Reducibility of joint relay positioning and flow optimization problem

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***Abstract*—This paper shows how to reduce the otherwise hard joint relay positioning and flow optimization problem into a sequence a two simpler decoupled problems. We consider a class of wireless multicast hypergraphs mainly characterized by their hyperarc rate functions, that are increasing and convex in power, and decreasing in distance between the transmit node and the farthest end node of the hyperarc. The set-up consists of a single multicast flow session involving a source, multiple destinations and a relay that can be positioned freely. The first problem formulates the relay positioning problem in a purely geometric sense, and once the optimal relay position is obtained the second problem addresses the flow optimization. Furthermore, we present simple and efficient algorithms to solve these problems.**

I. INTRODUCTION

We consider a version of network planning problem under a relatively simple construct of a single session consisting of a source s, a destination set T and an arbitrarily positionable relay r, all on a 2-D Euclidean plane. The problem can then be stated as: *What is optimal relay position that maximizes the multicast flow from s to T?* Similarly, we can also ask: *What is the optimal relay position that minimizes the cost of establishing the multicast session for a target flow F?*

A fairly general class of acyclic hypergraphs are considered. The hypergraph model is characterized by the following rules of construction of the hypergraph $\mathcal{G}(\mathcal{N}, \mathcal{A})$:

1) $\mathcal{G}(\mathcal{N}, \mathcal{A})$ consists of finite set of nodes $\mathcal{N}$ positioned on on a 2-D Euclidean plane and a finite set of hyperarcs $\mathcal{A}$.
2) Each hyperarc in $\mathcal{A}$ emanates from a transmit node and connects a set of receivers (or end nodes) in the system. Also, each hyperarc is associated with a rate function that is convex and increasing in transmit node power and decreasing in distance between the transmit node and the farthest node spanned by the hyperarc in the system.
3) Each end node spanned by the hyperarc can decode the information sent over the hyperarc equally reliably, i.e. all the end nodes of an hyperarc get equal rate.

In relation to the special case of our hypergraph model, the authors addressed the first question (max-flow) in the context of Low-SNR Broadcast Relay Channel in [1].

This paper has two major contributions. Firstly, we solve the general joint relay positioning and max-flow optimization problem for our hypergraph model. Secondly, we address the min-cost flow problem and establish a relation of duality between the max-flow and min-cost problems. An efficient algorithm that solves the joint relay positioning and max-flow problem is presented, in addition to an algorithm that solves an important special case of the min-cost problem.

The relay positioning problem has been studied in various settings [2]–[4]. In most cases, the problem is either heuristically solved due to inherent complexity or approximately solved using simpler methods but compromising accuracy. We reduce the non-convex joint problem into easily solvable sequence of two decoupled problems. The first problem solves for optimal relay position in a purely geometric sense with no flow optimization involved. Upon obtaining the optimal relay position, the second problem addresses the flow optimization. The decoupling of the joint problem comes as a consequence of the convexity (in power) of hyperarc rate functions.

The next section develops the wireless network model. Section III presents the key multicast flow concentration ideas for max-flow and min-cost flow that are central to the reducibility of the joint problem. In Section IV, we present the algorithms and Section V contains an example where the results of this paper are applied. Finally, we conclude in Section VI.

II. PRELIMINARIES AND MODEL

Consider a wireless network hypergraph $\mathcal{G}(\mathcal{N}, \mathcal{A})$ consisting of $|\mathcal{N}| = n+2$ nodes placed on a 2-D Euclidean plane with $|\mathcal{A}|$ number of hyperarcs and the only arbitrarily positionable node as the relay r. The node set $\mathcal{N} = \{s, r, t_1, .., t_n\}$ consists of a source node s, a relay r and an ordered destination set $T = \{t_1, .., t_n\}$ (in increasing distance from s). Their positions on the 2-D Euclidean plane are denoted by the set of two-tuple vector $\mathcal{Z} = \{z_i = (x_j, y_j) | \forall j \in \mathcal{N}\}$.

