Frames of exponentials: lower frame bounds for finite subfamilies and approximation of the inverse frame operator.

Ole Christensen, Alexander M. Lindner

Abstract

We give lower frame bounds for finite subfamilies of a frame of exponentials $\{e^{i\lambda_k(\cdot)}\}_{k\in\mathbb{Z}}$ in $L^2(-\pi,\pi)$. We also present a method for approximation of the inverse frame operator corresponding to $\{e^{i\lambda_k(\cdot)}\}_{k\in\mathbb{Z}}$, where knowledge of the frame bounds for finite subfamilies is crucial.

1 Introduction

A frame of exponentials allows every function in $L^2(-\pi,\pi)$ to be written as a superposition of exponentials. Knowledge of the frame bounds (see Section 1.1) is essential in many contexts, since the speed of convergence for algorithms involving frames usually depends on the frame bounds.
Clearly, every "real life" computation with exponentials has to be done with a finite system. Therefore it is very important to have estimates for the corresponding frame bounds. In the present paper, we present such estimates. We also discuss a method for approximation of the inverse frame operator, where knowledge of the lower frame bounds for finite sets of exponentials plays a crucial role.
The rest of Section 1 consists of background material. Then in Section 2 we estimate the lower frame bound for a finite set of exponential functions $\{e^{i\lambda_k(\cdot)}\}_{k=1}^N$. Section 3 is devoted to the question of approximation of the inverse frame operator using finite subsets of the frame. We show that the best performance is achieved if the frame contains a subfamily which is a Riesz basis.

1.1 Frames and Riesz bases

Let $\mathcal{H}$ be a separable Hilbert space and I a countable or finite index set. A family $\Phi = \{\varphi_k\}_{k\in I} \subseteq \mathcal{H}$ is a *frame* for $\mathcal{H}$, if

$$\exists A, B > 0 : A||f||^2 \leq \sum_{k\in I} |< f, \varphi_k >|^2 \leq B||f||^2, \ \forall f \in \mathcal{H}. \tag{1}$$

In particular, every finite set of elements in $\mathcal{H}$ is a frame for its span. The numbers A, B in (1) are called *lower* and *upper frame bounds*. The supremum of all lower frame bounds is again a frame bound, which will be denoted by A_I^{opt}.

If Φ is a frame, the *frame operator*

$$S : \mathcal{H} \to \mathcal{H}, \ Sf = \sum_{k\in I} < f, \varphi_k > \varphi_k$$

is bounded, positive, and invertible. Thus each $f \in \mathcal{H}$ has an expansion

$$f = SS^{-1}f = \sum_{k\in I} < f, S^{-1}\varphi_k > \varphi_k. \tag{2}$$

Recall that $\Phi = \{\varphi_k\}_{k\in I}$ is a *Riesz basis* for $\mathcal{H}$ if Φ is complete and

$$\exists A, B > 0 : \ A\sum |c_k|^2 \leq ||\sum c_k\varphi_k||^2 \leq B\sum |c_k|^2 \tag{3}$$

for all finite sequences $\{c_k\}$ of complex scalars. Note in particular, that if $\{\varphi_k\}_{k\in I}$ is a Riesz basis, then each subfamily is a Riesz basis for its closed linear span, with the same constants as bounds.

A Riesz basis is a frame, and the numbers A, B appearing in (3) and the frame bounds coincide. On the other hand, a frame $\Phi = \{\varphi_k\}_{k\in I}$ is a Riesz basis if and only if Φ is ω-*independent*, meaning that

$$\sum c_k\varphi_k = 0, \ \{c_k\} \in \ell^2(I) \Rightarrow c_k = 0, \ \forall k.$$

We shall give another characterization of Riesz bases:

Proposition 1.1 *Let $\Phi = \{\varphi_k\}_{k\in I} \subseteq \mathcal{H}$ be a frame, and let $\{I_n\}_{n=1}^{\infty}$ be a family of finite subsets of I such that*

$$I_1 \subseteq I_2 \subseteq \cdots \uparrow I.$$

Then the following are equivalent:

(a) Φ *is a Riesz basis for* $\mathcal{H}$.

(b) Φ *is linearly independent and* $\inf_{n\in\mathbb{N}} A_{I_n}^{opt} > 0$.

(c) Φ *is linearly independent and* $\lim_{n\to\infty} A_{I_n}^{opt}$ *exists and is positive.*

Proof: It is well known that (a) $\Rightarrow$ (b). That (b) $\Rightarrow$ (a) is proved by Kim and Lim [9] as a consequence of a series of Theorems. For the readers convenience, we include a short direct proof. If (b) is satisfied, then, for each $n \in \mathbb{N}$, $\{\varphi_k\}_{k\in I_n}$ is a Riesz basis for its span with lower bound $A := \inf_{n\in\mathbb{N}} A_{I_n}^{opt}$, meaning that

$$A \sum |c_k|^2 \leq || \sum c_k \varphi_k ||^2 \tag{4}$$

for all sequences $\{c_k\}_{k\in I_n}$. Thus $\{\varphi_k\}_{k\in I}$ is a Riesz basis for $\mathcal{H}$. That (b) $\Leftrightarrow$ (c) follows from the fact that the sequence of bounds $A_{I_n}^{opt}$, $n \in \mathbb{N}$, is decreasing by definition. $\square$

In the rest of this section we discuss a method for approximation of the inverse frame operator S^{-1} associated to a frame $\{\varphi_k\}_{k\in I}$. The idea is to approximate S^{-1} using finite subsets $\{\varphi_k\}_{k\in I_n}$ of the frame. We will consider two ways of choosing the finite index sets I_n:

