semigroup in V (see Section 3.2.5).

The following lemma shows that the smoothing time $\tilde{t}$ in ($S2$) and the absorbing time T_B in ($S1$) corresponding to the absorbing set B can be arbitrary. Previously, it was assumed that these times coincide (and are equal to 1). Moreover, if the family $\{C(t)|\ t \in \mathbb{T}_+\}$ satisfies the properties of a semigroup we show that it suffices that the operators are eventually strict contractions with contraction constant $\lambda < 1$.

Lemma 3.1. *(i) If $\{T(t)|\ t \in \mathbb{T}_+\}$ is a semigroup in the Banach space V such that Property ($S1$) is satisfied, then there exists a bounded absorbing set $\widetilde{B}$ which is positively semi-invariant and Properties ($S2$) and ($S3$) are valid when B is replaced by $\widetilde{B}$.*

(ii) We can replace Assumptions ($S2$) and ($S3$) by the following:

($\widetilde{S}2$) *The family $\{S(t)|\ t \in \mathbb{T}_+\}$ satisfies the smoothing property within the absorbing set: There exists $\tilde{t} \in \mathbb{T}_+\backslash\{0\}$ such that for all $t \geq \tilde{t}$*

$$\|S(t)u - S(t)v\|_V \leq \kappa_t \|u - v\|_W \qquad u, v \in B,$$

for some constant $\kappa_t > 0$.

($\widetilde{S}3$) *The family $\{C(t)|\ t \in \mathbb{T}_+\}$ forms a semigroup in V. Moreover, there exists $\hat{t} \in \mathbb{T}_+\backslash\{0\}$ such that $C(t)B \subset B$ for all $t \geq \hat{t}$, and the operators are strict contractions within the absorbing set:*

$$\|C(\hat{t})u - C(\hat{t})v\|_V \leq \lambda \|u - v\|_V \qquad u, v \in B,$$

where the constant $0 \leq \lambda < 1$.

Proof. (i) If we define

$$\widetilde{B} := \bigcup_{s \in \mathbb{T}_+,\, 0 \leq s < T_B} T(T_B + s)B,$$

it is a bounded absorbing set for the semigroup which is positively semi-invariant. Indeed, it is bounded since $T(T_B + s)B \subset B$ for all $s \in \mathbb{T}_+$, by Property ($S1$). Moreover, if $D \subset V$ is a bounded subset, Assumption ($S1$) implies that there exists $T_D \in \mathbb{T}_+$ such that $T(t)D \subset B$ for all $t \geq T_D$, and we obtain

$$T(t)D = T(t - T_D - T_B)T(T_B)T(T_D)D \subset T(t - T_D - T_B)T(T_B)B \subset \widetilde{B},$$

for all $t \geq T_D + T_B$. Finally, we observe

$$T(t)\widetilde{B} = \bigcup_{s \in \mathbb{T}_+,\, 0 \leq s < T_B} T(t)T(T_B + s)B = \bigcup_{s \in \mathbb{T}_+,\, 0 \leq s < T_B} T(T_B + s + t)B \subset \widetilde{B},$$

for all $t \in \mathbb{T}_+$. Since the set $\widetilde{B} \subset B$ Properties $(S2)$ and $(S3)$ are certainly satisfied for all $u, v \in \widetilde{B}$.
(ii) We choose $l \in \mathbb{N}$ sufficiently large such that $\lambda^l < \frac{1}{2}$ and $l\hat{t} \geq \tilde{t}$. The semigroup property and Assumption $(\widetilde{S}3)$ imply

$$\|C(l\hat{t})u - C(l\hat{t})v\|_V \leq \lambda^l \|u - v\|_V \qquad \text{for all } u, v \in B.$$

If necessary, we replace the contraction time $\hat{t}$ and smoothing time $\tilde{t}$ by $t_0 := l\hat{t}$. Then, Hypothesis $(S2)$ is satisfied with smoothing constant $\kappa := \kappa_{t_0}$ and Assumption $(S3)$ holds with contraction constant $\lambda := \lambda^l < \frac{1}{2}$.

□

The Discrete Case

We now consider discrete semigroups, where $\mathbb{T} = \mathbb{Z}$, and use the letter n instead of t to denote discrete times $n \in \mathbb{Z}_+$. The following theorem yields an existence result for exponential attractors of discrete semigroups in the Banach space V and estimates for the fractal dimension of the exponential attractor.

In the sequel, we denote the ball of radius $r > 0$ and center $a \in X$ in a metric space X by $B_r^X(a)$.

Theorem 3.3. *Let $\{T(n)|\ n \in \mathbb{Z}_+\}$ be a discrete semigroup in the Banach space V and the assumptions (H_0), $(S1)$, $(S2)$ and $(S3)$ be satisfied with $\mathbb{T} = \mathbb{Z}$. Then, for every $\nu \in (0, \frac{1}{2} - \lambda)$ there exists an exponential attractor $\mathcal{M} \equiv \mathcal{M}^\nu$ in V for the semigroup $\{T(n)|\ n \in \mathbb{Z}_+\}$, and its fractal dimension can be estimated by*

$$\dim_{\mathrm{f}}^V(\mathcal{M}^\nu) \leq \log_{\frac{1}{2(\nu+\lambda)}} \left(N^W_{\frac{\nu}{\kappa}}(B_1^V(0)) \right),$$

where λ and κ are the smoothing and contraction constants in $(S2)$ and $(S3)$.

Proof. By Lemma 3.1 without loss of generality we can assume that the absorbing set B is positively semi-invariant.
Step 1: Coverings of $T(n\tilde{n})B$
Let $\nu \in (0, \frac{1}{2} - \lambda)$ be fixed, $R > 0$ and $v_0 \in B$ be such that $B \subset B_R^V(v_0)$. Moreover, we choose elements $w_1, \dots w_N \in V$ such that

$$B_1^V(0) \subset \bigcup_{i=1}^N B^W_{\frac{\nu}{\kappa}}(w_i),$$

where $N := N^W_{\frac{\nu}{\kappa}}(B_1^V(0))$ (see Definition 3.4). We define the set $W^0 := \{v_0\}$ and construct by induction in $n \in \mathbb{N}$ the family of sets W^n with the following properties:

(W1) $W^n \subset T(n\tilde{n})B \subset B$,

(W2) $\sharp W^n \leq N^n$,

(W3) $T(n\tilde{n})B \subset \bigcup_{u \in W^n} B^V_{(2(\nu+\lambda))^n R}(u)$,

where $\sharp A$ denotes the cardinality of the subset $A \subset V$.

To construct a covering of the image $T(\tilde{n})B$ we note that $v \in B^V_R(v_0)$ implies

$$\frac{1}{R}(v - v_0) \in B^V_1(0) \subset \bigcup_{i=1}^{N} B^W_{\frac{\nu}{\kappa}}(w_i),$$

and consequently,

$$B \subset B^V_R(v_0) \subset \bigcup_{i=1}^{N} B^W_{R\frac{\nu}{\kappa}}(Rw_i + v_0).$$

Due to the smoothing property $(S2)$ we obtain

$$\|S(\tilde{n})\tilde{u} - S(\tilde{n})\tilde{v}\|_V \leq \kappa \|\tilde{u} - \tilde{v}\|_W < 2\nu R$$

for all $\tilde{u}, \tilde{v} \in B^W_{R\frac{\nu}{\kappa}}(Rw_i + v_0) \cap B$, which yields the covering

$$S(\tilde{n})B \subset \bigcup_{i=1}^{N} B^V_{2\nu R}(z_i),$$

with centres $z_1, \ldots, z_N \in S(\tilde{n})B$. In particular, there exist $y_i \in B$ such that $z_i = S(\tilde{n})y_i$ for $i = 1, \ldots, N$. The contraction property $(S3)$ implies

$$\|C(\tilde{n})u - C(\tilde{n})y_i\|_V \leq \lambda \|u - y_i\|_V < 2\lambda R \qquad \text{for all } u \in B,$$

and we conclude

$$C(\tilde{n})B \subset B^V_{2\lambda R}(C(\tilde{n})y_i) \qquad \text{for all } i = 1, \ldots, N.$$

Finally, we obtain the desired covering

$$T(\tilde{n})B = S(\tilde{n})B + C(\tilde{n})B \subset \bigcup_{i=1}^{N} \left(B^V_{2\nu R}(S(\tilde{n})y_i) + B^V_{2\lambda R}(C(\tilde{n})y_i)\right) \subset \bigcup_{i=1}^{N} B^V_{2(\nu+\lambda)R}(T(\tilde{n})y_i),$$

with centres $T(y_i) \in T(\tilde{n})B$ for $1 = 1, \ldots, N$. Denoting the set of centres by W^1 follows

$$T(\tilde{n})B \subset \bigcup_{u \in W^1} B^V_{2(\nu+\lambda)R}(u),$$

where the set $W^1 \subset T(\tilde{n})B \subset B$ and $\sharp W^1 \leq N$.

Let us assume the sets W^l are already constructed for $l \leq n$, which yields the covering

$$T(\tilde{n}n)B \subset \bigcup_{u \in W^n} B^V_{(2(\nu+\lambda))^n R}(u).$$

To construct a covering of the iterate

$$T(\tilde{n}(n+1))B = T(\tilde{n})T(\tilde{n}n)B \subset \bigcup_{u \in W^n} T(\tilde{n})B^V_{(2(\nu+\lambda))^n R}(u)$$

let $u \in W^n$. We use the covering of the unit ball $B^V_1(0)$ by $\frac{\nu}{\kappa}$-balls in the space W and the smoothing property $(S2)$ to conclude

$$\begin{aligned} S(\tilde{n})\left(T(\tilde{n}n)B \cap B^V_{(2(\nu+\lambda))^n R}(u)\right) &\subset S(\tilde{n})\Big(T(\tilde{n}n)B \cap \bigcup_{i=1}^N B^W_{(2(\nu+\lambda))^n R\frac{\nu}{\kappa}}((2(\nu+\lambda))^n R w_i + u)\Big) \\ &\subset \bigcup_{i=1}^N B^V_{(2(\nu+\lambda))^n 2\nu R}(S(\tilde{n})y_i^u), \end{aligned}$$

for some $y_1^u, \ldots, y_N^u \in S(\tilde{n})(T(\tilde{n}n)B \cap B^V_{(2(\nu+\lambda))^n R}(u))$. Furthermore, the contraction property $(S3)$ implies

$$C(\tilde{n})\left(T(\tilde{n}n)B \cap B^V_{(2(\nu+\lambda))^n R}(u)\right) \subset B^V_{(2(\nu+\lambda))^n 2\lambda R}(C(\tilde{n})y_i^u) \qquad \text{for all } i = 1, \ldots, N.$$

This yields the covering

$$\begin{aligned} T(\tilde{n})\left(T(\tilde{n}n)B \cap B^V_{(2(\nu+\lambda))^n R}(u)\right) &= \left(S(\tilde{n}) + C(\tilde{n})\right)\left(T(\tilde{n}n)B \cap B^V_{(2(\nu+\lambda))^n R}(u)\right) \\ &\subset \bigcup_{i=1}^N \Big(B^V_{(2(\nu+\lambda))^n 2\nu R}(S(\tilde{n})y_i^u) + B^V_{(2(\nu+\lambda))^n 2\lambda R}(C(\tilde{n})y_i^u)\Big) \\ &\subset \bigcup_{i=1}^N B^V_{(2(\nu+\lambda))^{n+1} R}(T(\tilde{n})y_i^u), \end{aligned}$$

with centres in the set $T(\tilde{n}(n+1))B$. Constructing in the same way for all $u \in W^n$ such a covering of $B^V_{(2(\lambda+\nu))^n R}(u)$ by balls of radius $(2(\nu+\lambda))^{n+1}R$ in V we obtain a covering of the image $T(\tilde{n}(n+1))B$ and denote the new set of centres by W^{n+1}. We observe $\sharp W^{n+1} \leq N \sharp W^n \leq N^{n+1}$, by construction the set of centres $W^{n+1} \subset T(\tilde{n}(n+1))B$, and

$$T(\tilde{n}(n+1))B \subset \bigcup_{u \in W^{n+1}} B^V_{(2(\nu+\lambda))^{n+1} R}(u),$$

which proves the properties $(W1)$-$(W3)$.

Step 2: Definition of the Exponential Attractor
To obtain a semi-invariant exponential attractor we set $E^0 := W^0$ and iteratively define the sets E^n, $n \in \mathbb{N}$, by

$$\begin{aligned} E^1 &:= W^1 \cup T(1)W^0 \cup T(2)W^0 \cup \cdots \cup T(\tilde{n})W^0 \\ E^2 &:= W^2 \cup T(1)W^1 \cup \cdots \cup T(\tilde{n})W^1 \cup T(\tilde{n}+1)W^0 \cup \cdots \cup T(2\tilde{n})W^0 \\ &\vdots \\ E^n &:= W^n \cup T(1)W^{n-1} \cup \cdots \cup T(\tilde{n})W^{n-1} \cup \cdots \cup T(\tilde{n}(n-1)+1)W^0 \cup \cdots \cup T(\tilde{n}n)W^0 \\ &= W^n \cup \bigcup_{k=1}^{n} \bigcup_{l=1}^{\tilde{n}} T((k-1)\tilde{n}+l)W^{n-k}. \end{aligned}$$

Since the absorbing set B is semi-invariant we observe

$$T(n)B \subset T(m)B \qquad \text{for all } n \geq m,$$

and consequently, the sets E^n, $n \in \mathbb{N}$, satisfy the properties:

(E1) $\quad E^0 \subset B, \quad E^n \subset T((n-1)\tilde{n})B \subset B, \quad T(1)E^n \subset E^n \cup E^{n+1}$,

(E2) $\quad \sharp E^n \leq \tilde{n}(n+1)N^n$,

(E3) $\quad T(n\tilde{n})B \subset \bigcup_{u \in E^n} B^V_{(2(\nu+\lambda))^n R}(u)$.

These relations are immediate consequences of the definition of the sets E^n, the properties of the sets W^n and the semi-invariance of the absorbing set B, and can be proved by induction. Moreover, from the first relation follows $T(k)E^n \subset E^n \cup E^{n+1} \cup \cdots \cup E^{n+k}$, for all $k \in \mathbb{N}$.

We finally define the set

$$\widetilde{\mathcal{M}} := \bigcup_{n \in \mathbb{N}_0} E^n$$

and show that it is a precompact exponential attractor for the semigroup.
Step 3: Semi-invariance, Precompactness and Finite dimensionality
By using Property $(E1)$ we obtain

$$T(k)\widetilde{\mathcal{M}} := \bigcup_{n \in \mathbb{N}_0} T(k)E^n \subset \bigcup_{n \in \mathbb{N}_0} (E^n \cup \cdots \cup E^{n+k}) \subset \bigcup_{n \in \mathbb{N}_0} E^n = \widetilde{\mathcal{M}},$$

for all $k \in \mathbb{N}_0$, which proves the semi-invariance of $\widetilde{\mathcal{M}}$. Furthermore, by Property $(E1)$ and the semi-invariance of the absorbing set the sets $E^n \subset T((m-1)\tilde{n})B$ for all $n \geq m$, $m \in \mathbb{N}$, and we conclude

$$\widetilde{\mathcal{M}} = \bigcup_{n=0}^{m} E^n \cup \bigcup_{n=m+1}^{\infty} E^n \subset \bigcup_{n=0}^{m} E^n \cup T(m\tilde{n})B.$$

Properties (E2) and (W3) now imply the estimate

$$\sharp(\bigcup_{n=0}^{m} E^n) \leq (m+1)\sharp E^m \leq (m+1)^2 \tilde{n} N^m,$$

and the covering $T(m\tilde{n})B \subset \bigcup_{u \in W^m} B^V_{(2(\nu+\lambda))^m R}(u)$. For arbitrary $\epsilon > 0$ we choose m sufficiently large such that

$$(2(\nu+\lambda))^m R \leq \epsilon < (2(\nu+\lambda))^{m-1} R \tag{3.3}$$

holds. An estimate for the number of ϵ-balls needed to cover the set $\widetilde{\mathcal{M}}$ is then given by

$$N^V_\epsilon(\widetilde{\mathcal{M}}) \leq \sharp(\bigcup_{n=0}^{m} E^n) + \sharp W^m \leq (m+1)^2 \tilde{n} N^m + N^m \leq 2(m+1)^2 \tilde{n} N^m,$$

where we used Properties (W2) and (E2). This proves the precompactness of $\widetilde{\mathcal{M}}$ in V. Furthermore, by Relation (3.3) follows

$$m < \frac{\ln \frac{1}{\epsilon} + \ln R}{\ln \frac{1}{2(\lambda+\nu)}} + 1 = \frac{\ln \frac{1}{\epsilon}}{\ln \frac{1}{2(\lambda+\nu)}} + C,$$

for some constant $C \geq 0$ depending on R, λ and ν, and we obtain for the fractal dimension of the set $\widetilde{\mathcal{M}}$

$$\begin{aligned}
\dim^V_{\mathrm{f}}(\widetilde{\mathcal{M}}) &= \limsup_{\epsilon \to 0} \frac{\ln(N^V_\epsilon(\widetilde{\mathcal{M}}))}{\ln \frac{1}{\epsilon}} \leq \limsup_{\epsilon \to 0} \frac{\ln(2) + 2\ln(m+1) + \ln(\tilde{n}) + m\ln(N)}{\ln \frac{1}{\epsilon}} \\
&\leq \limsup_{\epsilon \to 0} \frac{2\ln\big(\frac{\ln \frac{1}{\epsilon}}{\ln \frac{1}{2(\lambda+\nu)}} + C + 1\big) + \big(\frac{\ln \frac{1}{\epsilon}}{\ln \frac{1}{2(\lambda+\nu)}} + C\big)\ln(N)}{\ln \frac{1}{\epsilon}} \leq \log_{\frac{1}{2(\nu+\lambda)}}(N).
\end{aligned}$$

It remains to show that the set $\widetilde{\mathcal{M}}$ exponentially attracts all bounded subsets of V. By Assumption $(S1)$ there exists for every bounded set $D \subset V$ an absorbing time $n_D \in \mathbb{Z}_+$ such that $T(n)D \subset B$ for all $n \geq n_D$. If we take $n \geq n_D + \tilde{n}$, then $n = n_D + \tilde{n}k_0 + k$ for some $k_0, k \in \mathbb{Z}_+$, where $k_0 > 0$, and it follows

$$\begin{aligned}
\mathrm{dist}^V_{\mathrm{H}}(T(n)D, \widetilde{\mathcal{M}}) &= \mathrm{dist}^V_{\mathrm{H}}\big(T(k_0\tilde{n})T(n_D+k)D, \bigcup_{n=0}^{\infty} E^n\big) \leq \mathrm{dist}^V_{\mathrm{H}}(T(k_0\tilde{n})B, \bigcup_{n=0}^{\infty} E^n) \\
&\leq \mathrm{dist}^V_{\mathrm{H}}(T(k_0\tilde{n})B, W^{k_0}) \leq (2(\nu+\lambda))^{k_0} R = (2(\nu+\lambda))^{\frac{n-n_D-k}{\tilde{n}}} R = ce^{-\omega n},
\end{aligned}$$

for some constant $c \geq 0$, where $\omega := \ln(\frac{1}{2(\nu+\lambda)})^{\frac{1}{\tilde{n}}}$.

Step 4: Compactness of the Exponential Attractor
Since V is a Banach space taking the closure $\mathcal{M} := \overline{\widetilde{\mathcal{M}}}^{\|\cdot\|_V}$ of the precompact subset $\widetilde{\mathcal{M}}$ we obtain a compact set in V. By Proposition 3.1 the fractal dimension of $\mathcal{M}$ coincides with the fractal dimension of $\widetilde{\mathcal{M}}$,

$$\dim_{\mathrm{f}}^{V}(\mathcal{M}) = \dim_{\mathrm{f}}^{V}(\overline{\widetilde{\mathcal{M}}}^{\|\cdot\|_V}) = \dim_{\mathrm{f}}^{V}(\widetilde{\mathcal{M}}),$$

and is therefore bounded by the same value. Moreover, the exponential attraction property of $\mathcal{M}$ follows immediately, since the set $\widetilde{\mathcal{M}}$ exponentially attracts all bounded subsets of V and $\widetilde{\mathcal{M}} \subset \mathcal{M}$. To show the semi-invariance of $\mathcal{M}$ let $k \in \mathbb{N}_0$. By the continuity of the semigroup (see Definition 3.1) and the semi-invariance of the set $\widetilde{\mathcal{M}}$ we observe

$$T(k)\mathcal{M} = T(k)\overline{\widetilde{\mathcal{M}}}^{\|\cdot\|_V} \subset \overline{T(k)\widetilde{\mathcal{M}}}^{\|\cdot\|_V} \subset \overline{\widetilde{\mathcal{M}}}^{\|\cdot\|_V} = \mathcal{M},$$

which shows that the set $\mathcal{M}$ is an exponential attractor for the semigroup $\{T(n)|\ n \in \mathbb{Z}_+\}$ and concludes the proof of the theorem. □

The Time Continuous Case

We now consider time continuous semigroups, where $\mathbb{T} = \mathbb{R}$, and construct exponential attractors in a standard way (see [26] or [13]). This requires an additional regularity property in time of the semigroup. We later propose an alternative concept, so-called *pullback exponential attractors for time continuous semigroups* (see Section 3.2.4). In the discrete case they coincide with exponential attractors of semigroups and exist under more general assumptions in the time continuous case.

Let $\{T(t)|\ t \in \mathbb{R}_+\}$ be a continuous semigroup in the Banach space $(V, \|\cdot\|_V)$. In addition to the hypothesis $(S1)$-$(S3)$ we assume Hölder continuity in time of the semigroup. We remark that the interval where the semigroup is Hölder continuous is arbitrary.

$(S4)$ The semigroup $\{T(t)|\ t \in \mathbb{R}_+\}$ is Hölder continuous in time: There exist $0 \le t_1 < t_2$ such that

$$\|T(s_1)u - T(s_2)u\|_V \le \zeta |s_1 - s_2|^{\theta} \qquad \text{for all } u \in B,\ s_1, s_2 \in [t_1, t_2],$$

for some constant $\zeta \ge 0$ and exponent $0 < \theta \le 1$.

The following theorem extends Theorem 3.3 for time continuous semigroups.

Theorem 3.4. *We assume $\{T(t)|\ t \in \mathbb{R}_+\}$ is a continuous semigroup in the Banach space V and the properties (H_0), $(S1)$-$(S4)$ are satisfied. Then, for any $\nu \in (0, \frac{1}{2} - \lambda)$ there exists an exponential attractor $\mathcal{M} \equiv \mathcal{M}^{\nu}$ for the semigroup $\{T(t)|\ t \in \mathbb{R}_+\}$, and its fractal dimension is bounded by*

$$\dim_{\mathrm{f}}^{V}(\mathcal{M}^{\nu}) \le \frac{1}{\theta} + \log_{\frac{1}{2(\nu+\lambda)}} \left(N^{W}_{\frac{\nu}{\kappa}}(B_1^V(0)) \right),$$

where λ and κ denote the constants in Hypothesis $(S2)$ and $(S3)$ and θ is the Hölder exponent in $(S4)$.

Proof. By Lemma 3.1 without loss of generality we can assume that the absorbing set is positively semi-invariant.

Step 1: Construction of the Exponential Attractor

Let $\nu \in (0, \frac{1}{2} - \lambda)$ be fixed. The associated discrete semigroup $\{\widetilde{T}(n) |\ n \in \mathbb{Z}_+\}$ defined by $\widetilde{T}(n) := T(n\tilde{t}), n \in \mathbb{Z}_+$, satisfies the hypothesis of Theorem 3.3 with $\tilde{n} = 1$. An exponential attractor $\mathcal{M}_d$ for the semigroup $\{\widetilde{T}(n) |\ n \in \mathbb{Z}_+\}$ can be constructed as in the proof of Theorem 3.3. We recall that the exponential attractor was defined by $\mathcal{M}_d = \overline{\widetilde{\mathcal{M}_d}}^{\|\cdot\|_V}$, where

$$\widetilde{\mathcal{M}_d} = \bigcup_{n\in\mathbb{N}_0} E^n,$$

and refer to the proof of Theorem 3.3 for the construction of the family of sets E^n, $n \in \mathbb{N}_0$.

To obtain an exponential attractor for the time continuous semigroup we choose $k \in \mathbb{N}$ such that $k\tilde{t} \geq t_1$ and define $\mathcal{M} := \overline{\widetilde{\mathcal{M}}}^{\|\cdot\|_V}$, where

$$\widetilde{\mathcal{M}} := \bigcup_{t\in[k\tilde{t},(k+1)\tilde{t}]} T(t)\widetilde{\mathcal{M}_d}.$$

It suffices to prove that the set $\widetilde{\mathcal{M}}$ is a precompact exponential attractor for the semigroup $\{T(t) |\ t \in \mathbb{R}_+\}$. The proof can then be completed as in the discrete case by showing the corresponding properties for the set $\mathcal{M}$. First, we observe that

$$\widetilde{\mathcal{M}} = \bigcup_{t\in[k\tilde{t},(k+1)\tilde{t}]} T(t)\widetilde{\mathcal{M}_d} = \bigcup_{t\in[k\tilde{t},(k+1)\tilde{t}]} T(t) \bigcup_{n\in\mathbb{N}_0} E^n = \bigcup_{n\in\mathbb{N}_0} \bigcup_{t\in[k\tilde{t},(k+1)\tilde{t}]} T(t)E^n.$$

Step 2: Semi-invariance and Exponential Attraction Property

Let $t \in \mathbb{R}_+$ and $s \in [k\tilde{t}, (k+1)\tilde{t}]$ be arbitrary. Then, $t + s = (k+l)\tilde{t} + s_0$, for some $l \in \mathbb{N}_0$ and $s_0 \in [0, \tilde{t}[$. Using Property $(E1)$ in the proof of Theorem 3.3 we conclude

$$\begin{aligned} T(t) \bigcup_{n\in\mathbb{N}_0} T(s)E^n = \bigcup_{n\in\mathbb{N}_0} T((k+l)\tilde{t} + s_0)E^n &\subset \bigcup_{n\in\mathbb{N}_0} T(k\tilde{t} + s_0)E^{n+l} \\ &\subset \bigcup_{n\in\mathbb{N}_0} \bigcup_{t\in[k\tilde{t},(k+1)\tilde{t}]} T(t)E^n = \widetilde{\mathcal{M}}. \end{aligned}$$

Since $s \in [k\tilde{t}, (k+1)\tilde{t}]$ was arbitrary follows the semi-invariance of the set $\widetilde{\mathcal{M}}$.

To show the exponential attraction property we observe that the smoothing property $(S2)$, the contraction property $(S3)$ and the continuous embedding (H_0) imply

$$\|T(k\tilde{t})u - T(k\tilde{t})v\|_V \leq (\mu\kappa + \lambda)^k \|u - v\|_V \qquad u, v \in B.$$

Let $D \subset V$ be a bounded subset. By Assumption $(S1)$ there exists $T_D \in \mathbb{T}_+$ such that $T(t)D \subset B$ for all $t \geq T_D$. Moreover, if $t \geq T_D + (k+1)\tilde{t}$, then $t = T_D + (k+l)\tilde{t} + s_0$ for some $l \in \mathbb{N}$ and $s_0 \in [0, \tilde{t}[$, and we conclude

$$\begin{aligned}\operatorname{dist}_{\mathrm{H}}^V(T(t)D, \widetilde{\mathcal{M}})) &= \operatorname{dist}_{\mathrm{H}}^V(T(t - T_D - s_0)T(T_D + s_0)D, \bigcup_{t\in[k\tilde{t},(k+1)\tilde{t}[} T(t) \bigcup_{n\in\mathbb{N}_0} E^n) \\ &\leq \operatorname{dist}_{\mathrm{H}}^V(T(t - T_D - s_0)B, \bigcup_{n\in\mathbb{N}_0} T(k\tilde{t})E^n) \\ &\leq \operatorname{dist}_{\mathrm{H}}^V\big(T((k + l)\tilde{t})B, T(k\tilde{t})E^l\big) \leq (\mu\kappa + \lambda)^k \operatorname{dist}_{\mathrm{H}}^V(T(l\tilde{t})B, E^l) \\ &\leq (\mu\kappa + \lambda)^k(2(\nu + \lambda))^l R = (\mu\kappa + \lambda)^k(2(\lambda + \nu))^{\frac{t-T_D-s_0}{\tilde{t}}-k} R = ce^{-\omega t},\end{aligned}$$

for some constant $c \geq 0$ and $\omega := \ln(\frac{1}{2(\nu+\lambda)})^{\frac{1}{\tilde{t}}}$.

