OPTIMAL PORTFOLIOS WITH BOUNDED CAPITAL-AT-RISK *

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We consider some continuous-time Markowitz type portfolio problems that consist of maximizing expected terminal wealth under the constraint of an upper bound for the Capital-at-Risk. In a Black-Scholes setting we obtain closed form explicit solutions and compare their form and implications to those of the classical continuous-time mean-variance problem. We also consider more general price processes which allow for larger fluctuations in the returns.

Keywords: Black-Scholes model, Capital-at-Risk, generalized inverse Gaussian diffusion, jump diffusion, portfolio optimization, Value-at-Risk.

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1 Introduction

It seems to be common wisdom that long term stock investment leads to an almost sure gain over locally riskless bond investments. In the long run stock indices are growing faster than riskless rates, despite the repeated occurrence of stock market declines. The conventional wisdom therefore holds that the more distant the planning horizon, the greater should be one's wealth in risky assets. One of our main findings presented in this paper will be the demonstration that there is indeed a reasonable portfolio problem with a solution that supports this empirical observation.

Traditional portfolio selection as introduced by Markowitz (1959) and Sharpe (1964) is based on a mean-variance analysis. This approach cannot explain the above phenomenon: the use of the variance as a risk measure of an investment leads to a decreasing proportion of risky assets in a portfolio, when the planning horizon increases (see Example 2.10).

In recent years certain variants of the classical Markowitz mean-variance portfolio selection criterion have been suggested. Such alternatives are typically based on the notion of downside risk concepts such as *lower partial moments*. The lower partial moment of order n is defined as

$$\mathrm{LPM}_n(x) = \int_{-\infty}^{x} (x-r)^n dF(r)\,, \quad x \in \mathbb{R}\,, \tag{1.1}$$

where F is the distribution function of the portfolio return. Examples can be found in Fishburn (1977) or Harlow (1991), who suggested for instance the shortfall probability ($n=0$), the expected target shortfall ($n=1$), the target semi-variance ($n=2$), and target semi-skewness ($n=3$). Harlow (1991) also discusses some practical consequences of various downside risk measures.

In this paper we concentrate on the Capital-at-Risk (CaR) as a replacement of the variance in portfolio selection problems. We think of the CaR as the capital reserve in equity. The CaR is defined via the Value-at-Risk; i.e. a low quantile (typically the 5%- or 1%-quantile) of the profit-loss distribution of a portfolio; see e.g. Jorion (1997). The CaR of a portfolio is then commonly defined as the difference between the mean of the profit-loss distribution and the VaR. VaR has become the most prominent risk measure during recent years. Even more, the importance of VaR models continues to grow since regulators accept these models as a basis for setting capital requirements for market risk exposure. If the profit-loss distribution of a portfolio is normal with mean μ and variance σ^2, then the CaR of the portfolio based on the α-quantile (e.g., $\alpha = 0.05$ or $\alpha = 0.01$) is

$$\mathrm{CaR} = \mu - (\mu - \sigma z_\alpha)\,, \tag{1.2}$$

where z_α is the α-quantile of the standard normal distribution and σ is positive. In this paper we will use another definition of the CaR.

The crucial point in the application of CaR models for setting capital requirement is the determination of reliable and accurate figures for the VaR, especially for non-normal cases. Consequently, VaR has attracted attention from a statistical point of view; e.g., see Embrechts, Klüppelberg and Mikosch (1997) for estimation via extreme value methods and further references, see Emmer, Klüppelberg and Trüstedt (1998) for an example.

In the context of hedging, VaR has been considered as a risk measure by Föllmer and Leukert (1999); see also Cvitanic and Karatzas (1999). They replace the traditional "hedge without risk" (perfect hedge) which typically only works in a complete market setting by a "hedge with small remaining risk" (so-called quantile-hedging). This concept can also deal with incomplete markets. In contrast to our problem, their main task consists of approximating a given claim. Surprisingly, the existence of that target wealth makes their problem more tractable than ours.

In a discrete world Zagst and Kehrbaum (1998) investigate the problem of optimizing portfolios under a limited CaR from a practical point of view, they solve the problem by numerical approximation, and they present a case study. This work is continued in Scheuenstuhl and Zagst (1998). Under a mean-variance and shortfall preference structure for the investor, they obtain optimal portfolios consisting of stocks and options via an approximation method.

One aim of our paper is to show that a replacement of the variance by the CaR in a continuous-time Markowitz-type model resolves exactly the above-mentioned contradiction between theory and empirical facts. Furthermore, we aim at closed form solutions and an economic interpretation of our results. In a Gaussian world, represented by a Black-Scholes market, possibly enriched with a jump component, the mean-CaR selection procedure leads to rather explicit solutions for the optimal portfolio. It is, however, not surprising that as soon as we move away from the Gaussian world, the optimization problem becomes analytically untractable.

The paper is organized as follows. In Section 2 we highlight the consequences of the introduction of the CaR as risk measure in a simple Black-Scholes market where we can obtain explicit closed form solutions. We also examine consequences for the investor when introducing CaR in a portfolio optimization problem. This approach indeed supports the above-mentioned market strategy that one should always invest in stocks for long-term investment.

Section 3 is devoted to the study of the portfolio problem for more general models of the stock price. As prototypes of models to allow for larger fluctuations than pure Gaussian models, we study jump diffusions and generalized inverse Gaussian diffusion processes. This also shows how the solution of the problem becomes much more involved when the Black-Scholes assumptions are abandoned. In particular, we show how the optimal portfolio under a CaR constraint reacts to the possibility of jumps. In the generalized inverse Gaussian diffusion setting even the problem formulation becomes questionable as we cannot ensure a finite expected terminal wealth of the optimal portfolio. We give an approximate solution, which allows for some interpretation, and also a numerical algorithm.

2 Optimal portfolios and Capital-at-Risk in the Black-Scholes setting

In this section, we consider a standard Black-Scholes type market consisting of one *riskless bond* and several *risky stocks*. Their respective prices $(P_0(t))_{t\geq 0}$ and $(P_i(t))_{t\geq 0}$ for $i = 1,\dots,d$ evolve according to the equations

$$\begin{array}{rclrcl} dP_0(t) & = & P_0(t) r dt\,, & P_0(0) & = & 1\,, \\ dP_i(t) & = & P_i(t)\left(b_i dt + \sum_{j=1}^d \sigma_{ij} dW_j(t)\right)\,, & P_i(0) & = & p_i\,, \quad i = 1,\dots,d\,. \end{array}$$

Here $W(t) = (W_1(t),\dots,W_d(t))'$ is a standard d-dimensional Brownian motion, $r \in \mathbb{R}$ is the *riskless interest rate*, $b = (b_1,\dots,b_d)'$ the vector of *stock-appreciation rates* and $\sigma = (\sigma_{ij})_{1\leq i,j\leq d}$ is the matrix of *stock-volatilities*. For simplicity, we assume that σ is invertible and that $b_i \geq r$ for $i = 1,\dots,d$.

Let $\pi(t) = (\pi_1(t),\dots,\pi_d(t))' \in \mathbb{R}^d$ be an admissible portfolio process, i.e. $\pi_i(t)$ is the fraction of the wealth $X^\pi(t)$, which is invested in asset i (see Korn (1997), Section 2.1 for relevant definitions). Denoting by $(X^\pi(t))_{t\geq 0}$ the *wealth process*, it follows the dynamic

$$(2.1)\; dX^\pi(t) = X^\pi(t)\left\{((1-\pi(t)'\underline{1})r + \pi(t)'b)dt + \pi(t)'\sigma dW(t)\right\}\,, \quad X^\pi(0) = x\,,$$

where $x \in \mathbb{R}$ denotes the initial capital of the investor and $\underline{1} = (1,\dots,1)'$ denotes the vector (of appropriate dimension) having unit components. The fraction of the investment in the bond is $\pi_0(t) = 1 - \pi(t)'\underline{1}$. Throughout the paper, we restrict ourselves to constant portfolios $\pi(t) = \pi = (\pi_1,\dots,\pi_d)$ for all $t \in [0,T]$. This means that the fractions in the different stocks and the bond remain constant on $[0,T]$. The advantage of this is two-fold: first we obtain, at least in a Gaussian setting, explicit results; and furthermore, the economic interpretation of the mathematical results is comparably easy. Finally, let us mention that for many other portfolio problems the optimal portfolios are constant ones (see Sections 3.3. and 3.4 of Korn (1997)). It is also important to point out that following a constant portfolio process does not mean that there is no trading. As the stock prices evolve randomly one has to trade at every time instant to keep the fractions of wealth invested in the different securities constant. Thus, following a constant portfolio process still means one must follow a dynamic trading strategy.