All hyperarcs in $\mathcal{A}$ are denoted by (u, V_{k_u}), where u is the transmit node and $V_{k_u} = \{v_1, .., v_{k_u}\}$ is the ordered set (in increasing distance u) of end nodes of the hyperarc, and $V_{k_u} \subset \mathcal{N} \backslash \{u\}$. The hyperarcs emanating from a transmitter node are constructed in order of increasing distances of the receivers from the transmitter (refer Figure 1). This construction rule captures the distance based approach and is analogous to time sharing for broadcasting. Note that, this is one technique to construct the hypergraph $\mathcal{G}(\mathcal{N}, \mathcal{A})$, our model allows arbitrary styles of hypergraph construction that follow

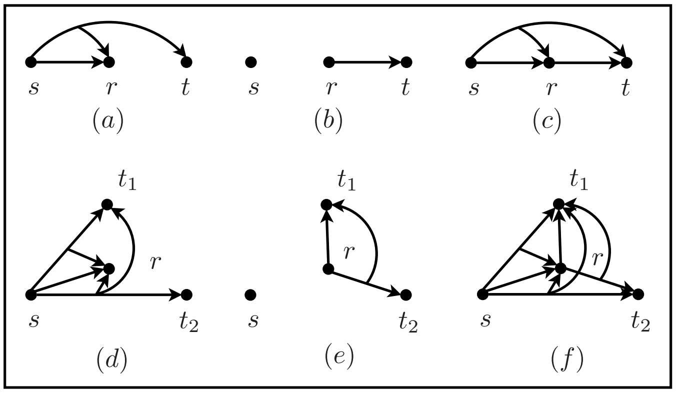

Fig. 1. Hyperarcs are constructed in increasing order of distance from the transmitter. (a)-(c): 3 node system. (a): Source hyperarc set - $\{(s,r),(s,rt)\}$. (b): Relay hyperarc set - $\{(r,t)\}$. (c): Hypergraph $\mathcal{G}(\mathcal{N},\mathcal{A})$. (d)-(f): 4 node system with $T=\{t_1,t_2\}$ such that $D_{sr}<D_{st_1}<D_{st_2}$ and $D_{rt_1}<D_{rt_2}$. (d) Source hyperarc set - $\{(s,r),(s,rt_1),(s,rt_1t_2)\}$. (e) Relay hyperarc set - $\{(r,t_1),(r,t_1t_2)\}$. (f): Hypergraph $\mathcal{G}(\mathcal{N},\mathcal{A})$.

the above three mentioned rules. Although, since time sharing is optimal for broadcasting we will stick to this technique as the main example in this paper. All the nodes in the set V_{k_u} receive the information transmitted over the hyperarc (u,V_{k_u}) equally reliably. Any hyperarc $(u,V_{k_u})\in\mathcal{A}$ is associated with a rate function $R^u_{v_{k_u}}=f(P^u_{v_{k_u}},D_{uv_{k_u}})$, where $P^u_{k_u}$ and $D_{uv_{k_u}}$ denotes the fraction of the total transmit node power allocated for the hyperarc and the Euclidean distance between transmit node u and the farthest end node v_{k_u}, respectively.

The hyperarc rate function $R^u_{v_{k_u}}$ is increasing and convex in power $P^u_{v_{k_u}}$ and decreasing in $D_{uv_{k_u}}$. Furthermore, without loss of generality, we write the hyperarc rate function into two separable functions of power and distance

$$R^u_{v_{k_u}}=\frac{g(P^u_{v_{k_u}})}{h(D_{uv_{k_u}})}\ \text{ or }\ R^u_{v_{k_u}}=g(P^u_{v_{k_u}})-h(D_{uv_{k_u}}),\quad (1)$$

where $g:\mathbf{R}^+\longrightarrow\mathbf{R}^+$ is increasing and convex and $h:\mathbf{R}^+\longrightarrow\mathbf{R}^+$ is increasing. Mainly, we will be concerned with the first equation in (1). Moreover, to comply with standard physical wireless channel models we assume that

$$\frac{\partial g(P^u_{v_{k_u}})}{\partial P^u_{v_{k_u}}}\leq\frac{\partial h(D_{uv_{k_u}})}{\partial D_{uv_{k_u}}},\quad (2)$$

$\forall(P^u_{v_{k_u}}=D_{uv_{k_u}})\in\mathbf{dom}(P^u_{v_{k_u}},D_{uv_{k_u}})$. If the functions g and h are not differentiable entirely in $\mathbf{dom}(P^u_{v_{k_u}},D_{uv_{k_u}})$, then Inequality 2 can be rewritten with partial sub-derivatives, implying that differentiability is not imperative.

