On the Limit Behavior of the Periodogram of High-Frequency Sampled Stable CARMA Processes

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In this paper we consider a continuous-time autoregressive moving average (CARMA) process $(Y_t)_{t\in\mathbb{R}}$ driven by a symmetric α-stable Lévy process with $\alpha \in (0,2]$ sampled at a high-frequency time-grid $\{0, \Delta_n, 2\Delta_n, \ldots, n\Delta_n\}$, where the observation grid gets finer and the last observation tends to infinity as $n \to \infty$. We investigate the normalized periodogram $I_{n,Y^{\Delta_n}}(\omega) = |n^{-1/\alpha}\sum_{k=1}^{n} Y_{k\Delta_n} \mathrm{e}^{-i\omega k}|^2$. Under suitable conditions on Δ_n we show the convergence of the finite-dimensional distribution of both $\Delta_n^{2-2/\alpha}[I_{n,Y^{\Delta_n}}(\omega_1\Delta_n), \ldots, I_{n,Y^{\Delta_n}}(\omega_m\Delta_n)]$ for $(\omega_1, \ldots, \omega_m) \in (\mathbb{R}\backslash\{0\})^m$ and of self-normalized versions of it to functions of stable distributions. The limit distributions differ depending on whether $\omega_1, \ldots, \omega_m$ are linearly dependent or independent over $\mathbb{Z}$. For the proofs we require methods from the geometry of numbers.

AMS Subject Classification 2010: Primary: 60F05, 62M15
Secondary: 62E20, 60E07, 60G10

Keywords: CARMA process, high-frequency data, lattice, Lévy process, periodogram, self-normalized periodogram, stable distribution.

1 Introduction

Continuous-time ARMA (CARMA) processes are the continuous-time versions of the well known ARMA processes in discrete time having short memory. The advantage of continuous-time modelling is that it allows handling of irregularly spaced time series and in particular of high-frequency data often appearing in turbulence and finance. In this paper we consider a CARMA process $Y = (Y_t)_{t\in\mathbb{R}}$ driven by a symmetric α-stable Lévy process $(L_t)_{t\in\mathbb{R}}$. Before we start with its definition, we recall that a real-valued random variable X is called symmetric α-stable ($S\alpha S$) with index of stability $\alpha \in (0,2]$, if its characteristic function is of the form

$$\Phi_X(z) = \mathbb{E}\left[\exp\{izX\}\right] = \exp\left\{-\sigma^\alpha |z|^\alpha\right\}, \quad z \in \mathbb{R},$$

for some $\sigma \geq 0$, and a real random vector $X = (X_1, \ldots, X_d)^T$ is $S\alpha S$, if all linear combinations $\sum_{i=1}^{d} a_i X_i$, $(a_1, \ldots, a_d)^T \in \mathbb{R}^d$ are $S\alpha S$; see the monograph of Samorodnitsky and Taqqu [33] for details on stable distributions. Then a symmetric α-stable Lévy process $(L_t)_{t\in\mathbb{R}}$ is a stochastic process with $L_0 = 0$ almost surely, independent and stationary increments which are $S\alpha S$ distributed with characteristic function

$$\Phi_{L_t}(z) = \mathbb{E}\left[\exp\{izL_t\}\right] = \exp\left\{-|t|\sigma_L^\alpha |z|^\alpha\right\}, \quad z, t \in \mathbb{R},$$

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for some $\sigma_L \geq 0$ and almost surely càdlàg sample paths (cf. the book of Sato [34] on Lévy processes). A symmetric α-stable CARMA process is then defined as follows. Let $(L_t)_{t\in\mathbb{R}}$ be a symmetric α-stable Lévy process. Assume that we have given $p, q \in \mathbb{N}$, $p > q$, and $a_1, \ldots, a_p, c_0, \ldots, c_q \in \mathbb{R}$, $a_p, c_0 \neq 0$, set

$$A := \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \ldots & 0 & 1 \\ -a_p & -a_{p-1} & \ldots & \ldots & -a_1 \end{pmatrix} \in \mathbb{R}^{p\times p}$$

and let $(X_t)_{t\in\mathbb{R}}$ be a strictly stationary solution to the stochastic differential equation

$$\mathrm{d}X_t = AX_t\,\mathrm{d}t + e_p\,\mathrm{d}L_t, \quad t \in \mathbb{R}, \tag{1.1a}$$

where e_p denotes the p-th unit vector in $\mathbb{R}^p$. Then the process

$$Y_t := c^T X_t, \quad t \in \mathbb{R}, \tag{1.1b}$$

with $c = (c_q, c_{q-1}, \ldots, c_{q-p+1})^T$ (where we use the convention $c_j = 0$ for $j < 0$) is said to be a *symmetric α-stable CARMA process* of order (p, q). Necessary and sufficient conditions for the existence of a strictly stationary CARMA process are given in [11]. A CARMA process can be interpreted as a solution to the formal p-th order stochastic differential equation

$$a(\mathrm{D})Y_t = c(\mathrm{D})\mathrm{D}L_t, \quad t \in \mathbb{R},$$

where D denotes the differential operator with respect to t and

$$a(z) := z^p + a_1 z^{p-1} + \ldots + a_p \qquad \text{and} \qquad c(z) := c_0 z^q + c_1 z^{q-1} + \ldots + c_q$$

are the autoregressive and the moving average polynomial, respectively. Hence, $S\alpha S$ CARMA processes can be seen as the continuous-time analog of $S\alpha S$ (discrete-time) ARMA processes. The representation (1.1) of a CARMA process is the *controller canonical state space representation* going back to [7]. Alternatively there exists also the *observer canonical form* of a CARMA process (see (2.8) below) as derived in [28] for multivariate CARMA models. For an overview and a comprehensive list of references on CARMA processes we refer to [8, 12]. CARMA processes are important for stochastic modelling in many areas of application as, e.g., signal processing and control (cf. [19, 27]), econometrics (cf. [3, 30]), high-frequency financial econometrics (cf. [38]) and financial mathematics (cf. [2]). Stable CARMA processes are particularly relevant in modelling energy markets (cf. [1, 18]).

The aim of this paper is to investigate the sampled sequence $Y^\Delta := (Y_{k\Delta})_{k\in\mathbb{Z}}$ of a *causal* (i.e., current values of the process only depend on *past* values of the driving process) stable CARMA process, meaning we only observe the underlying CARMA process $(Y_t)_{t\in\mathbb{R}}$ at equidistant time points $0, \Delta, 2\Delta, \ldots$ with $\Delta > 0$ small as used for modelling high-frequency data (cf. [10, 15]), and to study the asymptotic behavior of the sampled process Y^Δ in the frequency domain. In the time domain the autocovariance function

$$\gamma_Y(h) = \frac{\sigma_L^2}{\pi} \int_{-\infty}^{\infty} \mathrm{e}^{ih\omega} \frac{|c(i\omega)|^2}{|a(i\omega)|^2}\,\mathrm{d}\omega = c^T \mathrm{e}^{|h|A}\,\gamma_X(0)\,c, \quad h \in \mathbb{R}, \tag{1.2}$$

with $\gamma_X(0) = 2\sigma_L^2 \int_0^\infty \mathrm{e}^{sA} e_p e_p^T \mathrm{e}^{sA^T}\,\mathrm{d}s$, gives information about the dependence structure, whereas in the frequency domain the spectral density

$$f_Y(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \gamma_Y(h)\,\mathrm{e}^{-ih\omega}\,\mathrm{d}h = \frac{\sigma_L^2}{\pi} \cdot \frac{|c(i\omega)|^2}{|a(i\omega)|^2}, \quad \omega \in \mathbb{R}, \tag{1.3}$$

gives information about the periodicities of the CARMA process. Both the spectral density and the autocovariance function exist only for $\alpha = 2$. The spectral density of the sampled process Y^Δ is

$$f_\Delta(\omega) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \gamma_Y(k\Delta)\,\mathrm{e}^{-ik\omega} = \frac{1}{\Delta}\sum_{k=-\infty}^{\infty} f_Y\Big(\frac{\omega+2k\pi}{\Delta}\Big), \quad -\pi \le \omega \le \pi, \tag{1.4}$$

where the second equality follows from [6, p. 206]. It is related to f_Y by

$$\lim_{\Delta\to 0} \Delta f_\Delta(\omega\Delta)\,\mathbb{1}_{[-\frac{\pi}{\Delta},\frac{\pi}{\Delta}]}(\omega) = f_Y(\omega), \quad \omega\in\mathbb{R}, \tag{1.5}$$

(see p. 12 for a proof). Loosely spoken, this means that in the limit $\Delta\to 0$ we can identify every CARMA process from its equidistantly sampled observations. The question arises whether this is also true if we study the empirical version of the spectral density, the *periodogram*. We investigate normalized and self-normalized versions. The normalized periodogram of Y^Δ at frequency $\omega\in[-\pi,\pi]$ is given by

$$I_{n,Y^\Delta}(\omega) = \Big| n^{-1/\alpha}\sum_{k=1}^{n} Y_{k\Delta}\,\mathrm{e}^{-i\omega k}\Big|^2.$$

Equation (1.5) suggests that we obtain a non-trivial limit by studying the behavior of the properly rescaled periodogram I_{n,Y^Δ} of the sampled CARMA process at point $\omega\Delta$. More precisely, we will show that the finite-dimensional distribution of the periodogram $\Delta^{2-2/\alpha}[I_{n,Y^\Delta}(\omega_1\Delta),\dots,I_{n,Y^\Delta}(\omega_m\Delta)]$ for $(\omega_1,\dots,\omega_m)\in(\mathbb{R}\backslash\{0\})^m$ converges weakly to a function of stable distributions, if simultaneously the grid distance Δ goes to 0 with a suitable rate and the number of observations n goes to infinity (see Theorem 2.6). A small grid distance and a huge number of observations reflect the behavior of high-frequency data. A consequence of this is the fact that the normalized periodogram is not a consistent estimator of the so-called power transfer function $|c(i\cdot)|^2/|a(i\cdot)|^2$. Moreover, if $(L_t)_{t\in\mathbb{R}}$ is a Brownian motion then the limit distribution has independent components. In contrast, if $(L_t)_{t\in\mathbb{R}}$ is a $S\alpha S$-stable Lévy process with $\alpha\in(0,2)$ then the components are dependent. In both cases the limit distributions differ depending on whether $\omega_1,\dots,\omega_m$ are linearly dependent or independent over $\mathbb{Z}$. However, the one-dimensional distributions do not depend on ω. Our result is comparable to Brockwell and Davis [9, Chapter 10.3] for the finite variance and Klüppelberg and Mikosch [23, Theorem 2.4] for the stable case, respectively, of an ARMA process in discrete time; although the α-stable limit distributions are different in the discrete-time and the continuous-time model.

Since the normalized periodogram depends on α, which is in general an unknown parameter, we also analyze different normalizations. So-called *self-normalized periodogram* versions are given by

$$\widetilde{I}_{n,Y^\Delta}(\omega) = \frac{\big|\sum_{k=1}^{n} Y_{k\Delta}\,\mathrm{e}^{-i\omega k}\big|^2}{(\sum_{k=1}^{n} Y_{k\Delta})^2} \quad \text{and} \quad \widehat{I}_{n,Y^\Delta}(\omega) = \frac{\big|\sum_{k=1}^{n} Y_{k\Delta}\,\mathrm{e}^{-i\omega k}\big|^2}{\sum_{k=1}^{n} Y_{k\Delta}^2}, \quad -\pi\le\omega\le\pi, \tag{1.6}$$

having the obvious benefit that they only depend on the data and not on the index of stability α. Again the finite-dimensional distributions of $\widetilde{I}_{n,Y^\Delta}(\Delta\cdot)$ converge to functions of stable distributions and do not provide consistent estimators (cf. Theorem 2.10). The limit distribution has similar properties as for the normalized periodogram. The second version $\widehat{I}_{n,Y^\Delta}$ has to be rescaled with Δ as in (1.5) to derive a limit result (see Theorem 2.11). Our conclusions for the self-normalized periodogram are in analogy to those for ARMA models in discrete time obtained by Klüppelberg and Mikosch [24].

The paper is structured in the following way. We start with our main results in Section 2. The discrete-time sampled CARMA process Y^Δ has a representation as an MA process in discrete time where the noise sequence is p-dependent. In Section 2.1 we investigate this moving average structure in detail. Then the asymptotic behavior of the normalized and the self-normalized periodogram is topic of Sections 2.2 and 2.3. Finally, in Section 3 we derive results for the characterization of the limit distributions of the normalized and the self-normalized periodogram versions. These are based on the geometry of numbers and on manifolds. The proofs of the results are presented in Section 4.

Notation

We use $\mathbb{N}^*$ and $\mathbb{R}^*$ for the natural and real numbers, respectively, excluding zero and $\mathbb{Z}$ for the integers. For the minimum of two real numbers $a, b \in \mathbb{R}$ we write shortly $a \wedge b$ and for the maximum $a \vee b$. The real and imaginary part of a complex number $z \in \mathbb{C}$ is written as $\Re(z)$ and $\Im(z)$, respectively, and its complex conjugate as $\bar{z}$. For two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ we say $a_n \sim b_n$ as $n \to \infty$ if $\lim_{n\to\infty} a_n/b_n = 1$. The transpose of a matrix M is written as M^T and the m-dimensional identity matrix shall be denoted by I_m.

For a subset $S \subseteq \mathbb{N}$ and $k \in \mathbb{N}$ we set

$$\binom{S}{k} := \{B \subseteq S : |B| = k\}.$$

The orthogonal complement of $S \subseteq \mathbb{R}^m$ is denoted by $S^\perp$.

On $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ the Euclidean norm is denoted by $|\cdot|$ whereas on $\mathbb{K}^m$ it will be usually written as $\|\cdot\|$. A scalar product on a linear space is written as $\langle \cdot\,, \cdot \rangle$; in $\mathbb{R}^m$ and $\mathbb{C}^m$, we usually take the Euclidean one. If X and Y are normed linear spaces, let $B(X,Y)$ be the set of bounded linear operators from X into Y. On $B(X,Y)$ we will usually use the operator norm which, in the case of Y being a Banach space, turns $B(X,Y)$ itself into a Banach space. In particular we always equip $B(\mathbb{K}^m, \mathbb{K}^n)$ with the corresponding operator norm if not stated otherwise.

For two random variables X and Y the notation $X \stackrel{\mathscr{D}}{=} Y$ means equality in distribution. If we consider a sequence of random variables $(X_n)_{n\in\mathbb{N}}$, we denote convergence in probability to some random variable X by $X_n \stackrel{\mathbb{P}}{\to} X$ as $n \to \infty$ and convergence in distribution by $X_n \stackrel{\mathscr{D}}{\to} X$ as $n \to \infty$.

2 Main Results

Before stating the main results, we establish the moving average structure of the sampled sequence together with two auxiliary lemmata.

2.1 Moving average structure of the sampled process

The aim of this section is to better understand the structure of the discrete-time sampled process Y^Δ. Let $\lambda_1, \dots, \lambda_p$ denote the eigenvalues of A. By defining the filter $\Phi^\Delta(\mathrm{B}) := \prod_{j=1}^p \left(1 - \mathrm{e}^{\lambda_j \Delta}\mathrm{B}\right)$ where, as usual, B denotes the backward shift operator and applying it to the sampled sequence Y^Δ, we obtain (cf. [11, Lemma 2.1]), for any $k \in \mathbb{Z}$,

$$\widetilde{Z}_{k,\Delta} := \Phi^\Delta(\mathrm{B})\, Y_k^\Delta = \sum_{r=1}^{p} Z^r_{k-r+1,\Delta}, \tag{2.1}$$

where

$$Z^r_{k,\Delta} := \int_{(k-1)\Delta}^{k\Delta} c^T \left(-\sum_{j=0}^{r-1} \Phi_j^\Delta \, \mathrm{e}^{(r-1-j)\Delta A} \right) \mathrm{e}^{(k\Delta - s)A}\, e_p \,\mathrm{d}L_s, \quad r = 1, \dots, p, \tag{2.2a}$$

and

$$\Phi_j^\Delta := (-1)^{j+1} \cdot \sum_{\{i_1,\dots,i_j\} \in \binom{\{1,\dots,p\}}{j}} \mathrm{e}^{\Delta \cdot \sum_{m=1}^{j} \lambda_{i_m}}, \qquad j = 0, 1, \dots, p. \tag{2.2b}$$

It is easy to see that we can rewrite the filter as $\Phi^\Delta(z) = \prod_{j=1}^p \left(1 - \mathrm{e}^{\lambda_j \Delta} z\right) = -\sum_{j=0}^p \Phi_j^\Delta z^j$ for any $z \in \mathbb{C}$. In this paper we will suppose that the eigenvalues $\lambda_1, \dots, \lambda_p$ of A have strictly negative real parts (see Assumption 1 below). Under this assumption we observe that $\Phi^\Delta(z) \neq 0$ for all $|z| \leq 1$ and thus deduce, for any $|z| \leq 1$,

$$\Psi^\Delta(z) := (\Phi^\Delta(z))^{-1} = \sum_{j=0}^{\infty} \Psi_j^\Delta z^j \qquad \text{with} \qquad \Psi_j^\Delta = \sum_{\substack{j_1,\dots,j_p \in \{0,1,\dots,j\} \\ \sum_{m=1}^p j_m = j}} \mathrm{e}^{\Delta \cdot \sum_{m=1}^p \lambda_m j_m}, \;\; j \in \mathbb{N}.$$

We can hence rewrite eq. (2.1) as

$$Y_k^{\Delta} = \Psi^{\Delta}(\mathrm{B})\,\widetilde{Z}_{k,\Delta}, \quad k \in \mathbb{Z}, \tag{2.3}$$

showing that the sampled CARMA process Y^{Δ} is a (discrete-time) moving average process of the noise sequence $\widetilde{Z}^{\Delta} := (\widetilde{Z}_{k,\Delta})_{k\in\mathbb{Z}}$. A challenge is that $\widetilde{Z}^{\Delta}$ is not an i.i.d. sequence; it is p-dependent. For this reason we define, for any $k \in \mathbb{Z}$, $\omega \in \mathbb{R}$ and $m \in \{1, \ldots, p\}$, the auxiliary (random) functions

$$\tilde{\tilde{Z}}_{k,\Delta}(\omega) := \sum_{r=1}^{p} Z_{k,\Delta}^{r}\,\mathrm{e}^{-i\omega(r-1)} \quad \text{and} \quad f_{\Delta}^{(m)}(\omega) := \sum_{r=1}^{p} \mathrm{e}^{-i\omega(r-1)}\left(-\sum_{j=0}^{r-1} \Phi_j^{\Delta}\,\mathrm{e}^{(r-1-j)\Delta\lambda_m}\right). \tag{2.4}$$

In contrast to $\widetilde{Z}^{\Delta}$ we have now that $\tilde{\tilde{Z}}^{\Delta}(\omega) := (\tilde{\tilde{Z}}_{k,\Delta})_{k\in\mathbb{Z}}(\omega)$ is an i.i.d. sequence, and the idea is to rewrite the periodogram essentially by means of $\tilde{\tilde{Z}}^{\Delta}(\omega)$. Then the next auxiliary lemma holds.

Lemma 2.1.

(i) Under the assumption that the eigenvalues $\lambda_1, \ldots, \lambda_p$ of A are distinct, we have, for any $\Delta > 0$, $r \in \{1, \ldots, p\}$, $k \in \mathbb{Z}$ and $s \in \mathbb{R}$,

$$c^T\left(-\sum_{j=0}^{r-1} \Phi_j^{\Delta}\,\mathrm{e}^{(r-1-j)\Delta A}\right)\mathrm{e}^{(k\Delta-s)A} e_p = \sum_{m=1}^{p} \frac{c(\lambda_m)}{a'(\lambda_m)}\left(-\sum_{j=0}^{r-1} \Phi_j^{\Delta}\,\mathrm{e}^{(r-1-j)\Delta\lambda_m}\right)\mathrm{e}^{(k\Delta-s)\lambda_m}.$$

(ii) We have, for any $\lambda \in \mathbb{C}$,

$$\frac{1}{\Delta}\int_0^{\Delta} \left|\mathrm{e}^{(\Delta-s)\lambda} - 1\right|^{\alpha} \mathrm{d}s \to 0 \quad \text{as } \Delta \to 0.$$

(iii) Assume that the eigenvalues $\lambda_1, \ldots, \lambda_p$ of A possess non-vanishing real parts. We then have, for any $m \in \{1, \ldots, p\}$ and any $\omega \in \mathbb{R}$,

$$f_{\Delta}^{(m)}(\omega\Delta) \sim \Delta^{p-1}\,a(i\omega)\,\frac{1}{i\omega - \lambda_m} \quad \text{as } \Delta \to 0.$$

(iv) Assume that the eigenvalues $\lambda_1, \ldots, \lambda_p$ of A are distinct and possess non-vanishing real parts. Then we have, for any $\omega \in \mathbb{R}$,

$$\sum_{m=1}^{p} \frac{c(\lambda_m)}{a'(\lambda_m)} \cdot \frac{1}{i\omega - \lambda_m} = \frac{c(i\omega)}{a(i\omega)}.$$

By virtue of Lemma 2.1(i), eqs. (2.2a) and (2.4) we obtain that

$$\left(\tilde{\tilde{Z}}_{k,\Delta}\right)_{k\in\mathbb{Z}}(\omega) = \left(\int_{(k-1)\Delta}^{k\Delta} \sum_{m=1}^{p} \frac{c(\lambda_m)}{a'(\lambda_m)} f_{\Delta}^{(m)}(\omega)\,\mathrm{e}^{(k\Delta-s)\lambda_m}\,\mathrm{d}L_s\right)_{k\in\mathbb{Z}} =: \left(\int_{(k-1)\Delta}^{k\Delta} g_{\Delta,\omega}^{(k)}(s)\,\mathrm{d}L_s\right)_{k\in\mathbb{Z}} \tag{2.5}$$

is an i.i.d. sequence of complex $S\alpha S$ random variables since $g_{\Delta,\omega}^{(k)} : \mathbb{R} \to \mathbb{C}$ is complex-valued. Recall that integration of complex-valued deterministic functions with respect to a $S\alpha S$ Lévy process is well defined as a limit in probability for all functions in $L^{\alpha}(\mathbb{C}) := \{f : \mathbb{R} \to \mathbb{C} \text{ measurable}, \int_{\mathbb{R}} |f(x)|^{\alpha}\,\mathrm{d}x < \infty\}$ (for further details, see [33, Section 3.4 and Section 6.2]). The characteristic function of the stable integral $\int_{\mathbb{R}} g\,\mathrm{d}L$ is given by

$$\mathbb{E}\left[\exp\left\{i z_1 \int_{\mathbb{R}} \Re(g(s))\,\mathrm{d}L_s + i z_2 \int_{\mathbb{R}} \Im(g(s))\,\mathrm{d}L_s\right\}\right] = \exp\left\{-\sigma_L^{\alpha}\int_{\mathbb{R}} |z_1\Re(g(x)) + z_2\Im(g(x))|^{\alpha}\,\mathrm{d}x\right\} \tag{2.6}$$

for any $z_1, z_2 \in \mathbb{R}$ (cf. [33, Example 6.1.5 and Proposition 6.2.1 (i)]) such that $(\Re(\int_{\mathbb{R}} g\,\mathrm{d}L), \Im(\int_{\mathbb{R}} g\,\mathrm{d}L))$ is $S\alpha S$.

Finally, we require the following conclusions for $(\Psi_j^{\Delta})_{j\in\mathbb{N}}$ for the proofs of our results.