Standard Itô integration and the fact that $Ee^{sW(1)} = e^{s^2/2}$, $s \in \mathbb{R}$, yield the following explicit formulae for the wealth process for all $t \in [0,T]$ (see e.g. Korn and Korn (2000)).

$$X^\pi(t) = x \exp\left((\pi'(b - r\underline{1}) + r - \|\pi'\sigma\|^2/2)t + \pi'\sigma W(t)\right) , \tag{2.2}$$

$$E(X^\pi(t)) = x \exp\left((\pi'(b - r\underline{1}) + r)t\right) , \tag{2.3}$$

$$\operatorname{var}(X^\pi(t)) = x^2 \exp\left(2(\pi'(b - r\underline{1}) + r)t\right) \left(\exp(\|\pi'\sigma\|^2 t) - 1\right) . \tag{2.4}$$

The norm $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$.

Definition 2.1 (Capital-at-Risk)
Let x be the initial capital and T a given time horizon. Let z_α be the α-quantile of the standard normal distribution. For some portfolio $\pi \in \mathbb{R}^d$ and the corresponding terminal wealth $X^\pi(T)$, the α-quantile of $X^\pi(T)$ is given by

$$\rho(x, \pi, T) = x \exp\left((\pi'(b - r\underline{1}) + r - \|\pi'\sigma\|^2/2)T + z_\alpha \|\pi'\sigma\| \sqrt{T}\right) ,$$

i.e., $\rho(x,\pi,T) = \inf\{z \in \mathbb{R} : P(X^\pi(T) \le z) \ge \alpha\}$. Then we define

$$\begin{aligned} \mathrm{CaR}(x,\pi,T) &= x\exp(rT) - \rho(x,\pi,T) \\ &= x\exp(rT) \\ &\quad \times \left(1 - \exp((\pi'(b - r\underline{1}) - \|\pi'\sigma\|^2/2)T + z_\alpha\|\pi'\sigma\|\sqrt{T})\right) \end{aligned} \tag{2.5}$$

the Capital-at-Risk *of the portfolio π (with initial capital x and time horizon T).* □

Assumption 2.2 *To avoid (non-relevant) subcases in some of the following results we always assume $\alpha < 0.5$ which leads to $z_\alpha < 0$.*

Remark 2.3 (i) Our definition of the Capital-at-Risk limits the possibility of excess losses over the riskless investment.

(ii) We typically want to have a positive CaR (although it can be negative in our definition as the examples below will show) as the upper bound for the "likely losses" (in the sense that $(1-\alpha) \times 100\%$ of occurring "losses" are smaller than $\mathrm{CaR}(x,\pi,T)$) compared to the pure bond investment. Further, we concentrate on the actual amount of losses appearing at the time horizon T. This is in line with the mean-variance selection procedure enabling us to directly compare the results of the two approaches; see below.

In the following it will be convenient to introduce the function $f(\pi)$ for the exponent in (2.5), that is

$$f(\pi) := z_\alpha \|\pi'\sigma\|\sqrt{T} - \|\pi'\sigma\|^2 T/2 + \pi'(b - r\underline{1})T , \quad \pi \in \mathbb{R}^d . \tag{2.6}$$

By the obvious fact that

$$f(\pi) \overset{\|\pi'\sigma\|\to\infty}{\longrightarrow} -\infty$$

we have

$$\sup_{\pi\in\mathbb{R}^d} \mathrm{CaR}(x,\pi,T) = x\exp(rT)\,;$$

i.e., the use of extremely risky strategies (in the sense of a high norm $\|\pi'\sigma\|$) can lead to a CaR which is close to the total capital. The computation of the minimal CaR is done in the following proposition.

(iii) Note how crucial the definition of CaR depends on the assumption of a constant portfolio process. Moving away from this assumption makes the problem untractable. In particular, $\rho(x,\pi,T)$ is nearly impossible to obtain for a general random $\pi(.)$. $\square$

Proposition 2.4 *Let* $\theta = \|\sigma^{-1}(b-r1)\|$.
(a) If $b_i = r$ *for all* $i = 1,\dots,d$, *then* $f(\pi)$ *attains its maximum for* $\pi^* = 0$ *leading to a minimum Capital-at-Risk of* $\mathrm{CaR}(x,\pi^*,T) = 0$.
(b) If $b_i \neq r$ *for some* $i \in \{1,\dots,d\}$ *and*

$$\theta\sqrt{T} < |z_\alpha|\,, \tag{2.7}$$

then the minimal CaR *equals zero and is only attained for the pure bond strategy.*
(c) If $b_i \neq r$ *for some* $i \in \{1,\dots,d\}$ *and*

$$\theta\sqrt{T} \geq |z_\alpha|\,, \tag{2.8}$$

then the minimal CaR *is attained for*

$$\pi^* = \left(\theta - \frac{|z_\alpha|}{\sqrt{T}}\right)\frac{(\sigma\sigma)^{-1}(b-r\underline{1})}{\|\sigma^{-1}(b-r\underline{1})\|} \tag{2.9}$$

with

$$\mathrm{CaR}(x,\pi^*,T) = x\exp(rT)\left(1-\exp\left(\frac{1}{2}(\sqrt{T}\theta-|z_\alpha|)^2\right)\right) < 0. \tag{2.10}$$

Proof (a) follows directly from the explicit form of $f(\pi)$ under the assumption of $b_i = r$ for all $i = 1,\dots,d$ and the fact that σ is invertible.
(b),(c) Consider the problem of maximizing $f(\pi)$ over all π which satisfy

$$\|\pi'\sigma\| = \varepsilon \tag{2.11}$$

for a fixed positive ε. Over the (boundary of the) ellipsoid defined by (2.11) $f(\pi)$ equals

$$f(\pi) = z_\alpha \varepsilon \sqrt{T} - \varepsilon^2 T/2 + \pi'(b - r\underline{1})T \ .$$

Thus, the problem is just to maximize a linear function (in π) over the boundary of an ellipsoid. Such a problem has the explicit solution

$$\pi^*_\varepsilon = \varepsilon \frac{(\sigma\sigma')^{-1}(b - r\underline{1})}{\|\sigma^{-1}(b - r\underline{1})\|} \tag{2.12}$$

with

$$f(\pi^*_\varepsilon) = -\varepsilon^2 T/2 + \varepsilon \left(\theta T - |z_\alpha| \sqrt{T} \right) . \tag{2.13}$$

As every $\pi \in \mathbb{R}^d$ satisfies relation (2.11) with a suitable value of ε (due to the fact that σ is regular), we obtain the minimum CaR strategy π^* by maximizing $f(\pi^*_\varepsilon)$ over all non-negative ε. Due to the form of $f(\pi^*_\varepsilon)$ the optimal ε is positive if and only if the multiplier of ε in representation (2.13) is positive. Thus, condition (2.7) implies assertion (b). Under assumption (2.8) the optimal ε is given as

$$\varepsilon = \theta - \frac{|z_\alpha|}{\sqrt{T}}.$$

Inserting this into equations (2.12) and (2.13) yields the assertions (2.9) and (2.10) (with the help of equations (2.5) and (2.6)). □

Remark 2.5 (i) Part (a) of the proposition states that in a risk-neutral market the CaR of every strategy containing stock investment is bigger than the CaR of the pure bond strategy.
(ii) Part (c) states the (at first sight surprising) fact that the existence of at least one stock with a mean rate of return different from the riskless rate implies the existence of a stock and bond strategy with a negative CaR as soon as the time horizon T is large. Thus, even if the CaR would be the only criterion to judge an investment strategy the pure bond investment would not be optimal if the time horizon is far away. On one hand this fact is in line with empirical results on stock and bond markets. On the other hand this shows a remarkable difference between the behaviour of the CaR and the variance as risk measures. Independent of the time horizon and the market coefficients, pure bond investment would always be optimal with respect to the variance of the corresponding wealth process.
(iii) The decomposition method to solve the optimization problem in the proof of parts (b) and (c) of Proposition 2.4 will be crucial for some of the proofs later in this paper. Note how we use it to overcome the problem that $f(\pi)$ is not differentiable in $\pi = 0$. □

The rest of this section is devoted to setting up a Markowitz mean-variance type optimization problem where we replace the variance constraint by a constraint on the CaR of the terminal wealth. More precisely, we solve the following problem:

$$\boxed{\max_{\pi \in \mathbb{R}^d} E(X^\pi(T)) \quad \text{subject to} \quad \mathrm{CaR}(x, \pi, T) \leq C\,,} \tag{2.14}$$

where C is a given constant of which we assume that it satisfies

$$C \leq x \exp(rT)\,. \tag{2.15}$$

Due to the explicit representations (2.4), (2.5) and a variant of the decomposition method as applied in the proof of Proposition 2.4 we can solve problem (2.14) explicitly.

