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Telecommunication Traffic, Queueing Models, and Subexponential Distributions

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This article reviews various models within the queueing framework which have been suggested for teletraffic data. Such models aim to capture certain stylised features of the data, such as variability of arrival rates, heavy-tailedness of on- and off-periods and long-range dependence in teletraffic transmission. Subexponential distributions constitute a large class of heavy-tailed distributions, and we investigate their (sometimes disastrous) influence within teletraffic models. We demonstrate some of the above effects in an explorative data analysis of Munich Universities' intranet data.

Keywords: buffer overflow; fluid queue; GI/G/1 queue; heavy-tailed distribution function; Lindley's equation; long-range dependence; power-law tail; queue-length distribution; regularly varying functions; subexponential distributions; on/off process; stationary waiting time distribution; truncated power-law tail.

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1. Background and Terminology

Recent measurements of traffic both on local and wide area communications networks have shown some extraordinary behaviour which proves critical for understanding the performance of broad-band networks: the data collected (e.g. packets on Ethernet networks) at Bellcore [32], frames from Variable-Bit-Rate (VBR) video service [7,18], FTP data connections, NNTP, and WWW arrivals in wide area traffic show enormous *variability of arrival rates* indicating that a homogeneous Poisson process may be an insufficient model for packet traffic, see Paxson and Floyd [40].

Other measurements have indicated that CPU times and data file sizes (see [17,38,39]) follow *heavy-tailed distributions.*

Moreover, the time series of teletraffic data show a *long-range dependence effect*, meaning that the current state of the time series has a strong dependency on the remote past. Definitions vary from author to author, but a commonly accepted definition in a covariance stationary time series is that a process (X_n) has long-range dependence, if the correlation coefficients $\mathrm{corr}(X_0, X_n)$ decrease to 0 at a rate slower than exponential. Admittedly, many authors even require that the autocorrelation coefficients are not absolutely summable, but we want some more flexibility in modelling. The exponential as a reference rate is motivated by the fact that for linear models as for instance causal and invertible ARMA (autoregressive-moving average) processes the correlation coefficients decrease to 0 exponentially fast, hence long-range dependence in the above sense cannot be modelled in this traditional way.

Various models have been suggested to capture these effects. They range from traditional queueing models to sophisticated on/off models [23,22,24], Markov modulated queues [25,26], shot noise models [31] and fractional Brownian motion [32,49,50].

The aim of this article is

- to clarify the various notions of heavy-tailed distributions as used in the queueing and network area,

- to describe the consequences of subexponential input distributions to the distributional behaviour of the output processes,

- to discuss possible models where heavy-tailed or dependent input may explain the observed long-range dependence in teletraffic data.

Our paper is organised as follows. In Section 2 we summarise various notions and properties of heavy-tailed distributions, the outer frame being built by the class of subexponential distributions.

In Section 3 we indicate what disasters heavy-tailed input can result in classical queueing models. Such models have been taken as basis for more sophisticated models in teletraffic data transmission. For instance, buffer sizes correspond to workload processes. We want to gain some qualitative insight into the effect of heavy tails on performance measures like waiting time distribution, workload process, and queue length.

In Section 4 we discuss various models within the queueing context which have been suggested for teletraffic data. We derive certain performance measures for such models. Section 5 concludes the paper with an explorative data analysis of Munich Universities' intranet data, measured at a network access point of the Germany wide broadband research network (B-WiN).

2. Subexponential distributions

Intuitively, we consider heavy-tailed distributions as models for possibly very large values in a sample. There is a common agreement that the tail of a heavy-tailed distribution function (df) decreases to zero more slowly than any exponential tail, i.e. for a heavy-tailed random variable (rv) X

$$P(X > x)e^{\varepsilon x} \to \infty\,, \quad x \to \infty\,,$$

for all positive ε. This class includes *Pareto, lognormal* and *heavy-tailed Weibull distributions.* In certain applications, in particular in queueing theory, more structure for the distribution tail is needed, which leads to the definition of subexponential distributions.

In this section we summarise definitions and properties of subexponential dfs concentrating on those properties which we shall need later on. A more complete account on subexponential dfs can be found in Embrechts, Klüppelberg and Mikosch [13] or in the review article by Goldie and Klüppelberg [20], from which results are quoted freely. If possible, we refer to Karl Sigman's "A primer on heavy-tailed distributions" [46] in this issue.

We give two equivalent definitions of subexponential dfs. The first, analytic one is motivated by the Pollaczek–Khinchin formula (3.2) below, while the second, probabilistic one provides a more intuitive interpretation.

Definition 2.1. *(Subexponential distribution function)*
Let $(X_i)_{i\in\mathbb{N}}$ be iid positive rvs with common df F such that $F(x) < 1$ for all $x > 0$. Denote by

$$\overline{F}(x) = 1 - F(x)\,, \quad x \geq 0\,,$$

the tail of F and

$$\overline{F^{n*}}(x) = 1 - F^{n*}(x) = P(X_1 + \cdots + X_n > x)\,, \quad x \geq 0\,,$$

the tail of the n–fold convolution of F. F is a *subexponential df* ($F \in \mathcal{S}$) if one of the following equivalent conditions holds:

(a) $\displaystyle\lim_{x\to\infty} \frac{\overline{F^{n*}}(x)}{\overline{F}(x)} = n$ for some (all) $n \geq 2\,,$

(b) $\displaystyle\lim_{x\to\infty} \frac{P(X_1 + \cdots + X_n > x)}{P(\max(X_1, \ldots, X_n) > x)} = 1$ for some (equivalently all) $n \geq 2$. □

Remark 2.2. (i) Definition (b) provides a physical interpretation of subexponentiality: the sum of n iid subexponential rvs is likely to be large if and only if their maximum is. This accounts for large values in a subexponential sample.

(ii) An important (though much smaller) subclass of $\mathcal{S}$ is the class of *dfs with regularly varying tail.* We write $\overline{F} \in \mathcal{R}(-\alpha)$ if

$$\lim_{x\to\infty} \frac{\overline{F}(tx)}{\overline{F}(x)} = t^{-\alpha}, \quad t > 0 .$$

Since $\overline{F}$ is non-increasing, the index $\alpha \in [0,\infty)$. $\overline{F} \in \mathcal{R}(-\alpha)$ is equivalent to $\overline{F}(x) = x^{-\alpha}L(x)$ for some slowly varying function L ($L \in \mathcal{R}(0)$). Examples for L are constants, functions converging to a constant, logarithms, or iterated logarithms. If $\overline{F} \in \mathcal{R}(-\alpha)$ for $\alpha < 1$, then F has infinite mean; if $\alpha < 2$, then F has infinite variance. The class of regularly varying functions allows one to apply Abel-Tauber theorems, quite a common tool in applied probability (see e.g. Bingham, Goldie and Teugels [8] or Feller [16]). Unfortunately, there is no characterisation of a subexponential distribution in terms of its Laplace transforms.

(iii) Further examples of subexponential distributions include the lognormal and the heavy-tailed Weibull distributions $\overline{F}(x) \sim \exp(-x^{\beta})$ for $\beta \in (0,1)$. □

All subexponential dfs also belong to the following class; see [46], Remark 2.3.

Definition 2.3. The df F of a positive rv X such that $F(x) < 1$ for all $x > 0$ belongs to the class $\mathcal{L}$ if

$$\lim_{x\to\infty} P(X - x > y | X > x) = \lim_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = 1 \quad \forall\, y \in \mathbb{R}. \tag{2.1}$$

The convergence is then locally uniformly in y. □

For positive y, $P(X - x \le y | X > x)$ is the df of the overshoot over a threshold x. For the class $\mathcal{L}$, this overshoot degenerates as $x \to \infty$, i.e. it becomes infinite.

Define for a positive rv X with df F having finite mean μ its *equilibrium distribution* (or *integrated tail distribution*) by

$$F_I(x) = \frac{1}{\mu} \int_0^x \overline{F}(y)\, dy , \quad x \ge 0 . \tag{2.2}$$

3. Classical queueing models and subexponentials

The first papers to recognise the importance of subexponential dfs for queueing theory were Cohen [12], Pakes [37], and Smith [47]. Further (early) references on subexponential dfs in the context of insurance risk can be found in [13].

We consider an M/G/1 queue with arrival rate $\lambda > 0$, service time df F having finite mean μ and equilibrium df $F_I(x)$. We assume that the queue is stable, i.e. its traffic intensity $\rho = \lambda\mu < 1$.

Denote by W_n the waiting time of the nth customer. Then the sequence (W_n) satisfies Lindley's equation which is given by the following recursion

$$W_0 = 0\,, \quad W_{n+1} = (W_n + X_n - U_n)^+\,, \quad n \in \mathbb{N}, \tag{3.1}$$

where X_n is the service time of the nth customer and $U_n = T_{n+1} - T_n$ is the interarrival time between nth and $(n+1)$st customer. It can be shown (see e.g. Feller [16] or Resnick [41]) that

$$W_n \stackrel{d}{=} \max_{1\le k\le n} \sum_{i=1}^{k} (X_i - U_i)\,, \quad n \in \mathbb{N},$$

and

$$E(X_i - U_i) = \mu - \lambda^{-1} = \lambda^{-1}(\lambda\mu - 1) < 0\,, \quad i \in \mathbb{N}.$$

Then W_n is distributed as the maximum of a random walk with negative drift. Hence $W_n \to W_\infty$ *a.s.*, where W_∞ is a finite rv with df $\pi(t)$, $t \ge 0$. For (U_n) iid exponential the stationary waiting time distribution $\pi(t)$, $t \ge 0$, is given by the Pollaczek-Khinchin formula:

$$\pi(t) = (1-\rho)\sum_{n=0}^{\infty} \rho^n F_I^{n*}(t)\,, \quad t \ge 0\,, \tag{3.2}$$

where $F_I^{0*} = I_{[0,\infty)}$ is the df of Dirac (unit) measure at 0. In this representation ρF_I is the *ladder height df* of the embedded random walk. The infinite series on the rhs of (3.2) defines a defective renewal measure ($\rho F_I(x) \to \rho < 1$ as $x \to \infty$), and the corresponding renewal process is transient: the sequence of renewals (ladder heights) eventually stops, and at each ladder height $1-\rho$ is the probability of termination then and there. This is a consequence of the negative drift of the embedded random walk, which is ensured by $\rho < 1$. For details see Feller [16], Section VI.9.

