Maxima of stochastic processes driven by fractional Brownian motion

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Abstract

We study stationary processes given as solutions to SDEs driven by fractional Brownian motion (FBM). This class includes the fractional Ornstein-Uhlenbeck process (FOUP), but is a much richer class of processes, which can be obtained by state space transformations of the FOUP. An explicit formula in terms of Euler's Γ-function describes the asymptotic behaviour of the covariance function of FOUP near zero, which, by an application of Berman's condition, guarantees that the FOUP is in the maximum domain of attraction of the Gumbel distribution. Necessary and sufficient conditions on the state space transforms are stated to classify the maximum domain of attraction of solutions of FBM-driven SDEs.

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1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space carrying a two-sided fractional Brownian motion $(B_t^H)_{t\in\mathbb{R}}$ (FBM) with Hurst index $H \in (0,1)$, i.e., a centred Gaussian process with covariance function

$$EB_t^H B_s^H = \frac{1}{2}\Big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\Big) \qquad s,t \in \mathbb{R}\,. \tag{1.1}$$

FBM has stationary increments and it is selfsimilar, i.e. for all $c \in \mathbb{R}$,

$$(B_{ct}^H) \stackrel{d}{=} |c|^H (B_t^H)\,, \quad t \in \mathbb{R}\,,$$

in particular, $B_0^H = 0$. A Hurst index of $H = 1/2$ corresponds to standard Brownian motion. Further properties can be found in Samorodnitsky and Taqqu [14].

Our goal is to investigate the asymptotic behaviour of partial maxima of stationary solutions X given by a SDE of the form

$$X_t - X_s = \int_s^t \mu(X_u)\,du + \int_s^t \sigma(X_u)\,dB_u^H\,, \quad s \le t\,. \tag{1.2}$$

The integrals are interpreted pathwisely as Riemann-Stieltjes integrals. For an analytic treatment and conditions on μ and σ for the existence of such solutions we refer to Buchmann and Klüppelberg [5].

A prominent example is the Ornstein-Uhlenbeck model, which corresponds to linear μ and constant σ. More precisely, for $\gamma, \sigma > 0$ define the fractional Ornstein-Uhlenbeck process (FOUP) by

$$O_t^{H,\gamma,\sigma} = \sigma \int_{-\infty}^t e^{-\gamma(t-s)} dB_s^H\,, \quad t \in \mathbb{R}\,. \tag{1.3}$$

The process $O^{H,\gamma,\sigma} = (O_t^{H,\gamma,\sigma})_{t\in\mathbb{R}}$ is stationary and solves pathwisely the SDE

$$O_t - O_s = -\gamma \int_s^t O_u du + \sigma(B_t^H - B_s^H)\,, \quad s \le t\,. \tag{1.4}$$

As $O^{H,\gamma,\sigma}$ is a Gaussian process classical results due to Pickands [11] and Berman [1] apply giving a limit result for partial maxima. Standard references summarizing the extreme value theory of Gaussian processes are Berman [2], Leadbetter, Lindgren and Rootzén [10] and Piterbarg [12]. Explicit calculations concerning the FOUP are presented in Section 2 of the present paper.

As was shown in [5], under certain conditions on μ and σ, the solution X to (1.2) can be represented as a state space transform of the FOUP. Consequently, in Section 3 we investigate the full class of processes which can be obtained from FOUP by state space transforms. Necessary and sufficient conditions are given to characterize the maximum domain of attraction for such processes.

In Section 4 we return to the original problem. In the framework of [5] we obtain necessary and sufficient conditions to characterize the maximum domain of attraction for stationary solutions of (1.2). These results are based on asymptotic inversion results, whose proofs are found in Appendix C.

Our approach bears some similarity to Davis [8] and Borkovec and Klüppelberg [4], who investigated the extremal behavior of diffusion processes given as solutions to SDEs driven by Brownian motion. Whereas they used the classical OU process as a reference process to obtain the extreme behaviour of other families of diffusion processes, we use the fractional OU process instead. In that papers scale functions and time changes of the classical Ornstein-Uhlenbeck process are the core arguments. As such methods do not exist for processes driven by FBM we use some slightly different, but related approach.

2 Maxima of fractional Ornstein-Uhlenbeck processes

For any continuous time process $X = (X_t)_{t\geq 0}$ we say it belongs to the domain of attraction of some extreme value distribution G, and we write $X \in \mathrm{MDA}(G)$, if there exist norming constants $a_T > 0$ and $b_T \in \mathbb{R}$ ($T \geq 0$) such that

$$a_T^{-1}\left(\max_{0\leq t\leq T} X_t - b_T\right) \quad \stackrel{d}{\to} \quad G\,,$$

where throughout $\stackrel{d}{\to}$ denotes convergence in distribution as $T \to \infty$.

Possible extreme value distributions are the Fréchet distribution Φ_α ($\alpha > 0$), the Gumbel distribution Λ and the Weibull distribution Ψ_α ($\alpha > 0$). For details on standard extreme value theory we refer to Embrechts, Klüppelberg and Mikosch [9] or Leadbetter et al. [10].

In this section we derive the extreme behaviour of the FOUP given in (1.4). As it is a Gaussian process we can apply the theory of Pickands [11] and Berman [1, 2]. The behaviour of partial maxima of a Gaussian process can be related to the behaviour of the covariance function in zero and infinity. We define for any $t \in \mathbb{R}$ the covariance function

$$\rho_{H,\gamma,\sigma}(h) = EO_t^{H,\gamma,\sigma}O_{t+h}^{H,\gamma,\sigma}\,, \quad h \in \mathbb{R}\,.$$

As FOUP is stationary the function $\rho_{H,\gamma,\sigma}(\cdot)$ does not depend on t. Throughout this paper we write $O^H = O^{H,1,1}$ and $\rho_H = \rho_{H,1,1}$. In the following Lemma we summarize some properties of ρ (see Appendix A for a proof).

LEMMA 2.1. (a) *Symmetry:* $\rho_{H,\gamma,\sigma}(h) = \rho_{H,\gamma,\sigma}(|h|)$.

(b) *Scaling property:* $\rho_{H,\gamma,\sigma}(h) = \dfrac{\sigma^2}{\gamma^{2H}}\,\rho_H(\gamma h)$.

(c) *Asymptotic behaviour at infinity [Cheridito, Kawaguchi and Maejima [7]]:*

$$\rho_{H,\gamma,\sigma}(h) = \begin{cases} \dfrac{1}{2}\dfrac{\sigma^2}{\gamma}\ \exp(-\gamma|h|)\,, & H = 1/2\,, \\ O(h^{2H-2})\,, \quad h \to \infty\,, & H \neq 1/2\,. \end{cases} \tag{2.1}$$

(d) *Asymptotic behaviour for* $h \to 0$*:*

$$\rho_{H,\gamma,\sigma}(h) = \begin{cases} \dfrac{\Gamma(2H+1)}{2}\ \dfrac{\sigma^2}{\gamma^{2H}} - \dfrac{\sigma^2}{2}|h|^{2H} + o(|h|) & H < 1/2\,, \\ \dfrac{1}{2}\ \dfrac{\sigma^2}{\gamma}\ e^{-\gamma|h|} & H = 1/2\,, \\ \dfrac{\Gamma(2H+1)}{2}\ \dfrac{\sigma^2}{\gamma^{2H}} - \dfrac{\sigma^2}{2}|h|^{2H} + \dfrac{\Gamma(2H+1)}{4}\dfrac{\sigma^2}{\gamma^{2H-2}}|h|^2 + o(|h|^2) & H > 1/2\,. \end{cases}$$

Now we can formulate a result for the partial maxima of a FOUP.

THEOREM 2.2. *Let* $\gamma, \sigma > 0$*. Then*

$$(\sigma\, a_T^{H,\gamma})^{-1}\ \left\{\max_{0\le t\le T} O_t^{H,\gamma,\sigma}\ -\ \sigma\, b_T^{H,\gamma}\right\} \quad \stackrel{d}{\rightarrow} \quad \Lambda\,,$$

where

$$\begin{aligned} a_T^{H,\gamma} &= \gamma^{-H}\,\frac{\Gamma(2H+1)^{1/2}}{2(\log T)^{1/2}}\,, \\ b_T^{H,\gamma} &= \gamma^{-H}\,\frac{\Gamma(2H+1)^{1/2}}{\sqrt{2}} \\ &\qquad \left(2(\log T)^{1/2} + \frac{1-H}{2H}\ \frac{\log\log T}{(\log T)^{1/2}} + \frac{\mathcal{C}(H,\gamma)}{(\log T)^{1/2}}\right)\,, \\ \mathcal{C}(H,\gamma) &= \log\left(\gamma\,\Gamma(2H+1)^{-1/(2H)}\mathcal{H}_{2H}(2\pi)^{-1/2}2^{(1-H)/(2H)}\right)\,, \end{aligned}$$

and $\mathcal{H}_{2H}$ *is Pickands' number.*

Proof. We apply the following result on Gaussian processes due to Pickands [11] and Berman [1]; see e.g. Theorem 12.3.5 of Leadbetter et al. [10]. For any normal process $(X_t)_{t\ge0}$ such that Berman's conditions hold, i.e.,

$$EX_hX_0 = \begin{cases} 1 - d|h|^{2H} + o(|h|^{2H}) & h \to 0\,, \\ o((\log h)^{-1}) & h \to \infty\,. \end{cases} \tag{2.2}$$

for constants $d > 0$ and $H \in (0,1)$, we have

$$(2\log T)^{1/2}\left\{\max_{0\le t\le T} X_t - \beta_T(H,d)\right\} \quad \stackrel{d}{\rightarrow} \quad \Lambda$$

where

$$\begin{aligned}\beta_T(H,d) &= (2\,\log T)^{1/2} + \frac{1-H}{2H}\,\frac{\log\log T}{(2\,\log T)^{1/2}} + \frac{\psi(H,d)}{(2\,\log T)^{1/2}},\\ \psi(H,d) &= \log\left(d^{1/(2H)}\,\mathcal{H}_{2H}\,(2\pi)^{-1/2}2^{(1-H)/(2H)}\right).\end{aligned}$$