(i) In general, we just consider any set of finite index sets I_n for which

$$I_1 \subseteq I_2 \subseteq ... \subseteq I_n \uparrow I. \tag{5}$$

(ii) In the special case where $\{\varphi_k\}_{k\in I}$ contains a Riesz basis $\{\varphi_k\}_{k\in J}$ we also consider a choice of finite index sets I_n for which

$$I_1 \subseteq I_2 \subseteq ... \subseteq I_n \uparrow J. \tag{6}$$

In both cases, we define

$$\mathcal{H}_n := \overline{\text{span}}\{\varphi_k\}_{k\in I_n} \text{ and } P_n : \quad \text{orthogonal projection of } \mathcal{H} \text{ onto } \mathcal{H}_n. \tag{7}$$

Now we have:

Theorem 1.2 *Let $\{\varphi_k\}_{k\in I}$ be a frame. Choose the index sets I_n as in (5) or - if $\{\varphi_k\}_{k\in I}$ contains a Riesz basis - as in (6). Given $n \in \mathbb{N}$, let A_n denote a lower frame bound for $\{\varphi_k\}_{k\in I_n}$ and choose a finite set J_n containing I_n such that*

$$\sum_{k\notin J_n} |<\varphi_j, \varphi_k>|^2 \leq \frac{A_n}{n\cdot|I_n|}, \ \forall j \in I_n. \tag{8}$$

Let $V_n : \mathcal{H}_n \to \mathcal{H}_n$ denote the frame operator for the finite family $\{P_n\varphi_k\}_{k\in J_n}$. Then

$$V_n^{-1}P_n f \to S^{-1}f \ \text{as } n \to \infty, \ \forall f \in \mathcal{H}.$$

Proof: For the case where the index set is chosen as in (5) the proof is given in [4]. The proof in the second case is similar, so we only sketch it. Suppose that $\{\varphi_k\}_{k\in I}$ contains a Riesz basis $\{\varphi_k\}_{k\in J}$ with lower bound A and choose the index sets I_n as in (6). Let $n \in \mathbb{N}$. First, it can be proved that (8) implies that for all $f \in \mathcal{H}_n$,

$$\sum_{k\notin J_n} |<f, \varphi_k>|^2 \quad \leq \tfrac{1}{n}||f||^2.$$

and

$$<(P_nS - V_n)f, f>= \sum_{k\notin J_n} |<f, \varphi_k>|^2.$$

So $P_nS - V_n$ is a positive operator on $\mathcal{H}_n$ and $||(P_nS - V_n)_{|\mathcal{H}_n}|| \leq \frac{1}{n}$.

We leave it to the reader to prove that $A - \frac{1}{n}$ is a lower frame bound for $\{P_n\varphi_k\}_{i\in J_n}$; this implies that $||V_n^{-1}|| \leq \frac{1}{A-\frac{1}{n}}$. Now, for $f \in \mathcal{H}$ we obtain that

$$
\begin{aligned}
& ||S^{-1}f - V_n^{-1}P_nf|| \\
\leq\ & ||(I-P_n)S^{-1}f|| + ||P_nS^{-1}f - V_n^{-1}P_nf|| \\
\leq\ & ||(I-P_n)S^{-1}f|| + ||V_n^{-1}|| \cdot ||V_nP_nS^{-1}f - P_nf|| \\
\leq\ & ||(I-P_n)S^{-1}f|| + \frac{1}{A-\frac{1}{n}}(||V_nP_nS^{-1}f - P_nSP_nS^{-1}f|| + ||P_nSP_nS^{-1}f - P_nf||) \\
\leq\ & ||(I-P_n)S^{-1}f|| + \frac{1}{A-\frac{1}{n}}(||(V_n - P_nS)P_nS^{-1}f|| + ||SP_nS^{-1}f - f||) \\
\leq\ & ||(I-P_n)S^{-1}f|| + \frac{1}{A-\frac{1}{n}}(\frac{1}{n} \cdot ||P_nS^{-1}f|| + ||S|| \cdot ||P_nS^{-1}f - S^{-1}f||) \\
\leq\ & \frac{1}{nA(A-\frac{1}{n})} \cdot ||f|| + (\frac{B}{A-\frac{1}{n}} + 1)||(I-P_n)S^{-1}f||.
\end{aligned}
$$

□

Remark: Theorem 1.2 allows us to approximate the inverse frame operator using finite dimensional linear algebra. For a frame containing a Riesz basis, the choice (6) of index sets is much better than (5). The reason is that as we have seen in Proposition 1.1, the lower bound A_n for a frame which is not a Riesz basis will converge to zero as $n \to \infty$; this forces the index sets J_n to be large in order to satisfy (8). But if $\{\varphi_k\}_{k\in J}$ is a Riesz basis, a lower bound A for $\{\varphi_k\}_{k\in J}$ will also be a lower bound for each subfamily $\{\varphi_k\}_{k\in I_n}$. That is, for each $n \in \mathbb{N}$ we can choose $A_n = A$.

In order to be able to apply Theorem 1.2 we need estimates for the lower frame bounds of finite subsets of the given frame. In the following sections we estimate the lower frame bounds for a finite set of exponentials. Estimates for the lower bounds for finite wavelet systems can be found in [6].