We rewrite formula (3.2) in terms of the tails,

$$\overline{\pi}(t) = (1-\rho)\sum_{n=1}^{\infty} \rho^n \overline{F_I^{n*}}(t)\,, \quad t \geq 0\,.$$

Dividing both sides by $\overline{F}_I(t)$, we see that Definition 2.1(a) yields an asymptotic estimate for $\overline{\pi}(t)$ for large t, provided that one can safely interchange the limit and the infinite sum. This is ensured by Lemma 2.10 of [46] and Lebesgue's dominated convergence theorem.

It turns out that the asymptotic equivalence of $\overline{\pi}(t)$ and $\overline{F}_I(t)$ is not just a consequence but a characterisation of subexponentiality, as follows from the following theorem (see Embrechts and Veraverbeke [14]).

Theorem 3.1. *(Stationary waiting time in the M/G/1 queue)*

$$\pi \in \mathcal{S} \iff F_I \in \mathcal{S} \iff \lim_{t\to\infty} \frac{\overline{\pi}(t)}{\overline{F}_I(t)} = \frac{\rho}{1-\rho}\,. \qquad \square$$

This theorem can partly be generalised to a GI/G/1 queue, where the arrival process is an arbitrary renewal process.

Theorem 3.2. *(Stationary waiting time in the GI/G/1 queue)*

$$\pi \in \mathcal{S} \iff F_I \in \mathcal{S} \implies \lim_{t\to\infty} \frac{\overline{\pi}(t)}{\overline{F}_I(t)} = \frac{\rho}{1-\rho}\,. \qquad \square$$

The next question is how and when high workloads (i.e. large buffer contents or buffer overflows) happen in a classical queueing system such as M/G/1 or GI/G/1.

Recall that the sequence of waiting times (W_n) given by (3.1) defines a regenerative process with respect to the renewal process formed by the visits of

0. Wlog we assume that a regeneration cycle begins at $W_1 > 0$ (with $W_0 = 0$) and ends at

$$\tau = \inf\{n > 0 : W_n = 0\} = \inf\{n > 0 : S_n \le 0\}\,,$$

where $S_n = \sum_{i=1}^n (X_i - U_i)$ is the random walk starting in $S_0 = 0$ with generic increment $Y = X - U$. If we assume that the df F of X belongs to the class $\mathcal{L}$, then Y has distribution tail

$$\overline{G}(x) = P(Y > x) = \int_0^\infty P(X > x + y)dP(U \le y) \sim P(X > x) = \overline{F}(x)\,, \quad x \to \infty\,. \tag{3.3}$$

This implies also that the stationary distribution π is tail equivalent to the integrated tail distribution of F, more precisely, with $\nu = E(X - U)$,

$$\overline{\pi}(x) \sim \frac{1}{\nu}\int_x^\infty \overline{F}(y)dy = \frac{\mu}{\nu}\overline{F}_I(x)\,, \quad x \to \infty\,. \tag{3.4}$$

A refined probabilistic description of the maximum of a regenerative cycle is given in Asmussen [2], Section 2.1, providing some further intuition. Denote by

$$M_\tau = \max_{1 \le n \le \tau} S_n$$

the cycle maximum and by

$$\tau(x) = \inf\{n > 0 : S_n > x\}$$

the first hitting time of the boundary x by the random walk (S_n) and note that $P(M_\tau > x) = P(\tau(x) < \tau)$. Now we investigate how a large maximum of a cycle “happens”. To this end define for $0 < x_0 < x < \infty$ the quantities

$$\begin{aligned} N(x, x_0) &= \mathrm{card}\{n : 0 \le n < \tau, S_n \le x_0 \text{ and } S_{n+1} > x\}\,, \\ p_1(x, x_0) &= P(S_{n+1} > x \text{ for some } 0 \le n < \tau \text{ with } S_n \le x_0)\,, \\ p_2(x, x_0) &= P(\tau(x) < \tau \text{ and } x_0 \le S_{\tau(x)-1} \le x)\,. \end{aligned}$$

Theorem 3.3. With $m = E\tau$ and the notation introduced above we have,

(i) $p_1(x, x_0) \le P(M_\tau > x) \le p_1(x, x_0) + p_2(x, x_0)\,, \quad x > x_0\,,$

(ii) $EN(x,x_0) \sim p_1(x,x_0) \sim m\overline{\pi}(x_0)\overline{F}(x)\,, \quad x_0 > 0\,, x \to \infty\,,$

(iii) If π has a density π' such that $\pi'(x) \sim \overline{F}(x)/\nu$ as $x \to \infty$ (i.e. (3.4) holds for the densities), then $\lim\limits_{x_0\to\infty} \limsup\limits_{x\to\infty} \dfrac{p_2(x,x_0)}{\overline{F}(x)} = 0\,,$

(iv) $P(M_\tau > x) \sim m\overline{F}(x)\,, \quad x \to \infty\,.$

Proof (i) is obvious by inclusion of the events.

(ii) We first summarise some results from renewal theory and regenerative process theory (see e.g. [1]). Denote by R the renewal measure of the random walk, i.e. for any Borel set $A \subset \mathbb{R}$ the quantity $R(A) = \sum_{n=0}^\infty P(S_n \in A)$ denotes the expected number of points of $(S_n)_{n\ge 0}$ in A. Then there exist $a, b \in \mathbb{R}$ such that $R([x, x+y)) \le a + by$ holds for all $x, y \in \mathbb{R}$. Furthermore, with $I(\cdot)$ denoting the indicator function,

$$C(A) = E\sum_{n=0}^{\tau-1} I(S_n \in A) = m\pi(A)\,.$$

Now let $(Y_i)_{i\in\mathbb{N}}$ be iid rv with common df G, then

$$\begin{aligned} EN(x,x_0) &= E\sum_{n=0}^{\tau-1} I(S_n \le x_0, S_n + Y_{n+1} > x) = \int_0^{x_0} \overline{G}(x-y)dC(y) \\ &= m\int_0^{x_0} \overline{G}(x-y)d\pi(y) \sim m\pi(x_0)\overline{F}(x)\,, \quad x \to \infty\,, \end{aligned}$$

where we used (3.3) and the fact that $G \in \mathcal{L}$ (since $F \in \mathcal{L}$) and local uniform convergence in (2.1). For $p_1(x,x_0)$ we use the crude approximation for $k \ge 1$,

$$\begin{aligned} &P(N(x,x_0) \ge k+1 | N(x,x_0) \ge k) \le \sum_{n=0}^\infty P(S_n < x_0, S_{n+1} > x) \\ &= \sum_{n=0}^\infty P(S_n < x_0, S_n + Y_{n+1} > x) = \sum_{n=0}^\infty \int_0^{x_0} \overline{G}(x-y)dP(S_n \le y) \\ &\le \overline{G}(x-x_0)R([0,x_0)) \le \overline{G}(x-x_0)(a+bx_0) =: \alpha(x,x_0)\,. \end{aligned}$$

Since $P(N(x,x_0) \ge 1) = p_1(x,x_0)$ this implies that

$$P(N(x,x_0) \ge k+1) = P(N(x,x_0) \ge 1)\prod_{j=1}^k P(N(x,x_0) \ge j+1 | N(x,x_0) \ge j)$$

$$\leq p_1(x,x_0)\alpha(x,x_0)^k \,.$$

Hence

$$E(N(x,x_0)I(N(x,x_0)\geq 2)) = \textstyle\sum_{k=1}^{\infty} P(N(x,x_0)\geq k+1)$$
$$\leq \; p_1(x,x_0)\textstyle\sum_{k=1}^{\infty}\alpha(x,x_0)^k \; = p_1(x,x_0)\dfrac{\alpha(x,x_0)}{1-\alpha(x,x_0)}$$

This yields

$$p_1(x,x_0) \leq EN(x,x_0) \leq p_1(x,x_0) + p_1(x,x_0)\frac{\alpha(x,x_0)}{1-\alpha(x,x_0)}\,.$$

Now note that $\lim_{x\to\infty}\alpha(x,x_0)=0$, since G has finite mean.

(iii) Consider only the downcrossings of x within the regenerative cycle. Define by

$$D_\tau(x) = E\sum_{n=0}^{\tau-1} I(S_n > x, S_{n+1}\leq x)$$

the expected number of downcrossings of a threshold x within a cycle. By the assumtion on the density π' we obtain by regenerative process theory,

$$\begin{aligned}\frac{D_\tau(x)}{m} &= \lim_{n\to\infty} P(W_n > x, W_{n+1}\leq x) = \lim_{n\to\infty}\int_x^\infty P(Y\leq x-z)dP(W_n\leq z)\\ &= \int_x^\infty G(x-z)d\pi(z) \qquad\qquad = \int_x^\infty G(x-z)\pi'(z)dz\\ &= \pi'(x)\int_{-\infty}^0 \frac{\pi'(x-y)}{\pi'(x)}G(y)dy \sim \frac{\overline{F}(x)}{\nu}\int_{-\infty}^0 G(y)dy \;=\; \frac{\nu_-}{\nu}\overline{F}(x)\,,\end{aligned}$$

where $\nu_- = \int_{-\infty}^0 G(y)dy = E(X-U)_-$. Now split the regenerative cycle up according to the jumps from below x_0 over x and denote by

$$\begin{aligned}\sigma_1 &= \inf\{n>0 : S_n\leq x_0, S_{n+1}>x\}\,,\\ \sigma_2 &= \inf\{n>\sigma_1 : S_n\leq x_0, S_{n+1}>x\}\,,\ldots,\\ \sigma_{K-1} &= \sup\{n<\tau : S_n\leq x_0, S_{n+1}>x\}\\ \sigma_K &= \tau\,.\end{aligned}$$

Now write

$$D_\tau(x) = E \sum_{j=1}^{K} \sum_{n=0}^{\sigma_j} I(S_n > x, S_{n+1} \le x)\,.$$

Notice that $p_2(x, x_0) \le \sum_{n=0}^{\sigma_1} I(S_n > x, S_{n+1} \le x)$. Moreover, the overshoot over x after an upcrossing from a level $\le x_0$ converges in distribution to ∞ (as a consequence of $G \in \mathcal{L}$) and hence the expected subsequent number of downcrossings of level x before the process falls below x_0 is approximately ν_-/ν. Hence, asymptotically for $x \to \infty$ we get

$$D_\tau(x) \ge p_2(x,x_0) + EN(x,x_0)\frac{\nu_-}{\nu}(1+o(1)) \sim p_2(x,x_0) + m\pi(x_0)\overline{F}(x)\frac{\nu_-}{\nu}(1+o(1))\,.$$

This implies that

$$\limsup_{x\to\infty} \frac{p_2(x,x_0)}{\overline{F}(x)} \le m\overline{\pi}(x_0)\nu_-/\nu\,.$$

(iv) is a consequence of (i)-(iii). □

Conclusion. The process evolves in a typical way, with negative drift, until a very large service time causes an upcrossing over high threshold. After the overshoot the drift takes over again, but there may be some additional upcrossings on the way down which can be considered as aftershocks caused essentially by the preceding large service time. □

A continuous time version of $(W_n)_{n\ge 0}$ is the *workload process* $(V_t)_{t\ge 0}$ which denotes the sum of service times (whole or remaining) in the system at time t.