Step 3: Precompactness and Finite Fractal Dimension

First, we observe that the semigroup $\{T(t)|\ t \in \mathbb{R}_+\}$ is Hölder continuous in every interval $[t_1 + lh, t_1 + (l + 1)h]$, $l \in \mathbb{N}_0$, where $h := t_2 - t_1$,

$$\|T(s_1)u - T(s_2)u\|_V \leq \zeta|s_1 - s_2|^\theta \qquad \text{for all } u \in B,\ s_1, s_2 \in [t_1 + lh, t_1 + (l + 1)h]. \tag{3.4}$$

Indeed, let $l \in \mathbb{N}_0$ and $s_1, s_2 \in [t_1 + lh, t_1 + (l+1)h]$. Then, $s_1 = t_1 + lh + r_1$, $s_2 = t_1 + lh + r_2$ with $r_1, r_2 \in [0, h]$, and by Assumption $(S4)$ and the semi-invariance of the absorbing set follows

$$\|T(s_1)u - T(s_2)u\|_V = \|T(t_1 + r_1)\,(T(lh)u) - T(t_1 + r_2)\,(T(lh)u)\,\|_V \leq \zeta|s_1 - s_2|^\theta$$

for all $u \in B$.

To prove the precompactness we show that for arbitrary $\epsilon > 0$ the set $\widetilde{\mathcal{M}}$ can be covered by a finite number of ϵ-balls in V. Let $m \in \mathbb{N}$ and $s \in [k\tilde{t}, (k + 1)\tilde{t}]$. Then, the semi-invariance of the absorbing set implies

$$T(s + n\tilde{t})B = T(m\tilde{t})T((n - m)\tilde{t} + s)B \subset T(m\tilde{t})B \qquad \text{for all } n \geq m,$$

and we obtain

$$T(s)E^n \subset T(m\tilde{t})B \qquad \text{for all } n \geq m,\ s \in [k\tilde{t}, (k + 1)\tilde{t}],$$

where we used that the sets $E^n \subset T(n\tilde{t})B$. Consequently, we observe

$$\widetilde{\mathcal{M}} = \bigcup_{s\in[k\tilde{t},(k+1)\tilde{t}]} T(s)\big(\bigcup_{n=0}^{m} E^n \cup \bigcup_{n=m+1}^{\infty} E^n\big) \subset \big(\bigcup_{n=0}^{m} \bigcup_{s\in[k\tilde{t},(k+1)\tilde{t}]} T(s)E^n\big) \cup T(m\tilde{t})B,$$

for all $m \in \mathbb{N}$. If we choose $m \in \mathbb{N}$ sufficiently large such that

$$(2(\nu + \lambda))^m R \leq \epsilon < (2(\nu + \lambda))^{m-1} R$$

holds, the ϵ-balls with centres in the set W^m yield a covering of the iterate $T(m\tilde{t})B$,

$$T(m\tilde{t})B \subset \bigcup_{u\in W^m} B_\epsilon^V(u).$$

We can estimate the number of ϵ-balls needed to cover the set $\widetilde{\mathcal{M}}$ therefore by

$$N_\epsilon^V(\widetilde{\mathcal{M}}) \leq \sharp\Big(\bigcup_{n=0}^{m} \bigcup_{s\in[k\tilde{t},(k+1)\tilde{t}]} T(s)E^n\Big) + \sharp W^m \leq \sharp\Big(\bigcup_{n=0}^{m} \bigcup_{s\in[k\tilde{t},(k+1)\tilde{t}]} T(s)E^n\Big) + N^m,$$

where we used Property $(W2)$ in the proof of Theorem 3.3. It remains to estimate the number of ϵ-balls in V needed to cover the finite union of curves

$$\begin{aligned}\bigcup_{n=0}^{m} \bigcup_{s\in[k\tilde{t},(k+1)\tilde{t}]} T(s)E^n &= \bigcup_{s\in[k\tilde{t},(k+1)\tilde{t}]} T(s)\bigcup_{n=0}^{m} E^n = \bigcup_{s\in[k\tilde{t},(k+1)\tilde{t}]} T(s)\widetilde{E}^m \\ &= \bigcup_{u\in\widetilde{E}^m} \bigcup_{s\in[k\tilde{t},(k+1)\tilde{t}]} T(s)u = \bigcup_{u\in\widetilde{E}^m} T_u([k\tilde{t},(k+1)\tilde{t}],\end{aligned}$$

where the curves $T_u : [k\tilde{t},(k+1)\tilde{t}] \to V$ are defined by $T_u(s) := T(s)u$ for $u \in \widetilde{E}^m$ and the set $\widetilde{E}^m := \bigcup_{n=0}^m E^n$. Property $(E2)$ implies that

$$\sharp(\widetilde{E}^m) = \sharp(\bigcup_{n=0}^{m} E^n) \leq (m+1)\sharp E^m \leq (m+1)^2 N^m.$$

Since we chose $k \in \mathbb{N}$ such that $k\tilde{t} \geq t_1$, we can divide the interval $[k\tilde{t},(k+1)\tilde{t}]$ into at most $p_0 := \lfloor \frac{\tilde{t}}{h} \rfloor + 1$ subintervals I_j, $1 \leq j \leq p_0$, of length less than or equal to $h := t_2 - t_1$, where the semigroup satisfies the Hölder continuity property (3.4),

$$\|T(s_1)u - T(s_2)u\|_V \leq \zeta|s_1 - s_2|^\theta \qquad \text{for all } s_1, s_2 \in I_j,\ u \in B.$$

Here and in the sequel, $\lfloor x \rfloor$ denotes the largest integer less than or equal to $x \in \mathbb{R}$. Let $u \in \widetilde{E}^m$. To construct an ϵ-covering of the image of the curve $T_u([k\tilde{t},(k+1)\tilde{t}])$, if necessary, we further subdivide the intervals I_j into intervals of length less than $(\frac{\epsilon}{2\zeta})^{\frac{1}{\theta}}$, and obtain at most

$$p_1 := \lfloor h(\frac{2\zeta}{\epsilon})^{\frac{1}{\theta}} \rfloor + 1$$

such subintervals I_j^i, $1 \leq i \leq p_1$, for each interval I_j, $1 \leq j \leq p_0$. Choosing an arbitrary point s_j^i in each subinterval I_j^i follows

$$\begin{aligned}\|T_u(r_1) - T_u(r_2)\|_V &= \|T(r_1)u - T(s_j^i)u\|_V + \|T(s_j^i)u - T(r_2)u\|_V \\ &\leq \zeta(|r_1 - s_j^i|^\theta + |s_j^i - r_2|^\theta) < \epsilon,\end{aligned}$$

for all $r_1, r_2 \in I_j^i$, where $1 \leq j \leq p_1, 1 \leq i \leq p_0$. Consequently, we obtain a covering of the image of the curve T_u,

$$T_u([k\tilde{t},(k+1)\tilde{t}]) \subset \bigcup_{j=1}^{p_1}\bigcup_{i=1}^{p_0} B_\epsilon^V(T(s_j^i)u).$$

Constructing in the same way for all $u \in \widetilde{E}^m$ such an ϵ-cover of $T_u([k\tilde{t},(k+1)\tilde{t}])$ we conclude

$$\begin{aligned} N_\epsilon^V(\widetilde{\mathcal{M}}) &\leq \sharp\Big(\bigcup_{n=0}^{m} \bigcup_{s\in[k\tilde{t},(k+1)\tilde{t}]} T(s)E^n\Big) + \sharp W^m \leq \sharp\Big(\bigcup_{u\in\widetilde{E}^m} T_u([k\tilde{t},(k+1)\tilde{t}])\Big) + N^m \\ &\leq p_0p_1(m+1)^2N^m + N^m \leq 2p_0p_1(m+1)^2N^m \leq 2p_0(h(\frac{2\zeta}{\epsilon})^{\frac{1}{\theta}}+1)(m+1)^2N^m, \end{aligned}$$

which proves the precompactness of the set $\widetilde{\mathcal{M}}$ in V.

The choice of m implies

$$m-1 < \frac{\ln\frac{1}{\epsilon} + \ln R}{\ln\frac{1}{2(\nu+\lambda)}},$$

which allows to estimate the fractal dimension of $\widetilde{\mathcal{M}}$ in V,

$$\begin{aligned} \dim_{\mathrm{f}}^V(\widetilde{\mathcal{M}}) &= \limsup_{\epsilon\to 0} \frac{\ln(N_\epsilon^V(\widetilde{\mathcal{M}}))}{\ln\frac{1}{\epsilon}} \\ &\leq \limsup_{\epsilon\to 0} \frac{\ln(2p_0) + 2\ln(m+1) + m\ln(N) + \ln(h(\frac{2\zeta}{\epsilon})^{\frac{1}{\theta}}+1)}{\ln\frac{1}{\epsilon}} \\ &\leq \log_{\frac{1}{2(\nu+\lambda)}}(N) + \frac{1}{\theta}, \end{aligned}$$

and concludes the proof of the theorem. □

3.1.5. Consequences of the Construction and Properties of the Exponential Attractor

An immediate consequence of the existence of exponential attractors is the existence and finite dimensionality of the global attractor. Moreover, the covering method applied in the construction of exponential attractors can directly be used to estimate the fractal dimension of the global attractor.

Theorem 3.5. *Let $\{T(t)|\ t\in\mathbb{T}_+\}$ be a semigroup in the Banach space V, where $\mathbb{T}=\mathbb{Z}$ or $\mathbb{T}=\mathbb{R}$, and the assumptions (H_0) and $(S1)$-$(S3)$ be satisfied. Then, the global attractor $\mathcal{A}$ of the semigroup $\{T(t)|\ t\in\mathbb{T}_+\}$ exists, and its fractal dimension is bounded by*

$$\dim_{\mathrm{f}}^V(\mathcal{A}) \leq \log_{\frac{1}{2(\nu+\lambda)}}\left(N_{\frac{\nu}{\kappa}}^W(B_1^V(0))\right),$$

where $\nu\in(0,\frac{1}{2}-\lambda)$ is arbitrary.

Proof. Without loss of generality we can assume that the absorbing set is positively semi-invariant.

If $\{T(t)|\ t\in\mathbb{Z}_+\}$ is a discrete semigroup the statement follows immediately from Theorem 3.3 and Theorem 3.1. Indeed, the exponential attractor $\mathcal{M}^\nu$ constructed in the proof of Theorem 3.3 is a compact attracting set for the semigroup. Theorem 3.1 and the semi-invariance of the exponential attractor imply $\mathcal{A} = \omega(\mathcal{M}^\nu) \subset \mathcal{M}^\nu$. The bound for the fractal dimension of the global attractor now follows from Proposition 3.1.

If the semigroup $\{T(t)|\, t \in \mathbb{R}_+\}$ is continuous we define the associated discrete semigroup $\{\widetilde{T}(n)|\, n \in \mathbb{Z}_+\}$ by $\widetilde{T}(n) := T(n\tilde{t})$, $n \in \mathbb{Z}_+$. Theorem 3.3 implies the existence of the exponential attractor $\mathcal{M}^\nu_d$ for the semigroup $\{\widetilde{T}(n)|\, n \in \mathbb{Z}_+\}$, and the set $\mathcal{M}^\nu_d$ is a compact attracting set for the time continuous semigroup $\{T(t)|\, t \in \mathbb{R}_+\}$. Indeed, by Assumption $(S1)$ for every bounded set $D \subset V$ there exists $T_D \in \mathbb{R}_+$ such that $T(t)D \subset B$ for all $t \geq T_D$. Let $t > T_D + \tilde{t}$, then $t = k\tilde{t} + T_D + s_0$, for some $k \in \mathbb{N}$ and $s_0 \in [0, \tilde{t}[$, and we observe

$$\mathrm{dist}^V_{\mathrm{H}}(T(t)D, \widetilde{\mathcal{M}^\nu_d)}) = \mathrm{dist}^V_{\mathrm{H}}(T(k\tilde{t})T(T_D + s_0)D, \mathcal{M}^\nu_d) \leq \mathrm{dist}^V_{\mathrm{H}}(T(k\tilde{t})B, \mathcal{M}^\nu_d).$$

Since $\mathcal{M}^\nu_d$ is an exponential attractor for the discrete semigroup now follows the exponential attraction property of the set $\mathcal{M}^\nu_d$ for the time continuous semigroup.

Theorem 3.1 implies that the global attractor of the semigroup $\{T(t)|\, t \in \mathbb{R}_+\}$ exists and is given by the ω-limit set $\mathcal{A} = \omega(\mathcal{M}^\nu_d)$. By definition the global attractor is strictly invariant. To derive an estimate for its fractal dimension we replace the absorbing set B in the construction of the sets W^n, $n \in \mathbb{N}_0$, in the proof of Theorem 3.3 by the global attractor $\mathcal{A}$ and construct coverings of the iterates $T(n\tilde{t})\mathcal{A} = \mathcal{A}$, where $n \in \mathbb{N}_0$. This leads to a family of sets V^n, $n \in \mathbb{N}_0$, that satisfies the following properties:

(V1) $\quad V^n \subset T(n\tilde{n})\mathcal{A} = \mathcal{A}$,

(V2) $\quad \sharp V^n \leq N^n$,

(V3) $\quad \mathcal{A} = T(n\tilde{n})\mathcal{A} \subset \bigcup_{u \in V^n} B^V_{(2(\nu+\lambda))^n R}(u)$,

where $N := N^W_{\frac{\nu}{\kappa}}(B^V_1(0))$.

Let $\epsilon > 0$. To estimate the number of ϵ-balls in V needed to cover the global attractor $\mathcal{A}$ we choose $m \in \mathbb{N}$ sufficiently large such that the relation

$$(2(\nu+\lambda))^m R \leq \epsilon < (2(\nu+\lambda))^{m-1} R$$

holds. Property $(V3)$ then yields the covering

$$\mathcal{A} \subset \bigcup_{u \in V^m} B^V_\epsilon(u),$$

and $\sharp V^m \leq N^m$ by Property $(V2)$. The estimate for the fractal dimension of the global attractor now follows similarly as in the proof of Theorem 3.3,

$$\dim^V_{\mathrm{f}}(\mathcal{A}) = \limsup_{\epsilon \to 0} \frac{\ln(N^V_\epsilon(\mathcal{A}))}{\ln \frac{1}{\epsilon}} \leq \limsup_{\epsilon \to 0} \frac{m \ln(N)}{\ln \frac{1}{\epsilon}} \leq \log_{\frac{1}{2(\nu+\lambda)}}(N),$$

which concludes the proof of the theorem.

□

Even for time continuous semigroups the properties (H_0) and $(S1)$-$(S3)$ imply the existence and finite dimensionality of the global attractor, and the bound on its fractal dimension is the same in the discrete and continuous case (see Theorem 3.5). The Hölder continuity property $(S4)$ is only needed for the construction of the time continuous exponential attractor and not required to estimate the fractal dimension of the global attractor. We propose to weaken the semi-invariance property of exponential attractors for time continuous semigroups and consider pullback exponential attractors in Section 3.2.4. This avoids the artificial increase in the fractal dimension of the time continuous exponential attractor.

In the following proposition we illustrate the relationship between global and exponential attractors. For discrete semigroups the exponential attractor is obtained by adding to the global attractor an appropriate countable set of points such that all bounded subsets of the phase space are attracted exponentially fast (compare also with the construction of exponential attractors in [26], Chapter 2).

Proposition 3.2. *Let $\{T(n)|\ n \in \mathbb{Z}_+\}$ be a discrete semigroup in the Banach space V and the assumptions (H_0) and $(S1)$-$(S3)$ be satisfied. Then, the exponential attractor of Theorem 3.3 can be represented as*

$$\mathcal{M} = \mathcal{A} \cup \bigcup_{n\in\mathbb{N}_0} E^n,$$

where $\mathcal{A}$ denotes the global attractor of the semigroup. We refer to the proof of Theorem 3.3 for the definition and construction of the family of sets E^n, $n \in \mathbb{N}_0$.

Consequently, the set $\mathcal{A} \cup \bigcup_{n\in\mathbb{N}_0} E^n$ is closed.

Proof. We defined the exponential attractor for the semigroup $\{T(n)|\ n \in \mathbb{Z}_+\}$ in the proof of Theorem 3.3 by

$$\mathcal{M} = \overline{\bigcup_{n\in\mathbb{N}_0} E^n}^{\|\cdot\|_V}.$$

Consequently, the inclusion $\mathcal{A} \cup \bigcup_{n\in\mathbb{N}_0} E^n \subset \mathcal{M}$ follows immediately from the fact that any exponential attractor $\mathcal{M}$ contains the global attractor $\mathcal{A}$.

It remains to prove the relation $\mathcal{M} \subset \mathcal{A} \cup \bigcup_{n\in\mathbb{N}_0} E^n$. Theorem 3.1 states that the global attractor coincides with the ω-limit set of the exponential attractor, $\mathcal{A} = \omega(\mathcal{M})$. Moreover, the ω-limit set of a subset $A \subset V$ can be characterized by

$$\omega(A) = \Big\{x \in V|\ \text{there exist sequences } \{t_k\}_{k\in\mathbb{N}} \subset \mathbb{Z}_+, \lim_{k\to\infty} t_k = \infty,\ \{x_k\}_{k\in\mathbb{N}} \subset A \\ \text{such that } \lim_{k\to\infty} T(t_k)x_k = x\Big\}$$

(**see** [69], Chapter I, Section 1.1). Let $x \in \mathcal{M}$, then there exists a sequence $\{x_k\}_{k\in\mathbb{N}}$ in $\bigcup_{n\in\mathbb{N}_0} E^n$ such that $\lim_{k\to\infty} x_k = x$ in V. Furthermore, for every $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that $x_k \in E^{n_k}$. If $n_0 := \sup_{k\in\mathbb{N}}\{n_k\} < \infty$ the sequence $\{x_k\}_{k\in\mathbb{N}}$ is contained in the finite set $\bigcup_{n=0}^{n_0} E^n$ and consequently, the limit $x \in \bigcup_{n=0}^{n_0} E^n \subset \mathcal{A} \cup \bigcup_{n\in\mathbb{N}_0} E^n$.

Otherwise, if $\sup_{k\in\mathbb{N}}\{n_k\}=\infty$, there exists a subsequence $\{n_{k_l}\}_{l\in\mathbb{N}}$ such that $\lim_{l\to\infty} n_{k_l}=\infty$. By the definition of the sets E^n, $n\in\mathbb{N}$, for every n_{k_l} there exist $t_{k_l}\in\mathbb{Z}_+$ and $x_{k_l}\in\bigcup_{n\in\mathbb{N}_0} W^n\subset\bigcup_{n\in\mathbb{N}_0} E^n$ such that $n_{k_l}=T(t_{k_l})x_{k_l}$. Moreover, $\lim_{l\to\infty} t_{k_l}=\infty$, and we conclude by the characterization of the ω-limit set that $x\in\omega(\mathcal{M})=\mathcal{A}$. □

Remark 3.1. *Let $\{T(n)|\ n\in\mathbb{Z}_+\}$ be a discrete semigroup, the hypothesis of Theorem 3.3 be satisfied and $\mathcal{A}$ and $\mathcal{M}^\nu$ be the corresponding global and exponential attractors. Proposition 3.2 implies that $\bigcup_{n\in\mathbb{N}} E^n\cap\mathcal{A}$ is a countable dense subset of the global attractor.*

Moreover, the Hausdorff dimensions of the global attractor $\mathcal{A}$ and exponential attractors $\mathcal{M}^\nu$ coincide,

$$\dim_{\mathrm{H}}^V(\mathcal{M}^\nu)=\dim_{\mathrm{H}}^V(\mathcal{A}),$$

since the Hausdorff dimension of every countable set is zero (see Section 3.1.2). This indicates that the Hausdorff dimension is not an appropriate measure to control the size of exponential attractors. Requiring finite Hausdorff dimension for the exponential attractor we could add an arbitrary countable semi-invariant set to the global attractor without changing its dimension (see also [26], Chapter 7). The more points we add the faster is the rate of convergence to the attractor. This is impossible is we require finite fractal dimension for the exponential attractor. In the proof of Theorem 3.3 it is essential in the construction to control the number of points we add in each step, that is, the cardinality of the sets E^n, $n\in\mathbb{N}_0$.

Exponential attractors of semigroups that are asymptotically compact in the space V were not considered previously, except in [32] (Theorem 1.3), where the existence for discrete semigroups was shown, but under different and more restrictive assumptions which are difficult to verify in applications. We now discuss other settings for the semigroup to recover and generalize former results. In the particular case that $\lambda=0$ immediately follows the existence of exponential attractors for semigroups that satisfy the smoothing property. Moreover, we consider semigroups that are asymptotically compact in the weaker space W and prove the existence of exponential attractors in the space W. These situations were addressed previously (among others see [13], [19], [35], [32], [33]). In both cases, it suffices that the absorbing set is bounded in W and, if the semigroup is time continuous, that the Hölder continuity is satisfied with respect to the metric of W.

$(S4)'$ The semigroup $\{T(t)|\ t\in\mathbb{R}_+\}$ is Hölder continuous in time: There exist $0\le t_1<t_2$ such that

$$\|T(s_1)u-T(s_2)u\|_W\le\zeta|s_1-s_2|^\theta\qquad\text{for all } s_1,s_2\in[t_1,t_2],u\in B$$

for some constant $\zeta\ge 0$ and exponent $0<\theta\le 1$.

The following corollary generalizes the results in [19] (Corollary 2.6), in [32] (Theorem 1.1), and in [13] (Corollary 2.9), and improves the estimates on the fractal dimension of the attractor.

Corollary 3.1. *Let $\{S(t)|\ t \in \mathbb{T}_+\}$ be a semigroup in the Banach space V and the assumptions (H_0) and $(S2)$ be satisfied. Moreover, we assume that Property $(S1)$ holds with $\{T(t)|\ t \in \mathbb{T}_+\}$ replaced by $\{S(t)|\ t \in \mathbb{T}_+\}$, where it suffices that the absorbing set is bounded in the metric of W. If the semigroup is continuous, we additionally suppose that it satisfies the Hölder continuity property $(S4)'$. Then, for any $\nu \in (0, \frac{1}{2})$ there exists an exponential attractor $\mathcal{M} \equiv \mathcal{M}^\nu$ for the semigroup $\{S(t)|\ t \in \mathbb{T}_+\}$, and its fractal dimension is bounded by*

$$\dim_{\mathrm{f}}^V(\mathcal{M}) \leq \log_{\frac{1}{2\nu}} \left(N^W_{\frac{\nu}{\kappa}}(B^V_1(0)) \right)$$

in the discrete case and by

$$\dim_{\mathrm{f}}^V(\mathcal{M}) \leq \frac{1}{\theta} + \log_{\frac{1}{2\nu}} \left(N^W_{\frac{\nu}{\kappa}}(B^V_1(0)) \right)$$

in the time continuous case.

Moreover, the global attractor of the semigroup exists and an estimate for its fractal dimension is given by

$$\dim_{\mathrm{f}}^V(\mathcal{A}) \leq \log_{\frac{1}{2\nu}} \left(N^W_{\frac{\nu}{\kappa}}(B^V_1(0)) \right)$$

for both discrete and continuous semigroups. For the existence of the global attractor Assumption $(S4)'$ is not required.

Proof. If the absorbing set B is bounded in W the smoothing property $(S2)$ implies that the set $S(\tilde{t})B$ is a bounded absorbing set for the semigroup $\{S(t)|\ t \in \mathbb{T}_+\}$ in V. For discrete semigroups the corollary follows immediately from Theorem 3.3 and Theorem 3.5.

If the semigroup is time continuous and satisfies Assumption $(S4)'$ we observe

$$\begin{aligned} \|S(\tilde{t} + s_1)u - S(\tilde{t} + s_2)u\|_V &= \|S(\tilde{t})S(s_1)u - S(\tilde{t})S(s_2)u\|_V \\ &\leq \kappa \|S(s_1)u - S(s_2)u\|_W \leq \kappa\zeta |s_1 - s_2|^\theta \end{aligned}$$

for all $s_1, s_2 \in [t_1, t_2]$ and $u \in B$, where we used the smoothing property $(S2)$. Consequently, the semigroup $\{S(t)|\ t \in \mathbb{R}_+\}$ is Hölder continuous with respect to the metric in V and satisfies Property $(S4)$ in the interval $[\tilde{t} + t_1, \tilde{t} + t_2]$. Theorem 3.4 and Theorem 3.5 now imply the statement of the corollary in the time continuous case. □

The following theorem addresses attractors of asymptotically compact semigroups in the weaker space W and generalizes Proposition 2.7 in [13] for time continuous semigroups. In the discrete case we recover Proposition 1 in [33]. Such attractors are also called bi-space attractors or (V, W)-attractors. To this end we replace the assumptions accordingly.

$(S1)'$ There exists a bounded absorbing set $B \subset W$ for the semigroup $\{T(t)|\ t \in \mathbb{T}_+\}$ in W: For every bounded subset $D \subset W$ there exists $T_D \in \mathbb{T}_+$ such that

$$T(t)D \subset B \qquad \text{for all } t \geq T_D.$$

$(S3)'$ The family $\{C(t)|\ t \in \mathbb{T}_+\}$ is a contraction in W within the absorbing set:

$$\|C(\tilde{t})u - C(\tilde{t})v\|_W \leq \lambda \|u - v\|_W \qquad \text{for all } u, v \in B,$$

for some constant $0 \leq \lambda < \frac{1}{2}$.

Theorem 3.6. *Let $\{T(t)|\ t \in \mathbb{T}_+\}$ be a semigroup in the Banach space W and the assumptions (H_0), $(S1)'$, $(S2)$ and $(S3)'$ be satisfied. In the time continuous case, $\mathbb{T} = \mathbb{R}$, we additionally assume that the semigroup fulfils the Hölder continuity assumption $(S4)'$. Then, for any $\nu \in (0, \frac{1}{2} - \lambda)$ there exists an exponential attractor $\mathcal{M}^\nu \equiv \mathcal{M}$ for the semigroup $\{T(t)|\ t \in \mathbb{T}_+\}$ in W, and its fractal dimension can be estimated by*

$$\dim_f^W(\mathcal{M}) \leq \log_{\frac{1}{2(\nu+\lambda)}} \left(N^W_{\frac{\nu}{\kappa}}(B^V_1(0)) \right)$$

in the discrete case and by

$$\dim_f^W(\mathcal{M}) \leq \frac{1}{\theta} + \log_{\frac{1}{2(\nu+\lambda)}} \left(N^W_{\frac{\nu}{\kappa}}(B^V_1(0)) \right)$$

in the continuous case.

Moreover, the global attractor $\mathcal{A}$ of the semigroup exists, and its fractal dimension is bounded by

$$\dim_f^W(\mathcal{A}) \leq \log_{\frac{1}{2(\nu+\lambda)}} \left(N^W_{\frac{\nu}{\kappa}}(B^V_1(0)) \right)$$

for both discrete and time continuous semigroups. The Hölder continuity $(S4)'$ of the semigroup is not required for the existence of the global attractor.

Proof. Without loss of generality we can assume that the absorbing set is positively semi-invariant.

We indicate how to adapt the covering method in the proof of Theorem 3.3 to the different setting. Let $\nu \in (0, \frac{1}{2} - \lambda)$ be fixed, $R > 0$ and $v_0 \in B$ such that $B \subset B^W_R(v_0)$. Moreover, we choose $w_1, \ldots, w_N$ such that

$$B^V_1(0) \subset \bigcup_{i=1}^N B^W_{\frac{\nu}{\kappa}}(w_i),$$

where $N := N^W_{\frac{\nu}{\kappa}}(B^V_1(0))$. We construct by induction the family of sets W^n, $n \in \mathbb{N}_0$, with the following properties:

(W1) $\quad W^n \subset T(n\tilde{n})B \subset B$,

(W2) $\quad \sharp W^n \leq N^n$,

(W3) $\quad T(n\tilde{n})B \subset \bigcup_{u \in W^n} B^W_{(2(\nu+\lambda))^n R}(u)$.

