Fractional Brownian motion as a weak limit of Poisson shot noise processes – with applications to finance ⋆

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Abstract

We consider Poisson shot noise processes that are appropriate to model stock prices and provide an economic reason for long-range dependence in asset returns. Under a regular variation condition we show that our model converges weakly to a fractional Brownian motion. Whereas fractional Brownian motion allows for arbitrage, the shot noise process itself can be chosen arbitrage-free. Using the marked point process skeleton of the shot noise process we construct a corresponding equivalent martingale measure explicitly.

Key words: Shot noise process, alternative stock price models, functional limit theorems, fractional Brownian motion, arbitrage, non-explosiveness of point processes
1991 MSC: 60F17, 60G55, 60G15

⋆ The authors thank Yuri Kabanov, Thomas Mikosch, and an anonymous referee for valuable comments on this paper. In addition, financial support by the Deutsche Forschungsgemeinschaft through the Graduiertenkolleg "Angewandte Algorithmische Mathematik" at Munich University of Technology is gratefully acknowledged by the second author.
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1 Introduction

Whereas Lévy processes and stochastic volatility models are by now standard models for stock prices, more recently long memory processes like fractional Brownian motion (FBM) have attracted attention by stochastic analysts and mathematical finance researchers, cf. e.g. Hu and Øksendal (HuOks03) and the references therein. For an introduction to FBM see Samorodnitsky and Taqqu (ST94). Certain financial time series show long memory properties as observed since the 1980s; see Granger (Gra80), Granger and Joyeux (GJ80), and Mandelbrot (Man97). Such observation has led to an ongoing debate among econometricians and statisticians. It is obvious that any deterministic component like a small trend or business cycle can cause a *fictitious* long memory effect in a time series and it has been shown recently that also change-points in a time series can exhibit such a long memory effect (Mikosch and Stărică (MS03)). More recently, Brody, Syroka and Zervos (BrSyZe01) have investigated weather derivatives written on temperature-based indices, whose dynamics show long memory and can be modelled by fractional Ornstein-Uhlenbeck processes.

From the point of view of stochastic analysis FBM has the distinct disadvantage that it is not a semimartingale and allows for arbitrage; explicit arbitrage strategies have been found for FBM by Rogers (Rog97) and for geometric FBM by Cheridito (Ch03). But, as already mentioned there, the existence of an arbitrage possibility is no inherent property of long memory processes. It is rather a consequence of the *local* behaviour of FBM that is inconsistent with the properties of a semimartingale, whereas long-range dependence is a property of the long-run behaviour of a process.

In this paper we answer the natural question for a possible economic explanation of logarithmic stock price processes to follow FBM. In the case of Brownian motion, it is well-known that it appears as Donsker limit of a random walk for relative price changes; this applies to Lévy processes in general. Stochastic volatility models, on the other hand, have the obvious economic interpretation of a volatility changing in time depending on past prices, past volatilities, and market conditions. As for FBM, a first idea is to find a discrete skeleton, which may have an economic interpretation. The most natural one is obviously a long memory linear model, more precisely an ARIMA(p,d,q) process with autoregressive part of order p, moving average part of order q and fractional difference parameter $d \in (0\,,\,0.5)$. Such models converge in a Donsker sense to FBM, see Konstantopoulos and Sakhanenko (KS03) and Davydov (Da70). A special example is given in Sottinen (Sot01), who shows convergence of a special binary market model to FBM.

However, all this does not provide an economic reason, why to consider FBM or geometric FBM as a price model. More promising in this context seems to us an idea by Stute (Stu00) who suggested to enrich geometric Brownian motion by a geometric shot noise part. The model he suggests is given by

$$P(t) = \exp\{B(t) + S(t)\}, \quad t \geq 0, \tag{1.1}$$

where $(B(t))_{t\geq 0}$ is a Brownian motion and $(S(t))_{t\geq 0}$ is a shot noise model, which we define in a slight modification by

$$S(t) = \sum_{i=1}^{N(t)} X_i(t - T_i) + \sum_{i=-1}^{-\infty} [X_i(t - T_i) - X_i(-T_i)], \quad t \geq 0. \tag{1.2}$$

Here $X_i = (X_i(t))_{t\in\mathbb{R}}$, $i \in \mathbb{Z} \setminus \{0\}$, are i.i.d. stochastic processes on $\mathbb{R}$ such that $X_i(t) = 0$ for $t < 0$, independent of the two-sided homogeneous Poisson process N with rate $\alpha > 0$ and points $\ldots < T_{-2} < T_{-1} < T_1 < T_2 < \ldots$. For $t = 0$ we have $S(0) = 0$ as both sums on the right-hand side are 0.

The shot noise model S is interpreted as a model for information provided by various sources which enters the price at random Poisson times. The arrival of information acts like a shock to the market which may change the price quite drastically and may also have some influence on the future price movements. The reason for this is that a new piece of information that is relevant for the stock price of a firm (e.g. a political decision or some rumor concerning a merger) needs some time to spread among the market participants. That means some traders have information earlier than others (think for example of insider-trading). Therefore it needs some time until the news reaches its maximum effect on the market. Later on, some effects may fade away again, but it may as well happen that certain information has a long lasting influence on the price. In this way long memory is introduced into the economic model.

We obtain convergence to FBM. Moreover, we show that the model (1.2) itself can be chosen arbitrage-free (by the right choice of X_i near 0), only its limit model FBM allows for arbitrage. Recently, another economic foundation for models based on FBM has been given by Bayraktar, Horst and Sircar (BHS03). In that paper investor inertia leads to longe range dependence.

Shot noise processes were used in various branches of stochastic modelling; references can be found in Klüppelberg and Mikosch (KM95) and Klüppelberg, Mikosch and Schärf (KMS03). Whereas in those papers limits for non-stationary shot noise models of the form $S(t) = \sum_{i=1}^{N(t)} X_i(t - T_i)$, $t \geq 0$, were investigated with a view towards applications in insurance, in this paper we work with a version of the process possessing stationary increments, which requires the introduction of the second sum in (1.2).

Our paper is organized as follows. In section 2 we investigate some properties of the restricted process $S\big|_{[0,t]}$ which are important for applications in mathematical finance. In particular, we show how to construct an equivalent martingale measure. Hence, our model does not allow for arbitrage. In section 3 we show weak convergence of the rescaled process to a FBM when the time horizon tends to infinity.

2 Shot noise processes and long-range dependence

It is straightforward to see that S given by (1.2) has stationary increments. Throughout the paper, we restrict ourselves to the special model of multiplicative shots: for all $i \in \mathbb{Z} \setminus \{0\}$,

$$X_i(u) = g(u)Y_i\,, \quad u \geq 0, \tag{2.1}$$

where $g : \mathbb{R}_+ \to \mathbb{R}$ is a continuously differentiable function with

$$g'(u) = O\left(u^{-1/2-\varepsilon}\right), \quad u \to \infty, \tag{2.2}$$

for some $\varepsilon > 0$. The Y_i are i.i.d. innovations with $EY_1 = 0$ and $EY_1^2 \in (0, \infty)$.

Notice that the shots entering S are in general not absolutely summable. However, the multiplicative process with the above restrictions on g and the Y_i exists and has nice sample path properties.