Important information about a queueing system with heavy-tailed service time can be gained by considering high excursions of the workload process $(V_t)_{t\ge 0}$ just after a buffer overflow happened (see Figure 1). This would be the traffic being lost or to be stored elsewhere. For mathematical details and proofs we refer to Asmussen and Klüppelberg [4].

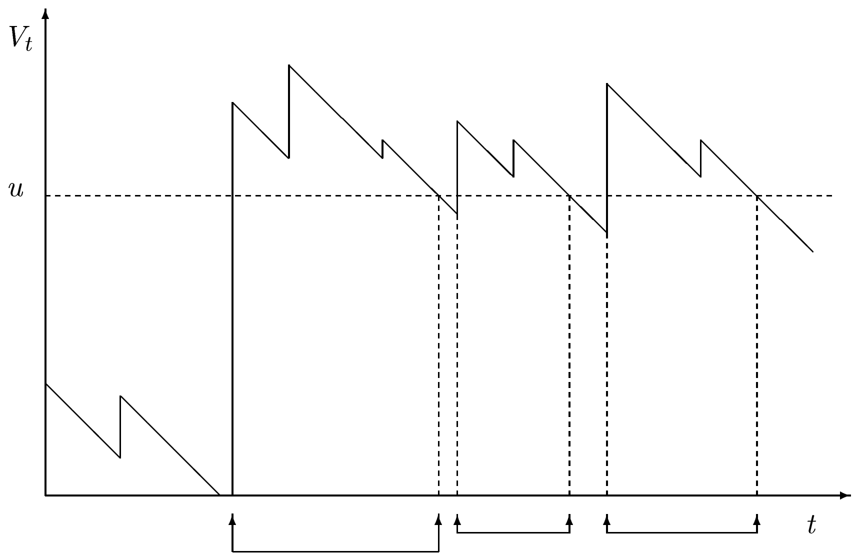

Figure 1. Sample path of the workload process (V_t) showing three high–level excursions.

Assume a GI/G/1 queue with service time X, which has df $F \in \mathcal{S}^*$ with finite mean μ, i.e.

$$\lim_{x\to\infty}\int_0^x \frac{\overline{F}(x-t)}{\overline{F}(x)}\overline{F}(t)dt = 2\mu\,. \tag{3.5}$$

For more details on the class $\mathcal{S}^*$ see Klüppelberg [27]. $F \in \mathcal{S}^*$ implies in particular that $F \in \mathcal{L}$ and $F_I \in \mathcal{S}$.

The stationary distribution of $(V_t)_{t\geq 0}$ is the stationary waiting time distribution π, which is linked by subexponentiality to the integrated service time df by Theorems 3.1 and 3.2. Let $P^{(x)}$ denote the distribution of a doubly infinite version $\{V_t\}_{-\infty<t<\infty}$ of the workload process for which a stationary excursion above level x starts at time 0; i.e. for any event A,

$$\begin{aligned} P^{(x)}(A) = \quad & P(A \mid V_{0-} \leq x, V_0 > x) \quad = \frac{P(A, V_{0-} \leq x, V_0 > x)}{P(V_{0-} \leq x, V_0 > x)} \\ = & \frac{P(A, V_{0-} \leq x, V_0 > x)}{\int_0^x P(V_{0-} = t, V_0 > x)d\pi(t)} = \frac{P(A, V_{0-} \leq x, V_0 > x)}{\int_0^x P(X > x-t)d\pi(t)}\,. \end{aligned}$$

The point 0 is of no importance, we simply describe a *stationary excursion* as a typical excursion in a stationary system, which starts with a jump from a pre-level $0 \le z = V_{0-} < x$, distributed as π conditioned to $[0,x)$, to a level $V_0 > x$, such that for $0 \le z < x$,

$$\begin{aligned} P^{(x)}(V_0 > x+y \mid V_{0-} = z) &= P(V_0 > x+y \mid V_{0-} = z, V_0 > x) \\ = \frac{P(V_0 > x+y, V_{0-} = z)}{P(V_0 > x, V_{0-} = z)} &= \frac{\overline{F}(x+y-z)}{\overline{F}(x-z)} \quad \to \quad 1\,, \quad x \to \infty\,, \end{aligned} \tag{3.6}$$

where the limit 1 is a consequence of $F \in \mathcal{L}$.

Denote the measure $\nu^{(x)}(A,B) = P^{(x)}(V_{0-} \in A, V_0 - x \in B)$ describing the joint distribution of the pre-level V_{0-} of the excursion and its initial overshoot $V_0 - x$ of level x. From (3.6) we get for $0 \le z < x,\ y \ge 0$ the formula

$$\begin{aligned} \nu^{(x)}([0,z),(y,\infty)) &= P^{(x)}(V_0 > x+y, V_{0-} < z) \\ &= \frac{\int_0^z P(V_0 > x+y, V_{0-} = t)d\pi(t)}{\int_0^x \overline{F}(x-t)d\pi(t)} \\ &= \frac{\int_0^z \overline{F}(x+y-t)d\pi(t)}{\int_0^x \overline{F}(x-t)d\pi(t)}\,. \end{aligned}$$

By the Markov property, the question of existence of a limit law for the excursion is equivalent to the convergence to a proper limit of $V_0 - x$ in $P^{(x)}$–distribution. One might intuitively expect that the limit, if it exists, would either be proper or 0, but in fact it is a defective df:

Theorem 3.4. For all $y \ge 0$,

$$\lim_{x\to\infty} P^{(x)}(V_0 - x > y) \;=\; \lim_{x\to\infty} \nu^{(x)}((0,\infty),(y,\infty)) = \rho \overline{F}_I(y)\,, \quad y \ge 0\,.$$

□

To formulate this more precisely, decompose $\nu^{(x)}$ as

$$\nu^{(x)}(A,B) \;=\; \nu_1^{(x)}(A,B) + \nu_2^{(x)}(x-A,B)\,,$$

where

$$\nu_1^{(x)}(A,B) \;=\; P^{(x)}(V_{0-} \in A, V_{0-} \le x/2, V_0 - x \in B)\,,$$

$$\nu_2^{(x)}(A,B) \;=\; P^{(x)}(x - V_{0-} \in A, x/2 < V_{0-} < x, V_0 - x \in B)\,.$$

Thus we distinguish excursions which start below $x/2$ and between $x/2$ and x. Properties of the subexponential distributions lead to the following result.

Theorem 3.5. For any $a, y > 0$ the following holds.

(i) $\lim_{x\to\infty} \nu_1^{(x)}((a,\infty),(y,\infty)) = \lim_{x\to\infty} P^{(x)}(a < V_{0-} \le x/2, V_0 > x+y)$
$= (1-\rho)\overline{\pi}(a)\,;$

(ii) $\lim_{x\to\infty} \nu_2^{(x)}((a,\infty),(y,\infty)) = \lim_{x\to\infty} P^{(x)}(x/2 < V_{0-} \le x - a, V_0 > x+y)$
$= \rho \overline{F}_I(a+y)\,.$ □

Conclusion. Thus, asymptotically, the first type of excursion, given by $\nu_1^{(x)}$, has pre-level V_{0-} distributed according to π, and the excess is ∞. The second type, given by $\nu_2^{(x)}$, has pre-level such that $x - V_{0-}$ is distributed according to $\overline{F}_I$, and the conditional distribution of the excess given $V_{0-} = x - z$ is just the overshoot distribution $F^{(z)}(y) = 1 - P(X > y + z \mid X > x)$, $y > 0$.

(1) With probability $1-\rho$ the excursion starts from $V_{0-} = O(1)$ and the excess is huge. There is one indicated in Figure 1, the first one.

(2) With probability ρ the excursion starts from pre–level $x - V_{0-} = O(1)$ and the excess $V_0 - x$ has df F_I. There are two indicated in Figure 1. □

Another quantity of interest is the *queue length in system* (stationary number of customers in system), we denote it by L. In an M/G/1 queue under FIFO (first in first out) and when the nth customer's sojourn time D_n in the system (total time spent in the system from arrival to departure) is independent of future interarrival times, then Little's law holds in distribution, meaning that

$$L \stackrel{d}{=} N_D\,, \tag{3.7}$$

where (N_t) denotes a stationary version of the renewal counting process (with first arrival time distributed according to the equilibrium distribution) and D

denotes the stationary sojourn time. Notice that $D = W + X$ (independent sum) is the sum of the stationary waiting time and the service time. If the service time X is subexponential, then by Lemma 3.1 of [46] $\overline{F}(x) = o(\overline{F}_I(x))$, hence W dominates X in the sum (Proposition 2.7 of [46]) and

$$P(D > x) \sim P(W > x)\,, \quad x \to \infty\,.$$

For heavy-tailed service times, the following result has been proved in Asmussen, Klüppelberg and Sigman [5].

Theorem 3.6. Consider an M/G/1 queue with arrival rate $\lambda > 0$ and traffic intensity $\rho < 1$. Denote the service time df by F and assume that the equilibrium df $F_I \in \mathcal{S}$. Let W_∞ denote the stationary waiting time. Assume that

$$\lim_{x\to\infty} \frac{\overline{F}(xe^{y/\sqrt{x}})}{\overline{F}(x)} = 1\,, \qquad \text{locally uniformly in } y \in \mathbb{R}\,. \tag{3.8}$$

Then the stationary queue length L satisfies

$$P(L > k) \sim P(\lambda W_\infty > k) \sim \frac{\rho}{1-\rho}\overline{F}_I(k/\lambda)\,, \quad k \to \infty\,. \tag{3.9}$$

□

The extra condition (3.8) is a tail condition guaranteeing that the tail decreases to zero more slowly than the Weibull tail $\exp(-\sqrt{x})$, hence the result holds for any $\overline{F} \in \mathcal{R}(-\alpha)$ for $\alpha > 1$, lognormal df and Weibull distributions with tail $\overline{F}(x) \sim \exp(-x^\beta)$ for $\beta < 0.5$. In these cases the queue length becomes large only by a large service time. The Poisson arrivals do not contribute substantially to the queue length (only via the arrival rate in ρ).