Defining the set $W^0 := \{v_0\}$ the properties are certainly satisfied for $n = 0$. We assume the sets W^l are already constructed for all $l \leq n$, $n \in \mathbb{N}$, which yields the covering

$$T(n\tilde{n})B \subset \bigcup_{u \in W^n} B^W_{(2(\nu+\lambda))^n R}(u).$$

To construct a covering of the iterate $T((n+1)\tilde{n})B$ let $u \in W^n$. The smoothing property $(S2)$ implies

$$\|S(\tilde{n})u - S(\tilde{n})v\|_V \leq \kappa\|u - v\|_W < \kappa(2(\nu+\lambda))^n R \qquad \text{for all } v \in B^W_{(2(\nu+\lambda))^n R}(u) \cap B,$$

and consequently,

$$\begin{aligned} S(\tilde{n})\big(B^W_{(2(\nu+\lambda))^n R}(u) \cap T(n\tilde{n})B\big) &\subset B^V_{(2(\nu+\lambda))^n \kappa R}(u) \\ &\subset \bigcup_{i=1}^{N} B^W_{(2(\nu+\lambda))^n \nu R}((2(\nu+\lambda))^n \kappa R w_i + S(\tilde{n})u). \end{aligned}$$

To shorten notations we define $y_i := (2(\nu+\lambda))^n \kappa R w_i + S(\tilde{n})u$, where $i = 1, \ldots, N$. The contraction property $(S3)'$ yields

$$\|C(\tilde{n})u - C(\tilde{n})v\|_W \leq \lambda\|u - v\|_W < \lambda(2(\nu+\lambda))^n R \qquad \text{for all } v \in B^W_{(2(\nu+\lambda))^n R}(u) \cap B,$$

and consequently, we obtain the covering

$$\begin{aligned} T(\tilde{n})\big(B^W_{(2(\nu+\lambda))^n R}(u) \cap T(n\tilde{n})B\big) &= \big(S(\tilde{n}) + C(\tilde{n})\big)\big(B^W_{(2(\nu+\lambda))^n R}(u) \cap T(n\tilde{n})B\big) \\ &\subset \bigcup_{i=1}^{N} B^W_{(2(\nu+\lambda))^n \nu R}(y_i) \cup B^W_{(2(\nu+\lambda))^n \lambda R}(C(\tilde{n})u) \\ &\subset \bigcup_{i=1}^{N} B^W_{(2(\nu+\lambda))^n (\nu+\lambda) R}(y_i + C(\tilde{n})u). \end{aligned}$$

If necessary, doubling the radii of the balls we can choose centres within the set

$$T(\tilde{n})\big(B^W_{(2(\nu+\lambda))^n R}(u) \cap T(n\tilde{n})B\big) \subset T((n+1)\tilde{n})B.$$

We construct in the same way for every $u \in W^n$ such a covering of

$$T(\tilde{n})\big(B^W_{(2(\nu+\lambda))^n R}(u) \cap T(n\tilde{n})B\big)$$

by balls with radius $(2(\nu+\lambda))^{n+1}R$ in W and denote the union of the new sets of centres by W^{n+1}. It follows

$$T((n+1)\tilde{n})B \subset T(\tilde{n})\big(\bigcup_{u \in W^n} B^W_{(2(\nu+\lambda))^n R}(u) \cap T(n\tilde{n})B\big) \subset \bigcup_{u \in W^{n+1}} B^W_{(2(\nu+\lambda))^{n+1} R}(u),$$

by construction the set $W^{n+1} \subset T((n+1)\tilde{n})B$, and $\sharp W^{n+1} \leq N \sharp W^n \leq N^{n+1}$. This concludes the proof of the properties $(W1)$-$(W3)$.

If the semigroup is discrete we set $E^0 := W^0$ and define the sets E^n, $n \in \mathbb{N}$, iteratively by $E^n := W^n \cup \bigcup_{k=1}^{n} \bigcup_{l=1}^{\tilde{n}} T((k-1)\tilde{n}+l)W^{n-k}$. Exactly as in the proof of Theorem 3.3 follows that the set $\mathcal{M} = \overline{\widetilde{\mathcal{M}}}^{\|\cdot\|_W}$, where

$$\widetilde{\mathcal{M}} = \bigcup_{n \in \mathbb{N}_0} E^n,$$

is an exponential attractor for the semigroup $\{T(t) |\, t \in \mathbb{Z}_+\}$ in W.

In the time continuous case we use the method above to construct the exponential attractor $\mathcal{M}_d$ for the associated discrete semigroup $\{\widetilde{T}(n) |\, n \in \mathbb{Z}_+\}$, where $\widetilde{T}(n) := T(n\tilde{t})$, $n \in \mathbb{Z}_+$. We choose $k \in \mathbb{N}_0$ sufficiently large such that $k\tilde{t} \geq t_1$, and define $\mathcal{M} := \overline{\widetilde{\mathcal{M}}}^{\|\cdot\|_W}$, where

$$\widetilde{\mathcal{M}} := \bigcup_{s \in [k\tilde{t}, (k+1)\tilde{t}]} T(s)\mathcal{M}_d.$$

Repeating the arguments in the proof of Theorem 3.4 implies that $\mathcal{M}$ is an exponential attractor in W for the time continuous semigroup $\{T(t) |\, t \in \mathbb{R}_+\}$.

The existence of the global attractor $\mathcal{A}$ and the bound on its fractal dimension can be shown as in the proof of Theorem 3.5, where the Hölder continuity $(S4)'$ was not applied. □

3.2. Non-Autonomous Evolution Equations

3.2.1. Evolution Processes and Non-Autonomous Global Attractors

We now analyse the existence of exponential attractors in non-autonomous problems. Since the solutions of non-autonomous initial value problems depend on both the elapsed time after starting and the initial time, the rule of time evolution of the associated dynamical system is described by a two-parameter family of operators. Here and in the sequel, (X, d_X) denotes a complete metric space and $\mathbb{T} = \mathbb{Z}$ or $\mathbb{T} = \mathbb{R}$.

Definition 3.7. *The family $\{U(t,s) |\, t \geq s\}_{t,s \in \mathbb{T}}$ of operators $U(t,s) : X \to X$ is called an **evolution process** in X if it satisfies the properties*

$$\begin{aligned} &U(t,s) \circ U(s,r) = U(t,r) && t \geq s \geq r, \\ &U(t,t) = Id && t \in \mathbb{T}, \\ &(t,s,x) \mapsto U(t,s)x && \textit{is continuous from } \mathcal{T} \times X \to X, \end{aligned}$$

where $\mathcal{T} := \{(t,s) \in \mathbb{T} \times \mathbb{T} \,|\, t \geq s\}$.

If the operators $U(t,s)$, $t \geq s$, depend only on the difference $t-s$,

$$U(t,s) = U(t-s,0) \qquad \text{for all } t \geq s,\ t,s \in \mathbb{T},$$

we call $\{U(t,s)|\ t \geq s\}$ an ***autonomous evolution process****.*

Moreover, if $\mathbb{T} = \mathbb{R}$ the family of operators $\{U(t,s)|\ t \geq s\}$ is called a ***time continuous evolution process*** *and in the case $\mathbb{T} = \mathbb{Z}$ a* ***discrete evolution process.***

Evolution processes extend the notion of semigroups. Indeed, if $\{T(t)|\ t \in \mathbb{T}_+\}$ is a semigroup in the metric space X, the operators $U(t,s) := T(t-s)$, $t \geq s$, form an autonomous evolution process in X. Conversely, if the evolution process $\{U(t,s)|\ t \geq s\}$ is autonomous, the operators $T(t-s) := U(t,s), t \geq s$, satisfy the properties of a semigroup in X.

While the theory of attractors of autonomous dynamical systems is well-established, its counterpart in the non-autonomous setting is far more complex and less understood. Different concepts were proposed to generalize the notion of global attractors of semigroups for evolution processes ([11], [12], [17]). One of the first attempts was to consider uniform attractors. Uniform attractors of evolution processes are fixed compact sets that attract all bounded subsets of the phase space uniformly with respect to initial time. This concept is well adapted for certain classes of non-autonomous functions and for small non-autonomous perturbations of autonomous problems (see [12] and [42]). However, for general non-autonomous terms in the equation the notion of uniform attractors is not appropriate what we illustrate in the following example.

Example 3.3. *The solution of the non-autonomous ODE*

$$\begin{aligned} \frac{d}{dt}x(t) &= -x(t) + t && t > s, \\ x(s) &= x_s && s \in \mathbb{R},\ x_s \in \mathbb{R}, \end{aligned}$$

is the function $x : [s,\infty[\to \mathbb{R}$, $x(t;s,x_s) = (x_s + 1 - s)e^{-(t-s)} + t - 1$. Since every solution becomes unbounded when time t tends to infinity there does not exist a fixed bounded subset of $\mathbb{R}$ that attracts all solutions.

On the other hand, the difference of two solutions satisfies the initial value problem

$$\begin{aligned} \frac{d}{dt}x(t) &= -(x(t) - y(t)) && t > s, \\ x(s) - y(s) &= x_s - y_s && s \in \mathbb{R},\ x_s, y_s \in \mathbb{R}, \end{aligned}$$

and consequently, $x(t;s,x_s) - y(t;s,y_s) = (x_s - y_s)e^{-(t-s)}$ for $t \geq s$. When time t tends to infinity all solutions approximate each other exponentially fast and converge to the solution $\tilde{x} : \mathbb{R} \to \mathbb{R}$, $\tilde{x}(t) = t - 1$. Consequently, in spite of the fact that no bounded attracting set exists the system satisfies a certain property of attraction.

To allow for more general non-autonomous terms in the equation, requires to weaken the concept of uniform attractors, which leads to families of time-dependent sets instead of a fixed bounded attracting set (for instance see [11] or [17]). In particular, the notion of pullback attraction turned out to be useful to study the longtime dynamics of evolution processes.

Definition 3.8. *The family of non-empty subsets* $\{\mathcal{A}(t)|\ t \in \mathbb{T}\}$ *of* X *is called the* ***(global) pullback attractor of the evolution process*** $\{U(t,s)|\ t \geq s\}$ *if the sets* $\mathcal{A}(t)$ *are compact, for all* $t \in \mathbb{T}$*, and the family* $\{\mathcal{A}(t)|\ t \in \mathbb{T}\}$ *is strictly invariant,*

$$U(t,s)\mathcal{A}(s) = \mathcal{A}(t) \qquad \text{for all } t \geq s.$$

Moreover, it pullback attracts all bounded subsets of X*; that is, for every bounded set* $D \subset X$ *and time* $t \in \mathbb{T}$

$$\lim_{s\to\infty} \mathrm{dist_H}(U(t,t-s)D, \mathcal{A}(t)) = 0,$$

and $\{\mathcal{A}(t)|\ t \in \mathbb{T}\}$ *is minimal within the families of closed subsets that pullback attract all bounded subsets of* X*.*

If an evolution process possesses the uniform attractor follows the existence of the pullback attractor. In particular, for certain classes of non-autonomous terms it was shown that the the pullback attractor $\{\mathcal{A}(t)|\ t \in \mathbb{T}\}$ reflects the structure of the uniform attractor $\mathcal{A}_{un}$,

$$\mathcal{A}_{un} = \bigcup_{t\in\mathbb{T}} \mathcal{A}(t)$$

(see [12], Theorem 6.2 in Chapter IV).

If we compare Definition 3.8 with the definition of global attractors for semigroups the minimality is an additional property which is needed to ensure uniqueness of the pullback attractor since non-autonomous invariance is a weaker concept than the invariance of a fixed set in the autonomous setting. This is illustrated the following example.

Example 3.4. *The initial value problem*

$$\begin{aligned} \frac{d}{dt}x(t) &= -x(t) && t > s, \\ x(s) &= x_s && s \in \mathbb{R},\ x_s \in \mathbb{R}, \end{aligned}$$

generates an evolution process $\{U(t,s)|\ t \geq s\}$ *in* $\mathbb{R}$*, which is defined by the operators* $U(t,s) : \mathbb{R} \to \mathbb{R}$*,* $x_s \mapsto x_s e^{-(t-s)}$*, where* $t \geq s$*. We observe that for every* $\alpha > 0$ *the family of compact sets* $\{\mathcal{A}_\alpha(t)|\ t \in \mathbb{R}\}$*, where* $\mathcal{A}_\alpha(t) := [-\alpha e^{-t}, \alpha e^{-t}]$*, is invariant and pullback attracts all bounded subsets of* $\mathbb{R}$*.*

If we replace the pullback attraction in the Definition 3.8 by forwards convergence; that is, for every bounded subset $D \subset X$ and $t \in \mathbb{T}$

$$\lim_{s\to\infty} \mathrm{dist_H}(U(t+s,t)D, \mathcal{A}(t+s)) = 0,$$

the family $\{\mathcal{A}(t)|\ t \in \mathbb{T}\}$ is called the ***forwards attractor*** for the evolution process $\{U(t,s)|\ t \geq s\}$. If the pullback (forwards) convergence to the attractor holds uniformly in time $t \in \mathbb{T}$, it implies the forwards (pullback) convergence and the attractors coincide. We then call the family $\{\mathcal{A}(t)|\ t \in \mathbb{T}\}$ a ***uniform forwards attractor*** or ***uniform pullback attractor*** for the process. However, these concepts are not related in general (see [11]). Some evolution processes possess the pullback but no forwards attractor and vice versa. In other cases both attractors exists, but do not coincide. Finally, we remark that for autonomous evolution processes the pullback convergence is equivalent to the forwards convergence. In this case the pullback attractor coincides with the global attractor of the associated semigroup.

Remark 3.2. *The pullback attractor of the evolution process generated by the initial value problem in Example 3.3 consists of the singleton sets $\mathcal{A}(t) = \{t-1\}$, $t \in \mathbb{R}$. It is also the forwards attractor of the evolution process.*

Similarly, we observe that the pullback attracting family of compact non-autonomous sets $\{\mathcal{A}_\alpha(t)|\ t \in \mathbb{R}\}$ in Example 3.4 attracts all bounded subsets of $\mathbb{R}$ in the forwards sense as well. It illustrates that for the uniqueness of non-autonomous attractors it is necessity to require the minimality property. The forwards attractor of the evolution process coincides with the pullback attractor and consists of the singleton set $\{\mathcal{A}(t)|\ t \in \mathbb{R}\} = \{0\}$.

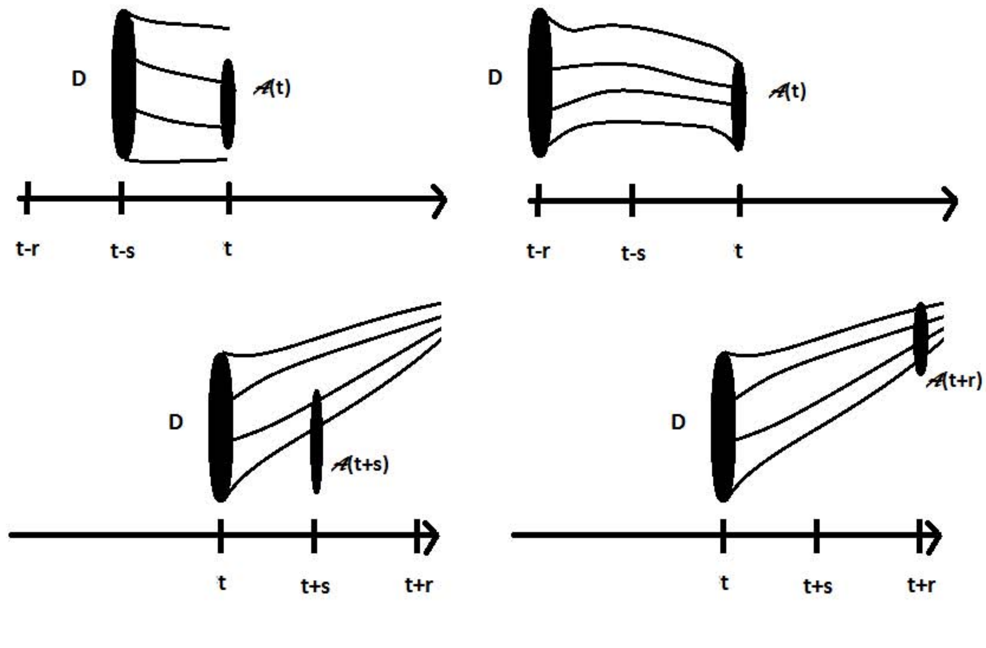


Figure 3.1.: Pullback and Forwards Attraction

Pullback attractors proved to be a useful concept to study the limiting dynamics of non-autonomous systems in various applications. Comparing with forwards attraction pullback attractors have the advantage that convergence to a fixed target is shown, not to a moving target which is generally difficult. However, to capture the complete asymptotic dynamics of non-autonomous systems it is necessary to take both the forwards and the pullback attraction into account. In a certain sense, the pullback attractor is related to the past of the system while the forwards attractor reflects the future limiting dynamics of the system. The pullback limit does not signify going backwards in time, it is the limit when the initial time tends to $-\infty$ as illustrated in Figure 3.1. If we are interested in the states of a non-autonomous system at a certain time $t \in \mathbb{T}$, all trajectories that have started in the distant past and have been evolving for a long time are well approximated by the states of the system within the section $\mathcal{A}(t)$ of the pullback attractor. The future asymptotic behaviour of the system however may be different and is described by the forwards attractor.

Global pullback attractors have the same nice properties and drawbacks as global attractors of semigroups, which motivates to generalize the notion of autonomous exponential attractors and to define pullback exponential attractors for evolution processes (see [19] and [49]). Like exponential attractors of semigroups pullback exponential attractors are not unique.

Definition 3.9. *Let $\{U(t,s)|\ t \geq s\}$ be an evolution process in the metric space (X, d_X). We call the family of non-autonomous sets $\mathcal{M} = \{\mathcal{M}(t)|\ t \in \mathbb{T}\}$ a* ***pullback exponential attractor for the evolution process*** *$\{U(t,s)|\ t \geq s\}$ if*

(i) for all $t \in \mathbb{T}$ the subset $\mathcal{M}(t) \subset X$ is non-empty and compact,

(ii) the family $\mathcal{M}$ is positively semi-invariant; that is,

$$U(t,s)\mathcal{M}(s) \subset \mathcal{M}(t) \qquad \text{for all } t \geq s,$$

(iii) the fractal dimension of the sections $\mathcal{M}(t)$, $t \in \mathbb{T}$, is uniformly bounded,

$$\sup_{t\in\mathbb{T}}\{\dim_{\mathrm{f}}^X(\mathcal{M}(t))\} < \infty,$$

(iv) and $\mathcal{M}$ exponentially pullback attracts all bounded subsets of X: There exists a positive constant $\omega > 0$ such that for every bounded subset $D \subset X$ and every $t \in \mathbb{T}$

$$\lim_{s\to\infty} e^{\omega s}\mathrm{dist}_{\mathrm{H}}(U(t,t-s)D, \mathcal{M}(t)) = 0.$$

The construction of exponential attractors for discrete semigroups in [33] was extended for discrete non-autonomous problems by using the concept of forwards attractors in [32]. An explicit algorithm for discrete evolution processes that satisfy the smoothing property was developed and in an application also a time continuous exponential attractor was constructed. Based on these results very recently, the construction was modified considering the pullback approach, and the algorithm was generalized for time continuous evolution processes in [19] and [49]. The constructions are similar, require strong regularity assumptions on the process and restrictive assumptions with respect to the pullback attraction.

In the following section we shortly summarize previous work before we present a different construction for time continuous pullback exponential attractors in Section 3.2.3. In [19] and [49] the existence of a fixed bounded pullback absorbing set was assumed. This allows the pullback attractor to be unbounded in the future, but it is always uniformly bounded in the past. In this case the theory of global (and exponential) pullback attractors essentially simplifies, and similar results as in the autonomous case are valid (see Section 3.2.2).

We modify the construction, show the existence of pullback exponential attractors under significantly weaker hypothesis and obtain better estimates for the fractal dimension of the sections of the attractor. Moreover, instead of a fixed bounded absorbing set we consider a family of time-dependent absorbing sets which can even grow in the past, and obtain a pullback exponential attractor with sections, that are not necessarily uniformly bounded in the past. If the pullback exponential attractor exists, it contains the global pullback attractor and immediately implies its existence and the finite dimensionality of its sections. Existence proofs for global pullback attractors of asymptotically compact processes often require the boundedness of the global pullback attractor in the past (see [8]). Our main theorem implies existence results for global pullback attractors. In particular, the finite dimensionality of pullback attractors that are not uniformly bounded in the past was an open problem (see Section 1 in [49] or Remark 3.2 in [50]).

3.2.2. Previous Results: Existence of Global and Exponential Pullback Attractors

Global Pullback Attractors

The following theorem characterizes the evolution processes possessing a global pullback attractor and generalizes Theorem 3.1 for evolution processes. For its proof we refer to [17].

Theorem 3.7. *Let $\{U(t,s)|\ t \geq s\}$ be an evolution process in a complete metric space X. Then, the following statements are equivalent:*

(a) The evolution process $\{U(t,s)|\ t \geq s\}$ possesses a global pullback attractor.

(b) There exists a family of compact subsets $\{K(t)|\ t \in \mathbb{T}\}$ of X such that for all $t \in \mathbb{T}$ the set $K(t)$ pullback attracts all bounded subsets of X at time t.

Furthermore, the pullback global attractor is given by

$$\mathcal{A}(t) = \overline{\bigcup_{\substack{D \subset X \\ \text{bounded}}} \omega(D,t)} \qquad t \in \mathbb{T},$$

where $\omega(D,t)$ denotes the pullback ω-limit set of the set $D \subset X$ at time instant $t \in \mathbb{T}$.

The ***pullback ω-limit set*** of the subset $D \subset X$ at time instant $t \in \mathbb{T}$ is defined by

$$\omega(D,t) := \bigcap_{r \geq 0} \overline{\bigcup_{s \geq r} U(t,t-s)D}.$$

Theorem 3.7 implies that if an evolution process possesses a pullback exponential attractor $\{\mathcal{M}(t)|\ t \in \mathbb{T}\}$ immediately follows the existence of the global pullback attractor $\{\mathcal{A}(t)|\ t \in \mathbb{T}\}$. Moreover, the global pullback attractor is contained in the pullback exponential attractor and possesses finite dimensional sections. Indeed, by the minimality property in Definition 3.8 we conclude $\mathcal{A}(t) \subset \mathcal{M}(t)$, for all $t \in \mathbb{T}$.

In applications the existence of global pullback attractors often follows from the existence of bounded absorbing sets. For evolution processes that are not eventually compact it is generally difficult to apply Theorem 3.7 directly. To establish the existence of the global pullback attractor in problems with asymptotically compact processes it is often assumed that the process satisfies a stronger pullback absorbing property (see [8]).

Definition 3.10. *Let $\{U(t,s)|\ t \geq s\}$ be an evolution process in the metric space X. A family of bounded subsets $\{B(t)|\ t \in \mathbb{T}\}$ in X is said to be* ***strongly pullback absorbing all bounded sets*** *of X, if for every bounded set $D \subset X$ and every $s \leq t$ there exits $T_{D,s} \in \mathbb{T}_+$ such that*

$$U(s, s-r)D \subset B(t) \qquad \text{for all } r \geq T_{D,s},\ s \leq t.$$

Evolution processes possessing a family of bounded strongly pullback absorbing sets are called ***pullback strongly bounded dissipative****.*

If the family of bounded subsets $\{B(t)|\ t \in \mathbb{T}\}$ is strongly pullback absorbing all bounded sets, the absorbing set $B(t)$ at a given time $t \in \mathbb{T}$ is also pullback absorbing for all earlier times $s \leq t$. Under this hypothesis the theory of pullback attractors simplifies. For instance, the minimality property in Definition 3.8 is not needed to ensure the uniqueness of the global pullback attractor. Moreover, if an evolution process is pullback asymptotically compact and pullback strongly bounded dissipative follows the existence of the global pullback attractor, and the sections of the attractor coincide with the pullback ω-limit sets of the absorbing family (see [8]).

Definition 3.11. *An evolution process $\{U(t,s)|\ t \geq s\}$ in a metric space X is called* ***pullback asymptotically compact*** *if for every time $t \in \mathbb{T}$, every sequence $\{s_n\}_{n\in\mathbb{N}} \subset \mathbb{T}_+$ and bounded sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ such that*

$$\lim_{n\to\infty} s_n = \infty \ \text{ and } \ \{S(t, t-s_n)x_n\}_{n\in\mathbb{N}} \text{ is bounded},$$

the sequence $\{S(t, t-s_n)x_n\}_{n\in\mathbb{N}}$ possesses a convergent subsequence.

Theorem 3.8. *We assume $\{U(t,s)|\ t \geq s\}$ is an evolution process in the complete metric space X that is pullback asymptotically compact and pullback strongly bounded dissipative. Then, the global pullback attractor $\{\mathcal{A}(t)|\ t \in \mathbb{T}\}$ exists, for every $t \in \mathbb{T}$ the union $\bigcup_{s\leq t} \mathcal{A}(s)$ is bounded and the global pullback attractor is given by*

$$\mathcal{A}(t) = \omega(B(t), t) \qquad t \in \mathbb{T}.$$

This theorem extends the corresponding result for semigroups (Theorem 3.1). We observe that the global pullback attractor of strongly bounded dissipative processes is always uniformly bounded in the past. To be more precise, for every time instant $t \in \mathbb{T}$ the union

$$\bigcup_{s \le t} \mathcal{A}(s)$$

is bounded.

Exponential Pullback Attractors

In this subsection we shortly summarize the results in [19] and [49], where time continuous pullback exponential attractors for evolution processes that satisfy the smoothing property were constructed. Both articles are based on the construction of discrete forwards exponential attractors in [32], modify the construction by using the pullback approach and extend the algorithm for time continuous evolution processes.

In the following we assume $\mathbb{T} = \mathbb{R}$ and $\{S(t,s)|\ t \ge s\}$ is an evolution process in the Banach space $(V, \|\cdot\|_V)$. The construction of exponential pullback attractors in [19] and [49] is based on the compact embedding (H_0) of the phase space into an auxiliary normed space $(W, \|\cdot\|_W)$ (see Section 3.1.4) and the smoothing property of the process. Moreover, it was essential for the proof that the evolution process is strongly bounded dissipative. To be more precise, for some $t_0 \in \mathbb{R}$ the following assumptions were made:

(H_1) There exists a bounded subset $B \subset V$, that uniformly pullback absorbs all bounded sets of V for all $t \le t_0$: For every bounded set $D \subset V$ there exists an absorbing time $T_D \ge 0$ such that

$$\bigcup_{t \le t_0} S(t, t-s)D \subset B \qquad \text{for all } s \ge T_D.$$

(H_2) The evolution process $\{S(t,s)|\ t \ge s\}$ satisfies the smoothing property within the absorbing set: There exists a constant $\kappa > 0$ such that

$$\|S(t, t-T_B)u - S(t, t-T_B)v\|_V \le \kappa \|u - v\|_W \qquad \text{for all } u, v \in B, t \le t_0,$$

where $T_B > 0$ denotes the absorbing time corresponding to the absorbing set B in Hypothesis (H_1).

(H_3) The evolution process $\{S(t,s)|\ t \ge s\}$ is Lipschitz continuous in V: For every $t \in \mathbb{R}$ and $s \le t$ there exists a constant $L_{t,s} \ge 0$ such that

$$\|S(t,s)u - S(t,s)v\|_V \le L_{t,s} \|u - v\|_V \qquad \text{for all } u, v \in B.$$

(H_4) The evolution process is Hölder continuous in time with respect to the metric in W: There exist constants $\zeta_1, \zeta_2 \ge 0$ and exponents $0 < \theta_1, \theta_2 \le 1$ such that

$$\sup_{t \le t_0} \|S(t, t-T_B)u - S(t-s, t-s-T_B)u\|_W \le \zeta_1 s^{\theta_1} \qquad \text{for all } s \in [0, T_B],$$

$$\sup_{t \le t_0} \|S(t, t-s_1)u - S(t, t-s_2)u\|_W \le \zeta_2 |s_1 - s_2|^{\theta_2} \qquad \text{for all } s_1, s_2 \in [T_B, 2T_B],$$

for every $u \in B$.

Remark 3.3. *1. In Hypothesis (H_1) it is not only assumed that the process is pullback strongly bounded dissipative, but also that the absorbing time T_D corresponding to a bounded subset $D \subset V$ is independent of the time instant $t \leq t_0$. This implies that the pullback exponential attractor may be unbounded in the future, but it is always uniformly bounded in the past, and the same applies to the global pullback attractor. Namely, for every $t \in \mathbb{R}$ the unions*

$$\bigcup_{s \leq t} \mathcal{A}(t) \subset \bigcup_{s \leq t} \mathcal{M}(t)$$

are bounded. We generalize these uniform assumptions regarding the pullback absorbing set in the next section.

2. *It follows from the smoothing property (H_2) that the process $\{S(t,s)|\ t \geq s\}$ is (eventually) compact. Furthermore, under the stated assumptions Theorem 3.8 implies the existence of the global pullback attractor and*

$$\mathcal{A}(t) = \omega(B,t) \qquad \textit{for all } t \leq t_0.$$

By definition the global pullback attractor is invariant and we obtain

$$\mathcal{A}(t) = S(t,t_0)\mathcal{A}(t_0) \qquad \textit{for all } t \geq t_0.$$

3. *The Hölder continuity in time of the process was important for the construction of time continuous pullback exponential attractors in [19] and [49] and is typical for parabolic problems. However, it is a restrictive assumption and generally not satisfied, for instance in hyperbolic problems. To apply the theory to evolution processes generated by hyperbolic equations also requires to extend the construction for asymptotically compact processes (see Section 3.2.5).*

4. *The assumptions (H_0)-(H_4) are taken from the article [19]. The hypothesis in [49] are very similar, but less general.*

5. *The absorbing time T_B corresponding to the bounded absorbing set B in (H_1), the smoothing time in (H_2) and the intervals, where the process is Hölder continuous coincide. This is not necessary for the construction of the pullback exponential attractor as we proved in Section 3.1.4 for semigroups.*

For further details and the proof of the following theorem we refer to [19] and [49].