Denote the convex hull of the nodes in $\mathcal{N}\backslash\{r\}$ by $\mathcal{C}$. For a given relay position $z_r\in\mathcal{C}$, let $L_i=\{l^i_1,..,l^i_{\tau_i}\}$ be the set of paths from s to a destination $t_i\in T$ and let $L=\{l_1,..,l_\tau\}$ be the set of paths from s that span all the destination set T, therefore $L\subset\bigcup_{i\in[1,n]}L_i$. Moreover, any path in the system consists of either a single hyperarc or at most two hyperarcs as there are only two transmitters in the system. Let μ and ν denote the total given power of source and relay, respectively, and $\gamma=\frac{\nu}{\mu}$ denote their ratio, where $\gamma\in(0,\infty)$. Denote with $F_{l^i_j}$ and F_i the flow over the path l^i_j (for $j\in[1,\tau_i]$) and the total flow to the destination $t_i\in T$, respectively, such that $F_i=\sum_{j\in[1,\tau_i]}F_{l^i_j}$. Define F to be the the multicast flow from s to the destination set T as the minimum among the total flows to each destination, then for a given relay position $z_r\in\mathcal{C}$ the multicast max-flow problem can be written as,

$$\text{Maximize}\qquad\left(F=\min_{i\in[1,n]}F_i\right)\qquad (A)$$

$$\text{subject to: }\ F_i\leq\sum_{j=1}^{\tau_i}F_{l^i_j},\forall i\in[1,n],\qquad (3)$$

$$0\leq F_{l^i_j}\in\mathfrak{C}(P,D),\ \ \forall j\in[1,\tau_i],\forall i\in[1,n].\qquad (4)$$

The hyperarc rate constraints and node sum-power constraints are denoted by the set $\mathfrak{C}(P,D)$ in Program (A) for simplicity. Program (A) in general is non-convex, as the path flow function $F_{l^i_j}$ can be non-convex, e.g. let the path $l^i_j\in L_i$ be $l^i_j=\{(s,V_{k_s}),(r,V_{k_r})\}$, ($l^{t_2}_1=\{(s,rt_1),(r,t_1t_2)\}$ in Figure 1(f)), then $F_{l^i_j}=\min(R^s_{v_{k_s}},R^r_{v_{k_r}})$.

Now we define the notion of cost for a given hyperarc rate $R^u_{v_{k_u}}=\frac{g(P^u_{v_{k_u}})}{h(D_{uv_{k_u}})}\geq 0$. The cost of rate $R^u_{v_{k_u}}$ is given by the total power consumed by the hyperarc to achieve $R^u_{v_{k_u}}$

$$P^u_{v_{k_u}}=g^{-1}\left(R^u_{v_{k_u}}h(D_{uv_{k_u}})\right),\qquad (5)$$

where $g^{-1}:\mathbf{R}^+\longrightarrow\mathbf{R}^+$ is the inverse function of g that maps its range to its domain. Therefore, the total cost of multicast flow F in the system is simply the sum of powers of all the hypearcs in the system. Note that the function g^{-1} is increasing and concave, and if h is convex then from Inequality (2), $g^{-1}\circ h$ increasing and convex. So for a given relay position $z_r\in\mathcal{C}$, the min-cost problem minimizing the total cost for setting up the multicast session (s,T) with a target flow F can be written as,

$$\text{Minimize}\qquad\left(P=\sum_{(u,V_{k_u})\in\mathcal{A}}P^u_{v_{k_u}}\right)\qquad (B)$$

$$\text{subject to: }\ F\leq F_i\leq\sum_{j=1}^{\tau_i}F_{l^i_j},\ \ \forall i\in[1,n],\qquad (6)$$

$$\mathfrak{C}(P,D)\ni F_{l^i_j}\geq 0,\forall j\in[1,\tau_i],\forall i\in[1,n].\qquad (7)$$

Constraint (6) makes sure that any destination $t_i\in T$ receives a minimum of flow F. Like in Program (A), we denote with the set $\mathfrak{C}(P,D)$ the hyperarc rate and power constraints.

Finally, define the point p^*, that will be crucial in developing algorithms in later sections, as

$$z_{p^*}=\arg\min_{z_p}(\max(\nu^*h(D_{z_ps}),\mu^*\max_{t_i\in T}(h(D_{z_pt_i})))),\quad (8)$$

where, $\mu^*=g(\mu)$ and $\nu^*=g(\nu)$. An easy way to understand p^* is that if $\mu^*=\nu^*=1$ then p^* is the circumcenter of two or more nodes in the set $\mathcal{N}\backslash\{r\}$. Note that the program in Equation (8) is a convex program. Also, denote the optimal value of the objective function in Equation (8) as D_{p^*}.

Hereafter, we represent with $(s,T,\mathcal{Z},\gamma)$ and $(s,T,\mathcal{Z},\gamma,F)$ the joint relay positioning and flow optimization problem instances that maximizes the multicast flow and minimizes the total cost for a the target flow F, and with $z^*_{\gamma\uparrow}$ and $z^*_{F\downarrow}$ denote the optimal relay positions, respectively.