When the service time is lighter than the tail of a Weibull distribution with parameter $\beta = 0.5$, the number of arriving customers comes into the picture as well. Then the combination of the number of customers and the likely large service time makes the queue-length large.

Example 3.7. (i) Let $\overline{F}(x) \sim \exp(-\sqrt{x})$, then

$$P(L > k) \sim \exp(1/(8\lambda)) \exp(-\sqrt{x/\lambda}) , \quad k \to \infty .$$

(ii) Let $\overline{F}(x) \sim \exp(-x^\beta)$ for $\beta \in (1/2, 2/3)$, then

$$P(L > k) \sim \exp\left(- \left(\frac{k}{\lambda}\right)^\beta + \frac{(1-\beta)\beta^2}{\lambda} \left(\frac{k}{\lambda}\right)^{2\beta - 1} \right) , \quad k \to \infty .$$

(iii) For $\beta > 2/3$ terms of higher order enter. □

4. Long-range dependence and heavy tails in teletraffic data

In recent years the question has been raised whether classical queueing and network models may not be too simplistic for modelling teletraffic networks. The heavy tails and dependence structure exhibited in explorative data analyses of teletraffic data cannot always be explained in the frame of such classical models, where heavy-tailed output is only possible by heavy-tailed input as we have seen in the preceding section. However, quite a variety of models has a regeneration structure leading to Lindley's equation (3.1) and hence, as in Section 3, to asymptotic results.

As an example consider the Asynchronous Transfer Mode (ATM) based broadband networks with statistical multiplexing (SMUX). Most of the multiplexed entities are calls originating from various sources. In order to operate properly, each of these calls has to satisfy some quality of service requirements (QoS). QoS requirements are usually bounds on performance measures characterising the dynamic behaviour of the multiplexed traffic. The most basic model of a SMUX is an infinite buffer single server queue with a work conserving scheduler. The fundamental performance measure is the tail of the stationary waiting time distribution $P(W > x)$.

Numerous investigations have shown that the arrival processes that arise in ATM networks (like voice and video) have a very complex statistical structure; an especially troublesome characteristic is the strong dependency. The modelling of this phenomenon usually leads to analytically very complex statistical characteristics, typically making the associated evaluation of the queue length distribution intractable. However, because of the stringent QoS requirements in ATM, the tail of the stationary waiting time distribution is needed in the domain of very small probabilities.

4.1. A simple on/off-model

A simple physical model is based on a sequence of points $(T_n)_{n\in\mathbb{N}}$ which constitute a stationary renewal process, i.e. $P(T_1 > t) = \overline{F}_I(t)$, $t \geq 0$, is the equilibrium df of F, where F is the df of all interarrival times after the first one. Assume further that the interarrival times are heavy-tailed in the sense that $F \in \mathcal{L}$, which implies that $\overline{F}(x) = o(\overline{F}_I(x))$ (see Lemma 3.1 of [46]).

Suppose that each of the interarrivals is either an on-period, where traffic is transmitted, or an off-period, where no traffic is transmitted. Assume furthermore that each interarrival is randomly chosen as on- or off-period by a Markov mechanism. Define the continuous-time fluid process $(A_t)_{t\geq 0}$ as being 1 during an on-period, and 0 during an off-period. The process (A_t) is sometimes called a "Markov chain embedded in a stationary renewal" process. This process gives rise to long-range dependence, as is seen from the following argument. Long-range dependence is defined by the property that $\operatorname{corr}(A_0, A_t)$ decreases to 0 more slowly than exponentially.

Theorem 4.1. Let $(T_n)_{n\in\mathbb{N}}$ be a stationary renewal point process whose interarrival times have df $F \in \mathcal{L}$ and $F_I \in \mathcal{S}$. Furthermore, let $(A_t)_{t\geq 0}$ be the embedded Markov chain with state space $E = \{0,1\}$, transition probabilities p_{ij} for $i,j = 0,1$, and stationary distribution π_i for $i = 0,1$. We consider the stationary version for $(A_t)_{t\geq 0}$. Then

$$\operatorname{cov}(A_0, A_t) \sim \operatorname{var}(A_0)\overline{F}_I(t)\,, \quad t \to \infty\,, \tag{4.1}$$

where F_I is the equilibrium df of F.

Proof

$$\begin{aligned}
\operatorname{cov}(A_0, A_t) &= E(A_0 A_t) - (EA_0)^2 \\
&= E[E(A_0 A_t | A_0)] - \pi_1^2 \\
&= \pi_1 P(A_t = 1 | A_0 = 1) - \pi_1^2 \\
&= \pi_1 \left(P(A_t = 1, T_1 > t | A_0 = 1) + \sum_{n=1}^{\infty} p_{11}^{(n)} P(T_n \le t < T_{n+1}) - \pi_1 \right) \\
&= \pi_1 \left(P(T_1 > t)(1 - \pi_1) + \sum_{n=1}^{\infty} (p_{11}^{(n)} - \pi_1) P(T_n \le t < T_{n+1}) \right) \qquad (4.2) \\
&\sim \pi_1 (1 - \pi_1) \overline{F}_I(t) \;=\; \operatorname{var}(A_0) \overline{F}_I(t), \quad t \to \infty .
\end{aligned}$$

For the estimation of the infinite sum in (4.2) we need several ingredients:

(i) Under very general conditions, there exist some $a > 0$ and some $\rho \in (0,1)$ such that for convergence of the $n$fold transition probabilities to the stationary distribution π the rate of convergence $|p_{ij}^{(n)} - \pi_j| \le a\rho^n$, $i, j \in E$, $n \in \mathbb{N}$, holds (see e.g. Lindvall [33]).

(ii) $P(T_n > t) = 1 - F_I * F^{(n-1)*}(t) \sim \overline{F}_I(t)$, $t \to \infty$, $n \in \mathbb{N}$, by Lemma 3.1 and Propositions 2.7 and 2.8 of [46] . This implies that $P(T_n \le t < T_{n+1}) = o(\overline{F}_I(t))$.

(iii) Furthermore, for all $\varepsilon > 0$ and all $n \in \mathbb{N}$ there exists some positive constant $K(\varepsilon)$ (independent of n) such that $1 - F_I * F^{(n-1)*}(t) \le K(\varepsilon)(1+\varepsilon)^n \overline{F}_I(t)$ for all $t \ge 0$ (cf. Remark 2.2(ii)).

(iv) Finally we use Lebesgue's dominated convergence theorem. □

This proof is a special case of the argument given in Jelenković and Lazar [26].

Remark 4.2. Observe that for large values of t, $\operatorname{corr}(A_0, A_t)$ is roughly proportional to the probability that the on-period that covers 0 is still active at t. The resulting distribution of the residual activity period has df F_I, i.e. has a heavier tail than the usual interarrival times. Consequently the autocorrelation function decreases like $\overline{F}_I$, implying long-range dependence. □

If F has a regularly varying tail, then we can apply Karamata's theorem and obtain the following.

Corollary 4.3. Let $\overline{F}(x) = x^{-\alpha}L(x)$, $x \geq 0$, for $\alpha > 1$ and $L \in \mathcal{R}(0)$. Then

$$\operatorname{cov}(A_0, A_t) \sim \operatorname{var}(A_0) \frac{1}{(\alpha - 1)\mu} t^{-(\alpha-1)} L(t) \,, \quad t \to \infty \,.$$

For $\alpha \in (1, 2)$ the autocorrelation function is not integrable. □

This model is a simple idealisation. It assumes that the tails for both, the on- and off-periods, have the same relative heaviness. As Willinger et al. [50] point out, this may not be consistent with telecommunication data. However, the example illustrates in a simple way how heavy tails can induce long-range dependence.

4.2. On/off models with different on-time and off-time distributions

In this section we present a generalisation of the above model with alternating on- and off-periods. We follow the presentation in Heath, Resnick and Samorodnitsky [23]. The non-negative iid rvs $(\xi_{\rm on}, \xi_n)_{n\in\mathbb{N}}$ represent the on-periods, and the non-negative iid rvs $(\eta_{\rm off}, \eta_n)_{n\in\mathbb{N}}$ the off-periods. On- and off-periods are assumed to be independent, the on-periods have common df $F_{\rm on}$, the off-periods have common df $F_{\rm off}$, both have finite mean $\mu_{\rm on}$ and $\mu_{\rm off}$ and we set $\mu = \mu_{\rm on} + \mu_{\rm off}$.

There exists a stationary renewal process with interarrival times distributed as $\xi_{\rm on} + \eta_{\rm off}$. This means that each renewal point is the starting point of an on-period, and each interarrival time consists of exactly one on- and one off-period. In a stationary version of the process we see in 0 either an on-period or an off-period. If we see an on-period, then an off-period follows before the renewal point T_1. If we see an off-period, then the renewal point T_1 follows immediately after this off-period. To capture the time interval $[0, T_1)$ we define independent rvs ξ_I, η_I, and B independent of $(\eta_{\rm off}, \xi_n, \eta_n)_{n\in\mathbb{N}}$, where ξ_I has df $F_{{\rm on},I}(x) = (1/\mu_{\rm on}) \int_0^x \overline{F}_{\rm on}(y)dy$, η_I has df $F_{{\rm off},I}(x) = (1/\mu_{\rm off}) \int_0^x \overline{F}_{\rm off}(y)dy$ and B

is a Bernoulli rv with success probability $P(B = 1) = \mu_{\text{on}}/\mu$. Then the stationary situation is modelled by

$$T_1 = B(\xi_I + \eta_{\text{off}}) + (1 - B)\eta_I\,.$$

The corresponding on/off process $(A_t)_{t\geq 0}$ which is equal to 1 if t falls in an on-period and 0 if t falls in an off-period can be defined in terms of $(T_n)_{n\in\mathbb{N}}$ as follows:

$$A_t = BI_{[0,\xi_I)}(t) + \sum_{n=1}^{\infty} I_{[T_n, T_n+\xi_{n+1})}(t)\,, \quad t > 0\,.$$