Theorem 3.9. *Let $\{S(t,s)|\ t \geq s\}$ be an evolution process in the Banach space V and the assumptions (H_0)-(H_4) be satisfied. Then, for every $\nu \in (0, \frac{1}{2})$ there exists a pullback exponential attractor $\{\mathcal{M}^\nu(t)|\ t \in \mathbb{R}\}$, and the fractal dimension of its sections is uniformly bounded,*

$$\sup_{t \in \mathbb{R}} \dim_f(\mathcal{M}^\nu(t)) \leq \max\{\frac{1}{\theta_1}, \frac{1}{\theta_2}\}(1 + \log_{\frac{1}{2\nu}}(1+\mu\kappa)) + \log_{\frac{1}{2\nu}}\left(N^W_{\frac{\nu}{\kappa}}(B^V_1(0))\right).$$

3.2.3. Existence Results for Pullback Exponential Attractors

In this section we present an algorithm for the construction of pullback exponential attractors which generalizes former results. In particular, we consider time-dependent pullback absorbing sets which possibly grow in the past, extend the construction of pullback exponential attractors for asymptotically compact processes and modify previous constructions in the time continuous case. This leads to pullback exponential attractors with sections that are not necessarily uniformly bounded in the past. Moreover, we prove the existence of pullback exponential attractors for time continuous evolution processes under significantly weaker hypothesis and obtain better estimates for the fractal dimension of the attractor.

Let $U = \{U(t,s) |\ t \geq s\}$ be an evolution process in the Banach space $(V, \|\cdot\|_V)$ and $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$. The construction of the pullback exponential attractor is based on the compact embedding (H_0) and the asymptotic smoothing property of the process. We assume the process U can be represented as $U = S + C$, where $\{S(t,s) |\ t \geq s\}$ and $\{C(t,s) |\ t \geq s\}$ are families of operators satisfying the following properties:

$(\mathcal{H}_1)$ There exists a family of bounded subsets $B(t) \subset V$, $t \in \mathbb{T}$, that pullback absorbs all bounded subsets of V: For every bounded set $D \subset V$ and every $t \in \mathbb{T}$ there exists a pullback absorbing time $T_{D,t} \in \mathbb{T}_+$ such that

$$U(t, t-s)D \subset B(t) \qquad \text{for all } s \geq T_{D,t}.$$

$(\mathcal{H}_2)$ The family $\{S(t,s) |\ t \geq s\}$ satisfies the smoothing property within the absorbing sets: There exists $\tilde{t} \in \mathbb{T}_+ \backslash \{0\}$ and a constant $\kappa > 0$ such that

$$\|S(t+\tilde{t}, t)u - S(t+\tilde{t}, t)v\|_V \leq \kappa \|u - v\|_W \qquad \text{for all } u, v \in B(t),\ t \in \mathbb{T}.$$

$(\mathcal{H}_3)$ The family $\{C(t,s) |\ t \geq s\}$ is a contraction within the absorbing sets:

$$\|C(t+\tilde{t}, t)u - C(t+\tilde{t}, t)v\|_V \leq \lambda \|u - v\|_V \qquad \text{for all } u, v \in B(t),\ t \in \mathbb{T},$$

where the contraction constant $0 \leq \lambda < \frac{1}{2}$.

$(\mathcal{H}_4)$ The process $\{U(t,s) |\ t \geq s\}$ is Lipschitz continuous within the absorbing sets: For all $t \in \mathbb{T}$ and $t \leq s \leq t + \tilde{t}$ there exists a constant $L_{t,s} > 0$ such that

$$\|U(s,t)u - U(s,t)v\|_V \leq L_{t,s} \|u - v\|_V \qquad \text{for all } u, v \in B(t),\ t \in \mathbb{T}.$$

The construction of pullback exponential attractors requires to impose additional assumptions on the pullback absorbing family in Hypothesis $(\mathcal{H}_1)$.

(A_1) The family of absorbing sets $\{B(t) |\ t \in \mathbb{T}\}$ is positively semi-invariant for the evolution process $\{U(t,s) |\ t \geq s\}$,

$$U(t,s)B(s) \subset B(t) \qquad \text{for all } t \geq s,\ t, s \in \mathbb{T}.$$

(A_2) For every bounded subset $D \subset V$ and time $t \in \mathbb{T}$ the corresponding absorbing times are bounded in the past: There exists $T_{D,t} \in \mathbb{T}_+$ such that

$$U(s, s-r)D \subset B(s) \qquad \text{for all } s \leq t,\ r \geq T_{D,t}.$$

The stated assumptions allow to construct pullback exponential attractors for the evolution process $\{U(t,s)|\ t \geq s\}$.

Theorem 3.10. *Let $\{U(t,s)|\ t \geq s\}$ be an evolution process in the Banach space V and the assumptions (H_0), $(\mathcal{H}_1)$-$(\mathcal{H}_4)$, (A_1) and (A_2) be satisfied. Moreover, we assume that the diameter of the family of absorbing sets $\{B(t)|\ t \in \mathbb{T}\}$ grows at most sub-exponentially in the past. Then, for every $\nu \in (0, \frac{1}{2} - \lambda)$ there exists a pullback exponential attractor $\{\mathcal{M}^\nu(t)|\ t \in \mathbb{T}\} = \{\mathcal{M}(t)|\ t \in \mathbb{T}\}$ for the evolution process $\{U(t,s)|\ t \geq s\}$, and the fractal dimension of its sections is uniformly bounded by*

$$\dim_{\mathrm{f}}^V(\mathcal{M}(t)) \leq \log_{\frac{1}{2(\nu+\lambda)}} \left(N^W_{\frac{\nu}{\kappa}}(B^V_1(0)) \right) \qquad \text{for all } t \in \mathbb{T}.$$

Remark 3.4. *1. The uniform pullback absorbing assumption (H_1) in Section 3.2.2 implies Hypothesis $(\mathcal{H}_1)$, (A_1) and (A_2).*

Indeed, let $t_0 \in \mathbb{T}$ be arbitrary and B be the uniformly pullback absorbing set in Assumption (H_1). A family of bounded pullback absorbing sets is given by

$$B(t) := \begin{cases} \bigcup_{s \geq T_B} U(t, t-s)B & \text{for } t \leq t_0 \\ U(t, t_0)B(t_0) & \text{for } t \geq t_0. \end{cases}$$

Moreover, the family $\{B(t)|\ t \in \mathbb{T}_+\}$ is positively semi-invariant for the evolution process $\{U(t,s)|\ t \geq s\}$, and the absorbing times are bounded in the past as required by Hypothesis (A_2).

2. *For our construction of time continuous pullback exponential attractors the Hölder continuity in time (H_4) of the evolution process is not needed. Moreover, we improve the estimates on the fractal dimension in Theorem 3.9 and obtain the same bound for the pullback exponential attractors of discrete and of time continuous evolution processes.*

3. *We generalize Theorem 3.9 for evolution processes that are asymptotically compact in the Banach space V. For evolution processes this setting was only considered in [32] (Theorem 2.3), where forwards exponential attractors were constructed, but for discrete processes and under hypothesis that are difficult to verify in applications. In [36] time continuous forwards exponential attractors for evolution processes that are asymptotically compact in the weaker space W were constructed.*

4. *Time-dependent absorbing sets were also considered in [36]. However, it was assumed that the diameter of the absorbing sets $\{B(t)\ t \in \mathbb{R}\}$ is uniformly bounded and the*

absorbing times are independent of the time instant. This implies that the union $\bigcup_{t\in\mathbb{T}} B(t)$ is a bounded pullback absorbing set for the evolution process and satisfies the uniform hypothesis in Section 3.2.2. Furthermore, the aim of this article was not to prove the existence of forwards exponential attractors in general, but knowing the existence of the uniform attractor for the evolution process, to show the existence of time-dependent forwards exponential attractors.

We remark that in applications the family of contraction operators often forms an evolution process in V. In this case, and if the contraction property $(\mathcal{H}_3)$ is globally satisfied, the smoothing time and the contraction time can be arbitrary, and it suffices that the evolution process C is a strict contraction. To be more precise we could replace Assumptions $(\mathcal{H}_2)$-$(\mathcal{H}_4)$ by the following:

$(\widetilde{\mathcal{H}}_2)$ The family $\{S(t,s)|\ t \geq s\}$ satisfies the following smoothing property within the absorbing sets: There exists $\tilde{t} \in \mathbb{T}_+\backslash\{0\}$ such that for all $s \geq \tilde{t}$

$$\|S(t+s,t)u - S(t+s,t)v\|_V \leq \kappa_s \|u-v\|_W \qquad \text{for all } u,v \in B(t),\ t \in \mathbb{T},$$

for some constant $\kappa_s > 0$.

$(\widetilde{\mathcal{H}}_3)$ The family $\{C(t,s)|\ t \geq s\}$ is an evolution process and a strict contraction in V: There exists $\hat{t} \in \mathbb{T}_+\backslash\{0\}$ such that

$$\|C(t+\hat{t},t)u - C(t+\hat{t},t)v\|_V \leq \lambda \|u-v\|_V \qquad \text{for all } u,v \in V,\ t \in \mathbb{T},$$

where the contraction constant $0 \leq \lambda < 1$.

$(\widetilde{\mathcal{H}}_4)$ The evolution process $\{U(t,s)|\ t \geq s\}$ satisfies the Lipschitz continuity in $(\mathcal{H}_4)$ for all $t \in \mathbb{T}$ and $t \leq s \leq t+\hat{t}$.

Indeed, let $k \in \mathbb{N}$ be such that $\lambda^k < \frac{1}{2}$ and $k\hat{t} \geq \tilde{t}$. Then, Property $(\widetilde{\mathcal{H}}_3)$ implies

$$\|C(t+\hat{t}k,t)u - C(t+\hat{t}k,t)v\|_V \ \leq\ \lambda^k \|u-v\|_V \qquad \text{for all } u,v \in B(t),\ t \in \mathbb{T}.$$

Furthermore, by the smoothing property $(\widetilde{\mathcal{H}}_2)$ follows

$$\|S(t+\hat{t}k,t)u - S(t+\hat{t}k,t)v\|_V \ \leq\ \kappa \|u-v\|_W \qquad \text{for all } u,v \in B(t),\ t \in \mathbb{T},$$

where $\kappa := \kappa_{\hat{t}k}$. Consequently, the assumptions $(\mathcal{H}_2)$-$(\mathcal{H}_3)$ are satisfied if we replace $\tilde{t}$ by $\hat{t}k$ and the smoothing and contraction constants by $\tilde{\lambda} = \lambda^k$ and $\tilde{\kappa} = \kappa_{\hat{t}k}$.

The Discrete Case

First, we construct pullback exponential attractors for discrete evolution processes. We assume $\mathbb{T} = \mathbb{Z}$ and $\{U(n,m)|\ n \geq m\}$ is a discrete evolution process in the Banach space V. Here and in the sequel, we use the letters n, m and k to denote discrete times $n, m, k \in \mathbb{Z}$. Without loss of generality we suppose that $\tilde{t} = \tilde{n} = 1$ in the hypothesis $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$. The general case $\tilde{n} \in \mathbb{N}$ follows as in the proof of Theorem 3.3 for semigroups. Properties (H_0), $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ then imply that the discrete process $\{U(n,m)|\ n \geq m\}$ is Lipschitz continuous and Assumption $(\mathcal{H}_4)$ is automatically satisfied.

Theorem 3.11. *Let $\{U(n,m)|\ n \geq m\}$ be a discrete evolution process in the Banach space V, and the assumptions (H_0), $(\mathcal{H}_1)$-$(\mathcal{H}_3)$, (A_1) and (A_2) be satisfied with $\tilde{n} = 1$. Moreover, we assume that the diameter of the family of absorbing sets $\{B(k)|\ k \in \mathbb{Z}\}$ grows at most sub-exponentially in the past. Then, for every $\nu \in (0, \frac{1}{2} - \lambda)$ there exists a pullback exponential attractor $\{\mathcal{M}(k)|\ k \in \mathbb{Z}\} = \{\mathcal{M}^\nu(k)|\ k \in \mathbb{Z}\}$ for the evolution process $\{U(n,m)|\ n \geq m\}$, and the fractal dimension of its sections is uniformly bounded by*

$$\dim_{\mathrm{f}}^V(\mathcal{M}(k)) \leq \log_{\frac{1}{2(\nu+\lambda)}} \left(N^W_{\frac{\nu}{\kappa}}(B^V_1(0)) \right) \qquad \text{for all } k \in \mathbb{Z}.$$

Proof. Step 1: Coverings of $U(k,k-n)B(k-n)$
Let $\nu \in (0, \frac{1}{2} - \lambda)$ be fixed, $R_k > 0$ and $v_k \in B(k)$ be such that $B(k) \subset B^V_{R_k}(v_k)$ for all $k \in \mathbb{Z}$. Moreover, we choose elements $w_1, \dots w_N \in V$ such that

$$B^V_1(0) \subset \bigcup_{i=1}^N B^W_{\frac{\nu}{\kappa}}(w_i),$$

where $N := N^W_{\frac{\nu}{\kappa}}(B^V_1(0))$. We define the sets $W^0(k) := \{v_k\}$, $k \in \mathbb{Z}$, and construct by induction in $n \in \mathbb{N}$ the family of time-dependent sets $W^n(k), n \in \mathbb{N}, k \in \mathbb{Z}$ that satisfies the properties:

(W1) $\quad W^n(k) \subset U(k,k-n)B(k-n) \subset B(k)$,

(W2) $\quad \sharp W^n(k) \leq N^n$,

(W3) $\quad U(k,k-n)B(k-n) \subset \bigcup_{u \in W^n(k)} B^V_{(2(\nu+\lambda))^n R_{k-n}}(u)$,

for all $k \in \mathbb{Z}$, $n \in \mathbb{N}_0$. To construct a covering of the image $U(k,k-1)B(k-1)$, $k \in \mathbb{Z}$, we note that $v \in B^V_{R_{k-1}}(v_{k-1})$ implies

$$\frac{1}{R_{k-1}}(v - v_{k-1}) \in B^V_1(0) \subset \bigcup_{i=1}^N B^W_{\frac{\nu}{\kappa}}(w_i)$$

and consequently,

$$B^V_{R_{k-1}}(v_{k-1}) \subset \bigcup_{i=1}^N B^W_{R_{k-1}\frac{\nu}{\kappa}}(R_{k-1}w_i + v_{k-1}).$$

Using the smoothing property $(\mathcal{H}_2)$ we obtain

$$\|S(k,k-1)\tilde{u} - S(k,k-1)\tilde{v}\|_V \leq \kappa \|\tilde{u} - \tilde{v}\|_W < 2\nu R_{k-1}$$

for all $\tilde{u}, \tilde{v} \in B^W_{R_{k-1}\frac{\nu}{\kappa}}(R_{k-1}w_i + v_{k-1}) \cap B(k-1)$, which yields the covering

$$S(k,k-1)\big(B^V_{R_{k-1}}(v_{k-1}) \cap B(k-1)\big) \subset \bigcup_{i=1}^N B^V_{2\nu R_{k-1}}(z_i),$$

for some $z_1, \dots, z_N \in S(k,k-1)B(k-1)$. In particular, we can choose $y_1, \dots, y_N \in B(k-1)$ such that $z_i = S(k,k-1)y_i$, where $i = 1, \dots, N$. For $u \in B(k-1)$ the contraction property $(\mathcal{H}_3)$ now implies

$$\|C(k,k-1)u - C(k,k-1)y_i\|_V \le \lambda \|u - y_i\|_V < 2\lambda R_{k-1},$$

for all $i = 1, \dots, N$, and we conclude

$$C(k,k-1)B(k-1) \subset B^V_{2\lambda R_{k-1}}(C(k,k-1)y_i).$$

Finally, we obtain the covering

$$\begin{aligned} U(k,k-1)B(k-1) &= (S(k,k-1) + C(k,k-1))B(k-1) \\ &\subset \bigcup_{i=1}^{N} B^V_{2\nu R_{k-1}}\Big((S(k,k-1)y_i) \cup B^V_{2\lambda R_{k-1}}(C(k,k-1)y_i)\Big) \\ &\subset \bigcup_{i=1}^{N} B^V_{2(\nu+\lambda)R_{k-1}}(U(k,k-1)y_i), \end{aligned}$$

with centres $U(k,k-1)y_i \in U(k,k-1)B(k-1)$, $i = 1, \dots, N$. Denoting the new set of centres by $W^1(k)$ follows

$$U(k,k-1)B(k-1) \subset \bigcup_{u \in W^1(k)} B^V_{2(\nu+\lambda)R_{k-1}}(u),$$

where the set $W^1(k) \subset U(k,k-1)B(k-1) \subset B(k)$ and $\sharp W^1(k) \le N$.

Let us assume that the sets $W^l(k)$ are already constructed for all $l \le n$ and $k \in \mathbb{Z}$, which yields the coverings

$$U(k,k-n)B(k-n) \subset \bigcup_{u \in W^n(k)} B^V_{(2(\nu+\lambda))^n R_{k-n}}(u) \qquad \text{for } k \in \mathbb{Z}.$$

In order to construct a covering of

$$\begin{aligned} U(k,k-(n+1))B(k-(n+1)) &= U(k,k-1)U(k-1,k-1-n)B(k-1-n) \\ &\subset \bigcup_{u \in W^n(k-1)} U(k,k-1)B^V_{(2(\nu+\lambda))^n R_{k-n-1}}(u) \end{aligned}$$

let $u \in W^n(k-1)$. We proceed as before and use the covering of the unit ball $B^V_1(0)$ by $\frac{\nu}{\kappa}$-balls in W to conclude

$$B^V_{(2(\nu+\lambda))^n R_{k-1-n}}(u) \subset \bigcup_{i=1}^{N} B^W_{(2(\nu+\lambda))^n R_{k-1-n}\frac{\nu}{\kappa}}((2(\nu+\lambda))^n R_{k-1-n} w_i + u).$$

By the smoothing property $(\mathcal{H}_2)$ then follows

$$S(k,k-1)\Big(U(k-1,k-1-n)B(k-1-n)\cap B^V_{(2(\nu+\lambda))^n R_{k-1-n}}(u)\Big)\\ \subset \bigcup_{i=1}^N B^V_{(2(\nu+\lambda))^n 2\nu R_{k-1-n}}(S(k,k-1)y_i^u),$$

for some $y_1^u,\dots,y_N^u \in U(k-1,k-1-n)B(k-1-n)$. Furthermore, the contraction property $(\mathcal{H}_3)$ implies

$$C(k,k-1)\Big(U(k-1,k-1-n)B(k-1-n)\cap B^V_{(2(\nu+\lambda))^n R_{k-1-n}}(u)\Big)\\ \subset B^V_{(2(\nu+\lambda))^n 2\lambda R_{k-1-n}}(C(k,k-1)y_i^u),$$

for all $i=1,\dots,N$. Consequently, we obtain the covering

$$\begin{aligned} &U(k,k-1)\Big(U(k-1,k-1-n)B(k-1-n)\cap B^V_{(2(\nu+\lambda))^n R_{k-1-n}}(u)\Big)\\ &=\big(S(k,k-1)+C(k,k-1)\big)\Big(U(k-1,k-1-n)B(k-1-n)\cap B^V_{(2(\nu+\lambda))^n R_{k-1-n}}(u)\Big)\\ &\subset \bigcup_{i=1}^N \Big(B^V_{(2(\nu+\lambda))^n 2\nu R_{k-1-n}}(S(k,k-1)y_i^u)+B^V_{(2(\nu+\lambda))^n 2\lambda R_{k-1-n}}(C(k,k-1)y_i^u)\Big)\\ &\subset \bigcup_{i=1}^N B^V_{(2(\nu+\lambda))^{n+1} R_{k-1-n}}\big(S(k,k-1)y_i^u+C(k,k-1)y_i^u\big)\\ &=\bigcup_{i=1}^N B^V_{(2(\nu+\lambda))^{n+1} R_{k-1-n}}\big(U(k,k-1)y_i^u\big), \end{aligned}$$

with centres $U(k,k-1)y_i^u\in U(k,k-1-n)B(k-1-n)$, for $1=1,\dots,N$. Constructing in the same way for every $u\in W^n(k-1)$ such a covering by balls with radius $(2(\nu+\lambda))^{n+1}R_{k-1-n}$ in V we obtain a covering of the set $U(k,k-(n+1))B(k-(n+1))$ and denote the new set of centres by $W^{n+1}(k)$. This yields $\sharp W^{n+1}(k)\le N\sharp W^n(k-1)\le N^{n+1}$, by construction the set of centres $W^{n+1}(k)\subset U(k,k-(n+1))B(k-(n+1))$, and

$$U(k,k-(n+1))B(k-(n+1))\subset \bigcup_{u\in W^{n+1}(k)} B^V_{(2(\nu+\lambda))^{n+1}R_{k-1-n}}(u),$$

which concludes the proof of the properties $(W1)$-$(W3)$.
Step 2: Definition of the Pullback Exponential Attractor
We define the sets $E^0(k):=W^0(k)$ for $k\in\mathbb{Z}$, and set

$$E^n(k):=W^n(k)\cup U(k,k-1)E^{n-1}(k-1)\qquad \text{for}\ \ n\in\mathbb{N}.$$

Then, the family of sets satisfies the properties:

(E1) $\quad U(k,k-1)E^n(k-1)\subset E^{n+1}(k),\quad E^n(k)\subset U(k,k-n)B(k-n)\subset B(k),$

(E2) $\quad E^n(k)=\bigcup_{l=0}^{n}U(k,k-l)W^{n-l}(k-l),\quad \sharp E^n(k)\leq \sum_{l=0}^{n}N^l,$

(E3) $\quad U(k,k-n)B(k-n)\subset \bigcup_{u\in E^n(k)}B^V_{(2(\nu+\lambda))^nR_{k-n}}(u),$

for all $n\in\mathbb{N}_0$ and $k\in\mathbb{Z}$. These relations are immediate consequences of the definition of the sets $E^n(k)$, the properties of the sets $W^n(k)$ and the semi-invariance of the absorbing family $\{B(k)|\ k\in\mathbb{Z}\}$, and can be proved by induction.

Using the family of sets $E^n(k)$, $n\in\mathbb{N}_0,k\in\mathbb{Z}$, we define

$$\widetilde{\mathcal{M}}(k):=\bigcup_{n\in\mathbb{N}_0}E^n(k)\qquad\text{for all } k\in\mathbb{Z},$$

and show that its closure $\{\mathcal{M}(k)|\ k\in\mathbb{Z}\}:=\{\overline{\widetilde{\mathcal{M}}(k)}^{\|\cdot\|_V}\ |\ k\in\mathbb{Z}\}$ is a pullback exponential attractor for the evolution process $\{U(n,m)|\ n\geq m\}$ in V.

Step 3: Semi-invariance of the Exponential Attractor

Primarily, we show that the family $\{\widetilde{\mathcal{M}}(k)|\ k\in\mathbb{Z}\}$ is positively semi-invariant. To this end let $l\in\mathbb{N}_0$ and $k\in\mathbb{Z}$. By Property $(E1)$ we obtain

$$U(k+l,k)\widetilde{\mathcal{M}}(k):=\bigcup_{n\in\mathbb{N}_0}U(k+l,k)E^n(k)\subset\bigcup_{n\in\mathbb{N}_0}E^{n+l}(k+l)\subset\bigcup_{n\in\mathbb{N}_0}E^n(k+l)=\widetilde{\mathcal{M}}(k+l).$$

The continuity of the process $\{U(n,m)|\ n\geq m\}$ now implies the semi-invariance of the family $\{\mathcal{M}(k)|\ k\in\mathbb{Z}\}$,

$$U(k+l,k)\mathcal{M}(k)=U(k+l,k)\overline{\widetilde{\mathcal{M}}(k)}^{\|\cdot\|_V}\subset\overline{U(k+l,k)\widetilde{\mathcal{M}}(k)}^{\|\cdot\|_V}\subset\overline{\widetilde{\mathcal{M}}(k+l)}^{\|\cdot\|_V}=\mathcal{M}(k+l),$$

for all $l\in\mathbb{N}_0$ and $k\in\mathbb{Z}$.

Step 4: Compactness and Finite Dimensionality of the Exponential Attractor

We first prove that the sets $\widetilde{\mathcal{M}}(k)$ are non-empty, precompact and of finite fractal dimension in V, for all $k\in\mathbb{Z}$. For every $m\in\mathbb{N}$ and $n\geq m$ we observe

$$\begin{aligned}E^n(k)&\subset U(k,k-n)B(k-n)=U(k,k-m)U(k-m,k-n)B(k-n)\\&\subset U(k,k-m)B(k-m),\end{aligned}$$

where we used the semi-invariance of the absorbing sets. Consequently, we obtain

$$\widetilde{\mathcal{M}}(k)=\bigcup_{n=0}^{m}E^n(k)\cup\bigcup_{n=m+1}^{\infty}E^n(k)\subset\bigcup_{n=0}^{m}E^n(k)\cup U(k,k-m)B(k-m).$$

Let $\epsilon>0$. If we choose $m\in\mathbb{N}$ sufficiently large such that

$$(2(\nu+\lambda))^mR_{k-m}\leq\epsilon<(2(\nu+\lambda))^{m-1}R_{k-m+1}$$

holds, Property $(W3)$ implies the covering

$$U(k,k-m)B(k-m) \subset \bigcup_{u \in W^m(k)} B^V_\epsilon(u).$$

We can therefore estimate the number of ϵ-balls in V needed to cover the set $\widetilde{\mathcal{M}}(k)$ by

$$\begin{aligned} N^V_\epsilon(\widetilde{\mathcal{M}}(k)) &\leq \sharp(\bigcup_{n=0}^{m} E^n(k)) + \sharp W^m(k) \leq (m+1)\sharp E^m(k) + N^m \\ &\leq (m+1)^2 N^m + N^m \leq 2(m+1)^2 N^m, \end{aligned}$$

for all $k \in \mathbb{Z}$, where we used Properties $(W2)$ and $(E2)$. This proves the precompactness of the sets $\widetilde{\mathcal{M}}(k)$, $k \in \mathbb{Z}$. Since V is a Banach space, taking the closure of the precompact sets $\widetilde{\mathcal{M}}(k)$ the subsets $\mathcal{M}(k) := \overline{\widetilde{\mathcal{M}}(k)}^{\|\cdot\|_V}$, $k \in \mathbb{Z}$, are compact in V.

Finally, for the fractal dimension of the sets $\widetilde{\mathcal{M}}(k)$ we obtain the estimate

$$\begin{aligned} \dim^V_{\mathrm{f}}(\widetilde{\mathcal{M}}(k)) &= \limsup_{\epsilon \to 0} \frac{\ln(N^V_\epsilon(\widetilde{\mathcal{M}}(k)))}{\ln \frac{1}{\epsilon}} \\ &\leq \limsup_{\epsilon \to 0} \frac{\ln(2) + 2\ln(m+1) + m\ln(N)}{\ln \frac{1}{\epsilon}} \leq \log_{\frac{1}{2(\nu+\lambda)}}(N), \end{aligned}$$

where we used that the family of absorbing sets grows at most sub-exponentially in the past. Proposition 3.1 implies that the fractal dimension of the family $\{\mathcal{M}(k) |\ k \in \mathbb{Z}\}$ is uniformly bounded by the same value,

$$\dim^V_{\mathrm{f}}(\mathcal{M}(k)) = \dim^V_{\mathrm{f}}(\overline{\widetilde{\mathcal{M}}(k)}^{\|\cdot\|_V}) = \dim^V_{\mathrm{f}}(\widetilde{\mathcal{M}}(k)) \qquad k \in \mathbb{Z}.$$

Step 5: Pullback Exponential Attraction

It remains to show that the set $\mathcal{M}(k)$ exponentially pullback attracts all bounded subsets of V at time $k \in \mathbb{Z}$. Let $D \subset V$ be bounded and $k \in \mathbb{Z}$. By Assumptions $(\mathcal{H}_1)$ and (A_2) there exists $n_{D,k} \in \mathbb{N}$ such that $U(l,l-n)D \subset B(l)$ for all $n \geq n_{D,k}$ and $l \leq k$. If $n \geq n_{D,k} + 1$, then $n = n_{D,k} + n_0$ for some $n_0 \in \mathbb{N}$, and we conclude

$$\begin{aligned} \mathrm{dist}^V_{\mathrm{H}}(U(k,k-n)D, \widetilde{\mathcal{M}}(k)) &\leq \mathrm{dist}^V_{\mathrm{H}}(U(k,k-n_0)U(k-n_0,k-n_0-n_{D,k})D, \bigcup_{n=0}^{\infty} E^n(k)) \\ &\leq \mathrm{dist}^V_{\mathrm{H}}(U(k,k-n_0)B(k-n_0), \bigcup_{n=0}^{\infty} E^n(k)) \\ &\leq \mathrm{dist}^V_{\mathrm{H}}(U(k,k-n_0)B(k-n_0), E^{n_0}(k)) \\ &\leq (2(\nu+\lambda))^{n_0} R_{k-n_0} \leq ce^{-\omega n}, \end{aligned}$$

for some constants $c \geq 0$ and $\omega > 0$. These estimates are valid since the family of pullback absorbing sets grows at most sub-exponentially in the past. This proves that the set $\widetilde{\mathcal{M}}(k)$ exponentially pullback attracts the set D at time $k \in \mathbb{Z}$. Since $\widetilde{\mathcal{M}}(k) \subset \mathcal{M}(k)$, for all $k \in \mathbb{Z}$, immediately follows the exponential pullback attraction property of the family $\{\mathcal{M}(k) |\ k \in \mathbb{Z}\}$.