Proposition 2.1 *The process S as defined in (1.2) with (2.1) possesses a càdlàg version and has finite variation. Therefore, it is a semimartingale with respect to its natural filtration.*

Proof. For fixed $t \geq 0$ the existence of $S(t)$ can be derived from Theorem 1 in Westcott (Wes76).
Note now that the first process $(\sum_{i=1}^{N(t)} Y_i g(t - T_i))_{t\geq 0}$ of the right-hand side of (1.2) has at most finitely many jumps in any compact time interval. Hence it is càdlàg and it has, as g is continuously differentiable, finite variation. Next consider the second sum of (1.2)

$$\tilde{S}(t) := \sum_{i=-1}^{-\infty} Y_i \left[g(t - T_i) - g(-T_i)\right], \quad t \geq 0.$$

For $u, h \geq 0$ we have

$$
\begin{aligned}
&E\left[\left(\widetilde{S}(u+h)-\widetilde{S}(u)\right)^2\right] \\
&= EY_1^2 E\left[\sum_{i=-1}^{-\infty}\left[g(u+h-T_i)-g(u-T_i)\right]^2\right] \\
&\le EY_1^2 h^2 E\left[\sum_{i=-1}^{-\infty}\sup_{s\in[u-T_i,u+h-T_i]}\left[g'(s)\right]^2\right]. \qquad (2.3)
\end{aligned}
$$

We obtain for $u, h, s_0 \ge 0$ (recall that $\alpha > 0$ is the rate of the Poisson process N),

$$
\begin{aligned}
&E\left[\sum_{i=-1}^{-\infty}\sup_{s\in[u-T_i,u+h-T_i]}\left[g'(s)\right]^2\right] \\
&= \sum_{i=-1}^{-\infty} E\left[I\left(|T_i| < s_0 \vee \frac{|i|}{2\alpha}\right)\sup_{s\in[u-T_i,u+h-T_i]}\left[g'(s)\right]^2\right] \\
&\qquad + \sum_{i=-1}^{-\infty} E\left[I\left(|T_i| \ge s_0 \vee \frac{|i|}{2\alpha}\right)\sup_{s\in[u-T_i,u+h-T_i]}\left[g'(s)\right]^2\right] \\
&=: I + II.
\end{aligned}
$$

We estimate

$$
I \le \sup_{0<s<\infty}\left[g'(s)\right]^2 \sum_{i=-1}^{-\infty} P\left(|T_i| < s_0 \vee \frac{|i|}{2\alpha}\right).
$$

The infinite sum on the right-hand side is finite, which follows from a general result for random walks with drift (see e.g. Theorem 3.3 in Chapter 3 of Gut (Gut88)) applied to the random walk $(|T_i| - |i|/(2\alpha))_{i=-1,-2,\ldots}$ (which has a positive drift and its increments are bounded from below). Hence, $I < \infty$ for all $s_0 \in \mathbb{R}_+$.
To estimate II recall from (2.2) that we have for s_0 large enough $|g'(s)| \le s^{-1/2-\varepsilon/2}$ for all $s \ge s_0$. Hence

$$
II \le \sum_{i=-1}^{-\infty} E\left[I\left(|T_i| \ge s_0 \vee \frac{|i|}{2\alpha}\right)\sup_{s\ge s_0\vee\frac{|i|}{2\alpha}}\left[g'(s)\right]^2\right] \le \sum_{i=-1}^{-\infty}\left(\frac{|i|}{2\alpha}\right)^{-1-\varepsilon} < \infty,
$$

giving $I + II =: A < \infty$. Thus, Kolmogorov's continuity theorem (see e.g. Protter (Pro90), Ch. IV, Theorem 53, p. 171) ensures the existence of a continuous version of $\widetilde{S}$ (or a càdlàg version of S).
To calculate the variation of $\widetilde{S}$ we approximate it by the variation of the process $\widetilde{S}$ on the dual grid $\{i2^{-n}t \mid i = 0, \ldots, 2^n\}$. Using Jensen's inequality and again (2.3) with the subsequent arguments we obtain for $t \ge 0$

$$
\begin{aligned}
&E\left[\sum_{k=0}^{2^n-1}\left|\widetilde{S}((k+1)2^{-n}t)-\widetilde{S}(k2^{-n}t)\right|\right] \\
&\leq \sum_{k=0}^{2^n-1}\sqrt{E\left[\left(\widetilde{S}((k+1)2^{-n}t)-\widetilde{S}(k2^{-n}t)\right)^2\right]} \\
&\leq t\sqrt{EY_1^2A}<\infty. \qquad (2.4)
\end{aligned}
$$

Due to monotone convergence, we get the assertion by letting $n\to\infty$. □

From now on we work with the completed stochastic basis $(\Omega, \mathcal{F}_t, (\mathcal{F}_s)_{0\le s\le t}, P)$, $t\geq 0$, where

$$
\mathcal{F}_s=\sigma\left((Y_i)_{i\in\mathbb{Z}_-},\ (T_i)_{i\in\mathbb{Z}_-},\ (S(u))_{0\le u\le s}\right). \qquad (2.5)
$$

Define f as *càglàd* modification of the process

$$
u\mapsto \sum_{i=N(u)}^{-\infty} Y_i g'(u-T_i).
$$

Similar to (2.3), by Kolmogorov's continuity theorem, there exists a continuous version of $\left(\sum_{i=-1}^{-\infty} Y_i g'(u-T_i)\right)_{u\geq 0}$, which we call f_c. In addition, we define

$$
Z(t):=g(0)\sum_{i=1}^{N(t)} Y_i, \quad t\geq 0. \qquad (2.6)
$$

Lemma 2.2 *With the quantities as defined above, S satisfies the stochastic differential equation (SDE)*

$$
dS(t)=dZ(t)+f(t)\,dt, \quad t\geq 0. \qquad (2.7)
$$

Proof. *Step 1:* We show first that we can interchange integration and summation, i.e. that for all $t,h\geq 0$,

$$
\sum_{i=-1}^{-\infty} Y_i\left[g(t+h-T_i)-g(t-T_i)\right]=\int_t^{t+h} f_c(u)\,du \quad P\text{-a.s.}. \qquad (2.8)
$$

Recall that the addends entering the sums in (2.8) need not be absolutely summable. On the grid points $\{t+hj2^{-n} \mid n\in\mathbb{N},\ j=1,\dots,2^n\}$, however, we have pointwise convergence by the martingale convergence theorem:

$$\lim_{m\to\infty} \sum_{i=-1}^{-m} Y_i g'(t+hj2^{-n}-T_i) = f_c(t+hj2^{-n}) \quad P\text{-a.s..} \tag{2.9}$$

Next we estimate the approximation error $G(m,n) := \sum_{i=-1}^{-m} Y_i a_i^{(n)}$, where

$$a_i^{(n)} := h2^{-n} \sum_{j=1}^{2^n} g'(t+hj2^{-n}-T_i) - \{g(t+h-T_i) - g(t-T_i)\}.$$

