Game contingent claims in complete and incomplete markets *

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Abstract

A game contingent claim is a contract which enables both buyer and seller to terminate it before maturity. For *complete* markets Kifer [8] shows a connection to a (zero-sum) Dynkin game whose value is the unique no-arbitrage price of the claim. But, for incomplete markets one needs a more general approach. We interpret the contract as a generalized non-zero-sum stopping game. For the complete case this leads to the same results as in Kifer [8]. For the general case we show the existence of an equilibrium point under the condition that both the seller and the buyer have an exponential utility function. For other utility functions such a point need not exist in the context of incomplete markets.

Keywords: Game contingent claims, incomplete markets, hedging, optimal stopping, nonzero-sum games, equilibrium points, exponential utility.

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1 Introduction

A game contingent claim (GCC) is a contract between a seller A and a buyer B which enables A to terminate it and B to exercise it at any time $t \in \{t_0, \ldots, t_k\}$ up to a maturity date $T = t_k$ when the contract is terminated anyway.

More precisely, let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in[0,T]})$ be a filtered probability space satisfying the usual conditions of right-continuity and completeness, and let $(X_{t_i})_{i=0,\ldots,k}$, $(Y_{t_i})_{i=0,\ldots,k}$, $(W_{t_i})_{i=0,\ldots,k}$ be sequences of real-valued random variables adapted to $(\mathcal{F}_{t_i})_{i=0,\ldots,k}$ with $Y_{t_i} \le W_{t_i} \le X_{t_i}$ for $i = 0, \ldots, k-1$ and $Y_{t_k} = W_{t_k} = X_{t_k}$. If A terminates the contract at time t_i before B exercises then A should pay B the amount X_{t_i}. The other way around, A should pay B only Y_{t_i}. If A terminates and B exercises at the same time, then A pays B the amount W_{t_i}.

Definition 1.1. *Let $\mathcal{S}_i$, $i = 0, \ldots, k$, be the sets of all stopping times resp. $(\mathcal{F}_t)_{t\in[0,T]}$ with values in $\{t_i, \ldots, t_k\}$.*

The above contract can be formulated as follows. If A selects a cancellation time $\sigma \in \mathcal{S}_0$ and B selects an exercise time $\tau \in \mathcal{S}_0$, then A pledges to pay B at time $\sigma \wedge \tau$ the amount

$$R(\sigma, \tau) = X_\sigma I(\sigma < \tau) + Y_\tau I(\tau < \sigma) + W_\tau I(\tau = \sigma).$$

The frictionless financial market consists of d risky assets whose discounted price processes are modeled by the $\mathbb{R}^d$-valued semimartingale S and one riskless asset with discounted price process equal to 1. We denote by Θ a suitable space of admissible trading strategies to be specified later.

Example 1.2 (Israeli call option). *An American style call option with strike price K where also the seller can terminate the contract, but at the expense of a penalty $\delta_{t_i} \ge 0$, i.e. $Y_{t_i} = (S^{(1)}_{t_i} - K)^+$, $X_{t_i} = (S^{(1)}_{t_i} - K)^+ + \delta_{t_i}$, and $W_{t_i} = (S^{(1)}_{t_i} - K)^+ + \delta_{t_i}/2$.*

Such a game version of an American option is safer for an investment company which issues it, and so it can be sold cheaper than the corresponding American option. As pointed out in Kifer [8], essentially any contract in modern life presumes explicitly or implicitly a cancellation option by each side which then has to pay some penalty, and so it is natural to introduce a buyback option to contingent claims, as well. An example which has already been traded on real markets is a Liquid Yield Option Note (LYON). It is discussed in McConnell and Schwarz [10] - on a rather heuristical level without indicating a connection to a Dynkin game.

In a complete market (i.e. Y, W, X are replicable by trading in S) one can solve our problem without letting enter the agents' preferences: A just wants to minimize $E_Q\left(R(\sigma,\tau)\right)$ whereas B wants to maximize the same expression (Q is the unique equivalent martingale measure). Thus, we have a zero-sum Dynkin stopping game. It is well-known that such a game has a *unique* value, cf. Ohtsubo [12]. Kifer [8] shows by hedging-arguments that this value is also the *unique* no-arbitrage price of the GCC. In other words, the expectation of the (discounted) payoff under the unique equivalent martingale measure is the variable to be maximized resp. minimized, and this ensures consistency with the principle of no-arbitrage. Consequently, one has to solve a classical Dynkin game.

In incomplete markets this argument fails because there is more than one equivalent martingale measure. It is possible to superhedge the claim and get an interval of no-arbitrage prices, but then the feature of a stochastic game gets lost.

We suggest a utility maximization approach that takes trading possibilities **explicitly** into consideration. This approach is very popular for valuating European style contingent claims in the context of incomplete markets; see e.g. Hodges and Neuberger [5], Delbaen et al. [3], or Davis [1]. For American style contingent claims see Davis and Zariphopoulou [2].

Let $U_1, U_2 : \mathbb{R} \longrightarrow \mathbb{R}$ be nondecreasing and concave; they are the utility functions of the seller resp. the buyer. Each "player" chooses a stopping time $\sigma \in \mathcal{S}_0$ (resp. $\tau \in \mathcal{S}_0$) and a trading strategy $\vartheta \in \Theta$, whose i-th component ϑ_t^i, $i = 1, \ldots, d$, represents the number of shares of asset i held in the portfolio at time $t \in [0,T]$. The seller wants to maximize

$$E_P\left(U_1\left(C_1 - R(\sigma,\tau) + \int_0^T \vartheta_t \, dS_t\right)\right), \tag{1.1}$$

while the buyer wants to maximize

$$E_P\left(U_2\left(C_2 + R(\sigma,\tau) + \int_0^T \vartheta_t \, dS_t\right)\right). \tag{1.2}$$

So, the agents are solely interested in *terminal* wealth. The rv $C_i \in \mathcal{F}_T$ $(i = 1, 2)$ is the exogenous endowment of the i-th player. This randomness especially makes sense for the buyer, who perhaps buys the claim to hedge against another risk in his portfolio.

