DENSITIES WITH GAUSSIAN TAILS

A. A. BALKEMA, C. KLÜPPELBERG, *and* S. I. RESNICK

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ABSTRACT

Consider densities $f_i(t)$, for $i = 1, \ldots, d$, on the real line which have thin tails in the sense that, for each i,

$$f_i(t) \sim \gamma_i(t)e^{-\psi_i(t)},$$

where γ_i behaves roughly like a constant and ψ_i is convex, C^2, with $\psi' \to \infty$ and $\psi'' > 0$ and $1/\sqrt{\psi''}$ is self-neglecting. (The latter is an asymptotic variation condition.) Then the convolution is of the same form

$$f_1 * \ldots * f_d(t) \sim \gamma(t)e^{-\psi(t)}.$$

Formulae for γ, ψ are given in terms of the factor densities and involve the conjugate transform and infimal convolution of convexity theory. The derivations require embedding densities in exponential families and showing that the assumed form of the densities implies asymptotic normality of the exponential families.

0. *Introduction*

The purpose of this paper is to explain the statement: 'The class of densities with Gaussian tails is closed under convolution'. For the moment we think of a density with a Gaussian tail as a density with a thin tail in the sense that

$$f(t) \sim \gamma(t)e^{-\psi(t)} \quad \text{as } t \to \infty,$$

where ψ is a convex C^2-function with ψ'' strictly positive and $\psi' \to \infty$, and where γ behaves more or less like a constant. The relation $\sim$ denotes asymptotic equality: the quotient of the two sides of the equation tends to 1 as $t \to \infty$. We shall shortly impose certain regularity conditions on the functions ψ and γ which will ensure that any finite convolution of such functions again has this form. Moreover, we shall see that the functions ψ and γ for the convolution can be expressed in terms of the functions ψ_i and γ_i of the factors.

Let us first say a few words about the terminology 'Gaussian tail'. Any distribution F with a moment-generating function generates an exponential family of distributions

$$dF_\lambda(x) = e^{\lambda x}\, dF(x)/C(\lambda)$$

where the norming factor $C(\lambda) = \int e^{\lambda x}\, dF(x)$ is nothing but the moment-generating function of F. If F has a density f with a thin tail as above, then the moment-generating function $C(\lambda)$ is finite for all $\lambda \geq 0$ and hence the density $f_\lambda(x) = e^{\lambda x}f(x)/C(\lambda)$ is well defined for all $\lambda \geq 0$. The conditions which we impose on γ and ψ imply that f_λ is asymptotically normal: there exist norming

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constants $a_\lambda > 0$ and $b_\lambda \in R$ such that the normalized densities g_λ satisfy

$$g_\lambda(x) = a_\lambda f_\lambda(b_\lambda + a_\lambda x) \to \frac{1}{\sqrt{(2\pi)}} e^{-x^2/2}, \quad \lambda \to \infty,$$

uniformly on R.

There is a second related reason for the name Gaussian tails. Originally we were interested in the distribution of extremes of linear combinations of independent identically distributed random variables. For distributions with thick tails there exists an elegant theory of subexponential distributions, which links sums and maxima. For thin tails there exist isolated results (see, for example, the excellent papers [**4**, **10**]) but no general theory. By using Laplace's principle for estimating the density of the convolution $f^{*2}(t) = \int f(t-x)f(x)\,dx$ for a density of the form above, we obtain

$$\begin{aligned} f^{*2}(t) &\sim \gamma^2(\tfrac{1}{2}t)\int e^{-(\psi(t/2+x)+\psi(t/2-x))}\,dx \\ &\sim \gamma^2(\tfrac{1}{2}t)e^{-2\psi(t/2)}\int e^{-x^2\psi''(t/2)}\,dx \\ &= \gamma^2(\tfrac{1}{2}t)(\sqrt{(\pi/\psi''(\tfrac{1}{2}t))})e^{-2\psi(t/2)}, \end{aligned}$$

at least if $\sigma^2 := 1/\psi''$ (and γ) behave like constants in an appropriate sense. Some reflection shows that a suitable condition on the functions σ and γ is