For $t\in\mathbb{R}$ define a normal process $X_t^{H,\gamma,\sigma} := (\rho_{H,\gamma,\sigma}(0))^{-1/2}\,O_t^{H,\gamma,\sigma}$. Condition (2.2) for $h\to\infty$ is ensured by (2.1). From Lemma 2.1 (d) we obtain for $d=\gamma^{2H}/\Gamma(2H+1)$

$$EX_h^{H,\gamma,\sigma}X_0^{H,\gamma,\sigma} = 1-(\rho_{H,\gamma,\sigma}(0))^{-1}\,\frac{\sigma^2}{2}\,|h|^{2H}+o(|h|^{2H}) = 1-d\,|h|^{2H}+o(|h|^{2H})$$

Hence for this value d,

$$(2\log T)^{1/2}\left\{\max_{0\le t\le T} X_t^{H,\gamma,\sigma}-\beta_T(H,d)\right\} \quad\xrightarrow{d}\quad \Lambda$$

and, therefore, defining $a_T^{H,\gamma}$ as stated in the theorem, we obtain

$$(\sigma a_T^{H,\gamma})^{-1}\left\{\max_{0\le t\le T} O_t^{H,\gamma,\sigma}-\frac{\sigma}{\gamma^H}\left(\frac{\Gamma(2H+1)}{2}\right)^{1/2}\beta_T(H,d)\right\}\quad\xrightarrow{d}\quad\Lambda\,.$$

Choosing $\mathcal{C}(H,\gamma)=\psi(H,\gamma^{2H}/\Gamma(2H+1))$ this proves the result. □

Remark 2.3. (a) We write from now on $O_t^{H,\gamma}=O_t^{H,\gamma,1}$. Setting $M_T^{H,\gamma}=\max_{0\le t\le T}O_t^{H,\gamma}$ we see that

$$\frac{b_T^{H,\gamma}}{a_T^{H,\gamma}}\left(\frac{M_T^{H,\gamma}}{b_T^{H,\gamma}}-1\right)\xrightarrow{d}\Lambda\,.$$

As $b_T^{H,\gamma}/a_T^{H,\gamma}\to\infty$ we conclude $M_T^{H,\gamma}/b_T^{H,\gamma}\xrightarrow{P}1$, hence the distribution of $M_T^{H,\gamma}$ becomes less spread around $b_T^{H,\gamma}$ as T gets large. Consequently, $b_T^{H,\gamma}$ describes the growth of the partial maxima for large T quite precisely.

(b) Observe that $a_T^{H,\gamma}b_T^{H,\gamma}\to 1/\delta^{H,\gamma}=\Gamma(2H+1)/(\gamma^{2H}\,2^{1/2})$. The convergence to types theorem (see Theorem A1.5 of Embrechts et al. [9]) allows for different scaling, namely,

$$\delta^{H,\gamma}\,b_T^{H,\gamma}\,(M_T^{H,\gamma}-b_T^{H,\gamma})\quad\xrightarrow{d}\quad\Lambda\,.$$

(c) The constant $\mathcal{H}_{2H}$ is Pickands' number. For the definition of the constant we refer to Leadbetter et al. [10]. The precise shape of the curve $H\mapsto\mathcal{H}_{2H}$ is unknown, a simulated curve can be found in Burnecki and Michna [6]. □

3 State space transforms and extremes

In this section we extend the maximum domain of attraction result for the FOUP to more general processes. We will use the notations of Remark 2.3 throughout; in particular, we set $\sigma = 1$.

In Buchmann and Klüppelberg [5] we denoted a function $f : \mathbb{R} \to \mathbb{R}$ a *state space transform* (SST), if f is continuous and strictly increasing. A SST f maps $\mathbb{R}$ to an open interval $I = (l, r) = f(\mathbb{R})$ which is called the *state space* of f. Defining $X_t^{H,\gamma,f} := f(O_t^{H,\gamma})$, $t \in \mathbb{R}$, this yields a rich class of stationary processes driven by FBM on arbitrary open intervals I.

The next theorem gives necessary and sufficient conditions on the SST f for $X^{H,\gamma,f} \in \mathrm{MDA}(\Lambda)$.

THEOREM 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a SST with $X_t^{H,\gamma,f} := f(O_t^{H,\gamma})$, $t \in \mathbb{R}$, as above.*
(a) Assume that

$$\lim_{y\to\infty} \frac{f(y + x/y) - f(y)}{f(y + 1/y) - f(y)} = x \quad \text{for all } x \in \mathbb{R}\,. \tag{3.1}$$

Then for $\delta^{H,\gamma}$ as in Remark 2.3 (b),

$$\frac{\delta^{H,\gamma}}{f(b_T^{H,\gamma} + 1/b_T^{H,\gamma}) - f(b_T^{H,\gamma})} \left\{ \max_{0\le t\le T} X_t^{H,\gamma,f} - f(b_T^{H,\gamma}) \right\} \quad \xrightarrow{d} \quad \Lambda\,.$$

(b) Assume there exist norming constants $\widetilde{a}_T > 0$ and $\widetilde{b}_T \in \mathbb{R}$ such that

$$\widetilde{a}_T^{-1} \left\{ \max_{0\le t\le T} X_t^{H,\gamma,f} - \widetilde{b}_T \right\} \quad \xrightarrow{d} \quad \Lambda\,,$$

then (3.1) holds and possible choices of the norming constants are

$$\widetilde{a}_T = \frac{1}{\delta^{H,\gamma}} \left(f\left(b_T^{H,\gamma} + 1/b_T^{H,\gamma}\right) - f(b_T^{H,\gamma}) \right)\,, \qquad \widetilde{b}_T = f(b_T^{H,\gamma})\,.$$

Proof. Let $M_T = M_T^{H,\gamma}$ and $\widetilde{M}_T = \max_{0\le t\le T} X_t^{H,\gamma,f}$. As f is increasing, $\widetilde{M}_T = f(M_T)$. We abbreviate $b_T = b_T^{H,\gamma}$ and $\delta = \delta^{H,\gamma}$ and recall that $b_T \to \infty$ for $T \to \infty$. Furthermore, observe that $T \mapsto b_T$ is strictly increasing for all sufficiently large T.

(a) For those T, $g_T : \mathbb{R} \to \mathbb{R}$ is well-defined, where

$$g_T(x) = \delta\, \frac{f\left(b_T + x/(\delta\, b_T)\right) - f(b_T)}{f\left(b_T + 1/b_T\right) - f(b_T)}\,.$$

Assumption (3.1) implies $\lim_{T\to\infty} g_T(x) = x$ for all $x \in \mathbb{R}$. Furthermore,

$$\begin{aligned} P\left(M_T \le b_T + \frac{x}{\delta\, b_T}\right) &= P\left(\frac{\delta}{f(b_T{+}1/b_T) - f(b_T)} \left(f(M_T) - f(b_T) \right) \le g_T(x)\right) \\ &= P\left(\frac{\delta}{f(b_T{+}1/b_T) - f(b_T)} \left(\widetilde{M}_T - f(b_T) \right) \le g_T(x)\right)\,. \end{aligned}$$

In particular, by Remark 2.3 (b), the left-hand side converges to $\Lambda(x)$ pointwisely. Thus, Lemma B.1 (a) of the Appendix applies.

(b) As above we write $P(M_T \le b_T + \frac{x}{\delta\, b_T}) = P(\widetilde{a}_T^{-1}\,(\widetilde{M}_T - \widetilde{b}_T) \le \widetilde{g}_T(x))$, where

$$\widetilde{g}_T(x) = \widetilde{a}_T^{-1}\,\left(f\,(b_T + x/(\delta\, b_T)) - \widetilde{b}_T \right). \tag{3.2}$$

By Lemma B.1 (b) we obtain $\widetilde{g}_T(x) \to x$ for all $x \in \mathbb{R}$. In particular,

$$\begin{aligned} \frac{f(b_T) - \widetilde{b}_T}{\widetilde{a}_T} &= \widetilde{g}_T(0) \to 0\,, \\ \frac{1}{\delta}\;\frac{f(b_T{+}1/b_T) - f(b_T)}{\widetilde{a}_T} &= \frac{1}{\delta}\;(\widetilde{g}_T(\delta) - g_T(0)) \;\to\; 1\,. \end{aligned}$$

By the convergence to types theorem we conclude that $(f(b_T{+}1/b_T){-}f(b_T))/\delta$ and $f(b_T)$ are a possible choice for $\widetilde{a}_T$ and $\widetilde{b}_T$, respectively. Plugging $\widetilde{a}_T = (f(b_T{+}1/b_T){-}f(b_T))/\delta$ and $\widetilde{b}_T = f(b_T)$ into (3.2) this yields

$$\left(f\left(b_T + \frac{x}{b_T}\right) - f(b_T)\right)\;\left(f\left(b_T{+}\,\frac{1}{b_T}\right){-}f(b_T)\right)^{-1} = \frac{1}{\delta}\,\widetilde{g}_T(\delta x)\,,$$

and the right-hand side converges to x for all $x \in \mathbb{R}$; thus, (3.1) holds. □

The following example illustrates condition (3.1).

EXAMPLE 3.2. Let $q \in (0,2]$. Let $f : \mathbb{R} \to \mathbb{R}$ be a SST given by $f(x) = \exp(x^q)$, $x > 0$.

(a) If $q \in (0,2)$ then for all $x \in \mathbb{R}$

$$\lim_{y\to\infty} \frac{f(y + x/y) - f(y)}{f(y+1/y) - f(y)} = \lim_{y\to\infty} \frac{\exp(\;y^q\;[\;(1+x/y^2)^q - 1\;]\;)\;-\;1}{\exp(\;y^q\;[\;(1+1/y^2)^q - 1\;]\;)\;-\;1} = x\,,$$

Therefore, $X^{H,\gamma,f} \in \mathrm{MDA}(\Lambda)$.