In the whole paper, the space Θ of admissible trading strategies has to satisfy the

Assumption 1.3. *All elements of Θ are $(\mathcal{F}_t)$-predictable and S-integrable. Θ is linear, and for all $t_i \in \{t_1, \ldots, t_{k-1}\}$, $A \in \mathcal{F}_{t_i}$ the following implication is valid:*

If $\vartheta^{(1)}, \vartheta^{(2)}, \vartheta^{(3)} \in \Theta$, then the compound strategy

$$\vartheta_t := \begin{cases} \vartheta_t^{(1)} & : \quad t \leq t_i, \\ \vartheta_t^{(2)} & : \quad t > t_i \quad \textit{and} \quad \omega \in A, \\ \vartheta_t^{(3)} & : \quad t > t_i \quad \textit{and} \quad \omega \notin A. \end{cases} \tag{1.3}$$

is also an element of Θ.

The latter is essential as it allows a successive optimization, first over all stategies $(\vartheta_t)_{t\in(t_i,T]}$ (fixing one strategy $(\vartheta_t)_{t\in(0,t_i]}$), and then over all $(\vartheta_t)_{t\in(0,t_i]}$. So, it is a quite natural assumption. But unfortunately, it is not as harmless as it looks like. For example, the set of all predictable trading strategies such that the discounted gain process $\int_0^t \vartheta_u \, dS_u$ is bounded from below (but not necessarily from above) does obviously not satisfy Assumption 1.3.

A permissible choice of Θ is for example

$$\Theta_1 = \left\{ \vartheta \in L(S) \,\middle|\, \int_0^t \vartheta_u \, dS_u \text{ is bounded uniformly in } t \text{ and } \omega \right\}. \tag{1.4}$$

Θ_1 is rather small, but in Delbaen et al. [3] resp. Kabanov and Stricker [7] it is shown for exponential utility that under the assumption that S is locally bounded and admits an equivalent local martingale measure with finite entropy the maximization problems (1.1) and (1.2) with $\Theta = \Theta_1$ have the same values as for much bigger Θ. Another permissible choice is

$$\Theta_2 = \Big\{ \vartheta \in L(S) \mid \int_0^t \vartheta_u \, dS_u \text{ is a martingale w.r.t. a special set } \mathcal{P} \text{ of absolutely continuous local martingale measures} \Big\}.$$

Remark 1.4. Analogously to Kühn [9], one can define from the seller's point of view a "still fair premium" for the GCC which coincides with the unique no-arbitrage price if the market is complete. But the main aim of this paper is not to determine a "premium" or "price" for the claim, but rather to describe the "game", defined above, that takes place after the premium has been paid till maturity - and compare the situations of complete and incomplete markets.

Definition 1.5. *We say that a pair $(\sigma^*, \tau^*) \in \mathcal{S}_0 \times \mathcal{S}_0$ is a Nash (or a non-cooperative) equilibrium point, if for all $(\sigma, \tau) \in \mathcal{S}_0 \times \mathcal{S}_0$*

$$\sup_{\vartheta\in\Theta} E_P\left(U_1\left(C_1 - R(\sigma^*, \tau^*) + \int_0^T \vartheta_t\, dS_t\right)\right) \geq \sup_{\vartheta\in\Theta} E_P\left(U_1\left(C_1 - R(\sigma, \tau^*) + \int_0^T \vartheta_t\, dS_t\right)\right),$$

and

$$\sup_{\vartheta\in\Theta} E_P\left(U_2\left(C_2 + R(\sigma^*, \tau^*) + \int_0^T \vartheta_t\, dS_t\right)\right) \geq \sup_{\vartheta\in\Theta} E_P\left(U_2\left(C_2 + R(\sigma^*, \tau) + \int_0^T \vartheta_t\, dS_t\right)\right).$$

Remark 1.6. To simplify the notation and to stress the point that the interdependence between the agents' decisions only takes place through the stopping times and not through the trading strategies, we have not explicitly taken the chosen trading strategies into the definition of a Nash equilibrium. But of course, the outcome would be the same.

Without a financial market, i.e. $\Theta = \{0\}$, we have a nonzero-sum extension of a Dynkin game. This has been thoroughly investigated by many authors, firstly and independently of each other by Ohtsubo [13] and Morimoto [11] for a discrete time space. Their results can be directly transfered to our model (1.1)/(1.2), when $\Theta = \{0\}$, and ensure the existence of equilibrium points. Nevertheless, the existence of a financial market makes things more complicated.

2 The case of exponential utility

In this section, we assume that both seller and buyer have an *exponential utility function*, i.e.

$$U_1(x) = 1 - e^{-\alpha_1 x}, \tag{2.1}$$

$$U_2(x) = 1 - e^{-\alpha_2 x}, \tag{2.2}$$

for some *risk aversion parameters* $\alpha_1, \alpha_2 > 0$. Now, we define stopping times $(\sigma_0, \tau_0) \in \mathcal{S}_0 \times \mathcal{S}_0$ that will turn out to be equilibrium points.