$$(0.1) \qquad \begin{aligned} \sigma(t+x\sigma(t))/\sigma(t) &\to 1 \quad \text{as } t\to\infty, \\ \gamma(t+x\sigma(t))/\gamma(t) &\to 1 \quad \text{as } t\to\infty, \end{aligned}$$

both uniformly on bounded x-intervals. (A function σ which satisfies relation (0.1) above is said to be self-neglecting.) This argument remains valid if the random variables X and Y do not have the same distribution (and even if they are dependent) as long as the bivariate density $h(x, y)$ has the form $e^{-\psi(x,y)}$ for a 'smooth' convex function ψ. In this paper we consider sums of a finite number of independent random variables whose densities satisfy the asymptotic relation above where the functions γ_i and ψ_i may depend on the index i. The restriction to independent variables allows us to obtain precise results. An interesting application of our results to saddlepoint approximations is contained in [**1**].

Feigin and Yashchin [**5**] relate the behaviour of the tail of a density function to the behaviour of the Laplace transform in the case where the Laplace transform is asymptotically Gaussian. Our results are analogous. Since we work with densities, the proofs are straightforward and do not make use of transform theory. Indeed an exact formulation of our result in terms of Laplace transforms might be quite difficult.

1. *Closure under convolution*

We introduce a class of probability densities with thin tails which is closed under convolution. Theorem 1.1 is the main result of this paper. It provides a flexible method for obtaining the tail behaviour of a finite convolution based on knowledge of the tail behaviour of the convolution factors.

THEOREM 1.1. *Let $X_1, \ldots, X_d$ be independent random variables with densities $f_1, \ldots, f_d$. We assume that the densities f_i are strictly positive on a left neighbourhood of their upper endpoints $t_{i\infty}$ and satisfy the asymptotic equality*

$$f_i(t) \sim \gamma_i(t) e^{-\psi_i(t)}, \quad \text{as } t \uparrow t_{i\infty}, \tag{1.1}$$

where the functions ψ_i satisfy

$$\psi_i \text{ is } C^2 \text{ and } \psi_i'' \text{ is positive} \tag{1.2}$$

(so that ψ_i is convex), and

$$\sigma_i := 1/\sqrt{\psi_i''} \text{ is self-neglecting.} \tag{1.3}$$

The functions γ_i satsify

$$\frac{\gamma_i(t + x\sigma_i(t))}{\gamma_i(t)} \to 1 \quad \text{as } t \uparrow t_{i\infty} \tag{1.4}$$

uniformly on bounded x-intervals. Furthermore, we assume that

$$\sup_t \psi_i'(t) =: \tau_\infty \leqslant \infty \text{ is independent of } i. \tag{1.5}$$

Then the density $f_0 = f_1 * \ldots * f_d$ of $X_0 := X_1 + \ldots + X_d$ has the same form

$$f_0(t) \sim \gamma_0(t) e^{-\psi_0(t)}, \quad \text{as } t \uparrow t_\infty = t_{1\infty} + \ldots + t_{d\infty}.$$

Explicit formulae for γ_0 and ψ_0 can be given in terms of the inverse functions $q_i = (\psi_i')^{\leftarrow}$ to the strictly increasing derivatives ψ_i' as follows. Write $t = q_1 + \ldots + q_d$. Then $t = t(\tau)$ is a continuous strictly increasing function of τ and $t(\tau) \uparrow t_\infty$ as $\tau \uparrow \tau_\infty$. Now one may choose

$$\psi_0(t) = \psi_1(q_1) + \ldots + \psi_d(q_d), \tag{1.6}$$

$$\sigma_0^2(t) = \sigma_1^2(q_1) + \ldots + \sigma_d^2(q_d), \tag{1.7}$$

$$(\sqrt{2\pi})\sigma_0(t)\gamma_0(t) = \prod_{1 \leqslant i \leqslant d} (\sqrt{2\pi})\sigma_i(q_i)\gamma_i(q_i). \tag{1.8}$$

Then $\sigma_0^2 = 1/\psi_0''$ and $\sup \psi_0'(t) = \tau_\infty$.