Proposition 2.6 *Let $\theta = \|\sigma^{-1}(b - r\underline{1})\|$ and assume that $b_i \neq r$ for at least one $i \in \{1, \ldots, d\}$. Assume furthermore that C satisfies*

$$0 \leq C \leq x \ \exp(rT) \quad \textit{if } \theta\sqrt{T} < |z_\alpha|, \tag{2.16}$$

$$x \ \exp(rT)\left(1 - \exp\left(\frac{1}{2}(\sqrt{T}\theta - |z_\alpha|)^2\right)\right) \leq C \leq x \ \exp(rT) \quad \textit{if } \theta\sqrt{T} \geq |z_\alpha|\,. \tag{2.17}$$

Then problem (2.14) will be solved by

$$\pi^* = \varepsilon^* \frac{(\sigma\sigma')^{-1}(b - r\underline{1})}{\|\sigma^{-1}(b - r\underline{1})\|}$$

with

$$\varepsilon^* = (\theta + z_\alpha/\sqrt{T}) + \sqrt{(\theta + z_\alpha/\sqrt{T})^2 - 2c/T}\,,$$

where $c = \ln\left(1 - \frac{C}{x}\exp(-rT)\right)$. The corresponding maximal expected terminal wealth under the CaR *constraint equals*

$$E\left(X^{\pi^*}(T)\right) = x \exp\left(\left(r + \varepsilon^* \|\sigma^{-1}(b - r\underline{1})\|\right) T\right)\ . \tag{2.18}$$

Proof The requirements (2.16) and (2.17) on C ensure that the CaR constraint in problem (2.14) cannot be ignored: in both cases C lies between the minimum and the maximum value that CaR can attain (see also Proposition 2.4). Every admissible π for problem (2.14) with $\|\pi'\sigma\| = \varepsilon$ satisfies the relation

$$(b - r\underline{1})'\pi T \geq c + \frac{1}{2}\varepsilon^2 T - z_\alpha \varepsilon \sqrt{T} \tag{2.19}$$

which is in this case equivalent to the CaR constraint in (2.14). But again, on the set given by $\|\pi'\sigma\| = \varepsilon$ the linear function $(b - r\underline{1})'\pi T$ is maximized by

$$\pi_\varepsilon = \varepsilon \frac{(\sigma\sigma')^{-1}(b - r\underline{1})}{\|\sigma^{-1}(b - r\underline{1})\|} . \tag{2.20}$$

Hence, if there is an admissible π for problem (2.14) with $\|\pi'\sigma\| = \varepsilon$ then π_ε must also be admissible. Further, due to the explicit form (2.3) of the expected terminal wealth, π_ε also maximizes the expected terminal wealth over the ellipsoid. Consequently, to obtain π for problem (2.14) it suffices to consider all vectors of the form π_ε for all positive ε such that requirement (2.19) is satisfied. Inserting (2.20) into the left-hand side of inequality (2.19) results in

$$(b - r\underline{1})'\pi_\varepsilon T = \varepsilon \|\sigma^{-1}(b - r\underline{1})\| T , \tag{2.21}$$

which is an increasing linear function in ε equalling zero in $\varepsilon = 0$. Therefore, we obtain the solution of problem (2.14) by determining the biggest positive ε such that (2.19) is still valid. But the right-hand side of (2.21) stays above the right-hand side of (2.19) until their largest positive point of intersection which is given by

$$\varepsilon^* = (\theta + z_\alpha/\sqrt{T}) + \sqrt{(\theta + z_\alpha/\sqrt{T})^2 - 2c/T} ,$$

The remaining assertion (2.18) can be verified by inserting π^* into equation (2.3). $\square$

Remark 2.7 (i) Note that the optimal expected value only depends on the stocks via the norm $\|\sigma^{-1}(b - r\underline{1})\|$. There is no explicit dependence on the number of stocks. We therefore interpret Proposition 2.4 as a kind of *mutual fund theorem* as there is no difference between investment in our multi-stock market and a market consisting of the bond and just one stock with appropriate market coefficients b and σ.

(ii) Consider for a general utility function $U(x)$ the problem of

$$\max_{\pi \in \mathbb{R}^d} E(U(X^\pi(T))) \quad \text{subject to} \quad \mathrm{CaR}(x, \pi, T) \le C.$$

The above method of solving the mean-CaR problem would still work as long as $E(U(X^\pi(T)))$ is of the form $f(x)\exp(h(\pi))$ with h a linear function. This is e.g. the case for the choice of the HARA function $U(x) = x^\gamma/\gamma$. It would also work for the log-utility case; i.e. $U(x) = \ln x$ as then we would have

$$E(U(X^\pi(T))) = \ln x + rT + (b - r\underline{1})'\pi T - \pi'\sigma\sigma'\pi T/2 .$$

Here, instead of looking at the exponent, we can also look at

$$\ln x + rT - (b - r\underline{1})'\pi t - \varepsilon^2 T/2\,,$$

which for all π with $\|\pi'\sigma\| = \varepsilon$ is a linear function in π. However, for reasons of comparison to the Markowitz type problems below we restrict ourselves to the mean-CaR problem.

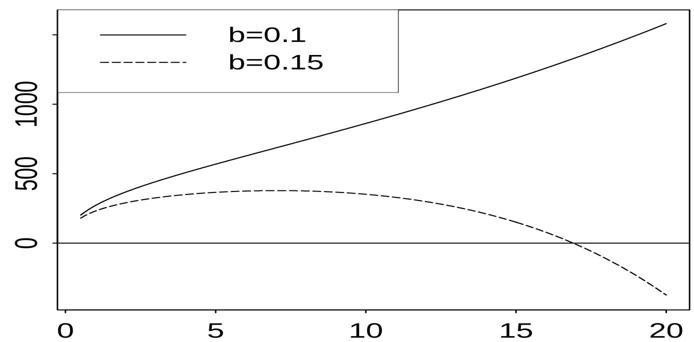

Figure 1: CaR(1 000, 1, T) of the pure stock portfolio (one risky asset only) for different appreciation rates as a function of the planning horizon T; $0 < T \leq 20$. The volatility is $\sigma = 0.2$. The riskless rate is $r = 0.05$.

Example 2.8 Figure 1 shows the dependence of CaR on the time horizon illustrated by CaR(1 000,1,T). Note that the CaR first increases and then decreases with time, a behaviour which was already indicated by Proposition 2.4. It differs substantially from the behaviour of the variance of the pure stock strategy, which increases with T. Figures 2 and 3 illustrate the behaviour of the optimal expected terminal wealth with varying time horizon corresponding to the pure bond strategy and the pure stock strategy as functions of the time horizon T. The expected terminal wealth of the optimal portfolio even exceeds the pure stock investment. The reason for this becomes clear if we look at the corresponding portfolios. The optimal portfolio always contains a short position in the bond as long as this is tolerated by the CaR constraint. This is shown in Figure 4 where we have plotted the optimal portfolio together with the pure stock portfolio as function of the time horizon. For $b = 0.15$ the optimal portfolio always contains a short position in the bond. For $b = 0.1$ and $T > 5$ the optimal portfolio (with the same CaR constraint as in Figures 2 and 3) again contains a long position in both bond and stock (with decreasing tendency of π as time increases!). This is an immediate consequence of the increasing CaR of the stock price. For the smaller appreciation rate of the stock it is simply not attractive enough to take the risk of a large stock investment. Figure 5 shows the mean-CaR efficient frontier for the above parameters with $b = 0.1$ and fixed time horizon $T = 5$. As expected it has a similar form as a typical mean-variance efficient frontier.