III. Multicast Flow Properties And Reduction

In this section we develop fundamental multicast flow properties that govern the multicast flow in the wireless network hypergraphs that we consider in this paper. First, we briefly note the main hurdles in jointly optimizing the problem. For a given problem instance different relay positions can result in different hypergraphs, which makes the use of standard graph-based flow optimization algorithms difficult. Moreover, the hyperarc rate function can be non-convex itself.

We will show that the joint problems $(s, T, \mathcal{Z}, \gamma)$ and $(s, T, \mathcal{Z}, \gamma, F)$ can be reduced to solving a sequence of two decoupled problems. The reduced problems are decoupled in the sense that the first problem is purely a geometric optimization problem and involves no flow optimization and vice versa for the second problem. At the same time, they are not entirely decoupled because the two problems need to be solved in succession and cannot be solved in parallel. Now we present a series of results that are fundamental to the reducibility of the joint problem.

Proposition 1: The optimal relay positions $z^*_{\gamma\uparrow}$ and $z^*_{F\downarrow}$ lie inside the convex hull $\mathcal{C}$.

Refer Appendix A in [5] for the proof. Proposition 1 tells us that only the points inside the polygon $\mathcal{C}$ need to be considered. This brings us to the following fundamental theorem.

Theorem 1 (Flow Concentration): Given $z_r \in \mathcal{C}$:

(i) the maximized multicast flow F^* concentrates over at most two paths from s to the destination set T.

(ii) for any target flow $F \in [0, F^*]$ the min-cost multicast flow concentrates over at most two paths from s to T.

The proof is detailed in Appendix B of [5]. Theorem 1 is central to the two questions we aim to answer and reduces the complexity of joint optimization greatly by considering only two paths instead of many. Essentially, Theorem 1 tells that for a given relay position $z_r \in \mathcal{C}$, the multicast flow F must go only over the paths that span all the destination set T, i.e. set L. Furthermore, among the paths in L, the maximized multicast flow F^* goes over only two paths, namely the path $\hat{l}_1 = \{(s, T_1), (r, T_2)\}$ that has the highest min-cut among all the paths through the relay r, and path $\hat{l}_2 = \{(s, t_1, .., t_n) = (s, T)\}$, which is the biggest hyperarc from s spanning all the destination set T, where $r \in T_1$ and $T_1 \cup T_2 = T$. The same holds for the min-cost case for a given relay position $z_r \in \mathcal{C}$. Consequently, it is also true for the optimal relay positions $z^*_{\gamma\uparrow}$ and $z^*_{F\downarrow}$. Hereafter, we only need to consider the flow over paths $\hat{l}_1$ and $\hat{l}_2$ (corresponding to the relay position in consideration).

A. Max-flow Problem - $(s, T, \mathcal{Z}, \gamma)$

Assuming that the transmitted signal propagates omnidirectionally, we can geometrically represent the hyperarcs of the path $\hat{l}_1 = \{(s, T_1), (r, T_2)\}$ by circles $C^s_{T_1}$ and $C^r_{T_2}$ centered at s and r with radii $\pi_s = D_{st_k}$ and $\pi_r = D_{rt_{k'}}$ (where $D_{st_k} = \max_{t_i \in T_1}(D_{st_i})$ and $D_{rt_{k'}} = \max_{t_j \in T_2}(D_{rt_j})$), respectively. Similarly, the path $\hat{l}_2 = \{(s, T)\}$ can be represented by the circle C^s_T with radius D_{st_n}. Also, $C_\cup = C^s_{T_1} \cup C^r_{T_2}$ denotes the union region of the two circles. Then using Theorem 1, Program (A) can be re-written as,

$$\underset{\substack{P^s_{T_1}+P^r_{T_2}\leq\mu,\\ P^s_T\leq\nu,\pi_s,\pi_r}}{\text{Maximize}} \left(\min\left(\frac{g(P^s_{T_1})}{h(\pi_s)}, \frac{g(P^r_{T_2})}{h(\pi_r)}\right) + \frac{g(P^s_T)}{h(D_{st_n})} \right) \quad \text{(C)}$$

where, $P^s_{T_1}, P^r_{T_2}$ and P^s_T are the powers for hyperarcs of the paths $\hat{l}_1 = \{C^s_{T_1}, C^r_{T_2}\}$ and $\hat{l}_2 = \{C^s_T\}$, respectively. The radii variables π_s and π_r correspond to path $\hat{l}_1$ for the relay position $z_r \in \mathcal{C}$ such that $z_r \in C^s_{T_1}$ and $\mathcal{Z} \in C_\cup$.