For all $i \in \{-1,\dots,-m\}$ and $n \in \mathbb{N}$, $|a_i^{(n)}| \le 2h \sup_{u\in(t-T_i,t+h-T_i)} |g'(u)| =: a_i$, and $E\left[\sum_{i=-1}^{-\infty} a_i^2\right] < \infty$, since the right-hand side of (2.3) is finite. On the other hand, as g' is continuous, we have by Riemann integration for all $i \in \mathbb{Z}_-$ that $\lim_{n\to\infty} a_i^{(n)} = 0$ P-a.s.. This yields $\lim_{n\to\infty} E\left[\sum_{i=-1}^{-\infty} (a_i^{(n)})^2\right] = 0$. Therefore, we obtain

(i) for fixed $n \in \mathbb{N}$, $G(m,n) \to G(\infty,n)$ as $m \to \infty$, both P-a.s. and in $L^2(P)$

(the former is by the martingale convergence theorem and the latter by the Cauchy criterion). Since

$$E[G(\infty,n)^2] = EY_1^2 E\left[\sum_{i=-1}^{-\infty} (a_i^{(n)})^2\right] \to 0\,, \quad n \to \infty\,,$$

we have $G(\infty,n) \xrightarrow{P} 0$ as $n \to \infty$ and

(ii) $P\left(\{\forall \delta > 0,\ G(\infty,n) \le \delta \text{ infinitely often}\}\right) = 1.$

Now, we are ready to proof Equation (2.8).
Take $\omega \in \Omega$ such that (i) and (ii) hold (due to completeness of $\mathcal{F}_t$ we can exclude all countably many null sets on which (i) or (ii) does not hold). Let $\varepsilon > 0$. As f_c is continuous the integral $\int_t^{t+h} f_c(u)\,du$ can be approximated (ω-wise) by the Riemann sums $h2^{-n}\sum_{j=1}^{2^n} f_c(t+hj2^{-n})$, $n \in \mathbb{N}$, i.e. there exists an n_1 such that for $n \ge n_1$

$$\left| h2^{-n} \sum_{j=1}^{2^n} f_c(t+hj2^{-n}) - \int_t^{t+h} f_c(u)\,du \right| \le \frac{\varepsilon}{3}. \tag{2.10}$$

Furthermore, since (i) holds, we can find $n_2 \ge n_1$ and $m_1 \in \mathbb{N}$ such that for all $m \ge m_1$

$$|G(m,n_2)| \le \frac{\varepsilon}{3}. \tag{2.11}$$

For this n_2 we use (2.9), i.e. convergence on the grid $\{t + hj2^{-n_2} \mid j = 1, \ldots, 2^{n_2}\}$, and we get for $m \geq m_2$

$$
\begin{aligned}
&\left| \sum_{i=-1}^{-m} Y_i h 2^{-n_2} \sum_{j=1}^{2^{n_2}} g'(t + hj2^{-n_2} - T_i) - h2^{-n_2} \sum_{j=1}^{2^{n_2}} f_c(t + hj2^{-n_2}) \right| \\
&= \left| h2^{-n_2} \sum_{j=1}^{2^{n_2}} \sum_{i=-1}^{-m} Y_i g'(t + hj2^{-n_2} - T_i) - h2^{-n_2} \sum_{j=1}^{2^{n_2}} f_c(t + hj2^{-n_2}) \right| \\
&\leq \frac{\varepsilon}{3}. \qquad (2.12)
\end{aligned}
$$

Putting (2.10), (2.11), and (2.12) together, we get for $m \geq m_1 \vee m_2$

$$
\left| \sum_{i=-1}^{-m} Y_i \left[g(t + h - T_i) - g(t - T_i) \right] - \int_t^{t+h} f_c(u)\, du \right| \leq \varepsilon,
$$

and therefore (2.8) holds.

Step 2: Using (2.8) Equation (2.7) follows from the calculation:

$$
\begin{aligned}
&S(t+h) - S(t) \\
&= \sum_{i=N(t)+1}^{N(t+h)} Y_i g(t + h - T_i) + \sum_{i=N(t)}^{-\infty} Y_i \left[g(t + h - T_i) - g(t - T_i) \right] \\
&= g(0) \sum_{i=N(t)+1}^{N(t+h)} Y_i + \sum_{i=N(t)+1}^{N(t+h)} Y_i \left[g(t + h - T_i) - g(0) \right] \\
&\qquad + \sum_{i=N(t)}^{-\infty} Y_i \left[g(t + h - T_i) - g(t - T_i) \right] \\
&= g(0) \sum_{i=N(t)+1}^{N(t+h)} Y_i + \sum_{i=N(t)+1}^{N(t+h)} Y_i \int_{T_i}^{t+h} g'(u - T_i)\, du + \int_t^{t+h} \sum_{i=N(t)}^{-\infty} Y_i g'(u - T_i)\, du \\
&= g(0) \sum_{i=N(t)+1}^{N(t+h)} Y_i + \int_t^{t+h} \sum_{i=N(t)+1}^{N(u)} Y_i g'(u - T_i)\, du + \int_t^{t+h} \sum_{i=N(t)}^{-\infty} Y_i g'(u - T_i)\, du \\
&= Z(t+h) - Z(t) + \int_t^{t+h} f(u)\, du. \qquad \square
\end{aligned}
$$

Theorem 2.3 *Let $t > 0$. If $g(0) \neq 0$ there exists a probability measure $Q \sim P$ such that $S\Big|_{[0,t]}$ is a local Q-martingale.*

To prove this theorem we need the following result.

Lemma 2.4 *Let $(\widehat{T}_n)_{n\in\mathbb{N}_0}$ be a sequence of random variables with $\widehat{T}_0 = 0$ and i.i.d. exponential increments. Let furthermore $(\widetilde{Y}_n)_{n\in\mathbb{N}_0}$ be an independent sequence of non-negative random variables, which are i.i.d. with $E\widetilde{Y}_1 < \infty$, and $P(\widetilde{Y}_0 > 0) = 1$. Define now recursively*

$$\widetilde{T}_0 := 0\,, \quad \widetilde{T}_n := \widetilde{T}_{n-1} + \frac{\widehat{T}_n - \widehat{T}_{n-1}}{\sum_{i=0}^{n-1} \widetilde{Y}_i}, \quad n \in \mathbb{N}.$$

Then, the sequence $(\widetilde{T}_n)_{n\in\mathbb{N}_0}$ is non-explosive, i.e. $\widetilde{T}_n \nearrow \infty$ P-a.s. as $n \to \infty$.