Thus, if $t \geq T_1$,

$$A_t = \begin{cases} 1 & \text{if } T_n \leq t < T_n + \xi_{n+1}\,, \\ 0 & \text{if } T_n + \xi_{n+1} \leq t < T_{n+1}\,, \end{cases}$$

while for $t \in [0, T_1)$ we have

$$A_t = \begin{cases} 1 & \text{if } B = 1 \text{ and } 0 \leq t < \xi_I\,, \\ 0 & \text{otherwise}\ . \end{cases}$$

With this construction, (A_t) is strictly stationary ((A_t) inherits the stationarity from the stationary renewal sequence (T_n)). Moreover, $P(A_t = 1) = \mu_{\text{on}}/\mu$. To see this, write

$$P(A_t = 1) = EA_t = P(B = 1)P(\xi_I > t) + \sum_{n=1}^{\infty} P(T_n \leq t < T_n + \xi_{n+1})\,. \quad (4.3)$$

Recall that the renewal function of the stationary sequence (T_n) is equal to

$$U(t) = \sum_{n=1}^{\infty} P(T_n \leq t) = \frac{t}{\mu}\,, \quad t > 0\,.$$

Now we can evaluate the infinite sum in (4.3) as

$$\begin{aligned} \sum_{n=1}^{\infty} \int_0^t \overline{F}_{\text{on}}(t-u)dP(T_n \leq u) &= \int_0^t \overline{F}_{\text{on}}(t-u)dU(u) \\ &= \frac{1}{\mu}\int_0^t \overline{F}_{\text{on}}(t-u)du = \frac{\mu_{\text{on}}}{\mu}F_{\text{on},I}(t)\,, t > 0\,. \end{aligned}$$

Hence,

$$EA_t = \frac{\mu_{\mathrm{on}}}{\mu}(P(\xi_I > t) + P(\xi_I \leq t)) = \frac{\mu_{\mathrm{on}}}{\mu}.$$

The main theorem in [23] describes the autocovariance function of the process $(A_t)_{t\geq 0}$.

Theorem 4.4. Assume that $\overline{F}_{on}(t) = t^{-\alpha}L(t)$, $t \geq 0$, where $L \in \mathcal{R}(0)$ and $\alpha \in (1,2)$. Assume also that $\overline{F}_{off}(t) = o(\overline{F}_{on}(t))$ as $t \to \infty$ and that $\xi_{\mathrm{on}} + \eta_{\mathrm{off}}$ is non-degenerate. Then

$$\mathrm{cov}(A_0, A_t) \sim \frac{\mu_{off}^2}{(\alpha-1)\mu^3} t^{-(\alpha-1)} L(t), \quad t \to \infty. \qquad \square$$

The proof in [23] is based on the following representation of the autocovariance function:

$$\mathrm{cov}(A_0, A_t) = c(\lim_{x\to\infty} z * U(x) - z * U(t)),$$

where $c > 0$ is a constant, $U = \sum_{n=0}^{\infty}(F_{on} * F_{off})^{n*}$ is the renewal function to the df $F_{on} * F_{off}$ and $z(t) = \int_0^t \overline{F}_{on}(t-u)F_{off}(t)du$.

The essential argument of the rather technical proof relies on the rate of convergence in Smith's key renewal theorem (see [16] or [41]) for heavy-tailed interarrival times.

Define by

$$A_t^* = \int_0^t A_u du, \quad t \geq 0,$$

the cumulative input to the system up to time t. Since $EA_t = \mu_{\mathrm{on}}/\mu$, we have by the SLLN $A_t^*/t \to \mu_{\mathrm{on}}/\mu$ *a.s.* as $t \to \infty$. Assume the system has a constant release rate $r > 0$ if the buffer is not empty. For stability we require $\mu_{\mathrm{on}}/\mu < r < 1$ (recall that 1 is the input rate of traffic into the system). If the buffer is empty, we set $r = 0$. The release rate of the system when the buffer content has level x is then $r(x) = r$ or 0 according as $x > 0$ or $x = 0$.

Define the *buffer contents process* $(V_t)_{t\geq 0}$ (which corresponds to the workload process) by the stochastic differential equation

$$dV_t = dA_t^* - r(V_t)dt\,, \tag{4.4}$$

for given initial rv V_0. During an on-period traffic enters at net rate $1-r$ and during an off-period the buffer content is released at rate r. The (T_n) are regeneration times of the contents process $(V_t)_{t\geq 0}$ which is stationary ergodic. Consider the change in the buffer between T_n and T_{n+1}. We see that

$$V_{T_{n+1}} = (V_{T_n} + (1-r)\xi_{n+1} - r\eta_{n+1})^+\,, \quad n \in \mathbb{N}\,, \tag{4.5}$$

where the increments have mean

$$E\big((1-r)\xi_{n+1} - r\eta_{n+1}\big) = (1-r)\mu_{\mathrm{on}} - r\mu_{\mathrm{off}} = \mu_{\mathrm{on}} - r\mu < 0\,.$$

Hence V_{T_n} satisfies Lindley's equation (3.1) and the limit variable is determined by

$$V_{T_n} \xrightarrow{d} V_{T_\infty} = \max_{n\geq 1} \sum_{i=1}^{n} ((1-r)\xi_{n+1} - r\eta_{n+1})\,. \tag{4.6}$$

In a subexponential regime the tail behaviour of the stationary waiting time distribution is given by Theorem 3.2. The rvs $(1-r)\xi_n$ correspond to the service times in this theorem. The constant ρ is given by

$\rho =$ rate of the arrival process / rate of the service process $= (1-r)\mu_{\mathrm{on}}/(r\mu_{\mathrm{off}})$.

Now we can reformulate Theorem 3.2 in our context and obtain the following result.

Proposition 4.5. Let π denote the df of V_{T_∞} and set $\rho = (1-r)\mu_{\mathrm{on}}/(r\mu_{\mathrm{off}})$. Then

$$\pi \in \mathcal{S} \iff F_{\mathrm{on},I} \in \mathcal{S} \Longrightarrow \overline{\pi}(x) \sim \frac{\rho}{1-\rho}\overline{F}_{\mathrm{on},I}\left(\frac{x}{1-r}\right)\,, \quad x \to \infty\,. \qquad \square$$

If F_{on} has regularly varying tail, then we can apply Karamata's theorem and obtain the following.

Corollary 4.6. Let $\overline{F}_{\text{on}}(x) = x^{-\alpha}L(x)$, $x \geq 0$, for $\alpha > 1$ and $L \in \mathcal{R}(0)$. Then

$$\overline{\pi}(x) \sim \frac{\rho}{1-\rho} \frac{(1-r)^{\alpha-1}}{\mu_{\text{on}}(\alpha-1)} x^{-(\alpha-1)} L(x) =: bx^{-(\alpha-1)} L(x)\,, \quad x \to \infty\,. \qquad \square$$

The buffer content process has its cycle maxima not at the points (T_n), but at the points $(T_n + \xi_n)$, hence the following result is not surprising. It shows that the distributional limit of V_t has even a heavier tail than that of V_{T_n}.

Proposition 4.7. Let $\overline{F}_{\text{on}}(x) = x^{-\alpha}L(x)$, $x \geq 0$, for $\alpha > 1$ and $L \in \mathcal{R}(0)$. Define b as in Corollary 4.6. Then $V_t \xrightarrow{d} V_\infty$ and

$$P(V_\infty > x) \sim \left(b + \frac{(1-r)^{\alpha-1}}{\mu(\alpha-1)} \right) x^{-(\alpha-1)} L(x)\,, \quad x \to \infty\,. \qquad \square$$

Remark 4.8. The buffer contents process as above has already been considered in Boxma [10], also in the context of telecommunication traffic. He requires (as Heath et al. [23] do) regularly varying tails for F_{on}. Boxma's model is more general in the sense that he allows for several ($N \in \mathbb{N}$) sources transmitting data to one buffer. But as he states, the analytical treatment becomes then rather complicated. Apart from the case of one source, he treats some examples for $N > 1$. The mean time to buffer overflow (as the buffer size tends to infinity) in a multisource model is given in [44]. $\square$

4.3. Dependent traffic transmission

The connections between heavy tails and long-range dependence may furthermore arise from a more complicated mechanism than in the above simple models. Various suggestions have been made, and we try to explain some of them within the queueing context.

Markov modulated queues

For modelling subexponential correlated arrivals, Jelenković and Lazar [25, 26] introduce the class of general "Markov chain embedded in a stationary renewal" processes. Consider the arrival sequence $(T_n)_{n\in\mathbb{N}}$ with iid subexponential interarrival times with df F. We take a stationary version by choosing the equilibrium df F_I for the first interarrival time T_1. Let J_n be an irreducible aperiodic Markov chain with finite state space E, transition matrix P and stationary distribution π. Now we construct the process

$$A_t = J_n \,, \quad T_n \le t < T_{n+1} \,.$$

The process (A_t) describes not only on- and off-periods, but allows for instance for different traffic transmission rates to and release rates from the buffer in different intervals $[T_n, T_{n+1})$. The environment $J_t = i$ for $i \in E$ corresponds to a certain df of the buffer input process $F_{\text{on},i}$ and the buffer output process $F_{\text{off},i}$. Moreover, buffer input and buffer output in $[T_n, T_{n+1})$ are conditionally independent, given J_{n-1} and J_n.

In general, it would be rather difficult to construct a stationary version of the process $(A_t)_{t\ge 0}$ as done for the simple case above, where $E = \{0,1\}$. Stationarity of $(J_n)_{n\in\mathbb{N}}$, however, guarantees stationarity of $(A_t)_{t\ge0}$ provided an overall stability condition holds. For $i \in E$ let $F_{\text{on},i}$ have finite mean μ_i and $F_{\text{off},i}$ finite mean $1/\lambda_i$. Denote by $\pi = (\pi_i)_{i\in E}$ the stationary distribution of $(J_i)_{i\in E}$. For stability we require that $\rho = \sum_{i\in E} \pi_i \lambda_i \mu_i < 1$.

A first result states that, due to the heavy-tailed interarrival times, the process $(A_t)_{t\ge0}$ approaches its stationary distribution π only at a very slow rate.