(b) If $q = 2$, then for all $x \in \mathbb{R}$

$$\lim_{y\to\infty} \frac{f(y + x/y) - f(y)}{f(y+1/y) - f(y)} = \lim_{y\to\infty} \frac{e^{2x+x^2/y^2} - 1}{e^{2+1/y^2} - 1} = \frac{e^{2x} - 1}{e^2 - 1}\,.$$

Thus, $X^{H,\gamma,f} \notin \mathrm{MDA}(\Lambda)$. More precisely, Theorem 3.6 below will show that $X^{H,\gamma,f} \in \mathrm{MDA}(\Phi_\alpha)$. □

Under the additional hypothesis of differentiability the next corollary provides an efficient method to calculate norming constants as is illustrated in Corollaries 3.4 and 3.5.

Corollary 3.3. *Let f be a SST, differentiable on (z_0,∞) with $f'(z)>0$ for all $z\in(z_0,\infty)$. Assume that*

$$\lim_{z\to\infty}\frac{f'(z+x/z)}{f'(z)}=1 \qquad \text{locally uniformly in } x\,. \tag{3.3}$$

Then

$$\frac{1}{a_T^{H,\gamma}\,f'(b_T^{H,\gamma})}\left\{\max_{0\le t\le T}X_t^{H,\gamma,f}-f(b_T^{H,\gamma})\right\} \qquad \xrightarrow{d} \qquad \Lambda\,. \tag{3.4}$$

Proof. Let $x\in\mathbb{R}$. For all $y>0$ sufficiently large we find $\theta_y\in[0,1]$ and $\widetilde{\theta}_y\in[0,1]$ such that

$$\frac{f(y+x/y)-f(y)}{f(y+1/y)-f(y)}=x\,\frac{f'(y+\theta_y x/y)}{f'(y)}\,\frac{f'(y)}{f'(y+\widetilde{\theta}_y/y)} \quad\to\quad x\,,\quad y\to\infty\,.$$

Therefore, (3.1) follows from (3.3); consequently, $X^{H,\gamma,f}\in\mathrm{MDA}(\Lambda)$.

Furthermore, for some $\bar\theta_T\in[0,1]$ and the quantity $\delta^{H,\gamma}$ in Remark 2.3 (b),

$$\frac{1}{\delta^{H,\gamma}}\,\frac{f(b_T^{H,\gamma}+1/b_T^{H,\gamma})-f(b_T^{H,\gamma})}{a_T^{H,\gamma}f'(b_T^{H,\gamma})}=\frac{1}{\delta^{H,\gamma}}\,\frac{1}{a_T^{H,\gamma}b_T^{H,\gamma}}\,\frac{f'(b_T^{H,\gamma}+\bar\theta_T/b_T^{H,\gamma})}{f'(b_T^{H,\gamma})} \quad\to\quad 1\,,$$

thus, (3.4) follows by the convergence to types theorem. □

Corollary 3.4. *Let ℓ be a slowly varying function on $[x_0,\infty)$ for some $x_0>0$; i.e., $\ell:[x_0,\infty)\to\mathbb{R}^+$ measurable and $\lim_{x\to\infty}\ell(tx)/\ell(x)=1$ for all $t>0$.*

If f is a SST with state space $I=(l,r)$, differentiable on (x_0,∞), such that for some $p\in\mathbb{R}$

$$f'(x)=x^p\,\ell(x) \qquad \text{for all } x>x_0\,,$$

then $X^{H,\gamma,f}\in\mathrm{MDA}(\Lambda)$.

Define

$$c_p^{H,\gamma}=2^{(p-2)/2}\,\gamma^{-H(p+1)}\,\Gamma(2H+1)^{(p+1)/2},\quad \widetilde{a}_T=c_p^{H,\gamma}\,(\log T)^{\frac12(p-1)}\,\ell((\log T)^{1/2})\,. \tag{3.5}$$

Then $\widetilde{a}_T$ and $\widetilde{b}_T=f(b_T^{H,\gamma})$ are a possible choice of normalizing constants.

Proof. By Theorem 1.5.2 of Bingham et al. [3] convergence in regular variation is locally uniform; thus, locally uniformly in x

$$\lim_{z\to\infty}\frac{f'(z+x/z)}{f'(z)}=\lim_{z\to\infty}\frac{\ell(z(1+x/z^2))}{\ell(z)}=1\,.$$

Consequently, $X^{H,\gamma,h}\in\mathrm{MDA}(\Lambda)$ by Corollary 3.3.

According to (3.4) we find $a_T^{H,\gamma}f'(b_T^{H,\gamma})\sim\widetilde{a}_T$ as given in (3.5); thus, $\widetilde{a}_T$ and $\widetilde{b}_T=f(b_T^{H,\gamma})$ are a possible choice of normalizing constants by the convergence to types theorem. □

Corollary 3.5. *Let ℓ be a slowly varying function on $[x_0,\infty)$ for some $x_0>0$.*

If f is a SST with state space $I=(l,r)$, differentiable on (x_0,∞), such that for some $p\in\mathbb{R}$, $q\in(0,2)$ and $\kappa\neq 0$

$$f'(x)=x^p\ \ell(x)\ \exp(\kappa x^q) \qquad \text{for all } x>x_0\,,$$

then $X^{H,\gamma,f}\in \mathrm{MDA}(\Lambda)$.

Let $c_p^{H,\gamma}$ be the quantity in (3.5) and define

$$\begin{aligned} \widetilde{c}_q^{H,\gamma} &= 2^{q/2}\,\gamma^{-qH}\,\Gamma(2H+1)^{q/2}\,,\\ \widetilde{a}_T &= c_p^{H,\gamma}\,(\log T)^{\frac{1}{2}(p-1)}\ \ell((\log T)^{1/2})\ \exp\left(\kappa\widetilde{c}_q^{H,\gamma}\,(\log T)^{q/2}\right). \end{aligned}$$

(a) If $\kappa>0$ then $r=\infty$ and $\widetilde{a}_T$ and $\widetilde{b}_T$ are a possible choice of the normalizing constants, where

$$\widetilde{b}_T=(q\kappa)^{-1}\ (b_T^{H,\gamma})^{p-q+1}\ \ell(b_T^{H,\gamma})\ \exp(\kappa(b_T^{H,\gamma})^q)\,.$$

(b) If $\kappa<0$ then $r<\infty$ and $\widetilde{a}_T$ and $\widetilde{b}_T$ are a possible choice of the normalizing constants, where

$$\widetilde{b}_T=r+(q\kappa)^{-1}\ (b_T^{H,\gamma})^{p-q+1}\ \ell(b_T^{H,\gamma})\ \exp(\kappa(b_T^{H,\gamma})^q)\,.$$

Proof. In view of Corollary 3.4, in order to prove (3.3), it suffices that for $0<q<2$, locally uniformly in x

$$(z+x/z)^q-z^q=q\ x\ z^{q-2}+o(z^{q-2})=o(1)\,,\quad z\to\infty\,.$$

Hence Corollary 3.3 applies and $X^{H,\gamma,f}\in\mathrm{MDA}(\Lambda)$. As $q<2$, observe that

$$a_T^{H,\gamma}\ f'(b_T^{H,\gamma})/\widetilde{a}_T\sim\exp\left(O\left((\log T)^{(q-2)/2}\log\log T\right)\right)\quad\to\quad 1\,.$$

If $\kappa>0$ then $f(x)\to\infty$ as $x\to\infty$, hence $r=\infty$. Without loss of generality suppose that $x_0>0$. For $x\geq x_0$ substitute $z=(\log\bar z)^{1/q}$; this yields

$$f(x)-f(x_0)=q^{-1}\int_{\exp(x_0^q)}^{\exp(x^q)}\ \bar z^{\kappa-1}\ \left(\log\bar z\right)^{(p-q+1)/q}\ \ell\left((\log\bar z)^{1/q}\right)\,d\bar z\,.$$

Karamata's theorem applies to $\kappa>0$ and for $\eta:=\exp(x^q)\to\infty$,

$$q^{-1}\int_{\eta_0}^{\eta}\bar z^{\kappa-1}\ \left(\log\bar z\right)^{(p-q+1)/q}\ \ell\left((\log\bar z)^{1/q}\right)\ d\bar z\sim\ (q\kappa)^{-1}\ \eta^\kappa\ \left(\log\eta\right)^{(p-q+1)/q}\ \ell\left((\log\eta)^{1/q}\right).$$

Thus, for $x\to\infty$,

$$f(x)-f(x_0)\sim\psi(x)\,,\quad \psi(x)=(q\kappa)^{-1}\ \ell(x)\ x^{p-q+1}\ \exp(\kappa x^q)\,.$$

Note that $\widetilde{a}_T \to \infty$ and $\widetilde{a}_T^{-1}\psi(b_T^{H,\gamma}) = O((\log T)^{(2-q)/2})$; thus,

$$\lim_{T\to\infty} \widetilde{a}_T^{-1}\left(f(b_T^{H,\gamma}) - \psi(b_T^{H,\gamma})\right) = 0\,.$$

An application of the convergence to types theorem implies (a).
The proof of (b) is similar. □

Now we want to derive an analogon of Theorem 3.1 for the domain of attraction of the Fréchet distribution. To this end we use the fact that by a logarithmic transformation

$$\widehat{a}_T^{-1}(H,\gamma) \max_{0\le t\le T} X_t^{H,\gamma,f} \quad \xrightarrow{d} \quad \Phi_\alpha\,,$$

for some $\alpha > 0$ if and only if

$$\alpha\left\{\max_{0\le t\le T} \log X_t^{H,\gamma,f} - \log \widehat{a}_T\right\} \quad \xrightarrow{d} \quad \Lambda\,.$$

Using this result, we can translate Theorem 3.1.