Lemma 2.2. *Suppose $\Delta = \Delta_n \to 0$ as $n \to \infty$ and that the eigenvalues $\lambda_1, \ldots, \lambda_p$ of A possess strictly negative real parts. Then we have:*

(i) There is a constant $C(p) > 0$ such that

$$\left|\Psi_j^{\Delta_n}\right| \le C(p)\,\Delta_n^{-(p-1)}\,\mathrm{e}^{\Delta_n \lambda_{\max} j} \quad \forall j \in \mathbb{N} \qquad \textit{where } \lambda_{\max} := \max_{k\in\{1,\ldots,p\}} \Re(\lambda_k) \in (-\infty, 0).$$

(ii) If $n\Delta_n^{1+\delta} \overset{n\to\infty}{\to} \infty$ for some $\delta > 0$, then we have

$$\sum_{j=n+1}^{\infty} \left|\Psi_j^{\Delta_n}\right| \overset{n\to\infty}{\to} 0 \quad \textit{and} \quad \frac{\Delta_n^{\alpha}}{n} \sum_{k=-\infty}^{-n-1} \left(\sum_{j=1-k}^{n-k} \left|\Psi_j^{\Delta_n}\right| \right)^{\alpha} \overset{n\to\infty}{\to} 0.$$

(iii) If $n\Delta_n \overset{n\to\infty}{\to} \infty$, then $\Delta_n^{\alpha p} n^{-1} \sum_{k=1-n}^{1-p} \left(\sum_{j=1-k}^{n} \left|\Psi_j^{\Delta_n}\right|\right)^{\alpha} \overset{n\to\infty}{\to} 0$.

(iv) If $n\Delta_n^{\alpha(p-1)+1-\alpha} \overset{n\to\infty}{\to} \infty$, then

$$\frac{\Delta_n^{\alpha}}{n} \sum_{k=2-p-n}^{-1} \left(\sum_{j=1\vee(2-p-k)}^{n\wedge(-k)} \left|\Psi_j^{\Delta_n}\right| \right)^{\alpha} \overset{n\to\infty}{\to} 0.$$

(v) If $n\Delta_n^{\alpha(p-1)} \overset{n\to\infty}{\to} \infty$, then $\Delta_n^{\alpha} n^{-1} \sum_{k=2-p}^{0} \left(\sum_{j=1}^{n} \left|\Psi_j^{\Delta_n}\right|\right)^{\alpha} \overset{n\to\infty}{\to} 0$.

2.2 Normalized periodogram

Before we formulate the main limit results for the normalized and the self-normalized periodogram, we introduce a random vector that will show up in the limits.

Let $m \in \mathbb{N}^*$, $\omega_1, \ldots, \omega_m \in \mathbb{R}^*$ and set $\underset{\sim}{\omega} = (\omega_1, \ldots, \omega_m)^T$. We define the $(2m+1)$-dimensional (stable) random vector $((S_j^{\Re}(\underset{\sim}{\omega}), S_j^{\Im}(\underset{\sim}{\omega}))_{j\in\{1,\ldots,m\}}, S_{m+1}(\underset{\sim}{\omega}))$ via its joint characteristic function

$$\mathbb{E}\left[\exp\left\{ i \left(\sum_{j=1}^{m} \theta_j S_j^{\Re}(\underset{\sim}{\omega}) + \nu_j S_j^{\Im}(\underset{\sim}{\omega}) + \tau S_{m+1}(\underset{\sim}{\omega}) \right) \right\}\right] = \exp\{-\sigma_L^{\alpha} \cdot K_{\underset{\sim}{\omega}}(\underset{\sim}{\theta}, \underset{\sim}{\nu}, \tau)\}, \quad \underset{\sim}{\theta}, \underset{\sim}{\nu} \in \mathbb{R}^m, \tau \in \mathbb{R}, \tag{2.7a}$$

with $K_{\underset{\sim}{\omega}}(\underset{\sim}{\theta}, \underset{\sim}{\nu}, \tau)$ given as follows:

(i) If $\omega_1, \ldots, \omega_m$ are linearly independent over $\mathbb{Z}$ (i.e. there is no $h \in \mathbb{Z}^m$, $h \neq 0$, such that $\langle h, \underset{\sim}{\omega}\rangle = 0$), then

$$K_{\underset{\sim}{\omega}}(\underset{\sim}{\theta}, \underset{\sim}{\nu}, \tau) = \int_{[0,1)^m} \left| \sum_{j=1}^{m} \theta_j \cos(2\pi x_j) + \nu_j \sin(2\pi x_j) + \tau \right|^{\alpha} \mathrm{d}(x_1, \ldots, x_m). \tag{2.7b}$$

(ii) If $\omega_1, \ldots, \omega_m$ are linearly dependent over $\mathbb{Z}$, then there is an $s \in \{1, \ldots, m-1\}$ such that

$$K_{\underset{\sim}{\omega}}(\underset{\sim}{\theta}, \underset{\sim}{\nu}, \tau) = \frac{1}{\mathscr{H}^{m-s}(\mathscr{M})} \int_{\mathscr{M}} \left| \sum_{j=1}^{m} \theta_j \cos(2\pi x_j) + \nu_j \sin(2\pi x_j) + \tau \right|^{\alpha} \mathrm{d}\mathscr{H}^{m-s}(x_1, \ldots, x_m), \tag{2.7c}$$

where $\mathscr{M} = \mathscr{M}(\omega_1, \ldots, \omega_m)$ is the $(m-s)$-dimensional linear manifold in $[0,1)^m$ defined in eq. (3.2) below and $\mathscr{H}^{m-s}$ is the $(m-s)$-dimensional Lebesgue (Hausdorff) measure on $\mathscr{M}(\omega_1, \ldots, \omega_m)$ (for a definition of manifolds, see, e.g., [29, pp. 200-201]).

We start to investigate the normalized periodogram in analogy to [9, 23]. Since we use Lemmata 2.1 and 2.2 for the proofs of the asymptotic behavior of the normalized periodogram we require

Assumption 1. *The eigenvalues $\lambda_1, \ldots, \lambda_p$ of A are distinct and possess strictly negative real parts.*

Moreover, we establish our limit results for the different periodogram versions in the asymptotic framework of high-frequency data within a long time interval using Lemma 2.2. Thus we need

Assumption 2. *There is some* $\delta > 0$ *such that, with* $\beta = \max\{1+\delta,\, \alpha(p-1)+\max\{0,\, 1-\alpha\}\}$*, we have* $\Delta = \Delta_n \to 0$ *whereas* $n\Delta_n^{\beta} \to \infty$ *as* $n \to \infty$*.*

Remark 2.3.

(i) Note that in the case of a symmetric α-stable Ornstein-Uhlenbeck process (i.e. $p = 1$), Assumption 2 becomes $\Delta_n \to 0$ and $n\Delta_n^{1+\delta} \to \infty$ as $n \to \infty$ for some $\delta > 0$ and does not depend on α.

(ii) Conversely, if $p \geq 2$, the convergence rate of Δ_n depends on α. However, one easily verifies that $\beta \leq 2p-1$ is always true and thus, if $\Delta_n \to 0$ and $n\Delta_n^{2p-1} \to \infty$ as $n \to \infty$ hold, Assumption 2 is satisfied as well. □

The following is an analog result to the discrete-time ones [9, Theorem 10.3.1] and [23, Proposition 2.1], respectively.

Proposition 2.4. *Let* $\Delta = \Delta_n$ *and* $Y^{\Delta_n} = (Y_{k\Delta_n})_{k\in\mathbb{Z}}$ *be the sampled SαS CARMA process. Under Assumption 1 the periodogram* $I_{n,Y^{\Delta_n}}$ *satisfies, for any* $\omega \in [-\pi, \pi]$*,*

$$I_{n,Y^{\Delta_n}}(\omega) = \left|\Psi^{\Delta_n}(\mathrm{e}^{-i\omega})\right|^2 I_{n,\widetilde{Z}^{\Delta_n}}(\omega) + R_{n,\Delta_n}(\omega)$$

with $\widetilde{Z}^{\Delta_n} := (\widetilde{Z}_{k,\Delta_n})_{k\in\mathbb{Z}}$ *as given in eq.* (2.1)*. If in addition Assumption 2 holds, then we have for any* $\omega \in \mathbb{R}^*$

$$\lim_{n\to\infty} \mathbb{P}\Big(\Delta_n^{2-\frac{2}{\alpha}} |R_{n,\Delta_n}(\omega\Delta_n)| > \varepsilon\Big) = 0 \quad \textit{for every } \varepsilon > 0.$$

This shows that we have to study the limit behavior of the periodogram of $\widetilde{Z}^{\Delta_n}$ in order to get insight into the asymptotic properties of $I_{n,Y^{\Delta_n}}$. The next theorem provides the key result therefore. Note that in terms of the discrete Fourier transform of $\widetilde{Z}^{\Delta_n}$,

$$J_{n,\widetilde{Z}^{\Delta_n}}(\omega) := n^{-1/\alpha} \sum_{k=1}^{n} \widetilde{Z}_{k,\Delta_n} \mathrm{e}^{-i\omega k}, \quad -\pi \leq \omega \leq \pi,$$

we can write $I_{n,\widetilde{Z}^{\Delta_n}}(\omega) = |J_{n,\widetilde{Z}^{\Delta_n}}(\omega)|^2$.

Theorem 2.5. *If Assumption 1 holds,* $\Delta = \Delta_n \to 0$ *and* $n\Delta_n^{1\vee\alpha(p-1)} \to \infty$ *as* $n \to \infty$*, then we have, for any* $m \in \mathbb{N}^*$ *and* $\underset{\sim}{\omega} = (\omega_1, \ldots, \omega_m)^T \in (\mathbb{R}^*)^m$*,*

$$\Delta_n^{1-p-\frac{1}{\alpha}} \Big[J_{n,\widetilde{Z}^{\Delta_n}}(\omega_j\Delta_n)\Big]_{j=1,\ldots,m} \xrightarrow{\mathscr{D}} \Big[c(i\omega_j)\cdot\Big(S_j^{\Re}(\underset{\sim}{\omega}) - iS_j^{\Im}(\underset{\sim}{\omega})\Big)\Big]_{j=1,\ldots,m} \quad \textit{as } n \to \infty.$$

The joint characteristic function of the $2m$*-dimensional stable random vector* $\big(S_j^{\Re}(\underset{\sim}{\omega}), S_j^{\Im}(\underset{\sim}{\omega})\big)_{j\in\{1,\ldots,m\}}$ *is given in eq.* (2.7) *(with* $\tau = 0$*).*

Combining now Proposition 2.4 and Theorem 2.5 together with the fact that

$$\left|\Psi^{\Delta_n}(\mathrm{e}^{-i\omega\Delta_n})\right|^2 \sim \Delta_n^{-2p} |a(i\omega)|^{-2} \quad \text{as } n \to \infty,$$

where the latter can be easily derived from the definition of Ψ^{Δ_n} together with the convergence of Δ_n to 0, we deduce the following main result for the limit behavior of the normalized periodogram.

Theorem 2.6. *Suppose* $\alpha \in (0,2]$ *and let* $Y^{\Delta_n} = (Y_{k\Delta_n})_{k\in\mathbb{Z}}$ *denote the sampled SαS CARMA(p,q) process. If Assumptions 1 and 2 hold, then* $I_{n,Y^{\Delta_n}}$ *satisfies for any* $m \in \mathbb{N}^*$ *and* $\underset{\sim}{\omega} = (\omega_1, \ldots, \omega_m)^T \in (\mathbb{R}^*)^m$

$$\Delta_n^{2-\frac{2}{\alpha}} \Big[I_{n,Y^{\Delta_n}}(\omega_j\Delta_n)\Big]_{j=1,\ldots,m} \xrightarrow{\mathscr{D}} \left[\frac{|c(i\omega_j)|^2}{|a(i\omega_j)|^2}\cdot\Big(\big[S_j^{\Re}(\underset{\sim}{\omega})\big]^2 + \big[S_j^{\Im}(\underset{\sim}{\omega})\big]^2\Big)\right]_{j=1,\ldots,m} \quad \textit{as } n \to \infty,$$

where the stable random vector $\left(S_j^{\Re}(\underset{\sim}{\omega}), S_j^{\Im}(\underset{\sim}{\omega})\right)_{j\in\{1,\ldots,m\}}$ *has joint characteristic function as given in eq.* (2.7) *(with* $\tau = 0$*).*

Remark 2.7.

(i) We highlight two important differences of our limit result to the one in [23] for ARMA models in discrete time. First, in our paper we do not have to distinguish between rational and irrational multiples of 2π in the frequency vector $\underset{\sim}{\omega}$ as it has been the case in discrete time (see, e.g., [23, Theorem 2.4]). The reason therefore is our asymptotic framework $\Delta_n \to 0$ as $n \to \infty$ which yields that in the proof of Proposition 3.4 the crucial eq. (4.33) holds for any $h \in \mathbb{Z}^m$, $h \neq 0$, whereas with $\Delta_n := \Delta$ constant and one frequency component being a rational multiple of 2π, (4.33) could not hold for all $h \in \mathbb{Z}^m$, $h \neq 0$. Secondly, the same equation explains why in our framework the limit distributions differ depending on whether or not the frequencies $\omega_1, \ldots, \omega_m$ are linearly dependent over $\mathbb{Z}$ (cf. eq. (2.7)). In discrete time they depend on whether or not $2\pi, \omega_1, \ldots, \omega_m$ (with $\omega_1, \ldots, \omega_m$ being irrational multiples of 2π) are linearly dependent over $\mathbb{Z}$ (see again [23, Theorem 2.4]). Note that the latter is also the reason why the manifold $\mathscr{M}(\omega_1, \ldots, \omega_m)$ in (3.2) is different from the manifold that appears in the discrete-time result.

(ii) Moreover, for linearly independent $\omega_1, \ldots, \omega_m$ the distribution of $(S_j^{\Re}(\underset{\sim}{\omega}), S_j^{\Im}(\underset{\sim}{\omega}))_{j\in\{1,\ldots,m\}}$ does not depend on $\underset{\sim}{\omega}$ anymore. In the dependent case, $\underset{\sim}{\omega}$ determines the manifold, and hence, has an influence on the limit distribution. The sequence of random variables $(S_j^{\Re}(\underset{\sim}{\omega}), S_j^{\Im}(\underset{\sim}{\omega}))_{j\in\{1,\ldots,m\}}$ is independent in the case $\alpha = 2$, whereas for $\alpha < 2$ it is dependent; in particular for $m = 1$ and $\underset{\sim}{\omega} = \omega \in \mathbb{R}^*$, the random variables $S_1^{\Re}(\omega)$ and $S_1^{\Im}(\omega)$ are dependent.

(iii) Investigating the special case $m = 1$, Theorem 2.6 gives for any $\omega \in \mathbb{R}^*$

$$\Delta_n^{2-\frac{2}{\alpha}} I_{n,Y^{\Delta_n}}(\omega\Delta_n) \xrightarrow{\mathscr{D}} \frac{|c(i\omega)|^2}{|a(i\omega)|^2} \cdot \left| \int_{[0,1)} e^{2\pi i s} \, dL_s \right|^2$$

as $n \to \infty$. Hence, the limit distribution factorizes in a parametric factor depending on ω (the so-called power transfer function) and a random factor, which does not depend on ω anymore. The limit distribution coincides with the limit distribution of the normalized periodogram of ARMA models if ω is an irrational multiple of 2π.

(iv) Let $\alpha = 2$. Then with $\omega \in \mathbb{R}^*$ as $n \to \infty$,

$$\Delta_n I_{n,Y^{\Delta_n}}(\omega\Delta_n) \xrightarrow{\mathscr{D}} 2\pi f_Y(\omega) \left(\frac{N_1^2}{2} + \frac{N_2^2}{2} \right) \stackrel{\mathscr{D}}{=} 2\pi f_Y(\omega) E,$$

where N_1 and N_2 are i.i.d. standard normal random variables and E is a standard exponential random variable. This limit result is the empirical counterpart to (1.5) with scaling factor Δ_n and in analogy to the results for ARMA models (cf. [9, Theorem 10.3.2]). It confirms, that $\Delta_n I_{n,Y^{\Delta_n}}(\omega\Delta_n)$ is not a consistent estimator for the spectral density.

(v) For any $h \in \mathbb{R}^*$, $(S_j^{\Re}(h\underset{\sim}{\omega}), S_j^{\Im}(h\underset{\sim}{\omega}))_{j\in\{1,\ldots,m\}} \stackrel{\mathscr{D}}{=} (S_j^{\Re}(\underset{\sim}{\omega}), S_j^{\Im}(\underset{\sim}{\omega}))_{j\in\{1,\ldots,m\}}$, such that as $n \to \infty$,

$$\Delta_n^{2-\frac{2}{\alpha}} \left[I_{n,Y^{\Delta_n}}(h\omega_j\Delta_n) \right]_{j=1,\ldots,m} \xrightarrow{\mathscr{D}} \left[\frac{|c(ih\omega_j)|^2}{|a(ih\omega_j)|^2} \cdot \left(\left[S_j^{\Re}(\underset{\sim}{\omega})\right]^2 + \left[S_j^{\Im}(\underset{\sim}{\omega})\right]^2 \right) \right]_{j=1,\ldots,m}.$$

On the other hand, if $\omega_1, \ldots, \omega_m$ are linearly independent over $\mathbb{Z}$, then there exists an $h \in \mathbb{R}$ with $h + \omega_1, \ldots, h + \omega_m$ linearly dependent over $\mathbb{Z}$ such that the limit distributions $(S_j^{\Re}(\underset{\sim}{\omega}), S_j^{\Im}(\underset{\sim}{\omega}))_{j\in\{1,\ldots,m\}}$ and $(S_j^{\Re}(h\underset{\sim}{1} + \underset{\sim}{\omega}), S_j^{\Im}(h\underset{\sim}{1} + \underset{\sim}{\omega}))_{j\in\{1,\ldots,m\}}$ are different. Consequently, there is no general result how a frequency shift influences the limit distribution. □

Remark 2.8. We conjecture that Assumption 2 is in this formulation not a necessary assumption for Theorem 2.6. However, it seems to be (close to) necessary for Proposition 2.4, but Proposition 2.4 is not necessary for Theorem 2.6. □

2.3 Self-normalized periodogram

Next we derive the limit behavior of the self-normalized periodogram $\widetilde{I}_{n,Y^{\Delta_n}}$ and $\widehat{I}_{n,Y^{\Delta_n}}$, respectively, as given in (1.6), which is comparable to those in [24, Section 3] for ARMA processes. As in the normalized case they converge to functions of stable distributions as the following two theorems show.

First, we have to state some notation. The observer canonical form of a CARMA process (cf. [28]) is given under Assumption 1 by the stationary and causal multivariate Ornstein-Uhlenbeck process

$$V_t = \int_{-\infty}^{t} \mathrm{e}^{(t-s)A}\beta \, \mathrm{d}L_s, \quad t \in \mathbb{R}, \tag{2.8a}$$

where the vector $\beta = (\beta_1, \ldots, \beta_p)^T \in \mathbb{R}^p$ is defined recursively by

$$\beta_{p-j} = -\sum_{i=1}^{p-1-j} a_i \beta_{p-j-i} + c_{q-j}, \quad j = 0, 1, \ldots, p-1,$$

(with the convention $c_j = 0$ for $j < 0$). Then

$$Y_t = e_1^T V_t, \quad t \in \mathbb{R}, \tag{2.8b}$$

where $e_1 = (1, 0, \ldots, 0)^T \in \mathbb{R}^p$. Hence, every $S\alpha S$ CARMA process can also be written as a Lévy-driven *moving average process* $Y_t = \int_{-\infty}^{\infty} g(t-s)\,\mathrm{d}L_s$, $t \in \mathbb{R}$, with kernel function

$$g(t) = e_1^T \, \mathrm{e}^{tA} \beta \, \mathbb{1}_{[0,\infty)}(t). \tag{2.9}$$

The following proposition is crucial for the asymptotic behavior of the different self-normalized periodogram versions.

Proposition 2.9. *Assume $\alpha \in (0,2]$ and let $Y^{\Delta_n} = (Y_{k\Delta_n})_{k\in\mathbb{Z}}$ denote the sampled $S\alpha S$ CARMA(p,q) process. Moreover, define $\Delta L(k\Delta_n) := L_{k\Delta_n} - L_{(k-1)\Delta_n}$ for $k \in \mathbb{Z}$, $n \in \mathbb{N}^*$. Suppose Assumption 1, $\Delta_n \to 0$ and $n\Delta_n \to \infty$ as $n \to \infty$ hold. Then*

(i) $\sum_{k=1}^{n} Y_{k\Delta_n} = \sum_{j=0}^{\infty} g(j\Delta_n) \cdot \sum_{k=1}^{n} \Delta L(k\Delta_n) + o_P\left(\Delta_n^{-1}(n\Delta_n)^{\frac{1}{\alpha}}\right)$ *as $n \to \infty$,*

(ii) $\sum_{k=1}^{n} Y_{k\Delta_n}^2 = \sum_{j=0}^{\infty} g^2(j\Delta_n) \cdot \sum_{k=1}^{n} \Delta L(k\Delta_n)^2 + o_P\left(\Delta_n^{-1}(n\Delta_n)^{\frac{2}{\alpha}}\right)$ *as $n \to \infty$.*

The main limit results are then:

Theorem 2.10. *Suppose $\alpha \in (0,2]$ and let $Y^{\Delta_n} = (Y_{k\Delta_n})_{k\in\mathbb{Z}}$ denote the sampled $S\alpha S$ CARMA(p,q) process. The self-normalized periodogram $\widetilde{I}_{n,Y^{\Delta_n}}$ is as in* (1.6). *If Assumptions 1 and 2 hold, and in addition $c_q \neq 0$, then we have for any $m \in \mathbb{N}^*$ and $\underset{\sim}{\omega} = (\omega_1, \ldots, \omega_m)^T \in (\mathbb{R}^*)^m$, as $n \to \infty$,*

$$\left[\widetilde{I}_{n,Y^{\Delta_n}}(\omega_j \Delta_n)\right]_{j=1,\ldots,m} \xrightarrow{\mathscr{D}} \left[\frac{|c(i\omega_j)|^2}{(\int_0^\infty g(s)\,\mathrm{d}s)^2 \cdot |a(i\omega_j)|^2} \cdot \frac{\left[S_j^{\Re}(\underset{\sim}{\omega})\right]^2 + \left[S_j^{\Im}(\underset{\sim}{\omega})\right]^2}{S_{m+1}^2(\underset{\sim}{\omega})}\right]_{j=1,\ldots,m},$$

where g is the kernel function of the CARMA process as given in eq. (2.9) *and the $(2m+1)$-dimensional stable random vector $\left(\left(S_j^{\Re}(\underset{\sim}{\omega}), S_j^{\Im}(\underset{\sim}{\omega})\right)_{j\in\{1,\ldots,m\}}, S_{m+1}(\underset{\sim}{\omega})\right)$ has joint characteristic function given by eq.* (2.7).

Theorem 2.11. *Suppose $\alpha \in (0,2]$ and let $Y^{\Delta_n} = (Y_{k\Delta_n})_{k\in\mathbb{Z}}$ denote the sampled $S\alpha S$ CARMA(p,q) process. The self-normalized periodogram $\widehat{I}_{n,Y^{\Delta_n}}$ is as in* (1.6). *If Assumptions 1 and 2 hold, then we have for any $m \in \mathbb{N}^*$ and $\underset{\sim}{\omega} = (\omega_1, \ldots, \omega_m)^T \in (\mathbb{R}^*)^m$, as $n \to \infty$,*

$$\Delta_n \left[\widehat{I}_{n,Y^{\Delta_n}}(\omega_j \Delta_n)\right]_{j=1,\ldots,m} \xrightarrow{\mathscr{D}} \left[\frac{|c(i\omega_j)|^2}{\int_0^\infty g^2(s)\,\mathrm{d}s \cdot |a(i\omega_j)|^2} \cdot \frac{\left[S_j^{\Re}(\underset{\sim}{\omega})\right]^2 + \left[S_j^{\Im}(\underset{\sim}{\omega})\right]^2}{S^2}\right]_{j=1,\ldots,m},$$

where g is again the kernel function of the CARMA process as given in eq. (2.9)*, the* $(2m)$*-dimensional stable random vector* $\left(S_j^{\Re}(\underset{\sim}{\omega}), S_j^{\Im}(\underset{\sim}{\omega})\right)_{j\in\{1,\ldots,m\}}$ *has joint characteristic function as given in eq.* (2.7) *(with* $\tau = 0$*) and* S^2 *is a positive* $\alpha/2$*-stable random variable.*

Remark 2.12.