We have verified all required properties in Definition 3.9 which shows that $\{\mathcal{M}(k) |\ k \in \mathbb{Z}\}$ is a pullback exponential attractor for the evolution process $\{U(n,m) |\ n \geq m\}$ in V. □

The Time Continuous Case

Using the results for discrete evolution processes we now construct pullback exponential attractors for time continuous evolution processes in V and prove Theorem 3.10 for the case $\mathbb{T} = \mathbb{R}$.

Proof of Theorem 3.10. Let $\mathbb{T} = \mathbb{R}$ and $\{U(t,s) |\ t \geq s\}$ be a time continuous evolution process satisfying the hypothesis of Theorem 3.10. We define the associated discrete evolution process $\{\widetilde{U}(n,m) |\ n \geq m\}$ by $\tilde{U}(n,m) := U(n\tilde{t}, m\tilde{t})$ for all $n \geq m$, $n, m \in \mathbb{Z}$. The discrete evolution process satisfies the hypothesis of Theorem 3.11, and we conclude that there exists a pullback exponential attractor $\{\mathcal{M}_d(k) |\ k \in \mathbb{Z}\}$ for process $\{\widetilde{U}(n,m) |\ n \geq m\}$. We recall that the pullback exponential attractor was defined by

$$\mathcal{M}_d(k) = \overline{\widetilde{\mathcal{M}}_d(k)}^{\|\cdot\|_V} = \overline{\bigcup_{n \in \mathbb{N}_0} E^n(k)}^{\|\cdot\|_V},$$

and we refer to the proof of Theorem 3.11 for the definition of the sets $E^n(k)$, $k \in \mathbb{Z}, n \in \mathbb{N}_0$.

To obtain a pullback exponential attractor $\{\mathcal{M}(t) |\ t \in \mathbb{R}\}$ for the time continuous process we define

$$\widetilde{\mathcal{M}}(t) := U(t, k\tilde{t})\widetilde{\mathcal{M}}_d(k) \qquad \text{for } t \in [k\tilde{t}, (k+1)\tilde{t}[,\ k \in \mathbb{Z},$$

and take its closure $\mathcal{M}(t) := \overline{\widetilde{\mathcal{M}}(t)}^{\|\cdot\|_V}$, $t \in \mathbb{R}$. We observe that $\mathcal{M}(k\tilde{t}) = \mathcal{M}_d(k)$ for all $k \in \mathbb{Z}$.

By Proposition 3.1 follows for the fractal dimension of the sections of the time continuous attractor

$$\begin{aligned}\dim_{\mathrm{f}}^V(\mathcal{M}(t)) &= \dim_{\mathrm{f}}^V(\overline{\widetilde{\mathcal{M}}(t)}^{\|\cdot\|_V}) = \dim_{\mathrm{f}}^V(\widetilde{\mathcal{M}}(t)) = \dim_{\mathrm{f}}^V(U(t, k\tilde{t})\widetilde{\mathcal{M}}(k\tilde{t})) \\ &\leq \dim_{\mathrm{f}}^V(\widetilde{\mathcal{M}}(k\tilde{t})) = \dim_{\mathrm{f}}^V(\widetilde{\mathcal{M}}_d(k)),\end{aligned}$$

for all $t \in [k\tilde{t}, (k+1)\tilde{t}[$, $k \in \mathbb{Z}$, where we used the Lipschitz-continuity $(\mathcal{H}_4)$ of the evolution process in the last estimate. Consequently, the bound for the fractal dimension in the time continuous case coincides with the bound for discrete pullback exponential attractors.

To show the semi-invariance of the family $\{\mathcal{M}(t) |\ t \in \mathbb{R}\}$ let $t, s \in \mathbb{R}$ such that $t \geq s$. Then, $s = k\tilde{t} + s_1$ and $t = l\tilde{t} + s_2$ for some $k, l \in \mathbb{Z}$, $k \leq l$ and $s_1, s_2 \in [0, \tilde{t}[$.

If $l \geq k+1$ we observe

$$\begin{aligned} U(t,s)\widetilde{\mathcal{M}}(s) &= U(l\tilde{t}+s_2, k\tilde{t}+s_1)\widetilde{\mathcal{M}}(k\tilde{t}+s_1) = U(l\tilde{t}+s_2, k\tilde{t}+s_1)U(k\tilde{t}+s_1, k\tilde{t})\widetilde{\mathcal{M}}(k\tilde{t}) \\ &= U(l\tilde{t}+s_2, l\tilde{t})U(l\tilde{t}, k\tilde{t})\widetilde{\mathcal{M}}(k\tilde{t}) \subset U(l\tilde{t}+s_2, l\tilde{t})\widetilde{\mathcal{M}}(l\tilde{t}) = \widetilde{\mathcal{M}}(l\tilde{t}+s_2) = \widetilde{\mathcal{M}}(t), \end{aligned}$$

where we used the semi-invariance of the family $\{\widetilde{\mathcal{M}}(k\tilde{t}) |\ k \in \mathbb{Z}\}$ under the action of the discrete process $\{\widetilde{U}(n,m) |\ n \geq m\}$.
On the other hand, if $l = k$, then $s = k\tilde{t} + s_1$ and $t = k\tilde{t} + s_2$ for some $s_1, s_2 \in [0, \tilde{t}[$ and we conclude

$$\begin{aligned} U(t,s)\widetilde{\mathcal{M}}(s) &= U(k\tilde{t}+s_2, k\tilde{t}+s_1)\widetilde{\mathcal{M}}(k\tilde{t}+s_1) = U(k\tilde{t}+s_2, k\tilde{t}+s_1)U(k\tilde{t}+s_1, k\tilde{t})\widetilde{\mathcal{M}}(k\tilde{t}) \\ &= U(k\tilde{t}+s_2, k\tilde{t})\widetilde{\mathcal{M}}(k\tilde{t}) = \widetilde{\mathcal{M}}(k\tilde{t}+s_2) = \widetilde{\mathcal{M}}(t). \end{aligned}$$

The semi-invariance of the family $\{\mathcal{M}(t) |\ t \in \mathbb{R}\}$ now follows by the continuity of the process and the semi-invariance of the sets $\{\widetilde{\mathcal{M}}(t) |\ t \in \mathbb{R}\}$ as in the discrete case.

It remains to prove that the set $\mathcal{M}(t)$ exponentially pullback attracts all bounded subsets of V at time $t \in \mathbb{R}$. To this end let $D \subset V$ be bounded, $t \in \mathbb{R}$ and $T_{D,t} \in \mathbb{R}_+$ be the corresponding pullback absorbing time in Assumption (A_2). Then, $t = k\tilde{t} + s_0$ for some $k \in \mathbb{Z}$ and $s_0 \in [0, \tilde{t}[$. Moreover, we assume $s \geq T_{D,t} + \tilde{t} + s_0$, which implies $s = l\tilde{t} + T_{D,t} + s_0 + s_1$, for some $l \in \mathbb{N}$ and $s_1 \in [0, \tilde{t}[$. We observe

$$\begin{aligned} U(t, t-s)D &= U(k\tilde{t}+s_0, (k-l)\tilde{t} - T_{D,t} - s_1)D \\ &= U(t, k\tilde{t})U(k\tilde{t}, (k-l)\tilde{t})U((k-l)\tilde{t}, (k-l)\tilde{t} - T_{D,t} - s_1)D \\ &\subset U(t, k\tilde{t})U(k\tilde{t}, (k-l)\tilde{t})B((k-l)\tilde{t}), \end{aligned}$$

and conclude

$$\begin{aligned} \operatorname{dist}_{\mathrm{H}}^{V}\big(U(t,t-s)D, \mathcal{M}(t)\big) &= \operatorname{dist}_{\mathrm{H}}^{V}\big(U(t,t-s)D, \overline{U(t,k\tilde{t})\widetilde{\mathcal{M}}(k\tilde{t})}^{\|\cdot\|_V}\big) \\ &\leq \operatorname{dist}_{\mathrm{H}}^{V}\big(U(t,t-s)D, U(t,k\tilde{t})\widetilde{\mathcal{M}}(k\tilde{t})\big) \\ &\leq \operatorname{dist}_{\mathrm{H}}^{V}\big(U(t,k\tilde{t})U(k\tilde{t},(k-l)\tilde{t})B((k-l)\tilde{t}), U(t,k\tilde{t})\widetilde{\mathcal{M}}(k\tilde{t})\big) \\ &\leq L_{t,k\tilde{t}} \operatorname{dist}_{\mathrm{H}}^{V}\big(U(k\tilde{t},(k-l)\tilde{t})B((k-l)\tilde{t}), \widetilde{\mathcal{M}}(k\tilde{t})\big) \\ &\leq L_{t,k\tilde{t}} \operatorname{dist}_{\mathrm{H}}^{V}\big(\widetilde{U}(k,k-l)B((k-l)\tilde{t}), \widetilde{\mathcal{M}}_d(k)\big), \end{aligned}$$

where we used Hypothesis $(\mathcal{H}_4)$, and $L_{t,k\tilde{t}} \geq 0$ denotes the corresponding Lipschitz constant. Consequently, it follows from the proof of Theorem 3.11 that $\mathcal{M}(t)$ exponentially pullback attracts the subset $D \subset V$ at time $t \in \mathbb{R}$. □

3.2.4. Consequences of the Construction and Properties of the Pullback Exponential Attractor

Consequences and Different Settings

An immediate consequence of Theorem 3.10 is the existence and finite dimensionality of the global pullback attractor.

Theorem 3.12. *Let $\mathbb{T} = \mathbb{Z}$ or $\mathbb{T} = \mathbb{R}$, $\{U(t,s)|\ t \geq s\}$ be an evolution process in the Banach space V and the assumptions (H_0), $(\mathcal{H}_1)$-$(\mathcal{H}_3)$, (A_1) and (A_2) be satisfied. Moreover, we assume that the diameter of the family of absorbing sets $\{B(t)|\ t \in \mathbb{T}\}$ grows at most sub-exponentially in the past. Then, the global pullback attractor $\{\mathcal{A}(t)|\ t \in \mathbb{T}\}$ of the evolution process $\{U(t,s)|\ t \geq s\}$ exists, and the fractal dimension of its sections is uniformly bounded by*

$$\dim_{\mathrm{f}}^V(\mathcal{A}(t)) \leq \log_{\frac{1}{2(\nu+\lambda)}} \left(N^W_{\frac{\nu}{\kappa}}(B^V_1(0)) \right) \qquad \text{for all } t \in \mathbb{T}.$$

Proof. For discrete evolution processes the statements follow from Theorem 3.10, Proposition 3.1 and the minimality property of the global pullback attractor (see Definition 3.8).

If $\mathbb{T} = \mathbb{R}$ we define the associated discrete evolution process $\{\widetilde{U}(n,m)|\ n \geq m\}$ by $\tilde{U}(n,m) := U(n\tilde{t}, m\tilde{t})$ for all $n \geq m$, $n,m \in \mathbb{Z}$. It satisfies the assumptions of Theorem 3.11, and we conclude that there exists a pullback exponential attractor $\{\mathcal{M}_d(k)|\ k \in \mathbb{Z}\}$ for the discrete evolution process $\{\widetilde{U}(n,m)|\ n \geq m\}$. We define the sets

$$\mathcal{M}'(t) := U(t, k\tilde{t})\mathcal{M}_d(k) \qquad \text{for } t \in [k\tilde{t}, (k+1)\tilde{t}[,\ k \in \mathbb{Z},$$

which implies $\mathcal{M}'(k\tilde{t}) = \mathcal{M}_d(k)$ for all $k \in \mathbb{Z}$. Since the operators $U(t,s) : V \to V$, $t \geq s$, are continuous and the sections $\mathcal{M}_d(k)$, $k \in \mathbb{Z}$, are compact, $\{\mathcal{M}'(t),|\ t \in \mathbb{R}\}$ is a family of compact subsets of V. Moreover, it follows as in the proof of Theorem 3.10 that the family $\{\mathcal{M}'(t)|\ t \in \mathbb{R}\}$ pullback attracts all bounded subsets of V. By Theorem 3.7 we conclude that the global pullback attractor $\{\mathcal{A}(t)|\ t \in \mathbb{R}\}$ of the time continuous process $\{U(t,s)|\ t \geq s\}$ exists, and the minimality property implies $\mathcal{A}(t) \subset \mathcal{M}'(t)$ for all $t \in \mathbb{R}$. By Proposition 3.1 and Theorem 3.11 the fractal dimension of the discrete global pullback attractor is uniformly bounded by

$$\dim_{\mathrm{f}}^V(\mathcal{A}(k\tilde{t})) \leq \log_{\frac{1}{2(\nu+\lambda)}} \left(N^W_{\frac{\nu}{\kappa}}(B^V_1(0)) \right) \qquad \text{for } k \in \mathbb{Z}.$$

It remains to estimate the fractal dimension of the time continuous sections. To this end let $r \in \mathbb{R}$ be arbitrary and the evolution process $\{U_r(t,s)|\ t \geq s\}$ be defined by $U_r(t,s) := U(t+r, s+r)$ for all $t \geq s, t, s \in \mathbb{R}$. The associated discrete evolution process $\{U_r(n,m)|\ n \geq m\}$ is given by $\tilde{U}_r(n,m) := U_r(n\tilde{t}, m\tilde{t})$ for all $n \geq m$, $n,m \in \mathbb{Z}$, and satisfies the hypothesis of Theorem 3.11. Consequently, there exists a pullback exponential attractor $\{\mathcal{M}^r_d(k)|\ k \in \mathbb{Z}\}$ for the discrete evolution process $\{\tilde{U}_r(n,m)|\ n \geq m\}$, and the fractal dimension of its sections satisfies the estimate stated in the theorem. We follow the previous arguments to conclude the existence of the global pullback attractor $\{\mathcal{A}_r(t)|\ t \in \mathbb{R}\}$ for the time continuous evolution process $\{U_r(t,s)|\ t \geq s\}$ and observe that

$$\mathcal{A}_r(t) = \mathcal{A}(t+r) \qquad \text{for all } t \in \mathbb{R}.$$

Moreover, the fractal dimension of the discrete sections of the global pullback attractor is uniformly bounded,

$$\dim_{\mathrm{f}}^V(\mathcal{A}_r(k\tilde{t})) \leq \dim_{\mathrm{f}}^V(\mathcal{M}^r_d(k)) \leq \log_{\frac{1}{2(\nu+\lambda)}} \left(N^W_{\frac{\nu}{\kappa}}(B^V_1(0)) \right) \qquad \text{for all } k \in \mathbb{Z}.$$

Finally, since $r \in \mathbb{R}$ was arbitrary and

$$\mathcal{A}_r(k\tilde{t}) = \mathcal{A}(k\tilde{t} + r) \qquad \text{for all } k \in \mathbb{Z},$$

follows the uniform bound for the fractal dimension of the time continuous global pullback attractor $\{\mathcal{A}(t)|\ t \in \mathbb{R}\}$. $\square$

Remark 3.5. *We remark that the Lipschitz continuity $(\mathcal{H}_4)$, which is essential for the construction of the time continuous pullback exponential attractor, is not required to establish the existence of the global pullback attractor and to derive estimates on its fractal dimension (see the hypothesis in Theorem 3.12).*

We now discuss different settings for the evolution process. Theorem 3.10 in the particular case that $\lambda = 0$ yields the existence of exponential pullback attractors for evolution processes that satisfy the smoothing property. It suffices to assume that the family of absorbing sets is bounded in the metric of W and, in the time continuous case, that the evolution process is Lipschitz continuous in W. The following theorem generalizes the previous results in [19] and [49], which were formulated in Theorem 3.9.

$(\mathcal{H}_4)'$ The evolution process $\{S(t,s)|\ t \geq s\}$ is Lipschitz continuous in W within the absorbing sets: For all $t \in \mathbb{T}$ and $t < s \leq t + \tilde{t}$ there exists a constant $L_{t,s} > 0$ such that

$$\|S(s,t)u - S(s,t)v\|_W \leq L_{t,s}\|u - v\|_W \qquad \text{for all } u, v \in B(t),\ t \in \mathbb{T}.$$

Corollary 3.2. *Let $\mathbb{T} = \mathbb{Z}$ or $\mathbb{T} = \mathbb{R}$, $\{S(t,s)|\ t \geq s\}$ be an evolution process in the Banach space V and the assumptions (H_0) and $(\mathcal{H}_2)$ be satisfied. We assume that Properties $(\mathcal{H}_1)$, (A_1) and (A_2) hold with $\{U(t,s)|\ t \geq s\}$ replaced by $\{S(t,s)|\ t \geq s\}$, where it suffices that the absorbing family is bounded in the metric of W. Moreover, the diameter of the family of absorbing sets $\{B(t)|\ t \in \mathbb{T}\}$ grows at most sub-exponentially in the past. If the evolution process is time continuous we additionally assume that $(\mathcal{H}_4)$ or $(\mathcal{H}_4)'$ is satisfied. Then, for any $\nu \in (0, \frac{1}{2})$ there exists a pullback exponential attractor $\{\mathcal{M}(t)|\ t \in \mathbb{T}\} = \{\mathcal{M}^\nu(t)|\ t \in \mathbb{T}\}$ for the evolution process $\{S(t,s)|\ t \geq s\}$, and the fractal dimension of its sections can be estimated by*

$$\dim_{\mathrm{f}}^V(\mathcal{M}(t)) \leq \log_{\frac{1}{2\nu}}\left(N^W_{\frac{\nu}{\kappa}}(B^V_1(0))\right) \qquad \textit{for all } t \in \mathbb{T}.$$

Moreover, the evolution process $\{S(t,s)|\ t \geq s\}$ possesses a global pullback attractor and the fractal dimension of its sections is uniformly bounded by the same value. For the existence of the global pullback attractor, the hypothesis $(\mathcal{H}_4)$ or $(\mathcal{H}_4)'$ are not required.

Proof. If the family of absorbing sets is bounded in the metric of W we define the sets

$$\widetilde{B}(t) := S(t, t - \tilde{t})B(t - \tilde{t}) \qquad t \in \mathbb{T},$$

which are pullback absorbing and bounded in the space V by the smoothing property $(\mathcal{H}_2)$. In the discrete case, $\mathbb{T} = \mathbb{Z}$, the corollary is an immediate consequence of Theorem 3.11 and Theorem 3.12.

If the evolution process is time continuous and Hypothesis $(\mathcal{H}_4)$ is satisfied the claim follows from Theorem 3.10 and Theorem 3.12. The same applies in the case that $\mathbb{T}=\mathbb{R}$ and Property $(\mathcal{H}_4)'$ holds. Indeed, by the smoothing property $(\mathcal{H}_2)$, the Lipschitz continuity $(\mathcal{H}_4)'$ and the continuous embedding (H_0) we observe

$$\begin{aligned}\|S(t+\tilde{t}+s,t)u-S(t+\tilde{t}+s,t)v\|_V &\le \kappa\|S(t+s,t)u-S(t+s,t)v\|_W\\ &\le \kappa L_{t,s}\|u-v\|_W \le \kappa L_{t,s}\mu\|u-v\|_V,\end{aligned}$$

for all $u,v\in B(t), t\in\mathbb{R}$ and $s\in[0,\tilde{t}]$. This proves the Lipschitz continuity of the evolution process in the space V and the results remain valid. □

We could also consider evolution processes that are asymptotically compact in the weaker phase space W. This setting was addressed in [32] for discrete evolution processes and in [36] for time continuous evolution processes, where forwards exponential attractors were constructed.

$(\mathcal{H}_1)'$ The family of bounded subsets $B(t)\subset W$, $t\in\mathbb{T}$, pullback absorbs all bounded subsets of W: For every bounded set $D\subset W$ and every $t\in\mathbb{T}$ there exists a pullback absorbing time $T_{D,t}\in\mathbb{T}_+$ such that

$$U(t,t-s)D\subset B(t) \qquad \text{for all } s\ge T_{D,t}.$$

$(\mathcal{H}_3)'$ The family $\{C(t,s)|\ t\ge s\}$ is a contraction in W within the absorbing sets:

$$\|C(t+\tilde{t},t)u-C(t+\tilde{t},t)v\|_W \le \lambda\|u-v\|_W \qquad \text{for all } u,v\in B(t),\ t\in\mathbb{T},$$

where the contraction constant $0\le\lambda<\frac{1}{2}$.

Theorem 3.13. *Let $\{U(t,s)|\ t\ge s\}$ be an evolution process in the Banach space W and the assumptions (H_0), $(\mathcal{H}_1)'$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)'$, (A_1) and (A_2) be satisfied. Moreover, we assume that the diameter of the family of absorbing sets $\{B(t)|\ t\in\mathbb{T}\}$ grows at most sub-exponentially in the past. In the time continuous case, $\mathbb{T}=\mathbb{R}$, we additionally assume that the process $\{U(t,s)|\ t\ge s\}$ satisfies the Lipschitz continuity property $(\mathcal{H}_4)'$. Then, for every $\nu\in(0,\frac{1}{2}-\lambda)$ there exists a pullback exponential attractor $\{\mathcal{M}^\nu(t)|\ t\in\mathbb{T}\}=\{\mathcal{M}(t)|\ t\in\mathbb{T}\}$ for the evolution process $\{U(t,s)|\ t\ge s\}$ in W, and the fractal dimension of its sections can be estimated by*

$$\dim_{\mathrm{f}}^W(\mathcal{M}(t))\le \log_{\frac{1}{2(\nu+\lambda)}}\left(N^W_{\frac{\nu}{\kappa}}(B^V_1(0))\right) \qquad \textit{for all } t\in\mathbb{T}.$$

Furthermore, the global pullback attractor of the evolution process in W exists, and the fractal dimension of its sections is uniformly bounded by the same value. For the existence of the global pullback attractor, the assumption $(\mathcal{H}_4)'$ is not required.

Proof. Generalizing the covering method in the proof of Theorem 3.6 for the non-autonomous setting as done in the proof of Theorem 3.11 for asymptotically compact processes in the phase space V yields discrete pullback exponential attractors in W. In the time continuous case let $\{\mathcal{M}_d(k)|\ k \in \mathbb{Z}\}$ be the pullback exponential attractor for the associated discrete evolution process. We define the time continuous sections of the pullback exponential attractor by $\mathcal{M}(t) = \overline{\widetilde{\mathcal{M}}(t)}^{\|\cdot\|_W}$, where

$$\widetilde{\mathcal{M}}(t) := U(t, k\tilde{t})\widetilde{\mathcal{M}}_d(k) \qquad t \in [k\tilde{t}, (k+1)\tilde{t}[,\ k \in \mathbb{Z}.$$

Following the arguments in the proof of Theorem 3.10 we conclude that the family of compact subsets $\{\mathcal{M}(t)|\ t \in \mathbb{R}\}$ is a pullback exponential attractor for the time continuous evolution process $\{U(t,s)|\ t \geq s\}$ in W.

The statements about the existence of the global pullback attractor follow as in the proof of Theorem 3.12, where the Lipschitz continuity of the evolution process is not required. □

Time Dependence of the Pullback Exponential Attractor and Forwards Attraction

Global pullback attractors are strictly invariant under the action of the evolution process, and the time dependence of the process is directly inherited by the attractor. To be more precise, let $\{U(t,s)|\ t \geq s\}$ be an evolution process in the Banach space V possessing a global pullback attractor $\{\mathcal{A}(t)|\ t \in \mathbb{T}\}$. Then, the invariance property

$$U(t,s)\mathcal{A}(s) = \mathcal{A}(t) \qquad \text{for all } t \geq s,\ t, s \in \mathbb{T},$$

immediately implies: If the evolution process is periodic, quasi-periodic or almost-periodic the pullback attractor exhibits the same property.

We analyse the respective property of the pullback exponential attractors constructed in Section 3.2.3. To this end we define the group of time shift operators or temporal translations acting on the space of evolution operators.

Definition 3.12. *Let $\{U(t,s)|\ t \geq s\}$ be an evolution process in the Banach space V. The action of the* ***group of time shift operators*** *$\{\mathcal{S}_r|\ r \in \mathbb{T}\}$ is defined by*

$$\mathcal{S}_r U(t,s) := U(t+r, s+r) \qquad t \geq s,\ t, s \in \mathbb{T},$$

where $r \in \mathbb{T}$.

Since pullback exponential attractors are not unique we could certainly construct for an evolution process U and the shifted process $\mathcal{S}_r U$, where $r \in \mathbb{T}$, pullback exponential attractors $\mathcal{M}_U$ and $\mathcal{M}_{\mathcal{S}_r U}$ that do not satisfy the cocycle property

$$\mathcal{M}_U(t+r) = \mathcal{M}_{\mathcal{S}_r U}(t) \qquad \text{for all } t, r \in \mathbb{T}.$$

However, if $\{\mathcal{M}_U(t)|\ t \in \mathbb{T}\}$ is a pullback exponential attractor for the evolution process U the translation of the attractor $\{\mathcal{M}_U(t+r)|\ t \in \mathbb{T}\}$ yields a pullback exponential attractor for the shifted process $\mathcal{S}_r U$, for every $r \in \mathbb{T}$.

Corollary 3.3. *Let $\{U(t,s)|\ t \geq s\}$ be an evolution process in the Banach space V. We assume that the hypothesis of Theorem 3.10 (or Theorem 3.11, if $\mathbb{T} = \mathbb{Z}$) are satisfied and denote by $\{\mathcal{M}_U(t)|\ t \in \mathbb{T}\}$ the pullback exponential attractor constructed in the proof of Theorem 3.10 (Theorem 3.11). Then, for every $r \in \mathbb{T}$ the family $\{\mathcal{M}_U(t+r)|\ t \in \mathbb{T}\}$ is a pullback exponential attractor for the evolution process $\{\mathcal{S}_r U(t,s)|\ t \geq s\}$, and the family of attractors satisfies the cocycle property*

$$\mathcal{M}_U(t+r) = \mathcal{M}_{\mathcal{S}_r U}(t) \qquad \text{for all } t, r \in \mathbb{T}.$$

In particular, if an evolution process is periodic, quasi-periodic or almost periodic the family of pullback exponential attractors $\{\mathcal{M}_{\mathcal{S}_r U}(t)|\ t \in \mathbb{T}\}_{r \in \mathbb{T}}$ exhibits the same property.

Proof. Let $r \in \mathbb{T}$ and $\{\mathcal{M}_U(t)|\ t \in \mathbb{T}\}$ be the pullback exponential attractor for the evolution process $\{U(t,s)|\ t \geq s\}$ constructed in the proof of Theorem 3.10, or Theorem 3.11 respectively. We define the sets

$$\mathcal{M}_{\mathcal{S}_r U}(t) := \mathcal{M}_U(t+r) \qquad \text{for all } t \in \mathbb{T}.$$

Then, the family $\{\mathcal{M}_{\mathcal{S}_r U}(t)|\ t \in \mathbb{T}\}$ is semi-invariant under the action of the evolution process $\{\mathcal{S}_r U(t,s)|\ t \geq s\}$. Moreover, the exponential pullback attraction property with respect to the process $\{\mathcal{S}_r U(t,s)|\ t \geq s\}$, the compactness of the sections and the uniform bound for the fractal dimension immediately follow from the corresponding properties of the family $\{\mathcal{M}_U(t)|\ t \in \mathbb{T}\}$, which proves that $\{\mathcal{M}_{\mathcal{S}_r U}(t)|\ t \in \mathbb{T}\}$ is a pullback exponential attractor for the shifted process.

□

Next, we formulate conditions such that the pullback exponential attractor also forwards attracts all bounded subsets exponentially.

Definition 3.13. *Let $\{U(t,s)|\ t \geq s\}$ be an evolution process in a metric space (X, d_X). We call the family $\{\mathcal{M}(t)|\ t \in \mathbb{T}\}$ a* ***forwards exponential attractor*** *for the evolution process if it satisfies Properties (i)-(iii) in Definition 3.9 and forwards exponentially attracts all bounded subset of X: There exists a constant $\omega > 0$ such that*

$$\lim_{s \to \infty} e^{\omega s} \operatorname{dist}_{\mathrm{H}}(U(t+s,t)D, \mathcal{M}(t+s)) = 0,$$

for every bounded set $D \subset X$ and time $t \in \mathbb{T}$.

