Autoregressions Generated by the Tent Map

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Abstract

It is well-known (see e.g. Tong, 1990, Gouriéroux, 1997) that if X_0 has the uniform distribution function U on $[0,1]$, then the sequence of iterates $\{X_n = g(X_{n-1})\}$ of the symmetric tent map g from $[0,1]$ onto $[0,1]$, is a strictly stationary Markov process with marginal distribution function U. It is also easy to show, using the symmetry of the map, that $\{X_n\}$ is white noise. In this note we show that if the symmetric tent map is replaced by a skewed tent map, then the sequence $\{X_n\}$ is a strictly stationary autoregression of order 1 with coefficient $\phi = (2/s) - 1$, where $s \in (1,\infty)$ is the right-derivative of the tent map at 0. An AR(1) process with uniform marginal distributions and arbitrary coefficient $\phi \in (-1,1)$ can thus be generated by computing the iterates with $s = 2/(\phi+1)$. For the symmetric map $s=2$ and $\phi = 0$.

Keywords: Non-linear dynamical system, chaos, nonlinear time series, linear prediction.

1 Introduction

The tent map with parameter $s \in (1,\infty)$ is the function

$$g(x) = sxI_{[0,1/s]}(x) + \frac{s}{s-1}(1-x)I_{[1/s,1]}(x), \quad x \in [0,1], \tag{1}$$

where I_A denotes the indicator function of the set A. It is easy to check that if X_0 has the uniform distribution on $[0,1]$ (denoted $X_0 \sim U$) and if

$$X_n = g(X_{n-1}),\ n = 1, 2, \dots\ , \tag{2}$$

then $\{X_n\}$ is a Markov chain and $X_n \sim U$ for all $n \in \{0, 1, 2, \dots\}$, so that $\{X_n\}$ is strictly (and weakly) stationary. Equation (2) can also be expressed as $X_n = g^{(n)}(X_0)$, where $g^{(n)}$ denotes the n^{th} iterate of the function g.

We shall show that the sequence $\{X_n\}$ is also an AR(1) process satisfying the equations,

$$X_n - 0.5 = \phi(X_{n-1} - 0.5) + Z_n, \tag{3}$$

where $\phi = (2/s) - 1$ and $\{Z_n\}$ is an uncorrelated sequence of random variables with mean zero and variance $(1 - \phi^2)/12$.

The minimum mean squared error *linear* predictor of X_{n+h} in terms of $1, X_0, \dots, X_n$ is thus $1/2 + \phi^h(X_n - 1/2)$, with mean squared error $(1 - \phi^{2h})/12$, while the corresponding minimum mean squared error predictor is $g^{(h)}(X_n)$, with zero mean squared error.

2 Properties of the Iterated Tent Map

The tent map has derivatives $g'(x) = s$ on $(0, 1/s)$ and $g'(x) = t = -s/(s-1)$ on $(1/s, 1)$. From the recursions

$$g^{(n)\prime}(x) = g'(x) g^{(n-1)\prime}(g(x))$$

(valid at points x where the derivatives exist) it follows that the graph of $y = g^{(k)}(x)$ is piecewise linear, consisting of 2^k lines, each joining points with $y = 0$ and $y = 1$. This set of 2^k lines is the union of disjoint subsets S_j, $j = 0, \dots, k$, with each of the $\binom{k}{j}$ lines in S_j having slope $s^{k-j} t^j$. These are the only properties of $g^{(k)}$ needed in the calculations which follow. Note that $1/s - 1/t = 1$ and $1/s + 1/t = (2/s) - 1 = \phi$, with ϕ defined as in Section 1.

In order to establish (3), we first compute the orthogonal projection $P(h|e)$ of the function $h(x) = g^{(k)}(x) - 0.5$ on the linear span of the function $e(x) = x - 0.5$ (in the Hilbert space $L^2([0,1], \mathcal{B}, U)$, where $\mathcal{B}$ denotes the Borel subsets of $[0, 1]$.) To do this we need the following proposition. The crucial (and somewhat surprising) feature of the result is that it depends only on the slope parameter m and not on the location parameter a.