Theorem 3.6. *Let $f:\mathbb{R}\to\mathbb{R}$ be a SST.*

(a) Assume that there exist $\kappa > 0$ and $z_0 \in \mathbb{R}$ such that for all $z \ge z_0$ both $f(z) > 0$ and

$$\log f(z) = \frac{1}{2}\,\kappa z^2 + h(z)\,, \tag{3.6}$$

where $h:\mathbb{R}\to\mathbb{R}$ satisfies

$$\lim_{z\to\infty} h(z + x/z) - h(z) = 0 \quad \textit{for all } x \in \mathbb{R}\,. \tag{3.7}$$

Then for $\alpha = \delta^{H,\gamma}/\kappa$ ($\delta^{H,\gamma}$ as in Remark 2.3 (b)),

$$\frac{1}{f(b_T^{H,\gamma})} \max_{0\le t\le T} X_t^{H,\gamma,f} \qquad \xrightarrow{d} \qquad \Phi_\alpha\,.$$

(b) Assume there exist norming constants $\widehat{a}_T > 0$ such that

$$\frac{1}{\widehat{a}_T} \max_{0\le t\le T} X_t^{H,\gamma,f} \quad \xrightarrow{d} \quad \Phi_\alpha\,,$$

then a possible choice of normalizing constants are $\widehat{a}_T = f(b_T^{H,\gamma})$.
Furthermore, there exist $h:\mathbb{R}\to\mathbb{R}$ satisfying (3.7) and $z_0 \in \mathbb{R}$ such that both $f(z) > 0$ and $\log f(z) = \frac{1}{2}\kappa z^2 + h(z)$ hold for all $z \ge z_0$, where $\kappa = \delta^{H,\gamma}/\alpha$.

Proof. (a) Let $\widetilde{M}_T = \max_{0\le t\le T} X_t^{H,\gamma,f}$ and $M_T = M_T^{H,\gamma}$. We abbreviate $b_T = b_T^{H,\gamma}$ and $\delta = \delta^{H,\gamma}$. Set $x = \alpha \log y$ for $y > 0$. Observe that,

$$\begin{aligned}
\Phi_\alpha(y) &= \Lambda(x) = \lim_{T\to\infty} P\left(M_T \le b_T + x/(\delta\, b_T)\right) \\
&= \lim_{T\to\infty} P\left(\frac{1}{f(b_T)}\,\widetilde{M}_T \le \frac{f\left(b_T + x/(\delta\, b_T)\right)}{f(b_T)}\right) \\
&= \lim_{T\to\infty} P\left(\frac{1}{f(b_T)}\,\widetilde{M}_T \le y\,\theta_T(\alpha\log y)\right), \qquad (3.8)
\end{aligned}$$

where we have set

$$\log\theta_T(x) = \frac{\kappa}{2}\left(\frac{x}{\delta\, b_T}\right)^2 + h\left(b_T + \frac{x}{\delta\, b_T}\right) - h(b_T)\,.$$

Assumption (3.7) implies $y\,\theta_T(\alpha\log y) \to y$ for all $y > 0$. Thus, Lemma B.1 (a) applies to the limit in (3.8) and $g_T : \mathbb{R}^+ \to \mathbb{R}$, $g_T(y) := y\,\theta_T(\alpha\log y)$.

(b) Let $y > 0$ and $x = \alpha\log y$. Replacing $f(b_T)$ by $\widehat{a}_T$ in the proof of (a) we obtain

$$\Phi_\alpha(y) = \Lambda(x) = \lim_{T\to\infty} P\left(\frac{1}{\widehat{a}_T}\widetilde{M}_T \le \widetilde{g}_T(y)\right),$$

where $\widetilde{g}_T : \mathbb{R}^+ \to \mathbb{R}$ is defined by

$$\widetilde{g}_T(y) = \frac{1}{\widehat{a}_T}\, f\left(b_T + \frac{\alpha}{\delta}\,\frac{\log y}{b_T}\right). \qquad (3.9)$$

Lemma B.1 (b) applies to g_T, i.e., $g_T(y) \to y$ for all $y \in \mathbb{R}^+$. Specializing to $y = 1$, this yields $f(b_T) \sim \widehat{a}_T$; thus, $f(b_T)$ is a possible choice for $\widehat{a}_T$ by the convergence to types theorem.

Plugging $\widehat{a}_T = f(b_T)$ and $\kappa = \delta/\alpha$ into (3.9), this yields for $T \to \infty$ and $y \in \mathbb{R}^+$

$$\frac{1}{f(b_T)}\, f\left(b_T + \frac{1}{\kappa}\frac{\log y}{b_T}\right) \;\to\; y\,.$$

Equivalently, for $x \in \mathbb{R}$,

$$\lim_{z\to\infty} \frac{f(z + x/z)}{f(z)} \;=\; \exp(\kappa x)\,.$$

As $f(b_T) \sim \widehat{a}_T$ where $\widehat{a}_T > 0$ there exists z_0 such that $f(z) > 0$ for all $z \ge z_0$. Set $h(z) = \log f(z) - \frac{1}{2}\kappa z^2$ for $z \ge z_0$ and $h(z) = 1$ for $z < z_0$. Observe

$$h(z + x/z) - h(z) = \log\frac{f(z + x/z)}{f(z)} - \kappa x - \frac{1}{2}\frac{x^2}{z^2}\,, \qquad z,\; z + x/z \ge z_0\,.$$

Thus, h is a function satisfying (3.7). □

REMARK 3.7. (a) Boundedness of h in (3.6) does not imply (3.7). To see this, let $h(x) = \sin(x^2)$, then

$$h(z + x/z) - h(x) = 2\cos(z^2 + x + x^2/(2z^2))\ \sin(x + x^2/(2z^2))\,.$$

Thus, for all $x \in \mathbb{R}\backslash(\pi\mathbb{Z})$ the limit in (3.7) for $z \to \infty$ does not exist.

(b) Observe that Theorem 3.6 covers Example 3.2 (b) with $\kappa = 2$ and $h \equiv 0$ in (3.6). Furthermore, suppose that h satisfies (3.7) and, in addition, $h(z) \to 0$ for $z \to \infty$. Then the scaling constants $\widehat{a}_T$ depend on κ and $b_T^{H,\gamma}$ only; i.e., we may choose

$$\widehat{a}_T = \exp\left(\frac{\kappa}{2}\,(b_T^{H,\gamma})^2\right) \quad \sim \quad f(b_T^{H,\gamma})\,.$$

(c) In general, knowledge of κ alone is not sufficent to calculate the scaling constants $\widehat{a}_T$. Therefore, observe that (3.7) holds for $h(x) = \kappa_p x^p$, $x > 0$, even when $p \in [0, 2)$ and $\kappa_p \neq 0$. But, for any choice of $\widehat{a}_T$ we must have that $\widehat{a}_T \sim \exp\big(\frac{1}{2}\kappa(b_T^{H,\gamma})^2 + \kappa_p(b_T^{H,\gamma})^p\big)$. As $b_T^{H,\gamma} \to \infty$ the scaling constants $\widehat{a}_T$ clearly depends on both κ_p and p. □

The following corollary complements Corollary 3.3.

COROLLARY 3.8. *Let f be a SST, differentiable on (z_0, ∞) for some $z_0 \in \mathbb{R}$. Assume that $f(z) > 0$ for all $z > z_0$ and*

$$\lim_{z\to\infty} \frac{(\log f)'(z + x/z)}{z} \quad = \quad \kappa \in (0, \infty)\,, \quad \textit{locally uniformly in } x\,.$$

Then for $\alpha = \delta^{H,\gamma}/\kappa$,

$$\frac{1}{f(b_T)} \max_{0\le t\le T} X_t^{H,\gamma,f} \quad \xrightarrow{d} \quad \Phi_\alpha\,.$$

Proof. Set $h(z) = \log f(z) - \frac{1}{2}\kappa z^2$. Then h is absolutely continuous on $[x_0, \infty)$ and we obtain

$$h(z + x/z) - h(z) = x \int_0^1 \frac{(\log f)'(z + \alpha x/z)}{z}\, d\alpha - \kappa x - \frac{1}{2}\,\kappa x^2/z^2\,,$$

the right-hand side tends to zero for $z \to \infty$ by dominated convergence. Theorem 3.6 applies. □

For completeness we state the analogous results for the Weibull distribution.

THEOREM 3.9. *Let $f : \mathbb{R} \to \mathbb{R}$ be a SST with state space $I = (l, r) \subseteq \mathbb{R}$, where $r < \infty$.*

(a) Suppose that there exists $\kappa > 0$ and $z_0 \in \mathbb{R}$ such that for all $z \geq z_0$ both $r - f(z) > 0$ and

$$\log(r - f(z)) = -\frac{1}{2}\,\kappa z^2 + h(z)\,,$$

where $h:\mathbb{R}\to\mathbb{R}$ satisfies (3.7). Then, for $\alpha=\delta^{H,\gamma}/\kappa$,

$$\frac{1}{r-f(b_T^{H,\gamma})}\left(\max_{0\le t\le T} X_t^{H,\gamma,f}-r\right) \quad \overset{d}{\to} \quad \Psi_\alpha\,.$$

(b) Assume there exist norming constants $\bar{a}_T>0$ such that

$$\frac{1}{\bar{a}_T}\left(\max_{0\le t\le T} X_t^{H,\gamma,f}-r\right) \quad \overset{d}{\to} \quad \Psi_\alpha\,,$$

then possible choices of norming constants are $\bar{a}_T=r-f(b_T^{H,\gamma})$.
Furthermore, there exist $h:\mathbb{R}\to\mathbb{R}$ satisfying (3.7) and $z_0\in\mathbb{R}$ such that both $r-f(z)>0$ and $\log(r-f(z))=-\frac{1}{2}\kappa z^2+h(z)$ hold for all $z\ge z_0$, where $\kappa=\delta^{H,\gamma}/\alpha$.

Proof. Set $x=-\alpha\log|z|$ for $z<0$ and $\alpha=\delta^{H,\gamma}/\kappa$. Observe that $\Psi_\alpha(z)=\Lambda(x)$. The result follows along the lines of the proof of Theorem 3.6. □

We collect results analogous to Remark 3.7 and Corollary 3.8.