Proof. W.l.o.g. we assume that $\widehat{T}_{i+1} - \widehat{T}_i$, $i \in \mathbb{N}_0$, are standard exponential. Define the filtration

$$\widetilde{\mathcal{F}}_i = \sigma\left(\widetilde{T}_j,\ j = 0, \ldots, i\,,\ (\widetilde{Y}_j)_{j\in\mathbb{N}_0},\right), \quad i \in \mathbb{N}_0\,.$$

As an immediate consequence of the strong law of large numbers, we have

$$\sum_{i=0}^{n} \widetilde{Y}_i \leq Cn \quad \forall n \in \mathbb{N} \quad P\text{-a.s.},$$

where C is an $\widetilde{\mathcal{F}}_0$-measurable real-valued random variable. Hence we obtain

$$P\left(\widetilde{T}_{i+1} - \widetilde{T}_i > \frac{\ln 2}{Ci} \mid \widetilde{\mathcal{F}}_i\right) \geq P\left(\widehat{T}_{i+1} - \widehat{T}_i > \ln 2\right) = \frac{1}{2} \quad P\text{-a.s.}. \tag{2.13}$$

Furthermore,

$$\widetilde{T}_n = \sum_{i=0}^{n-1} (\widetilde{T}_{i+1} - \widetilde{T}_i) \geq \sum_{i=0}^{n-1} I\left(\widetilde{T}_{i+1} - \widetilde{T}_i > \frac{\ln 2}{Ci}\right) \frac{\ln 2}{Ci}, \tag{2.14}$$

and

$$P\left(\sum_{i=0}^{\infty} \frac{\ln 2}{Ci} = \infty \mid \widetilde{\mathcal{F}}_0\right) = 1 \quad P\text{-a.s.}. \tag{2.15}$$

Putting (2.13), (2.14), and (2.15) together we obtain

$$
\begin{aligned}
&P\left(\sum_{i=0}^{\infty}(\widetilde{T}_{i+1}-\widetilde{T}_i)=\infty \mid \widetilde{\mathcal{F}}_0\right)\\
&\geq P\left(\sum_{i=0}^{\infty} I\left(\widetilde{T}_{i+1}-\widetilde{T}_i>\frac{\ln 2}{Ci}\right)\frac{\ln 2}{Ci}=\infty \mid \widetilde{\mathcal{F}}_0\right).
\end{aligned}
$$

If the indicator variables were (conditionally on $\widetilde{\mathcal{F}}_0$) i.i.d. and not vanishing, then by the Three-series theorem (cf. e.g. Feller (Fel71), Theorem IX.9.3), the right-hand probability would be equal to 1 P-a.s.. Using (2.13) and some conditioning argument, this also holds for the dependent indicator variables; for details see (Kueh02), Lemmas A.1.1 and A.1.2. Finally, by Fubini's theorem, $\widetilde{T}_n \nearrow \infty$ as $n \to \infty$ P-a.s. □

Proof of Theorem 2.3. *Step 1:* Let $t>0$. We construct a possible Q. Under Q, the process Z as defined in (2.6) should have drift rate $-f$. Then by (2.7) S becomes a local martingale. This can be achieved by applying Girsanov's theorem for point processes, cf. Brémaud (Bre81), Theorem T10, Ch. VIII. We interpret the double sequence $(T_i, Y_i)_{i\in\mathbb{N}}$ as a marked point process with points T_i having marks Y_i. Under P its intensity is $\lambda_P(t,dx)=\alpha P(Y_1\in dx)$. Translated to our notation Theorem T10 of Ch. VIII in (Bre81) gives the following recipe.
Choose a function $\phi:[0,t]\times\Omega\times\mathbb{R}\to(0,\infty)$, which is $\widetilde{\mathcal{P}}$-measurable ($\widetilde{\mathcal{P}}=\mathcal{P}\otimes\mathcal{B}(\mathbb{R})$ where $\mathcal{P}$ is the $\mathcal{F}$-predictable σ-algebra on $[0,t]\times\Omega$ and $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$) and satisfies

$$
\int_0^t\int_{\mathbb{R}}\phi(s,x)\alpha P(Y_1\in dx)\,ds<\infty \quad P\text{-a.s.}.
$$

Then define for $s\in[0,t]$

$L_s :=$

$$
\exp\left\{\int_0^s\int_{\mathbb{R}}\log\phi(u,x)\mu(du,dx)+\int_0^s\int_{\mathbb{R}}(1-\phi(u,x))\alpha P(Y_1\in dx)\,du\right\} \tag{2.16}
$$

or, equivalently,

$$
dL_s=L_{s-}\int_{\mathbb{R}}(\phi(s,x)-1)\left\{\mu(ds,dx)-\alpha P(Y_1\in dx)\,ds\right\},\ L_0=1,
$$

where $\mu((0,s]\times A) := \sum_{0<T_i\le s} I(Y_i\in A)$ for all $s>0$ and $A\in\mathcal{B}(\mathbb{R})$. If $EL_t=1$, then there exists a $Q\sim P$ defined by

$$\frac{dQ}{dP} = L_t,$$

such that under Q the marked point process $(T_i,Y_i)_{i\in\mathbb{N}}$ has intensity

$$\lambda_Q(s,dx) = \phi(s,x)\alpha P(Y_1\in dx), \quad s\ge 0,\ x\in\mathbb{R}\,. \tag{2.17}$$

To make S a local martingale we need that

$$g(0)\int_{\mathbb{R}} x\lambda_Q(s,dx) = -f(s), \quad s\ge 0. \tag{2.18}$$

This can be achieved by setting (recall $EY_1=0$)

$$\phi(s,x) := \begin{cases} 1+\dfrac{|f(s)|I(\{f(s)<0\})}{\alpha g(0)EY_1^+} & : \quad x\ge 0\,, \\[2ex] 1+\dfrac{f(s)I(\{f(s)\ge 0\})}{\alpha g(0)EY_1^-} & : \quad x<0\,. \end{cases} \tag{2.19}$$

Note that ϕ is $\widetilde{\mathcal{P}}$-measurable and strictly positive. Furthermore, $(L_s)_{0\le s\le t}$ is a local P-martingale and, due to positivity, a P-supermartingale, i.e. $EL_t\le 1$. To verify $EL_t=1$ we make a localization: as f is càglàd we can define a sequence $\tau_n := \inf\{s\ge 0 \mid |f(s+)|>n\}$, $n\in\mathbb{N}$, of stopping times with $|f^{\tau_n}|\le n$, where $f^{\tau_n}(\cdot)=f(\tau_n\wedge\cdot)$. As in the proof of Proposition 2.1 we obtain $P(\sup_{s\in[0,t]}|f(s)|<\infty)=1$, so that $P(\tau_n<t)\to 0$ as $n\to\infty$. Define for $s\in[0,t]$ and $x\in\mathbb{R}$

$$\phi^n(s,x) := \phi(s,x)I(\{s\le\tau_n\}) + I(\{s>\tau_n\}). \tag{2.20}$$

For the corresponding density processes $(L_s^n)_{s\in[0,t]}$ and $dQ^n/dP = L_t^n$ we have indeed $EL_t^n=1$ (cf. Brémaud (Bre81), Ch. VIII, Theorem T11) and therefore

$$\begin{aligned} 1 = E\left(L_t^n\right) &= E\left(L_tI(\{\tau_n\ge t\})\right) + E\left(L_t^nI(\{\tau_n<t\})\right) \\ &= E\left(L_tI(\{\tau_n\ge t\})\right) + Q^n(\tau_n<t). \end{aligned}$$