Define, for $0 = t_0 < t_1 < \cdots < t_k = T$ recursively (in reverse order of time):

$$\sigma_k = t_k, \quad \tau_k = t_k, \tag{2.3}$$

$$\sigma_{i-1} := \begin{cases} t_{i-1} & : \quad \omega \in A_{i-1}, \\ \sigma_i & : \quad \text{otherwise}, \end{cases} \tag{2.4}$$

$$\tau_{i-1} := \begin{cases} t_{i-1} & : \quad \omega \in B_{i-1}, \\ \tau_i & : \quad \text{otherwise}, \end{cases} \tag{2.5}$$

where A_{i-1} and B_{i-1} have to satisfy

$$\begin{aligned} A_{i-1} = \Big\{ & e^{\alpha_1 X_{t_{i-1}}} \operatorname*{ess\,inf}_{\vartheta \in \Theta} E_P \left(e^{-\alpha_1 \left(C_1 + \int_{(t_{i-1},T]} \vartheta_t \, dS_t \right)} \big| \mathcal{F}_{t_{i-1}} \right) \\ & \leq \operatorname*{ess\,inf}_{\vartheta \in \Theta} E_P \left(e^{-\alpha_1 \left(C_1 - R(\sigma_i, \tau_i) + \int_{(t_{i-1},T]} \vartheta_t \, dS_t \right)} \big| \mathcal{F}_{t_{i-1}} \right) \Big\} \setminus B_{i-1}, \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} B_{i-1} = \Big\{ & e^{-\alpha_2 Y_{t_{i-1}}} \operatorname*{ess\,inf}_{\vartheta \in \Theta} E_P \left(e^{-\alpha_2 \left(C_2 + \int_{(t_{i-1},T]} \vartheta_t \, dS_t \right)} \big| \mathcal{F}_{t_{i-1}} \right) \\ & \leq \operatorname*{ess\,inf}_{\vartheta \in \Theta} E_P \left(e^{-\alpha_2 \left(C_2 + R(\sigma_i, \tau_i) + \int_{(t_{i-1},T]} \vartheta_t \, dS_t \right)} \big| \mathcal{F}_{t_{i-1}} \right) \Big\} \setminus A_{i-1}. \end{aligned} \tag{2.7}$$

Remark 2.1. We have $A_{i-1} \cap B_{i-1} = \emptyset$ (i.e. the players never stop at the same time) and the system (2.6)/(2.7) has at least one solution.

Remark 2.2. Due to $Y_{t_{i-1}} \leq X_{t_{i-1}}$, for the seller it would be better that the buyer would stop the game as if he did it himself (and vice versa). This tends to result in a negative attitude towards stopping.

Theorem 2.3. *Let U_1, U_2 be the exponential utility functions (2.1) resp. (2.2), $Y_{t_i}, X_{t_i} \in L^\infty(\Omega, \mathcal{F}, P)$, $i = 0, \ldots, k$, and*

$$E_P \left(U_1 \left(C_1 + \int_0^T \vartheta_t^{(1)} \, dS_t \right) \right) > -\infty, \tag{2.8}$$

resp.

$$E_P\left(U_2\left(C_2+\int_0^T \vartheta_t^{(2)}\,dS_t\right)\right) > -\infty, \tag{2.9}$$

for some strategies $\vartheta^{(1)}, \vartheta^{(2)} \in \Theta$. *Then, each pair* $(\sigma_0, \tau_0) \in \mathcal{S}_0 \times \mathcal{S}_0$ *satisfying (2.3)-(2.7) is a Nash equilibrium in the sense of Definition 1.5.*

Proof. Let $(\sigma_i)_{i=0,\dots,k}$ and $(\tau_i)_{i=0,\dots,k}$ satisfy (2.3)-(2.7). To proof the optimality of σ_0 (for τ_0 the argumentation is analogous and therefore omitted) it is sufficient to show that for all $i = 0, \dots, k$ and $\sigma \in \mathcal{S}_i$ P-a.s.

$$\operatorname*{ess\,inf}_{\vartheta\in\Theta} E_P\left(e^{-\alpha_1\left(C_1-R(\sigma_i,\tau_i)+\int_{(t_i,T]}\vartheta_t\,dS_t\right)}\,|\mathcal{F}_{t_i}\right) \le \operatorname*{ess\,inf}_{\vartheta\in\Theta} E_P\left(e^{-\alpha_1\left(C_1-R(\sigma,\tau_i)+\int_{(t_i,T]}\vartheta_t\,dS_t\right)}\,|\mathcal{F}_{t_i}\right).$$

This is done by backward induction: for $i = k$ we have $\sigma = t_k = \sigma_k$. $i \rightsquigarrow i-1$: for all $A \in \mathcal{F}_{t_{i-1}}$ we have by definition of τ_{i-1} and σ_{i-1}

$$\begin{aligned}
&\int_A \operatorname*{ess\,inf}_{\vartheta\in\Theta} E_P\left(e^{-\alpha_1\left(C_1-R(\sigma_{i-1},\tau_{i-1})+\int_{(t_{i-1},T]}\vartheta_t\,dS_t\right)}\,\big|\mathcal{F}_{t_{i-1}}\right)\,dP \\
&= \int_{A\cap\{\tau_{i-1}=t_{i-1}\}} \operatorname*{ess\,inf}_{\vartheta\in\Theta} E_P\left(e^{-\alpha_1\left(C_1-Y_{i-1}+\int_{(t_{i-1},T]}\vartheta_t\,dS_t\right)}\,\big|\mathcal{F}_{t_{i-1}}\right)\,dP \\
&+ \int_{A\cap\{\tau_{i-1}>t_{i-1}\}} \min\left\{\operatorname*{ess\,inf}_{\vartheta\in\Theta} E_P\left(e^{-\alpha_1\left(C_1-X_{t_{i-1}}+\int_{(t_{i-1},T]}\vartheta_t\,dS_t\right)}\,\big|\mathcal{F}_{t_{i-1}}\right),\right. \\
&\left.\operatorname*{ess\,inf}_{\vartheta\in\Theta} E_P\left(e^{-\alpha_1\left(C_1-R(\sigma_i,\tau_i)+\int_{(t_{i-1},T]}\vartheta_t\,dS_t\right)}\,\big|\mathcal{F}_{t_{i-1}}\right)\right\}\,dP \\
&\le \int_{A\cap\{\tau_{i-1}=t_{i-1}\}} \operatorname*{ess\,inf}_{\vartheta\in\Theta} E_P\left(e^{-\alpha_1\left(C_1-Y_{t_{i-1}}+\int_{(t_{i-1},T]}\vartheta_t\,dS_t\right)}\,\big|\mathcal{F}_{t_{i-1}}\right)\,dP \qquad (2.10)\\
&+ \int_{A\cap\{\tau_{i-1}>t_{i-1}\}\cap\{\sigma=t_{i-1}\}} \operatorname*{ess\,inf}_{\vartheta\in\Theta} E_P\left(e^{-\alpha_1\left(C_1-X_{i-1}+\int_{(t_{i-1},T]}\vartheta_t\,dS_t\right)}\,\big|\mathcal{F}_{t_{i-1}}\right)\,dP \\
&+ \int_{A\cap\{\tau_{i-1}>t_{i-1}\}\cap\{\sigma>t_{i-1}\}} \operatorname*{ess\,inf}_{\vartheta\in\Theta} E_P\left(e^{-\alpha_1\left(C_1-R(\sigma_i,\tau_i)+\int_{(t_{i-1},T]}\vartheta_t\,dS_t\right)}\,\big|\mathcal{F}_{t_{i-1}}\right)\,dP.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&\operatorname*{ess\,inf}_{\vartheta\in\Theta} E_P\left(e^{-\alpha_1\left(C_1-R(\sigma_i,\tau_i)+\int_{(t_{i-1},T]}\vartheta_t\,dS_t\right)}\,\big|\mathcal{F}_{t_{i-1}}\right) \qquad (2.11)\\
&= \operatorname*{ess\,inf}_{\vartheta\in\Theta'} E_P\left[e^{-\alpha_1\int_{(t_{i-1},t_i]}\vartheta_t\,dS_t}\operatorname*{ess\,inf}_{\tilde\vartheta\in\Theta} E_P\left(e^{-\alpha_1\left(C_1-R(\sigma_i,\tau_i)+\int_{(t_i,T]}\tilde\vartheta_t\,dS_t\right)}\,|\mathcal{F}_{t_i}\right)\,\big|\mathcal{F}_{t_{i-1}}\right]
\end{aligned}$$