Theorem 3.14. *Let $\{U(t,s)|\ t \geq s\}$ be an evolution process in the Banach space V and the assumptions of Theorem 3.10 (or Theorem 3.11, if $\mathbb{T} = \mathbb{Z}$) be satisfied. Moreover, we assume that the absorbing time corresponding to a bounded subset $D \subset V$ in Hypothesis $(\mathcal{H}_1)$ is independent of the time instant $t \in \mathbb{T}$. Then, the pullback exponential attractor $\{\mathcal{M}(t)|\ t \in \mathbb{T}\}$ in Theorem 3.10 (Theorem 3.11) is also a forwards exponential attractor for the evolution process.*

Proof. It suffices to show the forwards exponential attraction property of the pullback exponential attractor $\{\mathcal{M}(t)|\ t \in \mathbb{T}\}$. If the absorbing time in Hypothesis $(\mathcal{H}_1)$ is independent of the time instant $t \in \mathbb{T}$, the family $\{B(t)|\ t \in \mathbb{T}\}$ is also forwards absorbing for the evolution process. Indeed, for a bounded subset $D \subset V$ there exists a pullback absorbing time $T_D \in \mathbb{T}_+$ such that

$$U(t, t-s)D \subset B(t) \qquad \text{for all} \quad s \geq T_D,\ t \in \mathbb{T},$$

which is equivalent to the forwards absorbing property

$$U(t+s, t)D \subset B(t+s) \qquad \text{for all} \quad s \geq T_D,\ t \in \mathbb{T}.$$

We recall that the pullback exponential attractor of the associated discrete evolution process $\{\widetilde{U}(n,m)|\ n \geq m\}$, where $\widetilde{U}(n,m) := U(n\tilde{t}, m\tilde{t})$, for all $n \geq m, n, m \in \mathbb{Z}$, was defined as $\mathcal{M}_d(k) = \overline{\widetilde{\mathcal{M}}_d(k)}^{\|\cdot\|_V}$, and

$$\widetilde{\mathcal{M}}_d(k) = \bigcup_{n \in \mathbb{N}_0} E^n(k) \qquad \text{for } k \in \mathbb{Z}$$

(see the proof of Theorem 3.11). We show that the family $\{\widetilde{\mathcal{M}}_d(k)|\ k \in \mathbb{Z}\}$ is forwards exponentially attracting for the discrete evolution process $\{\widetilde{U}(n,m)|\ n \geq m\}$. Let $D \subset V$ be bounded, $T_D \in \mathbb{Z}_+$ be the corresponding pullback absorbing time and $k \in \mathbb{Z}$. If $n \geq T_D + 1$, then $n = T_D + n_0$ for some $n_0 \in \mathbb{N}$, and we observe

$$\begin{aligned}
&\operatorname{dist}_{\mathrm{H}}^{V}\big(\widetilde{U}(k+n,k)D, \mathcal{M}_d(k+n)\big)\\
&\leq \operatorname{dist}_{\mathrm{H}}^{V}\big(\widetilde{U}(k+T_D+n_0, k+T_D)U(k+T_D,k)D, \widetilde{\mathcal{M}}_d(k+n)\big)\\
&\leq \operatorname{dist}_{\mathrm{H}}^{V}\big(\widetilde{U}(k+T_D+n_0, k+T_D)B(k+T_D), \widetilde{\mathcal{M}}_d(k+n)\big)\\
&\leq \operatorname{dist}_{\mathrm{H}}^{V}\big(\widetilde{U}(k+T_D+n_0, k+T_D)B(k+T_D), E^{n_0}(k+n)\big)\\
&\leq \operatorname{dist}_{\mathrm{H}}^{V}\big(\widetilde{U}(\tilde{k}, \tilde{k}-n_0)B(\tilde{k}-n_0), E^{n_0}(\tilde{k})\big),
\end{aligned}$$

where $\tilde{k} = k + n$. Consequently, the forwards exponential attraction property follows from the proof of Theorem 3.11.

For time continuous evolution processes the pullback exponential attractor was defined by $\mathcal{M}(t) = \overline{\widetilde{\mathcal{M}}(t)}^{\|\cdot\|_V}$, $t \in \mathbb{R}$, where

$$\widetilde{\mathcal{M}}(t) := U(t, k\tilde{t})\widetilde{\mathcal{M}}_d(k) \qquad \text{for } t \in [k\tilde{t}, (k+1)\tilde{t}[,\ k \in \mathbb{Z}$$

(see the proof of Theorem 3.10). To show the forwards exponential attraction property of the time continuous attractor let $t \in \mathbb{R}$, $D \subset V$ be a bounded subset and $T_D \in \mathbb{R}_+$ be the corresponding pullback absorbing time. If $s \geq T_D + 2\tilde{t}$, then $s \geq (l+1)\tilde{t} + T_D$ for some

$l \in \mathbb{N}$, and $t+s = n\tilde{t} + s_0$, for some $n \in \mathbb{Z}$ and $s_0 \in [0, \tilde{t}[$. We obtain

$$\begin{aligned}
\operatorname{dist}_{\mathrm{H}}^V\big(U(t+s,t)D, \mathcal{M}(t+s)\big) &\leq \operatorname{dist}_{\mathrm{H}}^V\big(U(t+s,t)D, \widetilde{\mathcal{M}}(t+s)\big) \\
&= \operatorname{dist}_{\mathrm{H}}^V\big(U(n\tilde{t}+s_0, n\tilde{t})U(n\tilde{t},t)D, U(n\tilde{t}+s_0, n\tilde{t})\widetilde{\mathcal{M}}_d(n)\big) \\
&\leq L\operatorname{dist}_{\mathrm{H}}^V\big(U(n\tilde{t},t)D, \widetilde{\mathcal{M}}_d(n)\big) \\
&\leq L\operatorname{dist}_{\mathrm{H}}^V\big(U(n\tilde{t},(n-l)\tilde{t})U((n-l)\tilde{t},t)D, \widetilde{\mathcal{M}}_d(n)\big) \\
&\leq L\operatorname{dist}_{\mathrm{H}}^V\big(U(n\tilde{t},(n-l)\tilde{t})B((n-l)\tilde{t}), \widetilde{\mathcal{M}}_d(n)\big) \\
&= L\operatorname{dist}_{\mathrm{H}}^V\big(\widetilde{U}(n,(n-l))B((n-l)\tilde{t}), \widetilde{\mathcal{M}}_d(n)\big),
\end{aligned}$$

where we used the Lipschitz continuity $(\mathcal{H}_4)$, and $L \geq 0$ denotes the corresponding Lipschitz constant. Now, it follows from the proof of Theorem 3.11 that the family $\{\mathcal{M}(t) |\ t \in \mathbb{R}\}$ exponentially forwards attracts all bounded subsets of V. □

A Pullback Exponential Attractor for Time Continuous Semigroups

We now apply our results to autonomous evolution processes. In the discrete case we recover the results we obtained in Section 3.1.4 for semigroups. They differ, however, in the time continuous case since the invariance of a family of subsets in the non-autonomous setting is a weaker concept than the invariance of a fixed set under the action of a semigroup.

The previous construction of pullback exponential attractors for time continuous processes in Theorem 3.9 is different (see also [19] and [49]). To obtain the time continuous attractor the union over a certain time interval of the image of the discrete attractor is taken. It requires additional regularity properties in time of the evolution process and leads to weaker estimates for the fractal dimension of the attractor. However, when applied to time continuous semigroups the construction yields an exponential attractor according to Definition 3.6. In the proof of Theorem 3.10 we take the time evolution of the discrete attractor instead and prove under significantly weaker assumptions the existence of a pullback exponential attractor for time continuous evolution processes. If the assumptions of Theorem 3.9 are satisfied, our pullback exponential attractor is contained in the pullback exponential attractor of Theorem 3.9. However, applying our method for autonomous time continuous evolution processes does not lead to a fixed semi-invariant subset of the phase space.

We therefore propose to consider pullback exponential attractors for time continuous semigroups instead of exponential attractors in the strict sense. They coincide with exponential attractors in the discrete case, and pullback exponential attractors for time continuous semigroups satisfy the same dimension estimates as exponential attractors of discrete semigroups. In other words, weakening the semi-invariance property in the definition of exponential attractors we avoid the artificial increase in the fractal dimension of the attractor (see Theorem 3.3 and Theorem 3.4 in Section 3.1.4). Moreover, the construction does not require the Hölder continuity in time $(S4)$ of the semigroup.

In the sequel let $\mathbb{T} = \mathbb{R}$ and $\{U(t,s) |\ t \geq s\}$ be an autonomous time continuous evolution process in the Banach space V. The family of operators $T(t-s) := U(t-s, 0)$, where

$t \geq s$, $t, s \in \mathbb{R}$, then forms a semigroup in V. We propose to weaken the semi-invariance property of time continuous exponential attractors and to consider pullback exponential attractors. For autonomous evolution processes Definition 3.9 leads to the following:

Definition 3.14. *We call the family $\{\mathcal{M}(t)|\ t \in \mathbb{R}\}$ a* ***pullback exponential attractor for the semigroup*** *$\{T(t)|\ t \in \mathbb{R}_+\}$ in V if there exists a constant $0 < a < \infty$ such that $\mathcal{M}(t) = \mathcal{M}(a+t)$ for all $t \in \mathbb{R}$,*

(i) the subsets $\mathcal{M}(t) \subset V$ are non-empty and compact in V for all $t \in \mathbb{R}$,

(ii) the family is positively semi-invariant,

$$T(t)\mathcal{M}(s) \subset \mathcal{M}(t+s) \qquad \text{for all } t \in \mathbb{R}_+, s \in \mathbb{R},$$

(iii) the fractal dimension of the sets $\mathcal{M}(t)$, $t \in \mathbb{R}$, is uniformly bounded and

(iv) the family exponentially attracts all bounded subsets of V uniformly in time: There exists a positive constant $\omega > 0$ such that

$$\lim_{s \to \infty} \sup_{0 \leq t \leq a} e^{\omega s} \mathrm{dist}_{\mathrm{H}}^V(T(s)D, \mathcal{M}(t)) = 0$$

for every bounded subset $D \subset V$.

The definition implies that the set $\mathcal{M}_d = \mathcal{M}(a)$ is an exponential attractor for the associated discrete semigroup $\{\widetilde{T}(n)|\ n \in \mathbb{Z}_+\}$, where $\widetilde{T}(n) := T(na)$ for all $n \in \mathbb{Z}_+$. Moreover, any member of the family $\{\mathcal{M}(t)|\ t \in \mathbb{R}\}$ satisfies the properties of an exponential attractor of the semigroup $\{T(t)|\ t \in \mathbb{R}_+\}$ except for the semi-invariance property.

Remark 3.6. *If a semigroup possesses an exponential attractor it implies the existence of the global attractor and its finite dimensionality. The same applies to pullback exponential attractors for time continuous semigroups: If the pullback exponential attractor exists any member of the family $\{\mathcal{M}(t)|\ t \in \mathbb{R}\}$ contains the global attractor of the semigroup and the fractal dimension of the global attractor is finite.*

We assume the semigroup $\{T(t)|\ t \in \mathbb{R}_+\}$ can be represented as $T(t) = S(t) + C(t)$ for all $t \in \mathbb{R}_+$, and the assumptions $(S1)$-$(S3)$ in Section 3.1.4 are satisfied. Instead of the Hölder continuity in time $(S4)$ we assume Lipschitz continuity of the semigroup.

$(S4)''$ The semigroup $\{T(t)|\ t \in \mathbb{R}_+\}$ is (eventually) Lipschitz continuous in V within the absorbing set: There exists $s_0 \in \mathbb{R}_+$ such that for all $t \geq s_0$

$$\|T(t)u - T(t)v\|_V \leq L_t \|u - v\|_V \qquad \text{for all } u, v \in B,$$

for some constant $L_t > 0$.

Theorem 3.15. *Let $\{T(t)|\ t \in \mathbb{R}_+\}$ be a semigroup in the Banach space V, and the assumptions (H_0), $(S1)$-$(S3)$ and $(S4)''$ be satisfied. Then, for any $\nu \in (0, \frac{1}{2} - \lambda)$ there exists a pullback exponential attractor $\{\mathcal{M}(t)|\ t \in \mathbb{R}\} = \{\mathcal{M}^\nu(t)|\ t \in \mathbb{R}\}$ for the semigroup $\{T(t)|\ t \in \mathbb{R}_+\}$, and the fractal dimension of its sections is uniformly bounded by*

$$\dim_{\mathrm{f}}^V(\mathcal{M}(t)) \leq \log_{\frac{1}{2(\nu+\lambda)}} \left(N^W_{\frac{\nu}{\kappa}}(B^V_1(0)) \right) \qquad \textit{for all } t \in \mathbb{R}.$$

Proof. By Lemma 3.1 without loss of generality we can assume that the absorbing set B is positively semi-invariant. The family of operators $\{U(t,s)|\ t \geq s\}$ defined by $U(t,s) := T(t-s)$ for all $t \geq s, t, s \in \mathbb{R}$, forms an autonomous evolution process in the Banach space V. The evolution process $\{U(t,s)|\ t \geq s\}$ certainly satisfies the absorbing assumptions $(\mathcal{H}_1)$, (A_1) and (A_2) in Section 3.2.3, where the pullback absorbing sets $B(t) = B$ for all $t \in \mathbb{R}$.

We apply the method in the proof of Theorem 3.10 and first construct a pullback exponential attractor $\mathcal{M}_d$ for the discrete evolution process $\{\widetilde{U}(n,m)|\ n \geq m\}$, where $\widetilde{U}(n,m) := U(n\tilde{t}, m\tilde{t})$ for all $n \geq m, n, m \in \mathbb{Z}$. For autonomous evolution processes the family of sets E^n, $n \in \mathbb{N}_0$, is independent of time and consequently, according to Definition 3.6 the set

$$\mathcal{M}_d = \overline{\widetilde{\mathcal{M}}_d}^{\|\cdot\|_V} = \overline{\bigcup_{n \in \mathbb{N}_0} E^n}^{\|\cdot\|_V}$$

is an exponential attractor for the associated discrete semigroup $\{\widetilde{T}(n)|\ n \in \mathbb{Z}_+\}$, where $\widetilde{T}(n) := T(n\tilde{t})$, $n \in \mathbb{Z}_+$. Moreover, Theorem 3.11 implies that the fractal dimension of the exponential attractor is bounded by

$$\dim_{\mathrm{f}}^V(\mathcal{M}_d) \leq \log_{\frac{1}{2(\nu+\lambda)}} \left(N^W_{\frac{\nu}{\kappa}}(B^V_1(0)) \right).$$

We take the iterate $T(s_0)\widetilde{\mathcal{M}}_d = U(s_0, 0)\widetilde{\mathcal{M}}_d$ and define the time continuous pullback exponential attractor for the evolution process $\{U(t,s)|\ t \geq s\}$ by $\mathcal{M}(t) := \overline{\widetilde{\mathcal{M}}(t)}^{\|\cdot\|_V}$, $t \in \mathbb{R}$, where

$$\widetilde{\mathcal{M}}(t) := U(t, k\tilde{t})U(s_0, 0)\widetilde{\mathcal{M}}_d = T(t - k\tilde{t} + s_0)\widetilde{\mathcal{M}}_d \qquad \text{for all } t \in [k\tilde{t}, (k+1)\tilde{t}[,\ k \in \mathbb{Z}.$$

By Assumption $(S4)''$ the semigroup is Lipschitz continuous within the absorbing set for $t \geq s_0$. Proposition 3.1 therefore implies

$$\dim_{\mathrm{f}}^V(\mathcal{M}(t)) = \dim_{\mathrm{f}}^V(\widetilde{\mathcal{M}}(t)) = \dim_{\mathrm{f}}^V(T(t - k\tilde{t} + s_0)\widetilde{\mathcal{M}}_d) \leq \dim_{\mathrm{f}}^V(\widetilde{\mathcal{M}}_d),$$

for all $t \in [k\tilde{t}, (k+1)\tilde{t}[,\ k \in \mathbb{Z}$, which proves the uniform bound on the fractal dimension of the family $\{\mathcal{M}(t)|\ t \in \mathbb{R}\}$. In Theorem 3.10 we showed that $\{\mathcal{M}(t)|\ t \in \mathbb{R}\}$ is a pullback exponential attractor for the autonomous evolution process $\{U(t,s)|\ t \geq s\}$, which implies that the family is a pullback exponential attractor for the time continuous semigroup $\{T(t)|\ t \in \mathbb{R}_+\}$, and satisfies the properties in Definition 3.14 with $a = \tilde{t}$. □

An immediate consequence is the existence of the global attractor and its finite fractal dimension.

Corollary 3.4. *We assume the hypothesis of Theorem 3.15 are satisfied. Then, the global attractor $\mathcal{A}$ of the semigroup $\{T(t)|\ t \in \mathbb{R}_+\}$ exists, is contained in any member of the pullback exponential attractor $\{\mathcal{M}(t)|\ t \in \mathbb{R}\}$, and its fractal dimension is bounded by*

$$\dim_{\mathrm{f}}^{V}(\mathcal{A}) \leq \log_{\frac{1}{2(\nu+\lambda)}} \left(N_{\frac{\nu}{\kappa}}^{W}(B_1^V(0)) \right).$$

Proof. It follows from the proof of Theorem 3.15 that the pullback exponential attractor for the time continuous semigroup $\{T(t)|\ t \in \mathbb{R}_+\}$ is a pullback exponential attractor for the autonomous evolution process $\{U(t,s)|\ t \geq s\}$, where $U(t,s) := T(t-s)$, $t \geq s, t, s \in \mathbb{R}$. Theorem 3.7 implies that the evolution process $\{U(t,s)|\ t \geq s\}$ possesses a global pullback attractor $\{\mathcal{A}(t)|\ t \in \mathbb{R}\}$, and the global pullback attractor is contained in the pullback exponential attractor, $\mathcal{A}(t) \subset \mathcal{M}(t)$ for all $t \in \mathbb{R}$. Consequently, the fractal dimensions of the sections of the global pullback attractor satisfy the uniform estimates in the Corollary.

Since the global pullback attractor $\{\mathcal{A}(t)|\ t \in \mathbb{R}\}$ of the autonomous evolution process $\{U(t,s)|\ t \geq s\}$ exists if and only if the associated semigroup $\{T(t)|\ t \in \mathbb{R}_+\}$ possesses a global attractor $\mathcal{A}$ and $\mathcal{A}(t) = \mathcal{A}$ for all $t \in \mathbb{R}$, the statement of the corollary follows from Theorem 3.15. □

Remark 3.7. *We proved a stronger version of Corollary 3.4 in Section 3.1.5 (see Theorem 3.5). However, if we apply Theorem 3.12 to autonomous time continuous evolution processes we recover Theorem 3.5 about the existence and finite dimensionality of global attractors of semigroups, where the Lipschitz continuity of the semigroup $(S4)''$ is not required.*

If we apply Corollary 3.2 to autonomous evolution processes follows the existence of pullback exponential attractors for time continuous semigroups that satisfy the smoothing property.

Corollary 3.5. *Let $\{S(t)|\ t \in \mathbb{R}_+\}$ be a time continuous semigroup in the Banach space V, and the properties (H_0) and $(S2)$ be satisfied. Moreover, we assume that $(S1)$ and $(S4)''$ hold with $\{T(t)|\ t \in \mathbb{R}_+\}$ replaced by $\{S(t)|\ t \in \mathbb{R}_+\}$. Here, it suffices that the absorbing set is bounded in the metric of W. Then, for any $\nu \in (0, \frac{1}{2})$ there exists a pullback exponential attractor $\{\mathcal{M}(t)|\ t \in \mathbb{R}\} = \{\mathcal{M}^\nu(t)|\ t \in \mathbb{R}\}$ for the semigroup $\{S(t)|\ t \in \mathbb{R}_+\}$, and the fractal dimension of its sections is uniformly bounded by*

$$\dim_{\mathrm{f}}^{V}(\mathcal{M}(t)) \leq \log_{\frac{1}{2\nu}} \left(N_{\frac{\nu}{\kappa}}^{W}(B_1^V(0)) \right) \qquad \text{for all } t \in \mathbb{R}.$$

Moreover, the semigroup possesses a global attractor $\mathcal{A}$, it is contained in any member of the exponential pullback attractor, $\mathcal{A} \subset \mathcal{M}(t)$ for all $t \in \mathbb{R}$, and its fractal dimension is bounded by the same value. To show the existence of the global attractor and to derive the estimate on its fractal dimension the Lipschitz continuity $(S4)''$ of the semigroup is not required.

Proof. We apply Corollary 3.2 to autonomous evolution processes and argue as in the proof of Corollary 3.4 to show the existence of the pullback exponential attractor $\{\mathcal{M}(t)|\ t \in \mathbb{R}\}$ for the semigroup $\{S(t)|\ t \in \mathbb{R}_+\}$. The statement about the global attractor for the semigroup follows as in the proof of Corollary 3.4. □

Finally, we formulate the result for semigroups that are asymptotically compact in the weaker phase space W.

Theorem 3.16. *Let $\{T(t)|\ t \in \mathbb{R}_+\}$ be a semigroup in the Banach space W, and the assumptions (H_0) $(S1)'$, $(S2)$ and $(S3)'$ be satisfied. Moreover, we assume that the Lipschitz continuity $(S4)''$ holds with V replaced by W. Then, for any $\nu \in (0, \frac{1}{2} - \lambda)$ there exists a pullback exponential attractor $\{\mathcal{M}^\nu(t)|\ t \in \mathbb{R}\} = \{\mathcal{M}(t)|\ t \in \mathbb{R}\}$ for the semigroup $\{T(t)|\ t \in \mathbb{R}_+\}$ in W, and the fractal dimension of its sections is uniformly bounded by*

$$\dim_{\mathrm{f}}^W(\mathcal{M}(t)) \leq \log_{\frac{1}{2(\nu+\lambda)}} \left(N^W_{\frac{\nu}{\kappa}}(B^V_1(0)) \right) \qquad \textit{for all } t \in \mathbb{R}.$$

Moreover, the semigroup possesses a global attractor $\mathcal{A}$, it is contained in any member of the exponential pullback attractor, $\mathcal{A} \subset \mathcal{M}(t)$ for all $t \in \mathbb{R}$, and its fractal dimension is bounded by the same value. To show the existence of the global attractor and to derive the estimate on its fractal dimension the Lipschitz continuity of the semigroup $(S4)''$ in W is not required.

Proof. We apply Theorem 3.13 to autonomous evolution processes. It follows as in the proof of Corollary 3.4 that the pullback exponential attractor $\{\mathcal{M}(t)|\ t \in \mathbb{R}\}$ in W for the time continuous semigroup $\{T(t)|\ t \in \mathbb{R}_+\}$ exists. Moreover, the existence and finite dimensionality of the global attractor for the semigroup can be shown by the same arguments, where the Lipschitz continuity of the semigroup is not required. □

Remark 3.8. *Let $\{T(t)|\ t \in \mathbb{R}_+\}$ be a time continuous semigroup in the Banach space V that possesses a pullback exponential attractor $\{\mathcal{M}(t)|\ t \in \mathbb{R}\}$. If the semigroup satisfies the Hölder continuity property $(S4)$, then*

$$\mathcal{M} := \bigcup_{t \in \mathbb{R}} \mathcal{M}(t)$$

is an exponential attractor for the time continuous semigroup in the sense of Definition 3.6 and coincides with the exponential attractor constructed in Section 3.1.4.

3.2.5. Applications

We now apply the theoretical results of the previous sections to show the existence of pullback exponential attractors for evolution processes generated by non-autonomous partial differential equations.

Non-Autonomous Chafee-Infante Equation

First, we analyse an initial value problem for a non-autonomous Chafee-Infante equation and show the existence of a pullback exponential attractor for the generated evolution process. In particular, we obtain an example for a finite dimensional pullback attractor which is unbounded in the past.

Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a bounded domain with smooth boundary $\partial\Omega$ and $s \in \mathbb{R}$. We consider the initial-/boundary value problem

$$\begin{aligned} \frac{\partial}{\partial t}u(x,t) &= \triangle u(x,t) + \lambda u(x,t) - \beta(t)\big(u(x,t)\big)^3 && x \in \Omega,\ t > s, \\ u(x,t) &= 0 && x \in \partial\Omega,\ t \geq s, \\ u(x,s) &= u_s(x) && x \in \Omega, \end{aligned} \tag{3.5}$$

where $\lambda \in \mathbb{R}$, Δ denotes the Laplace operator with respect to the spatial variable $x \in \Omega$ and $\frac{\partial}{\partial t}$ the partial derivative with respect to time $t > s$. The initial data u_s is a continuous function that vanishes on the boundary, $u_s \in C_0(\overline{\Omega})$. Moreover, we assume the non-autonomous term $\beta : \mathbb{R} \to \mathbb{R}_+$ is strictly positive, continuously differentiable and satisfies the properties

$$0 < \sup_{t\in\mathbb{R}} \beta(t) \leq \beta_0, \tag{3.6}$$

$$\lim_{t\to-\infty} \beta(t) = 0, \tag{3.7}$$

$$\sup_{t\in\mathbb{R}} \frac{|\beta'(t)|}{\beta(t)} \leq \beta_1, \tag{3.8}$$

$$\lim_{t\to-\infty} \frac{e^{\gamma t}}{\beta(t)} = 0 \qquad \text{for all } \gamma > 0, \tag{3.9}$$

where the constants $0 < \beta_0, \beta_1 < \infty$. We consider the evolution process generated by (3.5) in the phase space $W := C_0(\overline{\Omega})$, where the norm in W is defined by

$$\|u\|_W := \max_{x\in\overline{\Omega}} |u(x)| \qquad u \in W.$$

To show the existence of a positively semi-invariant family of absorbing sets we use the method of lower and upper solutions (see [58], Chapter 2).

Definition 3.15. *A function* $u^* \in C(\overline{\Omega} \times [s,\infty[) \cap C^{2,1}(\Omega\times]s,\infty[)$ *is called an* ***upper solution*** *for (3.5) if it satisfies the inequalities*

$$\begin{aligned} \frac{\partial}{\partial t}u^*(x,t) - \triangle u^*(x,t) &\geq \lambda u^*(x,t) - \beta(t)\big(u^*(x,t)\big)^3 && x \in \Omega,\ t > s, \\ u^*(x,t) &\geq 0 && x \in \partial\Omega,\ t \geq s, \\ u^*(x,s) &\geq u_s(x) && x \in \Omega. \end{aligned} \tag{3.10}$$

Analogously, the function $u_* \in C(\overline{\Omega} \times [s,\infty[) \cap C^{2,1}(\Omega\times]s,\infty[)$ *is a* ***lower solution*** *for (3.5) if it satisfies the reversed inequalities in (3.10).*

Lemma 3.2. *There exist constants $a, b \geq 0$ such that the function $c^* : [s, \infty[\to \mathbb{R}_+$,*

$$c^*(t) := \frac{a}{\sqrt{\beta(t)}} + b,$$

is an upper solution for (3.5) if the initial data satisfies $u_s(x) \leq c^*(s)$ for all $x \in \Omega$.

If the initial function fulfils $u_s(x) \geq -c^*(s)$ for all $x \in \Omega$, the function $c_* : [s, \infty[\to \mathbb{R}$, $c_*(t) := -c^*(t)$, is a lower solution for (3.5).

Proof. We define $c^*(t) := \frac{a}{\sqrt{\beta(t)}} + b$, where the constants $a, b \geq 0$ are chosen below, and obtain

$$\begin{aligned}
&\frac{\partial}{\partial t} c^*(t) - \triangle c^*(t) - \lambda c^*(t) + \beta(t)\big(c^*(t)\big)^3 \\
&= -\frac{a}{2}\frac{\beta'(t)}{\sqrt{\beta(t)}^3} - \lambda\big(\frac{a}{\sqrt{\beta(t)}} + b\big) + \beta(t)\big(\frac{a}{\sqrt{\beta(t)}} + b\big)^3 \\
&= -\frac{a}{2}\frac{\beta'(t)}{\sqrt{\beta(t)}^3} - \lambda\big(\frac{a}{\sqrt{\beta(t)}} + b\big) + \frac{a^3}{\sqrt{\beta(t)}} + b^3\beta(t) + 3a^2 b + 3\sqrt{\beta(t)}ab^2 \\
&= -\frac{a}{2}\frac{\beta'(t)}{\sqrt{\beta(t)}^3} + (3a^2 b - \lambda b) + \frac{a}{\sqrt{\beta(t)}}(a^2 - \lambda) + b^3\beta(t) + 3ab^2\sqrt{\beta(t)} \\
&= \frac{a}{\sqrt{\beta(t)}}\left(-\frac{\beta'(t)}{2\beta(t)} + b\sqrt{\beta(t)}(3a - \frac{\lambda}{a}) + (a^2 - \lambda) + \frac{b^3}{a}\sqrt{\beta(t)}^3 + 3b^2\beta(t)\right).
\end{aligned}$$

Since β vanishes slowly,

$$\sup_{t \in \mathbb{R}} \frac{|\beta'(t)|}{\beta(t)} \leq \beta_1 < \infty,$$

there exist positive constants $a, b > 0$ such that

$$\frac{\partial}{\partial t} c^*(t) - \triangle c^*(t) - \lambda c^*(t) + \beta(t)\big(c^*(t)\big)^3 \geq 0,$$

which proves that the function c^* is an upper solution for (3.5).