Note that we have replaced the intuitive condition that the derivatives ψ_i' are unbounded by the formal condition that $\psi_i'(t_{i\infty}) =: \tau_\infty \leqslant \infty$ is independent of the index i. If τ_∞ is finite then all the upper endpoints $t_{i\infty}$ are infinite (Proposition 5.5).

We give a sketch of the proof. Determine the minimum of the function

$$\psi(x_1, \ldots, x_d) = \psi_1(x_1) + \ldots + \psi_d(x_d)$$

on the hyperplane $x_1 + \ldots + x_d = t$ by Lagrange's method and evaluate the convolution integral by Laplace's method. The convexity of ψ supplies the necessary bounds on the tails of the integrand. The formal proof is presented in § 7. In the intervening §§ 3–5 we develop the theory of asymptotically parabolic functions. These are the functions ψ which satisfy the conditions (1.2) and (1.3) above. This class of functions is of sufficient interest to warrant some attention.

Section 4 uses the theory of conjugate convex functions to give two alternative descriptions of the function ψ_0 in terms of the functions ψ_i, viz.

$$\psi_0^* = \psi_1^* + \ldots + \psi_d^*,$$

$$\psi_0 = \psi_1 \square \ldots \square \psi_d,$$

where ψ_i^* is the conjugate function to ψ_i for $i = 0, \ldots, d$ and where the infimal convolution $\varphi_1 \square \varphi_2$ of two convex functions φ_1 and φ_2 is the (convex) function φ defined by $\varphi(t) = \inf\{\varphi_1(x) + \varphi_2(y)\colon x + y = t\}$.

Section 6 shows that for bounded densities $f \sim \gamma e^{-\psi}$ with ψ and γ as above, the associated exponential family of densities f_τ is asymptotically Gaussian for $\tau \to \tau_\infty$. The convexity of the exponent will ensure a very strong form of convergence of the normalized densities; see (6.1). This strong form of asymptotic normality not only gives Theorem 1.1, it also establishes asymptotic normality of the vector $X = (X_1, \ldots, X_d)$ conditioned on the sum $X_0 = t$ for $t \to t_\infty$. These conditional distributions are of interest in large deviation theory.

Section 2 treats the following question: when is the tail of a convolution determined (up to asymptotic equality) by the tails of the factors?

We now consider some remarks and examples which are intended to clarify the construction of the functions ψ_0 and γ_0 in the asymptotic expression for the density of the sum. We begin by checking the two relations $\sigma_0^2 = 1/\psi_0''$ and $\sup \psi_0'(t) = \tau_\infty$ given after relation (1.8).

PROPOSITION 1.2. *We have that*

$$\sigma_0^2 = 1/\psi_0'' \quad \textit{and} \quad \sup \psi_0'(t) = \tau_\infty.$$

Proof. It is convenient to take τ as the independent variable. Note that $\tau = \psi_i'(q_i)$. Hence $q_i'(\tau) = 1/\psi_i''(q_i) = \sigma_i^2(q_i)$. Differentiation of the equality (1.6) with respect to τ gives

$$\psi_0'(t)q_0'(\tau) = \psi_1'(q_1)q_1'(\tau) + \ldots + \psi_d'(q_d)q_d'(\tau) = \tau q_0'(\tau),$$

where we write $t = q_0(\tau)$. Hence $\psi_0'(t) = \tau$. This proves the second relation. Differentiation of this last expression with respect to τ gives $\psi_0''(t)q_0'(\tau) = 1$ and hence

$$1/\psi_0''(t) = q_0'(\tau) = q_1'(\tau) + \ldots + q_d'(\tau) = \sigma_1^2 + \ldots + \sigma_d^2 = \sigma_0^2(t).$$

Let us now look briefly at the conditions of the theorem. The densities f_i need not be continuous or bounded. Indeed it suffices that the distributions F_i have a density of the form $\gamma_i e^{-\psi_i}$ on a left neighbourhood of $t_{i\infty}$. In Proposition 5.7 we shall see that if a distribution has the form $F_i = 1 - e^{-\psi_i}$ on a left neighbourhood of its upper endpoint and ψ_i satisfies (1.2) and (1.3) then (1.1) holds and the theorem applies.