Step 2: It remains to show that $Q^n(\tau_n<t)\to 0$ as $n\to\infty$.
Note that by definition of ϕ in (2.19), if $f(s)<0$ the rate of positive jumps increases, whereas, if $f(s)>0$ the rate of the negative jumps increases. To describe the change of measure explicitly, we construct (possibly on a new

probability space, which is again called $(\Omega, \mathcal{F}, P)$) the sequence $(Y_i)_{i\in\mathbb{N}}$ in (2.1) as follows: let $(U_i^1)_{i\in\mathbb{N}}$, $(U_i^2)_{i\in\mathbb{N}}$, and $(I(A_i))_{i\in\mathbb{N}}$ be three independent i.i.d. sequences. U_i^1, U_i^2 are uniformly distributed on $(0,1)$ and the events A_i have probability $p := P(A_i) = F_Y(0)$. Define

$$Y_i := I(A_i)F_Y^{\leftarrow}(pU_i^1) + I(\Omega \setminus A_i)F_Y^{\leftarrow}(p + (1-p)U_i^2), \tag{2.21}$$

where $F_Y^{\leftarrow}(u) := \inf\{x \in \mathbb{R} \mid F_Y(x) > u\}$, $u \in (0,1)$, is the (right continuous) generalized inverse of F_Y. This construction guarantees that under the new measure Q^n, given by (2.16) and (2.20), the random variables U_i^1, U_i^2, $i \in \mathbb{N}$ remain independent and uniformly distributed on $(0,1)$. The density L_t^n only changes the distribution of $I(A_i)$. Moreover, since $pU_i^1 < p$, we have $F_Y^{\leftarrow}(pU_i^1) \leq 0$, hence $Y_i \leq 0$ on A_i and, similarly, $Y_i > 0$ on $\Omega \setminus A_i$. Based on the construction of the Y_i we define for later use

$$\widetilde{Y}_i := (-F_Y^{\leftarrow}(pU_i^1)) \vee F_Y^{\leftarrow}(p + (1-p)U_i^2) \geq |Y_i|, \quad n \in \mathbb{N}.$$

It is important to note that $\widetilde{Y}_i$ does not depend on A_i.
Denote

$$C_1 = \sup_{s\in[0,t]} \left| \sum_{i=-1}^{-\infty} Y_i g'(s - T_i) \right|, \; c_2 := \sup_{s\in[0,t]} |g'(s)|, \; c_3 := [g(0)(EY_1^+ \wedge EY_1^-)]^{-1}.$$

Define the increasing sequence $(\widetilde{T}_i)_{i\in\mathbb{N}}$ recursively by

$$\widetilde{T}_1 := \frac{\widehat{T}_1}{\alpha + c_3 C_1}, \quad \widetilde{T}_{n+1} := \widetilde{T}_n + \frac{\widehat{T}_{n+1} - \widehat{T}_n}{\alpha + c_3(C_1 + c_2 \sum_{i=1}^n \widetilde{Y}_i)}, \quad n \in \mathbb{N}, \tag{2.22}$$

where the increments of $(\widehat{T}_i)_{i\in\mathbb{N}}$ are (under P) i.i.d. standard exponential random variables. By Lemma 2.4 the point process $(\widetilde{T}_i)_{i\in\mathbb{N}}$ is non-explosive under P. Therefore

$$0 \leq \widetilde{f}(s) := C_1 + c_2 \sum_{\{i\in\mathbb{N}\,:\,\widetilde{T}_i < s\}} \widetilde{Y}_i \leq \widetilde{f}(t) < \infty \quad \forall s \in [0,t] \quad P-\text{a.s.}.$$

Consequently, for $\widetilde{\tau}_n := \inf\{s \geq 0 \mid \widetilde{f}(s) > n\}$, $n \in \mathbb{N}$, we have $P(\widetilde{\tau}_n < t) \to 0$ as $n \to \infty$. By construction $(\widetilde{T}_i)_{i\in\mathbb{N}}$ also does not depend on A_i. We shall show that for each $m \in \mathbb{N}$, $s \in [0,t]$

$$\begin{aligned} &P\left(\widetilde{T}_m > s \mid U_i^1, U_i^2,\; i = 1,\ldots,m, \mathcal{F}_0)\right) \\ &\leq Q^n\left(T_m > s \mid U_i^1, U_i^2,\; i = 1,\ldots,m, \mathcal{F}_0)\right) \quad P\text{-a.s.}, \end{aligned} \tag{2.23}$$

where $\mathcal{F}_0$ as defined in (2.5) is the information available at time 0.
We prove (2.23) by induction on m and start with the argument for $m = 1$. By definition of $\widetilde{T}_1$ and the fact that $\widehat{T}_1$ is standard exponential and independent of $U_1^1, U_1^2, \mathcal{F}_0$ we have for $s \in [0, t]$,

$$\begin{aligned} P(\widetilde{T}_1 > s \mid U_1^1, U_1^2, \mathcal{F}_0) &= P(\widehat{T}_1 > (\alpha + c_3 C_1)s \mid U_1^1, U_1^2, \mathcal{F}_0) \\ &= \exp(-(\alpha + c_3 C_1)s) \quad P\text{-a.s.}. \end{aligned} \tag{2.24}$$

Consider a new measure $\widetilde{Q}^n$ which is constructed similar to Q^n, but in (2.19) f is replaced by f_c; i.e., only jumps from the negative time axes enter. The rate $\lambda_{\widetilde{Q}^n}(u)$ of N under $\widetilde{Q}^n$ is then $\mathcal{F}_0$-measurable; i.e., N is under $\widetilde{Q}^n$ a doubly stochastic Poisson process (a Poisson process with random rate). On the other hand the (conditional) probability of the event $\{T_1 > s\}$ is invariant under Q^n and $\widetilde{Q}^n$ as the rates of N coincide on $[0, T_1]$ (no jump has yet occurred after 0). Using these two facts we obtain

$$\begin{aligned} Q^n(T_1 > s \mid U_1^1, U_1^2, \mathcal{F}_0) &= \widetilde{Q}^n(T_1 > s \mid U_1^1, U_1^2, \mathcal{F}_0) \\ &= \exp\left(- \int_0^s \lambda_{\widetilde{Q}^n}(u) du \right). \end{aligned} \tag{2.25}$$

Notice that the event $\{T_1 > s\}$ is $\widetilde{Q}^n$-independent of U_1^1, U_1^2. By definition of C_1, we have $|f_c(u)| \leq C_1$ for $u \in [0, t]$, and thus by (2.17)

$$\lambda_{\widetilde{Q}^n}(u) \leq \alpha + c_3 C_1 , \quad 0 \leq u \leq t ,$$

which, inserted in (2.25), gives the estimate (2.23).