The non-linearity is odd with respect to u, which implies

$$\begin{aligned}
&\frac{\partial}{\partial t} c_*(t) - \triangle c_*(t) - \lambda c_*(t) + \beta(t)\big(c_*(t)\big)^3 \\
&= \frac{\partial}{\partial t}\big(-c^*(t)\big) - \triangle\big(-c^*(t)\big) - \lambda\big(-c^*(t)\big) + \beta(t)\big(-c^*(t)\big)^3 \\
&= -\left(\frac{\partial}{\partial t} c^*(t) - \triangle c^*(t) - \lambda c^*(t) + \beta(t)\big(c^*(t)\big)^3\right).
\end{aligned}$$

Consequently, the function $c_* := -c^*$ is a lower solution for (3.5) if the initial data satisfies $u_s(x) \geq c_*(s)$ for all $x \in \Omega$. □

The linear heat equation

$$\begin{aligned} \frac{\partial}{\partial t}u(x,t) &= \triangle u(x,t) && x\in\Omega,\ t>0,\\ u(x,t) &= 0 && x\in\partial\Omega,\ t\geq 0,\\ u(x,0) &= u_0(x) && x\in\Omega, \end{aligned} \tag{3.11}$$

generates an analytic semigroup in the Banach space $W := (C_0(\overline{\Omega}), \|\cdot\|_W)$ (see [54]). We denote the semigroup corresponding to the linear problem (3.11) by $\{e^{\Delta t}|\ t\in\mathbb{R}_+\}$, and the associated fractional power spaces by X^α, $\alpha\geq 0$. The operators $e^{\Delta t}$ are linear and bounded from W to X^α, and the operator norm $\|\cdot\|_{\mathcal{L}(W;X^\alpha)}$ satisfies the estimate

$$\|e^{\Delta t}\|_{\mathcal{L}(W;X^\alpha)} \leq \frac{C_\alpha}{t^\alpha} \qquad \text{for all } t>0, \tag{3.12}$$

where the constant $C_\alpha\geq 0$. One can show that the semi-linear problem (3.5) generates an evolution process $\{U(t,s)|\ t\geq s\}$ in the phase space W, where the operators are defined by

$$U(t,s)u_s := u(\,\cdot\,,t;u_s,s) \qquad t\geq s,$$

and $u(\,\cdot\,,\,\cdot\,;u_s,s):\overline{\Omega}\times[s,\infty[\to\mathbb{R}$ denotes the unique solution of (3.11) corresponding to initial data $u_s\in C_0(\overline{\Omega})$ and initial time $s\in\mathbb{R}$. Moreover, the evolution process satisfies the variation of constants formula

$$U(t,s)u_s = e^{\Delta(t-s)}u_s + \int_s^t e^{\Delta(t-\tau)}f(\tau,U(\tau,s)u_s))d\tau.$$

For further details and the proof we refer to [54] and [61].

Lemma 3.2 and Theorem 4.1 in [58] imply the existence of a pullback absorbing family of bounded semi-invariant subsets.

Proposition 3.3. *The family of bounded subsets*

$$B(t) := \big\{v\in W \mid \|v\|_W \leq c^*(t)\big\} \qquad t\in\mathbb{R},$$

is positively semi-invariant for the evolution process $\{U(t,s)|\ t\geq s\}$ generated by the initial value problem (3.5) and pullback absorbs all bounded subsets of W.

Proof. Let $s\in\mathbb{R}$ and the initial data $u_s\in W$ satisfy $\|u_s\|_W\leq c^*(s)$. Lemma 3.2 implies that the functions c^* and c_* are upper and lower solutions for the initial-/boundary value problem (3.5). By Theorem 4.1 in Chapter 2 of [58] follows that there exists a unique classical solution $u(\,\cdot\,,\,\cdot\,;u_s,s):\overline{\Omega}\times[s,\infty[\to\mathbb{R}$ corresponding to the initial data u_s and initial time $s\in\mathbb{R}$, and the solution satisfies

$$c_*(t)\leq u(x,t;u_s,s)\leq c^*(t) \qquad \text{for all } x\in\overline{\Omega}, t\geq s.$$

Consequently, the associated evolution process satisfies $U(t,s)u_s\in B(t)$ for all $u_s\in B(s)$ and $t\geq s$, which shows the semi-invariance of the family $\{B(t)|\ t\in\mathbb{R}\}$.

It remains to prove that the family $\{B(t)|\ t \in \mathbb{R}\}$ is pullback absorbing. To this end let $D \subset W$ be bounded and $t \in \mathbb{R}$. Then, the set $D \subset B_R^W(0)$ if $R > 0$ is sufficiently large. By Assumption (3.7) there exists $t_0 \in \mathbb{R}$ such that $R \leq \frac{a}{\beta(t)}$ for all $t \leq t_0$, and consequently, $D \subset B(t)$ for all $t \leq t_0$. Finally, we observe that the pullback absorbing time is bounded in the past, $T_{D,s} \leq t - t_0$ for all $s \leq t$. □

The following lemma states that the evolution process $\{U(t,s)|\ t \geq s\}$ satisfies the smoothing property with respect to the Banach spaces $V := C_0^1(\overline{\Omega})$ and W, where the norm in V is defined by

$$\|u\|_V := \|u\|_W + \sum_{j=1}^{n} \|\frac{\partial u}{\partial x_j}\|_W.$$

Lemma 3.3. *Let $\{U(t,s)|\ t \geq s\}$ be the evolution process generated by the initial value problem (3.5). Then, there exists a positive constant $\kappa > 0$ such that*

$$\|U(t+1,t)u - U(t+1,t)v\|_V \leq \kappa \|u - v\|_W \qquad \text{for all} \ \ u, v \in B(t),\ t \in \mathbb{R}.$$

Proof. Let $s \in \mathbb{R}$ and the initial data $u, v \in B(s)$. We denote the corresponding solutions by $u(t) := U(t,s)u$ and $v(t) := U(t,s)v$, where $t \geq s$. It was shown in [54] (Theorem 2.4) that the continuous embedding $X^\alpha \hookrightarrow V$ exists for all $\alpha > \frac{1}{2}$. Moreover, we use the variation of constants formula and obtain

$$\begin{aligned}
\|u(t) - v(t)\|_V &\leq c\|u(t) - v(t)\|_{X^\alpha} \\
&\leq c\Big(\|e^{\Delta(t-s)}(u - v)\|_{X^\alpha} + \int_s^t \|e^{\Delta(t-\tau)}(f(\tau, u(\tau) - f(\tau, v(\tau))\|_{X^\alpha} d\tau\Big) \\
&\leq c\|e^{\Delta(t-s)}\|_{\mathcal{L}(W;X^\alpha)}\|u - v\|_W \\
&\quad + c\int_s^t \|e^{\Delta(t-\tau)}\|_{\mathcal{L}(W;X^\alpha)}\|f(\tau, u(\tau)) - f(\tau, v(\tau))\|_W d\tau,
\end{aligned}$$

where $c \geq 0$ denotes the embedding constant. By Proposition 3.3 we conclude

$$\begin{aligned}
&\|f(\tau, u(\tau)) - f(\tau, v(\tau))\|_W \\
&\leq \lambda\|u(\tau) - v(\tau)\|_W + \|\beta(\tau)\big(u(\tau) - v(\tau)\big)\big(u(\tau)^2 + u(\tau)v(\tau) + v(\tau)^2\big)\|_W \\
&\leq \lambda\|u(\tau) - v(\tau)\|_W + 2\|\big(u(\tau) - v(\tau)\big)\beta(\tau)\big(u(\tau)^2 + v(\tau)^2\big)\|_W \\
&\leq \lambda\|u(\tau) - v(\tau)\|_W + 4\|\big(u(\tau) - v(\tau)\big)\beta(\tau)\big(\frac{a}{\sqrt{\beta(\tau)}} + b\big)^2\|_W \\
&\leq (\lambda + C)\|u(\tau) - v(\tau)\|_W,
\end{aligned} \tag{3.13}$$

for some constant $C \geq 0$, where we used Assumption (3.6) in the last estimate. The bound on the operator norm (3.12) and the continuous embedding $V \hookrightarrow W$ imply

$$\begin{aligned}
\|u(t) - v(t)\|_V &\leq cC_\alpha\Big(\frac{1}{(t-s)^\alpha}\|u - w\|_W + (\lambda + C)\int_s^t \frac{1}{(t-\tau)^\alpha}\|u(\tau) - v(\tau)\|_W d\tau\Big) \\
&\leq cC_\alpha\Big(\frac{1}{(t-s)^\alpha}\|u - w\|_W + (\lambda + C)\mu\int_s^t \frac{1}{(t-\tau)^\alpha}\|u(\tau) - v(\tau)\|_V d\tau\Big),
\end{aligned}$$

where $\mu > 0$ denotes the embedding constant. Finally, we set $t = s + 1$ and

$$y(s+1) := \|U(s+1,s)u - U(s+1,s)v\|_V,$$

which implies

$$y(s+1) \leq cC_\alpha\Big(\|u-v\|_W + (\lambda + C)\mu \int_s^{s+1} \frac{1}{(s+1-\tau)^\alpha} y(\tau)d\tau\Big).$$

From a generalized Gronwall inequality (see Theorem 1.26 in [75]) we conclude

$$y(s+1) \leq \kappa\|u-v\|_W,$$

for some constant $\kappa > 0$, which concludes the proof of the lemma. □

Corollary 3.2 now implies the existence of a pullback exponential attractor in V for the evolution process $\{U(t,s)|\ t \geq s\}$.

Theorem 3.17. *Let $\{U(t,s)|\ t \geq s\}$ be the evolution process in the Banach space W generated by the initial-/boundary value problem (3.5). Moreover, we assume that the non-autonomous term satisfies Properties (3.6)-(3.9). Then, for every $\nu \in (0, \frac{1}{2})$ there exists a pullback exponential attractor $\{\mathcal{M}^\nu(t)|\ t \in \mathbb{R}\}$ in V for the evolution process $\{U(t,s)|\ t \geq s\}$, and the fractal dimension of its sections is uniformly bounded by*

$$\dim_f^V(\mathcal{M}^\nu(t)) \leq \log_{\frac{1}{2\nu}}\Big(N^W_{\frac{\nu}{\kappa}}(B^V_1(0))\Big) \qquad \text{for all } t \in \mathbb{R},$$

where $\kappa > 0$ denotes the smoothing constant in Lemma 3.3. Furthermore, the global pullback attractor exists and is contained in the pullback exponential attractor.

Proof. The family of pullback absorbing sets $\{B(t)|\ t \in \mathbb{R}\}$ defined in Lemma 3.3 satisfies the hypothesis (A_1) and (A_2) in Section 3.2.3. Since the diameter of the absorbing sets is bounded by

$$\|B(t)\|_W \leq 2\big(\frac{a}{\sqrt{\beta(t)}} + b\big) \qquad t \in \mathbb{R},$$

and the non-autonomous term satisfies Property (3.9), the absorbing sets grow at most sub-exponentially in the past. Moreover, the embedding $V \hookrightarrow\hookrightarrow W$ is compact, and the smoothing property with respect to the spaces V and W was shown in Lemma 3.3. To apply Corollary 3.2 it remains to verify the Lipschitz continuity of the evolution process. The variation of constants formula implies

$$\begin{aligned}
&\|U(t,s)u - U(t,s)v\|_W \\
&\leq \|e^{\Delta(t-s)}(u-v)\|_W + \int_s^t \|e^{\Delta(t-\tau)}\big(f(\tau, U(\tau,s)u) - f(\tau, U(\tau,s)v)\big)\|_W d\tau \\
&\leq C_0\|u-v\|_W + C_0\int_s^t \|f(\tau,U(\tau,s)u) - f(\tau,U(\tau,s)v)\|_W d\tau \\
&\leq C_0\|u-v\|_W + C_0(\lambda + C)\int_s^t \|U(\tau,s)u - U(\tau,s)v\|_W d\tau,
\end{aligned}$$

for some constant $C_0 \geq 0$, where we used the estimate (3.13) in Lemma 3.3. By Gronwalls Lemma follows the Lipschitz continuity of the evolution process in W.

□

Non-Autonomous Damped Wave Equation

The following initial value problem for the non-autonomous dissipative wave equation generates an evolution process that is asymptotically compact,

$$\begin{aligned} \frac{\partial^2}{\partial t^2}u(x,t) + \beta(t)\frac{\partial}{\partial t}u(x,t) &= \Delta u(x,t) + f(u(x,t)) && x \in \Omega,\ t > s, \\ u(x,s) &= u_s(x) && x \in \Omega, \\ \frac{\partial}{\partial t}u(x,s) &= v_s(x) && x \in \Omega, \\ u(x,t) &= 0 && x \in \partial\Omega,\ t \geq s, \end{aligned} \tag{3.14}$$

where $s \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 3$, is a bounded domain with smooth boundary $\partial\Omega$. We assume that the non-linearity $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and satisfies

$$|f'(z)| \leq c(1 + |z|^p) \qquad z \in \mathbb{R}, \tag{3.15}$$

$$\limsup_{|z|\to\infty} \frac{f(z)}{z} \leq 0, \tag{3.16}$$

for some constant $c > 0$ and $0 < p < \frac{2}{n-2}$. Furthermore, the function $\beta : \mathbb{R} \to \mathbb{R}_+$ is Hölder continuous and bounded from above and below by positive constants $0 < b_0 \leq b_1 < \infty$,

$$b_0 \leq \beta(t) \leq b_1 \qquad \text{for all } t \in \mathbb{R}. \tag{3.17}$$

We apply Theorem 3.10 to show that the evolution process generated by (3.14) possesses a pullback exponential attractor. Setting $v := \frac{\partial}{\partial t}u$ and $w := \begin{pmatrix} u \\ v \end{pmatrix}$ we rewrite Equation (3.14) in the abstract form

$$\begin{aligned} \frac{\partial}{\partial t}w &= A_\beta(t)w + F(w) && t > s, \\ w|_{t=s} &= w_s && w_s \in V, \end{aligned} \tag{3.18}$$

where the initial data $w_s = \begin{pmatrix} u_s \\ v_s \end{pmatrix}$, and the phase space is $V := H_0^1(\Omega) \times L^2(\Omega)$. The norm in V is given by

$$\|w\|_V := \left(\|u\|^2_{H_0^1(\Omega)} + \|v\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} \qquad \text{for } w = (u,v) \in V.$$

Furthermore, the operators are defined by $A_\beta(t) = A_1 + A_2(t)$,

$$A_1 := \begin{pmatrix} 0 & Id \\ -A & 0 \end{pmatrix}, \qquad A_2(t) := \begin{pmatrix} 0 & 0 \\ 0 & -\beta(t)Id \end{pmatrix}, \qquad F(w) := \begin{pmatrix} 0 \\ \widetilde{F}(u) \end{pmatrix},$$

where $A = -\Delta$ denotes the Laplace operator with homogeneous Dirichlet boundary conditions and domain $\mathcal{D}(A) = H_0^1(\Omega) \cap H^2(\Omega)$ in $L^2(\Omega)$. The domain of the operator A_1 in V is $\mathcal{D}(A_1) = (H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega)$, and $\widetilde{F}$ denotes the Nemytskii operator

$$\widetilde{F} : H_0^1(\Omega) \to L^2(\Omega), \quad \widetilde{F}(u) := f(u(\cdot)).$$

The initial value problem (3.18) generates an evolution process $\{U(t,s)|\ t \geq s\}$ in the Banach space V, which is asymptotically compact and pullback strongly bounded dissipative. In the sequel, we only present a sketch of the proof and refer to [42] (Chapter 4), [12] (Section VI.4), [3] and [8] for details.

We first consider the linear homogeneous problem

$$\begin{aligned} \frac{\partial}{\partial t} w &= A_\beta(t) w \qquad && t > s, \\ w|_{t=s} &= w_s && w_s \in V, \end{aligned} \tag{3.19}$$

and denote the generated evolution process in V by $\{C(t,s)|\ t \geq s\}$. The following lemma was proved in [8] and yields the exponential decay of the solutions of the linear homogeneous equation.

Lemma 3.4. *Let $\{C(t,s)|\ t \geq s\}$ be the evolution process in the Banach space V generated by (3.19). Then, there exist constants $C \geq 0$ and $\omega > 0$ such that the norm of the operators is bounded by*

$$\|C(t,s)\|_{\mathcal{L}(V;V)} \leq Ce^{-\omega(t-s)} \qquad \text{for all } t \geq s,\ t,s \in \mathbb{R}.$$

Sketch of the proof. We consider the Hilbert space $H_0^1(\Omega)$ with the norm and scalar product

$$\langle u, v \rangle_{H_0^1(\Omega)} := \int_\Omega \nabla u(x) \cdot \nabla v(x) dx, \quad \|u\|_{H_0^1(\Omega)} := \langle u, u \rangle_{H_0^1(\Omega)}^{\frac{1}{2}} \qquad u, v \in H_0^1(\Omega),$$

which is equivalent to the standard norm and scalar product in $H_0^1(\Omega)$ by Poincaré's inequality. We define the functional $\mathcal{F} : V \to \mathbb{R}$ by

$$\mathcal{F}(\phi, \psi) := \frac{1}{2}\|\phi\|_{H_0^1(\Omega)}^2 + \frac{1}{2}\|\psi\|_{L^2(\Omega)}^2 + 2b\langle \phi, \psi \rangle_{L^2(\Omega)},$$

where the constant $b > 0$ will be chosen below. If $w(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$ is a smooth solution of (3.19) we observe

$$\begin{aligned} 0 &= \langle v, v_t \rangle_{L^2(\Omega)} + \langle u, v \rangle_{H_0^1(\Omega)} + \beta(t)\|v\|_{L^2(\Omega)}^2, \\ 0 &= \langle u, v_t \rangle_{L^2(\Omega)} + \|u\|_{H_0^1(\Omega)}^2 + \beta(t)\langle u, v \rangle_{L^2(\Omega)}. \end{aligned}$$

Using these identities and Poincaré's inequality follows

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(u,v) &= \langle u,v\rangle_{H_0^1(\Omega)} + \langle v,v_t\rangle_{L^2(\Omega)} + 2b\langle u_t,v\rangle_{L^2(\Omega)} + 2b\langle u,v_t\rangle_{L^2(\Omega)}\\ &= -2b\|u\|^2_{H_0^1(\Omega)} - \big(\beta(t)-2b\big)\|v\|^2_{L^2(\Omega)} - 2b\beta(t)\langle u,v\rangle_{L^2(\Omega)}\\ &\le -2b\|u\|^2_{H_0^1(\Omega)} - \big(\beta(t)-2b\big)\|v\|^2_{L^2(\Omega)} + b\beta(t)\big(\epsilon\|u\|^2_{L^2(\Omega)} + \frac{1}{\epsilon}\|v\|^2_{L^2(\Omega)}\big)\\ &\le -2b\|u\|^2_{H_0^1(\Omega)} - \big(\beta(t)-2b\big)\|v\|^2_{L^2(\Omega)} + bb_1\big(\frac{\epsilon}{\lambda_1}\|u\|^2_{H_0^1(\Omega)} + \frac{1}{\epsilon}\|v\|^2_{L^2(\Omega)}\big)\\ &\le -b(2-\frac{b_1\epsilon}{\lambda_1})\|u\|^2_{H_0^1(\Omega)} - (b_0-2b-\frac{bb_1}{\epsilon})\|v\|^2_{L^2(\Omega)},\end{aligned}$$

where we used Young's inequality, and λ_1 denotes the first eigenvalue of the Laplace operator A. If we chose $\epsilon = \frac{\lambda_1}{b_1}$ and $b = \frac{b_0}{2(2+\frac{b_1}{\epsilon})}$ follows

$$\frac{d}{dt}\mathcal{F}(u,v) \le -b\big(\|u\|^2_{H_0^1(\Omega)} + \|v\|^2_{L^2(\Omega)}\big) = -b\|(u,v)\|^2_V.$$

Next, we prove that the functional $\mathcal{F}$ defines an equivalent norm on V, if the constant $b>0$ is sufficiently small. Let $(\phi,\psi)\in V$, then

$$\begin{aligned}\mathcal{F}(\phi,\psi) &\le \frac{1}{2}\|\phi\|^2_{H_0^1(\Omega)} + \frac{1}{2}\|\psi\|^2_{L^2(\Omega)} + b(\|\phi\|^2_{L^2(\Omega)} + \|\psi\|^2_{L^2(\Omega)})\\ &\le (\frac{1}{2}+\frac{b}{\lambda_1})\|\phi\|^2_{H_0^1(\Omega)} + (\frac{1}{2}+b)\|\psi\|^2_{L^2(\Omega)} \le \frac{3}{4}\|(\phi,\psi)\|^2_V,\end{aligned} \tag{3.20}$$

and on the other hand

$$\begin{aligned}\mathcal{F}(\phi,\psi) &\ge \frac{1}{2}\|\phi\|^2_{H_0^1(\Omega)} + \frac{1}{2}\|\psi\|^2_{L^2(\Omega)} - b(\|\phi\|^2_{L^2(\Omega)} + \|\psi\|^2_{L^2(\Omega)})\\ &\ge (\frac{1}{2}-\frac{b}{\lambda_1})\|\phi\|^2_{H_0^1(\Omega)} + (\frac{1}{2}-b)\|\psi\|^2_{L^2(\Omega)} \ge \frac{1}{4}\|(\phi,\psi)\|^2_V,\end{aligned} \tag{3.21}$$

if $b < \frac{1}{4}\min\{1,\lambda_1\}$. Setting $\alpha = \min\{\frac{1}{4}, \frac{\lambda_1}{4}, \frac{b_0}{2(2+\frac{b_1}{\epsilon})}\}$ we obtain

$$\frac{d}{dt}\mathcal{F}(u,v) \le -\alpha\|(u,v)\|^2_V \le -\frac{4}{3}\alpha\mathcal{F}(u,v).$$

Gronwall's Lemma now implies

$$\mathcal{F}(u,v) \le \mathcal{F}(u_s,v_s)e^{-\alpha\frac{4}{3}(t-s)} \qquad t\ge s,$$

and using the equivalence of the norms follows the exponential decay of the solutions,

$$\|C(t,s)w_s\|_V \le \sqrt{3}e^{-\frac{2}{3}\alpha(t-s)}\|w_s\|_V \qquad t\ge s.$$

□

The semi-linear problem (3.18) generates an evolution process $\{U(t,s) |\ t \geq s\}$ in V, that satisfies the integral equation

$$\begin{aligned} U(t,s)w_s &= C(t,s)w_s + \int_s^t C(t,\tau)F(U(\tau,s)w_s)d\tau \\ &= C(t,s)w_s + S(t,s)w_s \end{aligned}$$

(see [8] and [42]). Next, we show that the evolution process $\{U(t,s) |\ t \geq s\}$ is pullback strongly bounded dissipative.

Lemma 3.5. *Let $\{U(t,s) |\ t \geq s\}$ be the evolution process in the Banach space V generated by the initial value problem (3.18). Then, there exists a bounded subset $B \subset V$ that uniformly pullback absorbs all bounded sets of V: For every bounded set $D \subset V$ there exists $T_D \geq 0$ such that*

$$U(t,t-s)D \subset B \qquad \textit{for all } s \geq T_D,\ t \in \mathbb{R}.$$

Sketch of the proof. We only indicate the ideas of the proof and refer to [12] (Section VI.4) and [42] (Chapter 4) for details. We define the functional $\widetilde{\mathcal{F}} : V \to \mathbb{R}$ by

$$\widetilde{\mathcal{F}}(\phi,\psi) := \frac{1}{2}\|\phi\|^2_{H^1_0(\Omega)} + \frac{1}{2}\|\psi\|^2_{L^2(\Omega)} + 2\tilde{b}\langle\phi,\psi\rangle_{L^2(\Omega)} - \int_\Omega G(\phi(x))dx,$$

where $G(s) := \int_0^s f(r)dr$ and $\tilde{b} > 0$ will be chosen appropriately. If $w = w(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$ is a smooth solution of (3.18) we observe

$$\begin{aligned} 0 &= \langle v, v_t\rangle_{L^2(\Omega)} + \langle u, v\rangle_{H^1_0(\Omega)} + \beta(t)\|v\|^2_{L^2(\Omega)} - \langle\widetilde{F}(u), v\rangle_{L^2(\Omega)}, \\ 0 &= \langle u, v_t\rangle_{L^2(\Omega)} + \|u\|^2_{H^1_0(\Omega)} + \beta(t)\langle u, v\rangle_{L^2(\Omega)} - \langle\widetilde{F}(u), u\rangle_{L^2(\Omega)}. \end{aligned}$$

Using these identities and the growth restriction (3.15) on the non-linearity one can prove, similarly as in Lemma 3.4, that the functional satisfies

$$\frac{d}{dt}\widetilde{\mathcal{F}}(u,v) \leq -\tilde{b}\widetilde{\mathcal{F}}(u,v) + \tilde{c}_1,$$

for some constant $\tilde{c}_1 \geq 0$, if we choose $\tilde{b} > 0$ sufficiently small. Gronwall's Lemma and the norm equivalence in the proof of Lemma 3.4 now imply

$$\begin{aligned} \widetilde{\mathcal{F}}(u(t),v(t)) &\leq \widetilde{\mathcal{F}}(u_s,v_s)e^{-\tilde{b}(t-s)} + \frac{\tilde{c}_1}{\tilde{b}}(1-e^{-\tilde{b}(t-s)}) \\ &\leq \Big(\|(u_s,v_s)\|^2_V - \int_\Omega G(u_s(x))dx\Big)e^{-\tilde{b}(t-s)} + \frac{\tilde{c}_1}{\tilde{b}}. \end{aligned}$$

Furthermore, the growth restriction (3.15) and the continuous embedding $H^1_0(\Omega) \hookrightarrow L^{p+2}(\Omega)$ (see (3.22) below) allow to estimate the integral

$$\Big|\int_\Omega G(u_s(x))dx\Big| \leq \tilde{c}_2\Big(\int_\Omega |u_s(x)|^{p+2}dx + 1\Big) \leq \tilde{c}_2\tilde{c}_3\big(\|u_s\|^{p+2}_{H^1_0(\Omega)} + 1\big),$$

where the constants $\tilde{c}_2, \tilde{c}_3 \geq 0$. On the other hand, one can show using Assumption (3.16) and the estimates in the proof of Lemma 3.4 that there exits a constant $\tilde{c}_4 \geq 0$ such that

$$\widetilde{\mathcal{F}}(u(t), v(t)) \geq \frac{1}{8}\|(u(t), v(t))\|_V^2 - \tilde{c}_4.$$

Combining all estimates we conclude

$$\|U(t,s)(u_s, v_s)\|_V^2 \leq \tilde{c}_5\big(\|(u_s, v_s)\|_V^2 + \|u_s\|_{H_0^1(\Omega)}^{p+2} + 1\big)e^{-\alpha(t-s)} + \tilde{c}_6,$$

for some constants $\tilde{c}_5, \tilde{c}_6 \geq 0$.

This shows that the set $B := \{w \in V|\ \|w\|_V \leq 2\tilde{c}_6\}$ is a fixed bounded pullback absorbing set for the evolution process $\{U(t,s)|\ t \geq s\}$. Moreover, for a bounded subset $D \subset V$ the corresponding pullback absorbing time $T_D \geq 0$ is independent of the time instant $t \in \mathbb{R}$. □

To show that the family of operators $\{S(t,s)|\ t \geq s\}$ satisfies the smoothing property we establish several auxiliary results. We denote by X^α, $\alpha \in \mathbb{R}$, the fractional power spaces associated to the operator A with domain $\mathcal{D}(A) = X^1 = H_0^1(\Omega) \cap H^2(\Omega)$ in $X := L^2(\Omega)$ (see [69] or [61]). Furthermore, let $H^s(\Omega)$, $s \in \mathbb{R}_+$, be the fractional Sobolev spaces obtained by interpolation between the spaces $H^m(\Omega)$ and $L^2(\Omega)$, $m \in \mathbb{N}$ (see [1] or Section II.1.3 in [69]). Since the domain Ω is bounded we have the following continuous embeddings

$$H_0^s(\Omega) \hookrightarrow H^s(\Omega) \hookrightarrow L^{p'}(\Omega) \hookrightarrow L^2(\Omega) \qquad \text{if } \frac{1}{2} \geq \frac{1}{p'} \geq \frac{1}{2} - \frac{s}{n} > 0, \tag{3.22}$$

where $H_0^s(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$ (see [1] or [12] Theorem 1.1 in Chapter 2). If the second inequality in (3.22) is strict, the embedding $H^s(\Omega) \hookrightarrow L^{p'}(\Omega)$ is compact. Moreover, for all $s > 0$ the embeddings

$$H_0^s(\Omega) \hookrightarrow X^{\frac{s}{2}} \hookrightarrow H^s(\Omega),$$

are continuous (this follows by Theorem 16.1 in [75]). By duality we conclude

$$L^2(\Omega) \hookrightarrow L^{q'}(\Omega) \hookrightarrow X^{-\frac{s}{2}} \qquad \text{if } \frac{1}{p'} + \frac{1}{q'} = 1,\ \frac{1}{2} \geq \frac{1}{p'} \geq \frac{1}{2} - \frac{s}{n} > 0, \tag{3.23}$$

and the embedding $L^{q'}(\Omega) \hookrightarrow X^{-\frac{s}{2}}(\Omega)$ is compact if the second inequality in (3.23) is strict.