Although Program (C) is reduced, it is still a non-convex optimization problem. The objective function is non-convex and different positions of the relay $z_r \in \mathcal{C}$ result in different end node sets T_1 and T_2 for the hyperarcs of path $\hat{l}_1$.

On the other hand, we know that the relay position is sensitive only to the flow over path $\hat{l}_1$. In addition, as there always exist a relay position $z_r \in \mathcal{C}$ such that the min-cut of path $\hat{l}_1$ is higher than that of path $\hat{l}_2$, then this also holds true for $z^*_{\gamma\uparrow}$. Therefore, optimizing the relay position to maximize the flow over path $\hat{l}_1$ results in global optimal relay position solving the original problem $(s, T, \mathcal{Z}, \gamma)$. This motivates the decoupling of computation of optimal relay position from the flow maximization over the path $\hat{l}_1$.

Proposition 2: For a given problem instance $(s, T, \mathcal{Z}, \gamma)$, if $g(\nu)h(D_{sp^*}) = D_{p^*}$, then $z^*_{\gamma\uparrow} = z_{p^*}$.

Refer Appendix C in [5] for the detailed proof. At point p^*, the following holds $\frac{g(\mu)}{h(\pi^{p^*}_s)} \geq \frac{g(\nu)}{h(\pi^{p^*}_r)}$ (from Equation (8)), thus making it naturally a good candidate for $z^*_{\gamma\uparrow}$. Proposition 2, essentially proves that if the relay is positioned at p^*, and if maximizing the flow over the path $\hat{l}_1$ results in no spare source power (i.e. $g(\nu)h(D_{sp^*}) = D_{p^*}$), then $z^*_{\gamma\uparrow} = z_{p^*}$ and $F^* = \frac{g(\mu)}{h(\pi^{p^*}_s)}$. Furthermore, the joint problem in Program (C) can be reduced to solving in sequence the computation of the optimal relay position p^* by solving Equation (8) and then calculating the max-flow F^*. But this is not true when $\frac{g(\mu)}{h(\pi^{p^*}_s)} > \frac{g(\nu)}{h(\pi^{p^*}_r)}$. We cover this case in the section of algorithms.

Let us now see the problem in a different way. Consider the radius $\pi_s \in (0, D_{st_n})$ and construct the hyperarc $C^s_{\pi_s}$. Denote with $T' = \{t_j \in T | D_{st_j} > \pi_s\}$, the set of destination nodes

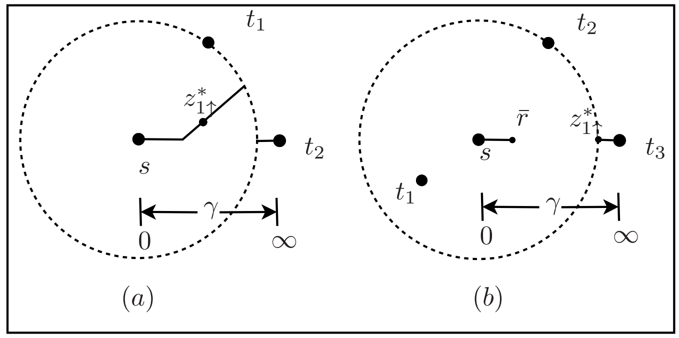

Fig. 2. The solid piecewise linear segment in examples (a) and (b) marks the set of points $\hat{r}$ for different values of $\pi_s \in (0, D_{st_2})$. Each point $\hat{r}$ corresponds to $z^*_{\gamma\uparrow}$ for some $\gamma \in (0, \infty)$. The piecewise linear segment breaks beyond the dashed circle as $z_1 \in C^s_{T_1}$. (a): E.g. C^s_r with $0 < \pi_s < D_{st_1}$, $z_{\hat{r}} = \arg\min_{\hat{z_r} \in C^s_r} \max(D_{\hat{z_r}t_1}, D_{\hat{z_r}t_2})$. Same goes for the example in (b).