Remark 3.10. (a) If h satisfies (3.7) and in addition $h(z)\to 0$ for $z\to\infty$ then we may choose

$$\bar{a}_T=\exp\left(-\frac{1}{2}\,\kappa(b_T^{H,\gamma})^2\right) \quad \sim \quad r-f(b_T^{H,\gamma})\,.$$

(b) For $p\in[0,2)$ and $\kappa_p\neq 0$ and $h(x)=\kappa_p x^p$, $x>0$, we obtain

$$\bar{a}_T\sim\exp\left(-\frac{1}{2}\,\kappa(b_T^{H,\gamma})^2+\kappa_p(b_T^{H,\gamma})^p\right).$$

□

Corollary 3.11. *Let f be a SST with state space $I=(l,r)\subseteq\mathbb{R}$, where $r<\infty$.*
Let f be differentiable on (z_0,∞) for some $z_0\in\mathbb{R}$.
Assume that $f(z)>0$ for all $z>z_0$ and locally uniformly in x

$$\lim_{z\to\infty}\frac{(\log(r-f))'(z+x/z)}{z} \quad = \quad -\kappa\ \in(-\infty,0)\,,$$

then for $\alpha=\delta^{H,\gamma}/\kappa$,

$$\frac{1}{r-f(b_T^{H,\gamma})}\left(\max_{0\le t\le T} X_t^{H,\gamma,f}-r\right) \quad \overset{d}{\to} \quad \Psi_\alpha\,.$$

Remark 3.12. We have here only considered SSTs of the FOUP. But, of course, SSTs are more generally applicable to any stationary Gaussian process. □

4 MDAs of solutions to fractional integral equations

In this section we return to the MDA problem for a family of processes defined as solutions to SDEs (1.2). Therefore let $I = (l, r) \subseteq \mathbb{R}$ be an open non-empty interval and $\mu, \sigma : I \to \mathbb{R}$ be some continuous functions, where σ is non-negative.

In [5] conditions on μ and σ were obtained such that a stationary solution X for (1.2) exists and is of the form $X = X^{H,\gamma,f}$ for some $\gamma > 0$ and a SST f. Those conditions were summarized into the concept of *H-proper triples* (I, μ, σ) (see Definition 3.4 in [5]). For such triples the ratio μ/σ possesses a unique absolutely continuous extension $\psi : I \to \mathbb{R}$ which determines the SST f and the so-called *friction coefficient* (FC) γ by the relations

$$\gamma \;=\; -\sigma\psi' \quad \text{Lebesgue-a.e. on } I\,, \qquad f^{-1} \;=\; -\frac{\psi}{\gamma}\,. \tag{4.1}$$

The number $\xi := f(0)$ is called the *center* of the H-proper triple (I, μ, σ).

For the reader's convenience, we recall that for H-proper triples the function $z \mapsto 1/\sigma(z)$ is necessarily locally integrable and the following formula holds for the inverse function $f^{-1} : I \to \mathbb{R}$:

$$f^{-1}(z) \;=\; \int_\xi^z \frac{dw}{\sigma(w)}\,, \quad z \in I\,. \tag{4.2}$$

We start with a simple example.

Example 4.1. [Fractional Vasicek model]
For σ_0, $\gamma > 0$ let $\mu(x) = -\gamma(x - \xi)$ and $\sigma(x) \equiv \sigma_0$, $x \in \mathbb{R}$. Define a SST $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \xi + \sigma_0 x$. The triple $(\mathbb{R}, \mu, \sigma)$ is H-proper for all $H \in (0, 1)$ with FC γ, $SST\ f$ and center ξ. For this choice of μ and σ observe that $X = X^{H,\gamma,f}$ is a solution of (1.2) and therefore serves as a natural extension of the usual Vasicek model driven by ordinary Brownian motion to the fractional world. It is a mean reverting stationary Gaussian process. Theorem 2.2 implies $X \in \text{MDA}(\Lambda)$, more precisely,

$$(\sigma_0 a_T^{H,\gamma})^{-1}\left\{\max_{0\le t\le T} X_t \;-(\xi +\; \sigma_0 b_T^{H,\gamma})\right\} \quad \xrightarrow{d} \quad \Lambda\,. \qquad \square$$

Although Example 4.1 shows that H-proper triples may exist for certain models for all $H \in (0, 1)$, they indeed only exist for the full range for Vasicek models (see Remark 3.3 (vii) in [5]). When considering more general models we restrict ourselves to a choice of $H \in (1/2, 1)$, which is uncritical for most models.

Formulas (4.1) and (4.2) provide us with two different representations for f^{-1}, the first is based on the ratio μ/σ and the second on the integral representation (4.2). Using the results of Section 3 and asymptotic inversion rules yield different characterizations of the MDA.

We start with the MDA of the Gumbel distribution. The proof of Theorem 4.2 is found in Appendix C.1. The equivalence of the conditions (i), (ii) and (iii) is a direct consequence of the formulas (4.1) and (4.2).

THEOREM 4.2. *Let $H \in (1/2, 1)$. Suppose (I, μ, σ) to be H-proper with FC γ, SST f and center ξ. Let ψ be the absolutely continuous extension of μ/σ to I.*

The following assertions are equivalent.

- *(a)* $X^{H,\gamma,f} \in \mathrm{MDA}(\Lambda)$.
- *(b) There exist $z_0 \in I$ and $g : (z_0, r) \to \mathbb{R}^+$ such that $\forall x \in \mathbb{R}\ \exists z_1 \in (z_0, r)$ satisfying $z + xg(z) \in I$ for all $z \in (z_1, r)$ and one of the following equivalent conditions holds for all $x \in \mathbb{R}$.*
  - *(i)* $\lim\limits_{z\uparrow r} f^{-1}(z) \left[f^{-1}\big(z+xg(z)\big) - f^{-1}(z)\right] = x.$
  - *(ii)* $\lim\limits_{z\uparrow r} \gamma^{-2}\, \psi(z)\, \big(\psi(z+xg(z)) - \psi(z)\big) = x.$
  - *(iii)* $\lim\limits_{z\uparrow r} \displaystyle\int_\xi^z \frac{dw}{\sigma(w)} \int_z^{z+xg(z)} \frac{dw}{\sigma(w)} = x.$

Concerning $r = \infty$ the proof of the following corollary illustrates a possible construction of g as in Theorem 4.2(b). Analogous results hold for $r < \infty$.

COROLLARY 4.3. *Let $H \in (1/2, 1)$. Suppose (I, μ, σ) to be H-proper with FC γ, SST f and center ξ. Suppose $r = \infty$ and there exists $z_0 \in I$ such that $\ell : (z_0, \infty) \to \mathbb{R}^+$ is a slowly varying function.*

- *(a) If there exists $p < 1$ such that $\sigma(z) = z^p \ell(z)$ for all $z > z_0 > \max\{0, \xi\}$, then* $X^{H,\gamma,f} \in \mathrm{MDA}(\Lambda)$.
- *(b) If there exists $q < 1/2$ such that $\sigma(z) = z\,(\log z)^q \ell(\log z)$ for all $z > z_0 > \max\{1, e^\xi\}$, then* $X^{H,\gamma,f} \in \mathrm{MDA}(\Lambda)$.

Proof. In both cases we check condition (iii) of Theorem 4.2 (b). (a) Define $g : (z_0, \infty) \to \mathbb{R}^+$ by $g(z) = \sigma(z)/\int_\xi^z \frac{dw}{\sigma(w)}$. Karamata's theorem implies

$$\lim_{z\to\infty} \frac{g(z)}{z} = (1-p) \lim_{z\to\infty} \ell^2(z) z^{2p-2} = 0\,.$$

Thus, for all $x \in \mathbb{R}$, we find $z_1 > z_0$ such that $z + g(z)x \subseteq (z_0, \infty)$ for all $z > z_1$. In particular, as ℓ is strictly positive and $\sigma : I \to \mathbb{R}$ is continuous, also $1/\sigma$ is continuous on

(z_1,∞). Consequently, for $z > z_1$ the mean value theorem provides a $\theta(z) \in [0,1]$ such that

$$\int_z^{z+xg(z)} \frac{dw}{\sigma(w)} = \frac{xg(z)}{\sigma(z+\theta(z)xg(z))}\,.$$

On the other hand, by definition,

$$\int_\xi^z \frac{dw}{\sigma(w)} \int_z^{z+xg(z)} \frac{dw}{\sigma(w)} = \frac{x\sigma(z)}{\sigma\big(z\,(1+\theta(z)xg(z)z^{-1})\big)}\,.$$

The right-hand side tends to x for $z \to \infty$ as $g(z)/z \to 0$ and convergence in regular variation is locally uniformly on $(0,\infty)$ (c.f. Theorem 1.5.2 of Bingham et al. [3]).

(b) Define $g : (z_0,\infty) \to \mathbb{R}^+$ by $g(z) = \sigma(z)/\int_\xi^z \sigma^{-1}(w)\,dw$ as in (a). Substituting $y = \log w$ yields $\int_{z_0}^z \sigma^{-1}(w)\,dw = \int_{\log z_0}^{\log z} 1/(y^q\ell(y)(\,dy$; Karamata's theorem implies

$$\lim_{z\to\infty} \frac{g(z)}{z} = (1-q)\ \lim_{z\to\infty} \ell(\log z)^2\,(\log z)^{2q-1} = 0\,.$$

Thus, for all $x \in \mathbb{R}$, we find $z_1 > z_0$ such that $z + g(z)x \subseteq (z_0,\infty)$ for all $z > z_1$. The remaining part of the proof follows the same lines as in (a). □

Theorem 3.6 yields a characterization of $\mathrm{MDA}(\Phi_\alpha)$ in the following theorem; see Appendix C.2 for a proof. The equivalence of (i), (ii) and (iii) is a direct consequence of (4.1) and (4.2).

THEOREM 4.4. *Let $H \in (1/2,1)$ and (I,μ,σ) be H-proper with FC γ, SST f and center ξ. Let ψ be the absolutely continuous extension of μ/σ to I.*

The following assertions are equivalent.