Proposition 4.9. Let $(T_n)_{n\in\mathbb{N}}$ be a stationary renewal point process whose interarrival times have df $F \in \mathcal{L}$ and $F_I \in \mathcal{S}$. Define $(A_t)_{t\ge0}$ as above and denote $P_i(\cdot) = P(\cdot | A_0 = i)$. Then

$$P_i(A_t = j) - \pi_j \sim (\delta_{ij} - \pi_j)\overline{F}_I(t) \,, \quad t \to \infty \,.$$

Proof Denote $\delta_{ij} = 1$ for $i = j$ and 0 otherwise. Then

$$\begin{aligned}
P_i(A_t = j) - \pi_j &= P(A_t = j | A_0 = i) - \pi_j \\
&= (P(A_t = j, T_1 > t | A_0 = i) - \pi_j P(T_1 > t)) \\
&\qquad +(P(A_t = j, T_1 \le t | A_0 = i) - \pi_j P(T_1 \le t)) \\
&= P(T_1 > t)(\delta_{ij} - \pi_j) + \sum_{n=1}^{\infty} (p_{ij}^{(n)} - \pi_j) P(T_n \le t < T_{n+1}) \\
&\sim (\delta_{ij} - \pi_j)\overline{F}_I(t) , \quad t \to \infty .
\end{aligned}$$

The last line follows exactly as (4.2). □

Another characteristic of these processes is that their autocovariance function $\mathrm{cov}(A_t, A_0)$ is asymptotically proportional to the tail of the equilibrium df of the interarrival times.

Proposition 4.10. If $F \in \mathcal{L}$ and $F_I \in \mathcal{S}$, then

$$\mathrm{cov}(A_0, A_t) \sim \mathrm{var}(A_0)\overline{F}_I(t) , \quad t \to \infty .$$

Proof We use Proposition 4.9 and obtain

$$\begin{aligned}
\mathrm{cov}(A_0, A_t) &= E(A_0 A_t) - (EA_0)^2 \\
&= E[E(A_0 A_t | A_0)] - (EA_0)^2 \\
&= \sum_{i,j \in E} a_i a_j (\pi_i P_i(A_t = a_j) - \pi_i \pi_j) \\
&\sim \sum_{i,j \in E} a_i a_j \pi_i (\delta_{ij} - \pi_j) \overline{F}_I(t) \\
&= \left(\sum_{i \in E} \pi_i a_i^2 - \sum_{i,j \in E} \pi_i \pi_j a_i a_j \right) \overline{F}_I(t) \\
&= \mathrm{var}(A_0)\overline{F}_I(t) , \quad t \to \infty .
\end{aligned}$$

□

In this situation, with initial rv W_0, Lindley's equation holds, i.e.

$$W_{n+1} = (W_n + \xi_n - \eta_n)^+ , \quad n \geq 0 .$$

Here (ξ_n, η_n) denotes the pair of buffer input and buffer output during the time interval $[T_n, T_{n+1})$. Recall that (ξ_n) and (η_n) are conditionally independent, given J_{n-1} and J_n, but their dfs change with the environment.

As is immediate from Lindley's equation, η_n can be interpreted as the interarrival time between customer n and $n+1$, and ξ_n can be considered as the nth customer's service requirement in a Markov modulated queue.

As proved in Asmussen, Fløe Henriksen and Klüppelberg [3], if at least for one $i \in E$ the df $F_{\text{on},i}$ is subexponential, then Theorem 3.2 extends to the Markov modulated model in the sense that the subexponential environment determines the asymptotic behaviour of the stationary waiting time.

Theorem 4.11. Assume that for all $i \in E$,

$$\lim_{x\to\infty} \frac{\overline{F}_{\text{on},i}(x)}{\overline{H}(x)} = c_i$$

for some df $H \in \mathcal{L}$, $H_I \in \mathcal{S}$ and constants $c_i \in [0, \infty)$. Assume furthermore that

$$c = \sum_{i\in E} \pi_i \lambda_i c_i > 0 .$$

Then, for $i \in E$,

$$P_i(W_\infty > t) \sim \frac{c}{1-\rho} \int_t^\infty \overline{H}(u)\, du , \quad t \to \infty . \qquad \square$$

The independence of the proportionality constant of the initial state $i \in E$ has been shown by Jelenković and Lazar [26]. The result generalises to semi-Markov models as is shown in Asmussen, Schmidli and Schmidt [6].

Exponential mixture model

Can light-tailed input also cause heavy-tailed output? We answer this question by the following example taken from Greiner, Jobmann and Lipsky [21].

Let X_n be an exponential rv with mean γ^n for some $\gamma > 1$. Let furthermore N be a geometric rv with parameter $\theta > 0$. Then for $x \geq 0$

$$P(X_N > x) = (1-\theta)\sum_{n=1}^{\infty} \theta^{n-1} P(X_n > x)$$
$$= (1-\theta)\sum_{n=1}^{\infty} \theta^{n-1} e^{-x/\gamma^n}\,.$$

Notice first that for $k \in \mathbb{N}$

$$EX_N^k = \frac{1-\theta}{\theta}\int_0^\infty \sum_{n=1}^{\infty} \theta^n \gamma^{-n} x^k e^{-x/\gamma^n}\,dx$$
$$= \frac{1-\theta}{\theta}\int_0^\infty \sum_{n=1}^{\infty} \theta^n \gamma^{nk} x^k e^{-x}\,dx$$
$$= \frac{1-\theta}{\theta}\sum_{n=1}^{\infty} \theta^n \gamma^{nk} \mu_k\,,$$

where μ_k is the kth moment of a standard exponential rv. Notice that the infinite series on the rhs is finite if and only if $\theta\gamma^k < 1$ or, equivalently, if

$$k < -\frac{\ln\theta}{\ln\gamma} =: \alpha\,.$$

Take (X_n) as an iid sequence of service times in a GI/G/1 model and $X_1 \stackrel{d}{=} X_N$. If we choose $k \geq -\ln\theta/\ln\gamma$, then the moment of order k is infinite, hence we are in a heavy-tailed regime. However, if we truncate the geometric rv N, say at $\nu \in \mathbb{N}$, and denote by $X_{N(\nu)}$ the corresponding rv, then

$$P(X_{N(\nu)} > x) = \frac{1-\theta}{1-\theta^\nu}\sum_{n=1}^{\nu} \theta^{n-1} P(X_n > x)\,, \quad x \geq 0\,.$$

This rv has finite exponential moments, and we are in a light-tailed regime, so that the tail of the stationary waiting time distribution decreases to zero exponentially fast. Notice that $X_{N(\nu)} \stackrel{\text{a.s.}}{\to} X_N$ as $\nu \to \infty$ and hence a result of Borovkov [9], p.118, applies for $\alpha > 1$; see also Feldmann and Whitt [15]. The queueing models GI/G$_\nu$/1 (with service times distributed as $X_{N(\nu)}$) converge weakly to the original model GI/G/1. Then the stationary waiting time distribution for the truncated model converges to the stationary waiting time distribution of the original model. This means that, by truncation, we approximate a heavy-tailed regime by a light-tailed one.

In Greiner et al. [21] such an approximation has been applied, not to the service times, but to the interarrival times. They show for small traffic intensity ρ that the mean of the queue length may become arbitrarily large for α close to 1 where truncation points are as low as $\nu = 10$. For more details see [21].

Furthermore, by the use of these truncated rvs $X_{N(\nu)}$ instead of X_N analytic evaluation techniques based on matrix algebraic approaches (see e.g. Lipsky [34] or Neuts [36]) can be applied to those queueing models for large degrees of truncations (close to the heavy-tailed regime in the above sense).

Remark 4.12. In a similar spirit, though much more sophisticated, Resnick and Samorodnitsky [43] suggest a stationary sequence of light-tailed interarrivals, which are not independent, quite contrary, their sequence has a long-range dependence pattern. The structure is determined by a sequence (ξ_n) of interarrival times which is a moving average process built from the increments of the gamma process Y:

$$\xi_n = \sum_{j=0}^{\infty} (Y(j-n+a_j) - Y(j-n)) \,, \quad n \in \mathbb{N} \,,$$

where (a_j) is a sequence of non-negative numbers such that $\sum_{j=0}^{\infty} a_j = 1$. It can be shown that

$$\xi_k \stackrel{d}{=} \sum_{i=0}^{k} \sum_{j=k}^{n-1} \Gamma(i,j) \,, \quad k = 0, \ldots, n-1 \,,$$

where $\Gamma(i,j)$ are independent Gamma rvs, all with scale parameter 1 and shape parameter given in terms of (a_j). Then it can be shown that

$$\operatorname{cov}(\xi_m, \xi_{m+n}) = \sum_{i=n+1}^{\infty} a_i \,, \quad n \in \mathbb{N} \,.$$

Furthermore, if a_n decreases to 0 slowly enough, the process (ξ_n) has long-range dependence. $\square$

Remark 4.13. A more realistic model assumes that the buffer receives input from several access lines and is emptied by a single high-speed backbone communication link. There are various approaches to describe this model and give meaningful performance measures. In [22] the total input rate at time t is given by the sum of the input rates of the different sources, each described as an on/off process as in Section 4.2. As a performance measure the mean time to buffer overflow is determined.

Choudhury and Whitt [11] follow a different approach. They base a fluid model on a environment process (A_t) with finitely many states, as for the Markov modulated model of Section 4.3. When $A_t = i$, there is a constant net fluid flow into the buffer at rate r_i. Then bounds can be found e.g. for the distribution tail of the stationary buffer content. □

4.4. *Selfsimilarity*

Another important feature of network traffic has been observed from the data, which we cannot present in detail in this paper. Since it is an important observation we want at least to mention it and give further references on the topic. Network traffic looks roughly the same when measured over a large range of time scales ranging from milliseconds to minutes to hours to days etc. (see especially [50]). Data traffic of this type is said to be *selfsimilar*.

One of the models in teletraffic for capturing the observed selfsimilarity is fractional Brownian motion, the only selfsimilar Gaussian process with stationary increments (see e.g. Samorodnitsky and Taqqu [45]). This process may appear as the weak limit process of the aggregated workload of a sequence of on/off models, i.e. during on-periods a source is active, traffic is transmitted, during off-periods no transmission happens. First letting the number of sources tend to ∞ and then time, the limit process is fractional Brownian motion (see Taqqu, Willinger and Sherman [48], Taqqu and Levy [49]). This is an extension of a model first introduced by Mandelbrot [35].

Another class of models, also with a selfsimilar limit process, has been considered by Kurtz [31] as an alternative to the classical workload process, where each arrival brings at the instant of its arrival a required random service time into the system (see Section 3). Assume that the input of work into the system comes from a large number of sources. Each source turns on at a random time and inputs work into the system for some period of time. We assume that the time-points, where on-times start, constitute a Poisson process $(N_t)_{t\geq 0}$. While a source is active it feeds work into the system, and we model the workprocess of the ith activation by a stochastic process $(X_t^{(i)})_{t\geq 0}$ with cadlag and non-decreasing sample paths. We furthermore assume that the stochastic processes $(X_t^{(i)})$ are iid and independent of (N_t).