P-a.s., where

$$\Theta' = \left\{ \vartheta \in \Theta \,\middle|\, E_P\left(e^{-\alpha_1\left(C_1 + \int_{(t_{i-1},T]} \vartheta_t\, dS_t\right)} \,\middle|\, \mathcal{F}_{t_{i-1}} \right) < \infty \quad P\text{-a.s.} \right\}.$$

The restriction to Θ' ensures dominated convergence and is possible due to (2.8) (notice that $R(\sigma_i, \tau_i)$ is bounded), cf. the proof of Theorem A.2. We can now apply the induction assumption for $\sigma' = \sigma \vee t_i \in \mathcal{S}_i$ to the last expression in (2.10). Then, we again make use of (2.11) for σ' instead of σ_i. Finally, we obtain as $Y_{t_{i-1}} \le W_{t_{i-1}}$ that

$$\int_A \operatorname*{ess\,inf}_{\vartheta \in \Theta} E_P\left(e^{-\alpha_1\left(C_1 - R(\sigma_{i-1},\tau_{i-1}) + \int_{(t_{i-1},T]} \vartheta_t\, dS_t\right)} \,\middle|\, \mathcal{F}_{t_{i-1}} \right) dP$$
$$\le \int_A \operatorname*{ess\,inf}_{\vartheta \in \Theta} E_P\left(e^{-\alpha_1\left(C_1 - R(\sigma,\tau_{i-1}) + \int_{(t_{i-1},T]} \vartheta_t\, dS_t\right)} \,\middle|\, \mathcal{F}_{t_{i-1}} \right) dP.$$

□

Remark 2.4. We want to construct an example for which no Nash equilibrium exists. We take *logarithmic* utility functions, i.e. $U_i = \log$ (i=1,2), and a discrete two-period binomial model. There are a riskless bond with value identical to 1, a tradeable risky asset with $S_0 = 1$ and

$$S_2 = S_1 = \begin{cases} 3 & : \text{ with probability } 1/2 \\ 0 & : \text{ with probability } 1/2 \end{cases}$$

(so trading in the second period can be ignored and the trading strategy consists of the number $\vartheta \in \mathbb{R}$ of risky assets held in the first period), and another random source H, *stochastically independent* of S, with

$$H = \begin{cases} 1.7522 & : \text{ with probability } 1/2 \\ 0 & : \text{ with probability } 1/2 \end{cases}$$

$X_2 = Y_2 = H$ is the final payoff. If A cancels at time 1 before B he has to pay a constant amount $X_1 = 1$ and vice versa B gets the smaller constant payoff $Y_1 = 0.9$ (stopping at time 0 is excluded by prohibitive disadvantageous payoffs). A has initial capital $c_1 = 5$ whereas B has the random endowment $c_2 = 10.692 - H$.

At time 1, having the information S_1, both players can decide whether to stop or not. As S_1 can take two different values, each player can choose between four possible stopping times, symbolized by $\{\delta^{11}, \delta^{12}, \delta^{21}, \delta^{22}\}$ resp. $\{\varepsilon^{11}, \varepsilon^{12}, \varepsilon^{21}, \varepsilon^{22}\}$ (where "ij" means: stopping at time i if $S_1 = 3$ and at time j if $S_1 = 0$).

The example is constructed in such a way that no stopping-strategy δ^{ij}, $i, j = 1, 2$ can be part of an equilibrium: given δ^{ij}, there are uniquely determined optimal responses $\varepsilon^{i'j'}$ and $\delta^{i''j''}$. And, we always have $\delta^{ij} \neq \delta^{i''j''}$, indeed:

$$\delta^{11} \rightsquigarrow \varepsilon^{22} \rightsquigarrow \delta^{21}, \quad \delta^{12} \rightsquigarrow \varepsilon^{22} \rightsquigarrow \delta^{21}, \quad \delta^{21} \rightsquigarrow \varepsilon^{12} \rightsquigarrow \delta^{22}, \quad \delta^{22} \rightsquigarrow \varepsilon^{22} \rightsquigarrow \delta^{21}.$$

Remark 2.5. *Why does Theorem 2.3 fail in Remark 2.4 ?*

The exponential utility function has for every initial capital $x \in \mathbb{R}$ the same *risk aversion* $\alpha = -U''(x)/U'(x)$. Therefore, for each player there exists - given the "state of the world" at time 1 (here: $S_1 = 3$ resp. $S_1 = 0$) and the chosen stopping decision of the other player - an optimal stopping decision that is *independent* of the capital $\vartheta(S_1 - S_0)$ gained until 1, and thus independent of his trading strategy $\vartheta \in \mathbb{R}$. As a consequence, the optimal stopping decision for one "state of the world" does not depend on things that happen on other "states of the world". That is in contrast to other utility functions: due to the varying risk aversion the interdependence arises through the choice of ϑ.