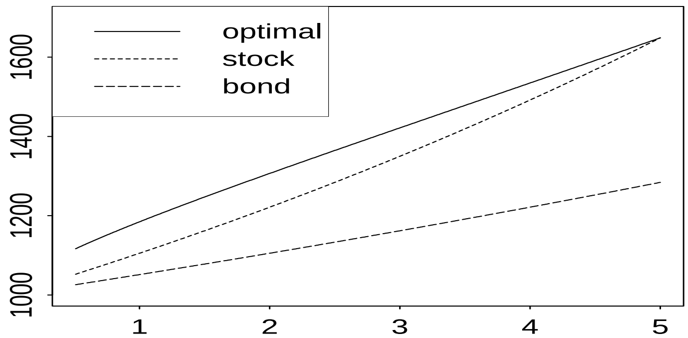

Figure 2: Expected terminal wealth of different investment strategies depending on the time horizon T, $0 < T \leq 5$. The parameters are $d = 1$, $r = 0.05$, $b = 0.1$, $\sigma = 0.2$, and $\alpha = 0.05$. As the upper bound C of the CaR we used $\mathrm{CaR}(1\,000, 1, 5)$, the CaR of the pure stock strategy with time horizon $T = 5$.

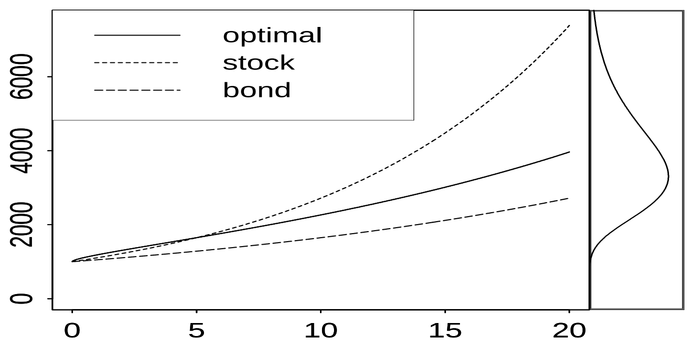

Figure 3: Expected terminal wealth of different investment strategies depending on the time horizon T, $0 \leq T \leq 20$. The parameters are $d = 1$, $r = 0.05$, $b = 0.1$, $\sigma = 0.2$, and $\alpha = 0.05$. As the upper bound C of the CaR we used $\mathrm{CaR}(1\,000, 1, 5)$, the CaR of the pure stock strategy with time horizon $T = 5$. On the right border we have plotted the density function of the wealth for the optimal portfolio. It is always between 0 and 0.0004.

We will now compare the behaviour of the optimal portfolios for the mean-CaR with solutions of a corresponding mean-variance problem. To this end we consider the following simpler optimization problem:

$$\boxed{(2.22) \qquad \max_{\pi \in \mathbb{R}^d} E(X^\pi(T)) \quad \text{subject to} \quad \mathrm{var}(X^\pi(T)) \leq C\ .}$$

By using the explicit form (2.4) of the variance of the terminal wealth, we can rewrite the

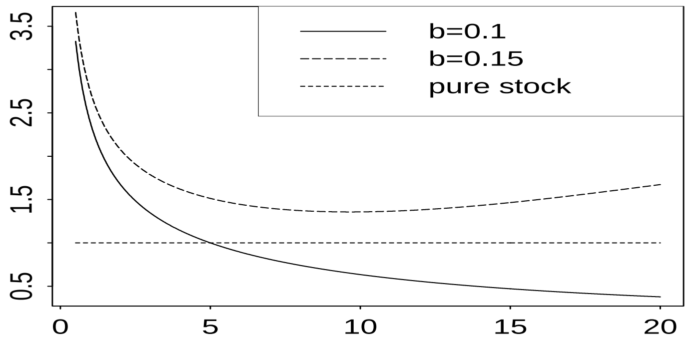

Figure 4: For the same parameters as in Figure 2 and different appreciation rates the figure shows the optimal portfolio and the pure stock portfolio.

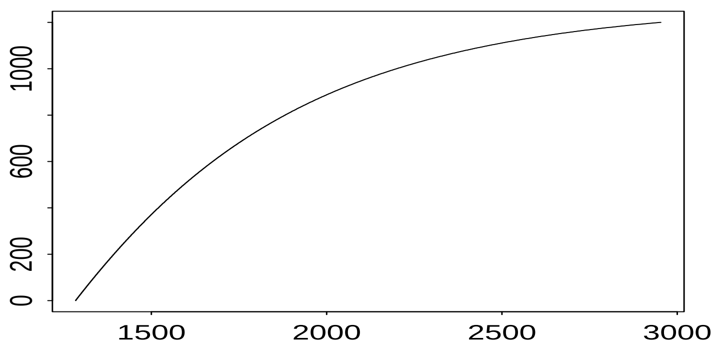

Figure 5: Mean-CaR efficient frontier with the mean on the horizontal axis and the CaR on the vertical axis. The parameters are the same as in Figure 2.

variance constraint in problem (2.22) as

$$(b - r\underline{1})'\pi T \leq \frac{1}{2} \ln \left(\frac{C}{x^2(\exp(\varepsilon^2 T) - 1)} \right) - rT =: h(\varepsilon), \qquad \|\pi'\sigma\| = \varepsilon \tag{2.23}$$

for $\varepsilon > 0$. More precisely, if $\pi \in \mathbb{R}^d$ satisfies the constraints in (2.23) for one $\varepsilon > 0$ then it also satisfies the variance constraint in (2.22) and vice versa. Noting that $h(\varepsilon)$ is strictly decreasing in $\varepsilon > 0$ with

$$\lim_{\varepsilon \downarrow 0} h(\varepsilon) = \infty \qquad \lim_{\varepsilon \to \infty} h(\varepsilon) = -\infty$$

we see that left-hand side of (2.23) must be smaller than the right-hand one for small values of $\varepsilon > 0$ if we plug in π_ε as given by equation (2.20). Recall that this was the portfolio with the highest expected terminal wealth of all portfolios π satisfying $\|\pi'\sigma\| = \varepsilon$. It even maximizes $(b - r\underline{1})'\pi T$ over the set given by $\|\pi'\sigma\| \leq \varepsilon$. If we have equality

$$(b - r\underline{1})'\pi_{\widehat{\varepsilon}} T = h(\widehat{\varepsilon}) \tag{2.24}$$

for the first time with increasing $\varepsilon > 0$ then this determines the optimal $\widehat{\varepsilon} > 0$. To see this, note that we have

$$E(X^{\pi}(T)) \leq E(X^{\pi_{\widehat{\varepsilon}}}(T)) \quad \text{for all } \pi \text{ with} \quad \|\pi'\sigma\| \leq \widehat{\varepsilon}\ ,$$

and for all admissible π with $\varepsilon = \|\pi'\sigma\| > \widehat{\varepsilon}$ we obtain

$$(b - r\underline{1})'\pi T \leq h(\varepsilon) < h(\widehat{\varepsilon}) = (b - r\underline{1})'\pi_{\widehat{\varepsilon}} T\ .$$

By solving the non-linear equation (2.24) for $\widehat{\varepsilon}$ we have thus completely determined the solution of problem (2.22):