Now consider $m > 1$. Define

$$\widetilde{\mathcal{F}}_m := \sigma(T_i,\ i = 1, \dots, m-1,\ U_i^1, U_i^2,\ i = 1, \dots, m, \mathcal{F}_0).$$

Similar to (2.24) we have for all $s \in [0, t]$

$$\begin{aligned} &P(\widetilde{T}_m > s \mid \widetilde{T}_i,\ i = 1, \dots, m-1,\ U_i^1, U_i^2,\ i = 1, \dots, m, \mathcal{F}_0) \\ &= \exp\left(-(\alpha + c_3(C_1 + c_2 \sum_{i=1}^{m-1} \widetilde{Y}_i))((s - \widetilde{T}_{m-1}) \vee 0) \right). \end{aligned} \tag{2.26}$$

Consider now a measure $\widetilde{Q}^{n,m}$, again constructed like Q^n, but in (2.19) f is replaced by

$$f_m(u) := \sum_{\{i=m-1, m-2, \dots, 1, -1, \dots \ :\ T_i < u\}} Y_i g'(u - T_i), \quad u \in [0, t] ,$$

i.e., only the jumps from the negative time axes and the first $m-1$ jumps from the positive time axes enter. The point process $(N(T_{m-1}+u)-m+1)_{u\geq 0}$ is under $\widetilde{Q}^{n,m}$ a doubly stochastic Poisson process with rate $(\lambda_{\widetilde{Q}^{n,m}}(T_{m-1}+u))_{u\geq 0}$ which is measurable with respect to $\widetilde{\mathcal{F}}_m$. As the rates of Q^n and $\widetilde{Q}^{n,m}$ coincide on $[0,T_m]$ we obtain for $s\in[0,t]$

$$
\begin{aligned}
Q^n(T_m>s \mid \widetilde{\mathcal{F}}_m) &= \widetilde{Q}^{n,m}(T_m>s \mid \widetilde{\mathcal{F}}_m)\\
&= \exp\left(-\int_0^{(s-T_{m-1})\vee 0}\lambda_{\widetilde{Q}^{n,m}}(T_{m-1}+u)du\right). \qquad (2.27)
\end{aligned}
$$

On $[0,(t-T_{m-1})\vee 0]$ we have $|f_m(T_{m-1}+\cdot)|\leq C_1+c_2\sum_{i=1}^{m-1}|Y_i|$ and thus by (2.17)

$$
|\lambda_{\widetilde{Q}^{n,m}}(T_{m-1}+\cdot)|\leq \alpha+c_3(C_1+c_2\sum_{i=1}^{m-1}|Y_i|)\leq \alpha+c_3(C_1+c_2\sum_{i=1}^{m-1}\widetilde{Y}_i),
$$

which, inserted in (2.27), implies the estimate

$$
\begin{aligned}
&Q^n(T_m>s \mid \widetilde{\mathcal{F}}_m)\\
&\geq \exp\left(-(\alpha+c_3(C_1+c_2\sum_{i=1}^{m-1}\widetilde{Y}_i))((s-T_{m-1})\vee 0)\right). \qquad (2.28)
\end{aligned}
$$

As

$$
\begin{aligned}
&Q^n(T_m>s) \mid U_i^1,U_i^2,\ i=1,\dots,m,\mathcal{F}_0)\\
&=\int_0^\infty Q^n(T_m>s \mid U_i^1,U_i^2,\ i=1,\dots,m,\mathcal{F}_0;\ T_{m-1}\in du),
\end{aligned}
$$

(2.23) follows from (2.26) and (2.28) by induction. Confer also Lemma A.1.1 in Kühn (Kueh02).

As $\widetilde{T}_m$ is P-independent of U_i for $i\geq m$ and, because of the special form of ϕ in (2.19) and the construction of the Y_i in (2.21),the random variable T_m is Q^n-independent of U_i for all $i\geq m$. Therefore (2.23) implies

$$
P\left(\widetilde{T}_m>t \mid U_i^1,U_i^2,\ i\in\mathbb{N},\mathcal{F}_0)\right)\leq Q^n\left(T_m>t \mid U_i^1,U_i^2,\ i\in\mathbb{N},\mathcal{F}_0)\right),
$$

P-a.s.. As the joint distribution of $(U_i^1,U_i^2)_{i\in\mathbb{N}}$ and the random variables generating $\mathcal{F}_0$ does not change under the measure transformation from P to Q^n

and, moreover, $\widetilde{Y}_i \geq |Y_i|$ we obtain that $Q^n(\sup_{0 \leq s \leq t} |f_s| > n) \leq P(\widetilde{f}_t > n)$ and, therefore, $Q^n(\tau_n < t) \leq P(\widetilde{\tau}_n < t) \to 0$. $\square$

Remark 2.5 A heuristic explanation, why S is arbitrage-free, goes as follows. Although an investor could profit by the stochastic drift $f(s)\,ds$, there remains the risk dZ_s that consists of random jumps. Therefore, it cannot be controlled as effectively as for FBM, which has continuous sample paths. In contrast, the fractional binary market model in Sottinen (Sot01) obviously allows for arbitrage as – given the history of the process – it can happen that the discounted stock price increases with probability one. The same phenomena occurs in our model, when setting $g(0) = 0$.

Remark 2.6 Q is obviously not unique. The unities in (2.19) can be replaced by any other element of $\mathbb{R}_+ \setminus \{0\}$.

Coming back to model (1.2) we add to S an independent Brownian motion σB, $\sigma \geq 0$. To transfer an additive to a geometric model, there are two common approaches in mathematical finance: the Doléans-Dade-exponential and the ordinary exponential of the process S. In the first case the price process of the asset satisfies the SDE

$$dP(t) = P(t-)\left(dS(t) + \sigma\, dB(t)\right), \quad t \geq 0, \qquad P(0) = p_0 > 0. \tag{2.29}$$

$(S(s) + \sigma B(s))_{s \in [0,t]}$ is a local Q-martingale and hence $(P(s))_{s \in [0,t]}$, cf. e.g. Protter (Pro90) Ch. III, Theorem 17. If $\Delta S > -1$ then $(P(s))_{s \in [0,t]}$ is positive. In the second case, i.e. setting

$$\widetilde{P}(t) = p_0 \exp\left\{S(t) + \sigma B(t) - \frac{\sigma^2}{2} t\right\}, \quad t \geq 0, \tag{2.30}$$

by Itô's formula, the price process satisfies the SDE

$$d\widetilde{P}(t) = \widetilde{P}(t-)\left(dS(t) + e^{\Delta S(t)} - 1 - \Delta S(t) + \sigma\, dB(t)\right), \quad t \geq 0,$$

and, by (2.7),

$$d\widetilde{P}(t) = \widetilde{P}(t-)\left(e^{\Delta Z(t)} - 1 + df(t) + \sigma\, dB(t)\right), \quad t \geq 0.$$

Thus, condition (2.18) has to be replaced by

$$\int_{\mathbb{R}} \left(e^{g(0)x} - 1\right) \lambda_Q(s, dx) = -f(s), \quad s \geq 0. \tag{2.31}$$

Assume that $Ee^{|g(0)Y_1|} < \infty$. Then, (2.31) can be achieved by setting

$$\widetilde{\phi}(s,x) := \begin{cases} 1 & : f(s) < 0,\ g(0)x < 0 \\ \dfrac{1 - Ee^{-[g(0)Y_1]^-}}{Ee^{[g(0)Y_1]^+} - 1} - \dfrac{f(s)}{\alpha\left(Ee^{[g(0)Y_1]^+} - 1\right)} & : f(s) < 0,\ g(0)x \geq 0 \\ 1 + \dfrac{f(s)}{\alpha\left(1 - Ee^{-[g(0)Y_1]^-}\right)} & : f(s) \geq 0,\ g(0)x < 0 \\ \dfrac{1 - Ee^{-[g(0)Y_1]^-}}{Ee^{[g(0)Y_1]^+} - 1} & : f(s) \geq 0,\ g(0)x \geq 0 \end{cases}$$

With the same arguments as in the proof of Theorem 2.3 one verifies that by plugging $\widetilde{\phi}$ into (2.16) one obtains a measure $\widetilde{Q} \sim P$, under which $\widetilde{P}$ becomes a local martingale.