Then the total work input into the system up to time t is given by the shot noise process

$$V_t = \sum_{i=1}^{N_t} X_{t-T_i}^{(i)}, \quad t \geq 0, \tag{4.7}$$

where $T_1, T_2, \ldots$ denote the points of the process N, i.e. the points when a source becomes active. When the ith source becomes inactive, the process $X_t^{(i)}$ remains constant from this time on, hence does not increase the workload in the system any more.

Such models have also been considered in Klüppelberg and Mikosch [28,29] and in Klüppelberg and Schärf [30] within the framework of insurance mathematics. These models give rise to selfsimilar limits, under appropriate moment conditions to selfsimilar Gaussian processes ([28,29]) or to selfsimilar stable limits ([30]).

5. An explorative data analysis

The appropriateness of on/off models is shown by an application to telecommunication traffic observed at a network access point (Customer Service Switch) of the Munich Universities' intranet to the Germany wide broadband research network (B-WiN) at the Leibniz Research and Supercomputing Center of the Bavarian Academy of Sciences. For more details on how the data were collected see Gogl [19].

The outgoing pure IP (Internet Protocol) data traffic is presented in Figure 2. From the frequency and size of the peaks the data seem to be heavy-tailed. This opens the way to fit models as presented in this paper.

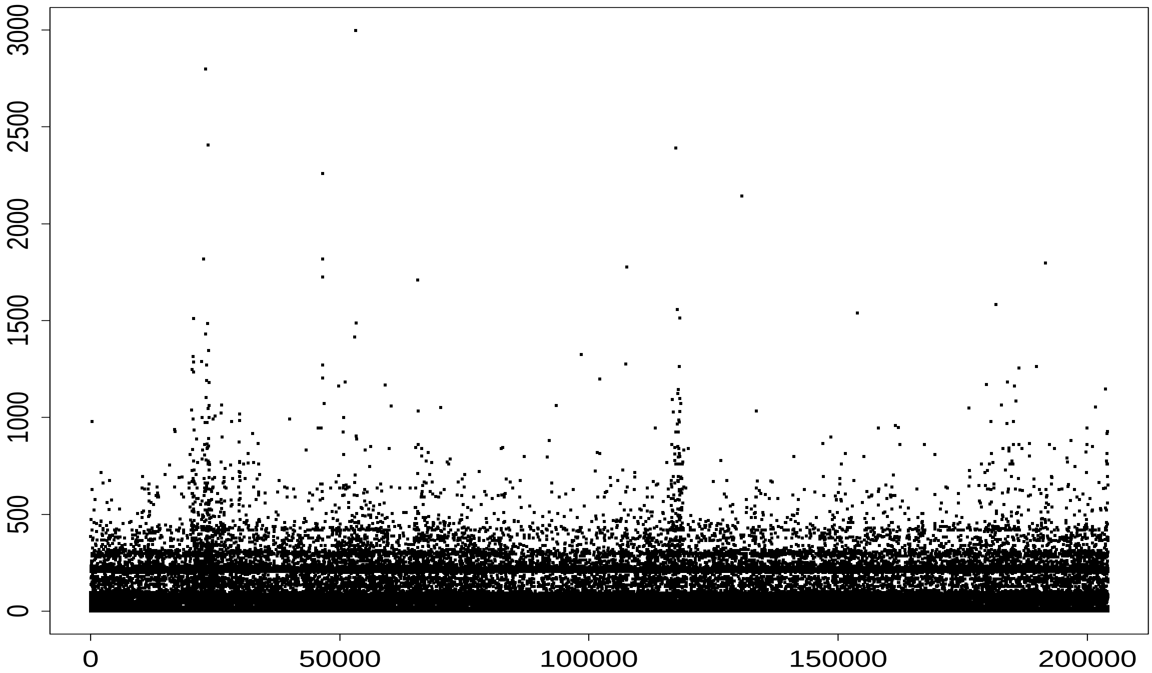

Figure 2. Length of on-periods in μs; i.e. lengths of cell bursts, extracted from a total of 1 690 730 ATM cells. This comprises 247 995 IP packets which were captured within approximately two minutes in the afternoon of December 23, 1997.

Various graphical tools are presented in the literature to investigate the tail behaviour of a distribution. For details we refer to Section 6.2 of [13]. For such methods applied to telecommunication data see Resnick [42].

An excellent explorative method to discriminate distribution tails is the *mean excess function*. For a non-negative rv X it is defined as

$$e(u) = E(X - u | X > u)\,, \quad u > 0\,.$$

This function is constant for exponential X, it tends to 0 for light-tailed dfs and to ∞ for heavy-tailed dfs. The Pareto distribution is characterised by a linearly increasing mean excess function.

Figure 3 shows the empirical mean excess function of the on- and off-periods of the above traffic. A Pareto tail (or regularly varying tail) seems to fit extremely well to the on-periods, whereas the off-periods have apparently a lighter tail than the on-periods.

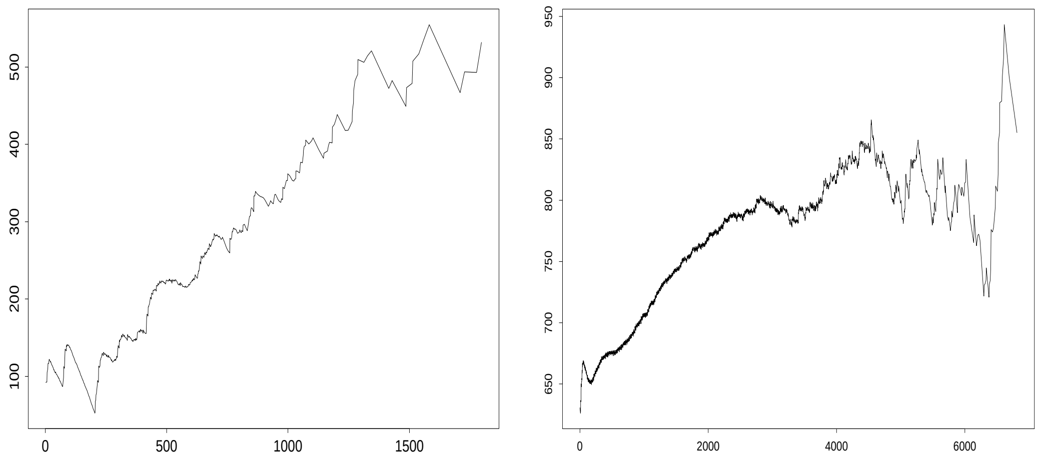

Figure 3. Mean excess function of the on-periods (lhs) and off-periods (rhs).

This indicates that an on/off model as presented in Section 4 may be appropriate for our data. Notice that for the moment we ignore the clusters around the peaks and assume that the lenghts of on-periods can be modelled by iid rvs. We investigate the corresponding fluid process $(A_t)_{t\geq 0}$, which is defined as 1 during an on-period and 0 during an off-period. We estimate the autocorrelation function of this process for our data. The result is shown in Figure 4, where we plot the estimated autocorrelation function from lag 5 to lag 7 000. We only start from lag 5 to make the function visible at larger lags (of course the estimated autocorrelation at lag 0 is equal to 1). Comparing the estimated autocorrelation function to the 95% confidence bounds, we see a long-range dependence effect.

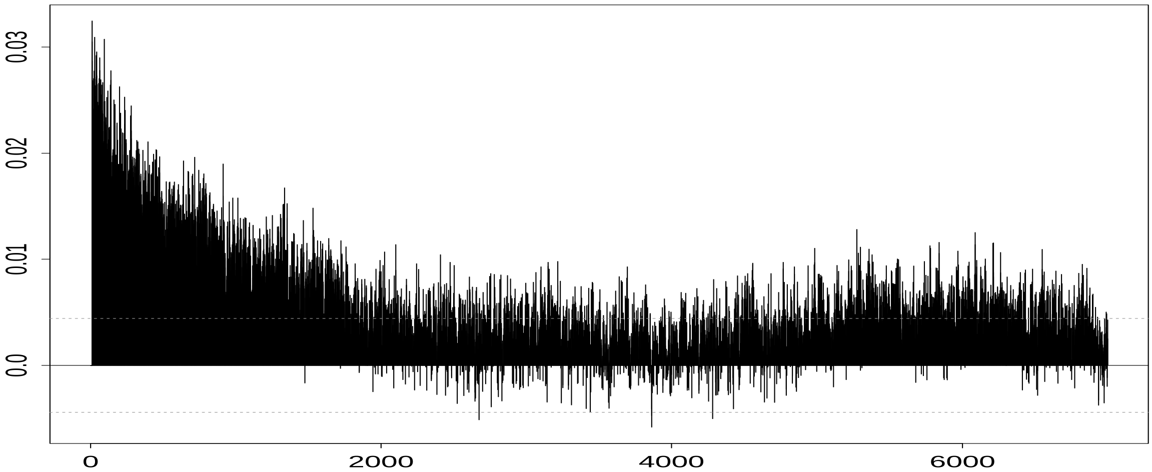

Figure 4. The empirical autocorrelation function of $(A_t)_{t\geq 0}$ from lag 5 up to lag 7 000.

We plot the estimated autocorrelation function on log-scale, i.e. we use logarithmic axes (to the basis 10). Now we estimate the slope of the curve by linear regression for the lags in the interval $(20, 7\,000)$. The slope is estimated by -0.82, hence we estimate

$$\widehat{\operatorname{corr}(A_0, A_t)} = c\, t^{-0.82}\,,$$

for some positive constant c. Notice that for this model we have estimated an autocorrelation function, which is heavy-tailed and not integrable over t.

We should like to say that this first very crude model fitting is by no means the end of the story. On the contrary, we only consider it as a promising start.