1.2 Frames of exponentials

From now on we specialize to frames $\{e^{i\lambda_k(\cdot)}\}_{k\in I}$ of exponentials in $L^2(-\pi,\pi)$, where $\{\lambda_k\}_{k\in I} \subseteq \mathbb{R}$ and $L^2(-\pi,\pi)$ is equipped with the inner product

$$< f, g >= \int_{-\pi}^{\pi} f(x)\overline{g(x)}dx, \ \ f, g \in L^2(-\pi,\pi).$$

A set $\{\lambda_k\}_{k\in I} \subseteq \mathbb{R}$ is *separated*, if there is some $\delta > 0$ such that for all

$k, j \in I,\ k \neq j,$

$$|\lambda_k - \lambda_j| \geq \delta.$$

The constant δ is called a *separation constant.* If $\{\lambda_k\}_{k\in I}$ is a finite union of separated sets, i.e., if

$$\{\lambda_k\}_{k\in I} = \cup_{n=1}^{K} \{\lambda_k\}_{k\in I_n}, \tag{9}$$

where each set $\{\lambda_k\}_{k\in I_n}$ is separated, we say that $\{\lambda_k\}_{k\in I}$ is *relatively separated.* Usually we need to keep track of the number K of separated sets and the separation constants for the sets $\{\lambda_k\}_{k\in I_n}$: if $\delta \in \mathbb{R}^+$ is a separation constant for all the sets $\{\lambda_k\}_{k\in I_n}$, we say that $\{\lambda_k\}_{k\in I}$ is (K, δ)-*relatively separated.*

Given a relatively separated sequence $\Lambda = \{\lambda_k\}_{k\in I}$, we define for $r > 0$,

$$n^-(r) := \min_{r_0\in\mathbb{R}} \text{card}\,\{k \in I : \lambda_k \in (r_0, r_0 + r)\}.$$

Thus $n^-(r)$ is the smallest number of points from Λ to be found in an interval of length r. The *lower density* of Λ is defined as

$$D^-(\Lambda) := \lim_{r\to\infty} \frac{n^-(r)}{r}.$$

It is well known that $\{e^{i\lambda_k(\cdot)}\}_{k\in I}$ satisfies the upper frame condition if and only if $\{\lambda_k\}_{k\in I}$ is relatively separated. Seip proved that under this condition, $\{e^{i\lambda_k(\cdot)}\}_{k\in I}$ is a frame if $D^-(\Lambda) > 1$, and furthermore that in this case $\{e^{i\lambda_k(\cdot)}\}_{k\in I}$ contains a subsequence which is a Riesz basis, cf. [12]. Seip's paper also shows that there exist frames $\{e^{i\lambda_k(\cdot)}\}_{k\in I}$ for which no subfamily $\{e^{i\lambda_k(\cdot)}\}_{k\in J}$, $J \subseteq I$, is a Riesz basis.

If $\{\lambda_k\}_{k\in\mathbb{Z}}$ is close to k for all $k \in \mathbb{Z}$, Kadec's celebrated 1/4 theorem can be used to prove that $\{e^{i\lambda_k(\cdot)}\}_{k\in\mathbb{Z}}$ is a Riesz basis for $L^2(-\pi, \pi)$. The extension below to frames was proven independently by Balan [2] and Christensen [5].

Theorem 1.3 *Let $\{\lambda_k\}_{k\in\mathbb{Z}}, \{\mu_k\}_{k\in\mathbb{Z}} \subseteq \mathbb{R}$. Suppose that $\{e^{i\mu_k x}\}_{k\in\mathbb{Z}}$ is a frame for $L^2(-\pi, \pi)$ with bounds A, B. If there exists a constant $L < 1/4$ such that*

$$|\mu_n - \lambda_n| \leq L \quad \textit{and} \quad 1 - cos\pi L + sin\pi L < \sqrt{\frac{A}{B}},$$

then $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$ *is a frame for* $L^2(-\pi,\pi)$ *with bounds*

$$A(1-\sqrt{\frac{B}{A}}(1-cos\pi L+sin\pi L))^2, B(2-cos\pi L+sin\pi L)^2$$

Typically, Theorem 1.3 is used with $\mu_k = k,\ k\in\mathbb{Z}$.

2 Lower bounds for finite exponential frames

Let $\lambda_1,\dots,\lambda_N$ be a finite sequence of distinct real numbers. Then, $\{e^{i\lambda_k(\cdot)}\}_{k=1}^N$ is a Riesz basis for its linear span in $L^2(-\pi,\pi)$. The purpose of this section is to obtain a lower frame bound for this frame.

The assumption that $\lambda_1,\dots,\lambda_N$ consists of distinct numbers is not a restriction: if $\lambda_1,\dots,\lambda_N$ contains repetitions, then the lower frame bound for $\{e^{i\lambda_k(\cdot)}\}_{k=1}^N$ is at least as big as the lower bound for the corresponding family without repetitions. Our asumption implies that $\lambda_1,\dots,\lambda_N$ is separated; we will choose a separation constant $\delta\le 1$.

In the proof, a special class of entire functions plays an important role. An entire function Φ of exponential type is called *sine type function*, if its zeroes are simple and separated and there are positive constants C_1, C_2 and τ such that

$$C_1\cdot e^{\pi|y|}\le|\Phi(x+iy)|\le C_2\cdot e^{\pi|y|}\ \ \forall\ x,y\in\mathbb{R}:|y|\ge\tau$$

holds. We shall say that Φ has *growth constants* (C_1,C_2,τ). For example, $\sin\pi(\cdot)$ is a sine-type-function with growth constants $(1/4,1,1)$, which follows easily from the triangle inequality.

Levin and Golovin have shown that the family of exponentials, derived from the zero set of a sine type function, is a Riesz basis for $L^2(-\pi,\pi)$ (cf. Young [17, Ch. 4, Th. 2]). The following lemma gives an estimate for the lower bound, involving an additional constant C_3 appearing in (10). However, Levin has shown that for any sine type function there is some $C_3>0$ such that inequality (10) holds (cf. Young [17, Ch. 4, Cor. 1 to Th. 10]).