The decomposition $f_i(t) = c_i(t)\gamma_i(t)e^{-\psi_i(t)}$ with $c_i(t) \to 1$ for $t \to t_{i\infty}$ is far from unique. If desired, we may choose $\gamma_i \equiv 1$. Even in the independent identically distributed case, this does not lead to a substantial simplification of the expression for γ_0. See (1.11) below. Condition (1.5) is indispensable, as will be shown by an example in § 2.

The function class of self-neglecting functions used in condition (1.3) is discussed in greater detail in § 3. Recall that the function σ is self-neglecting with endpoint t_∞ if

(1.9) $$\lim_{t\uparrow t_\infty} \sigma(t+x\sigma(t))/\sigma(t)=1 \quad \text{locally uniformly in } x.$$

For now, think of self-neglecting functions as functions whose derivative goes to zero. In a similar spirit we observe that condition (1.4) is satisfied if σ_i is self-neglecting and γ_i has density γ_i' satisfying

(1.10) $$\gamma_i'(t)\sigma_i(t)/\gamma_i(t)\to 0.$$

To see this, drop the subscript for typographical ease. The mean value theorem with $s=t+\theta x\sigma(t)$ gives $\log\gamma(t+x\sigma(t))-\log\gamma(t)=x\sigma(t)(\log\gamma)'(s)\to 0$ by (1.9) and (1.10).

The following three examples will help to clarify the theorem's statement.

EXAMPLE 1.1. If the random variables X_i are independent and identically distributed with common density $f\sim\gamma e^{-\psi}$ then the sum has density

(1.11) $$f_0(t)\sim\frac{1}{\sqrt{d}}\left(\frac{2\pi}{\psi''(t/d)}\right)^{(d-1)/2} f\left(\frac{t}{d}\right)^d \quad \text{as } t\to t_\infty.$$

EXAMPLE 1.2. If the random variables X_i are independent with densities $f_i\sim\gamma_i e^{-\psi}$ then the density of the sum satisfies

$$f_0(t)\sim\frac{1}{\sqrt{d}}\left(\frac{2\pi}{\psi''(t/d)}\right)^{(d-1)/2} e^{-\psi(t/d)}\prod_{i=1}^{d}\gamma_i(t/d).$$

EXAMPLE 1.3. Let $f(t)\sim Ct^\beta e^{-ct^\alpha}$ with $\alpha>1$, c and $C>0$, $\beta\in R$. Take $\psi(t)=ct^\alpha$. Then $\sigma(t)=1/\sqrt{(\psi''(t))}$ has a derivative which vanishes in ∞ and $\gamma(t)=Ct^\beta$ satisfies (1.10).

Now suppose $f_i(t)\sim C_i t^{\beta_i}\exp(-c_i t^{\alpha_i})$ for $i=1,\dots,d$. For simplicity assume $c_i\alpha_i=1$ and write $\alpha_i=1+1/\rho_i$. Then $\psi_i'(t)=t^{\alpha_i-1}=t^{1/\rho_i}$ and therefore $q_i=\tau^{\rho_i}$. Expressions are simplest in terms of the variable τ. Note that $t=t(\tau)=\sum_{i=1}^d\tau^{\rho_i}$ with $\rho_i>0$. This allows us to treat τ as a function of the variable t in the expressions below.

We have $\sigma_i^2(q_i)=\rho_i\tau^{\rho_i-1}$ and $\gamma_i(q_i)=C_i\tau^{\rho_i\beta_i}$. Theorem 1.1 gives

$$\psi_0(t)=\sum_{i=1}^d c_i\tau^{\rho_i+1},\quad \sigma_0^2(t)=\sum_{i=1}^d\rho_i\tau^{\rho_i-1},\quad \gamma_0(t)=\frac{C\tau^{s-d/2}}{\sqrt{(\sum_{i=1}^d\rho_i\tau^{\rho_i-1})}},$$

where $s=\sum_{i=1}^d\rho_i(\beta_i+\frac{1}{2})$ and $C=(2\pi)^{(d-1)/2}\prod_{i=1}^d C_i\sqrt{\rho_i}$. In the independent identically distributed case these types of densities have been analysed by Rootzen **[10]**.