(i) Theorems 2.10 and 2.11 show that also the self-normalized periodogram versions do not yield consistent estimators for the (normalized) power transfer function. However, based on that paper we will show in [16] that applying a smoothing filter to the self-normalized periodogram gives such a consistent estimate. Since the model parameters influence the power transfer function and, causality and invertibility of the CARMA process preconditioned, the latter uniquely determines those parameters, it is possible to use that consistent estimator of the normalized power transfer function for statistical inference on the CARMA parameters.

(ii) We have not specified explicitly the joint characteristic function of the random vector that determines the limit in Theorem 2.11. However, it is uniquely identifiable from the calculated Laplace transform in eq. (4.31). Note that the limit distributions in Theorems 2.10 and 2.11 are not the same.

(iii) Moreover, we have to multiply $(\widehat{I}_{n,Y^{\Delta_n}}(\omega_j\Delta_n))_{j\in\{1,\ldots,m\}}$ in Theorem 2.11 by Δ_n to obtain an asymptotic limit result. This normalization is not necessary for $(\widetilde{I}_{n,Y^{\Delta_n}}(\omega_j\Delta_n))_{j\in\{1,\ldots,m\}}$ in Theorem 2.10. Observing (1.5) the rescaling with Δ_n seems to be natural in some way. The point is that with Proposition 2.9 we have for the different normalizations

$$\frac{\Delta_n\left(\sum_{k=1}^n Y_{k\Delta_n}\right)^2}{\sum_{k=1}^n Y_{k\Delta_n}^2} = \frac{\left(\Delta_n \sum_{j=0}^{\infty} g(j\Delta_n)\right)^2}{\Delta_n \sum_{j=0}^{\infty} g(j\Delta_n)^2}\cdot\frac{\left(\sum_{k=1}^n \Delta L(k\Delta_n)\right)^2}{\sum_{k=1}^n \Delta L(k\Delta_n)^2} + o_P(1) \xrightarrow{\mathscr{D}} \frac{\left(\int_0^\infty g(s)\,\mathrm{d}s\right)^2}{\int_0^\infty g(s)^2\,\mathrm{d}s}\cdot\frac{L_1^2}{[L,L]_1}$$

as $n\to\infty$, where $([L,L]_t)_{t\geq 0}$ is the quadratic variation process of $(L_t)_{t\geq 0}$. For this reason Δ_n appears in Theorem 2.11. □

3 Lattices in $\mathbb{R}^m$ and the manifolds $\mathscr{M}(\omega_1,\ldots,\omega_m)$

In this section we recall some basic facts about lattices in $\mathbb{R}^m$ and use them to construct the manifolds $\mathscr{M}(\omega_1,\ldots,\omega_m)$ in eq. (2.7c). For more details concerning the theory of lattices we refer the reader to [14, 20].

Definition 3.1 (Lattice)**.** *For* $S\subseteq\mathbb{R}^m$ *let* $\operatorname{span}^{\mathbb{Z}}(S)$ *and* $\operatorname{span}^{\mathbb{R}}(S)$*, respectively, denote the integer and linear hull of S. For any linearly independent vectors* $b_1,\ldots,b_d\in\mathbb{R}^m$ *the additive subgroup of* $\mathbb{R}^m$

$$\mathscr{L} := \mathscr{L}(b_1,\ldots,b_d) := \operatorname{span}^{\mathbb{Z}}(\{b_1,\ldots,b_d\})$$

is said to be a lattice *and* $b_1,\ldots,b_d$ *is called a* basis *of* $\mathscr{L}$*. The* dimension *of the lattice* $\mathscr{L}$ *is given by*

$$\dim(\mathscr{L}) := \dim\left(\operatorname{span}^{\mathbb{R}}(\mathscr{L})\right) = d.$$

We call a subset S in $\mathbb{R}^m$ *discrete* if S has no accumulation point in $\mathbb{R}^m$. It is a classical result that discreteness characterizes lattices among additive subgroups in $\mathbb{R}^m$.

Theorem 3.2 (cf. [20], § 3.2)**.** *A subset* $S\subseteq\mathbb{R}^m$ *is a lattice if and only if it is a discrete, additive subgroup of* $\mathbb{R}^m$*. In either case the dimension of the lattice is equal to the maximal number of linearly independent vectors in S.*

Suppose that we have given $\omega_1,\ldots,\omega_m\in\mathbb{R}^*$ which are linearly dependent over $\mathbb{Z}$. Let $\underset{\sim}{\omega} = (\omega_1,\ldots,\omega_m)^T = 2\pi\underset{\sim}{\eta}$. Note that all lattices as well as the manifolds $\mathscr{M}(\omega_1,\ldots,\omega_m)$ in this paper depend on the frequency vector $\underset{\sim}{\omega}$ and $\underset{\sim}{\eta}$, respectively. We neglect, however, that dependency for ease of notation. We define

$$\widetilde{\mathscr{L}} := \{\underset{\sim}{\eta}\}^{\perp}\cap\mathbb{Z}^m.$$

Then $\widetilde{\mathscr{L}}$ constitutes a discrete, additive subgroup of $\mathbb{R}^m$ and since the maximal possible number of linearly independent vectors in $\widetilde{\mathscr{L}}$ is $m-1$, we apply Theorem 3.2 and obtain an $s \in \{1, \ldots, m-1\}$ and a basis $b_{m-s+1}, \ldots, b_m \in \mathbb{Z}^m$ of the lattice $\widetilde{\mathscr{L}}$. Now

$$\mathscr{L} := \widetilde{\mathscr{L}}^{\perp} \cap \mathbb{Z}^m \tag{3.1}$$

is a discrete, additive subgroup in $\mathbb{R}^m$ as well and hence, again due to Theorem 3.2, it is a lattice generated by a basis $b_1, \ldots, b_{m-s} \in \mathbb{Z}^m$. That the dimension of $\mathscr{L}$ is indeed $m-s$ (i.e. the maximal possible dimension of the orthogonal complement of $\widetilde{\mathscr{L}}$) can be seen from the following fact: let

$$H := \begin{pmatrix} b_{m-s+1}^T \\ \vdots \\ b_m^T \end{pmatrix} \in \mathbb{Z}^{s \times m}$$

and note that there has to be an $s \times s$-block with non-vanishing determinant. W.l.o.g. this block is given by the first s columns of H, denoted by $H^{[s]}$. We can solve, for any $j \in \{s+1, \ldots, m\}$, the linear systems $H^{[s]} x_j = -h_j$ where h_j is the j-th column of H and obtain, using Cramer's rule, solutions $x_j \in \mathbb{Q}^s$ with common denominator $\det\left(H^{[s]}\right) \in \mathbb{Z}$. Hence, the vectors

$$v_j := \det\left(H^{[s]}\right) \cdot \left[\begin{pmatrix} x_j \\ 0 \\ \vdots \\ 0 \end{pmatrix} + e_j\right] \in \mathbb{Z}^m, \qquad j \in \{s+1, \ldots, m\},$$

with e_j being the j-th unit vector in $\mathbb{R}^m$, are linearly independent and $Hv_j = 0$ for all $j \in \{s+1, \ldots, m\}$. This shows that $v_j \in \{b_{m-s+1}, \ldots, b_m\}^{\perp} \cap \mathbb{Z}^m = \mathscr{L}$ for any $j \in \{s+1, \ldots, m\}$, and hence, the dimension of the lattice $\mathscr{L}$ has to be $m-s$ as claimed above. Let

$$B := \begin{pmatrix} b_1 & b_2 & \ldots & b_{m-s} \end{pmatrix} \in \mathbb{Z}^{m \times (m-s)}$$

and

$$T : (\mathbb{R} \bmod 1)^{m-s} \to (\mathbb{R} \bmod 1)^m, \quad x = (x_1, \ldots, x_{m-s})^T \mapsto Bx \bmod 1 = \left(\sum_{j=1}^{m-s} x_j b_j\right) \bmod 1,$$

where the mod-operator has to be applied componentwise. We then define

$$\mathscr{M} := T\left((\mathbb{R} \bmod 1)^{m-s}\right), \tag{3.2}$$

the Gram matrix $G := B^T B$ and the set of functions on $\mathscr{M}$

$$\mathscr{T} := \left\{ f_h : \mathscr{M} \to \mathbb{C} : \ f_h = \mathrm{e}^{2\pi i \langle h, \cdot \rangle} \circ T \circ G^{-1} \circ T^{-1} \text{ for an } h \in \mathscr{L} \right\}. \tag{3.3}$$

$\mathscr{T}$ is well-defined due to the injectivity of T (see the proof of the upcoming Theorem 3.3(i)). Moreover, it can be shown that all the functions in $\mathscr{T}$ are continuous (mod 1) on $\mathscr{M}$. The following theorem holds.

Theorem 3.3.

(i) $\mathscr{M}$ *is an* $(m-s)$*-dimensional* C^1*-manifold in* $[0,1)^m$.

(ii) Let $\underset{\sim}{\mu} \in \mathbb{R}^{m-s}$ *be the coordinates of* $\underset{\sim}{\eta}$ *in the basis B, i.e.* $\underset{\sim}{\eta} = B\underset{\sim}{\mu}$. *Then* $\langle z, \underset{\sim}{\mu} \rangle \neq 0$ *for all* $z \in \mathbb{Z}^{m-s}$, $z \neq 0$.

(iii) For any $f_h \in \mathscr{T}$ with $h \in \mathscr{L}$, $h \neq 0$, we have

$$\frac{1}{\mathscr{H}^{m-s}(\mathscr{M})} \int_{\mathscr{M}} f_h(x)\, \mathscr{H}^{m-s}(\mathrm{d}x) = 0,$$

where $\mathscr{H}^{m-s}$ is the $(m-s)$-dimensional Lebesgue measure on $\mathscr{M}$.

(iv) For any $x, y \in \mathscr{M}$, $x \neq y$, there is an $h \in \mathscr{L}$ such that $f_h(x) \neq f_h(y)$.

Since $(\mathbb{R} \bmod 1)^m$ and $(\mathbb{R} \bmod 1)^{m-s}$ are compact Hausdorff spaces, one immediately obtains that also $\mathscr{M}$ is a compact Hausdorff space. Note that the subalgebra $\mathrm{span}^{\mathbb{C}}(\mathscr{T})$ of the algebra $C(\mathscr{M})$ of all continuous complex-valued functions on $\mathscr{M}$ contains the constant function 1 (take $h = 0$). Moreover, $\mathrm{span}^{\mathbb{C}}(\mathscr{T})$ is closed under complex conjugation and separates points (see Theorem 3.3(iv)). Applying the Stone-Weierstraß Theorem (cf. [32, p. 122] or [35, p. 161]), this yields that $\mathrm{span}^{\mathbb{C}}(\mathscr{T})$ is dense in $C(\mathscr{M})$ with respect to the topology of uniform convergence.

An application of Theorem 3.3 as given in the next proposition characterizes the limit distributions of the normalized and the first version of the self-normalized periodogram, respectively, by random vectors with characteristic functions as given in (2.7).

Proposition 3.4.
Suppose $\Delta = \Delta_n \to 0$ and $n\Delta_n \to \infty$ as $n \to \infty$. Moreover, define for any $z_1, z_2 \in \mathbb{R}$ the function $\Xi_{z_1,z_2} : \mathbb{C} \to \mathbb{R}$ by $\Xi_{z_1,z_2}(x) := z_1 \Re(x) + z_2 \Im(x)$. Then, for any $m \in \mathbb{N}^*$, $\omega_1, \ldots, \omega_m \in \mathbb{R}^*$ and $\underset{\sim}{\theta}, \underset{\sim}{\nu} \in \mathbb{R}^m$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n-p+1} \left| \sum_{j=1}^{m} \Xi_{\theta_j,\nu_j}\left(\mathrm{e}^{-i\omega_j\Delta_n k} c(i\omega_j)\right)\right|^{\alpha} = K_{\underset{\sim}{\omega}}\left(\left(\Xi_{\theta_j,\nu_j}(c(i\omega_j))\right)_{j\in\{1,\ldots,m\}}, \left(\Xi_{-\nu_j,\theta_j}(c(i\omega_j))\right)_{j\in\{1,\ldots,m\}}, 0\right),$$

where $K_{\underset{\sim}{\omega}}$ is given by eqs. (2.7b) *and* (2.7c)*, respectively.*

For $\omega_1, \ldots, \omega_m$ linearly independent over $\mathbb{Z}$ a similar result was derived in [25, Corollary 4].

Finally, we shall require Proposition 3.5 from below for the limit result of the second version of the self-normalized periodogram. The proof of this proposition is based on Theorem 3.3 as well.

Proposition 3.5. *Suppose $\Delta = \Delta_n \to 0$ and $n\Delta_n \to \infty$ as $n \to \infty$. Let $m \in \mathbb{N}^*$, $\omega_1, \ldots, \omega_m \in \mathbb{R}^*$ and write $\underset{\sim}{\omega} = (\omega_1, \ldots, \omega_m)^T = 2\pi(\eta_1, \ldots, \eta_m)^T = 2\pi\underset{\sim}{\eta}$. Moreover, suppose that $(N_k)_{k\in\mathbb{N}^*}$ are i.i.d. standard normal random variables.*

(i) If $\omega_1, \ldots, \omega_m$ are linearly independent over $\mathbb{Z}$, we assume that we have given a random variable $\underset{\sim}{U}$, uniformly distributed on $[0,1)^m$ and independent of $(N_k)_{k\in\mathbb{N}^*}$, and a function $f : (\mathbb{R} \bmod 1)^m \times \mathbb{R} \to \mathbb{R}$ such that $\mathbb{E}[f^2(\underset{\sim}{U}, N_1)] < \infty$ and $g^{(k)}(x) := \mathbb{E}[f^k(x, N_1)]$, $k = 1, 2$, is continuous on $(\mathbb{R} \bmod 1)^m$.

(ii) If $\omega_1, \ldots, \omega_m$ are linearly dependent over $\mathbb{Z}$, we assume that we have given a random variable $\underset{\sim}{V}$, uniformly distributed on $[0,1)^{m-s}$ and independent of $(N_k)_{k\in\mathbb{N}^*}$, and a function $f : \mathscr{M} \times \mathbb{R} \to \mathbb{R}$ such that $\mathbb{E}[f^2(\underset{\sim}{U}, N_1)] < \infty$ and $g^{(k)}(x) := \mathbb{E}[f^k(x, N_1)]$, $k = 1, 2$, is continuous on $\mathscr{M}$, where $\underset{\sim}{U} := T(\underset{\sim}{V})$ and T is the parametrization of $\mathscr{M}$.

Then in either case

$$\frac{1}{n} \sum_{k=1}^{n} f(k\Delta_n \underset{\sim}{\eta} \bmod 1, N_k) \xrightarrow{\mathbb{P}} \mathbb{E}\left[f(\underset{\sim}{U}, N_1)\right] \quad \text{as } n \to \infty. \tag{3.4}$$

4 Proofs

4.1 Proofs of Section 1

Proof of Equation (1.5)**.**
Fix an arbitrary $\omega \in \mathbb{R}$ and assume that Δ is sufficiently small such that $\omega\Delta \in [-\pi, \pi]$. Then

$$\Delta f_\Delta(\omega\Delta) \overset{(1.4)}{=} \frac{\Delta}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_Y(k\Delta)\, \mathrm{e}^{-ik\omega\Delta} \overset{(1.2)}{=} \frac{1}{2\pi} c^T \left(\Delta \cdot \sum_{k=-\infty}^{\infty} \mathrm{e}^{|k|\Delta A} \mathrm{e}^{-ik\omega\Delta}\right) \gamma_X(0)\, c. \tag{4.1}$$

For any $\varepsilon > 0$, there exist an $N_0 \in \mathbb{N}$ and $\Delta_0 > 0$ such that

$$
\begin{aligned}
&\Big\| \int_{-\infty}^{\infty} \mathrm{e}^{|h|A} \mathrm{e}^{-ih\omega} \,\mathrm{d}h - \Delta \cdot \sum_{k=-\infty}^{\infty} \mathrm{e}^{|k|\Delta A} \mathrm{e}^{-ik\omega\Delta} \Big\| \\
&\leq \int_{|h|\geq N_0} \big\| \mathrm{e}^{|h|A} \big\| \,\mathrm{d}h + \Big\| \int_{-N_0}^{N_0} \mathrm{e}^{|h|A} \mathrm{e}^{-ih\omega} \,\mathrm{d}h - \Delta \cdot \sum_{|k|\leq \lfloor N_0/\Delta \rfloor} \mathrm{e}^{|k|\Delta A} \mathrm{e}^{-ik\omega\Delta} \Big\| + \Delta \cdot \sum_{|k|\geq \lfloor N_0/\Delta \rfloor + 1} \big\| \mathrm{e}^{|k|\Delta A} \big\| \\
&\leq \frac{\varepsilon}{3} + \Big\| \int_{-N_0}^{N_0} \mathrm{e}^{|h|A} \mathrm{e}^{-ih\omega} \,\mathrm{d}h - \Delta \cdot \sum_{|k|\leq \lfloor N_0/\Delta \rfloor} \mathrm{e}^{|k|\Delta A} \mathrm{e}^{-ik\omega\Delta} \Big\| + \frac{\varepsilon}{3} \qquad (4.2)
\end{aligned}
$$

for all $0 < \Delta \leq \Delta_0$. The second addend on the right-hand side converges to 0 as $\Delta \to 0$ (Riemann sums!), i.e. there is a $\Delta_1 > 0$ such that (4.2) is less or equal to ε for any $\Delta \leq \Delta_1$. Hence, the right-hand side of eq. (4.1) converges, as $\Delta \to 0$, to

$$
\frac{1}{2\pi} c^T \Big(\int_{-\infty}^{\infty} \mathrm{e}^{|h|A} \mathrm{e}^{-ih\omega} \,\mathrm{d}h \Big) \gamma_X(0)\, c = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{c^T \mathrm{e}^{|h|A} \gamma_X(0)\, c}_{\overset{(1.2)}{=} \gamma_Y(h)} \cdot \mathrm{e}^{-ih\omega} \,\mathrm{d}h \overset{(1.3)}{=} f_Y(\omega).
$$

□

4.2 Proofs of Section 2.1

Proof of Lemma 2.1. *(i)* By virtue of [4, Proposition 11.2.1] we have, for any $t \in \mathbb{R}$,

$$
\mathrm{e}^{tA} = \frac{1}{2\pi i} \int_{\rho} (z\mathrm{I}_p - A)^{-1} \mathrm{e}^{tz} \,\mathrm{d}z,
$$

where ρ is a simple closed curve in the complex plane enclosing the spectrum of A. Moreover, from [13, Lemma 3.1] we immediately obtain

$$
c^T (z\mathrm{I}_p - A)^{-1} e_p = \frac{c(z)}{a(z)}
$$

for any $z \in \mathbb{C} \backslash \{\lambda_1, \ldots, \lambda_p\}$. Hence,

$$
\begin{aligned}
c^T \Big(-\sum_{j=0}^{r-1} \Phi_j^{\Delta} \mathrm{e}^{(r-1-j)\Delta A} \Big) \mathrm{e}^{(k\Delta - s)A} e_p &= -\sum_{j=0}^{r-1} \Phi_j^{\Delta} c^T \Big(\frac{1}{2\pi i} \int_{\rho} (z\mathrm{I}_p - A)^{-1} \mathrm{e}^{(r-1-j)\Delta z + (k\Delta - s)z} \,\mathrm{d}z \Big) e_p \\
&= -\sum_{j=0}^{r-1} \Phi_j^{\Delta} \cdot \frac{1}{2\pi i} \int_{\rho} \frac{c(z)}{a(z)} \mathrm{e}^{(r-1-j)\Delta z + (k\Delta - s)z} \,\mathrm{d}z \\
&= \sum_{m=1}^{p} \frac{c(\lambda_m)}{a'(\lambda_m)} \Big(-\sum_{j=0}^{r-1} \Phi_j^{\Delta} \mathrm{e}^{(r-1-j)\Delta \lambda_m} \Big) \mathrm{e}^{(k\Delta - s)\lambda_m},
\end{aligned}
$$

where the last equality follows from the Residue Formula (see, e.g., [26, Chapter VI, Theorem 1.2 and Lemma 1.3] or [17, Theorem III.6.3 and Remark III.6.4]) and the fact that the eigenvalues $\lambda_1, \ldots, \lambda_p$ of A are supposed to be distinct.

(ii) We obviously have

$$
\begin{aligned}
\frac{1}{\Delta} \int_0^{\Delta} \big| \mathrm{e}^{(\Delta - s)\lambda} - 1 \big|^{\alpha} \mathrm{d}s &= \frac{1}{\Delta} \int_0^{\Delta} \big| \mathrm{e}^{s\lambda} - 1 \big|^{\alpha} \mathrm{d}s \\
&\leq \frac{2^{\alpha}}{\Delta} \int_0^{\Delta} \big| \mathrm{e}^{s\Re(\lambda)} \cos\big(s\Im(\lambda)\big) - 1 \big|^{\alpha} + \big| \mathrm{e}^{s\Re(\lambda)} \sin\big(s\Im(\lambda)\big) \big|^{\alpha} \mathrm{d}s.
\end{aligned}
$$

Due to the Mean Value Theorem there exists an $\varepsilon(\Delta) \in [0,\Delta]$ such that

$$\frac{1}{\Delta}\int_0^\Delta \left|\mathrm{e}^{s\Re(\lambda)}\cos\big(s\Im(\lambda)\big)-1\right|^\alpha \mathrm{d}s = \left|\mathrm{e}^{\varepsilon(\Delta)\cdot\Re(\lambda)}\cos\big(\varepsilon(\Delta)\Im(\lambda)\big)-1\right|^\alpha. \tag{4.3}$$

Since $\varepsilon(\Delta)\to 0$ as $\Delta\to 0$, we immediately obtain that the right-hand side of (4.3) converges to 0 as $\Delta\to 0$. Likewise we deduce that

$$\frac{1}{\Delta}\int_0^\Delta \left|\mathrm{e}^{s\Re(\lambda)}\sin\big(s\Im(\lambda)\big)\right|^\alpha \mathrm{d}s \to 0 \quad \text{as } \Delta\to 0$$

and hence, *(ii)* follows.