Proposition 2.9 *If $b_i \neq r$ for at least one $i \in \{1, \dots, d\}$, then the optimal solution of the mean-variance problem (2.22) is given by*

$$\widehat{\pi} = \widehat{\varepsilon}\,\frac{(\sigma\sigma')^{-1}(b - r\underline{1})}{\|\sigma^{-1}(b - r\underline{1})\|}\ ,$$

where $\widehat{\varepsilon}$ is the unique positive solution of the non-linear equation

$$\|\sigma^{-1}(b - r\underline{1})\|\varepsilon T - \frac{1}{2}\ln\left(\frac{C}{x^2(\exp(\varepsilon^2 T) - 1)}\right) + rT = 0\ .$$

The corresponding maximal expected terminal wealth under the variance constraint equals

$$E(X^{\widehat{\pi}}(T)) = x\exp\big((r + \widehat{\varepsilon}\,\|\sigma^{-1}(b - r\underline{1})\|)T\big)\ . \qquad \square$$

Example 2.10 Figure 6 below compares the behaviour of $\widehat{\varepsilon}$ and ε^* as functions of the time horizon. We have used the same data as in Example 2.8. To make the solutions of problems (2.14) and (2.22) comparable we have chosen C differently for the variance and the CaR risk measures in such a way that $\widehat{\varepsilon}$ and ε^* concide for $T = 5$. Notice that C for the variance problem is roughly the square of C for the CaR problem taking into account that the variance measures an L^2-distance, whereas CaR measures an L^1-distance. The (of course expected) bottom line of Figure 6 is that with increasing time the variance constraint demands a smaller fraction of risky securities in the portfolio. This is also true for the CaR constraint for small time horizons. For larger time horizon T ($T \geq 20$) ε^* increases again due to the fact that the CaR decreases. In contrast to that, $\widehat{\varepsilon}$ decreases to 0, since the variance increases. $\square$

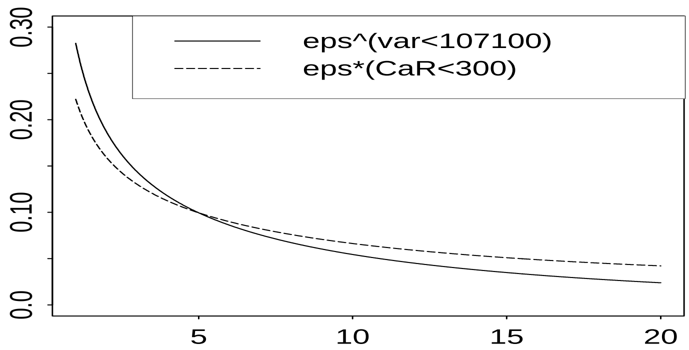

Figure 6: $\widehat{\varepsilon}$ and ε^* as functions of the time horizon; $0 < T \leq 20$. The parameters are the same as in Figure 2.

3 Capital-at-Risk portfolios and more general price processes

In this section we consider again the mean-CaR problem (2.14) but drop the assumption of log-normality of the stock price process. The self-financing condition, however, will still manifest itself in the form of the wealth equation

$$\frac{dX^\pi(t)}{X^\pi(t-)} = (1-\pi'\underline{1})\frac{dP_0(t)}{P_0(t-)} + \sum_{i=1}^{d} \pi_i \frac{dP_i(t)}{P_i(t-)}\,, \quad t>0\,, \qquad X^\pi(0)=x\,,$$

where P_i is the price process for stock i. Of course, the explicit form of the stochastic process P_i is crucial for the computability of the expected terminal wealth $X^\pi(T)$. To concentrate on these tasks we simplify the model in assuming $d=1$, a bond price given by $P_0(t)=e^{rt}$, $t \geq 0$, as before, and a risky asset price satisfying

$$\frac{dP(t)}{P(t-)} = b\,dt + dY(t)\,, \quad t>0\,, \qquad P(0)=p\,, \tag{3.1}$$

where $b \in \mathbb{R}$ and Y is a semimartingale with $Y(0)=0$. Under these assumptions the choice of the portfolio π leads to the following explicit formula for the wealth process

$$\begin{aligned} X^\pi(t) &= x\exp((r+\pi(b-r))t)\mathcal{E}(\pi Y(t)) \\ &= x\exp((r+\pi(b-r))t)\exp\left(\pi Y^c(t) - \tfrac{1}{2}\pi^2\langle Y^c\rangle_t\right) \\ &\quad \times \prod_{0<s\leq t}(1+\pi\Delta Y(s))\,,\ t\geq 0\,, \end{aligned} \tag{3.2}$$

where Y^c denotes the continuous part and ΔY the jump part of the process Y (more precisely, $\Delta Y(t)$ is the height of a (possible) jump at time t). This means that the wealth process is a product of a deterministic process and the stochastic exponential $\mathcal{E}(\pi Y)$ of πY (see Protter (1990)). Analogously to Definition 2.1 we define the CaR in this more general context.

Definition 3.1 *Consider the market given by a riskless bond with price* $P_0(t) = e^{rt}$, $t \geq 0$, *for* $r \in \mathbb{R}$ *and one stock with price process* P *satisfying (3.1) for* $b \in \mathbb{R}$ *and a semimartingale* Y *with* $Y(0) = 0$. *Let* x *be the initial capital and* T *a given time horizon. For some portfolio* $\pi \in \mathbb{R}$ *and the corresponding terminal wealth* $X^\pi(T)$ *the* α*-quantile of* $X^\pi(T)$ *is given by*

$$\widetilde{\rho}(x, \pi, T) = x \exp((\pi(b - r) + r)T) \cdot \widetilde{z}_\alpha \,,$$

where $\widetilde{z}_\alpha$ *is the* α*-quantile of* $\mathcal{E}(\pi Y(T))$, *i.e.* $\widetilde{z}_\alpha = \inf\{z \in \mathbb{R} : P(\mathcal{E}(\pi Y(T)) \leq z) \geq \alpha\}$. *Then we call*

$$\text{CaR}(x, \pi, T) = x \exp(rT)(1 - \exp(\pi(b - r)T) \cdot \widetilde{z}_\alpha) \tag{3.3}$$

the Capital-at-Risk *of the portfolio* π *(with initial capital* x *and time horizon* T*).* □

One of our aims of this section is to explore the behaviour of the solutions to the mean-CaR problem (2.14) if we model the returns of the price process by processes having heavier tails than the Brownian motion. We present some specific examples in the following subsections.

3.1 The Black-Scholes model with jumps

We consider a stock price process P, where the random fluctuations are generated by both a Brownian motion and a compound jump process, i.e., we consider the model (3.1) with

$$dY(t) = \sigma dW(t) + \sum_{i=1}^{n} (\beta_i dN_i(t) - \beta_i \lambda_i dt) \,, \quad t > 0 \,, \quad Y(0) = 0 \,, \tag{3.4}$$

where $n \in \mathbb{N}$, and for $i = 1, \ldots, n$ the process N_i is a homogeneous Poisson process with intensity λ_i. It counts the number of jumps of height β_i of Y. In order to avoid negative stock prices we assume

$$-1 < \beta_1 < \cdots < \beta_n < \infty \,.$$

An application of Itô's formula results in the explicit form

$$P(t) = p \exp\left(\left(b - \frac{1}{2}\sigma^2 - \sum_{i=1}^{n} \beta_i \lambda_i\right) t + \sigma W(t) + \sum_{i=1}^{n} (N_i(t) \ln(1 + \beta_i))\right) \,, \; t \geq 0 \,. \tag{3.5}$$

In order to avoid the possibility of negative wealth after an "unpleasant" jump we have to restrict the portfolio π as follows

$$\pi \in \begin{cases} \left[-\frac{1}{\beta_n}, -\frac{1}{\beta_1}\right) & \text{if} \quad \beta_n > 0 > \beta_1 \,, \\ \left(-\infty, -\frac{1}{\beta_1}\right] & \text{if} \quad \beta_n < 0 \,, \\ \left[-\frac{1}{\beta_n}, \infty\right) & \text{if} \quad \beta_1 > 0 \,. \end{cases} \tag{3.6}$$

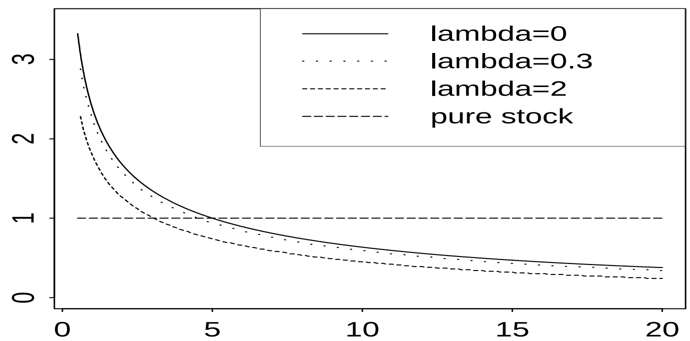

Figure 7: Optimal portfolios for Brownian motion with and without jumps depending on the time horizon T, $0 < T \leq 20$. The basic parameters are the same as in Figure 2. The possible jump size is $\beta = -0.1$.