Lemma 2.1 *Let* $\{\lambda_k\}_{k\in\mathbb{Z}}$ *be a* δ*-separated sequence of real numbers, which is the zero set of a sine type function* Φ *with growth constants* (C_1,C_2,τ)*. Let* C_3 *be a positive constant such that*

$$|\Phi'(\lambda_k)|\ge C_3\ \ \forall\,k\in\mathbb{Z}. \tag{10}$$

Then $\{e^{i\lambda_k(\cdot)}\}_{k\in\mathbb{Z}}$ *is a Riesz basis for* $L^2(-\pi,\pi)$ *with lower bound*

$$\frac{\delta C_1^2 C_3^2}{8C_2^4} e^{-8\pi\tau-\delta\pi}. \tag{11}$$

Proof: The proof follows by explicating the proof of Lemma 7 of Katsnel'son [8]: Denoty by PW the Paley–Wiener–space, consisting of all entire functions of exponential type at most π, whose restriction to $\mathbb{R}$ belongs to $L^2(\mathbb{R})$. Katsnel'son's proof of Lemma 7 in [8] shows that $\forall F \in PW$:

$$\int_{-\infty}^{\infty} |F(t-i\tau)|^2\, dt \le \frac{2\pi C_2^2}{C_1 C_3} \left(\sum_{k=-\infty}^{\infty} |F(\lambda_k)|^2 \right)^{1/2} \left(\sum_{k=-\infty}^{\infty} |F(\lambda_k - 2i\tau)|^2 \right)^{1/2}. \tag{12}$$

By two theorems of Plancherel and Pólya (cf. Boas [1, Th. 6.7.15, Th. 6.7.18]), we have

$$\left(\sum_{k=-\infty}^{\infty} |F(\lambda_k - 2i\tau)|^2 \right)^{1/2} \le 2\sqrt{\frac{e^{\pi\delta}}{\pi\delta}} \left(\int_{-\infty}^{\infty} |F(t-2i\tau)|^2\, dt \right)^{1/2} \le$$

$$\le 2e^{2\pi\tau} \sqrt{\frac{e^{\pi\delta}}{\pi\delta}} \left(\int_{-\infty}^{\infty} |F(t)|^2\, dt \right)^{1/2} \tag{13}$$

and

$$\int_{-\infty}^{\infty} |F(t)|^2\, dt \le e^{2\pi\tau} \int_{-\infty}^{\infty} |F(t-i\tau)|^2\, dt. \tag{14}$$

Combining (12), (13) and (14) shows

$$\left(\int_{-\infty}^{\infty} |F(t)|^2\, dt \right)^{1/2} \le 2e^{4\pi\tau} \sqrt{\frac{e^{\pi\delta}}{\pi\delta}} \cdot \frac{2\pi C_2^2}{C_1 C_3} \left(\sum_{k=-\infty}^{\infty} |F(\lambda_k)|^2 \right)^{1/2}.$$

By the Paley–Wiener–theorem, it hence follows that $\{e^{i\lambda_k(\cdot)}\}_{k\in\mathbb{Z}}$ is a frame for $L^2(-\pi,\pi)$ with lower bound given by (11). For the proof that $L^2(-\pi,\pi)$ is in fact a Riesz basis, we refer to Katsnel'son [8]. □

Return to the given finite δ-separated sequence $\lambda_1,\ldots,\lambda_N$ of real numbers, where $0 < \delta \le 1$. We shall construct a sequence $\{\lambda_k\}_{k\in\mathbb{Z}\setminus\{1,\ldots,N\}}$ of real numbers, such that $\{\lambda_k\}_{k\in\mathbb{Z}}$ is the zero set of some sine type function Φ. Furthermore, we shall obtain the growth constants of Φ and some C_3 occuring

in inequality (10). An application of Lemma 2.1 will give a Riesz basis $\{e^{i\lambda_k(\cdot)}\}_{k\in\mathbb{Z}}$ for $L^2(-\pi,\pi)$ and a lower frame bound. By restriction, we shall then obtain a lower bound for $\{e^{i\lambda_k(\cdot)}\}_{k=1}^N$ in $\mathcal{H}_N$.
For the construction of $\{\lambda_k\}_{k\in\mathbb{Z}\setminus\{1,\dots,N\}}$, we proceed as follows:
W.l.o.g., we may suppose

$$\operatorname{dist}(\lambda_1,\mathbb{Z}) \le \operatorname{dist}(\lambda_2,\mathbb{Z}) \le \ldots \le \operatorname{dist}(\lambda_N,\mathbb{Z}).$$

Construct inductively a sequence $\mu_1,\dots,\mu_N$ of different integers, such that

$$|\lambda_k - \mu_k| = \inf_{\mu\in\mathbb{Z}\setminus\{\mu_1,\dots,\mu_{k-1}\}} |\lambda_k - \mu|, \;\; \forall\, k = 1,\dots,N.$$