(a) There exists an $\alpha > 0$ such that $X \in \mathrm{MDA}(\Phi_\alpha)$.

(b) $r = \infty$ and there exist $\kappa > 0$ and $\widetilde{h} : (\max\{1,l\},\infty) \to \mathbb{R}$ such that

$$\lim_{z\to\infty} (\log z)^{1/2}\,\big(\widetilde{h}(xz) - \widetilde{h}(z)\big) = 0 \quad \text{for all } x > 0\,, \tag{4.3}$$

and one of the following equivalent representions holds for all $z > \max\{1,l\}$.

(i) $f^{-1}(z) = (2/\kappa)^{1/2}\,(\log z)^{1/2} + \widetilde{h}(z)$,

(ii) $\psi(z) = -\gamma\big((2/\kappa)^{1/2}\,(\log z)^{1/2} + \widetilde{h}(z)\big)$,

(iii) $\displaystyle\int_\xi^z \frac{dw}{\sigma(w)} = (2/\kappa)^{1/2}\,(\log z)^{1/2} + \widetilde{h}(z)$.

If one of the conditions (a) or (b) is satisfied, then $\alpha = \delta^{H,\gamma}/\kappa$, where $\delta^{H,\gamma}$ is the quantity in Remark 2.3 (b).

As an application of Corollary 4.3 and Theorem 4.4 we present a family of models, which belong to MDA(Λ) or MDA(Φ_α), depending on the choice of parameters.

EXAMPLE 4.5. Let $H \in (1/2, 1)$, $q \in ((1-H), 1)$, $\sigma_0 > 0$, $a < 0$ and $b \geq 0$.

Calculations similar to those of Section 5 in [5] show that (I, μ, σ) is H-proper, where

$$I = \mathbb{R}^+ , \qquad \mu(z) = az \log z + bz |\log z|^q , \qquad \sigma(z) = \sigma_0 \, z \, |\log z|^q .$$

Furthermore, formula (4.1) shows that $\gamma = (1-q)|a|$. We obtain two cases.
For $q = \frac{1}{2}$ we observe

$$\psi(z) = \frac{a}{\sigma_0} (\log z)^{1/2} + \frac{b}{\sigma_0} , \quad z > 1 .$$

Set $\kappa = \frac{1}{2}\sigma_0^2$ and $\widetilde{h}(z) \equiv b/\sigma_0$. Theorem 4.4 (b) applies to ψ; thus, $X^{H,\gamma,f} \in$ MDA(Φ_α) for $\alpha = 2\delta^{H,\gamma}/\sigma_0^2$.
For $q < \frac{1}{2}$ Corollary 4.3 (b) implies $X^{H,\gamma,f} \in$ MDA(Λ).
As the SST f can be explicitly calculated as

$$f(z) = \exp\Big(\mathrm{sign}(\sigma_0 (1-q)\, z - b/a) \, \big|\sigma_0 (1-q)\, z - b/a\big|^{1/(1-q)} \Big) , \quad z \in \mathbb{R} ,$$

we could also have argued with the theory given Section 3. In this case, Example 3.2 shows that $X^{H,\gamma,f} \notin$ MDA(G) for any extreme value distribution G and any $q \in (1/2, 1)$.
□

For completeness, we conclude the section with the corresponding results for MDA(Ψ_α). The following theorem is based on Theorem 3.9; its proof can be found in Appendix C.3.

THEOREM 4.6. *Let $H \in (1/2, 1)$ and (I, μ, σ) be H-proper with FC γ, SST f and center ξ. Let ψ be the absolutely continuous extension of μ/σ to I.*

The following assertions are equivalent.

(a) There exists an $\alpha > 0$ such that $X \in$ MDA(Ψ_α).

(b) $r < \infty$ and there exist $\kappa > 0$ and $\bar{h} : (0, r-l) \to \mathbb{R}$ such that

$$\lim_{z \downarrow 0} |\log z|^{1/2} \left(\bar{h}(xz) - \bar{h}(z) \right) = 0 \quad \text{for all } x > 0 , \tag{4.4}$$

and one of the following equivalent representions holds for all $0 < z < \min\{1, r-l\}$.

(i) $f^{-1}(r - z) = (2/\kappa)^{1/2} \, |\log z|^{1/2} + \bar{h}(z)$

(ii) $\psi(r-z) = -\gamma \, (2/\kappa)^{1/2} \, |\log z|^{1/2} + \bar{h}(z)$

(iii) $\displaystyle\int_{\xi}^{r-z} \frac{dw}{\sigma(w)} = (2/\kappa)^{1/2}\,|\log z|^{1/2} + \bar{h}(z)$

If one of the conditions (a) or (b) is satisfied, then $\alpha = \delta^{H,\gamma}/\kappa$ *where* $\delta^{H,\gamma}$ *is the quantity in Remark 2.3 (b).*

Appendix

A Proof of Lemma 2.1

It remains to show (b) and (d).

(b) By selfsimilarity of FBM we obtain for $h \in \mathbb{R}$

$$\begin{aligned}\rho_{H,\gamma,\sigma}(h) &= \sigma^2\, E\int_{-\infty}^{0} e^{\gamma s} dB_s^H \int_{-\infty}^{h} e^{-\gamma(h-s)} dB_s^H \\ &= \sigma^2\, e^{-\gamma h}\, E\int_{-\infty}^{0} e^{s} dB_{s/\gamma}^H \int_{-\infty}^{\gamma h} e^{s} dB_{s/\gamma}^H = \frac{\sigma^2}{\gamma^{2H}}\, \rho_H(\gamma h).\end{aligned}$$

(d) The closed formula stated for $H = 1/2$ is well-known (e.g. Cheridito et al. [7]). Thus, by (a) and (b), it suffices to investigate the case $\gamma = \sigma = 1$, $H \neq 1/2$ and $h \downarrow 0$.

By partial integration applied to (1.3), we observe

$$\int_{-\infty}^{t} e^{-(t-s)}\, dB_s^H = B_t^H - \int_{-\infty}^{t} e^{-(t-s)} B_s^H\, ds\,, \quad t \in \mathbb{R}\,, \tag{A.1}$$

where the integral on the right-hand side is interpreted as Lebesgue integral (c.f. Buchmann and Klüppelberg [5], Proposition 2.3).

By formula (A.1) and Fubini's theorem we have

$$\begin{aligned}\rho_H(0) &= \int_{-\infty}^{0}\int_{-\infty}^{0} e^{s_1+s_2} E(B_{s_1}^H B_{s_2}^H)\, ds_1\, ds_2 \\ &= \frac{1}{2}\int_{-\infty}^{0}\int_{-\infty}^{0} e^{s_1+s_2}\Big\{|s_1|^{2H} + |s_2|^{2H} - |s_1 - s_2|^{2H}\Big\}\, ds_1\, ds_2 \\ &= \Gamma(2H+1) - \frac{1}{2}\int_{0}^{\infty}\int_{0}^{\infty} e^{-(s_1+s_2)}|s_1 - s_2|^{2H}\, ds_1 ds_2.\end{aligned}$$

The second integral can be interpreted as multiple of the expectation $E|S_1 - S_2|^{2H}$ where S_1 and S_2 are independent standard exponential random variables. As $S_1 - S_2$ is a two-sided exponential random variable we obtain

$$\int_{0}^{\infty}\int_{0}^{\infty} e^{-(s_1+s_2)}|s_1 - s_2|^{2H}\, ds_1 ds_2 = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|s|}\, |s|^{2H}\, ds = \Gamma(2H+1).$$

Hence

$$\rho_H(0) = \frac{\Gamma(2H+1)}{2}. \tag{A.2}$$

Now let $h \geq 0$. Set

$$\begin{aligned}
\phi_H(h) &= \Gamma(2H+1) - \int_0^\infty e^{-s}(h+s)^{2H}\, ds \\
\psi_H(h) &= \frac{1}{2}\left\{\Gamma(2H+1)\int_0^h e^s\, ds + \int_0^h s^{2H} e^s\, ds - \int_0^h e^{s_1}\int_0^\infty e^{-s_2}(s_1+s_2)^{2H}\, ds_2\, ds_1\right\}.
\end{aligned}$$

By Fubini's theorem and (1.1),

$$EB_h^H \int_{-\infty}^0 e^s B_s^H\, ds \;=\; \frac{1}{2}\int_{-\infty}^0 e^s\left(h^{2H} - s^{2H} - (h-s)^{2H}\right)\, ds \;=\; \frac{1}{2}\left(h^{2H} + \phi_H(h)\right)\,,$$

and, similarly, $\psi_H(h) = E\int_{-\infty}^0 e^s B_s^H\, ds \int_0^h e^s B_s^H\, ds$.