Under these preliminary conditions we obtain explicit representations of the expected terminal wealth and the CaR corresponding to a portfolio π similar to the equations (2.3) and (2.5).

Lemma 3.2 *With a stock price given by equation (3.5) let X^π be the wealth process corresponding to the portfolio π satisfying (3.6). Then for initial capital x and finite time horizon T,*

$$\begin{aligned} X^\pi(T) &= x\exp((r + \pi(b-r) - \sum_{i=1}^n \pi\beta_i\lambda_i - \frac{1}{2}\pi^2\sigma^2)T + \pi\sigma W(T) + \sum_{i=1}^n N_i(T)\ln(1+\pi\beta_i))\,, \\ E(X^\pi(T)) &= x\exp((r+\pi(b-r))T)\,, \\ \mathrm{CaR}(x,\pi,T) &= x\exp(rT)\left(1 - \exp\left(\left(\pi(b-r) - \sum_{i=1}^n \pi\beta_i\lambda_i - \frac{1}{2}\pi^2\sigma^2\right)T + \widetilde{z}_\alpha\right)\right)\,, \end{aligned}$$

where $\widetilde{z}_\alpha$ is the α-quantile of

$$\pi\sigma W(T) + \sum_{i=1}^n (N_i(T)\ln(1+\pi\beta_i))\,,$$

i.e. the real number $\widetilde{z}_\alpha$ satisfying

$$\begin{aligned}\alpha &= P\left(\pi\sigma W(T) + \sum_{i=1}^{n}(N_i(T)\ln(1+\pi\beta_i)) \le \widetilde{z}_\alpha\right)\\ &= \sum_{n_1,\ldots,n_n=0}^{\infty}\left(\Phi\left(\frac{1}{|\pi\sigma|\sqrt{T}}\left(\widetilde{z}_\alpha - \sum_{i=1}^{n}(n_i\ln(1+\pi\beta_i))\right)\right)\right.\\ &\qquad\left.\times\exp\left(-T\sum_{i=1}^{n}\lambda_i\right)\prod_{i=1}^{n}\frac{(T\lambda_i)^{n_i}}{n_i!}\right)\,. \end{aligned} \tag{3.7}$$

Proof $X^\pi(T)$ is a result of an application of Itô's formula. To obtain the expected value simply note that the two processes

$$\exp\left(-\frac{1}{2}\sigma^2 t + \sigma W(t)\right) \quad \text{and} \quad \exp\left(-\sum_{i=1}^{n}\beta_i\lambda_i t + \sum_{i=1}^{n}\sum_{j=1}^{N_i(t)}\ln(1+\beta_i)\right)$$

are both martingales with unit expectation and that they are independent. Regarding the representation of the CaR, only equation (3.7) has to be commented on. But this is a consequence of conditioning on the number of jumps of the different jump heights in $[0,T]$. □

Unfortunately, $\widetilde{z}_\alpha$ cannot be represented in such an explicit form as in the case without jumps. However, due to the explicit form of $E(X^\pi(T))$, it is obvious that the corresponding mean-CaR problem (2.14) will be solved by the largest π that satisfies both the CaR constraint and requirement (3.6). Thus for an explicit example we obtain the optimal mean-CaR portfolio by a simple numerical iteration procedure, where we approximated the infinite sum in (3.7) by the finite sum of its first $2[\lambda T]+1$ summands, if we set $n=1$ and $\lambda=\lambda_1$. Comparisons of the solutions for the Brownian motion with and without jumps are given in Figure 7.

We have used the same parameters as in the examples of Section 2, but have included the possibility of a jump of height $\beta = -0.1$, occuring with different intensities. For $\lambda = 0.3$ one would expect a jump approximately every three years, for $\lambda = 2$ even two jumps per year. Notice that the stock has the same expected terminal value in both cases! To explain this we rewrite equation (3.5) as follows:

$$\frac{dP(t)}{P(t-)} = \left(b - \sum_{i=1}^{n}\beta_i\lambda_i\right)dt + \sigma W(t) + \sum_{i=1}^{n}\beta_i dN_i(t)\,, \quad t>0\,, \qquad P(0)=p\,.$$

Whereas a jump occurs for instance for $\lambda = 0.3$ on average only every three years, meaning that with rather high probability there may be no jump within two years, the drift has a permanent influence on the dynamic of the price process. Despite this additional stock

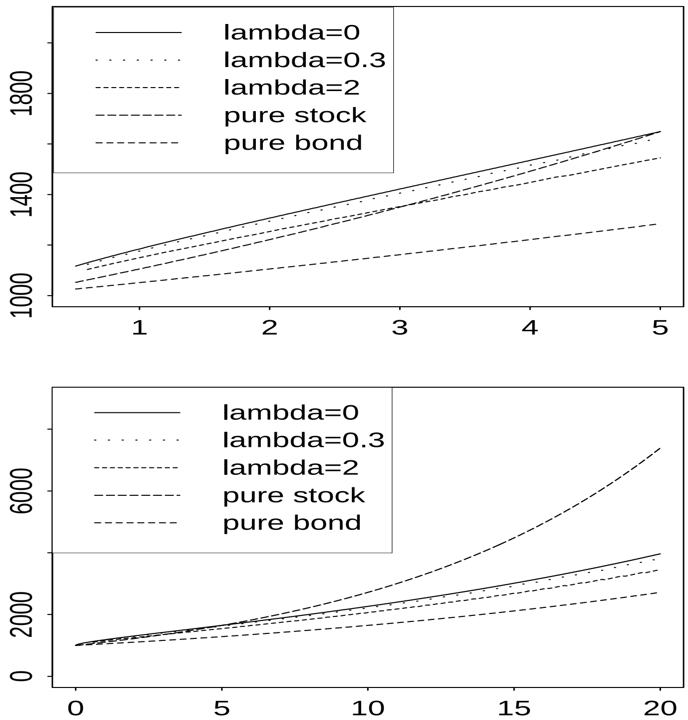

Figure 8: Wealth corresponding to the optimal portfolios for Brownian motion with and without jumps depending on the time horizon T, $0 < T \leq 5$ (top) and $0 < T \leq 20$ (bottom). The parameters are the same as in Figure 7. The possible jump size is again $\beta = -0.1$.

drift of $-\beta'\lambda$ the optimal portfolio for stock prices following a geometric Brownian motion with jumps is always below the optimal portfolio of the geometric Brownian motion (solid line). This means that the threat of a downwards jump of 10% leads an investor to a less risky behaviour, and the higher λ is, the less risky is the investor‘s behaviour.

3.2 Generalized inverse Gaussian diffusion

Moving away from the Black-Scholes model towards more general diffusion models is a rather obvious generalization. It is also desirable, since marginal distributions of the log-returns of stock prices are often heavier tailed than normal. This has been shown very convincingly, for instance, by a data analysis in Eberlein and Keller (1995). Various models have been suggested: a simple hyperbolic model has been investigated by Bibby and Sørensen (1997); a more general class of models has been suggested by Barndorff-Nielsen (1998).

We consider a generalized inverse Gaussian diffusion model (for brevity we write GIG diffusion) for the log-returns of stock prices. This class of diffusions has been introduced in Borkovec and Klüppelberg (1998) and we refer to this source for details.