(iii) By virtue of eq. (2.2b) we have, for any $r\in\{1,\ldots,p\}$,

$$\begin{aligned}
-\sum_{j=0}^{r-1}\Phi_j^\Delta\,\mathrm{e}^{(r-1-j)\Delta\lambda_m}
&= -\mathrm{e}^{(r-1)\Delta\lambda_m}\,\Phi_0^\Delta - \mathrm{e}^{(r-2)\Delta\lambda_m}\,\Phi_1^\Delta - \mathrm{e}^{(r-3)\Delta\lambda_m}\,\Phi_2^\Delta - \ldots - \Phi_{r-1}^\Delta\\
&= (-1)^2\cdot\mathrm{e}^{(r-1)\Delta\lambda_m} - \mathrm{e}^{(r-2)\Delta\lambda_m}\cdot(-1)^2\cdot\sum_{\{i_1\}\in\binom{\{1,\ldots,p\}}{1}}\mathrm{e}^{\Delta\lambda_{i_1}} - \mathrm{e}^{(r-3)\Delta\lambda_m}\,\Phi_2^\Delta - \ldots - \Phi_{r-1}^\Delta\\
&= (-1)^3\cdot\mathrm{e}^{(r-2)\Delta\lambda_m}\cdot\sum_{\{i_1\}\in\binom{\{1,\ldots,p\}\setminus\{m\}}{1}}\mathrm{e}^{\Delta\lambda_{i_1}} - \mathrm{e}^{(r-3)\Delta\lambda_m}\cdot(-1)^3\cdot\sum_{\{i_1,i_2\}\in\binom{\{1,\ldots,p\}}{2}}\mathrm{e}^{\Delta(\lambda_{i_1}+\lambda_{i_2})}\\
&\qquad - \ldots - \Phi_{r-1}^\Delta\\
&= (-1)^4\cdot\mathrm{e}^{(r-3)\Delta\lambda_m}\cdot\sum_{\{i_1,i_2\}\in\binom{\{1,\ldots,p\}\setminus\{m\}}{2}}\mathrm{e}^{\Delta(\lambda_{i_1}+\lambda_{i_2})} - \ldots - (-1)^r\cdot\sum_{\{i_1,\ldots,i_{r-1}\}\in\binom{\{1,\ldots,p\}}{r-1}}\mathrm{e}^{\Delta\sum_{s=1}^{r-1}\lambda_{i_s}}\\
&= \ldots = (-1)^{r+1}\cdot\sum_{\{i_1,\ldots,i_{r-1}\}\in\binom{\{1,\ldots,p\}\setminus\{m\}}{r-1}}\mathrm{e}^{\Delta\sum_{s=1}^{r-1}\lambda_{i_s}}
\end{aligned}\tag{4.4}$$

and hence, due to eq. (2.4),

$$\begin{aligned}
f_\Delta^{(m)}(\omega\Delta) &= \sum_{r=1}^{p}\mathrm{e}^{-i\omega\Delta(r-1)}\Bigg(-\sum_{j=0}^{r-1}\Phi_j^\Delta\,\mathrm{e}^{(r-1-j)\Delta\lambda_m}\Bigg) \overset{(4.4)}{=} \sum_{r=0}^{p-1}(-1)^r\,\mathrm{e}^{-i\omega\Delta r}\sum_{\{i_1,\ldots,i_r\}\in\binom{\{1,\ldots,p\}\setminus\{m\}}{r}}\mathrm{e}^{\Delta\sum_{s=1}^{r}\lambda_{i_s}}\\
&= \sum_{r=0}^{p-1}(-1)^r\sum_{\{i_1,\ldots,i_r\}\in\binom{\{1,\ldots,p\}\setminus\{m\}}{r}}\mathrm{e}^{\Delta\left(\sum_{s=1}^{r}\lambda_{i_s}-i\omega r\right)}\\
&= \sum_{j=0}^{p-1}\frac{\Delta^j}{j!}\sum_{r=0}^{p-1}(-1)^r\sum_{\{i_1,\ldots,i_r\}\in\binom{\{1,\ldots,p\}\setminus\{m\}}{r}}\Bigg(\sum_{s=1}^{r}\lambda_{i_s}-i\omega r\Bigg)^j + o(\Delta^{p-1}) \quad \text{as } \Delta\to 0.
\end{aligned}\tag{4.5}$$

Now, since the eigenvalues of A are also the zeros of the autoregressive polynomial $a(z)$, we observe that in order to show Lemma 2.1(iii) it remains to prove the following

$$\sum_{r=0}^{p-1}(-1)^r\sum_{\{i_1,\ldots,i_r\}\in\binom{\{1,\ldots,p\}\setminus\{m\}}{r}}\Bigg(\sum_{s=1}^{r}\lambda_{i_s}-i\omega r\Bigg)^j = \begin{cases} 0 & \text{if } j=0,1,\ldots,p-2,\\ (p-1)!\cdot\prod_{\substack{s=1\\ s\neq m}}^{p}(i\omega-\lambda_s) & \text{if } j=p-1.\end{cases} \tag{4.6}$$

If $p=1$, one immediately verifies that (4.6) holds since both sides are equal to 1. Hence, we assume

$p > 1$ in the following.

For $j = 0$, due to the Binomial Theorem, the left-hand side of (4.6) is equal to

$$\sum_{r=0}^{p-1} (-1)^r \binom{p-1}{r} = \big(1 + (-1)\big)^{p-1} = 0.$$

For $j \in \{1, \dots, p-1\}$ we obtain

$$\sum_{r=0}^{p-1} (-1)^r \sum_{\{i_1,\dots,i_r\} \in \binom{\{1,\dots,p\}\setminus\{m\}}{r}} \left(\sum_{s=1}^{r} \lambda_{i_s} - i\omega r \right)^j = \sum_{r=1}^{p-1} (-1)^r \sum_{\{i_1,\dots,i_r\} \in \binom{\{1,\dots,p\}\setminus\{m\}}{r}} \left(\sum_{s=1}^{r} (\lambda_{i_s} - i\omega) \right)^j$$

$$= \sum_{r=1}^{p-1} (-1)^r \sum_{t=1}^{j} \binom{p-1-t}{r-t} \sum_{k_1=1}^{p-1-(t-1)} \sum_{k_2=1}^{p-1-(t-2)-k_1} \cdots \sum_{k_{t-1}=1}^{p-1-(t-(t-1))-\sum_{h=1}^{t-2} k_h} \binom{j}{k_1} \binom{j-k_1}{k_2}$$

$$\times \cdots \times \binom{j - \sum_{h=1}^{t-2} k_h}{k_{t-1}} \sum_{\substack{u_1,\dots,u_t \in \{1,\dots,p\}\setminus\{m\} \\ u_1 < u_2 < \dots < u_t}} (\lambda_{u_1} - i\omega)^{j - \sum_{h=1}^{t-1} k_h} \prod_{s=2}^{t} (\lambda_{u_s} - i\omega)^{k_{t+1-s}}$$

$$= \sum_{t=1}^{j} \sum_{k_1=1}^{p-1-(t-1)} \sum_{k_2=1}^{p-1-(t-2)-k_1} \cdots \sum_{k_{t-1}=1}^{p-2-\sum_{h=1}^{t-2} k_h} \binom{j}{k_1} \binom{j-k_1}{k_2} \cdots \binom{j - \sum_{h=1}^{t-2} k_h}{k_{t-1}}$$

$$\times \sum_{\substack{u_1,\dots,u_t \in \{1,\dots,p\}\setminus\{m\} \\ u_1 < u_2 < \dots < u_t}} (\lambda_{u_1} - i\omega)^{j - \sum_{h=1}^{t-1} k_h} \prod_{s=2}^{t} (\lambda_{u_s} - i\omega)^{k_{t+1-s}} \sum_{r=1}^{p-1} (-1)^r \binom{p-1-t}{r-t}. \quad (4.7)$$

Since $\binom{n}{j} = 0$ for all $n \in \mathbb{N}$ and $j < 0$, we get

$$\sum_{r=1}^{p-1} (-1)^r \binom{p-1-t}{r-t} = (-1)^t \cdot \sum_{r=0}^{p-1-t} (-1)^r \binom{p-1-t}{r} = (-1)^t \cdot \big(1 + (-1)\big)^{p-1-t}$$
$$= \begin{cases} 0 & \text{if } t = 1, \dots, p-2, \\ (-1)^{p-1} & \text{if } t = p-1, \end{cases}$$

where we used again the Binomial Theorem. Consequently, for any $j \in \{1, \dots, p-2\}$, the right-hand side of (4.7) vanishes, whereas for $j = p-1$ it becomes

$$(-1)^{p-1} \binom{p-1}{1} \binom{p-2}{1} \cdots \binom{2}{1} \prod_{\substack{s=1 \\ s \neq m}}^{p} (\lambda_s - i\omega) = (p-1)! \cdot \prod_{\substack{s=1 \\ s \neq m}}^{p} (i\omega - \lambda_s),$$

which completes the proof of eq. (4.6) and hence, *(iii)* is shown.

(iv) It is a simple consequence of Liouville's Theorem (see, for instance, [26, Chapter III, Theorem 7.5]) that any rational function $f(z) = \frac{q(z)}{p(z)}$ with $\deg(q) < \deg(p)$ can be written as

$$f(z) = h_f(z; \lambda_1) + \dots + h_f(z; \lambda_r)$$

where $\lambda_1, \dots, \lambda_r$ are the distinct zeros of $p(z)$ and $h_f(z; \lambda_m)$ is the principal part of the Laurent series expansion of f at the point λ_m.

Again, the eigenvalues of A are also the zeros of the autoregressive polynomial $a(z)$. Consequently, we can apply the above result to the rational function $c(z)/a(z)$ (note that $\deg(a) = p > q = \deg(c)$) and obtain

$$\frac{c(z)}{a(z)} = h_{c/a}(z; \lambda_1) + \dots + h_{c/a}(z; \lambda_p).$$

Since $\lambda_1, \ldots, \lambda_p$ are distinct, every λ_m, $m \in \{1, \ldots, p\}$, is a pole of order 1 of the rational function c/a. In this case, it is well known (see, e.g., [26, p. 174]) that the principal part of the Laurent series expansion of c/a at the point λ_m reduces to

$$\frac{c(\lambda_m)}{a'(\lambda_m)} \cdot \frac{1}{z - \lambda_m}.$$

Since $\lambda_1, \ldots, \lambda_p$ are supposed to have non-vanishing real parts, we have $a(i\omega) \neq 0$ for any $\omega \in \mathbb{R}$. Hence, Lemma 2.1(iv) holds for any $\omega \in \mathbb{R}$. □

Proof of Lemma 2.2. *(i)* This statement follows easily by induction over p from the definition of the $\Psi_j^{\Delta_n}$.

(ii) We deduce from *(i)* that

$$\begin{aligned} \sum_{j=n+1}^{\infty} \left|\Psi_j^{\Delta_n}\right| &\leq C(p)\,\Delta_n^{-(p-1)} \sum_{j=n+1}^{\infty} \mathrm{e}^{\Delta_n \lambda_{\max} j} = C(p)\,\Delta_n^{-(p-1)} \frac{\mathrm{e}^{(n+1)\Delta_n\lambda_{\max}}}{1-\mathrm{e}^{\Delta_n\lambda_{\max}}} \\ &\sim -\frac{C(p)}{\lambda_{\max}}\,\mathrm{e}^{n\Delta_n\left(\lambda_{\max} - p\frac{\log(\Delta_n)\cdot\Delta_n^{\delta}}{n\Delta_n^{1+\delta}}\right)} \overset{n\to\infty}{\longrightarrow} 0, \end{aligned} \tag{4.8}$$

since $\Delta_n \to 0$ and $n\Delta_n^{1+\delta} \to \infty$ as $n \to \infty$.

If $0 < \alpha \leq 1$, we have (cf. also [23, Proof of Proposition 2.1])

$$\frac{\Delta_n^{\alpha}}{n} \sum_{k=-\infty}^{-n-1} \left(\sum_{j=1-k}^{n-k} \left|\Psi_j^{\Delta_n}\right| \right)^{\alpha} \leq \Delta_n^{\alpha} \sum_{j=n+2}^{\infty} \left|\Psi_j^{\Delta_n}\right|^{\alpha},$$

and analogously to (4.8) it can be shown that the right-hand side converges to 0 as $n \to \infty$. Otherwise, if $1 < \alpha \leq 2$, we set $\widetilde{\Psi}_j^{\Delta_n} := \Psi_j^{\Delta_n} / \sum_{j=n+2}^{\infty} \left|\Psi_j^{\Delta_n}\right|$ and obtain

$$\begin{aligned} \frac{\Delta_n^{\alpha}}{n} \sum_{k=-\infty}^{-n-1} \left(\sum_{j=1-k}^{n-k} \left|\Psi_j^{\Delta_n}\right| \right)^{\alpha} &= \left(\sum_{j=n+2}^{\infty} \left|\Psi_j^{\Delta_n}\right| \right)^{\alpha} \cdot \frac{\Delta_n^{\alpha}}{n} \sum_{k=-\infty}^{-n-1} \left(\sum_{j=1-k}^{n-k} \left|\widetilde{\Psi}_j^{\Delta_n}\right| \right)^{\alpha} \\ &\leq \left(\sum_{j=n+2}^{\infty} \left|\Psi_j^{\Delta_n}\right| \right)^{\alpha-1} \cdot \Delta_n^{\alpha} \sum_{j=n+2}^{\infty} \left|\Psi_j^{\Delta_n}\right| \overset{n\to\infty}{\longrightarrow} 0 \end{aligned}$$

due to eq. (4.8).

(iii) We use again *(i)* to derive

$$\begin{aligned} \frac{\Delta_n^{\alpha p}}{n} \sum_{k=1-n}^{1-p} \left(\sum_{j=1-k}^{n} \left|\Psi_j^{\Delta_n}\right| \right)^{\alpha} &\leq \frac{C(p)^{\alpha}\,\Delta_n^{\alpha}}{n} \sum_{k=1}^{n} \left(\sum_{j=k}^{n} \mathrm{e}^{\Delta_n\lambda_{\max} j} \right)^{\alpha} \leq \frac{C(p)^{\alpha}\,\Delta_n^{\alpha}}{n\left(1-\mathrm{e}^{\Delta_n\lambda_{\max}}\right)^{\alpha}} \sum_{k=1}^{n} \mathrm{e}^{\alpha\Delta_n\lambda_{\max} k} \\ &\leq \frac{C(p)^{\alpha}\,\Delta_n^{\alpha}}{n\left(1-\mathrm{e}^{\Delta_n\lambda_{\max}}\right)^{\alpha}} \cdot \frac{1}{1-\mathrm{e}^{\alpha\Delta_n\lambda_{\max}}} \sim \frac{C(p)^{\alpha}}{(-\lambda_{\max})^{\alpha}} \cdot \frac{1}{-\alpha\lambda_{\max} n\Delta_n} \to 0 \end{aligned}$$

as $n \to \infty$, since we suppose $n\Delta_n \to \infty$.

(iv) We have, once again due to *(i)*,

$$\begin{aligned} \frac{\Delta_n^{\alpha}}{n} \sum_{k=2-p-n}^{-1} \left(\sum_{j=1\vee(2-p-k)}^{n\wedge(-k)} \left|\Psi_j^{\Delta_n}\right| \right)^{\alpha} &\leq \frac{\Delta_n^{\alpha}}{n} \left[\sum_{k=1}^{p-2} \left(\sum_{j=1}^{k} \left|\Psi_j^{\Delta_n}\right| \right)^{\alpha} + \sum_{k=p-1}^{n+p-2} \left(\sum_{j=k+2-p}^{k} \left|\Psi_j^{\Delta_n}\right| \right)^{\alpha} \right] \\ &\leq \frac{\Delta_n^{\alpha}}{n} \left[(p-2)\cdot(p-1)^{\alpha p} + \left(C(p)\cdot(p-1)\cdot\Delta_n^{-p+1}\right)^{\alpha} \cdot \sum_{k=p-1}^{n+p-2} \mathrm{e}^{\alpha\Delta_n\lambda_{\max}(k+2-p)} \right] \\ &\leq \frac{\Delta_n^{\alpha}}{n} \left[(p-2)\cdot(p-1)^{\alpha p} + \left(C(p)\cdot(p-1)\cdot\Delta_n^{-p+1}\right)^{\alpha} \cdot \frac{1}{1-\mathrm{e}^{\alpha\Delta_n\lambda_{\max}}} \right], \end{aligned}$$

where the first summand obviously vanishes as $n \to \infty$. The second term is asymptotically equivalent to

$$\frac{(C(p)\cdot(p-1))^{\alpha}}{-\alpha\lambda_{\max}}\cdot\frac{1}{n\Delta_n^{\alpha(p-2)+1}} \to 0$$

as $n \to \infty$ by assumption.

(v) It is once more *(i)* that gives

$$\begin{aligned}\frac{\Delta_n^{\alpha}}{n}\sum_{k=2-p}^{0}\left(\sum_{j=1}^{n}\left|\Psi_j^{\Delta_n}\right|\right)^{\alpha} &\le (p-1)\frac{\Delta_n^{\alpha}}{n}\left(\sum_{j=1}^{\infty}\left|\Psi_j^{\Delta_n}\right|\right)^{\alpha} \le (p-1)\frac{\Delta_n^{\alpha}}{n}\cdot C(p)^{\alpha}\left(\frac{\Delta_n^{-p+1}}{1-\mathrm{e}^{\Delta_n\lambda_{\max}}}\right)^{\alpha}\\ &\sim \frac{C(p)^{\alpha}\cdot(p-1)}{(-\lambda_{\max})^{\alpha}}\cdot\frac{1}{n\Delta_n^{\alpha(p-1)}} \to 0\end{aligned}$$

as $n \to \infty$, since we assume that $n\Delta_n^{\alpha(p-1)} \to \infty$ as $n \to \infty$. □

4.3 Proofs of Section 2.2

Since the proof of Proposition 2.4 is based on Theorem 2.5, we prove first Theorem 2.5 and then Proposition 2.4. For the proof of Theorem 2.5 we need the following additional result:

Proposition 4.1. *If Assumption 1 holds, $\Delta = \Delta_n \to 0$ and $n\Delta_n^{\alpha(p-1)} \to \infty$ as $n \to \infty$, then, for any $\omega \in \mathbb{R}$,*

$$J_{n,\widetilde{Z}^{\Delta_n}}(\omega\Delta_n) = J^{(2)}_{n,\Delta_n}(\omega\Delta_n) + o_P\left(\Delta_n^{\frac{1}{\alpha}+p-1}\right) \quad \text{as } n \to \infty$$

with $J^{(2)}_{n,\Delta_n}(\omega\Delta_n) := n^{-1/\alpha}\sum_{k=1}^{n-p+1}\tilde{\tilde{Z}}_{k,\Delta_n}(\omega\Delta_n)\,\mathrm{e}^{-i\omega\Delta_n k}$ and $(\tilde{\tilde{Z}}_{k,\Delta_n})_{k\in\mathbb{Z}}$ as given in eq. (2.4).

Proof. We first observe that

$$\begin{aligned}J_{n,\widetilde{Z}^{\Delta_n}}(\omega\Delta_n) &= n^{-1/\alpha}\sum_{k=1}^{n}\widetilde{Z}_{k,\Delta_n}\,\mathrm{e}^{-i\omega\Delta_n k} \overset{(2.1)}{=} n^{-1/\alpha}\sum_{k=1}^{n}\left(\sum_{r=1}^{p}Z^r_{k-r+1,\Delta_n}\right)\mathrm{e}^{-i\omega\Delta_n k}\\ &= n^{-1/\alpha}\sum_{k=2-p}^{n}\sum_{r=1\vee(2-k)}^{p\wedge(n+1-k)}Z^r_{k,\Delta_n}\,\mathrm{e}^{-i\omega\Delta_n(k+r-1)} = J^{(1)}_{n,\Delta_n}(\omega\Delta_n)+J^{(2)}_{n,\Delta_n}(\omega\Delta_n)+J^{(3)}_{n,\Delta_n}(\omega\Delta_n) \quad (4.9)\end{aligned}$$

with

$$\begin{aligned}J^{(1)}_{n,\Delta_n}(\omega\Delta_n) &:= n^{-1/\alpha}\sum_{k=2-p}^{0}\sum_{r=2-k}^{p}Z^r_{k,\Delta_n}\,\mathrm{e}^{-i\omega\Delta_n(k+r-1)},\\ J^{(2)}_{n,\Delta_n}(\omega\Delta_n) &:= n^{-1/\alpha}\sum_{k=1}^{n-p+1}\sum_{r=1}^{p}Z^r_{k,\Delta_n}\,\mathrm{e}^{-i\omega\Delta_n(k+r-1)} \overset{(2.4)}{=} n^{-1/\alpha}\sum_{k=1}^{n-p+1}\mathrm{e}^{-i\omega\Delta_n k}\,\tilde{\tilde{Z}}_{k,\Delta_n}(\omega\Delta_n) \quad \text{and}\\ J^{(3)}_{n,\Delta_n}(\omega\Delta_n) &:= n^{-1/\alpha}\sum_{k=n-p+2}^{n}\sum_{r=1}^{n+1-k}Z^r_{k,\Delta_n}\,\mathrm{e}^{-i\omega\Delta_n(k+r-1)}.\end{aligned}$$

Moreover, we define, for any $z_1, z_2 \in \mathbb{R}$, the function $\Xi_{z_1,z_2} : \mathbb{C} \to \mathbb{R}$, $\Xi_{z_1,z_2}(x) := z_1\,\Re(x) + z_2\,\Im(x)$. Then we have, due to eq. (2.2a) and Lemma 2.1(i),

$$\begin{aligned}&J^{(1)}_{n,\Delta_n}(\omega\Delta_n)\\ &= n^{-1/\alpha}\sum_{k=2-p}^{0}\mathrm{e}^{-i\omega\Delta_n k}\sum_{r=2-k}^{p}\mathrm{e}^{-i\omega\Delta_n(r-1)}\int_{(k-1)\Delta_n}^{k\Delta_n}c^T\left(-\sum_{j=0}^{r-1}\Phi_j^{\Delta_n}\,\mathrm{e}^{(r-1-j)\Delta_n A}\right)\mathrm{e}^{(k\Delta_n-s)A}e_p\,\mathrm{d}L_s\\ &= n^{-1/\alpha}\sum_{k=2-p}^{0}\mathrm{e}^{-i\omega\Delta_n k}\sum_{m=1}^{p}\frac{c(\lambda_m)}{a'(\lambda_m)}\sum_{r=2-k}^{p}\mathrm{e}^{-i\omega\Delta_n(r-1)}\left(-\sum_{j=0}^{r-1}\Phi_j^{\Delta_n}\,\mathrm{e}^{(r-1-j)\Delta_n\lambda_m}\right)\int_{(k-1)\Delta_n}^{k\Delta_n}\mathrm{e}^{(k\Delta_n-s)\lambda_m}\,\mathrm{d}L_s\end{aligned}$$

$$= n^{-1/\alpha} \sum_{k=2-p}^{0} \int_{(k-1)\Delta_n}^{k\Delta_n} \mathrm{e}^{-i\omega\Delta_n k}\, \zeta^{(k)}_{\Delta_n,\,\omega\Delta_n}(s)\,\mathrm{d}L_s, \tag{4.10}$$

where, for any $\omega \in \mathbb{R}$ and $\Delta > 0$,

$$\zeta^{(k)}_{\Delta,\,\omega}(s) := \sum_{m=1}^{p} \frac{c(\lambda_m)}{a'(\lambda_m)} f_{\Delta}^{(m;2-k)}(\omega)\,\mathrm{e}^{(k\Delta-s)\lambda_m} \ \text{ and } \ f_{\Delta}^{(m;2-k)}(\omega) := \sum_{r=2-k}^{p} \mathrm{e}^{-i\omega\Delta(r-1)} \left(-\sum_{j=0}^{r-1} \Phi_j^{\Delta}\, \mathrm{e}^{(r-1-j)\Delta\lambda_m} \right).$$

Hence, the joint characteristic function of the complex $S\alpha S$ random variable $\Delta_n^{1-p-1/\alpha} J^{(1)}_{n,\Delta_n}(\omega\Delta_n)$ is given by (cf. (2.6))

$$\Phi_{J^{(1)}_{n,\Delta_n}}(z_1,z_2) = \exp\left\{ -\sigma_L^{\alpha} \cdot \frac{1}{n\Delta_n^{1+\alpha(p-1)}} \sum_{k=2-p}^{0} \int_{(k-1)\Delta_n}^{k\Delta_n} \left| \Xi_{z_1,z_2}\left(\mathrm{e}^{-i\omega\Delta_n k}\, \zeta^{(k)}_{\Delta_n,\,\omega\Delta_n}(s) \right) \right|^{\alpha} \mathrm{d}s \right\}, \quad z_1, z_2 \in \mathbb{R}.$$

With the same arguments as in eqs. (4.4) and (4.5) we further obtain, as $n \to \infty$,

$$f_{\Delta_n}^{(m;2-k)}(\omega\Delta_n) = \sum_{r=1-k}^{p-1} (-1)^r \binom{p-1}{r} + O(\Delta_n) \tag{4.11}$$

and hence, $\left| f_{\Delta_n}^{(m;2-k)}(\omega\Delta_n) \right| \le 2^{p-1}$ for any $m \in \{1,\dots,p\}$ and $k = 2-p, 3-p, \dots, 0$, if only n is sufficiently large. Thus,

$$\begin{aligned} \frac{1}{n\Delta_n^{1+\alpha(p-1)}} \sum_{k=2-p}^{0} \int_{(k-1)\Delta_n}^{k\Delta_n} \left| \Xi_{z_1,z_2}\left(\mathrm{e}^{-i\omega\Delta_n k}\, \zeta^{(k)}_{\Delta_n,\,\omega\Delta_n}(s) \right) \right|^{\alpha} \mathrm{d}s &\le \frac{(|z_1|+|z_2|)^{\alpha}}{n\Delta_n^{1+\alpha(p-1)}} \sum_{k=2-p}^{0} \int_{(k-1)\Delta_n}^{k\Delta_n} \left| \zeta^{(k)}_{\Delta_n,\,\omega\Delta_n}(s) \right|^{\alpha} \mathrm{d}s \\ &\le (p-1) \frac{(|z_1|+|z_2|)^{\alpha}}{n\Delta_n^{\alpha(p-1)}} \left(2^{p-1} \sum_{m=1}^{p} \frac{|c(\lambda_m)|}{|a'(\lambda_m)|} \right)^{\alpha} \end{aligned}$$

and the right-hand side converges to 0 as $n \to \infty$, since we suppose $n\Delta_n^{\alpha(p-1)} \to \infty$. This obviously yields $J^{(1)}_{n,\Delta_n}(\omega\Delta_n) = o_P\big(\Delta_n^{1/\alpha+p-1}\big)$ as $n \to \infty$.