3 Asymptotic theory

From now on we assume that g is normalized regularly varying in ∞ with index $\gamma \in (-1/2\,,\,1/2)$, i.e. $g : \mathbb{R}_+ \to \mathbb{R}_+$ is continuously differentiable and $\lim_{u\to\infty} ug'(u)/g(u) = \gamma$, cf. Bingham, Goldie, and Teugels (BGT87). This implies that g' is regularly varying with index $\gamma - 1$ and thus (2.2) is satisfied for any $\varepsilon \in (0, 1/2 - \gamma)$.

Example 3.1 Examples for such normalized regularly varying functions are
$g(u) = (u+1)^\gamma$, $u \geq 0$, and
$g(u) = (u+1)^\gamma \ln(u+2)$ or $g(u) = (u+1)^\gamma / \ln(u+2)$, $u \geq 0$.

We introduce for $t > 0$ the rescaled process

$$S_x(t) = \frac{S(xt)}{\sigma(t)}, \quad x \in [0,\infty),$$

where $\sigma^2(t) = \mathrm{Var}\,(S(t))$ and show weak convergence to a FBM.

Theorem 3.2 *Let B^H be a FBM with Hurst parameter $H = \gamma + 1/2$ for $\gamma \in (0\,,\,1/2)$. Then*

$$S_{\cdot}(t) \xrightarrow{d} B^H, \quad t \to \infty,$$

where the convergence holds in $D[0,\infty)$ equipped with the metric of uniform convergence on compacta.

Remark 3.3 If $\gamma \in (-1/2\,,\,0]$ we still have convergence of the finite-dimensional distributions. This follows from Steps 1-2 in the proof of Theorem 3.2, which go through for all $\gamma \in (-1/2\,,\,1/2)$.
On the other hand, for $\gamma < 0$, by taking innovations with $E|Y_1|^{2+\varepsilon} = \infty$ for $\varepsilon > 0$ arbitrarily small, it is easy to construct an example with $\max_{x\in[0,1]} |S_x(t)| \xrightarrow{d} \infty, \quad t \to \infty$. Therefore $(S_.(t))_{t\in\mathbb{R}_+\setminus\{0\}}$ need not be tight for such γ.
In the case $\gamma \le -1/2$ the shots $g(t-T_i)Y_i$ regress very fast when time is going on. Therefore the effect of a single shot on the accumulated process can be significant. The Lindeberg condition is violated and, in general, there is no Gaussian limit.

Proof of Theorem 3.2. The limit process has continuous sample paths. Therefore, by Theorem 6.6 in Billingsley (Bil99) we can equivalently consider weak convergence with respect to the Skorohod d^o_∞-metric on $D[0,\infty)$. For a definition of d^o_∞ see e.g. (16.4) in Billingsley (Bil99). By Billingsley (Bil99), Theorem 16.7 and Theorem 13.1, we have to show weak convergence of the finite-dimensional distributions and tightness of $\left(S_.(t)|_{[0,M]}\right)_{t\in\mathbb{R}_+}$ for each $M \in \mathbb{R}_+$.

Step 1: By Campbell's theorem (cf. Daley and Vere-Jones (DVJ88)) we have for $0 \le s \le t$

$$\begin{aligned}\operatorname{Cov}(S(s),S(t)) &= \alpha\int_0^s EX_1(u)X_1(u+t-s)\,du\\ &+\alpha\int_0^\infty E\left[X_1(s+u)-X_1(u)\right]\left[X_1(t+u)-X_1(u)\right]\,du.\end{aligned}\tag{3.1}$$

For the model of multiplicative shots we obtain and $0 \le x \le y$ we obtain

$$\frac{\operatorname{Cov}(S(xt),S(yt))}{\sigma^2(t)} = \frac{\displaystyle\int_0^x \frac{g(ut)g((u+y-x)t)}{g(t)^2}\,du + \int_0^\infty \frac{[g((x+u)t)-g(ut)]\,[g((y+u)t)-g(ut)]}{g(t)^2}\,du}{\displaystyle\int_0^1 \frac{g(ut)^2}{g(t)^2}\,du + \int_0^\infty \frac{[g((1+u)t)-g(ut)]^2}{g(t)^2}\,du}.$$

We show that the right-hand side converges as $t \to \infty$ to

$$\frac{\int_0^x u^\gamma (u+y-x)^\gamma \, du + \int_0^\infty [(x+u)^\gamma - u^\gamma]\,[(y+u)^\gamma - u^\gamma] \, du}{\frac{1}{2\gamma+1} + \int_0^\infty [(1+u)^\gamma - u^\gamma]^2 \, du},$$

which is the covariance function of $B^{\gamma+1/2}$. For fixed $u \in \mathbb{R}_+ \setminus \{0\}$ convergence is obvious and by Potter bounds the integrals on compacta converge. But, for the integrals on $(0,\infty)$ we need integrable dominating functions. Let $\varepsilon \in (0, 1/2-\gamma)$. As g' is regularly varying with index $\gamma - 1$, the function $h(s) := g'(s)s^{1-\gamma-\varepsilon}$, $s \geq 0$, is regularly varying with index $-\varepsilon$. Therefore, $h(\lambda t)/h(t)$ converges to $\lambda^{-\varepsilon}$ *uniformly* in $\lambda \in [1,\infty)$ as $t \to \infty$, cf. Bingham, Goldie, and Teugels (BGT87), Theorem 1.5.2. This implies that for all $\lambda \in [1,\infty)$ and t large enough

$$\frac{g'(\lambda t)}{g'(t)} = \lambda^{-1+\gamma+\varepsilon} \frac{h(\lambda t)}{h(t)} \leq \lambda^{-1+\gamma+\varepsilon}\left(1+\lambda^{-\varepsilon}\right) \leq 2\lambda^{-1+\gamma+\varepsilon}.$$

Therefore, we have for some $\xi \in (u, x+u)$

$$\left|\frac{g((x+u)t) - g(ut)}{g(t)}\right| = \left| xt\frac{g'(\xi t)}{g(t)}\right| = \left| x\frac{tg'(t)}{g(t)}\frac{g'(\xi t)}{g'(t)}\right| \leq 2x|\gamma| u^{-1+\gamma+\varepsilon},$$

for all $u \geq 1, t \geq t_0$. As $-1+\gamma+\varepsilon < -1/2$ we have the required integrable dominating function. Hence we have shown for $0 \leq x \leq y$ that

$$\lim_{t\to\infty} \frac{\mathrm{Cov}(S(xt), S(yt))}{\sigma^2(t)} = \mathrm{Cov}(B^{\gamma+1/2}(x), B^{\gamma+1/2}(y)). \tag{3.2}$$