By formula (A.1),

$$\begin{aligned}
&\rho_H(h) - \rho_H(0) \\
&\quad= -E\int_{-\infty}^0 e^s B_s^H\, ds\left(B_h^H - e^{-h}\int_{-\infty}^h e^s B_s^H\, ds + \int_{-\infty}^0 e^s B_s^H\, ds\right) \\
&\quad= -EB_h^H \int_{-\infty}^0 e^s B_s^H\, ds + (e^{-h}-1)E\int_{-\infty}^h e^s B_s^H\, ds \int_{-\infty}^0 e^s B_s^H\, ds \\
&\qquad + E\int_{-\infty}^0 e^s B_s^H\, ds \int_0^h e^s B_s^H\, ds \\
&\quad= -\frac{1}{2}\left(h^{2H} + \phi_H(h)\right) + (e^{-h}-1)\left(\frac{\Gamma(2H+1)}{2} + \psi_H(h)\right) + \psi_H(h)\,.
\end{aligned} \tag{A.3}$$

For $H < 1/2$ we can differentiate both ϕ_H and $\psi_H(h)$ under the integral sign by dominated convergence. We obtain

$$\begin{aligned}
\phi_H(h) &= \phi_H(0) + \phi_H'(0+)h + o(h) \;=\; -\Gamma(2H+1)\, h + o(h)\,, \\
\psi_H(h) &= \psi_H(0) + \psi_H'(0+)\, h + o(h) \;=\; o(h)\,.
\end{aligned}$$

Equation (A.3) yields

$$\rho_H(h) - \rho_H(0) \;=\; -\frac{1}{2}h^{2H} + \frac{\Gamma(2H+1)}{2}\, h - \frac{\Gamma(2H+1)}{2}\, h + o(h) \;=\; -\frac{1}{2}h^{2H} + o(h).$$

By (A.2) and (b) we find

$$\rho_{H,\gamma,\sigma}(h) \;=\; \frac{\sigma^2}{\gamma^{2H}}\left(\rho_H(0) + \rho_H(\gamma h) - \rho_H(0)\right) \;=\; \frac{\Gamma(2H+1)}{2}\,\frac{\sigma^2}{\gamma^{2H}} - \frac{1}{2}\,\sigma^2\, h^{2H} + o(h).$$

For $H > 1/2$, both ϕ_H and ψ_H are twice differentiable under the integral sign, i.e.,

$$\phi_H(h) \;=\; -\Gamma(2H+1)\, h - \Gamma(2H+1)\,\frac{h^2}{2} + o(h^2)\,, \quad \psi_H(h) \;=\; -\frac{\Gamma(2H+1)}{4}\, h^2 + o(h^2)\,,$$

thus, as $(e^{-h}-1) = -h + \frac{1}{2}h^2 + o(h^2)$, (A.3) implies

$$\rho_H(h) - \rho_H(0) = -\frac{1}{2}h^{2H} + \frac{\Gamma(2H+1)}{4}\, h^2 + o(h^2).$$

With the same arguments as above,

$$\rho_{H,\gamma,\sigma}(h) = \frac{\Gamma(2H+1)}{2}\,\frac{\sigma^2}{\gamma^{2H}} - \frac{1}{2}\,\sigma^2\, h^{2H} + \frac{\Gamma(2H+1)}{4}\,\frac{\sigma^2}{\gamma^{2H-2}}\, h^2 + o(h^2). \qquad \square$$

B A general convergence to types lemma

In this section a result is stated which forms the core of section 3. For a probability distribution function $F:\mathbb{R}\to[0,1]$

$$D_<(F) = \{x\in\mathbb{R}:\ \forall\epsilon>0\ F(x-\epsilon) < F(x) < F(x+\epsilon)\}\,.$$

Set $x_L := -\infty$ if $F(x)>0$ for all $x\in\mathbb{R}$; otherwise, set $x_L = \sup\{x\in\mathbb{R}:\ F(x)=0\}$. Set $x_R := \infty$ if $F(x)<1$ for all $x\in\mathbb{R}$; otherwise, set $x_R = \inf\{x\in\mathbb{R}:\ F(x)=1\}$.

Lemma B.1. *Let $F, F_n:\mathbb{R}\to[0,1]$, $n\in\mathbb{N}$, be probability distribution functions on $\mathbb{R}$, where F is continuous.*

(a) Let $M=(x_L,x_R)$ and $g_n: M\to\mathbb{R}$, $n\in\mathbb{N}$. Let $G_n = F_n\circ g_n: M\to[0,1]$.
If $\lim_{n\to\infty} g_n(x) = x$ and $\lim_{n\to\infty} G_n(x) = F(x)$ for all $x\in M$ then $\lim_{n\to\infty} F_n(x) = F(x)$ for all $x\in\mathbb{R}$.

(b) Let $M = D_<(F)$, $g_n: M\to\mathbb{R}$, $n\in\mathbb{N}$. Let $G_n = F_n\circ g_n: M\to[0,1]$.
If $\lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} G_n(x) = F(x)$ for all $x\in M$ then $g_n(x)\to x$ for all $x\in M$.

Proof. (a) It suffices to show $\lim_{n\to\infty} F_n(x) = F(x)$ for all $x\in M$. Contradicting the hypothesis, suppose that there exist $x_0\in M$ and $y_0\in[0,1]$ and, as $F_n(x)$ is bounded, a subseqence n' such that

$$\lim_{n'\to\infty} F_{n'}(x_0) = y_0 \neq F(x_0)\,. \tag{B.1}$$

Without loss of generality suppose that $n=n'$. By Helly's selection theorem we find a subsequence n' and a non-decreasing right-continuous function $\widetilde{F}:\mathbb{R}\to[0,1]$ such that $\lim_{n'\to\infty} F_{n'}(x) = \widetilde{F}(x)$ for all continuity points x of $\widetilde{F}$. Let $C(\widetilde{F})$ be the set of continuity points of $\widetilde{F}$ and let $x\in C(\widetilde{F})\cap(x_L,x_R)$. Then for all $x'\in(x,x_R)\cap C(\widetilde{F})$

$$F(x) = \lim_{n'\to\infty} G_{n'}(x) = \lim_{n'\to\infty} F_{n'}(g_{n'}(x)) \le \lim_{n'\to\infty} F_{n'}(x') = \widetilde{F}(x')$$

and hence $F(x)\le \lim_{x'\downarrow x,\, x'\in C(\widetilde{F})} \widetilde{F}(x') = \widetilde{F}(x)$. Analogously, for all $x'\in(x_L,x)\cap C(\widetilde{F})$

$$\widetilde{F}(x') = \lim_{n'\to\infty} F_{n'}(x') \le \lim_{n'\to\infty} F_{n'}(g_{n'}(x)) = \lim_{n'\to\infty} G_{n'}(x) = F(x),$$

Hence $\widetilde{F}(x) = \lim_{x'\uparrow x,\, x'\in C(\widetilde{F})} \widetilde{F}(x') \le F(x)$. Thus, $\widetilde{F}(x) = F(x)$ for all $x \in (x_L, x_R) \cap C(\widetilde{F})$. As $C(\widetilde{F})$ is dense in (x_L, x_R) and F is continuous we find $x_0 \in (x_L, x_R) \subseteq C(\widetilde{F})$; contradicting (B.1).

(b) Suppose the contrary is true. Then there exists $x \in D_<(F)$ and a subsequence n' such that $g_{n'}(x) \to y \in \overline{\mathbb{R}}$ where $y \neq x$. Without loss of generality suppose that $y \in [-\infty, x)$. As F is continuous, uniform convergence of $F_n \to F$ holds. Set $F(y) = 0$ whenever $y = -\infty$. Then

$$F(y) = \lim_{n'\to\infty} F_{n'}(g_{n'}(x)) = \lim_{n'\to\infty} G_{n'}(x) = F(x)\,,$$

contradicting $x \in D_<(F)$. □

C Results on asymptotic inversion

C.1 Proof of Theorem 4.2

Theorem 4.2 is a consequence of Theorem 3.1 and the following lemma.

Lemma C.1. *Let f be a SST with state space $I = (l, r)$. The following assertions are equivalent.*

(a) f satisfies (3.1).

(b) There exist $z_0 \in I$ and $g : (z_0, r) \to \mathbb{R}^+$ satisfying the following properties.

(i) For all $x \in \mathbb{R}$ there exists $z_1 \in (z_0, r)$ with $z + xg(z) \in I$ for all $z \in (z_1, r)$.

(ii) $\lim_{z\uparrow r} f^{-1}(z)\,\big[f^{-1}\big(z+xg(z)\big) - f^{-1}(z)\big] = x$ for all $x \in \mathbb{R}$.

Proof. (a) ⇒ (b) f has representation $f(z) = v \circ h(z)$ for all $z > 0$, where $h(z) = \exp(z^2/2)$. Combining Exercises 0.4.3.7 and 0.4.3.8 in Resnick [13], we find a function $a : (1, \infty) \to \mathbb{R}^+$ such that $\lim_{z\to\infty}[v(zx) - v(z)]/a(z) = \log x$ for all $x > 0$. As both f and h are strictly increasing, also v is; moreover, $\lim_{z\uparrow r} v^{-1}(z) = \infty$. Let $z_0 = f(0)$. By Proposition 0.9 (b) in [13], we find $g : (z_0, r) \to \mathbb{R}^+$ satisfying (i) such that $\lim_{z\uparrow r}[v^{-1}(z + xg(z))]/v^{-1}(z) = e^x$ for all $x \in \mathbb{R}$. Property (ii) follows from the following calculation: for $x \in \mathbb{R}$,

$$\begin{aligned}
&\lim_{z\uparrow r} f^{-1}(z)\,\big[f^{-1}\big(z+xg(z)\big) - f^{-1}(z)\big] \\
&\quad= \lim_{z\uparrow r} 2\left[\log v^{-1}(z)\right]\left[\left(1 + \frac{\log[v^{-1}(z + xg(z))/v^{-1}(z)]}{\log v^{-1}(z)}\right)^{1/2} - 1\right].
\end{aligned}$$

Now a Taylor expansion of $\sqrt{1+z}$ yields assertion (b)(ii).