that lie outside the hyperarc circle $C^s_{\pi_s}$. Then compute the point $\widehat{r}$ such that

$$z_{\widehat{r}} = \arg\min_{z_p \in C^s_{\pi_s}} (\max_{t_j \in T'} (D_{r' t_j})),$$

and position the relay at $\widehat{r}$ (here $\widehat{r}$ is the point in $C^s_{\pi_s}$ that minimizes the maximum among the distance to the nodes in the set T' from itself). If $D_{s\widehat{r}} < \pi_s$, then we contract the hyperarc $C^s_{\pi_s}$ to $C^s_{\widehat{r}}$, else we simply re-denote it with $C^s_{\widehat{r}}$. Finally, we can construct the hyperarc $C^{\widehat{r}}_{t_n}$ (note that $\mathcal{Z} \in C'_{\cup} = C^s_{\widehat{r}} \cup C^{\widehat{r}}_{t_n}$). The set $\mathcal{R}'$ of points $\widehat{r}$ computed in this way for different values of $\pi_s \in (0, D_{st_n})$ are the optimal relay positions $z^*_{\gamma\uparrow}$ solving $(s, T, \mathcal{Z}, \gamma)$ for some $\gamma \in (0, \infty)$. Figure 2(a) captures this interesting insight of the relationship between the points $\widehat{r}$ and $z^*_{\gamma\uparrow}$. Note that the set $\widehat{\mathcal{R}}$ of points $\widehat{r}$ is a discontinuous piecewise linear segment.

B. Min-cost Problem $(s, T, \mathcal{Z}, \gamma, F)$ And Duality

The min-cost problem $(s, T, \mathcal{Z}, \gamma, F)$ can be written as

$$\text{Minimize} \quad (P^s_{T_1} + P^r_{T_2} + P^s_{T}) \tag{D}$$

$$\text{subject to: } F \le \min\left(\frac{g(P^s_{T_1})}{h(\pi_s)}, \frac{g(Pr_{T_2})}{h(\pi_r)}\right) + \frac{g(P^s_T)}{h(D_{st_n})}, \tag{9}$$

$$P^s_{T_1} + P^r_{T_2} \le \mu, \quad P^s_T \le \nu. \tag{10}$$

In the non-convex Program (D), the path $\hat{l}_1 = \{C^s_{T_1}, C^r_{T_2}\}$ correspond to the relay position $z_r \in \mathcal{C}$ which is implicitly represented in the distance variables π_s and π_r. From Theorem 1, we know that paths $\hat{l}_1$ and $\hat{l}_2$ carry all the min-cost target multicast flow F. In this sub-section we refer the path $\hat{l}_1$ as the cheapest path for a unit flow among all the paths through r in L for given position of relay.

Now, we claim that $z^*_{F\downarrow} \in \widehat{\mathcal{R}}$. This is true because given the hyperarc $C^s_{T_1}$ of path $\hat{l}_1$ with optimal radius π^*_s, the second hyperarc $C^r_{T_2}$ must be centered at the point that minimizes the maximum among the distances to all the destination nodes not spanned by the hyperarc $C^s_{T_1}$ from itself, as this minimizes the cost over the hyperarc $C^r_{T_2}$. Therefore, $z^*_{F\downarrow}$ (like $z^*_{\gamma\uparrow}$) always lie on on the curve $\widehat{\mathcal{R}}$. This observation motivates an interesting fundamental relationship between $z^*_{F\downarrow}$ and $z^*_{\gamma\uparrow}$.

Theorem 2 (Max-flow/Min-cost Duality):

$$z^*_{F\downarrow} = z^*_{\widehat{\gamma}\uparrow}, \tag{11}$$

where $\widehat{\gamma} \in [\min(\overline{\gamma}, \gamma), \max(\overline{\gamma}, \gamma)]$, $F \in [0, F^*]$ and $z^*_{1\downarrow} = z^*_{\overline{\gamma}\uparrow}$.

Theorem 2 establishes the underlying duality relation between the max-flow problem $(s, T, \mathcal{Z}, \gamma)$ and the min-cost problem $(s, T, \mathcal{Z}, \gamma, F)$ and says that the point $z^*_{F\downarrow}$ (or $z^*_{\widehat{\gamma}\uparrow}$) lies on the segment $z^*_{1\downarrow} - z^*_{F^*\downarrow}$ ($z^*_{\overline{\gamma}\uparrow} - z^*_{\gamma\uparrow}$, respectively) of the curve $\widehat{\mathcal{R}}$. Implying that the optimal relay position $z^*_{F\downarrow}$ solving the problem $(s, T, \mathcal{Z}, \gamma, F)$ is also the optimal relay position $z^*_{\widehat{\gamma}\uparrow}$ solving the problem $(s, T, \mathcal{Z}, \gamma)$ for some $\widehat{\gamma}$. The proof of Theorem 2 is presented in Appendix D of [5].