Step 2: For $d \in \mathbb{N}$ let $\lambda_i \in \mathbb{R}$, $i = 1,\ldots,d$, $0 \leq x_1 < \ldots < x_d < \infty$, and consider

$$Z(t) := \sum_{i=1}^d \lambda_i S_{x_i}(t), \quad t \geq 0.$$

By (3.2) the variance of $Z(t)$ converges to those of $\sum_{i=1}^d \lambda_i B^{\gamma+1/2}(x_i)$. Now we verify a condition (similar to the Lindeberg condition) for $Z(t)$ to ensure that it converges to a normal limit. $Z(t)$ has zero mean and, if not all λ_i vanish, $\mathrm{Var}(Z(t)) \to c$ as $t \to \infty$ for some $c > 0$. Hence by Theorem 3 in Lane (Lan84) we have to show that for every $\varepsilon > 0$

$$\sigma(t)^{-2} \int\limits_{\varepsilon\sigma(t)}^{\infty} y \left[\int\limits_{0}^{x_d t} P\left(\left| Y_1 \sum_{i=1}^{d} \lambda_i g(x_i t - u) \right| > y \right) du \right.$$
$$\left. + \int\limits_{0}^{\infty} P\left(\left| Y_1 \sum_{i=1}^{d} \lambda_i \left(g(u + x_i t) - g(u) \right) \right| > y \right) du \right] dy \to 0$$

as $t \to \infty$. It is sufficient to verify that for every λ, $\varepsilon > 0$ and $t \to \infty$

$$\sigma(t)^{-2} \int\limits_{\varepsilon\sigma(t)}^{\infty} y \int\limits_{0}^{\infty} P\left(\lambda \left| Y_1 \left(g(u+t) - g(u) \right) \right| > y \right) du\, dy \to 0 \tag{3.3}$$

and

$$\sigma(t)^{-2} \int\limits_{\varepsilon\sigma(t)}^{\infty} y \int\limits_{0}^{t} P\left(\lambda \left| Y_1 g(u) \right| > y \right) du\, dy \to 0. \tag{3.4}$$

Ad (3.3): we have

$$\begin{aligned}
&\sigma(t)^{-2} \int\limits_{\varepsilon\sigma(t)}^{\infty} y \int\limits_{0}^{\infty} P\left(\lambda \left| Y_1 \left(g(u+t) - g(u) \right) \right| > y \right) du\, dy \\
&= \sigma(t)^{-2} \int\limits_{0}^{\infty} \int\limits_{\varepsilon\sigma(t)}^{\infty} y P\left(\lambda \left| Y_1 \left(g(u+t) - g(u) \right) \right| > y \right) dy\, du \\
&= \frac{1}{2} \sigma(t)^{-2} \int\limits_{0}^{\infty} E\left(\left(\lambda \left| Y_1 \left(g(u+t) - g(u) \right) \right| - \varepsilon\sigma(t) \right)^{+} \right)^{2} du \\
&= \frac{1}{2} \frac{t g(t)^2}{\sigma(t)^2} \int\limits_{0}^{\infty} E\left(\left(\lambda \left| Y_1 \frac{g((u+1)t) - g(ut)}{g(t)} \right| - \varepsilon \frac{\sigma(t)}{g(t)} \right)^{+} \right)^{2} du. \qquad (3.5)
\end{aligned}$$

Since

$$\begin{aligned}
&\int\limits_{0}^{\infty} E\left(\lambda Y_1 \frac{g((u+1)t) - g(ut)}{g(t)} \right)^{2} du \\
&= \lambda^2 E Y_1^2 \int\limits_{0}^{\infty} \left(\frac{g((u+1)t) - g(ut)}{g(t)} \right)^{2} du < \infty,
\end{aligned}$$

we have an integrable function that dominates the integrand in the last line of (3.5). From (3.1) we see that $\sigma^2(t) = c g(t)^2 t \,(1 + o(1))$ as $t \to \infty$ for some $c > 0$. Therefore, dominated convergence implies that the last line of (3.5) converges to zero as $t \to \infty$. (3.4) can be proven in a similar way. Thus $Z(t) \xrightarrow{d} \sum_{i=1}^{d} \lambda_i B^{\gamma+1/2}(x_i)$ as $t \to \infty$, and the Cramér-Wold device yields the convergence of the finite-dimensional distributions.

Step 3: Finally, we check tightness.
The family of processes $((\sigma(t)^{-1} g(0) \sum_{i=1}^{N(\cdot t)} Y_i) \,|_{[0,M]})_{t\geq 0}$ is obviously tight. Thus we can replace g by $\tilde{g} = g - g(0)$. Since the increments of S are stationary we have for $0 \leq x \leq y$ and all $t \geq 0$

$$E\left(S_y(t) - S_x(t)\right)^2 = \frac{\sigma^2\left((y-x)t\right)}{\sigma^2(t)}.$$

Due to (3.2) σ^2 is regularly varying with index $1 + 2\gamma$. Therefore, $h(s) := \sigma^2(s)/s^{1+\gamma}$ is regularly varying with index $\gamma > 0$, and h is bounded near zero, which can be seen from the following calculation:

$$\begin{aligned}
\sigma^2(s) &= EY_1^2 \left\{ \int_0^s \tilde{g}^2(u)\,du + \int_0^\infty \left[\tilde{g}(s+u) - \tilde{g}(u)\right]^2 du \right\} \\
&\leq EY_1^2 \left\{ \int_0^s \left(\int_0^u \tilde{g}'(v)\,dv \right)^2 du + s^2 \int_0^\infty \sup_{\xi \geq u} \left[\tilde{g}'(\xi)\right]^2 du \right\} \\
&\leq EY_1^2 \left\{ \frac{1}{3} s^3 \sup_{0<u<\infty} \left[\tilde{g}'(u)\right]^2 + s^2 \int_0^\infty \sup_{\xi \geq u} \left[\tilde{g}'(\xi)\right]^2 du \right\}.
\end{aligned}$$

Therefore, $h((y-x)t)/h(t)$ converges to $(y-x)^\gamma$ for $t \to \infty$ *uniformly* in $x < y$ on compact subsets of $\mathbb{R}_+$, cf. Bingham, Goldie, and Teugels (BGT87), Theorem 1.5.2. This implies that for each $M > 0$ and $t \geq t_M$

$$\frac{\sigma^2\left((y-x)t\right)}{\sigma^2(t)} \leq (M^\gamma + 1)(y-x)^{1+\gamma}, \quad 0 \leq x < y \leq M.$$

This (together with Cauchy-Schwarz's inequality) ensures the tightness condition (13.14) in Billingsley (Bil99). $\square$

4 Conclusion

We have constructed Poisson shot noise processes whose finite-dimensional distributions are close to those of FBM, but which lead to arbitrage-free models for stock prices. By way of contrast, if the shots $(X_i)_{i\in\mathbb{Z}\setminus\{0\}}$ have no jumps at zero and there is no additional Brownian noise σB in (2.29) and (2.30), respectively, our model obviously allows for arbitrage, even with so-called "simple" trading strategies.

These results can also be considered as supplements to recent work of Cheridito (Ch00),(Ch01). He has excluded arbitrage from FBM by changing slightly the convolution kernel in the Mandelbrot-Van Ness representation of FBM or, alternatively, he considered, for $H \in (3/4\,,\,1]$, the process $B^H + \varepsilon B^{1/2}$ ($\varepsilon > 0$ arbitrary small) instead of just B^H. This leads to complete models whereas our models are incomplete.

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