We then have

$$|\lambda_k - \mu_k| \le k - 1/2 \;\; \forall\, k \in \{1,\dots,N\}. \tag{15}$$

Define a sequence $\{\lambda_k\}_{k\in\mathbb{Z}\setminus\{1,\dots,N\}}$ such that

$$\{\lambda_k\}_{k\in\mathbb{Z}\setminus\{1,\dots,N\}} = \mathbb{Z}\setminus\{\mu_1,\dots,\mu_N\}$$

and no element in the sequence $\{\lambda_k\}_{k\in\mathbb{Z}\setminus\{1,\dots,N\}}$ occurs twice. Put

$$\Phi(z) := \sin\pi z \cdot \prod_{n=1}^{N} \frac{z-\lambda_n}{z-\mu_n}.$$

Lemma 2.2 *The function Φ constructed above is a sine type function with growth constants*

$$C_1 := \frac{1}{4(N+1)!},\; C_2 := (N+1)!,\; \tau := 1.$$

The sequence $\{\lambda_k\}_{k\in\mathbb{Z}}$ is $\delta/2$-separated and the zero-set of Φ. Furthermore, inequality (10) *holds with*

$$C_3 := \frac{2(\delta/2)^N}{(N+1)!}.$$

Proof: Since

$$|\Phi(z)| = |\sin\pi z| \cdot \prod_{n=1}^{N} \left| 1 + \frac{\mu_n - \lambda_n}{z-\mu_n} \right|,$$

we have for $|\Im z| \geq 1$ from the growth constants for $\sin \pi(\cdot)$ and from (15):

$$|\Phi(z)| \leq e^{\pi|\Im z|} \prod_{n=1}^{N} \left(1 + \frac{n-1/2}{1}\right) \leq e^{\pi|\Im z|}\,(N+1)!.$$

Similarly,

$$|\Phi(z)| \geq \frac{1}{4}\, e^{\pi|\Im z|} \prod_{n=1}^{N} \left|1 + \frac{\lambda_n - \mu_n}{z - \lambda_n}\right|^{-1} \geq \frac{e^{\pi|\Im z|}}{4(N+1)!}.$$

Hence, Φ is a sine-type-function with growth constants (C_1, C_2, τ).

That $\{\lambda_k\}_{k\in\mathbb{Z}}$ is the zero sequence of Φ and $\delta/2$-separated follows easily from the construction and from $\delta \leq 1$. Let $k \in \mathbb{Z}$. To estimate $|\Phi'(\lambda_k)|$ from below, we distinguish between two cases:
Case 1: $\lambda_k \in \mathbb{Z}$.
In particular, this is fulfilled if $k \notin \{1,\ldots,N\}$. If $k \in \{1,\ldots,N\}$ and $\lambda_k \in \mathbb{Z}$, then $\mu_k = \lambda_k$, and the occuring factor $\frac{\lambda_k-\lambda_k}{\lambda_k-\mu_k}$ in the following calculation is to be interpreted as 1. We have, using $\sin \pi\lambda_k = 0$,

$$\Phi'(\lambda_k) = \pi \cos \pi\lambda_k \prod_{n=1}^{N} \frac{\lambda_k - \lambda_n}{\lambda_k - \mu_n} = \pm\pi \prod_{n=1}^{N} \frac{\lambda_k - \lambda_n}{\lambda_k - \mu_n}.$$

From

$$\left|\frac{\lambda_k - \mu_n}{\lambda_k - \lambda_n}\right| = \left|1 + \frac{\lambda_n - \mu_n}{\lambda_k - \lambda_n}\right| \leq 1 + \frac{n-1/2}{\delta/2} \leq \frac{n+1/2}{\delta/2}$$

we conclude

$$|\Phi'(\lambda_k)| \geq \pi \frac{(\delta/2)^N}{(N+1)!},$$

hence (10) for $\lambda_k \in \mathbb{Z}$ with C_3 as defined.
Case 2: $\lambda_k \notin \mathbb{Z}$.

In this case, we must have $k \in \{1, \ldots, N\}$. Taking derivates, we obtain

$$\begin{aligned}
\Phi'(\lambda_k) &= \pi \cos \pi\lambda_k \underbrace{\prod_{n=1}^{N} \frac{\lambda_k - \lambda_n}{\lambda_k - \mu_n}}_{=0} + \sin \pi\lambda_k \left(\prod_{n=1}^{N} \frac{z - \lambda_n}{z - \mu_n} \right)' \Bigg|_{z=\lambda_k} \\
&= \sin \pi\lambda_k \prod_{n\in\{1,\ldots,N\}\setminus\{k\}} \frac{\lambda_k - \lambda_n}{\lambda_k - \mu_n} \cdot \frac{1}{\lambda_k - \mu_k} \\
&= \pm\pi \frac{\sin \pi(\lambda_k - \mu_k)}{\pi(\lambda_k - \mu_k)} \prod_{n\in\{1,\ldots,N\}\setminus\{k\}} \frac{\lambda_k - \lambda_n}{\lambda_k - \mu_n}.
\end{aligned}$$

As in the first case, we obtain

$$|\Phi'(\lambda_k)| \geq \pi \cdot \left| \frac{\sin \pi(\lambda_k - \mu_k)}{\pi(\lambda_k - \mu_k)} \right| \cdot \prod_{n\in\{1,\ldots,N\}\setminus\{k\}} \left(\frac{\delta/2}{n + 1/2} \right).$$

Now, if $\operatorname{dist}(\lambda_k, \mathbb{Z}) < \delta/2$, then we have (since $\delta \leq 1$)

$$|\mu_k - \lambda_k| = \operatorname{dist}(\lambda_k, \mathbb{Z}) \leq \frac{\delta}{2} \leq \frac{1}{2},$$

and from

$$\left| \frac{\sin x}{x} \right| \geq \frac{\sin \pi/2}{\pi/2} = \frac{2}{\pi} \quad \text{for} \quad x \in [-\pi/2, \pi/2],$$

we obtain

$$\left| \frac{\sin \pi(\lambda_k - \mu_k)}{\pi(\lambda_k - \mu_k)} \right| \geq \frac{2}{\pi}.$$

On the other hand, if $\operatorname{dist}(\lambda_k, \mathbb{Z}) \geq \delta/2$, then

$$\left| \frac{\sin \pi(\lambda_k - \mu_k)}{\pi(\lambda_k - \mu_k)} \right| \geq \frac{\sin \pi\delta/2}{\pi(k - 1/2)} = \frac{\sin \pi\delta/2}{\pi\delta/2} \cdot \frac{\delta/2}{k - 1/2} \geq \frac{2}{\pi} \cdot \frac{\delta/2}{k - 1/2}.$$