Likewise we obtain $J^{(3)}_{n,\Delta_n}(\omega\Delta_n) = o_P\big(\Delta_n^{1/\alpha+p-1}\big)$ as $n \to \infty$ which completes the proof of Proposition 4.1. □

Proof of Theorem 2.5. We prove that $\Delta_n^{1-p-1/\alpha}\big[J^{(2)}_{n,\Delta_n}(\omega_j\Delta_n)\big]_{j=1,\dots,m} \overset{\mathscr{D}}{\to} \big[c(i\omega_j)\cdot\big(S_j^{\Re}(\underset{\sim}{\omega}) - iS_j^{\Im}(\underset{\sim}{\omega})\big)\big]_{j=1,\dots,m}$ as $n \to \infty$ and then conclude with Proposition 4.1. By virtue of (2.5) we have

$$J^{(2)}_{n,\Delta_n}(\omega_j\Delta_n) = n^{-1/\alpha} \sum_{k=1}^{n-p+1} \int_{(k-1)\Delta_n}^{k\Delta_n} \mathrm{e}^{-i\omega_j\Delta_n k}\, g^{(k)}_{\Delta_n,\,\omega_j\Delta_n}(s)\,\mathrm{d}L_s \tag{4.12}$$

for any $j \in \{1,\dots,m\}$ and the joint characteristic function of the complex $S\alpha S$ random vector $\Delta_n^{1-p-1/\alpha}\big[J^{(2)}_{n,\Delta_n}(\omega_j\Delta_n)\big]_{j=1,\dots,m}$ is given by

$$\Phi_{J^{(2)}_{n,\Delta_n}}\big(\underset{\sim}{\theta},\underset{\sim}{\nu}\big) = \exp\left\{ -\sigma_L^{\alpha} \cdot \frac{1}{n\Delta_n^{1+\alpha(p-1)}} \sum_{k=1}^{n-p+1} \int_{(k-1)\Delta_n}^{k\Delta_n} \left| \sum_{j=1}^{m} \Xi_{\theta_j,\nu_j}\left(\mathrm{e}^{-i\omega_j\Delta_n k}\, g^{(k)}_{\Delta_n,\,\omega_j\Delta_n}(s) \right) \right|^{\alpha} \mathrm{d}s \right\} \tag{4.13}$$

with arbitrary $\underset{\sim}{\theta}, \underset{\sim}{\nu} \in \mathbb{R}^m$. Hence, due to Lévy's Continuity Theorem, we have to show for any $\underset{\sim}{\theta}, \underset{\sim}{\nu} \in \mathbb{R}^m$

$$\begin{aligned} &\frac{1}{n\Delta_n^{1+\alpha(p-1)}} \sum_{k=1}^{n-p+1} \int_{(k-1)\Delta_n}^{k\Delta_n} \left| \sum_{j=1}^{m} \Xi_{\theta_j,\nu_j}\left(\mathrm{e}^{-i\omega_j\Delta_n k}\, g^{(k)}_{\Delta_n,\,\omega_j\Delta_n}(s) \right) \right|^{\alpha} \mathrm{d}s \\ &\qquad \overset{n\to\infty}{\longrightarrow} K_{\underset{\sim}{\omega}}\left(\Big(\Xi_{\theta_j,\nu_j}\big(c(i\omega_j)\big) \Big)_{j\in\{1,\dots,m\}}, \Big(\Xi_{-\nu_j,\theta_j}\big(c(i\omega_j)\big) \Big)_{j\in\{1,\dots,m\}}, 0 \right), \end{aligned} \tag{4.14}$$

where K_{ω} has been defined in (2.7b) and (2.7c), respectively.

We first claim

$$\left| \frac{1}{n} \sum_{k=1}^{n-p+1} \left(\frac{1}{\Delta_n} \int_{(k-1)\Delta_n}^{k\Delta_n} \left| \sum_{j=1}^{m} \Xi_{\theta_j, \nu_j} \left(\mathrm{e}^{-i\omega_j \Delta_n k} \frac{g^{(k)}_{\Delta_n, \omega_j \Delta_n}(s)}{\Delta_n^{p-1}} \right) \right|^{\alpha} \mathrm{d}s - \left| \sum_{j=1}^{m} \Xi_{\theta_j, \nu_j} \left(\mathrm{e}^{-i\omega_j \Delta_n k} c(i\omega_j) \right) \right|^{\alpha} \right) \right| \overset{n\to\infty}{\to} 0. \tag{4.15}$$

To this end, we use $\left||x|^{\alpha} - |y|^{\alpha}\right| \le (|x|^{\alpha/2} + |y|^{\alpha/2}) \cdot |x-y|^{\alpha/2}$ for $\alpha \in (0,2]$ together with the Cauchy-Schwarz inequality and obtain

$$\begin{aligned}
&\left| \frac{1}{n\Delta_n} \sum_{k=1}^{n-p+1} \int_{(k-1)\Delta_n}^{k\Delta_n} \left| \sum_{j=1}^{m} \Xi_{\theta_j, \nu_j} \left(\mathrm{e}^{-i\omega_j \Delta_n k} \frac{g^{(k)}_{\Delta_n, \omega_j \Delta_n}(s)}{\Delta_n^{p-1}} \right) \right|^{\alpha} - \left| \sum_{j=1}^{m} \Xi_{\theta_j, \nu_j} \left(\mathrm{e}^{-i\omega_j \Delta_n k} c(i\omega_j) \right) \right|^{\alpha} \mathrm{d}s \right| \\
&\quad \le \frac{1}{n\Delta_n} \sum_{k=1}^{n-p+1} \int_{(k-1)\Delta_n}^{k\Delta_n} \left(\left| \sum_{j=1}^{m} \Xi_{\theta_j, \nu_j} \left(\mathrm{e}^{-i\omega_j \Delta_n k} \frac{g^{(k)}_{\Delta_n, \omega_j \Delta_n}(s)}{\Delta_n^{p-1}} \right) \right|^{\frac{\alpha}{2}} + \left| \sum_{j=1}^{m} \Xi_{\theta_j, \nu_j} \left(\mathrm{e}^{-i\omega_j \Delta_n k} c(i\omega_j) \right) \right|^{\frac{\alpha}{2}} \right) \\
&\qquad\qquad \times \left| \sum_{j=1}^{m} \Xi_{\theta_j, \nu_j} \left(\mathrm{e}^{-i\omega_j \Delta_n k} \left(\frac{g^{(k)}_{\Delta_n, \omega_j \Delta_n}(s)}{\Delta_n^{p-1}} - c(i\omega_j) \right) \right) \right|^{\frac{\alpha}{2}} \mathrm{d}s \\
&\quad \le \left[\frac{1}{n} \sum_{k=1}^{n-p+1} \frac{1}{\Delta_n} \int_{(k-1)\Delta_n}^{k\Delta_n} \left(\sum_{j=1}^{m} \left(|\theta_j| + |\nu_j| \right)^{\frac{\alpha}{2}} \cdot \left(\left| \frac{g^{(k)}_{\Delta_n, \omega_j \Delta_n}(s)}{\Delta_n^{p-1}} \right|^{\frac{\alpha}{2}} + \left| c(i\omega_j) \right|^{\frac{\alpha}{2}} \right) \right)^2 \mathrm{d}s \right]^{\frac{1}{2}} \\
&\qquad\qquad \times \left[\frac{1}{n} \sum_{k=1}^{n-p+1} \frac{1}{\Delta_n} \int_{(k-1)\Delta_n}^{k\Delta_n} \left| \sum_{j=1}^{m} \Xi_{\theta_j, \nu_j} \left(\mathrm{e}^{-i\omega_j \Delta_n k} \left(\frac{g^{(k)}_{\Delta_n, \omega_j \Delta_n}(s)}{\Delta_n^{p-1}} - c(i\omega_j) \right) \right) \right|^{\alpha} \mathrm{d}s \right]^{\frac{1}{2}} \\
&\quad =: I_1 \times I_2,
\end{aligned}$$

where, due to Assumption 1, eq. (2.5) and Lemma 2.1(iii), there are constants $C(\omega_j) > 0$ such that for all sufficiently large n

$$\begin{aligned}
I_1^2 &\le 2m^2 \sum_{j=1}^{m} \left(|\theta_j| + |\nu_j| \right)^{\alpha} \cdot \frac{1}{n} \sum_{k=1}^{n-p+1} \frac{1}{\Delta_n} \int_{(k-1)\Delta_n}^{k\Delta_n} \left| \frac{g^{(k)}_{\Delta_n, \omega_j \Delta_n}(s)}{\Delta_n^{p-1}} \right|^{\alpha} + \left| c(i\omega_j) \right|^{\alpha} \mathrm{d}s \\
&\le 2m^2 \sum_{j=1}^{m} \left(|\theta_j| + |\nu_j| \right)^{\alpha} \cdot \left(\left(C(\omega_j) \sum_{l=1}^{p} \frac{|c(\lambda_l)|}{|a'(\lambda_l)|} \right)^{\alpha} + \left| c(i\omega_j) \right|^{\alpha} \right) < \infty
\end{aligned}$$

and hence, I_1 is bounded. Setting

$$h^{(k)}_{\Delta_n, \omega}(s) := \sum_{l=1}^{p} \frac{c(\lambda_l)}{a'(\lambda_l)} \frac{a(i\omega)}{i\omega - \lambda_l} \mathrm{e}^{(k\Delta_n - s)\lambda_l}, \quad k \in \{1, \ldots, p\},$$

we obtain for the second term

$$I_2^2 \le m^{\alpha} \sum_{j=1}^{m} \left(|\theta_j| + |\nu_j| \right)^{\alpha} \cdot \frac{1}{n} \sum_{k=1}^{n-p+1} \frac{1}{\Delta_n} \int_{(k-1)\Delta_n}^{k\Delta_n} \left| \frac{g^{(k)}_{\Delta_n, \omega_j \Delta_n}(s)}{\Delta_n^{p-1}} - c(i\omega_j) \right|^{\alpha} \mathrm{d}s$$

$$\leq (2m)^{\alpha} \sum_{j=1}^{m} \big(|\theta_j| + |\nu_j|\big)^{\alpha} \frac{1}{n} \sum_{k=1}^{n-p+1} \left[\frac{1}{\Delta_n} \int_{(k-1)\Delta_n}^{k\Delta_n} \left| \frac{g^{(k)}_{\Delta_n, \omega_j \Delta_n}(s)}{\Delta_n^{p-1}} - h^{(k)}_{\Delta_n, \omega_j}(s) \right|^{\alpha} + \left| h^{(k)}_{\Delta_n, \omega_j}(s) - c(i\omega_j) \right|^{\alpha} \mathrm{d}s \right]. \tag{4.16}$$

Then, for any $j \in \{1, \dots, m\}$,

$$\frac{1}{n} \sum_{k=1}^{n-p+1} \frac{1}{\Delta_n} \int_{(k-1)\Delta_n}^{k\Delta_n} \left| \frac{g^{(k)}_{\Delta_n, \omega_j \Delta_n}(s)}{\Delta_n^{p-1}} - h^{(k)}_{\Delta_n, \omega_j}(s) \right|^{\alpha} \mathrm{d}s \leq \left(\sum_{l=1}^{p} \frac{|c(\lambda_l)|}{|a'(\lambda_l)|} \cdot \left| \frac{f^{(l)}_{\Delta_n}(\omega_j \Delta_n)}{\Delta_n^{p-1}} - \frac{a(i\omega_j)}{i\omega_j - \lambda_l} \right| \right)^{\alpha} \overset{n\to\infty}{\to} 0 \tag{4.17}$$

by virtue of Lemma 2.1(iii). Moreover,

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^{n-p+1} \frac{1}{\Delta_n} \int_{(k-1)\Delta_n}^{k\Delta_n} \left| h^{(k)}_{\Delta_n, \omega_j}(s) - c(i\omega_j) \right|^{\alpha} \mathrm{d}s &= \frac{1}{n} \sum_{k=1}^{n-p+1} \frac{1}{\Delta_n} \int_{(k-1)\Delta_n}^{k\Delta_n} \left| \sum_{l=1}^{p} \frac{c(\lambda_l)}{a'(\lambda_l)} \frac{a(i\omega_j)}{i\omega_j - \lambda_l} \left(\mathrm{e}^{(k\Delta_n - s)\lambda_l} - 1 \right) \right|^{\alpha} \mathrm{d}s \\ &\leq p^{\alpha} \cdot \sum_{l=1}^{p} \left(\frac{|c(\lambda_l)|}{|a'(\lambda_l)|} \cdot \frac{|a(i\omega_j)|}{|i\omega_j - \lambda_l|} \right)^{\alpha} \frac{1}{\Delta_n} \int_0^{\Delta_n} \left| \mathrm{e}^{(\Delta_n - s)\lambda_l} - 1 \right|^{\alpha} \mathrm{d}s \overset{n\to\infty}{\to} 0, \end{aligned} \tag{4.18}$$

where we used Lemma 2.1(ii) and (iv). Hence, by eqs. (4.17) and (4.18) the right-hand side of (4.16) converges to 0 as $n \to \infty$ and thus, (4.15) is shown, as well.

In order to obtain (4.14) and hence, $\Delta_n^{1-p-1/\alpha} \big[J^{(2)}_{n,\Delta_n}(\omega_j \Delta_n)\big]_{j=1,\dots,m} \overset{\mathscr{D}}{\to} \big[c(i\omega_j) \cdot \big(S^{\mathfrak{R}}_j(\underset{\sim}{\omega}) - iS^{\mathfrak{I}}_j(\underset{\sim}{\omega})\big)\big]_{j=1,\dots,m}$ as $n \to \infty$, it remains to prove that

$$\frac{1}{n} \sum_{k=1}^{n-p+1} \left| \sum_{j=1}^{m} \Xi_{\theta_j, \nu_j} \left(\mathrm{e}^{-i\omega_j \Delta_n k} c(i\omega_j) \right) \right|^{\alpha} \overset{n\to\infty}{\to} K_{\underset{\sim}{\omega}} \left(\left(\Xi_{\theta_j, \nu_j}\big(c(i\omega_j)\big) \right)_{j \in \{1,\dots,m\}}, \left(\Xi_{-\nu_j, \theta_j}\big(c(i\omega_j)\big) \right)_{j \in \{1,\dots,m\}}, 0 \right).$$

Since we suppose in particular $n\Delta_n \to \infty$ as $n \to \infty$, this follows from Proposition 3.4.

Finally, since also $n\Delta_n^{\alpha(p-1)} \to \infty$ as $n \to \infty$ holds, Proposition 4.1 yields $J^{(1)}_{n,\Delta_n}(\omega\Delta_n) + J^{(3)}_{n,\Delta_n}(\omega\Delta_n) = o_P\big(\Delta_n^{1/\alpha+p-1}\big)$ for any $\omega \in \mathbb{R}$ and hence, $\Delta_n^{1-p-1/\alpha} \big[J_{n,\widetilde{Z}^{\Delta_n}}(\omega_j \Delta_n)\big]_{j=1,\dots,m} \overset{\mathscr{D}}{\to} \big[c(i\omega_j) \cdot \big(S^{\mathfrak{R}}_j(\underset{\sim}{\omega}) - iS^{\mathfrak{I}}_j(\underset{\sim}{\omega})\big)\big]_{j=1,\dots,m}$ as $n \to \infty$. This completes the proof. □

Proof of Proposition 2.4. We immediately obtain

$$\begin{aligned} J_{n,Y^{\Delta_n}}(\omega) &= n^{-1/\alpha} \sum_{k=1}^{n} Y_{k\Delta_n} \mathrm{e}^{-i\omega k} \overset{(2.3)}{=} n^{-1/\alpha} \sum_{k=1}^{n} \left(\sum_{j=0}^{\infty} \Psi^{\Delta_n}_j \widetilde{Z}_{k-j,\Delta_n} \right) \mathrm{e}^{-i\omega k} \\ &= n^{-1/\alpha} \sum_{j=0}^{\infty} \Psi^{\Delta_n}_j \mathrm{e}^{-i\omega j} \left(\sum_{k=1}^{n} \widetilde{Z}_{k,\Delta_n} \mathrm{e}^{-i\omega k} + U_{n,j,\Delta_n}(\omega) \right) = \Psi^{\Delta_n}(\mathrm{e}^{-i\omega}) J_{n,\widetilde{Z}^{\Delta_n}}(\omega) + W_{n,\Delta_n}(\omega), \end{aligned}$$

where

$$U_{n,j,\Delta_n}(\omega) = \sum_{k=1-j}^{n-j} \widetilde{Z}_{k,\Delta_n} \mathrm{e}^{-i\omega k} - \sum_{k=1}^{n} \widetilde{Z}_{k,\Delta_n} \mathrm{e}^{-i\omega k} \quad \text{and} \quad W_{n,\Delta_n}(\omega) = n^{-1/\alpha} \sum_{j=0}^{\infty} \Psi^{\Delta_n}_j \mathrm{e}^{-i\omega j} U_{n,j,\Delta_n}(\omega).$$

Hence,

$$I_{n,Y^{\Delta_n}}(\omega) = \left| \Psi^{\Delta_n}(\mathrm{e}^{-i\omega}) \right|^2 I_{n,\widetilde{Z}^{\Delta_n}}(\omega) + R_{n,\Delta_n}(\omega),$$

with

$$R_{n,\Delta_n}(\omega) = \Psi^{\Delta_n}(\mathrm{e}^{-i\omega}) J_{n,\widetilde{Z}^{\Delta_n}}(\omega) \overline{W_{n,\Delta_n}(\omega)} + \overline{\Psi^{\Delta_n}(\mathrm{e}^{-i\omega}) J_{n,\widetilde{Z}^{\Delta_n}}(\omega)} W_{n,\Delta_n}(\omega) + |W_{n,\Delta_n}(\omega)|^2.$$

For the rest of the proof suppose that Assumption 2 holds and fix an arbitrary $\omega \in \mathbb{R}^*$. We have to show that $\Delta_n^{2-2/\alpha} |R_{n,\Delta_n}(\omega\Delta_n)| \overset{\mathbb{P}}{\to} 0$ as $n \to \infty$.

Since $\Psi^{\Delta_n}(\mathrm{e}^{-i\omega\Delta_n}) \sim \Delta_n^{-p} a(i\omega)^{-1}$ as $n \to \infty$ and since in particular $n\Delta_n^{1\vee\alpha(p-1)} \to \infty$ if Assumption 2 holds, it follows from Theorem 2.5 that $\Delta_n^{1-1/\alpha}\Psi^{\Delta_n}(\mathrm{e}^{-i\omega\Delta_n})J_{n,\widetilde{Z}^{\Delta_n}}(\omega\Delta_n)\xrightarrow{\mathscr{D}}\frac{c(i\omega)}{a(i\omega)}\big(S_1^{\Re}(\omega)-iS_1^{\Im}(\omega)\big)$ as $n\to\infty$, where the joint characteristic function of $\big(S_1^{\Re}(\omega),S_1^{\Im}(\omega)\big)$ is given by eq. (2.7) (with $m=1$ and $\tau=0$). Hence, in order to show $\Delta_n^{2-2/\alpha}|R_{n,\Delta_n}(\omega\Delta_n)|\xrightarrow{\mathbb{P}}0$ as $n\to\infty$, it is sufficient to prove that

$$\Delta_n^{1-\frac{1}{\alpha}}W_{n,\Delta_n}(\omega\Delta_n)\xrightarrow{\mathbb{P}}0\qquad\text{as } n\to\infty. \tag{4.19}$$

We shall prove (4.19) by an appropriate decomposition of the sum $W_{n,\Delta_n}(\omega\Delta_n)$, analogously to the one in [23, Proof of Proposition 2.1]. We write

$$\begin{aligned}W_{n,\Delta_n}(\omega\Delta_n)&=n^{-1/\alpha}\sum_{j=n+1}^{\infty}\Psi_j^{\Delta_n}\mathrm{e}^{-i\omega\Delta_n j}U_{n,j,\Delta_n}(\omega\Delta_n)+n^{-1/\alpha}\sum_{j=0}^{n}\Psi_j^{\Delta_n}\mathrm{e}^{-i\omega\Delta_n j}U_{n,j,\Delta_n}(\omega\Delta_n)\\&=:W_{n,\Delta_n}^{(1)}(\omega\Delta_n)+W_{n,\Delta_n}^{(2)}(\omega\Delta_n)\end{aligned}$$

and

$$\begin{aligned}W_{n,\Delta_n}^{(1)}(\omega\Delta_n)&=n^{-1/\alpha}\sum_{j=n+1}^{\infty}\Psi_j^{\Delta_n}\mathrm{e}^{-i\omega\Delta_n j}\bigg(-\sum_{k=1}^{n}\widetilde{Z}_{k,\Delta_n}\mathrm{e}^{-i\omega\Delta_n k}\bigg)+n^{-1/\alpha}\sum_{j=n+1}^{\infty}\Psi_j^{\Delta_n}\mathrm{e}^{-i\omega\Delta_n j}\sum_{k=1-j}^{n-j}\widetilde{Z}_{k,\Delta_n}\mathrm{e}^{-i\omega\Delta_n k}\\&=:W_{n,\Delta_n}^{(11)}(\omega\Delta_n)+W_{n,\Delta_n}^{(12)}(\omega\Delta_n).\end{aligned}$$

We have

$$\Delta_n^{1-\frac{1}{\alpha}}\Big|W_{n,\Delta_n}^{(11)}(\omega\Delta_n)\Big|\le\Delta_n^{1-p-\frac{1}{\alpha}}\big|J_{n,\widetilde{Z}^{\Delta_n}}(\omega\Delta_n)\big|\cdot\Delta_n^{p}\sum_{j=n+1}^{\infty}\big|\Psi_j^{\Delta_n}\big|$$

and it is again Theorem 2.5 together with the Continuous Mapping Theorem (see, for instance, [21, Theorem 13.25]) showing $\Delta_n^{1-p-1/\alpha}|J_{n,\widetilde{Z}^{\Delta_n}}(\omega\Delta_n)|\xrightarrow{\mathscr{D}}|c(i\omega)|\cdot\big|S_1^{\Re}(\omega)-iS_1^{\Im}(\omega)\big|$ as $n\to\infty$. Since we have $\sum_{j=n+1}^{\infty}\big|\Psi_j^{\Delta_n}\big|\to0$ by virtue of Lemma 2.2(ii), we immediately deduce $\Delta_n^{1-1/\alpha}W_{n,\Delta_n}^{(11)}(\omega\Delta_n)\xrightarrow{\mathbb{P}}0$ as $n\to\infty$.