To construct a Nash equilibrium for exponential utility let (for example) the seller determine his optimal cancellation strategy assuming that the buyer never stops. Then, on the set A_1 where the seller cancels the optimal responding buyer does not terminate (as $W_1 \leq X_1$). Here the seller's hypothesis is self-fulfilling. On the set $\Omega \setminus A_1$ where the seller does not cancel the optimal responding buyer can terminate (cross the seller's hypothesis), but as $W_1 \geq Y_1$ this does not motivate the seller to change his initial strategy and to stop on this set, as well. As for the exponential utility the optimal decision for one "state of the world" does not depend on things that happen on other "states of the world", this "state-wise" argumentation is valid. Therefore, the seller need not change his stopping strategy *at all* and we have an equilibrium. For over utility function this "state-wise" argumentation fails and the seller could change his stopping-stategy on another state where his hypothesis was actually right. This is visible in Remark 2.4:

$$\varepsilon^{22} \rightsquigarrow \delta^{21} \rightsquigarrow \varepsilon^{12} \rightsquigarrow \delta^{22}.$$

3 The case of a complete market

If the financial market is complete, i.e. there exists a unique equivalent martingale measure Q, we get for general utility functions a result similar to Theorem 2.3. In addition, the values of the game for seller and buyer are unique. So, we have a similar property as in a zero-sum stopping game.

We can define a corresponding zero-sum stopping game which has the unique value V_0

$$V_0 = \inf_{\sigma \in \mathcal{S}_0} \sup_{\tau \in \mathcal{S}_0} E_Q \left(R(\sigma, \tau) \right) = \sup_{\tau \in \mathcal{S}_0} \inf_{\sigma \in \mathcal{S}_0} E_Q \left(R(\sigma, \tau) \right). \tag{3.1}$$

Analogously to Kifer [8], it turns out that $(\sigma_0, \tau_0) \in \mathcal{S}_0 \times \mathcal{S}_0$, defined as in (2.3)-(2.7), but taking

$$A_{i-1} = \left\{ X_{t_{i-1}} \leq E_Q \left(R(\sigma_i, \tau_i) \,\middle|\, \mathcal{F}_{t_{i-1}} \right) \right\}, \tag{3.2}$$

and

$$B_{i-1} = \left\{ Y_{t_{i-1}} \geq E_Q \left(R(\sigma_i, \tau_i) \,\middle|\, \mathcal{F}_{t_{i-1}} \right) \right\}, \tag{3.3}$$

is a saddlepoint of (3.1).

Lemma 3.1. *Let $\Theta = \Theta_2$ with $\mathcal{P} = \{Q\}$, let U be a utility function, $H \in L^1(\Omega, \mathcal{F}, Q)$, and $C \in \mathcal{F}_T$, then we have*

$$\sup_{\vartheta \in \Theta} E_P \left(U \left(C + H + \int_0^T \vartheta_t \, dS_t \right) \right) = \sup_{\vartheta \in \Theta} E_P \left(U \left(C + E_Q(H) + \int_0^T \vartheta_t \, dS_t \right) \right).$$

Proof. Due to the completeness (cf. e.g. Jacka [6]), H can be represented by a constant plus a stochastic integral, i.e. there exists a $\widehat{\vartheta} \in \Theta$ such that P-a.s.

$$H = E_Q(H) + \int_{(0,T]} \widehat{\vartheta}_t \, dS_t,$$

and due to the linearity of Θ, the mapping $\vartheta \mapsto \vartheta + \widehat{\vartheta}$ is a bijection of Θ into itself. □

Theorem 3.2. *Let $Y_{t_i}, X_{t_i} \in L^1(\Omega, \mathcal{F}, Q)$, i=0, ... ,k, and $\Theta = \Theta_2$ with $\mathcal{P} = \{Q\}$. Then*

(i) the pair (σ_0, τ_0) according to (3.2)/(3.3) is a Nash equilibrium in the sense of Definition 1.5, and

(ii) if in addition

$$-\infty < \sup_{\vartheta\in\Theta} E_P \left(U_1 \left(C_1 - V_0 + \int_0^T \vartheta_t \, dS_t \right)\right) < U_1(\infty), \tag{3.4}$$

and

$$-\infty < \sup_{\vartheta\in\Theta} E_P \left(U_2 \left(C_2 + V_0 + \int_0^T \vartheta_t \, dS_t \right)\right) < U_2(\infty), \tag{3.5}$$

then all other Nash equilibria (σ^*, τ^*) *have the same pair of values, i.e.*

$$\sup_{\vartheta\in\Theta} E_P \left(U_1 \left(C_1 - R(\sigma^*, \tau^*) + \int_0^T \vartheta_t \, dS_t \right)\right) = \sup_{\vartheta\in\Theta} E_P \left(U_1 \left(C_1 - R(\sigma_0, \tau_0) + \int_0^T \vartheta_t \, dS_t \right)\right),$$

and

$$\sup_{\vartheta\in\Theta} E_P \left(U_2 \left(C_2 + R(\sigma^*, \tau^*) + \int_0^T \vartheta_t \, dS_t \right)\right) = \sup_{\vartheta\in\Theta} E_P \left(U_2 \left(C_2 + R(\sigma_0, \tau_0) + \int_0^T \vartheta_t \, dS_t \right)\right).$$

Proof. (i) follows immediately from the respective assertions for the zero-sum game (3.1) and Lemma 3.1. For (ii) one needs in addition the fact that the mappings

$$u_i : \quad \mathbb{R} \longrightarrow \mathbb{R} \cup \{\pm\infty\}, \quad x \mapsto \sup_{\vartheta\in\Theta} E_P \left(U_i \left(C_i + x + \int_0^T \vartheta_t \, dS_t \right)\right), \quad i = 1, 2,$$

satisfy $u_1(x) < u_1(-V_0)$, for $x < -V_0$, resp. $u_2(x) < u_2(V_0)$, for $x < V_0$. So (σ^*, τ^*) is an equilibrium for (1.1)/(1.2) if and *only if* it is an equilibrium for (3.1).