Thus, we obtain in the case $\lambda_k \notin \mathbb{Z}$:

$$|\Phi'(\lambda_k)| \geq \pi \cdot \frac{2}{\pi} \cdot \frac{\delta/2}{k - 1/2} \cdot \prod_{n\in\{1,\ldots,N\}\setminus\{k\}} \frac{\delta/2}{n + 1/2} \geq 2 \cdot \frac{(\delta/2)^N}{(N+1)!} = C_3. \ \square$$

With the sequence $\{\lambda_k\}_{k\in\mathbb{Z}\setminus\{1,\ldots,N\}}$ constructed above, it follows:

Proposition 2.3 $\{e^{i\lambda_k(\cdot)}\}_{k\in\mathbb{Z}}$ *is a Riesz basis for* $L^2(-\pi,\pi)$ *with lower frame bound*

$$A_N := 1.6 \cdot 10^{-14} \cdot (\delta/2)^{2N+1} \cdot ((N+1)!)^{-8}.$$

Proof: From Lemmas 2.1 and 2.2 follows that $\{e^{i\lambda_k(\cdot)}\}_{k\in\mathbb{Z}}$ is a Riesz basis for $L^2(-\pi,\pi)$ with lower bound

$$\begin{aligned} & \frac{\delta}{16} \cdot \frac{1}{4^2} \cdot ((N+1)!)^{-8} \cdot 4 \cdot (\delta/2)^{2N} \cdot e^{-9\pi} \\ \geq \ & 1.6 \cdot 10^{-14} \cdot (\delta/2)^{2N+1} \cdot ((N+1)!)^{-8}. \ \square \end{aligned}$$

By restriction, we now obtain:

Theorem 2.4 *Let* $\lambda_1, \ldots, \lambda_N$ *be a finite sequence of distinct real numbers. Choose a separation constant* $\delta \leq 1$. *Then* $\{e^{i\lambda_k(\cdot)}\}_{k=1}^N$ *is a Riesz basis for its linear span in* $L^2(-\pi,\pi)$ *with lower frame bound*

$$A_N = 1.6 \cdot 10^{-14} \cdot (\delta/2)^{2N+1} \cdot ((N+1)!)^{-8}.$$

Thus, the lower bound can be expressed entirely in terms of the number of elements in the set and the separation constant. Also, observe that the bound does not reflect how the sequence $\{\lambda_k\}$ is ordered. That is, in the following we can use Theorem 2.4 for *arbitrary* orderings of $\{\lambda_k\}_{k=1}^N$. Note, however, that the bounds are very small. In special cases, better estimates can be obtained using Theorem 1.3:

Example: Given $\{\lambda_k\}_{k\in\mathbb{Z}}$, suppose that for a certain N,

$$|\lambda_k - k| \leq L < \frac{1}{4}, \ k = 1, \ldots, N.$$

Let $\mu_k = k, \ k \in \mathbb{Z}$. By Theorem 1.3 the family $\{e^{i\lambda_k(\cdot)}\}_{k=1}^N \cup \{e^{ik(\cdot)}\}_{k\in\mathbb{Z}\setminus\{1,\ldots,N\}}$ is a Riesz basis for $L^2(-\pi,\pi)$ with lower bound $A = 2\pi(\cos\pi L - \sin\pi L)^2 = 2\pi(1 - \sin 2\pi L)$. Thus $\{e^{i\lambda_k(\cdot)}\}_{k=1}^N$ is a Riesz sequence (i.e., a Riesz basis for its span) with lower frame bound $2\pi(1 - \sin 2\pi L)$.

Consider in particular the case where $\lambda_k = k\delta, \ k \in \mathbb{Z}$. For $N \in \mathbb{N}$ chosen such that $L := |N - N\delta| < \frac{1}{4}$, we conclude that $\{e^{i\lambda_k(\cdot)}\}_{k=1}^N$ is a Riesz sequence with lower bound $2\pi(1 - \sin 2\pi L)$. For example, if $\delta = 0.96$, we can choose $N = 6$. Then $L = 0.24$, and we conclude that $\{e^{i0.96k(\cdot)}\}_{k=1}^6$ has the lower frame bound $2\pi(1 - \sin(2\pi \cdot 0.96)) > 0.012$.

3 Approximation of S^{-1}

In this section we assume that $\{e^{i\lambda_k(\cdot)}\}_{k\in\mathbb{Z}}$ is a frame for $L^2(-\pi,\pi)$. Thus $\{\lambda_k\}_{k\in\mathbb{Z}}$ is (K,δ)-relatively separated for appropriate choices of K,δ. We suppose that $\{\lambda_k\}_{k\in\mathbb{Z}}$ is ordered such that

$$\cdots \lambda_{-1} \leq \lambda_0 \leq \lambda_1 \leq \cdots$$

The purpose of the section is to obtain more concrete versions of Theorem 1.2 for exponential frames. First, we need a Lemma:

Lemma 3.1 *Let $n \in \mathbb{N}$. For $m \in \mathbb{N}, m \geq K$, and all $j \in \{-n,-n+1,...,n\}$, we have*

$$\sum_{|k|>n+m} |< e^{i\lambda_k(\cdot)}, e^{i\lambda_j(\cdot)} >|^2 \leq \frac{8K^2}{\delta^2(m-K+1)}.$$