The following equations determine a general diffusion market.

$$
\begin{array}{lrcll}
(3.8) & dP_0(t) & = & P_0(t)r\,dt\,, & P_0(0)=1\,,\\
& dP(t) & = & P(t)(bdt+dY(t))\,, & P(0)=p\,,\\
& Y(t) & = & U(t)-u\,, & Y(0)=0\,,
\end{array}
$$

In our case we now choose U as a GIG diffusion given by the SDE

$$
\begin{array}{lrcl}
(3.9) & dU(t) & = & \frac{1}{4}\sigma^2U^{2\gamma-2}(t)\left(\psi+2(2\gamma+\lambda-1)U(t)-\chi U^2(t)\right)dt\\
& & & +\sigma U^{\gamma}(t)dW(t),\; U(0)=u>0\,,
\end{array}
$$

where W is standard Brownian motion. The parameter space is given by $\sigma>0$, $\gamma\geq 1/2$, $\chi,\psi\geq 0$, $\max(\chi,\psi)>0$, and

$$
\begin{array}{llll}
(3.10) & \lambda\in\mathbb{R} & \text{if} & \chi,\psi>0\,,\\
& \lambda\leq\min(0,2(1-\gamma)) & \text{if} & \chi=0,\psi>0\,,\\
& \lambda\geq\min(0,2(1-\gamma)) & \text{if} & \chi>0,\psi=0\,.
\end{array}
$$

The GIG model is a formal extension of the Black-Scholes model, which corresponds to the choice of parameters $\gamma=\psi=0$, $\lambda=1,\chi=0$. It also contains the (generalized) Cox-Ingersoll-Ross model as a special case. The advantage of our construction lies in the structural resemblance of the resulting price process to the geometric Brownian motion model. We can decompose the stock price into a drift term multiplied by a local martingale:

$$
\begin{aligned}
P(t) \;=\;& p\exp\left(bt+\frac{1}{4}\sigma^2\int_0^t U^{2\gamma-2}(s)\left(\psi+2(2\gamma+\lambda-1)U(s)-\chi U^2(s)\right)ds\right)\\
&\times\exp\left(\sigma\int_0^t U^{\gamma}(s)dW(s)-\frac{1}{2}\sigma^2\int_0^t U^{2\gamma}(s)ds\right),\quad t\geq 0\,.
\end{aligned}
$$

The following lemma shows another property of the process U that is useful, when describing the wealth process.

Lemma 3.3 *Let U be the GIG diffusion given by (3.9) and $\pi>0$. Then the process $\widetilde{U}=\pi U$ is again a GIG diffusion with $\widetilde{U}(0)=\pi U(0)$ and parameters*

$$
(3.11)\qquad \widetilde{\sigma}=\sigma\pi^{1-\gamma}\,,\quad \widetilde{\psi}=\psi\pi\,,\quad \widetilde{\chi}=\chi/\pi\,.
$$

The parameters γ and λ remain the same.

Proof Notice first that all parameters of $\widetilde{U}$ satisfy the necessary non-negativity assumptions and (3.9). The assertion now follows by calculating $d\widetilde{U}(t)=d(\pi U(t))=\pi dU(t)$, $t\geq 0$. □

Remark 3.4 As a consequence of Lemma 3.3 the wealth process X^π has a very nice explicit form. Indeed it is of a similar form as the stock price process P:

$$X^\pi(t) \quad = \quad x \exp\left((1-\pi)rt + \widetilde{b}t + \widetilde{Y}(t) - \frac{1}{2}\langle \widetilde{Y}\rangle_t\right), \quad t \geq 0\,, \tag{3.12}$$

where

$$\widetilde{b} = \pi b \quad \text{and} \quad \widetilde{Y}(t) = \widetilde{U}(t) - \pi u\,, \quad t \geq 0\,,$$

for any positive portfolio π. □

According to Definition 3.1 for the $\mathrm{CaR}(x,\pi,T)$ we have to determine the α-quantile of $\widetilde{Y}(T) - \frac{1}{2}\langle \widetilde{Y}\rangle_T$. Here we see one of the big advantages of the CaR as a risk measure: it does not depend on the existence of moments. Even for an infinite mean it is well-defined.

However, if we want to solve the mean-CaR problem, we have to ensure that $X^\pi(T)$ has a finite mean. In general, it is not always possible to easily decide if this is the case. A natural assumption is to assume $U(T)$ or $\widetilde{U}(T)$ to have the stationary distribution of the process U or $\widetilde{U}$ respectively. This is certainly justified if the time horizon T is chosen sufficiently large. As in Bibby and Sørensen (1998) we therefore make this simplifying assumption which helps us to give a result about the existence of $E(X^\pi(T))$.

Proposition 3.5 *Assume that $U(T)$ and $\widetilde{U}(T)$ are GIG distributed with parameters ψ, χ, λ and $\widetilde{\psi}$, $\widetilde{\chi}$, λ respectively, i.e. they have the stationary distributions of the processes $U(\cdot)$ and $\widetilde{U}(\cdot)$ respectively. Assume that π is a positive portfolio. Then $X^\pi(T)$ has a finite mean if $\widetilde{\chi} = \chi/\pi > 2$.*

Proof As $\widetilde{U}$ is always positive, we estimate

$$X^\pi(T) \leq x \exp\left((1-\pi)rT + \widetilde{b}T + \widetilde{U}(T) - \pi u\right).$$

If $E\exp(\widetilde{U}(T)) < \infty$, then $EX^\pi(T) < \infty$. By Jørgensen (1982) we know the explicit form of the moment generating function of the GIG distribution leading to

$$E\left(\exp(\widetilde{U}(T))\right) = \frac{K_\lambda\left(\sqrt{\chi\psi}(1-2/\widetilde{\chi})\right)}{K_\lambda\left(\sqrt{\chi\psi}\right)(1-2/\widetilde{\chi})^{\lambda/2}}\,, \tag{3.13}$$

where $K_\lambda(\cdot)$ denotes the generalized Bessel function of the third kind. The rhs of equation (3.13) is finite for $\widetilde{\chi} > 2$. □

Thus if the original parameters satisfy $\chi > 2$ and $\pi \in [0,1]$, then also $\widetilde{\chi} > 2$ and in this case $X^\pi(T)$ has a finite mean. In this case the mean-CaR problem is well-defined and can be solved, however one cannot hope for an analytic solution. In the following example we show how the mean-CaR problem can be solved using analytic properties of the process as far as possible, and then present a simple simulation procedure to solve the problem numerically.

Example 3.6 (Generalized Cox-Ingersoll-Ross model (GCIR))
As an example we consider the *generalized Cox-Ingersoll-Ross model*, i.e., the GIG market model with parameters $\gamma = 1$, $\chi = 0$. This results in the following explicit form for U:

$$U(t) = \exp\left(\frac{1}{2}\sigma^2\lambda t + \sigma W(t)\right)\left\{u + \frac{1}{4}\sigma^2\psi\int_0^t \exp\left(-\frac{1}{2}\sigma^2\lambda s - \sigma W(s)\right)ds\right\},\ t \geq 0\,,$$

which has mean

$$EU(t) = \begin{cases} \exp\left((\lambda+1)\dfrac{\sigma^2}{2}t\right)\left(u + \dfrac{\psi}{2(\lambda+1)}\left(1 - \exp\left(-(\lambda+1)\dfrac{\sigma^2}{2}t\right)\right)\right) & \text{if } \lambda \neq -1\,, \\ u + \frac{1}{2}\sigma^2\psi t & \text{if } \lambda = -1\,, \end{cases}$$

(see e.g. Borkovec and Klüppelberg (1998)). Further, note that we have

$$Y(t) = U(t) - u = \frac{1}{4}\sigma^2\psi t + \frac{1}{2}(1+\lambda)\sigma^2\int_0^t U(s)ds + \sigma\int_0^t U(s)dW(s) \tag{3.14}$$

and we obtain the same representations for $\widetilde{U}(t)$ and $\widetilde{Y}(t)$ if we substitute ψ by $\widetilde{\psi} = \pi\psi$. An explicit solution of the mean-CaR problem does not seem to be possible. What remains are Monte-Carlo simulations and numerical approximations.