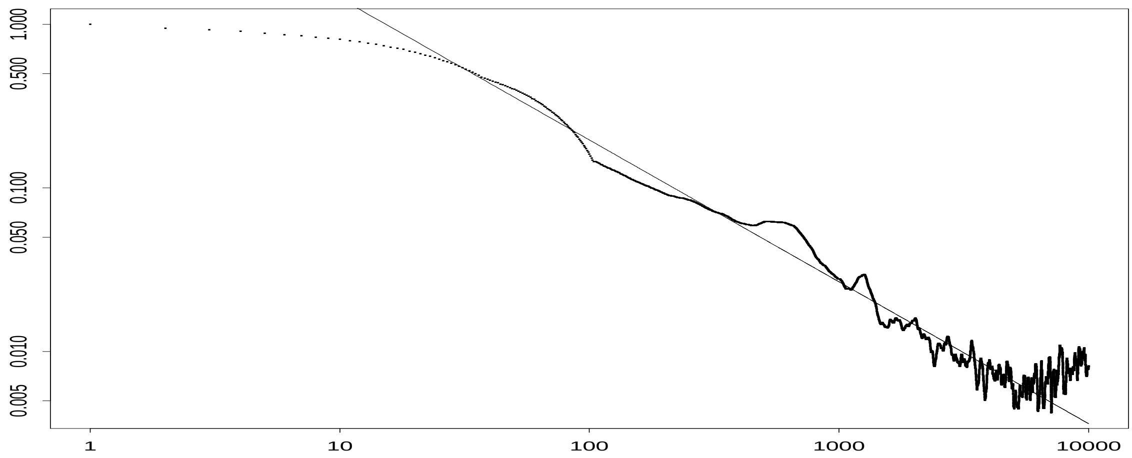

Figure 5. The log-log plot of the empirical autocovariance function of (A_t). Both axes have been log-transformed to the basis 10.

The data exhibit many structural features which have to be explored in much greater detail. We only mention here the clusters around the peaks in Figure 2. The data will be analysed and modelled much more carefully and the results will be presented in future work.

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References

[1] S. Asmussen. *Applied Probability and Queues.* Wiley, New York, 1987.

[2] S. Asmussen. Subexponential asymptotics for stochastic processes: extremal behaviour, stationary distributions and first passage probabilities. *Ann. Appl. Probab.*, 8:354–374, 1998.

[3] S. Asmussen, L. Fløe Henriksen, and C. Klüppelberg. Large claims approximations for risk processes in a Markovian environment. *Stoch. Proc. Appl.*, 54:29–43, 1994.

[4] S. Asmussen and C. Klüppelberg. Stationary M/G/1 excursions in the presence of heavy tails. *J. Appl. Probab.*, 34:208–212, 1997.

[5] S. Asmussen, C. Klüppelberg, and K. Sigman. Sampling at subexponential times, with queueing applications. *Stoch. Proc. Appl.* 1999. To appear. Preprint available at http://www.ma.tum.de/stat/Papers/papers.html.

[6] S. Asmussen, H. Schmidli, and V. Schmidt. Tail probabilities for non-standard risk and queueing processes with subexponential jumps. *Adv. Appl. Probab.* **31**, 1999. To appear.

[7] J. Beran, R. Sherman, M.S. Taqqu, and W. Willinger. Long-range dependence in variable-bit-rate video traffic. *IEEE Transactions on Communications*, 43, 1995.

[8] N.H. Bingham, C.M. Goldie, and J. Teugels. *Regular Variation.* Oxford University Press, 1987.

[9] A.A. Borovkov. *Stochastic Processes in Queueing Theory.* Springer, New York, 1976.

[10] O.J. Boxma. Fluid queues and regular variation. *Performance Evaluation*, 29:699–712, 1996.

[11] G.L. Choudhury and W. Whitt. Long-tail buffer-content distributions in broadband networks. *Performance Evaluation*, 30:177–190, 1997.

[12] J.W. Cohen. Some results on regular variation for distributions in queueing and fluctuation theory. *J. Appl. Probab.*, 10:343–353, 1973.

[13] P. Embrechts, C. Klüppelberg, and T. Mikosch. *Modelling Extremal Events for Insurance and Finance.* Springer, Berlin, 1997.

[14] P. Embrechts and N. Veraverbeke. Estimates for the probability of ruin with special emphasis on the possibility of large claims. *Insurance: Mathematics and Economics*, 1:55–72, 1982.

[15] A. Feldmann and W. Whitt. Fitting mixtures of exponentials to long-tailed distributions to analyze network performance models. *Performance Evaluation*, 31:247–278, 1998.

[16] W. Feller. *An Introduction to Probability Theory and its Applications, Vol. II.* Wiley, New York, 1971.

[17] R. Garg. Characterization of video traffic. Technical report, International Computer Science Institute, Berkeley, CA, 1947 Center Street, Berkeley, CA 94704-1198, 1995.

[18] M. W. Garret and W. Willinger. Analysis, modeling and generation of self-similar VBR video traffic. In *ACM SigComm'94.* ACM, London, September 1994.

[19] H. Gogl. Data acquisition and data analysis in ATM based networks. Tech-

nical report, Department of Computer Science. Munich University of Technology, D-80290 Munich, Germany, 1998. http://wwwjessen.informatik.tu-muenchen.de/forschung/leistung/ATM_Project/ atm_index.html.

[20] C. M. Goldie and C. Klüppelberg. Subexponential Distributions. In R. Adler, R. Feldman, and M.S. Taqqu, editors, *A Practical Guide To Heavy Tails: Statistical Techniques for Analyzing Heavy Tailed Distributions*, pages 435–459. Birkhäuser, Boston, 1998.

[21] M. Greiner, M. Jobmann, and L. Lipsky. The importance of power-tail distributions for modeling queueing systems. Operations Research: Telecommunications Area. To appear, 1999.

[22] D. Heath, S. Resnick, and G. Samorodnitsky. Patterns of buffer overflow in a class of queues with long memory in the input stream. *The Annals of Applied Probability*, 7:1021–1057, 1997.

[23] D. Heath, S. Resnick, and G. Samorodnitsky. Heavy tails and long range dependence in on/off processes and associated fluid models. Technical report, 1998.

[24] P. R. Jelenković and A. A. Lazar. Asymptotic results for multiplexing subexponential on-off sources. Submitted for publication to Adv. Appl. Probab., 1999.

[25] P.R. Jelenković and A.A. Lazar. A network multiplexer with multiple time scale and subexponential arrivals. In *Stochastic Networks, Stability and Rare Events*, pages 215–235. Lecture Notes in Statistics, Springer, New York, 1996.

[26] P.R. Jelenković and A.A. Lazar. Subexponential asymptotics of a markov-modulated random walk with queueing applications. *J. Appl. Probab.*, 35, 1998. To appear.

[27] C. Klüppelberg. Subexponential distributions and integrated tails. *J. Appl. Probab.*, 25:132–141, 1988.

[28] C. Klüppelberg and T. Mikosch. Delay in claim settlement and ruin probability approximations. *Scand. Actuar. J.*, pages 154–168, 1995.

[29] C. Klüppelberg and T. Mikosch. Explosive poisson shot noise processes with applications to risk reserves. *Bernoulli*, 1:125–147, 1995.

[30] C. Klüppelberg and A. Schärf. Stable limits of poisson shot noise processes. Preprint, 1997. Center for Mathematical Sciences, Munich University of Technology, D-80290 Munich, Germany. http://www.ma.tum.de/stat/Papers/papers.html.

[31] T. Kurtz. Limit theorems for workload input models. In F.P. Kelly, S. Zachary, and I. Ziedins, editors, *Stochastic Networks: Theory and Applications.* Royal Statistical Society Lecture Note Series 4, 1996.

[32] W. E. Leland, M. S. Taqqu, W. Willinger, and D. V. Wilson. On the self-similiar nature

of ethernet traffic (extended version). *IEEE/ACM Transactions on Networking*, 2(1):1–15, February 1994.

[33] T. Lindvall. *Lectures on the Coupling Method.* Wiley, New York, 1992.

[34] L. Lipsky. *Queueing Theory: A Linear Algebraic Approach.* MacMillan and Company, New York, 1992.

[35] B.B. Mandelbrot. Self-similar error clusters in communication systems and the concept of conditional stationarity. *IEEE Transactions on Communictions Technology*, 13:71–90, 1965.

[36] M. F. Neuts. *Matrix-Geometric Solutions in Stochastic Models. An Algorithmic Approach.* Johns Hopkins Series in the Mathematical Sciences. The Johns Hopkins University Press, Baltimore, 1981.

[37] A.G. Pakes. On the tails of waiting time distributions. *J. Appl. Probab.*, 12:555–564, 1975.

[38] K. Park, G. Kim, and M. Crovella. On the relationship between file sizes, transport protocols, and self-similar network traffic. Technical Report BU-CS-96-016, Computer Science Department, Boston University, 1996.

[39] K. Park, G. Kim, and M. Crovella. On the effect of traffic self-similarity on network performance. In *Proceedings of the 1997 SPIE International Conference on Performance and Control of Network Systems*, 1997.

[40] V. Paxson and S. Floyd. Wide area traffic: The failure of poisson modeling. *IEEE/ACM Transactions on Networking*, 3(3):226–244, June 1995.

[41] S.I. Resnick. *Adventures in Stochastic Processes.* Birkhäuser, Boston, 1992.

[42] S.I. Resnick. Heavy tail modelling in teletraffic data. *Ann. Statist.*, 25:1805–1848, 1997. With discussion.

[43] S.I. Resnick and G. Samorodnitsky. Performance decay in a single server exponential queueing model with long range dependence. *Operations Research*, 45:235–243, 1997.

[44] S.I. Resnick and G. Samorodnitsky. Fluid queues, leaky buckets, on-off processes and teletraffic modeling with highly variable and correlated inputs. Preprint, Cornell University, 1998.

[45] G. Samorodnitsky and M.S. Taqqu. *Stable Non-Gaussian Random Processes.* Chapman and Hall, New York, 1994.

[46] K. Sigman. A primer on heavy-tailed distributions. *Queueing Systems*, 1999. This issue.

[47] W.L. Smith. On the tails of queueing time distributions. Technical report, Mimeo Series No. 830, Dept. of Statistics, University of North Carolina, Chapel Hill, 1972.

[48] M. S. Taqqu, W. Willinger, and R. Sherman. Proof of a Fundamental Result in Self-Similar Traffic Modeling. *Computer Communication Review*, 27:5–23, 1997.

[49] M.S. Taqqu and J. Levy. Using renewal processes to generate long-range dependence and high variability. In E. Eberlein and M.S. Taqqu, editors, *Dependence in Probability and Statistics*, pages 73–89. Birkhäuser, Boston, 1986.

[50] W. Willinger, M. S. Taqqu, R. Sherman, and D. V. Wilson. Self-Similarity Through High-Variability: Statistical Analysis of Ethernet LAN Traffic at the Source Level. *IEEE/ACM Transactions on Networking*, 5(1):71–86, April 1997.