Concerning the term $W_{n,\Delta_n}^{(12)}(\omega\Delta_n)$ we write

$$\begin{aligned}W_{n,\Delta_n}^{(12)}(\omega\Delta_n)&=n^{-1/\alpha}\sum_{j=n+1}^{\infty}\Psi_j^{\Delta_n}\mathrm{e}^{-i\omega\Delta_n j}\sum_{k=1-j}^{n-j}\widetilde{Z}_{k,\Delta_n}\mathrm{e}^{-i\omega\Delta_n k}\\&=n^{-1/\alpha}\sum_{k=-n}^{-1}\widetilde{Z}_{k,\Delta_n}\mathrm{e}^{-i\omega\Delta_n k}\sum_{j=n+1}^{n-k}\Psi_j^{\Delta_n}\mathrm{e}^{-i\omega\Delta_n j}+n^{-1/\alpha}\sum_{k=-\infty}^{-n-1}\widetilde{Z}_{k,\Delta_n}\mathrm{e}^{-i\omega\Delta_n k}\sum_{j=1-k}^{n-k}\Psi_j^{\Delta_n}\mathrm{e}^{-i\omega\Delta_n j}\\&=:W_{n,\Delta_n}^{(121)}(\omega\Delta_n)+W_{n,\Delta_n}^{(122)}(\omega\Delta_n)\end{aligned}$$

and obtain for arbitrary $\varepsilon>0$

$$\begin{aligned}\mathbb{P}\Big(\Delta_n^{1-\frac{1}{\alpha}}\big|W_{n,\Delta_n}^{(121)}(\omega\Delta_n)\big|>\varepsilon\Big)&\le\sum_{r=1}^{p}\mathbb{P}\bigg(\Delta_n^{1-\frac{1}{\alpha}}n^{-\frac{1}{\alpha}}\bigg|\sum_{k=-n}^{-1}Z_{k-r+1,\Delta_n}^{r}\sum_{j=n+1}^{n-k}\Psi_j^{\Delta_n}\mathrm{e}^{-i\omega\Delta_n(k+j)}\bigg|>\frac{\varepsilon}{p}\bigg)\\&\le\sum_{r=1}^{p}\bigg[\mathbb{P}\bigg(\Delta_n^{1-\frac{1}{\alpha}}n^{-\frac{1}{\alpha}}\bigg|\sum_{k=-n}^{-1}Z_{k-r+1,\Delta_n}^{r}\cdot\Re\bigg(\sum_{j=n+1}^{n-k}\Psi_j^{\Delta_n}\mathrm{e}^{-i\omega\Delta_n(k+j)}\bigg)\bigg|>\frac{\varepsilon}{2p}\bigg)\\&\qquad+\mathbb{P}\bigg(\Delta_n^{1-\frac{1}{\alpha}}n^{-\frac{1}{\alpha}}\bigg|\sum_{k=-n}^{-1}Z_{k-r+1,\Delta_n}^{r}\cdot\Im\bigg(\sum_{j=n+1}^{n-k}\Psi_j^{\Delta_n}\mathrm{e}^{-i\omega\Delta_n(k+j)}\bigg)\bigg|>\frac{\varepsilon}{2p}\bigg)\bigg].\end{aligned}\tag{4.20}$$

Since, for any $r\in\{1,\dots,p\}$ and $n\in\mathbb{N}^*$, the random variables Z_{k-r+1,Δ_n}^{r}, $k\in\{-n,-n+1,\dots,-1\}$, are independent and symmetric we apply [37, Theorem 1.2] and the right-hand side of (4.20) can be bounded by

$$4\sum_{r=1}^{p}\mathbb{P}\bigg(\Delta_n^{1-\frac{1}{\alpha}}n^{-\frac{1}{\alpha}}\cdot\sum_{j=n+1}^{2n}\big|\Psi_j^{\Delta_n}\big|\cdot\bigg|\sum_{k=-n}^{-1}Z_{k-r+1,\Delta_n}^{r}\bigg|>\frac{\varepsilon}{2p}\bigg). \tag{4.21}$$

By virtue of (2.2a), (4.4) and Lemma 2.1(i), the characteristic function of $\Delta_n^{1-1/\alpha}n^{-1/\alpha}\cdot\sum_{j=n+1}^{2n}\big|\Psi_j^{\Delta_n}\big|\cdot\sum_{k=-n}^{-1}Z_{k-r+1,\Delta_n}^{r}$ is given by

$$\Phi(z_1,z_2)=\exp\bigg\{-\sigma_L^{\alpha}\cdot\frac{\Delta_n^{\alpha}}{n\Delta_n}\bigg(\sum_{j=n+1}^{2n}\big|\Psi_j^{\Delta_n}\big|\bigg)^{\alpha}$$
$$\times\sum_{k=-n}^{-1}\int_{(k-r)\Delta_n}^{(k-r+1)\Delta_n}\bigg|\Xi_{z_1,z_2}\bigg(\sum_{m=1}^{p}\frac{c(\lambda_m)}{a'(\lambda_m)}\cdot\sum_{\{i_1,\ldots,i_{r-1}\}\in\binom{\{1,\ldots,p\}\setminus\{m\}}{r-1}}\mathrm{e}^{\Delta_n\sum_{h=1}^{r-1}\lambda_{i_h}}\cdot\mathrm{e}^{((k-r+1)\Delta_n-s)\lambda_m}\bigg)\bigg|^{\alpha}\mathrm{d}s\bigg\}$$

for any $z_1,z_2\in\mathbb{R}$ (see proof of Proposition 4.1 for the definition of Ξ_{z_1,z_2}). We then obtain with $\lambda_{\max}:=\max_{k\in\{1,\ldots,p\}}\Re(\lambda_k)<0$

$$\bigg|-\frac{1}{\sigma_L^{\alpha}}\log\Phi(z_1,z_2)\bigg|\leq\Delta_n^{\alpha}\bigg(\sum_{j=n+1}^{2n}\big|\Psi_j^{\Delta_n}\big|\bigg)^{\alpha}\cdot(|z_1|+|z_2|)^{\alpha}\cdot\bigg(\binom{p-1}{r-1}\mathrm{e}^{\Delta_n\lambda_{\max}(r-1)}\sum_{m=1}^{p}\frac{|c(\lambda_m)|}{|a'(\lambda_m)|}\bigg)^{\alpha}$$

and the right-hand side converges to 0 as $n\to\infty$ due to Lemma 2.2(ii). Thus, (4.21) converges to 0 as well and $\Delta_n^{1-1/\alpha}W_{n,\Delta_n}^{(121)}(\omega\Delta_n)\xrightarrow{\mathbb{P}}0$ is shown.

In order to get $\Delta_n^{1-1/\alpha}W_{n,\Delta_n}^{(122)}(\omega\Delta_n)\xrightarrow{\mathbb{P}}0$, we prove, for any $r\in\{1,\ldots,p\}$,

$$\Delta_n^{1-1/\alpha}n^{-1/\alpha}\sum_{k=-\infty}^{-n-1}Z_{k-r+1,\Delta_n}^{r}\sum_{j=1-k}^{n-k}\Psi_j^{\Delta_n}\,\mathrm{e}^{-i\omega\Delta_n(k+j)}\xrightarrow{\mathbb{P}}0.$$

Therefore it is sufficient (using the same arguments as above via characteristic functions) to show that

$$\frac{\Delta_n^{\alpha}}{n}\sum_{k=-\infty}^{-n-1}\bigg(\sum_{j=1-k}^{n-k}\big|\Psi_j^{\Delta_n}\big|\bigg)^{\alpha}\to0$$

as $n\to\infty$. This can be found in Lemma 2.2(ii) and hence, $\Delta_n^{1-1/\alpha}W_{n,\Delta_n}^{(122)}(\omega\Delta_n)\xrightarrow{\mathbb{P}}0$. All together we have shown that $\Delta_n^{1-1/\alpha}W_{n,\Delta_n}^{(1)}(\omega\Delta_n)$ converges to 0 in probability.

It remains to prove that also $\Delta_n^{1-1/\alpha}W_{n,\Delta_n}^{(2)}(\omega\Delta_n)\xrightarrow{\mathbb{P}}0$ as $n\to\infty$. To this end, we define

$$\begin{aligned}W_{n,\Delta_n}^{(21)}(\omega\Delta_n):=n^{-1/\alpha}\sum_{j=1}^{n}\Psi_j^{\Delta_n}\,\mathrm{e}^{-i\omega\Delta_nj}\bigg[\bigg(&\sum_{k=2-p-j}^{-j}\sum_{r=2-j-k}^{p}+\sum_{k=2-p}^{0}\sum_{r=1}^{1-k}-\sum_{k=n+2-p-j}^{n-j}\sum_{r=n+2-j-k}^{p}\\&-\sum_{k=n-p+2}^{n}\sum_{r=1}^{n+1-k}\bigg)Z_{k,\Delta_n}^{r}\,\mathrm{e}^{-i\omega\Delta_n(k+r-1)}\bigg]\\=:W_{n,\Delta_n}^{(211)}(\omega\Delta_n)+W_{n,\Delta_n}^{(212)}(\omega\Delta_n)&-W_{n,\Delta_n}^{(213)}(\omega\Delta_n)-W_{n,\Delta_n}^{(214)}(\omega\Delta_n)\end{aligned}$$

and write

$$\begin{aligned}W_{n,\Delta_n}^{(2)}(\omega\Delta_n)&=n^{-1/\alpha}\sum_{j=1}^{n}\Psi_j^{\Delta_n}\,\mathrm{e}^{-i\omega\Delta_nj}\bigg(\sum_{k=1-j}^{0}\sum_{r=1}^{p}Z_{k-r+1,\Delta_n}^{r}\,\mathrm{e}^{-i\omega\Delta_nk}-\sum_{k=n-j+1}^{n}\sum_{r=1}^{p}Z_{k-r+1,\Delta_n}^{r}\,\mathrm{e}^{-i\omega\Delta_nk}\bigg)\\&=n^{-1/\alpha}\sum_{j=1}^{n}\Psi_j^{\Delta_n}\,\mathrm{e}^{-i\omega\Delta_nj}\bigg(\sum_{k=2-p-j}^{0}\sum_{r=1\vee(2-j-k)}^{p\wedge(1-k)}Z_{k,\Delta_n}^{r}\,\mathrm{e}^{-i\omega\Delta_n(k+r-1)}\end{aligned}$$

$$-\sum_{k=n+2-p-j}^{n}\sum_{r=1\vee(n+2-j-k)}^{p\wedge(n+1-k)} Z^r_{k,\Delta_n}\,\mathrm{e}^{-i\omega\Delta_n(k+r-1)}\Bigg)$$

$$= W^{(21)}_{n,\Delta_n}(\omega\Delta_n)+n^{-1/\alpha}\sum_{j=1}^{n}\Psi^{\Delta_n}_j\,\mathrm{e}^{-i\omega\Delta_n j}\Bigg[\Bigg(\sum_{k=1-j}^{1-p}-\sum_{k=n-j+1}^{n-p+1}\Bigg)\tilde{\tilde{Z}}_{k,\Delta_n}(\omega\Delta_n)\,\mathrm{e}^{-i\omega\Delta_n k}\Bigg]$$

$$=: W^{(21)}_{n,\Delta_n}(\omega\Delta_n)+W^{(22)}_{n,\Delta_n}(\omega\Delta_n)-W^{(23)}_{n,\Delta_n}(\omega\Delta_n).$$

By virtue of eq. (2.5) we have

$$\Delta_n^{1-\frac{1}{\alpha}}W^{(22)}_{n,\Delta_n}(\omega\Delta_n)=\Delta_n^{1-\frac{1}{\alpha}}\,n^{-\frac{1}{\alpha}}\sum_{k=1-n}^{1-p}\tilde{\tilde{Z}}_{k,\Delta_n}(\omega\Delta_n)\,\mathrm{e}^{-i\omega\Delta_n k}\sum_{j=1-k}^{n}\Psi^{\Delta_n}_j\,\mathrm{e}^{-i\omega\Delta_n j}$$

$$=\frac{\Delta_n}{(n\Delta_n)^{1/\alpha}}\sum_{k=1-n}^{1-p}\sum_{j=1-k}^{n}\Psi^{\Delta_n}_j\,\mathrm{e}^{-i\omega\Delta_n(k+j)}\int_{(k-1)\Delta_n}^{k\Delta_n}\sum_{m=1}^{p}\frac{c(\lambda_m)}{a'(\lambda_m)}f^{(m)}_{\Delta_n}(\omega\Delta_n)\,\mathrm{e}^{(k\Delta_n-s)\lambda_m}\,\mathrm{d}L_s.$$

Since, due to Lemma 2.1(iii), $f^{(m)}_{\Delta_n}(\omega\Delta_n)\sim\Delta_n^{p-1}a(i\omega)\frac{1}{i\omega-\lambda_m}$ as $n\to\infty$ for all $m\in\{1,\ldots,p\}$, it is easy to see by calculating the characteristic function of $\Delta_n^{1-1/\alpha}W^{(22)}_{n,\Delta_n}(\omega\Delta_n)$ that it is enough to show that

$$\frac{\Delta_n^{\alpha p}}{n}\sum_{k=1-n}^{1-p}\Big(\sum_{j=1-k}^{n}\big|\Psi^{\Delta_n}_j\big|\Big)^{\alpha}\overset{n\to\infty}{\to}0.$$

This follows immediately from Lemma 2.2(iii) and hence also $\Delta_n^{1-1/\alpha}W^{(22)}_{n,\Delta_n}(\omega\Delta_n)\overset{\mathbb{P}}{\to}0$ as $n\to\infty$ holds.

Since the complex $S\alpha S$ random variables $\big(\tilde{\tilde{Z}}_{k,\Delta_n}\big)_{k\in\mathbb{Z}}(\omega\Delta_n)$ are i.i.d. (cf. eq. (2.5)), we easily derive

$$\Delta_n^{1-\frac{1}{\alpha}}W^{(23)}_{n,\Delta_n}(\omega\Delta_n)=\mathrm{e}^{-i\omega\Delta_n n}\cdot\Delta_n^{1-\frac{1}{\alpha}}\,n^{-\frac{1}{\alpha}}\sum_{j=1}^{n}\Psi^{\Delta_n}_j\,\mathrm{e}^{-i\omega\Delta_n j}\sum_{k=1-j}^{1-p}\tilde{\tilde{Z}}_{k+n,\Delta_n}(\omega\Delta_n)\,\mathrm{e}^{-i\omega\Delta_n k}$$

$$\overset{\mathscr{D}}{=}\mathrm{e}^{-i\omega\Delta_n n}\cdot\Delta_n^{1-\frac{1}{\alpha}}W^{(22)}_{n,\Delta_n}(\omega\Delta_n)$$

and thus $\Delta_n^{1-1/\alpha}W^{(23)}_{n,\Delta_n}(\omega\Delta_n)\overset{\mathbb{P}}{\to}0$ as $n\to\infty$, as well.

Finally, we have to prove that $\Delta_n^{1-1/\alpha}W^{(21)}_{n,\Delta_n}(\omega\Delta_n)\overset{\mathbb{P}}{\to}0$. Therefore, observe that

$$\Delta_n^{1-\frac{1}{\alpha}}W^{(211)}_{n,\Delta_n}(\omega\Delta_n)=\frac{\Delta_n}{(n\Delta_n)^{1/\alpha}}\sum_{k=2-p-n}^{-1}\mathrm{e}^{-i\omega\Delta_n k}\sum_{j=1\vee(2-p-k)}^{n\wedge(-k)}\Psi^{\Delta_n}_j\,\mathrm{e}^{-i\omega\Delta_n j}\sum_{r=2-j-k}^{p}\mathrm{e}^{-i\omega\Delta_n(r-1)}Z^r_{k,\Delta_n}$$

$$=\frac{\Delta_n}{(n\Delta_n)^{1/\alpha}}\sum_{k=2-p-n}^{-1}\sum_{j=1\vee(2-p-k)}^{n\wedge(-k)}\Psi^{\Delta_n}_j\,\mathrm{e}^{-i\omega\Delta_n(k+j)}$$

$$\times\int_{(k-1)\Delta_n}^{k\Delta_n}\sum_{m=1}^{p}\frac{c(\lambda_m)}{a'(\lambda_m)}f^{(m;2-j-k)}_{\Delta_n}(\omega\Delta_n)\,\mathrm{e}^{(k\Delta_n-s)\lambda_m}\,\mathrm{d}L_s \tag{4.22}$$

(cf. eq. (4.10)). Using eq. (4.11) and its upper bound (see proof of Proposition 4.1), the joint characteristic function of the right-hand side of eq. (4.22), denoted once more by Φ, satisfies

$$\Big|-\frac{1}{\sigma_L^{\alpha}}\log\Phi(z_1,z_2)\Big|\le(|z_1|+|z_2|)^{\alpha}\Bigg(2^{p-1}\sum_{m=1}^{p}\frac{|c(\lambda_m)|}{|a'(\lambda_m)|}\Bigg)^{\alpha}\cdot\frac{\Delta_n^{\alpha}}{n}\sum_{k=2-p-n}^{-1}\Bigg(\sum_{j=1\vee(2-p-k)}^{n\wedge(-k)}\big|\Psi^{\Delta_n}_j\big|\Bigg)^{\alpha}.$$

By virtue of Lemma 2.2(iv) we then have

$$\frac{\Delta_n^\alpha}{n}\sum_{k=2-p-n}^{-1}\Bigg(\sum_{j=1\vee(2-p-k)}^{n\wedge(-k)}\left|\Psi_j^{\Delta_n}\right|\Bigg)^\alpha \overset{n\to\infty}{\to} 0,$$

and hence, $\Delta_n^{1-1/\alpha}W_{n,\Delta_n}^{(211)}(\omega\Delta_n)\overset{\mathbb{P}}{\to}0$ as $n\to\infty$.

Likewise, we get

$$\Delta_n^{1-\frac{1}{\alpha}}W_{n,\Delta_n}^{(212)}(\omega\Delta_n)=\frac{\Delta_n}{(n\Delta_n)^{1/\alpha}}\sum_{k=2-p}^{0}\mathrm{e}^{-i\omega\Delta_n k}\sum_{j=1}^{n}\Psi_j^{\Delta_n}\mathrm{e}^{-i\omega\Delta_n j}\sum_{r=1}^{1-k}\mathrm{e}^{-i\omega\Delta_n(r-1)}Z_{k,\Delta_n}^r$$

and, as before, one derives that it is sufficient to show that $\frac{\Delta_n^\alpha}{n}\sum_{k=2-p}^{0}\left(\sum_{j=1}^n\left|\Psi_j^{\Delta_n}\right|\right)^\alpha\overset{n\to\infty}{\to}0$. This has been done in Lemma 2.2(v).

One can show analogously to $W_{n,\Delta_n}^{(211)}$ that also $\Delta_n^{1-1/\alpha}W_{n,\Delta_n}^{(213)}(\omega\Delta_n)\overset{\mathbb{P}}{\to}0$ and analogously to $W_{n,\Delta_n}^{(212)}$ it follows that $\Delta_n^{1-1/\alpha}W_{n,\Delta_n}^{(214)}(\omega\Delta_n)\overset{\mathbb{P}}{\to}0$ as $n\to\infty$. Hence, $\Delta_n^{1-1/\alpha}W_{n,\Delta_n}^{(21)}(\omega\Delta_n)\overset{\mathbb{P}}{\to}0$ and $\Delta_n^{1-1/\alpha}W_{n,\Delta_n}^{(2)}(\omega\Delta_n)\overset{\mathbb{P}}{\to}0$ as $n\to\infty$, as well. This completes the proof. □

4.4 Proofs of Section 2.3

Proof of Proposition 2.9. *(i)* We first observe that the state vector in eq. (2.8a) can be written as

$$V_{k\Delta_n}=\sum_{j=0}^{\infty}\mathrm{e}^{j\Delta_n A}\xi_{n,k-j}\qquad\forall n\in\mathbb{N}^*,\,k\in\mathbb{Z},$$

where $\xi_{n,k}:=\int_{(k-1)\Delta_n}^{k\Delta_n}\mathrm{e}^{(k\Delta_n-s)A}\beta\,\mathrm{d}L_s$ (cf. [15, Proof of Theorem 4.2]). Thus, the Beveridge-Nelson decomposition (cf. [5]) has the form

$$V_{k\Delta_n}=\left(\sum_{j=0}^{\infty}\mathrm{e}^{j\Delta_n A}\right)\xi_{n,k}+\widetilde{V}_{n,k-1}-\widetilde{V}_{n,k}\qquad\forall n\in\mathbb{N}^*,\,k\in\mathbb{Z},$$

with $\widetilde{V}_{n,k}:=\sum_{j=0}^{\infty}\left(\sum_{l=j+1}^{\infty}\mathrm{e}^{l\Delta_n A}\right)\xi_{n,k-j}$ (see also [15, Proof of Theorem 2.2]). Hence,

$$\sum_{k=1}^{n}V_{k\Delta_n}=\left(\mathrm{I}_p-\mathrm{e}^{\Delta_n A}\right)^{-1}\sum_{k=1}^{n}\xi_{n,k}+\widetilde{V}_{n,0}-\widetilde{V}_{n,n},$$

where $\widetilde{V}_{n,0}-\widetilde{V}_{n,n}=\left(\mathrm{I}_p-\mathrm{e}^{\Delta_n A}\right)^{-1}\mathrm{e}^{\Delta_n A}(V_0-V_{n\Delta_n})$. Since $\Delta_n\left(\mathrm{I}_p-\mathrm{e}^{\Delta_n A}\right)^{-1}\overset{n\to\infty}{\to}-A^{-1}$ and $V_0\overset{\mathscr{D}}{=}V_{n\Delta_n}$ for any $n\in\mathbb{N}^*$, we obviously get $\widetilde{V}_{n,0}-\widetilde{V}_{n,n}=o_P(\Delta_n^{-1}(n\Delta_n)^{1/\alpha})$ as $n\to\infty$. By analog calculations via characteristic functions (as used in the proofs of Theorem 2.5 and Proposition 4.1), we further obtain $\sum_{k=1}^n\xi_{n,k}=\beta\sum_{k=1}^n\Delta L(k\Delta_n)+o_P((n\Delta_n)^{1/\alpha})$ as $n\to\infty$. Putting all this together, we have

$$\begin{aligned}\sum_{k=1}^{n}Y_{k\Delta_n}&\overset{(2.8b)}{=}e_1^T\sum_{k=1}^{n}V_{k\Delta_n}=e_1^T\left(\mathrm{I}_p-\mathrm{e}^{\Delta_n A}\right)^{-1}\left(\beta\sum_{k=1}^{n}\Delta L(k\Delta_n)+o_P\left((n\Delta_n)^{\frac{1}{\alpha}}\right)\right)+o_P\left(\Delta_n^{-1}(n\Delta_n)^{\frac{1}{\alpha}}\right)\\&=\sum_{j=0}^{\infty}g(j\Delta_n)\cdot\sum_{k=1}^{n}\Delta L(k\Delta_n)+o_P\left(\Delta_n^{-1}(n\Delta_n)^{\frac{1}{\alpha}}\right)\quad\text{as } n\to\infty\end{aligned}$$

and *(i)* is shown.

(ii) Let $(0,\Sigma_L,\nu_L)$ denote the characteristic triplet of the underlying Lévy process L. As in the proof of [15, Proposition 3.3(c)]), we first factorize the Lévy measure ν_L into two Lévy measures

$$\nu_{L^{(1)}}(A):=\nu_L(A\backslash\{x\in\mathbb{R}:|x|\le1\})\quad\text{and}\quad\nu_{L^{(2)}}(A):=\nu_L(A\cap\{x\in\mathbb{R}:|x|\le1\}),\ \text{for any Borel set } A\subseteq\mathbb{R}^*,$$

such that $\nu_L = \nu_{L^{(1)}} + \nu_{L^{(2)}}$. We decompose L into two independent Lévy processes $L = L^{(1)} + L^{(2)}$ where $L^{(1)}$ has characteristic triplet $(0, 0, \nu_{L^{(1)}})$ and $L^{(2)}$ has characteristic triplet $(0, \Sigma_L, \nu_{L^{(2)}})$.