This strict monotonicity can be derived as follows: the monotonicity and concavity of U_i imply the respective properties of u_i (for the latter implication one makes use of the fact that a convex combination of admissible strategies is again an admissible strategy). By $u_1(-V_0) > -\infty$ resp. $u_2(V_0) > -\infty$ and dominated convergence we conclude that $u_i(\infty) = U_i(\infty)$. Therefore, (3.4) resp. (3.5) implies the required strict monotonicity. □

Remark 3.3. The uniqueness of the values is due to the fact that in the *complete* market there is never an incentive for *both* players to stop. Only if both A and B are indifferent, i.e. on $\{X_{t_{i-1}} = E_Q\left(R(\sigma_i, \tau_i)|\mathcal{F}_{t_{i-1}}\right) = Y_{t_{i-1}}\}$ the behaviour can be different for different Nash equilibria, but that has no influence on the expected utility.

So, we have a characteristic of a zero-sum game. In a certain sense, this gives a different argument for Kifer's approach in [8].

A Appendix

We want to give full details about the iterative application of the essential infimum in (2.11).

Definition A.1. *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space and let* $\mathcal{X}$ *be a nonempty family of random variables defined on* $(\Omega, \mathcal{F}, P)$*. The* essential infimum *of* $\mathcal{X}$*, denoted by* $\operatorname{ess\,inf} \mathcal{X}$*, is a random variable* X^* *satisfying*

(i) $\forall X \in \mathcal{X}$, $X^* \leq X$ *P-a.s., and*

(ii) if Y *is a random variable satisfying* $Y \leq X$ *P-a.s. for all* $X \in \mathcal{X}$*, then* $Y \leq X^*$ *P-a.s.*

The essential infimum exists (for a proof see Gihman and Skorohod [4]) and is obviously unique P-a.s.

Theorem A.2. *Under the conditions of Theorem 2.3, we have for every* $(\sigma, \tau) \in \mathcal{S}_0 \times \mathcal{S}_0$, $i = 1, \dots, k$

$$\begin{aligned}
&\operatorname*{ess\,inf}_{\vartheta \in \Theta} E_P \left(e^{-\alpha_1 \left(C_1 - R(\sigma,\tau) + \int_{(t_{i-1},T]} \vartheta_t \, dS_t \right)} \big| \mathcal{F}_{t_{i-1}} \right) \\
&= \operatorname*{ess\,inf}_{\vartheta \in \Theta'} E_P \left[e^{-\alpha_1 \int_{(t_{i-1},t_i]} \vartheta_t \, dS_t} \operatorname*{ess\,inf}_{\widetilde{\vartheta} \in \Theta} E_P \left(e^{-\alpha_1 \left(C_1 - R(\sigma,\tau) + \int_{(t_i,T]} \widetilde{\vartheta}_t \, dS_t \right)} \big| \mathcal{F}_{t_i} \right) \big| \mathcal{F}_{t_{i-1}} \right]
\end{aligned} \tag{1.6}$$

P-a.s., where

$$\Theta' := \left\{ \vartheta \in \Theta \,\middle|\, E_P \left(e^{-\alpha_1 \left(C_1 + \int_{(t_{i-1},T]} \vartheta_t \, dS_t \right)} \big| \mathcal{F}_{t_{i-1}} \right) < \infty \quad P\text{-a.s.} \right\}.$$

Proof. Due to Assumption 1.3 one can rewrite Θ as a product space consisting of strategies $\vartheta \in \Theta$ coming into effect on $(t_{i-1}, t_i]$ and strategies $\widetilde{\vartheta} \in \Theta$ coming into effect on $(t_i, T]$, i.e.

$$\begin{aligned}
&\operatorname*{ess\,inf}_{\vartheta \in \Theta} E_P \left(e^{-\alpha_1 \left(C_1 - R(\sigma,\tau) + \int_{(t_{i-1},T]} \vartheta_t \, dS_t \right)} \big| \mathcal{F}_{t_{i-1}} \right) \\
&= \operatorname*{ess\,inf}_{(\vartheta,\widetilde{\vartheta}) \in \Theta \times \Theta} E_P \left(e^{-\alpha_1 \left(C_1 - R(\sigma,\tau) + \int_{(t_{i-1},t_i]} \vartheta_t \, dS_t + \int_{(t_i,T]} \widetilde{\vartheta}_t \, dS_t \right)} \big| \mathcal{F}_{t_{i-1}} \right) \quad P\text{-a.s.}
\end{aligned} \tag{1.7}$$

Then, one can split the essential infimum over the product space into two essential infima (using the same arguments as for the infimum in $\mathbb{R}$):

$$\begin{aligned}
&\operatorname*{ess\,inf}_{(\vartheta,\widetilde{\vartheta}) \in \Theta \times \Theta} E_P \left(e^{-\alpha_1 \left(C_1 - R(\sigma,\tau) + \int_{(t_{i-1},t_i]} \vartheta_t \, dS_t + \int_{(t_i,T]} \widetilde{\vartheta}_t \, dS_t \right)} \big| \mathcal{F}_{t_{i-1}} \right) \\
&= \operatorname*{ess\,inf}_{\vartheta \in \Theta} \operatorname*{ess\,inf}_{\widetilde{\vartheta} \in \Theta} E_P \left(e^{-\alpha_1 \left(C_1 - R(\sigma,\tau) + \int_{(t_{i-1},t_i]} \vartheta_t \, dS_t + \int_{(t_i,T]} \widetilde{\vartheta}_t \, dS_t \right)} \big| \mathcal{F}_{t_{i-1}} \right) \quad P\text{-a.s.}
\end{aligned} \tag{1.8}$$