A simple algorithm to solve the mean-CaR problem would be the following:
For large N and $i = 1, \ldots, N$:

- Simulate sample paths $(W_i(t))_{t\in[0,T]}$ of the Brownian motion $(W(t))_{t\in[0,T]}$.
- Compute realisations $U_i(T)$ and $\int_0^T U_i^2(t)dt$ of $U(T)$ and $\int_0^T U^2(t)dt$, respectively, from the simulated sample paths of $(W_i(t))_{t\in[0,T]}$.
- For "all" $\pi \in \mathbb{R}$ compute
$$\widetilde{Z}_i^\pi(T) = \pi U_i(T) - \frac{1}{2}\pi^2\sigma^2\int_0^T U_i^2(t)dt - \pi u.$$
- Get estimators $\widehat{\mu}(\pi)$ for $E(X^\pi(T))$ and $\widehat{\nu}(x,\pi,T)$ for $\text{CaR}(x,\pi,T)$:
$$\begin{aligned} \widehat{\mu}(\pi) &:= \frac{x}{N}\sum_{i=1}^N \exp\left((r+(b-r)\pi)T + \widetilde{Z}_i^\pi(T)\right) \\ \widehat{\nu}(x,\pi,T) &:= x\exp(rT)\left(1 - \exp\left(\pi(b-r)T + \widehat{z}_\alpha(\pi)\right)\right)\,, \end{aligned}$$
where $\widehat{z}_\alpha(\pi)$ is the α-quantile of the empirical distribution of the $\widetilde{Z}_i^\pi(T)$ with the convention we already used in Definition 3.1.

- Choose the portfolio π with the largest value of $\widehat{\mu}(\pi)$ such that $\widehat{\nu}(x, \pi, T)$ is below the upper bound C for the CaR.

Of course, it is not possible to compute the quantities $\widehat{\mu}(\pi)$ and $\widehat{z}_\alpha(\pi)$ for all $\pi \in \mathbb{R}$ explicitly. A practical method consists in choosing $K = 100$ values of π in a bounded interval of interest and derive functions $\mu(\pi), z_\alpha(\pi)$ via interpolation. One then chooses that value of π that solves the mean-CaR problem corresponding to these functions.

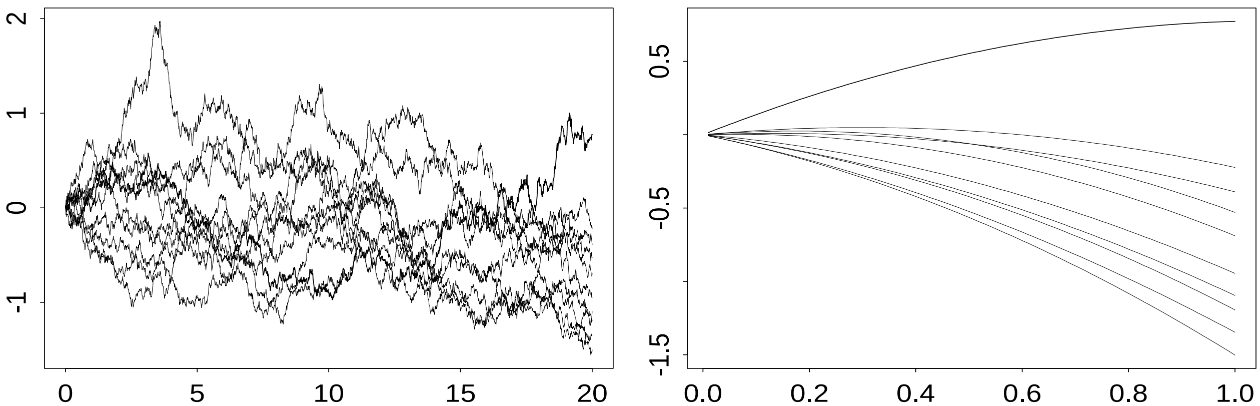

Figure 9: Ten sample paths of $(\widetilde{Z}(t))_{0 \le t \le 20}$ for $\pi = 1$ (left) and ten sample paths of $(\widetilde{Z}^\pi(20))_{\pi \in (0,1)}$ (right) for parameter values $x = 1000, r = 0.05, b = 0.10, \psi = 4, \lambda = 0, \sigma = 0.05$ and $u = 5$.

To give an impression of the behaviour of $\widetilde{Z}(t)$ the first diagram in Figure 9 shows ten sample paths for the parameter values $x = 1000, r = 0.05, b = 0.10, \psi = 4, \lambda = 0, \sigma = 0.05$ and $u = 5$. The second diagram depicts the behaviour of $\widetilde{Z}(20)$ as a function of π. Figure 10 shows a result of the simulation algorithm described above. It is the result of $N = 100$ simulations for $T = 20$ and the remaining parameters chosen as those of Figure 9. As expected, both the mean terminal wealth and the CaR increase with π. Therefore the problem can be solved by identifying that portfolio π in the right side diagram that corresponds to the given upper bound C for the CaR.

4 Conclusion

We have investigated some simple portfolio problems containing an upper bound on the CaR as an additional constraint. As long as we were able to calculate expectations and quantiles of the stock prices in explicit form we could also solve the problems explicitly. This can be done within a Gaussian world, but very little beyond. The Black-Scholes model with jumps is just feasible and easily understood. As soon as one moves away

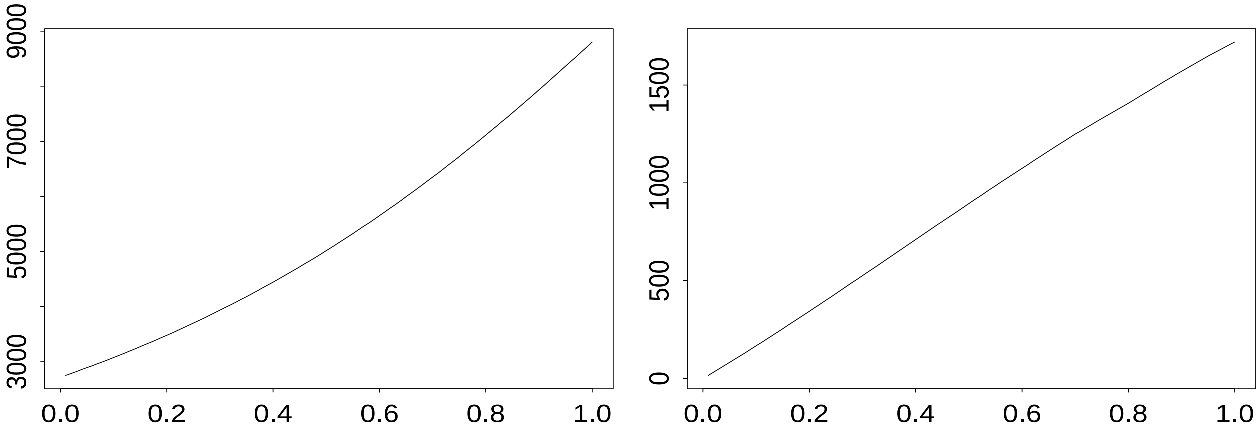

Figure 10: Estimated expected terminal wealth (left) and the corresponding CaR (right) as functions of the portfolio π for the GCIR model for T=20 and the same parameters as in Figure 9 (based on N=100 simulations). The expected terminal wealth and the CaR for the GCIR model increase for all $\pi \in (0,1)$.

from such simple models the solution of the mean-CaR problems becomes less tractable and Monte Carlo simulation and numerical solutions are called for. As an example we treated the generalized Cox-Ingersoll-Ross model, which gave us a first impression of the complexity of the problem.

In this sense the paper should be understood as the starting point of a larger research project. We indicate some of the problems we want to deal with in future work:

– A deeper analysis should investigate the influence of the parameters of the generalized inverse Gaussian; also other models should be investigated as for instance hyperbolic and normal inverse Gaussian models (see Eberlein, Keller and Prause (1998) and Barndorff-Nielsen (1998)).

– Investigate the optimization problem for other downside risk measures; replace for instance the quantile in Definition 2.1 by the expected shortfall. Comparisons of results for the CaR with respect to the quantile and the shortfall can be found in Emmer, Klüppelberg and Korn (2000).

– Replace the constant portfolio by a general portfolio process. Then we have to bring in much more sophisticated techniques to deal with the quantiles of the wealth process, and our method of solving the optimization problem explicitly will no longer work.

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