However, the max-flow is not always reducible to a sequence of decoupled problems. This is mainly due to the fact that the path $\hat{l}_2$ can be cheaper than path $\hat{l}_1$ for a unit flow corresponding to the optimal position $z^*_{F\downarrow}$, i.e.

$$g^{-1}(h(\pi^*_s)) + g^{-1}(h(\pi^*_r)) \ge g^{-1}(h(D_{st_n})).$$

This information is not easy to get a priori. In contrast, we can safely assume that

$$g^{-1}(h(\pi^*_s)) + g^{-1}(h(\pi^*_r)) \le g^{-1}(h(D_{st_n})), \tag{12}$$

as almost all wireless network models that comply with our model result in the hyperarc cost function $g^{-1}(h(D_{uv_{k_u}}))$ being the increasing convex function of distance $D_{uv_{k_u}}$ that satisfy Inequality (12). If Inequality (12) holds, then similar to the Max-flow problem the joint optimal relay positioning and min-cost flow optimization problem in Program (D) can be reduced to a sequence of decoupled problems of computing the optimal relay position and then optimizing the hyperarc powers to achieve the min-cost flow F in the network using the similar arguments as in previous subsection. For a special of the min-cost problem $(s, T, \mathcal{Z}, \gamma, F)$, we present the Min-cost Algorithm that sequentially solves and outputs the optimal relay position and powers to achieve the target flow $F \in [0, F^*]$ in Section IV-B.

IV. ALGORITHMS

In this section we present the general max-flow and the special case min-cost algorithms that solve the sequence of decoupled problems.

A. Max-flow Algorithm

Input: Problem instance $(s, T, \mathcal{Z}, \gamma)$.

1: Compute p^*, if $g(\nu)h(D_{sp^*}) = g(\mu)h(D_{p^* t_n})$, output $z^*_{\gamma\uparrow} = z_{p^*}$, $F^* = g(\nu)h(D_{sp^*})$ and quit, else go to 2.
2: Construct the set $T' = \{t'_j \in T | D_{st'_j} < D_{p^* t'_j}\} = \{t'_1, .., t'_{j'}\}$ (ordered in increasing distance from s) and compute $p^*_{T\setminus T'}$. If $D_{st'_{j'}} \le D_{sp^*_{T\setminus T'}}$, declare $z^*_{\gamma\uparrow} = z_{p^*_{T\setminus T'}}$ and $F^* = g(\nu)h(D_{sp^*_{T\setminus T'}})$ and quit, else go to Step 3.
3: Compute the points z^*_1 and z^*_2, and maximized multicast flow F^*_1 and F^*_2, respectively. Declare before quitting,

$$z^*_{\gamma\uparrow} = \begin{cases} z^*_1 & \text{if } F^*_1 > F^*_2, \\ z^*_2 & \text{if } F^*_1 < F^*_2. \end{cases}$$

Output: $z^*_{\gamma\uparrow}$ and F^*.

Fig. 3. Max-flow Algorithm.

The Max-flow Algorithm in Figure 3, is a simple and non-iterative 3 step algorithm that outputs the optimal relay position and the maximized multicast flow. The first step is essentially Proposition 2, in case it is not satisfied the second step filters the redundant nodes that are too close to the source and can be ignored. If the conditions of first or second step are not met, then the third step divides the computation of $z^*_{\gamma\uparrow}$ into two regions of $\mathcal{C}$ and computes the optimal relay position z^*_1 and z^*_2 for these two regions and outputs the better one. The proof of optimality is provided in Appendix E of [5].

B. Min-cost Algorithm

In this subsection, we assume that the Inequality (12) is satisfied and the target flow $F \in [0, F^*]$ goes over the path $\hat{l}_1$ (corresponding to the optimal relay position $z^*_{F\downarrow}$) only. Min-cost Algorithm in Figure 4, unlike the Max-flow algorithm, is an iterative algorithm. In the first step the geometric feasibility region is constructed and in the second step this region is divided into at most $n-1$ sub-regions. The optimal relay position is computed for all the sub-regions and the one minimizing the cost among them is declared global optimal. Computing the optimal relay position for the sub-regions is a simple geometric convex program that can be solved efficiently and the number of such iterations are upper bounded by $n-1$. The proof of optimality is presented in Appendix F of [5].

Input: Problem instance $(s, T, \mathcal{Z}, \gamma, F)$ and $C'_\cap$.

1: Compute $\widehat{p} = \arg\min_{p \in C'_\cap}(h(D_{sp}) + \max_{i\in[1,n]}(h(D_{pt_i})))$, and build the set $\widehat{T} = \{\widehat{t} \in T | D_{s\widehat{t}} \leq D_{s\widehat{p}}\}$. If $\widehat{T} \neq \{\emptyset\}$, then recompute $\widehat{p} = \arg\min_{p\in C'_\cap}(h(D_{sp}) + \max_{t\in T\setminus\{\widehat{T}\}}(h(D_{pt})))$, calculate $\Psi_{\widehat{p}} = h(D_{s\widehat{p}}) + D_{\widehat{p}t_n}$ and to go to Step 2.