For every fixed strategy $\widehat{\vartheta} \in \Theta$ we have of course that

$$E_P\left(e^{-\alpha_1\left(C_1-R(\sigma,\tau)+\int_{(t_i,T]}\widehat{\vartheta}_t\,dS_t\right)}|\mathcal{F}_{t_i}\right) \geq \operatorname*{ess\,inf}_{\widetilde{\vartheta}\in\Theta} E_P\left(e^{-\alpha_1\left(C_1-R(\sigma,\tau)+\int_{(t_i,T]}\widetilde{\vartheta}_t\,dS_t\right)}|\mathcal{F}_{t_i}\right)$$

P-a.s., and general properties of the essential infimum (cf. e.g. Gihman and Skorohod [4]) guarantee that the essential infimum can be approximated by a countable set of elements of Θ, i.e. there exists a sequence $(\widetilde{\vartheta}^{(n)})_{n\in\mathbb{N}} \subset \Theta$ s.t.

$$\begin{aligned}&\inf_{n\in\mathbb{N}} E_P\left(e^{-\alpha_1\left(C_1-R(\sigma,\tau)+\int_{(t_i,T]}\widetilde{\vartheta}^{(n)}_t\,dS_t\right)}|\mathcal{F}_{t_i}\right)\\ &= \operatorname*{ess\,inf}_{\widetilde{\vartheta}\in\Theta} E_P\left(e^{-\alpha_1\left(C_1-R(\sigma,\tau)+\int_{(t_i,T]}\widetilde{\vartheta}_t\,dS_t\right)}|\mathcal{F}_{t_i}\right) \quad P\text{-a.s.},\end{aligned}$$

where the inf is understood pointwise. For two strategies $\widetilde{\vartheta}^{(1)}, \widetilde{\vartheta}^{(2)} \in \Theta$ define

$$\widetilde{\vartheta}^{(3)}_t = \begin{cases} \mathbf{1}(t>t_i)\widetilde{\vartheta}^{(1)}_t & : \quad E_P\left(e^{-\alpha_1\left(C_1-R(\sigma,\tau)+\int_{(t_i,T]}\widetilde{\vartheta}^{(1)}_t\,dS_t\right)}|\mathcal{F}_{t_i}\right) \\ & \qquad \leq E_P\left(e^{-\alpha_1\left(C_1-R(\sigma,\tau)+\int_{(t_i,T]}\widetilde{\vartheta}^{(2)}_t\,dS_t\right)}|\mathcal{F}_{t_i}\right), \\ \mathbf{1}(t>t_i)\widetilde{\vartheta}^{(2)}_t & : \quad \text{otherwise.} \end{cases}$$

Due to Assumption 1.3 we have $\widetilde{\vartheta}^{(3)} \in \Theta$, and in addition

$$\begin{aligned}&E_P\left(e^{-\alpha_1\left(C_1-R(\sigma,\tau)+\int_{(t_i,T]}\widetilde{\vartheta}^{(3)}_t\,dS_t\right)}|\mathcal{F}_{t_i}\right)\\ &= \min\left\{E_P\left(e^{-\alpha_1\left(C_1-R(\sigma,\tau)+\int_{(t_i,T]}\widetilde{\vartheta}^{(1)}_t\,dS_t\right)}|\mathcal{F}_{t_i}\right), E_P\left(e^{-\alpha_1\left(C_1-R(\sigma,\tau)+\int_{(t_i,T]}\widetilde{\vartheta}^{(2)}_t\,dS_t\right)}|\mathcal{F}_{t_i}\right)\right\},\end{aligned}$$

and therefore inf-stability. Hence, there exists a sequence $(\widetilde{\vartheta}^n)_{n\in\mathbb{N}} \in \Theta$ such that

$$\begin{aligned}&E_P\left(e^{-\alpha_1\left(C_1-R(\sigma,\tau)+\int_{(t_i,T]}\widetilde{\vartheta}^{(n)}_t\,dS_t\right)}|\mathcal{F}_{t_i}\right)\\ &\searrow \operatorname*{ess\,inf}_{\widetilde{\vartheta}\in\Theta} E_P\left(e^{-\alpha_1\left(C_1-R(\sigma,\tau)+\int_{(t_i,T]}\widetilde{\vartheta}_t\,dS_t\right)}|\mathcal{F}_{t_i}\right) \quad P\text{-a.s., } n\to\infty,\end{aligned}$$

resp.

$$\begin{aligned}&e^{-\alpha_1\int_{(t_{i-1},t_i]}\vartheta_t\,dS_t}\min\left\{E_P\left(e^{-\alpha_1\left(C_1-R(\sigma,\tau)+\int_{(t_i,T]}\widetilde{\vartheta}^{(n)}_t\,dS_t\right)}|\mathcal{F}_{t_i}\right),\right.\\ &\left.E_P\left(e^{-\alpha_1\left(C_1-R(\sigma,\tau)+\int_{(t_i,T]}\vartheta_t\,dS_t\right)}|\mathcal{F}_{t_i}\right)\right\}\\ &\searrow e^{-\alpha_1\int_{(t_{i-1},t_i]}\vartheta_t\,dS_t}\operatorname*{ess\,inf}_{\widetilde{\vartheta}\in\Theta} E_P\left(e^{-\alpha_1\left(C_1-R(\sigma,\tau)+\int_{(t_i,T]}\widetilde{\vartheta}_t\,dS_t\right)}|\mathcal{F}_{t_i}\right) \quad P\text{-a.s.,} \qquad (1.9)\end{aligned}$$

as $n \to \infty$, where $\vartheta \in \Theta'$. Due to (2.8) Θ' is nonempty and the sequence in (1.9) is dominated by the random variable