The solution theory of the linear homogeneous problem can be extended to the fractional power spaces $X^\alpha \times X^{\alpha-\frac{1}{2}}$, $\alpha \in \mathbb{R}$ (see [69] Section IV.1.1).

Lemma 3.6. *Let $\epsilon > 0$ and the space $V^\epsilon := X^{\frac{1}{2}-\epsilon} \times X^{-\epsilon}$. Then, for every initial data $w_s = \binom{u_s}{v_s} \in V^\epsilon$ there exists a unique solution $w \in C([s, s+T]; V^\epsilon)$ of the homogeneous problem*

$$\begin{aligned} w_t &= A_\beta(t)w & t &> s, \\ w|_{t=s} &= w_s & w_s &\in V^\epsilon, \end{aligned}$$

where $T > 0$ is arbitrary. Moreover, the generated evolution process is uniformly bounded in V^ϵ,

$$\|C(t,s)\|_{\mathcal{L}(V^\epsilon;V^\epsilon)} < d \qquad t \geq s,\ t,s \in \mathbb{R},$$

for some constant $d \geq 0$.

Proof. We consider the operator

$$A_\beta(t) = A_1 + A_2(t) = \begin{pmatrix} 0 & Id \\ -\overline{A} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -\beta(t)Id \end{pmatrix},$$

in V^ϵ, where the operators $A_2(t) : V^\epsilon \to V^\epsilon$ are linear and uniformly bounded in t by Assumption (3.17). Here, $\overline{A}$ denotes the extension of the operator A to an operator in $X^{-\epsilon}$ with domain $\mathcal{D}(\overline{A})$. Since $\overline{A}$ is selfadjoint the operator A_1 is dissipative in V^ϵ. Indeed, let $w = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{D}(A_1) = \mathcal{D}(\overline{A}) \times X^{\frac{1}{2}-\epsilon}$, then

$$\begin{aligned} \langle w, A_1 w\rangle_{V^\epsilon} &= \Big\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} v \\ -\overline{A}u \end{pmatrix} \Big\rangle_{V^\epsilon} = \langle A^{\frac{1}{2}-\epsilon}u, A^{\frac{1}{2}-\epsilon}v\rangle_X + \langle A^{-\epsilon}v, A^{-\epsilon}(-\overline{A}u)\rangle_X \\ &= \langle \overline{A}^{\frac{1}{2}-\epsilon}u, \overline{A}^{\frac{1}{2}-\epsilon}v\rangle_X - \langle \overline{A}^{\frac{1}{2}-\epsilon}v, \overline{A}^{\frac{1}{2}-\epsilon}u\rangle_X = 0. \end{aligned}$$

By [61] (Corollary 4.4 in Chapter 1) the operator A_1 generates a strongly continuous semigroup of contractions in V^ϵ. The lemma now follows by Theorem 1.2, Chapter 6 in [61]. □

Lemma 3.7. *There exists $0 < \epsilon < 1$ such that the Nemytskii operator $\widetilde{F}$ is uniformly Lipschitz continuous from $H^{1-\epsilon}(\Omega)$ to $L^2(\Omega)$ within bounded subsets of $H_0^1(\Omega)$,*

$$\|\widetilde{F}(u) - \widetilde{F}(v)\|_{L^2(\Omega)} \leq c_f \|u - v\|_{H^{1-\epsilon}(\Omega)} \qquad \textit{for all } u, v \in D,$$

where the constant $c_f \geq 0$ and the subset $D \subset H_0^1(\Omega)$ is bounded.

Proof. Let the subset $D \subset H_0^1(\Omega)$ be bounded, $u, v \in D$ and $R > 0$ such that $D \subset B_0$, where $B_0 := B_R^{H_0^1(\Omega)}(0)$. By assumption, $p < \frac{2}{n-2}$ and consequently, $p = (1-\epsilon)\frac{2}{n-2}$ for some $0 < \epsilon < 1$. The growth restriction (3.15) and Hölder's inequality with $p' = \frac{n}{2-2\epsilon}$ and $q' = \frac{n}{n-2+2\epsilon}$ imply

$$\begin{aligned} \|F(u) - F(v)\|_{L^2(\Omega)} &\leq c\|(1 + |\zeta|^p)(u - v)\|_{L^2(\Omega)} \\ &\leq c\big(\|u - v\|_{L^2(\Omega)} + \||\zeta|^p\|_{L^{2p'}(\Omega)}\|u - v\|_{L^{2q'}(\Omega)}\big) \\ &\leq c\big(C_1\|u - v\|_{H^{1-\epsilon}(\Omega)} + C_2\|\zeta\|^p_{L^{2pp'}(\Omega)}\|u - v\|_{H^{1-\epsilon}(\Omega)}\big), \end{aligned}$$

for some $\zeta \in B_0$. Here, we used the continuous embeddings $H^{1-\epsilon}(\Omega) \hookrightarrow L^2(\Omega)$ and $H^{1-\epsilon}(\Omega) \hookrightarrow L^{2q'}(\Omega)$ in (3.22), and $C_1, C_2 \geq 0$ are the corresponding embedding constants. Since the set $D \subset B_0 \subset H_0^1(\Omega)$ is bounded, the embedding $H_0^1(\Omega) \hookrightarrow L^{2pp'}(\Omega) = L^{\frac{2n}{n-2}}(\Omega)$ in (3.22) yields the uniform bound on the norm $\|\zeta\|^p_{L^{2pp'}(\Omega)}$, and concludes the proof of the lemma.

□

Next, we show that the evolution process $\{U(t,s)|\ t \geq s\}$ restricted to the bounded pullback absorbing set B is uniformly Lipschitz continuous in $V^{\frac{\epsilon}{2}} = X^{\frac{1-\epsilon}{2}} \times X^{-\frac{\epsilon}{2}}$, where $\epsilon = 1 - \frac{p}{2}(n-2)$ was defined in the proof of Lemma 3.7.

Lemma 3.8. *Let* $\epsilon := 1 - \frac{p}{2}(n-2)$ *and the initial data* $w_s = \begin{pmatrix} u_s \\ v_s \end{pmatrix} \in B$, *where* $B \subset V$ *denotes the uniformly pullback absorbing set in Lemma 3.5. Then, the evolution process* $\{U(t,s)|\ t \geq s\}$ *generated by the initial value problem (3.18) is Lipschitz continuous with respect to the norm of* $V^{\frac{\epsilon}{2}}$.

Proof. We assume $u, v \in B$. We proved in Lemma 3.7 that the Nemytskii operator $\widetilde{F}$ is uniformly Lipschitz continuous from $H^{1-\epsilon}$ to $L^2(\Omega)$ in bounded subsets of $H_0^1(\Omega)$. Moreover, using the continuous embeddings $L^2(\Omega) = X \hookrightarrow X^{-\frac{\epsilon}{2}}$ and $X^{\frac{1-\epsilon}{2}} \hookrightarrow H^{1-\epsilon}(\Omega)$ we obtain

$$\|\widetilde{F}(u) - \widetilde{F}(v)\|_{X^{-\frac{\epsilon}{2}}} \leq c_1\|\widetilde{F}(u) - \widetilde{F}(v)\|_X \leq c_f c_1 \|u - v\|_{H^{1-\epsilon}(\Omega)} \leq c_f c_1 c_2 \|u - v\|_{X^{\frac{1-\epsilon}{2}}},$$

for some constants $c_1, c_2 \geq 0$. This shows that the operator $\widetilde{F}$ is uniformly Lipschitz continuous from $X^{\frac{1-\epsilon}{2}}$ to $X^{-\frac{\epsilon}{2}}$ in bounded subsets of $H_0^1(\Omega)$.

Let the initial data $w_s, z_s \in B$. We recall that the solution of the semi-linear problem (3.18) satisfies the integral identity

$$U(t,s)w_s = C(t,s)w_s + \int_s^t C(t,\tau)F(U(\tau,s)w_s)d\tau \qquad t \geq s,$$

and the evolution process $\{U(t,s)|\ t \geq s\}$ is bounded in V by Lemma 3.5. We can estimate the difference of the solutions $w(t) = \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} = U(t,s)w_s$ and $z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = U(t,s)z_s$ in the space $V^{\frac{\epsilon}{2}}$ by

$$\begin{aligned}
\|w(t) - z(t)\|_{V^{\frac{\epsilon}{2}}} &\leq \|C(t,s)\|_{\mathcal{L}(V^{\frac{\epsilon}{2}};V^{\frac{\epsilon}{2}})}\|w_s - z_s\|_{V^{\frac{\epsilon}{2}}} + \\
&\quad + \int_s^t \|C(t,\tau)\|_{\mathcal{L}(V^{\frac{\epsilon}{2}};V^{\frac{\epsilon}{2}})}\|F(U(\tau,s)w_s) - F(U(\tau,s)z_s)\|_{V^{\frac{\epsilon}{2}}} d\tau \\
&\leq d\Big(\|w_s - z_s\|_{V^{\frac{\epsilon}{2}}} + \int_s^t \|\widetilde{F}(w_1(\tau)) - \widetilde{F}(z_1(\tau))\|_{X^{-\frac{\epsilon}{2}}} d\tau\Big) \\
&\leq d\Big(\|w_s - z_s\|_{V^{\frac{\epsilon}{2}}} + \int_s^t c_1c_2c_f\|w_1(\tau) - z_1(\tau)\|_{X^{\frac{1-\epsilon}{2}}} d\tau\Big) \\
&\leq d\Big(\|w_s - z_s\|_{V^{\frac{\epsilon}{2}}} + \int_s^t c_1c_2c_f\|w(\tau) - z(\tau)\|_{V^{\frac{\epsilon}{2}}} d\tau\Big),
\end{aligned}$$

where we used the above estimate and Lemma 3.7. The Lipschitz continuity now follows by Gronwall's Lemma,

$$\|U(t,s)w_s - U(t,s)z_s\|_{V^{\frac{\epsilon}{2}}} = \|w(t) - z(t)\|_{V^{\frac{\epsilon}{2}}} \leq d\|w_s - z_s\|_{V^{\frac{\epsilon}{2}}} e^{c_3(t-s)}, \qquad (3.24)$$

where the constant $c_3 = dc_1c_2c_f$.

□

Combining the previous results we prove the smoothing property of the family of operators $\{S(t,s)|\ t \geq s\}$ with respect to the Banach space V and the auxiliary normed space $W := V^{\frac{\epsilon}{2}}$.

Lemma 3.9. *Let $\epsilon = 1 - \frac{p}{2}(n-2)$ and $W := V^{\frac{\epsilon}{2}}$. Then, the embedding $V \hookrightarrow\hookrightarrow W$ is compact, and for every $t_0 > 0$ there exists a positive constant $\kappa_{t_0} > 0$ such that*

$$\|S(t+t_0,t)w - S(t+t_0,t)z\|_V \leq \kappa_{t_0}\|w-z\|_W \qquad \text{for all } w,z \in B,\ t \in \mathbb{R},$$

where B denotes the uniformly pullback absorbing set defined in Lemma 3.5.

Proof. Let $t \in \mathbb{R}$, $t_0 > 0$ and the initial data $w, z \in B$. We denote the corresponding solutions of (3.18) by $U(\tau,t)w = \begin{pmatrix} U_1(\tau,t)w \\ U_2(\tau,t)w \end{pmatrix}$ and $U(\tau,t)z = \begin{pmatrix} U_1(\tau,t)z \\ U_2(\tau,t)z \end{pmatrix}$, where $\tau \geq t$. By the definition of the operators $\{S(t,s)|\ t \geq s\}$, Lemma 3.4 and Lemma 3.8 we obtain

$$\begin{aligned}
&\|S(t+t_0,t)w - S(t+t_0,t)z\|_V \leq \int_t^{t+t_0} \|C(t+t_0,\tau)\big(F(U(\tau,t)w) - F(U(\tau,t)z)\big)\|_V d\tau \\
&\leq C\int_t^{t+t_0} e^{-\omega(t+t_0-\tau)}\|\widetilde{F}(U_1(\tau,t)w - \widetilde{F}(U_1(\tau,t)z)\|_X d\tau \\
&\leq c_f C\int_t^{t+t_0} \|U_1(\tau,t)w - U_1(\tau,t)z\|_{H^{1-\epsilon}(\Omega)} d\tau \\
&\leq c_f c_4 C\int_t^{t+t_0} \|U_1(\tau,t)w - U_1(\tau,t)z\|_{X^{\frac{1-\epsilon}{2}}} d\tau \leq c_f c_4 C\int_t^{t+t_0} \|U(\tau,t)w - U(\tau,t)z\|_{V^{\frac{\epsilon}{2}}} d\tau \\
&\leq c_f c_4 C\int_t^{t+t_0} de^{c_3(\tau-t)}\|w-z\|_{V^{\frac{\epsilon}{2}}} d\tau \leq \kappa_{t_0}\|w-z\|_W,
\end{aligned}$$

for some constants $c_4 \geq 0$ and $\kappa_{t_0} > 0$. In the estimate we used the continuous embedding $X^{\frac{1-\epsilon}{2}} \hookrightarrow H^{1-\epsilon}(\Omega)$ and the Lipschitz continuity (3.24) of the process $\{U(t,s)|\ t \geq s\}$ in V^ϵ, which was proved in Lemma 3.8. The compactness of the embedding $V \hookrightarrow\hookrightarrow W$ follows by (3.23). □

Theorem 3.10 now implies the existence of a pullback exponential attractor for the evolution process $\{U(t,s)|\ t \geq s\}$.

Theorem 3.18. *Let $\{U(t,s)|\ t \geq s\}$ be the evolution process in the Hilbert space $V = H_0^1(\Omega) \times L^2(\Omega)$ generated by the initial value problem (3.18). We set $\epsilon = 1 - \frac{p}{2}(n-2)$ and consider the space $W = X^{\frac{1-\epsilon}{2}} \times X^{-\frac{\epsilon}{2}}$. Moreover, for arbitrary $\lambda < \frac{1}{2}$ we define $t_0 := \frac{1}{\omega}\ln\frac{C}{\lambda}$, where the constants $C \geq 0$ and $\omega > 0$ are determined by the estimate in Lemma 3.4.*

Then, for every $\nu \in (0, \frac{1}{2}-\lambda)$ there exists a pullback exponential attractor $\{\mathcal{M}^\nu(t)|\ t \in \mathbb{R}\}$ for the evolution process $\{U(t,s)|\ t \geq s\}$, and the fractal dimension of its sections is uniformly bounded by

$$\dim_f^V(\mathcal{M}^\nu(t)) \leq \log_{\frac{1}{2(\nu+\lambda)}}\left(N_{\frac{\nu}{\kappa}}^W(B_1^V(0))\right) \qquad \text{for all } t \in \mathbb{R},$$

where $\kappa = \kappa_{t_0} > 0$ denotes the smoothing constant in Lemma 3.9.

Furthermore, the global pullback attractor exists and is contained in the pullback exponential attractor $\{\mathcal{M}^\nu(t)|\ t \in \mathbb{R}\}$.

Proof. In Lemma 3.5 we proved the existence of a fixed bounded uniformly pullback absorbing set $B \subset V$ for the evolution process $\{U(t,s)|\ t \geq s\}$, and by Remark 3.4 in Section 3.2.3 the pullback absorbing assumptions $(\mathcal{H}_1)$, (A_1) and (A_2) are satisfied. If $\lambda \in (0, \frac{1}{2})$ and $t_0 = \frac{1}{\omega} \ln \frac{C}{\lambda}$, Lemma 3.4 implies that the linear operators $C(t+t_0,t)$, $t \in \mathbb{R}$, are contractions in V with contraction constant $\lambda < \frac{1}{2}$, which verifies Hypothesis $(\mathcal{H}_3)$. Moreover, the smoothing property $(\mathcal{H}_2)$ of the family of operators $\{S(t,s)|\ t \geq s\}$ is valid within the absorbing set B by Lemma 3.9. It remains to show the Lipschitz continuity $(\mathcal{H}_4)$ of the evolution process. To this end we recall that the Nemytskii operator $\widetilde{F}$ is uniformly Lipschitz continuous from $H^{1-\epsilon}(\Omega)$ to $L^2(\Omega)$ in bounded subsets of $H^1_0(\Omega)$ (see Lemma 3.7). If the subset $D \subset H^1_0(\Omega)$ is bounded we use the continuous embedding $H^1_0(\Omega) \hookrightarrow H^{1-\epsilon}(\Omega)$ and obtain

$$\|\widetilde{F}(u) - \widetilde{F}(v)\|_{L^2(\Omega)} \leq c_f \|u-v\|_{H^{1-\epsilon}(\Omega)} \leq C_f \|u-v\|_{H^1_0(\Omega)} \qquad \text{for all } u,v \in D, \tag{3.25}$$

where the constant $C_f \geq 0$. The Lipschitz continuity of the process $\{U(t,s)|\ t \geq s\}$ in V now follows as in the proof of Lemma 3.7 by replacing the space $V^{\frac{\epsilon}{2}}$ by V and using the estimate (3.25).

Consequently, all required hypothesis are verified and the existence of the pullback exponential attractor and the uniform estimates for the fractal dimension of its sections follow from Theorem 3.10. The global pullback attractor of the evolution process exists by Theorem 3.12 and is contained in the pullback exponential attractor. □

3.3. Concluding Remarks

We constructed pullback exponential attractors for asymptotically compact evolution processes assuming that the process possesses a family of time-dependent pullback absorbing sets that possibly grow in the past. In Section 3.2.5 we applied the theoretical results to show the existence of pullback exponential attractors for a non-autonomous Chafee-Infante equation and a non-autonomous damped wave equation. We hope our results are applicable in various other cases such as the non-autonomous Navier-Stokes equation or more general non-autonomous wave equations.

Another interesting problem is whether the theory of exponential attractors can be applied to study the longtime behaviour of degenerate parabolic equations such as the biofilm models discussed in Chapter 1. This requires the construction of exponential attractors in a generalized setting, which was developed in [20] for semigroups. For non-autonomous degenerate parabolic problems it is necessary to extend this construction for evolution processes.

An important property of exponential attractors is its stability under perturbations. For semigroups it was proved in [26] (Theorem 4.1) that the *Hölder continuity of exponential* *attractors up to a time shift* follows from the exponential attraction property. The proof can be adapted and extended to show the Hölder continuity (up to a time shift) of the pullback exponential attractor constructed in Theorem 3.10. However, it is desirable to establish a stronger version of Hölder continuity. In the autonomous case this was obtained in [35], and similarly in [32] where the stability was also shown for discrete non-autonomous forwards exponential attractors.

A. Function Spaces

We collect and explain in this appendix frequently used notation for function spaces. For details and properties of the spaces we refer to [1] and [69].

Spaces of Continuous Functions

Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a bounded domain and $u : \Omega \to \mathbb{R}$ be a scalar function. We denote partial derivatives of u by

$$\partial_{x_i} = \frac{\partial}{\partial x_i} \qquad \text{for } i = 1, \dots, n,$$

and use the multi-index notation for partial derivatives of higher order $m \in \mathbb{N}$,

$$\partial^\beta = \partial_{x_n}^{\beta_n} \dots \partial_{x_1}^{\beta_1}, \qquad |\beta| = \sum_{i=1}^n \beta_i = m,$$

where $\beta = (\beta_1, \dots, \beta_n)$, and $\beta_i \in \mathbb{Z}_+$ for $i = 1, \dots, n$.

The space $C(\overline{\Omega})$ consists of continuous functions $u : \overline{\Omega} \to \mathbb{R}$, and the norm is defined by

$$\|u\|_{C(\overline{\Omega})} := \max\left\{ |u(x)| \;\middle|\; x \in \overline{\Omega} \right\} \qquad u \in C(\overline{\Omega}).$$

We denote by $C^m(\Omega)$, $m \in \mathbb{N}$, the functions $u : \Omega \to \mathbb{R}$ that are m-times continuously differentiable on Ω. The space $C^m(\overline{\Omega})$ contains all functions in $C^m(\Omega)$ such that the function and all partial derivatives up to order m can be continuously extended to $\overline{\Omega}$. The norm in $C^m(\overline{\Omega})$ is given by

$$\|u\|_{C^m(\overline{\Omega})} := \sum_{|\beta| \le m} \|\partial^\beta u\|_{C(\overline{\Omega})} \qquad u \in C^m(\overline{\Omega}).$$

Finally, the space $C_0^m(\Omega)$ consists of the functions in $C^m(\overline{\Omega})$ that have compact support in Ω.

For $0 < \alpha < 1$ the Hölder space $C^\alpha(\Omega)$ contains all functions in $C(\overline{\Omega})$ such that

$$|u|_{\alpha,\Omega} := \sup\left\{ \frac{|u(x) - u(y)|}{|x - y|^\alpha} \;\middle|\; x, y \in \Omega, x \neq y \right\}$$

is finite. The norm in $C^\alpha(\Omega)$ is defined by

$$\|u\|_{C^\alpha(\Omega)} := \|u\|_{C(\overline{\Omega})} + |u|_{\alpha,\Omega} \qquad u \in C^\alpha(\Omega).$$

Let $T > 0$ and the parabolic cylinder be defined by $Q_T := \Omega \times (0, T)$. In the sequel we consider functions $u : Q_T \to \mathbb{R}$ depending on the spatial variable $x \in \Omega$ and time variable $t \in (0, T)$. The space $C^{k,m}(Q_T)$, where $k, m \in \mathbb{N}$, consists of functions $u : Q_T \to \mathbb{R}$ that are k-times continuously differentiable with respect to x and l-times continuously differentiable with respect to t. Analogously, the spaces $C^{k,m}(\overline{Q_T})$ are defined.

Furthermore, we denote by $C^{\alpha,\beta}(Q_T)$ the functions in $C(\overline{Q_T})$ that are Hölder continuous with exponent $0 < \alpha < 1$ with respect to x and Hölder continuous with exponent $0 < \beta < 1$ with respect to time t.

Lebesgue Spaces

For $1 \le p < \infty$ the Lebesgue space $L^p(\Omega)$ consists of measurable functions $u : \Omega \to \mathbb{R}$ such that the norm

$$\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u(x)|^p dx \right)^{\frac{1}{p}}$$

is finite. The space of essentially bounded functions $L^\infty(\Omega)$ consists of measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{L^\infty(\Omega)} := \operatorname{esssup}\big\{ |u(x)| \;\big|\; x \in \Omega \big\}$$

is finite. The local Lebesgue spaces $L^p_{loc}(\mathbb{R}^n)$, where $1 \le p \le \infty$, contain the measurable functions $u : \mathbb{R}^n \to \mathbb{R}$ such that for every bounded subset $K \subset \mathbb{R}^n$ the restriction $f|_K$ belongs to the space $L^p(K)$.

If $p = 2$ the space $L^2(\Omega)$ is a Hilbert space, and the inner product is defined by

$$\langle u, v \rangle_{L^2(\Omega)} := \int_\Omega u(x)v(x)dx \qquad u, v \in L^2(\Omega).$$

For vector valued functions $u : \Omega \to \mathbb{R}^k$, where $k \in \mathbb{N}$, the Hilbert space $L^2(\Omega; \mathbb{R}^k)$ consists of functions $u = (u_1, \ldots, u_k)$ such that $u_i \in L^2(\Omega)$ for all $i = 1, \ldots, n$. The inner product in $L^2(\Omega; \mathbb{R}^k)$ is defined by

$$\langle u, v \rangle_{L^2(\Omega;\mathbb{R}^k)} := \sum_{i=1}^{k} \langle u_i, v_i \rangle_{L^2(\Omega)} \qquad u, v \in L^2(\Omega; \mathbb{R}^k).$$

Sobolev Spaces

We denote the Sobolev spaces by $W^{m,p}(\Omega)$, where $m \in \mathbb{N}$ and $1 \le p \le \infty$. The norm in $W^{m,p}(\Omega)$ is defined by

$$\|u\|_{W^{m,p}(\Omega)} := \sum_{|\beta| \le m} \|\partial^\beta u\|_{L^p(\Omega)} \qquad u \in W^{m,p}(\Omega).$$

For $p = 2$ the Sobolev spaces are Hilbert spaces that we denote by $H^m(\Omega) := W^{m,2}(\Omega)$. The inner product in $H^m(\Omega)$ is defined by

$$\langle u, v\rangle_{H^m(\Omega)} := \sum_{|\beta|\leq m} \langle \partial^\beta u, \partial^\beta v\rangle_{L^2(\Omega)} \qquad u, v \in H^m(\Omega).$$

For non-integer $s \in \mathbb{R}_+$ the spaces $H^s(\Omega)$ are defined by interpolation between $L^2(\Omega)$ and $H^m(\Omega)$, $m \in \mathbb{N}$. Moreover, for $s \in \mathbb{R}_+$ we denote by $H_0^s(\Omega)$ the completion of the space $C_0^\infty(\Omega)$ in $H^s(\Omega)$, and by $H^{-s}(\Omega)$ the dual spaces of $H_0^s(\Omega)$.

Banach Space Valued Functions

Let $(V, \|\cdot\|_V)$ be a Banach space and $T > 0$. We denote by $C([0,T]; V)$ the space of continuous functions $u : [0,T] \to V$, where the norm is defined by

$$\|u\|_{C([0,T];V)} := \max\big\{\|u(t)\|_V | \ t \in [0,T]\big\} \qquad u \in C([0,T]; V).$$

The Bochner spaces $L^p((0,T); V)$, where $1 \leq p < \infty$, consist of measurable functions $u : (0,T) \to V$ such that the norm

$$\|u\|_{L^p((0,T);V)} := \Big(\int_0^T \|u(t)\|_V^p dt\Big)^{\frac{1}{p}}$$

is finite. Similarly, the Bochner space $L^\infty((0,T); V)$ contains all measurable functions $u : (0,T) \to V$ such that

$$\|u\|_{L^\infty((0,T);\Omega)} := \operatorname{esssup}\big\{\|u(t)\|_V \ \big| \ t \in (0,T)\big\}$$

is finite.

B. An Auxiliary Lemma

The following result is needed in the proof of Lemma 1.3. Its proof was indicated by M.A. Efendiev.

Lemma B.1. *Let the function $f \in C^2(\mathbb{R};\mathbb{R})$ satisfy $C_1|u|^{p-1} \leq f'(u) \leq C_1|u|^{p-1}$, where $p > 1$ and the constants C_1 and C_2 are positive. Then, for every $s \in (0,1)$ and $1 < q \leq \infty$, we have*

$$\|u\|_{W^{s/p,pq}(\Omega)} \leq C_p \|f(u)\|^{1/p}_{W^{s,q}(\Omega)}$$

where the constant $C_p \geq 0$ is independent of u.

Proof. Let f^{-1} denote the inverse of the function f. The conditions on f imply that the function $G(v) := \operatorname{sgn}(v)|f^{-1}(v)|^p$ is non-degenerate and satisfies

$$C_2 \leq G'(v) \leq C_1,$$

for some positive constants C_1 and C_2. Consequently, we obtain

$$|f^{-1}(v_1) - f^{-1}(v_2)|^p \leq C_p|G(v_1) - G(v_2)| \leq C_p'|v_1 - v_2|,$$

for all $v_1, v_2 \in \mathbb{R}$ and some constant $C_p' \geq 0$. Finally, according to the characterization of fractional Sobolev spaces (see [27]) follows

$$\begin{aligned} \|f^{-1}(v)\|^{pq}_{W^{s/p,qp}(\Omega)} &:= \|f^{-1}(v)\|^{pq}_{L^{pq}(\Omega)} + \int_\Omega \int_\Omega \frac{|f^{-1}(v(x)) - f^{-1}(v(y)|^{pq}}{|x-y|^{n+sq}}\, dx\, dy \\ &\leq C\|v\|^q_{L^q(\Omega)} + C_p' \int_\Omega \int_\Omega \frac{|v(x) - v(y)|^q}{|x-y|^{n+sq}}\, dx\, dy = C_p''\|v\|^q_{W^{s,q}(\Omega)}, \end{aligned}$$

for some constant $C_p'' \geq 0$, where we implicitly used that $f^{-1}(v) \sim \operatorname{sign}(v)|v|^{1/p}$. □

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