(b) $\Rightarrow$ (a) Observe that for all $x \in \mathbb{R}$,

$$\lim_{z\uparrow r} [f^{-1}(z)]^2 \left[\frac{f^{-1}\big(z+xg(z)\big)}{f^{-1}(z)} - 1\right] = x\,.$$

Consequently, $\lim_{z\uparrow r} f^{-1}\big(z+xg(z)\big)/f^{-1}(z) = 1$ as $\lim_{z\uparrow r} f^{-1}(z) = \infty$. Now define $u(z) = \exp[(f^{-1}(z))^2/2]$, $z \in I$. Then u is strictly increasing on $(f(0), r)$ and a mapping from $(f(0), r)$ onto $(u(f(0)), \infty)$. For all $x \in \mathbb{R}$, we get

$$\begin{aligned} &\lim_{z\uparrow r} \frac{u(z+g(z)x)}{u(z)} \\ &= \lim_{z\uparrow r} \exp\left[\frac{1}{2}\ \frac{f^{-1}\big(z+xg(z)\big)+f^{-1}(z)}{f^{-1}(z)}\ f^{-1}(z)\ \left[f^{-1}\big(z+xg(z)\big)-f^{-1}(z)\right]\right] = e^x\,. \end{aligned}$$

Proposition 0.9 (a) in [13] applies to u. There exist $z_1 > u(f(0))$ and a function $a : (z_1, \infty) \to \mathbb{R}^+$ such that $\lim_{z\to\infty}(u^{-1}(zx) - u^{-1}(z))/a(z) = \log x$ for all $x > 0$. By monotonicity, the convergence holds locally uniformly in x on $\mathbb{R}^+$. In particular, for all $x \in \mathbb{R}$

$$\begin{aligned} &\lim_{z\to\infty} \frac{f(z+xz^{-1})-f(z)}{a\big(\exp[z^2/2]\big)} \\ &= \lim_{z\to\infty} \frac{u^{-1}\big(\exp[z^2/2]\exp[x+x^2/(2z^2)]\big)-u^{-1}(\exp[z^2/2])}{a\big(\exp[z^2/2]\big)} = x\,. \end{aligned}$$

Therefore, for all $x \in \mathbb{R}$

$$\lim_{z\to\infty} \frac{f(z+xz^{-1})-f(z)}{f(z+z^{-1})-f(z)} = \lim_{z\to\infty} \frac{f(z+xz^{-1})-f(z)}{a\big(\exp[z^2/2]\big)}\ \frac{a\big(\exp[z^2/2]\big)}{f(z+z^{-1})-f(z)} = x\,. \qquad \square$$

C.2 Proof of Theorem 4.4

We prepare the result with a technical lemma.

Lemma C.2. *If $h : (x_0, \infty) \to \mathbb{R}$ with $\lim_{z\to\infty} z^\alpha \left[h(z+xz^{-\beta})-h(z)\right] = 0$ locally uniformly in $x \in \mathbb{R}$ for some $x_0 \in \mathbb{R}$, $\alpha \in [0,1)$ and $\beta \geq 0$, then $\lim_{z\to\infty} z^{\alpha-1-\beta}h(z) = 0$.*

Proof. We use the convention $\sum_k^l = 0$ for $l < k$. Let $\epsilon > 0$ and define a sequence (z_n) as follows. Choose $z_0 > \max\{1, x_0\}$ such that for all $z \geq z_0$ and $x \in [0,1]$

$$|h(z+xz^{-\beta})-h(z)| < \epsilon\ (z+z^{-\beta})^{-\alpha}\,.$$

For $n \geq 1$ set $z_n = z_{n-1} + z_{n-1}^{-\beta}$. Observe that $z_n = z_0 + \sum_{l=0}^{n-1} z_l^{-\beta} \geq n z_n^{-\beta}$. Thus, $z_n \geq n^{1/(1+\beta)}$. In particular, $z_n \to \infty$ and $|h(z_{n+1})-h(z_n)| < \epsilon\ (n+1)^{-\alpha/(1+\beta)}$ for all $n \geq 0$.

Let $z \geq z_0$ arbitrary. Set $n_1 = \max\{n : z_n \leq z\}$, clearly, $n_1 \leq z^{1+\beta}$. By choice of n_1, $z \in [z_{n_1}, z_{n_1} + z_{n_1}^{-\beta})$ and hence $|h(z) - h(z_{n_1})| < \epsilon$. Finally, summing and subtracting terms

$$|h(z)| \leq \epsilon + |h(z_0)| + \epsilon \sum_{k=0}^{n_1-1} (1+k)^{-\alpha/(1+\beta)} \leq \epsilon + |h(z_0)| + \epsilon \left[1 + \frac{1+\beta}{1-\alpha+\beta} z^{1-\alpha+\beta}\right].$$

Thus, $\limsup_{z\to\infty} z^{\alpha-1-\beta}|h(z)| \leq \epsilon$. □

Corollary C.3. *(a) If $x_0 \in \mathbb{R}$ and $h : (x_0, \infty) \to \mathbb{R}$ with $\lim_{z\to\infty} \left[h(z+xz^{-1}) - h(z)\right] = 0$ locally uniformly in $x \in \mathbb{R}$ then $\lim_{z\to\infty} z^{-2}h(z) = 0$.*

(b) If $x_0 \in \mathbb{R}$ and $h : (x_0, \infty) \to \mathbb{R}$ with $\lim_{z\to\infty} (\log z)^{1/2}\left[h(zx) - h(z)\right] = 0$ locally uniformly in $x \in \mathbb{R}^+$ then $\lim_{z\to\infty} (\log z)^{-1/2}h(z) = 0$.

Proof. For the choice of $\beta = 1$ and $\alpha = 0$ Lemma C.2 implies (a). To show (b) set $g = h \circ \exp$. Then $\lim_{z\to\infty} z^{1/2}(g(z+x) - g(z)) = 0$ locally uniformly in $x \in \mathbb{R}$; Lemma C.2 yields $\lim_{z\to\infty} z^{-1/2}g(z) = 0$; equivalently, $\lim_{z\to\infty}(\log z)^{-1/2}h(z) = 0$. □

Proof of Theorem 4.4 (a) ⇒ (b). Observe that for all $x \in \mathbb{R}$

$$\lim_{z\to\infty} \frac{f(z+x/z)}{f(z)} = \exp(\kappa x). \tag{C.1}$$

This convergence strengthens to locally uniform convergence by Proposition 3.10.2 in Bingham et al. [3]. By Theorem 3.6 $X^{H,\gamma,f} \in \mathrm{MDA}(\Phi_\alpha)$ for $\alpha > 0$ is equivalent to the existence of $z_0 \in \mathbb{R}$ and $\kappa > 0$, $h : (z_0, \infty) \to \mathbb{R}$ satisfying (3.7) such that both $f(z) > 0$ and $\log f(z) = \frac{\kappa}{2}z^2 + h(z)$ for all $z > z_0$ holds. Consequently, $h(z+x/z) - h(z) \to 0$ as $z \to \infty$ locally uniformly in $x \in \mathbb{R}$; thus, $z^{-2}h(z) \to 0$ by Corollary C.3 (a). In particular, $f^{-1}(z) \sim (2/\kappa)^{1/2}(\log z)^{1/2}$ for $z \to \infty$.

By Theorem 3.10.4 in [3], (C.1) implies $\lim_{z\to\infty} f^{-1}(z)\left[f^{-1}(zx) - f^{-1}(z)\right] = \kappa^{-1} \log x$ for all $x > 0$. Equivalently, $\lim_{z\to\infty}(\log z)^{1/2}\left[f^{-1}(zx) - f^{-1}(z)\right] \to (2\kappa)^{-1/2}\log x$ for all $x > 0$. Finally, set $\widetilde{h}(z) = f^{-1}(z) - (2/\kappa)^{1/2}(\log z)^{1/2}$ for $z > \max\{1, l\}$. Then $\widetilde{h}$ is a function satisfying (4.3).

(b) ⇒ (a). Observe that $\lim_{z\to\infty}(\log z)^{1/2}\left(f^{-1}(xz) - f^{-1}(z)\right) = (2\kappa)^{-1/2}\log x$ for all $x > 0$. Now let $x > 0$ and $x(z) \to x$ for $z \to \infty$. By monotonicity, for all $0 < \epsilon < x$,

$$\begin{aligned}(2\kappa)^{-1/2}\log(x-\epsilon) &\leq \liminf_{z\to\infty}(\log z)^{1/2}\left(f^{-1}(xz) - f^{-1}(z)\right)\\ &\leq \limsup_{z\to\infty}(\log z)^{1/2}\left(f^{-1}(xz) - f^{-1}(z)\right) \leq (2\kappa)^{-1/2}\log(x+\epsilon).\end{aligned}$$

Consequently, $\lim_{z\to\infty}(\log z)^{1/2}\left(f^{-1}(xz) - f^{-1}(z)\right) = (2\kappa)^{-1/2}\log x$ holds locally uniformly in $x > 0$. This implies $\lim_{z\to\infty}(\log z)^{1/2}\left(\widetilde{h}(xz) - \widetilde{h}(z)\right) = 0$ uniformly in $x > 0$. Corollary C.3 (b) implies $f^{-1}(z) \sim (2/\kappa)^{1/2}(\log z)^{1/2}$; thus, for all $x > 0$,

$$\lim_{z\to\infty} f^{-1}(z)\left[f^{-1}(xz) - f^{-1}(z)\right] = \kappa^{-1}\log x. \tag{C.2}$$

By Theorem 3.10.4 in [3], (C.2) implies $\lim_{z\to\infty} f(z+x/z)/f(z) = \exp(\kappa z)$. Set $h(z) = \log f(z) - \frac{\kappa}{2}z^2$ for $z \in \mathbb{R}$ with $f(z) > 0$; then h extends to a function satisfying (3.7). $\square$

C.3 Proof of Theorem 4.6

Proof of Theorem 4.6 To show the equivalence of (a) and (b) set $\widetilde{f}(z) = 1/(r - f(z))$. Then $\widetilde{f}$ is a SST with state space $J = ((r-l)^{-1}, \infty)$. By Theorem 3.9 $X^{H,\gamma,f} \in \mathrm{MDA}(\Psi_\alpha)$ for some $\alpha > 0$ is equivalent to the existence of $z_0 > \max\{1, l\}$ and $\kappa > 0$, $h : (z_0, r) \to \mathbb{R}$ satisfying (3.6) such that both $\widetilde{f}(z) > 0$ and $\log \widetilde{f}(z) = \frac{\kappa}{2}z^2 + h(z)$ for all $z > z_0$. As in the proof of Theorem 4.4, this holds if and only if there exists $\widetilde{h}$ satisfying (4.3) such that $\widetilde{f}^{-1}(z) = (2/\kappa)^{1/2}(\log z)^{1/2} + \widetilde{h}(z)$. As $\widetilde{f}^{-1}(1/z) = f^{-1}(r-z)$, $0 < z < r-l$ this is equivalent to (b) where $\bar{h} = \widetilde{h}(1/z)$ satisfies (4.4). $\square$

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