Then one can show, as in the proof of [15, Theorem 4.5], that

$$\sum_{k=1}^{n} V_{k\Delta_n} V_{k\Delta_n}^T = \sum_{j=0}^{\infty} \mathrm{e}^{j\Delta_n A} \left(\sum_{k=1}^{n} \xi_{n,k}^{(1)} \left(\xi_{n,k}^{(1)}\right)^T \right) \mathrm{e}^{j\Delta_n A^T} + o_P\left(\Delta_n^{-1} (n\Delta_n)^{\frac{2}{\alpha}}\right)$$

as $n \to \infty$, where $V_{k\Delta_n}$ is the state vector in eq. (2.8a), $\xi_{n,k}^{(1)} := \int_{(k-1)\Delta_n}^{k\Delta_n} \mathrm{e}^{(k\Delta_n - s)A} \beta \, \mathrm{d}L_s^{(1)}$ if $\alpha \in (0,2)$ and $\xi_{n,k}^{(1)} := \xi_{n,k}$ if $\alpha = 2$ where $\xi_{n,k} := \int_{(k-1)\Delta_n}^{k\Delta_n} \mathrm{e}^{(k\Delta_n - s)A} \beta \, \mathrm{d}L_s$. Next we claim that, also for $\alpha \in (0,2)$,

$$\sum_{k=1}^{n} \xi_{n,k}^{(1)} \left(\xi_{n,k}^{(1)}\right)^T = \sum_{k=1}^{n} \xi_{n,k} \xi_{n,k}^T + o_P\left((n\Delta_n)^{\frac{2}{\alpha}}\right) \tag{4.23}$$

as $n \to \infty$. Together with $\lim_{n\to\infty} \Delta_n \sum_{j=0}^{\infty} \mathrm{e}^{j\Delta_n A} B_n \mathrm{e}^{j\Delta_n A^T} = \int_0^\infty \mathrm{e}^{sA} B \mathrm{e}^{sA^T} \mathrm{d}s$ for all matrices $B_n, B \in \mathbb{R}^{p\times p}$ with $\lim_{n\to\infty} B_n = B$, this yields

$$\sum_{k=1}^{n} V_{k\Delta_n} V_{k\Delta_n}^T = \sum_{j=0}^{\infty} \mathrm{e}^{j\Delta_n A} \left(\sum_{k=1}^{n} \xi_{n,k} \xi_{n,k}^T \right) \mathrm{e}^{j\Delta_n A^T} + o_P\left(\Delta_n^{-1} (n\Delta_n)^{\frac{2}{\alpha}}\right) \tag{4.24}$$

as $n \to \infty$. As to (4.23), we observe with $\xi_{n,k}^{(2)} := \xi_{n,k} - \xi_{n,k}^{(1)} = \int_{(k-1)\Delta_n}^{k\Delta_n} \mathrm{e}^{(k\Delta_n - s)A} \beta \, \mathrm{d}L_s^{(2)}$ that

$$\sum_{k=1}^{n} \xi_{n,k} \xi_{n,k}^T = \sum_{k=1}^{n} \xi_{n,k}^{(1)} \left(\xi_{n,k}^{(1)}\right)^T + \sum_{k=1}^{n} \xi_{n,k}^{(1)} \left(\xi_{n,k}^{(2)}\right)^T + \sum_{k=1}^{n} \xi_{n,k}^{(2)} \left(\xi_{n,k}^{(1)}\right)^T + \sum_{k=1}^{n} \xi_{n,k}^{(2)} \left(\xi_{n,k}^{(2)}\right)^T$$

and thus, by virtue of Hölder's Inequality and taking the norm $\|M\| := \|\mathrm{vec}(M)\|$, we obtain

$$\left\| \sum_{k=1}^{n} \xi_{n,k} \xi_{n,k}^T - \sum_{k=1}^{n} \xi_{n,k}^{(1)} \left(\xi_{n,k}^{(1)}\right)^T \right\| \leq 2 \left(\sum_{k=1}^{n} \left\| \xi_{n,k}^{(1)} \right\|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{k=1}^{n} \left\| \xi_{n,k}^{(2)} \right\|^2 \right)^{\frac{1}{2}} + \sum_{k=1}^{n} \left\| \xi_{n,k}^{(2)} \right\|^2 .$$

Note that the second Lévy component $L^{(2)}$ has finite moments of any order (cf. [34, Corollary 25.8]) and hence, we can apply [15, Proposition 3.3(a)] and deduce for some $C > 0$ and all sufficiently large n

$$\mathbb{E}\left[(n\Delta_n)^{-\frac{2}{\alpha}} \sum_{k=1}^{n} \left\| \xi_{n,k}^{(2)} \right\|^2 \right] = (n\Delta_n)^{-\frac{2}{\alpha}} \sum_{k=1}^{n} \mathbb{E}\left[\left\| \xi_{n,k}^{(2)} \right\|^2 \right] \leq C \cdot (n\Delta_n)^{1-\frac{2}{\alpha}},$$

where the right-hand side converges to 0, since we suppose $n\Delta_n \to \infty$ and $1 - 2/\alpha \in (-\infty, 0)$ for any $\alpha \in (0,2)$. We further obtain by combining [15, Proposition 3.4(a,c)] and [31, Theorem 7.1] that $(n\Delta_n)^{-2/\alpha} \sum_{k=1}^{n} \left\| \xi_{n,k}^{(1)} \right\|^2$ converges weakly as $n \to \infty$ (note that $L^{(1)}$ is a compound Poisson process). This completes the proof of (4.23) and hence also eq. (4.24) is shown.

Now also

$$\sum_{k=1}^{n} \xi_{n,k} \xi_{n,k}^T = \beta \sum_{k=1}^{n} \Delta L(k\Delta_n)^2 \beta^T + o_P\left((n\Delta_n)^{\frac{2}{\alpha}}\right) \quad \text{as } n \to \infty \tag{4.25}$$

holds. For, the (i,j)-th component of $\sum_{k=1}^{n} \xi_{n,k} \xi_{n,k}^T - \beta \sum_{k=1}^{n} \Delta L(k\Delta_n)^2 \beta^T$ can be bounded, again due to Hölder's Inequality, by

$$\left| \left[\sum_{k=1}^{n} \xi_{n,k} \xi_{n,k}^T - \beta \sum_{k=1}^{n} \Delta L(k\Delta_n)^2 \beta^T \right]_{i,j} \right| \leq \left(\sum_{k=1}^{n} \left[\xi_{n,k}\right]_i^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{k=1}^{n} \left(\left[\xi_{n,k}\right]_j - \beta_j \Delta L(k\Delta_n) \right)^2 \right)^{\frac{1}{2}}$$

$$+\left(\sum_{k=1}^{n}\left(\beta_j\,\Delta L(k\Delta_n)\right)^2\right)^{\frac{1}{2}}\cdot\left(\sum_{k=1}^{n}\left(\left[\xi_{n,k}\right]_i-\beta_i\,\Delta L(k\Delta_n)\right)^2\right)^{\frac{1}{2}}$$

with $\left[\xi_{n,k}\right]_i$ and β_j being the i-th and j-th component of $\xi_{n,k}$ and β, respectively. Similar arguments as used above for $\sum_{k=1}^{n}\left\|\xi_{n,k}^{(1)}\right\|^2$ yield that $(n\Delta_n)^{-2/\alpha}\sum_{k=1}^{n}\left[\xi_{n,k}\right]_i^2$ as well as $(n\Delta_n)^{-2/\alpha}\sum_{k=1}^{n}\left(\beta_j\,\Delta L(k\Delta_n)\right)^2$ converge weakly to positive $\alpha/2$-stable random variables. In order to obtain eq. (4.25), it hence remains to prove that, for any $i\in\{1,\ldots,p\}$, the sum $(n\Delta_n)^{-2/\alpha}\sum_{k=1}^{n}\left(\left[\xi_{n,k}\right]_i-\beta_i\,\Delta L(k\Delta_n)\right)^2$ converges to 0 in probability. This is indeed true, since the random variables $\left[\xi_{n,k}\right]_i-\beta_i\,\Delta L(k\Delta_n)$, $k\in\{1,\ldots,n\}$, are i.i.d. symmetric α-stable with scale parameter $\sigma_L\left(\int_0^{\Delta_n}\left|e_i^T(\mathrm{e}^{(\Delta_n-s)A}-\mathrm{I}_p)\beta\right|^\alpha\mathrm{d}s\right)^{1/\alpha}$ and $\Delta_n^{-1}\int_0^{\Delta_n}\left|e_i^T(\mathrm{e}^{(\Delta_n-s)A}-\mathrm{I}_p)\beta\right|^\alpha\mathrm{d}s\to 0$ as $n\to\infty$ (cf. Lemma 2.1(ii)). We thus deduce

$$\begin{aligned}\sum_{k=1}^{n}Y_{k\Delta_n}^2&\overset{(2.8b)}{=}e_1^T\left(\sum_{k=1}^{n}V_{k\Delta_n}V_{k\Delta_n}^T\right)e_1\overset{(4.24)}{=}e_1^T\left(\sum_{j=0}^{\infty}\mathrm{e}^{j\Delta_nA}\left(\sum_{k=1}^{n}\xi_{n,k}\xi_{n,k}^T\right)\mathrm{e}^{j\Delta_nA^T}\right)e_1+o_P\left(\Delta_n^{-1}(n\Delta_n)^{\frac{2}{\alpha}}\right)\\&\overset{(4.25)}{=}e_1^T\left(\sum_{j=0}^{\infty}\mathrm{e}^{j\Delta_nA}\left(\beta\sum_{k=1}^{n}\Delta L(k\Delta_n)^2\beta^T+o_P\left((n\Delta_n)^{\frac{2}{\alpha}}\right)\right)\mathrm{e}^{j\Delta_nA^T}\right)e_1+o_P\left(\Delta_n^{-1}(n\Delta_n)^{\frac{2}{\alpha}}\right)\\&=\sum_{j=0}^{\infty}g^2(j\Delta_n)\cdot\sum_{k=1}^{n}\Delta L(k\Delta_n)^2+o_P\left(\Delta_n^{-1}(n\Delta_n)^{\frac{2}{\alpha}}\right)\quad\text{as }n\to\infty\end{aligned}$$

and *(ii)* is shown. □

Proof of Theorem 2.10. Assume that $c_q\neq 0$. By virtue of [13, Lemma 3.1], the integrated kernel function $\int_0^\infty g(s)\,\mathrm{d}s$ is equal to $\int_0^\infty e_1^T\mathrm{e}^{sA}\beta\,\mathrm{d}s=-e_1^TA^{-1}\beta=c_q\,a_p^{-1}$. Due to Proposition 2.4 we immediately obtain, for any $\omega\in\mathbb{R}^*$ and n sufficiently large

$$\widetilde{I}_{n,Y^{\Delta_n}}(\omega\Delta_n)=\left|\Psi^{\Delta_n}(\mathrm{e}^{-i\omega\Delta_n})\right|^2\frac{I_{n,\widetilde{Z}^{\Delta_n}}(\omega\Delta_n)}{\left(n^{-1/\alpha}\sum_{k=1}^{n}Y_{k\Delta_n}\right)^2}+\widetilde{R}_{n,\Delta_n}(\omega\Delta_n)$$

with $\widetilde{R}_{n,\Delta_n}(\omega\Delta_n)=R_{n,\Delta_n}(\omega\Delta_n)\cdot\left(n^{-1/\alpha}\sum_{k=1}^{n}Y_{k\Delta_n}\right)^{-2}$. Since $R_{n,\Delta_n}(\omega\Delta_n)=o_P\left(\Delta_n^{2/\alpha-2}\right)$ as $n\to\infty$ (see again Proposition 2.4) and since $\left(\Delta_n\,(n\Delta_n)^{-1/\alpha}\sum_{k=1}^{n}Y_{k\Delta_n}\right)^2\overset{\mathscr{D}}{\to}\left(\int_0^\infty g(s)\,\mathrm{d}s\right)^2\cdot S^2=c_q^2\,a_p^{-2}\cdot S^2$ as $n\to\infty$ with S being a $S\alpha S$ random variable with scale parameter σ_L (cf. [15, Theorem 5.2(a)]), we have

$$\widetilde{R}_{n,\Delta_n}(\omega\Delta_n)=o_P(1)\quad\text{as }n\to\infty.\tag{4.26}$$

Since $\left|\Psi^{\Delta_n}(\mathrm{e}^{-i\omega\Delta_n})\right|^2\sim\Delta_n^{-2p}\,|a(i\omega)|^{-2}$ and $\Delta_n\sum_{j=0}^{\infty}g(j\Delta_n)\to\int_0^\infty g(s)\,\mathrm{d}s$ as $n\to\infty$, we combine eq. (4.26), Proposition 4.1 and Proposition 2.9(i), and observe that, in order to show Theorem 2.10, it remains to prove

$$\left(\Delta_n^{1-p-\frac{1}{\alpha}}\left[J_{n,\widetilde{Z}^{\Delta_n}}^{(2)}(\omega_j\Delta_n)\right]_{j\in\{1,\ldots,m\}},\,(n\Delta_n)^{-\frac{1}{\alpha}}\sum_{k=1}^{n}\Delta L(k\Delta_n)\right)\overset{\mathscr{D}}{\to}\left(\left[c(i\omega_j)\cdot\left(S_j^{\Re}(\underset{\sim}{\omega})-iS_j^{\Im}(\underset{\sim}{\omega})\right)\right]_{j\in\{1,\ldots,m\}},\,S_{m+1}(\underset{\sim}{\omega})\right)$$

as $n\to\infty$ and to apply the Continuous Mapping Theorem (see, e.g., [21, Theorem 13.25]). However, this weak convergence result can be shown along the lines of the proof of Theorem 2.5. □

Proof of Theorem 2.11. Assume w.l.o.g. that $\int_0^\infty g^2(s)\,\mathrm{d}s\neq 0$ (otherwise the CARMA process would be trivial). Furthermore, we obtain as in the proof of Theorem 2.10 for all sufficiently large n

$$\widehat{I}_{n,Y^{\Delta_n}}(\omega\Delta_n)=\left|\Psi^{\Delta_n}(\mathrm{e}^{-i\omega\Delta_n})\right|^2\frac{I_{n,\widetilde{Z}^{\Delta_n}}(\omega\Delta_n)}{n^{-2/\alpha}\sum_{k=1}^{n}Y_{k\Delta_n}^2}+\widehat{R}_{n,\Delta_n}(\omega\Delta_n)$$

with $\widehat{R}_{n,\Delta_n}(\omega\Delta_n)=R_{n,\Delta_n}(\omega\Delta_n)\cdot\left(n^{-2/\alpha}\sum_{k=1}^{n}Y_{k\Delta_n}^2\right)^{-1}$. Since $R_{n,\Delta_n}(\omega\Delta_n)=o_P\left(\Delta_n^{2/\alpha-2}\right)$ as $n\to\infty$ (see Proposition 2.4) and since $\Delta_n\,(n\Delta_n)^{-2/\alpha}\sum_{k=1}^{n}Y_{k\Delta_n}^2\overset{\mathscr{D}}{\to}\int_0^\infty g^2(s)\,\mathrm{d}s\cdot[L,L]_1$ as $n\to\infty$ with $([L,L]_t)_{t\geq 0}$ being the quadratic variation process of $(L_t)_{t\geq 0}$ (cf. [15, Theorem 5.5(a)]), we get

$$\Delta_n \widehat{R}_{n,\Delta_n}(\omega\Delta_n) = o_P(1) \quad \text{as } n\to\infty. \tag{4.27}$$

Since $\left|\Psi^{\Delta_n}(\mathrm{e}^{-i\omega\Delta_n})\right|^2 \sim \Delta_n^{-2p}\,|a(i\omega)|^{-2}$ and $\Delta_n \sum_{j=0}^{\infty} g^2(j\Delta_n) \to \int_0^\infty g^2(s)\,\mathrm{d}s$ as $n\to\infty$, we combine (4.27), Proposition 4.1 and Proposition 2.9(ii), and observe that

$$\Delta_n \widehat{I}_{n,Y^{\Delta_n}}(\omega\Delta_n) = |a(i\omega)|^{-2}\cdot\left(\int_0^\infty g^2(s)\,\mathrm{d}s\right)^{-1}\cdot\frac{\Delta_n^{2-2p-\frac{2}{\alpha}}\left|J^{(2)}_{n,\Delta_n}(\omega\Delta_n)\right|^2}{(n\Delta_n)^{-\frac{2}{\alpha}}\sum_{k=1}^n \Delta L(k\Delta_n)^2}\cdot\big(1+o_P(1)\big) \quad \text{as } n\to\infty. \tag{4.28}$$

In the proof of Theorem 2.5 it has been shown that, for any $\omega\in\mathbb{R}^*$,

$$\Delta_n^{1-p-\frac{1}{\alpha}} J^{(2)}_{n,\Delta_n}(\omega\Delta_n) - \frac{c(i\omega)}{(n\Delta_n)^{\frac{1}{\alpha}}}\sum_{k=1}^n \Delta L(k\Delta_n)\,\mathrm{e}^{-i\omega\Delta_n k} \xrightarrow{\mathbb{P}} 0 \quad \text{as } n\to\infty$$

(cf. eqs. (4.12), (4.13) and (4.16) to (4.18)). Hence, (4.28) becomes

$$\Delta_n \widehat{I}_{n,Y^{\Delta_n}}(\omega\Delta_n) = \frac{|c(i\omega)|^2}{\int_0^\infty g^2(s)\,\mathrm{d}s\cdot|a(i\omega)|^2}\cdot\frac{\left|\sum_{k=1}^n \Delta L(k\Delta_n)\,\mathrm{e}^{-i\omega\Delta_n k}\right|^2}{\sum_{k=1}^n \Delta L(k\Delta_n)^2}\cdot\big(1+o_P(1)\big) \quad \text{as } n\to\infty.$$

We introduce an i.i.d. sequence $(Z_k)_{k\in\mathbb{N}^*}$ of symmetric α-stable random variables with scale parameter σ_L and observe that $\big(\Delta L(k\Delta_n)\big)_{k\in\mathbb{N}^*} \overset{\mathscr{D}}{=} (\Delta_n)^{1/\alpha}\cdot(Z_k)_{k\in\mathbb{N}^*}$. Consequently, to finish the proof of Theorem 2.11, it is sufficient to show that

$$\left[\frac{\left|\sum_{k=1}^n Z_k\,\mathrm{e}^{-i\omega_j\Delta_n k}\right|^2}{\sum_{k=1}^n Z_k^2}\right]_{j\in\{1,\ldots,m\}} \xrightarrow{\mathscr{D}} \left[\frac{\big[S_j^{\Re}(\underset{\sim}{\omega})\big]^2+\big[S_j^{\Im}(\underset{\sim}{\omega})\big]^2}{S^2}\right]_{j\in\{1,\ldots,m\}} \quad \text{as } n\to\infty. \tag{4.29}$$

Since $n^{-2/\alpha}\left|\sum_{k=1}^n Z_k\,\mathrm{e}^{-i\omega_j\Delta_n k}\right|^2 \xrightarrow{\mathscr{D}} \big[S_j^{\Re}(\underset{\sim}{\omega})\big]^2+\big[S_j^{\Im}(\underset{\sim}{\omega})\big]^2$ as $n\to\infty$, which follows implicitly from the proofs of Proposition 3.4 and Theorem 2.5, and since $n^{-2/\alpha}\sum_{k=1}^n Z_k^2 \xrightarrow{\mathscr{D}} S^2$ as $n\to\infty$ with S^2 being a positive $\alpha/2$-stable random variable, which can be easily derived from, e.g., [31, Theorem 7.1], we will show that also the random vector

$$\big(\gamma^2_{n,Z},\,\alpha^2_{n,Z}(\omega_j\Delta_n),\,\beta^2_{n,Z}(\omega_j\Delta_n)\big)_{j\in\{1,\ldots,m\}}, \tag{4.30}$$

with

$$\gamma^2_{n,Z} := n^{-2/\alpha}\sum_{k=1}^n Z_k^2,\;\; \alpha_{n,Z}(\omega_j\Delta_n) := n^{-1/\alpha}\sum_{k=1}^n Z_k\cos(\omega_j\Delta_n k) \text{ and } \beta_{n,Z}(\omega_j\Delta_n) := n^{-1/\alpha}\sum_{k=1}^n Z_k\sin(\omega_j\Delta_n k),$$

converges weakly. Note that this implies eq. (4.29).

We take the same approach as in the proof of [24, Proposition 2.2] (which can be found in [22]). Let $(N_k)_{k\in\mathbb{N}^*}, P_1, P_2, \ldots, P_m, M_1, M_2, \ldots, M_m$ be i.i.d. standard normal random variables, independent of $(Z_k)_{k\in\mathbb{N}^*}$. Then, with $\varphi\geq 0$ and $\underset{\sim}{\theta},\,\underset{\sim}{\nu}\in[0,\infty)^m$, the Laplace transform of the random vector in (4.30) is given by

$$\begin{aligned} f_{n,\Delta_n}(\varphi,\underset{\sim}{\theta},\underset{\sim}{\nu}) &= \mathbb{E}\left[\exp\left\{-\frac{\varphi^2}{2}\gamma^2_{n,Z}-\sum_{j=1}^m\left(\frac{\theta_j^2}{2}\alpha^2_{n,Z}(\omega_j\Delta_n)+\frac{\nu_j^2}{2}\beta^2_{n,Z}(\omega_j\Delta_n)\right)\right\}\right] \\ &= \mathbb{E}\left(\mathbb{E}\left[\exp\left\{i\varphi n^{-\frac{1}{\alpha}}\sum_{k=1}^n Z_k N_k + i\sum_{j=1}^m\big(\theta_j P_j\,\alpha_{n,Z}(\omega_j\Delta_n)+\nu_j M_j\,\beta_{n,Z}(\omega_j\Delta_n)\big)\right\}\,\middle|\,(Z_k)_{k\in\mathbb{N}^*}\right]\right) \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\left[\exp\left\{ i\varphi n^{-\frac{1}{\alpha}} \sum_{k=1}^{n} Z_k N_k + i \sum_{j=1}^{m} \left(\theta_j P_j \, \alpha_{n,Z}(\omega_j\Delta_n) + \nu_j M_j \, \beta_{n,Z}(\omega_j\Delta_n)\right)\right\}\right] \\
&= \mathbb{E}\left[\exp\left\{ i n^{-\frac{1}{\alpha}} \sum_{k=1}^{n} Z_k \left(\varphi N_k + \sum_{j=1}^{m} \left(\theta_j P_j \cos(\omega_j\Delta_n k) + \nu_j M_j \sin(\omega_j\Delta_n k)\right)\right)\right\}\right] \\
&= \mathbb{E}\left[\exp\left\{ i n^{-\frac{1}{\alpha}} Z_1 \left(\sum_{k=1}^{n} \left|\varphi N_k + \sum_{j=1}^{m} \left(\theta_j P_j \cos(\omega_j\Delta_n k) + \nu_j M_j \sin(\omega_j\Delta_n k)\right)\right|^{\alpha}\right)^{\frac{1}{\alpha}}\right\}\right] \\
&= \mathbb{E}\left[\exp\left\{ -\frac{\sigma_L^{\alpha}}{n} \sum_{k=1}^{n} \left|\varphi N_k + \sum_{j=1}^{m} \left(\theta_j P_j \cos(\omega_j\Delta_n k) + \nu_j M_j \sin(\omega_j\Delta_n k)\right)\right|^{\alpha}\right\}\right] \\
&=: \mathbb{E}\left[\exp\left\{-\sigma_L^{\alpha} \cdot K_{n,\Delta_n}(\varphi, \underset{\sim}{\theta}, \underset{\sim}{\nu})\right\}\right]
\end{aligned}
$$

with $K_{n,\Delta_n}(\varphi, \underset{\sim}{\theta}, \underset{\sim}{\nu}) := 1/n \cdot \sum_{k=1}^{n} \left|\varphi N_k + \sum_{j=1}^{m} \left(\theta_j P_j \cos(\omega_j\Delta_n k) + \nu_j M_j \sin(\omega_j\Delta_n k)\right)\right|^{\alpha}$. We define the function $h(x, y) := \left|\varphi y + \sum_{j=1}^{m} \left(\theta_j P_j \cos(2\pi x_j) + \nu_j M_j \sin(2\pi x_j)\right)\right|^{\alpha}$, $x \in \mathbb{R}^m$, $y \in \mathbb{R}$. Note that h satisfies the assumptions of Proposition 3.5 for every realization of $\underset{\sim}{P} = (P_1, \dots, P_m)^T$ and $\underset{\sim}{M} = (M_1, \dots, M_m)^T$.

Now, if $\omega_1, \dots, \omega_m$ are linearly independent over $\mathbb{Z}$ we obtain by virtue of Proposition 3.5

$$
\begin{aligned}
f_{n,\Delta_n}(\varphi, \underset{\sim}{\theta}, \underset{\sim}{\nu}) &\overset{n\to\infty}{\to} \mathbb{E}\left[\exp\left\{-\sigma_L^{\alpha} \cdot \mathbb{E}\left[h(\underset{\sim}{U}, N_1) | \underset{\sim}{P}, \underset{\sim}{M}\right]\right\}\right] \\
&= \mathbb{E}\left[\exp\left\{-\sigma_L^{\alpha} \cdot \mathbb{E}\left[\left|\varphi N_1 + \sum_{j=1}^{m} \left(\theta_j P_j \cos(2\pi U_j) + \nu_j M_j \sin(2\pi U_j)\right)\right|^{\alpha} \middle| \underset{\sim}{P}, \underset{\sim}{M}\right]\right\}\right] \\
&=: f(\varphi, \underset{\sim}{\theta}, \underset{\sim}{\nu}). \qquad (4.31)
\end{aligned}
$$

Here $U_1, \dots, U_m$ are i.i.d. $[0, 1)$-uniform random variables independent of $P_1, \dots, P_m, M_1, \dots, M_m$ and N_1.