Proposition. *If e and f are the functions in $L^2([0,1], \mathcal{B}, U)$ defined by $e(x) = x - 0.5$ and $f(x) = [-0.5 + m(x - a)] I_{[a, a+1/m]}$, where $0 \le a < a + 1/m \le 1$, then the orthogonal projection of f on the span of e is*

$$P(f|e) = e/m^2.$$

Proof. A straightforward calculation shows that $\int_0^1 (f(x) - m^{-2}e(x))e(x)dx = 0$.

Corollary. *If* $f(x) = [0.5 - m(x-a)]I_{[a,a+1/m]}$, *where* $0 \le a < a+1/m \le 1$, *then* $P(f|e) = -e/m^2$.

Now consider the 2^k lines which together constitute the graph of the piecewise linear function $g^{(k)}$ on $[0,1]$. The $\binom{k}{j}$ lines in S_j all have slope $s^{k-j}t^j,\ j = 0, \ldots, k$. Corresponding to this decomposition of the graph of $g^{(k)}$ we can express the function $h = g^{(k)} - 0.5$ as the sum of 2^k functions having the form of f as in either the proposition or the corollary (depending on whether the slope $s^{k-j}t^j$ is positive or negative, i.e. on whether j is even or odd). By the linearity of the projection operator, the projection of the function h onto the span of e is the sum of the projections of these 2^k functions. But by the proposition and corollary these are $e/(s^{2(k-j)}t^{2j})$ for j even and $-e/(s^{2(k-j)}t^{2j})$ for j odd. Hence

$$P(h|e) = \left[\sum_{j=0}^{k} \binom{k}{j} \left(\frac{1}{s^2}\right)^{k-j} \left(-\frac{1}{t^2}\right)^{j} \right] e = (s^{-2} - t^{-2})^k e.$$

Since $1/s - 1/t = 1$, we conclude that

$$P(h|e) = \phi^k e, \tag{4}$$

where $\phi = 1/s + 1/t = (2/s) - 1$.

3 The Autoregression

From (4) it follows at once that if $X_0 \sim U$ and $X_k = g^{(k)}(X_0),\ k = 1, 2, \ldots,$ then

$$X_k - 0.5 = \phi^k (X_0 - 0.5) + W_k,$$

where $E[W_k(X_0 - 0.5)] = 0$. Hence

$$\mathrm{cov}(X_k, X_0) = \phi^k \mathrm{var}(X_0) = \phi^k/12, \quad k = 0, 1, 2, \ldots, \tag{5}$$

and, since $\{X_n\}$ is weakly stationary, it has the autocorrelation function,

$$\rho(h) = \mathrm{corr}(X_{n+h}, X_n) = \phi^{|h|},\ h = 0, \pm 1, \pm 2, \ldots. \tag{6}$$

The latter equation implies (see e.g. Brockwell and Davis, 1991) that the best linear predictor of X_{n+1} in terms of $1, X_0, \ldots, X_n$ is $0.5 + \phi(X_n - 0.5)$

and hence that the sequence of prediction errors $\{X_{n+1} - 0.5 - \phi(X_n - 0.5)\}$ is an uncorrelated, zero-mean sequence with constant variance. Defining $Z_n := X_n - 0.5 - \phi(X_{n-1} - 0.5),\ n = 1, 2, \ldots$, we immediately obtain the autoregressive representation,

$$X_n - 0.5 = \phi(X_{n-1} - 0.5) + Z_n, \tag{7}$$

where $\{Z_n\}$ is an uncorrelated sequence of zero-mean random variables with variance σ^2. Since the variance of the stationary process defined by (7) is $\sigma^2/(1 - \phi^2)$ and $\mathrm{var}(X_n) = \mathrm{var}(X_0) = 1/12$, the variance of Z_n is $\sigma^2 = (1 - \phi^2)/12$ as claimed in (3).

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