$$E_P\left(e^{-\alpha_1\left(C_1 - R(\sigma,\tau) + \int_{(t_{i-1},T]} \vartheta_t \, dS_t\right)} \big| \mathcal{F}_{t_i}\right),$$

which has $P-a.s.$ finite $P\left(\bullet \,\middle|\, \mathcal{F}_{t_{i-1}}\right)$-expectation (notice that $R(\sigma,\tau)$ is bounded). So, for every $\vartheta \in \Theta'$, we can apply the dominated convergence theorem for conditional expectations to (1.9). Then, we take the essential infimum over all $\vartheta \in \Theta'$ on both sides:

$$\begin{aligned} &\operatorname*{ess\,inf}_{\vartheta\in\Theta'} \operatorname*{ess\,inf}_{\widetilde{\vartheta}\in\Theta} E_P\left(e^{-\alpha_1\left(C_1 - R(\sigma,\tau) + \int_{(t_{i-1},t_i]} \vartheta_t \, dS_t + \int_{(t_i,T]} \widetilde{\vartheta}_t \, dS_t\right)} \big| \mathcal{F}_{t_{i-1}}\right) \qquad (1.10) \\ &= \operatorname*{ess\,inf}_{\vartheta\in\Theta'} E_P\left[e^{-\alpha_1 \int_{(t_{i-1},t_i]} \vartheta_t \, dS_t} \operatorname*{ess\,inf}_{\widetilde{\vartheta}\in\Theta} E_P\left(e^{-\alpha_1\left(C_1 - R(\sigma,\tau) + \int_{(t_i,T]} \widetilde{\vartheta}_t \, dS_t\right)} \big| \mathcal{F}_{t_i}\right) \big| \mathcal{F}_{t_{i-1}}\right] \end{aligned}$$

P-a.s. It remains to show that it makes no difference whether the essential infimum in the *first* expression of (1.10) is taken over all $\vartheta \in \Theta$ or only over all $\vartheta \in \Theta'$. Take at first an arbitrary $\vartheta \in \Theta$ and define

$$A = \left\{\operatorname*{ess\,inf}_{\widetilde{\vartheta}\in\Theta} E_P\left(e^{-\alpha_1\left(C_1 - R(\sigma,\tau) + \int_{(t_{i-1},t_i]} \vartheta_t \, dS_t + \int_{(t_i,T]} \widetilde{\vartheta}_t \, dS_t\right)} \big| \mathcal{F}_{t_{i-1}}\right) < \infty\right\} \qquad (1.11)$$

The essential infimum in (1.11) can be monotonously approximated by a sequence $(\widetilde{\vartheta}^{(n)})_{n\in\mathbb{N}} \subset \Theta$. That implies

$$A^{(n)} := \left\{E_P\left(e^{-\alpha_1\left(C_1 - R(\sigma,\tau) + \int_{(t_{i-1},t_i]} \vartheta_t \, dS_t + \int_{(t_i,T]} \widetilde{\vartheta}_t^{(n)} \, dS_t\right)} \big| \mathcal{F}_{t_{i-1}}\right) < \infty\right\} \nearrow A \quad P\text{-a.s.},$$

as $n \to \infty$. Let $\widehat{\vartheta} \in \Theta' \neq \emptyset$ and define

$$\vartheta_t^{(n)} := \begin{cases} \vartheta_t & : \ t \le t_i \quad \text{and} \quad \omega \in A^{(n)}, \\ \widetilde{\vartheta}_t^{(n)} & : \ t > t_i \quad \text{and} \quad \omega \in A^{(n)}, \\ \widehat{\vartheta}_t & : \ \text{otherwise.} \end{cases}$$

$\vartheta^{(n)}$ are by construction elements of Θ'. Furthermore, $A^{(n)} \cup (\Omega \setminus A) \nearrow \Omega$, P-a.s., as $n \to \infty$, and on $A^{(n)} \cup (\Omega \setminus A)$ we have

$$\begin{aligned} &\operatorname*{ess\,inf}_{\widetilde{\vartheta}\in\Theta} E_P\left(e^{-\alpha_1\left(C_1 - R(\sigma,\tau) + \int_{(t_{i-1},t_i]} \vartheta_t^{(n)} \, dS_t + \int_{(t_i,T]} \widetilde{\vartheta}_t \, dS_t\right)} \big| \mathcal{F}_{t_{i-1}}\right) \\ &\le \operatorname*{ess\,inf}_{\widetilde{\vartheta}\in\Theta} E_P\left(e^{-\alpha_1\left(C_1 - R(\sigma,\tau) + \int_{(t_{i-1},t_i]} \vartheta_t \, dS_t + \int_{(t_i,T]} \widetilde{\vartheta}_t \, dS_t\right)} \big| \mathcal{F}_{t_{i-1}}\right). \end{aligned}$$

Therefore,

$$
\begin{aligned}
&\operatorname*{ess\,inf}_{\vartheta\in\Theta'} \operatorname*{ess\,inf}_{\widetilde{\vartheta}\in\Theta} E_P\left(e^{-\alpha_1\left(C_1-R(\sigma,\tau)+\int_{(t_{i-1},t_i]}\vartheta_t\,dS_t+\int_{(t_i,T]}\widetilde{\vartheta}_t\,dS_t\right)}\Big|\mathcal{F}_{t_{i-1}}\right) \qquad (1.12)\\
&= \operatorname*{ess\,inf}_{\vartheta\in\Theta} \operatorname*{ess\,inf}_{\widetilde{\vartheta}\in\Theta} E_P\left(e^{-\alpha_1\left(C_1-R(\sigma,\tau)+\int_{(t_{i-1},t_i]}\vartheta_t\,dS_t+\int_{(t_i,T]}\widetilde{\vartheta}_t\,dS_t\right)}\Big|\mathcal{F}_{t_{i-1}}\right) \quad P\text{-a.s.}
\end{aligned}
$$

Putting (1.7), (1.8), (1.10), and (1.12) together, this implies the assertion. □

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