Proof: When $\lambda_k \neq \lambda_j$,

$$\begin{aligned} |< e^{i\lambda_k(\cdot)}, e^{i\lambda_j(\cdot)} >| &= |\int_{-\pi}^{\pi} e^{i(\lambda_k-\lambda_j)x}dx| \\ &= \frac{1}{|\lambda_k-\lambda_j|}\cdot|e^{i(\lambda_k-\lambda_j)\pi} - e^{-i(\lambda_k-\lambda_j)\pi}| \\ &\leq \frac{2}{|\lambda_k-\lambda_j|}. \end{aligned}$$

In case $\{\lambda_k\}_{k\in\mathbb{Z}}$ is δ-separated, we immediately get that

$$|\lambda_k-\lambda_j| \geq \delta|k-j|, \ \forall k,j.$$

If $\{\lambda_k\}_{k\in\mathbb{Z}}$ is (K,δ)-relatively separated, the same idea gives that when $|k-j| \geq K$, then

$$|\lambda_k-\lambda_j| \geq \frac{\delta}{K}(|k-j|-K+1), \ \forall k,j.$$

Thus, if $j \in \{-n,-n+1,...,n\}$ and $m \geq K$,

$$\begin{aligned} \sum_{|k|>n+m} |< e^{i\lambda_k(\cdot)}, e^{i\lambda_j(\cdot)} >|^2 &\leq \sum_{|k|>n+m} \frac{4}{|\lambda_k-\lambda_j|^2} \\ &\leq \frac{8K^2}{\delta^2}\sum_{k=n+m+1}^{\infty}\frac{1}{(|k-j|-K+1)^2} \\ &\leq \frac{8K^2}{\delta^2}\sum_{k=m-K+2}^{\infty}\frac{1}{k^2}. \end{aligned}$$

By the integral criteria,

$$\sum_{k=m-K+2}^{\infty} \frac{1}{k^2} \leq \int_{m-K+1}^{\infty} \frac{1}{x^2} dx = \frac{1}{m-K+1},$$

from which the lemma follows. □

Now consider the general choice of index set (5). For $n \in \mathbb{N}$, let $I_n := \{-n, -n+1, ..., n\}$ and define $\mathcal{H}_n$ and P_n as in (7). Note that $|I_n| = 2n+1$. Applying Theorem 1.2 we now have:

Theorem 3.2 *For $n \in \mathbb{N}$, let A_n denote a lower frame bound for $\{e^{i\lambda_k(\cdot)}\}_{k=-n}^{n}$, and choose*

$$m(n) \geq \frac{8K^2 n(2n+1)}{\delta^2 A_n} + K - 1.$$

Let $V_n : \mathcal{H}_n \to \mathcal{H}_n$ denote the frame operator for the finite family $\{P_n e^{i\lambda_k(\cdot)}\}_{|k| \leq n+m(n)}$. Then for all $f \in L^2(-\pi, \pi)$,

$$V_n^{-1} P_n f \to S^{-1} f, \ \text{as } n \to \infty.$$

Unfortunately, $m(n)$ is forced to be large when A_n is small. We now show how a better result can be obtained when the choice of index set (6) is available. Suppose that $\{e^{i\lambda_k(\cdot)}\}_{k\in\mathbb{Z}}$ contains a subfamily $\{e^{i\lambda_k(\cdot)}\}_{k\in J}$ which is a Riesz basis for $L^2(-\pi, \pi)$ with lower bound A. By choosing the subfamilies $\{I_n\}_{n=1}^{\infty}$ as in (6) we obtain that each family $\{e^{i\lambda_k(\cdot)}\}_{k\in I_n}$ is a Riesz basis for its span with lower frame bound A. For $n \in \mathbb{N}$, define again $\mathcal{H}_n$ and P_n as in (7). Also, let $\tilde{n} := \max_{k\in I_n} |k|$ and let

$$J_n = \{\lambda_k\}_{|k| \leq m(n)+\tilde{n}};$$

here $m(n)$ is a natural number, which has to be chosen such that the condition (8) is satisfied.

With the above definitions we have

Theorem 3.3 *Suppose that the frame $\{e^{i\lambda_k(\cdot)}\}_{k\in\mathbb{Z}}$ contains a Riesz basis $\{e^{i\lambda_k(\cdot)}\}_{k\in J}$ with lower bound A. Choose*

$$m(n) \geq \frac{8K^2 \cdot n \cdot |I_n|}{\delta^2 A} + K - 1$$

and let $V_n : \mathcal{H}_n \to \mathcal{H}_n$ *denote the frame operator for the finite family* $\{P_n e^{i\lambda_k(\cdot)}\}_{k\in J_n}$. *Then for all* $f \in L^2(-\pi,\pi)$,

$$V_n^{-1}P_nf \to S^{-1}f, \text{ as } n \to \infty.$$

The proof of Theorem 3.3 follows by Lemma 3.1 and Theorem 1.2 .

Example: Let $\delta \in]0, \frac{1}{2}[$ and consider

$$\lambda_k := \delta k, \ k \in \mathbb{Z}.$$

Then $\{e^{i\lambda_k(\cdot)}\}_{k\in\mathbb{Z}}$ is a frame for $L^2(-\pi,\pi)$, cf. [13]. For each $k \in \mathbb{Z}$, there exists $l(k) \in \mathbb{Z}$ such that

$$|\lambda_{l(k)} - k| \le \frac{1}{2}\delta < \frac{1}{4}.$$

By Theorem 1.3, $\{e^{i\lambda_{l(k)}(\cdot)}\}_{k\in\mathbb{Z}}$ is a Riesz basis for $L^2(-\pi,\pi)$ with lower frame bound