If $\omega_1, \dots, \omega_m$ are linearly dependent over $\mathbb{Z}$, then also by virtue of Proposition 3.5 $f_{n,\Delta_n}(\varphi, \underset{\sim}{\theta}, \underset{\sim}{\nu}) \to f(\varphi, \underset{\sim}{\theta}, \underset{\sim}{\nu})$ as $n \to \infty$ but now $\underset{\sim}{U} = T(V_1, \dots, V_{m-s})$ with T being the parametrization of the $(m-s)$-dimensional manifold $\mathscr{M}(\omega_1, \dots, \omega_m)$ (cf. (3.2)) and $V_1, \dots, V_{m-s}$ are i.i.d. $[0, 1)$-uniform random variables independent of $\underset{\sim}{P}$, $\underset{\sim}{M}$ and N_1.

Hence, in both cases the Laplace transform $f_{n,\Delta_n}(\varphi, \underset{\sim}{\theta}, \underset{\sim}{\nu})$ of the random vector (4.30) converges to a function that is continuous in the origin. This implies that $\left(\gamma_{n,Z}^2, \alpha_{n,Z}^2(\omega_j\Delta_n), \beta_{n,Z}^2(\omega_j\Delta_n)\right)_{j\in\{1,\dots,m\}}$ converges weakly and completes the proof. □

4.5 Proofs of Section 3

Proof of Theorem 3.3. For the proof we identify the equivalence classes in $(\mathbb{R} \bmod 1)^{m-s}$ and $(\mathbb{R} \bmod 1)^m$, respectively, by their representatives in $[0, 1)^{m-s}$ and $[0, 1)^m$.

(i) Define

$$
\begin{aligned}
N := \{x = (x_1, \dots, x_{m-s})^T \in [0, 1)^{m-s} :\ & \exists j \in \{1, \dots, m-s\}, i \in \{1, \dots, m\} \text{ such that} \\
& x_j = k \cdot \left|b_j^{(i)}\right|^{-1} \text{ for some } k \in \{0, 1, \dots, \left|b_j^{(i)}\right| - 1\}\},
\end{aligned}
$$

where $b_j^{(i)}$ denotes the i-th component of the vector b_j. Clearly $\mathscr{H}^{m-s}(T(N)) = 0$ and $T|_{[0,1)^{m-s}\setminus N}$ is continuously differentiable with $\operatorname{rank}(D\,T|_{[0,1)^{m-s}\setminus N}(x)) = \operatorname{rank}(B) = m-s$ for all $x \in [0, 1)^{m-s}\setminus N$. Moreover, T is injective. The reason is the following. Suppose that $T(x_1, \dots, x_{m-s}) = T(y_1, \dots, y_{m-s})$ for some $(x_1, \dots, x_{m-s})^T, (y_1, \dots, y_{m-s})^T \in [0, 1)^{m-s}$. Then

$$
\left(\sum_{j=1}^{m-s} x_j b_j\right) \bmod 1 = \left(\sum_{j=1}^{m-s} y_j b_j\right) \bmod 1 \quad \Longleftrightarrow \quad \sum_{j=1}^{m-s} (x_j - y_j) b_j \in \mathbb{Z}^m.
$$

Since $\sum_{j=1}^{m-s}(x_j-y_j)b_j \in \operatorname{span}^{\mathbb{R}}(\{b_1,\ldots,b_{m-s}\})\cap\mathbb{Z}^m \subseteq \widetilde{\mathscr{L}}^{\perp}\cap\mathbb{Z}^m = \mathscr{L} = \operatorname{span}^{\mathbb{Z}}(\{b_1,\ldots,b_{m-s}\})$, there exist integers z_j, $j\in\{1,\ldots,m-s\}$, such that $\sum_{j=1}^{m-s}(x_j-y_j-z_j)b_j = 0$ and hence, $(x_j-y_j) = z_j \in \mathbb{Z}$ for all $j\in\{1,\ldots,m-s\}$. Since $x_j-y_j\in(-1,1)$ we must have $x_j = y_j$ for all $j\in\{1,\ldots,m-s\}$. This shows that T is indeed injective. Note that T^{-1} is continuous (mod 1) on $\mathscr{M}$ and thus, $T([0,1)^{m-s}\backslash N)$ is an $(m-s)$-dimensional C^1-manifold in $[0,1)^m$ (for a definition of manifolds, see, e.g., [29, pp. 200-201]). Since $\mathscr{H}^{m-s}(T(N)) = 0$, also $\mathscr{M}$ is an $(m-s)$-dimensional C^1-manifold and integration over $\mathscr{M}$ is the same as integration over $T([0,1)^{m-s}\backslash N) = \mathscr{M}\backslash T(N)$ (note that $T(N)$ itself is a manifold in $[0,1)^m$ from lower dimension than $m-s$).

(ii) Suppose there is a $z=(z_1,\ldots,z_{m-s})^T\in\mathbb{Z}^{m-s}$, $z\neq 0$, such that $\langle z, \underset{\sim}{\mu}\rangle = 0$. W.l.o.g. $z_1\neq 0$. Then

$$\mu_1 = -\sum_{i=2}^{m-s}\frac{z_i}{z_1}\mu_i \qquad \text{and} \qquad \underset{\sim}{\eta} = \sum_{i=2}^{m-s}\mu_i\cdot\left(-\frac{z_i}{z_1}b_1+b_i\right).$$

The vectors $\widetilde{b}_i := -\frac{z_i}{z_1}b_1+b_i\in\mathbb{Q}^m$, $i=2,\ldots,m-s$, are obviously linearly independent. Thus,

$$\left(\operatorname{span}^{\mathbb{R}}\left\{\widetilde{b}_2,\ldots,\widetilde{b}_{m-s}\right\}\right)^{\perp}\subseteq\{\underset{\sim}{\eta}\}^{\perp}\Rightarrow\left(\operatorname{span}^{\mathbb{R}}\left\{\widetilde{b}_2,\ldots,\widetilde{b}_{m-s}\right\}\right)^{\perp}\cap\mathbb{Z}^m\subseteq\{\underset{\sim}{\eta}\}^{\perp}\cap\mathbb{Z}^m = \widetilde{\mathscr{L}},$$

and since the dimension of $\widetilde{\mathscr{L}}$ is s whereas the dimension of $\operatorname{span}^{\mathbb{R}}\{\widetilde{b}_2,\ldots,\widetilde{b}_{m-s}\}^{\perp}\cap\mathbb{Z}^m$ is $s+1$ (the latter can be obtained as in the proof of $\dim(\mathscr{L}) = m-s$ on p. 11), we have a contradiction. Hence, $\langle z,\underset{\sim}{\mu}\rangle\neq 0$ for all $z\in\mathbb{Z}^{m-s}$, $z\neq 0$.

(iii) We have, with $h = Bz$ and $z\in\mathbb{Z}^{m-s}$, $z\neq 0$,

$$\begin{aligned}\frac{1}{\mathscr{H}^{m-s}(\mathscr{M})}\int_{\mathscr{M}} f_h(x)\,\mathscr{H}^{m-s}(\mathrm{d}x) &= \int_{[0,1)^{m-s}} f_h(T(x))\,\mathrm{d}x = \int_{[0,1)^{m-s}} \mathrm{e}^{2\pi i\langle h, T(G^{-1}x)\rangle}\,\mathrm{d}x\\ &= \int_{[0,1)^{m-s}} \mathrm{e}^{2\pi i\langle h, BG^{-1}x \bmod 1\rangle}\,\mathrm{d}x = \int_{[0,1)^{m-s}} \mathrm{e}^{2\pi i\langle z, B^TBG^{-1}x\rangle}\,\mathrm{d}x\\ &= \prod_{j=1}^{m-s}\int_0^1 \mathrm{e}^{2\pi i z_j x_j}\,\mathrm{d}x_j. \end{aligned}\tag{4.32}$$

Since $z\neq 0$ there is a $j\in\{1,\ldots,m-s\}$ with $z_j\in\mathbb{Z}\backslash\{0\}$, and the right-hand side of (4.32) has to be zero.

(iv) Let $T(x), T(y)\in\mathscr{M}$, $T(x)\neq T(y)$. Since T is injective, there is some $j_0\in\{1,\ldots,m-s\}$ such that $x_{j_0}\neq y_{j_0}$. For $h = Be_{j_0} = b_{j_0}$ we have

$$f_h(T(x))\cdot f_h(T(y))^{-1} = \mathrm{e}^{2\pi i\langle b_{j_0}, T(G^{-1}x)-T(G^{-1}y)\rangle} = \mathrm{e}^{2\pi i\langle Be_{j_0}, BG^{-1}(x-y)\rangle} = \mathrm{e}^{2\pi i(x_{j_0}-y_{j_0})}\neq 1,$$

since $x_{j_0}-y_{j_0}\in(-1,1)\backslash\{0\}$. □

Proof of Proposition 3.4. Letting $\underset{\sim}{\omega} = (\omega_1,\ldots,\omega_m)^T = 2\pi(\eta_1,\ldots,\eta_m)^T = 2\pi\underset{\sim}{\eta}$, we immediately get

$$\begin{aligned}&\frac{1}{n}\sum_{k=1}^{n-p+1}\left|\sum_{j=1}^{m}\Xi_{\theta_j,\nu_j}\left(\mathrm{e}^{-i\omega_j\Delta_n k}c(i\omega_j)\right)\right|^{\alpha}\\ &\quad\overset{n\to\infty}{\sim}\frac{1}{n}\sum_{k=1}^{n}\left|\sum_{j=1}^{m}\cos\left(2\pi\{\eta_j\Delta_n k\}\right)\cdot\Xi_{\theta_j,\nu_j}\left(c(i\omega_j)\right)+\sin\left(2\pi\{\eta_j\Delta_n k\}\right)\cdot\Xi_{-\nu_j,\theta_j}\left(c(i\omega_j)\right)\right|^{\alpha}.\end{aligned}$$

Let us first consider the case where $\omega_1,\ldots,\omega_m$ are linearly independent over $\mathbb{Z}$. We claim that, for any $h\in\mathbb{Z}^m$, $h\neq 0$,

$$\frac{1}{n}\sum_{k=1}^{n}\mathrm{e}^{2\pi i\langle h,\underset{\sim}{\eta}\rangle\Delta_n k}\to 0 \quad \text{as } n\to\infty. \tag{4.33}$$

To this end, note that for n sufficiently large

$$\left|\frac{1}{n}\sum_{k=1}^{n} \mathrm{e}^{2\pi i\langle h,\underset{\sim}{\eta}\rangle\Delta_n k}\right| = \frac{1}{n}\cdot\frac{\left|\mathrm{e}^{2\pi i\langle h,\underset{\sim}{\eta}\rangle\Delta_n n}-1\right|}{\left|\mathrm{e}^{2\pi i\langle h,\underset{\sim}{\eta}\rangle\Delta_n}-1\right|} \le \frac{1}{\left|\langle h,\underset{\sim}{\eta}\rangle\right|}\cdot\frac{1}{n\Delta_n}$$

and the right-hand side converges to 0 as $n\to\infty$ since $n\Delta_n\to\infty$ by assumption and since $\omega_1,\ldots,\omega_m$ are supposed to be linearly independent over $\mathbb{Z}$.

However, (4.33) already implies that

$$\frac{1}{n}\sum_{k=1}^{n} f(\Delta_n k\underset{\sim}{\eta}) \overset{n\to\infty}{\to} \int_{[0,1)^m} f(x)\,\mathrm{d}x \tag{4.34}$$

for any continuous function $f:\mathbb{R}^m\to\mathbb{C}$ with period 1 in each component variable (more precisely, f should be seen as a function, mapping from the compact Hausdorff space $(\mathbb{R} \bmod 1)^m$ to the complex numbers). An explanation is the following. If we fix $\varepsilon>0$, we know from the Weierstrass Approximation Theorem (cf. [36, Theorem 17]) that there exists a trigonometrical polynomial Ψ_ε, i.e. a finite linear combination of functions of the type $\mathrm{e}^{2\pi i\langle h,\,\cdot\,\rangle}$, $h\in\mathbb{Z}^m$, such that $\sup_{x\in\mathbb{R}^m}|f(x)-\Psi_\varepsilon(x)|\le\varepsilon$. This yields

$$\begin{aligned}&\left|\int_{[0,1)^m} f(x)\,\mathrm{d}x-\frac{1}{n}\sum_{k=1}^{n} f(\Delta_n k\underset{\sim}{\eta})\right|\\ &\quad\le \underbrace{\left|\int_{[0,1)^m}(f(x)-\Psi_\varepsilon(x))\,\mathrm{d}x\right|}_{\le\varepsilon}+\left|\int_{[0,1)^m}\Psi_\varepsilon(x)\,\mathrm{d}x-\frac{1}{n}\sum_{k=1}^{n}\Psi_\varepsilon(\Delta_n k\underset{\sim}{\eta})\right|+\underbrace{\left|\frac{1}{n}\sum_{k=1}^{n}\Psi_\varepsilon(\Delta_n k\underset{\sim}{\eta})-f(\Delta_n k\underset{\sim}{\eta})\right|}_{\le\varepsilon}.\end{aligned} \tag{4.35}$$

Since $\int_{[0,1)^m}\mathrm{e}^{2\pi i\langle h,x\rangle}\,\mathrm{d}x=0$ for any $h\in\mathbb{Z}^m$, $h\neq0$, eq. (4.33) implies that the second term on the right-hand side of (4.35) converges to 0 as $n\to\infty$. This shows that (4.33) already implies (4.34).

We conclude the first part of the proof by applying (4.34) to the function

$$f(x_1,\ldots,x_m):=\left|\sum_{j=1}^{m}\cos(2\pi x_j)\cdot\Xi_{\theta_j,\nu_j}\big(c(i\omega_j)\big)+\sin(2\pi x_j)\cdot\Xi_{-\nu_j,\theta_j}\big(c(i\omega_j)\big)\right|^{\alpha}. \tag{4.36}$$

In the case where $\omega_1,\ldots,\omega_m$ are linearly dependent over $\mathbb{Z}$, we first observe that for any $f_h\in\mathscr{T}$ with $h\in\mathscr{L}$, $h\neq0$,

$$\frac{1}{n}\sum_{k=1}^{n} f_h(\Delta_n k\underset{\sim}{\eta} \bmod 1)\to 0 \quad \text{as } n\to\infty \tag{4.37}$$

(where the mod-operator is defined componentwise; for the definition of $\mathscr{T}$ and $\mathscr{L}$ see (3.1) and (3.3), respectively). Therefore note that $\Delta_n k\underset{\sim}{\eta} \bmod 1\in\mathscr{M}$ for any $n\in\mathbb{N}^*$, $k\in\{1,\ldots,n\}$, since (cf. Theorem 3.3)

$$\Delta_n k\underset{\sim}{\eta} \bmod 1 = B(\Delta_n k\underset{\sim}{\mu}) \bmod 1 = B(\underbrace{\Delta_n k\underset{\sim}{\mu} \bmod 1}_{\in[0,1)^{m-s}}) \bmod 1 = T(\Delta_n k\underset{\sim}{\mu} \bmod 1)\in\mathscr{M}. \tag{4.38}$$

Then, with $h=Bz\in\mathscr{L}$, $z\in\mathbb{Z}^{m-s}\backslash\{0\}$,

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n} f_h(\Delta_n k\underset{\sim}{\eta} \bmod 1) &= \frac{1}{n}\sum_{k=1}^{n}\mathrm{e}^{2\pi i\langle Bz,BG^{-1}T^{-1}(\Delta_n k\underset{\sim}{\eta} \bmod 1)\rangle} = \frac{1}{n}\sum_{k=1}^{n}\mathrm{e}^{2\pi i\langle z,T^{-1}(\Delta_n k\underset{\sim}{\eta} \bmod 1)\rangle}\\ &\overset{(4.38)}{=} \frac{1}{n}\sum_{k=1}^{n}\mathrm{e}^{2\pi i\langle z,\underset{\sim}{\mu}\rangle\Delta_n k},\end{aligned}$$

and since $\langle z,\underset{\sim}{\mu}\rangle\neq0$ for all $z\in\mathbb{Z}^{m-s}\backslash\{0\}$ (see Theorem 3.3(ii)), we obtain eq. (4.37) in the same way as we have shown (4.33) in the linearly independent case.

Now, in the linearly dependent case (4.37) already implies

$$\frac{1}{n}\sum_{k=1}^{n} f(\Delta_n k \underset{\sim}{\eta} \bmod 1) \overset{n\to\infty}{\to} \frac{1}{\mathscr{H}^{m-s}(\mathscr{M})}\int_{\mathscr{M}} f(x)\,\mathscr{H}^{m-s}(\mathrm{d}x) \tag{4.39}$$

for any continuous function $f:\mathscr{M}\to\mathbb{C}$. Indeed, $\mathrm{span}^{\mathbb{C}}(\mathscr{T})$ is a dense subalgebra in $C(\mathscr{M})$, the algebra of all continuous complex-valued functions on the compact Hausdorff space $\mathscr{M}$, with respect to the topology of uniform convergence (cf. also comments after Theorem 3.3). Hence, for any continuous function $f:\mathscr{M}\to\mathbb{C}$ and any fixed $\varepsilon>0$ there is a finite linear combination Ψ_ε of functions in $\mathscr{T}$ such that $\sup_{x\in\mathscr{M}}|f(x)-\Psi_\varepsilon(x)|\le\varepsilon$. This yields, analogously to (4.35),

$$\begin{aligned}&\left|\frac{1}{\mathscr{H}^{m-s}(\mathscr{M})}\int_{\mathscr{M}} f(x)\,\mathscr{H}^{m-s}(\mathrm{d}x)-\frac{1}{n}\sum_{k=1}^{n} f(\Delta_n k \underset{\sim}{\eta} \bmod 1)\right|\\ &\qquad\le 2\varepsilon+\left|\frac{1}{\mathscr{H}^{m-s}(\mathscr{M})}\int_{\mathscr{M}} \Psi_\varepsilon(x)\,\mathscr{H}^{m-s}(\mathrm{d}x)-\frac{1}{n}\sum_{k=1}^{n} \Psi_\varepsilon(\Delta_n k \underset{\sim}{\eta} \bmod 1)\right|,\end{aligned}$$

and the second term on the right-hand side converges to 0 as $n\to\infty$ by virtue of Theorem 3.3(iii) and eq. (4.37). This shows (4.39).

We conclude the linearly dependent case by applying eq. (4.39) to the function $f|_{\mathscr{M}}$ with the same f as in the linearly independent case in (4.36). □

Proof of Proposition 3.5. We have

$$\begin{aligned}&\frac{1}{n}\sum_{k=1}^{n} f(k\Delta_n \underset{\sim}{\eta} \bmod 1, N_k)-\mathbb{E}\left[f(\underset{\sim}{U},N_1)\right]\\ &=\frac{1}{n}\sum_{k=1}^{n}\Big(f(k\Delta_n\underset{\sim}{\eta} \bmod 1,N_k)-\mathbb{E}\Big[f(k\Delta_n\underset{\sim}{\eta} \bmod 1,N_1)\Big]\Big)+\frac{1}{n}\sum_{k=1}^{n}\mathbb{E}\Big[f(k\Delta_n\underset{\sim}{\eta} \bmod 1,N_1)\Big]-\mathbb{E}\left[f(\underset{\sim}{U},N_1)\right]\\ &=: I_1+I_2.\end{aligned}$$

We consider first the case where $\omega_1,\dots,\omega_m$ are linearly independent over $\mathbb{Z}$. Then, by virtue of eq. (4.34) and the assumption that $g^{(1)}$ is continuous on $(\mathbb{R} \bmod 1)^m$, we have

$$\begin{aligned}I_2=\frac{1}{n}\sum_{k=1}^{n} g^{(1)}(k\Delta_n\underset{\sim}{\eta} \bmod 1)-\mathbb{E}\left[f(\underset{\sim}{U},N_1)\right] &\overset{n\to\infty}{\to}\int_{[0,1)^m} g^{(1)}(x)\,\mathrm{d}x-\mathbb{E}\left[f(\underset{\sim}{U},N_1)\right]\\ &=\int_{[0,1)^m}\mathbb{E}\left[f(x,N_1)\right]\mathrm{d}x-\mathbb{E}\left[f(\underset{\sim}{U},N_1)\right]=0.\end{aligned}$$

With Chebyshev's Inequality and the assumption that $g^{(2)}$ is continuous on $(\mathbb{R} \bmod 1)^m$, we further obtain

$$\begin{aligned}\mathbb{P}(|I_1|>\varepsilon) &\le \frac{1}{\varepsilon^2\cdot n^2}\sum_{k=1}^{n}\mathbb{E}\left[\Big(f(k\Delta_n\underset{\sim}{\eta} \bmod 1,N_1)-\mathbb{E}\Big[f(k\Delta_n\underset{\sim}{\eta} \bmod 1,N_1)\Big]\Big)^2\right]\\ &\le \frac{1}{\varepsilon^2\cdot n^2}\sum_{k=1}^{n}\mathbb{E}\Big[f^2(k\Delta_n\underset{\sim}{\eta} \bmod 1,N_1)\Big]=\frac{1}{\varepsilon^2\cdot n^2}\sum_{k=1}^{n} g^{(2)}(k\Delta_n\underset{\sim}{\eta} \bmod 1)\\ &=\frac{1}{\varepsilon^2\cdot n}\int_{[0,1)^m} g^{(2)}(x)\,\mathrm{d}x\cdot(1+o(1))=\frac{1}{\varepsilon^2\cdot n}\mathbb{E}[f^2(\underset{\sim}{U},N_1)]\cdot(1+o(1))\overset{n\to\infty}{\to}0,\end{aligned}$$

where we used once more (4.34). Hence, eq. (3.4) is shown in the linearly independent case.

Suppose now that $\omega_1,\dots,\omega_m$ are linearly dependent over $\mathbb{Z}$. As above, now due to eq. (4.39),

$$I_2\overset{n\to\infty}{\to}\frac{1}{\mathscr{H}^{m-s}(\mathscr{M})}\int_{\mathscr{M}} g^{(1)}(x)\,\mathscr{H}^{m-s}(\mathrm{d}x)-\mathbb{E}\left[f(\underset{\sim}{U},N_1)\right]=\int_{[0,1)^{m-s}} g^{(1)}(T(x))\,\mathrm{d}x-\mathbb{E}\left[f(T(\underset{\sim}{V}),N_1)\right]$$

$$= \int_{[0,1)^{m-s}} \mathbb{E}\left[f(T(x),N_1)\right] \mathrm{d}x - \mathbb{E}\left[f(T(\underset{\sim}{V}),N_1)\right] = 0$$

and

$$\mathbb{P}(|I_1| > \varepsilon) \leq \frac{1}{\varepsilon^2 \cdot n^2} \sum_{k=1}^{n} g^{(2)}(k\Delta_n \underset{\sim}{\eta} \bmod 1) = \frac{1}{\varepsilon^2 \cdot n} \cdot \underbrace{\frac{1}{\mathscr{H}^{m-s}(\mathscr{M})} \int_{\mathscr{M}} g^{(2)}(x)\, \mathscr{H}^{m-s}(\mathrm{d}x)}_{=\mathbb{E}[f^2(T(\underset{\sim}{V}),N_1)]} \cdot (1+o(1)) \overset{n\to\infty}{\to} 0.$$

Thus, also in the linearly dependent case (3.4) holds. □

Acknowledgments

Financial support by the Deutsche Forschungsgemeinschaft through the research grants DFG 2005/1-1 and LI 1026/4-1 is gratefully acknowledged. Moreover, the second author thanks ETH Zurich for its hospitality during his research stays. We are also indebted to an associate editor and a referee for valuable comments.

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