2: Build the set $\overline{T} = \{t \in T\setminus\{\widehat{T}, t_n\} | D_{st} > \pi_s^{\widehat{p}}, D_{st} \leq \pi'_s\} = \{\bar{t}_1, .., \bar{t}_l\}$ (ordered in increasing distance from s), compute the points

$$\widehat{p}_j = \arg\min_{p\in\overline{C}^s_j}(\max(h(D_{sp}), h(D_{s\bar{t}_{j-1}})) + \max_{t\in\overline{T}_j}(h(D_{pt}))),$$

and calculate the cost of unit flow $\overline{\Psi}_j = h(D_{s\widehat{p}}) + h(\max_{t\in\overline{T}_j}(D_{\widehat{p}t}))$ over the path $\hat{l}_2$ corresponding to the relay position $\widehat{p}_j$, $\forall j \in [1, l]$. Declare

$$z^*_{F\downarrow} = \begin{cases} z_{\widehat{p}} & \text{if } \Psi_{\widehat{p}} \leq \overline{\Psi}_m, \\ z_{\overline{p}_m} & \text{if } \Psi_{\widehat{p}} \geq \overline{\Psi}_m, \end{cases}$$

where $\overline{\Psi}_m = \min_{j\in[1,n]}(\overline{\Psi}_j)$, ${P^s_{T_1}}^* = g^{-1}(h(\pi^*_s)F)$ and ${P^r_{T_2}}^* = g^{-1}(h(\pi^*_r)F)$ and quit.

Output: $z^*_{F\downarrow}$, ${P^s_{T_1}}^*$ and ${P^r_{T_2}}^*$.

Fig. 4. Min-Cost Algorithm.

V. Example: Low-SNR Achievable Network Model

In this section we present an example from the interference delimited network model that was originally presented in [1].

A. Low-SNR Broadcast and MAC Channel Model

Consider the AWGN Low-SNR (wideband) Broadcast Channel with a single source s and multiple destinations $T = \{t_1, .., t_n\}$ (arranged in the order of increasing distance from s). From [6] and [7], we know that the superposition coding is equivalent to time sharing, which is optimal. Implying that the broadcast communication from a single source to multiple receivers can be decomposed into communication over n hyperarcs sharing the common source power. Therefore, we get the set of hyperarcs $\mathcal{A}_{bc} = \{(s, t_1), (s, t_1t_2), .., (s, t_1t_2..t_n)\}$.

Similarly, in the Low-SNR (wideband) regime, interference becomes negligible with respect to noise, and all sources can achieve their point-to-point capacities analogous to Frequency Division Multiple Access (FDMA). In general, the MAC Channel consisting from n sources $s_1, ..., s_n$ transmitting to a common destination t can be interpreted as n point-to-point arcs each having point-to-point capacities. Thus, we get $\mathcal{A}_{mac} = \{(s_1, t), .., (s_n, t)\}$. Each hyperarc $(s, t_1..t_j) \in \mathcal{A}_{bc} \cup \mathcal{A}_{mac}$ is associated with the rate function

$$R^s_{t_j} = \frac{P^s_{t_j}}{N_0 D^\alpha_{st_j}}, \forall j \in [1, n], \quad (13)$$

where $\alpha \geq 2$ is the path loss exponent.

B. Low-SNR Achievable Hypergraph Model

By concatenating the Low-SNR Broadcast Channel and MAC Channel models we obtain an Achievable Hypergraph Broadcast Model. For example the Broadcast Relay Channel consisting of a single source, n destinations and a relay. Although, the time sharing and FDMA are capacity achieving optimal schemes in the respective models, the Achievable Hypergraph Model is not necessarily capacity achieving. In contrast and more importantly for practical use, this model is easy to scale to larger and more complex networks.

The above Low-SNR Achievable Hypergraph Model also incorporates fading [1]. The rate function in Equation (13) is linear in transmitter power and convex in hyperarc distance, hence the results from this paper can be directly applied.

VI. Conclusion

We present simple and efficient geometry based algorithms for solving joint relay positioning and flow (max-flow/min-cost) optimization problems for a fairly general class of hypergraphs. Any application that satisfies the hypergraph construction rules and can be modeled under the classical multicommodity framework can use the results presented here.

As a part of future work it would be of interest to extend the work presented here to the general multicommodity setting where multiple sessions use a common relay.

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