$$2\pi(1-\sin(2\pi\frac{1}{2}\delta)) = 2\pi(1-\sin\pi\delta).$$

Given n, let $I_n = \{l(k)\}_{k=-n}^{n}$ and choose

$$m(n) \ge \frac{4n(2n+1)}{\pi\delta^2(1-\sin\pi\delta)}.$$

Then by Theorem 3.3 we conclude that for all $f \in L^2(-\pi,\pi)$,

$$V_n^{-1}P_nf \to S^{-1}f, \text{ as } n \to \infty.$$

Note, that the example easily can be generalized to cover all frames $\{e^{i\lambda_k(\cdot)}\}_{k\in\mathbb{Z}}$ for which $\{\lambda_k\}_{k\in\mathbb{Z}}$ has a subsequence $\{\lambda_{l(k)}\}_{k\in\mathbb{Z}}$ such that

$$|\lambda_{l(k)} - k| \le L < \frac{1}{4}, \ \forall k \in \mathbb{Z}.$$

Remark: As mentioned before, Seip showed that for a relatively separated sequence $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$, $D^-(\Lambda) > 1$ is a sufficient condition for $\{e^{i\lambda_k(\cdot)}\}_{k\in\mathbb{Z}}$ to contain a Riesz basis $\{e^{i\lambda_k(\cdot)}\}_{k\in J}$, where $J \subset \mathbb{Z}$. Thus, Theorem 3.3 can in principle be used for all frames satisfying his conditions. However, we need to know an estimate for the lower bound for the Riesz basis, which is not included in Seip's article. As far as we know, the only bounds for $\{e^{i\lambda_k(\cdot)}\}_{k\in J}$ that are given so far are by Lindner [11],[10]. They apply if Λ is separated and if there exist $\rho > 1$ and $L \geq 0$ such that $|\lambda_k - k/\rho| \leq L \;\forall\, k \in \mathbb{Z}$; this is a classical condition by Duffin and Schaeffer, that can be found already in the paper [7]. It should be observed that any lower frame bound for $\{e^{i\lambda_k(\cdot)}\}_{k\in J}$ is also a lower bound for $\{e^{i\lambda_k(\cdot)}\}_{k\in\mathbb{Z}}$, so the bounds in [11],[10] are also bounds for the whole sequence $\{e^{i\lambda_k(\cdot)}\}_{k\in\mathbb{Z}}$. However, for the latter sequence better bounds have been obtained by Voß [16],[15],[14].

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References

[1] Boas, R. P., Jr.: *Entire Functions.* Academic Press, New York (1954).

[2] Balan, R.: *Stability theorems for Fourier frames and wavelet Riesz bases.* J. Fourier Anal. Appl. **3** no.5 (1997), p. 499-504

[3] Casazza, P.G. and Christensen, O.: *Approximation of the inverse frame operator and applications to Weyl-Heisenberg frames.* J. Approx. Theory **99** (2000). 20 p.

[4] Christensen, O.: *Finite-dimensional approximation of the inverse frame operator and applications to Weyl-Heisenberg frames and wavelet frames.* J. Fourier Anal. Appl. **6** no. 1 (2000), p. 79-91.

[5] Christensen, O.: *Perturbations of frames and applications to Gabor frames.* In "Gabor Analysis and Algorithms: Theory and Applications", p. 193- 209. Eds. H. G. Feichtinger and T. Strohmer. Birkhauser, 1997.

[6] Christensen, O. and Lindner, A. M.: *Lower bounds for finite Gabor and wavelet systems.* Submitted, 2000.

[7] Duffin, R.J. and Schaeffer, A.C.: *A class of nonharmonic Fourier series.* Trans. Amer. Math. Soc. **72** (1952) p. 341-366.

[8] Katsnel'son, V. É.: *Exponential bases in* L^2. Funktsional'nyi Analiz i Ego Prilozheniya , **5** no. 1 (1971), p. 37-47 (russian); engl. transl. in: Funct. Anal. Appl., **5**, p. 31-38. 1971.

[9] Kim, H.O. and Lim, J.K.: *New characterizations of Riesz bases.* Appl. Comp. Harm. Anal. **4** (1997), p. 222-229.

[10] Lindner, A.: *Über exponentielle Rahmen, insbesondere exponentielle Riesz-Basen.* Thesis, Univ. Erlangen-Nuremberg, 1999.

[11] Lindner, A.M.: *On lower bounds of exponential frames.* J. Fourier. Anal. Appl. **5** no. 2 (1999), p.187-194.

[12] Seip, K.: *On the connection between exponential bases and certain related sequences in $L^2(-\pi, \pi)$.* J. Funct. Anal. **130** (1995), p.131-160.

[13] Seip, K.: *A simple construction of exponential bases in L^2 of the union of several invertals.* Proc. Edinburgh. Math. Soc. **38** (1995), p. 171-177.

[14] Voß, J., *Irreguläres Abtasten: Fehleranalyse, Anwendungen und Erweiterungen*, Thesis, Univ. Erlangen-Nuremberg (1999).

[15] Voß, J., *Irregular sampling: error analysis, applications, and extensions*, Mitt. Math. Sem. Giessen **238** (1999).

[16] Voß, J.J., *On discrete and continuous norms in Paley-Wiener spaces and consequences for exponential frames.* J. Fourier Anal. Appl. **5** no. 2 (1999), p.195-203.

[17] Young, R.M.: *An introduction to nonharmonic Fourier series.* Academic Press, New York, 1980.

Ole Christensen
Technical University of Denmark
Department of Mathematics
Building 303
2800 Lyngby
Denmark
email: Ole.Christensen@mat.dtu.dk

Alexander M. Lindner
Zentrum Mathematik
Technische Universität München
D-80290 München
Germany
email: lindner@mathematik.tu-muenchen.de