2. *The tail of a convolution*

The conditions of Theorem 1.1 imply that the upper tail of the convolution $f=f_1*\dots*f_d$ is determined (up to asymptotic equality) by the upper tails of the densities f_i of the summands. This aspect of Theorem 1.1 will form the subject of the present section. There is considerable literature on this subject for regularly varying and subexponential tails. See, for example, [2]. We shall consider distributions F_i with upper endpoint $\sup\{F_i < 1\} = \infty$ and with the property that the density $f_i \sim e^{-\psi_i}$ exists and is strictly positive on a neighbourhood of ∞. The next example shows what happens if condition (1.5) is violated.

EXAMPLE. Let $f(x) \sim e^{(\sqrt{x})-x}$. Write $\psi(x) = x - \sqrt{x}$. Then $\psi''(x) = \frac{1}{4}x^{-\frac{3}{2}}$ and $\sigma(x) = 2x^{\frac{3}{4}}$ is self-neglecting since the derivative vanishes in ∞. Let $g(x) \sim x^{-m}e^{(\sqrt{x})-x}$. The function $\gamma(x) = x^{-m}$ satisfies (1.10). Example 1.2 implies that

$$(f * g)(t) \sim 2(\sqrt{\pi})(\tfrac{1}{2}t)^{\frac{3}{4}}(\tfrac{1}{2}t)^{-m}e^{(\sqrt{2t})-t} \quad \text{as } t \to \infty.$$

Now take a density g with a slightly thinner tail, so that

$$g(x) = O(e^{-x}) \quad \text{and} \quad \int_0^\infty e^x g(x)\,dx < \infty.$$

Then

$$(f * g)(t) \sim Af(t) \quad \text{as } t \to \infty \tag{2.1}$$

with $A = \int e^x\,dG(x)$. To verify this, write the convolution as the sum of three terms (for $t > c \geq 0$):

$$(f * g)(t) = \int_0^{t-c} f(t-x)g(x)\,dx + \int_{-\infty}^0 f(t-x)\,dG(x) + \int_{-\infty}^c g(t-x)\,dF(x).$$

For fixed $x \in R$, one has $f(t-x)/f(t) \to e^x$ and one can choose $c \geq 0$ so large that $\frac{1}{2} < f(t)/e^{(\sqrt{t})-t} < 2$ for $t \geq c$. This implies that $f(t-x)/f(t) \leq 4e^x$ for $x \geq 0$, $t - x \geq c$ and $f(t-x)/f(t) \leq 4$ for $t \geq c$ and $x \leq 0$. Lebesgue's theorem on dominated convergence with majorizing function $4 \vee 4e^x$ gives

$$\frac{1}{f(t)}\int_{-\infty}^{t-c} f(t-x)\,dG(x) \to \int_{-\infty}^\infty e^x\,dG(x) = A \quad \text{as } t \to \infty.$$

The third integral above is bounded by $F(c)O(e^{-t+c}) = O(e^{-t}) = o(f(t))$ for $t \to \infty$. This establishes (2.1).

In the situation sketched here we do obtain an asymptotic expression for the tail of the density of the convolution, but of a less symmetric form than the expression in Theorem 1.1. In particular, we see that the upper tail of the convolution depends on the whole distribution function G and not only on the asymptotic behaviour of the density $g(x)$ for $x \to \infty$. The same argument establishes the following result:

PROPOSITION 2.1. *Let the distributions F and G have densities $f(x) \sim e^{-\psi(x)}$ and $g(x) = o(e^{-bx})$ on a neighbourhood of ∞. Assume that $\psi'(x) \leq b$ for $x > x_0$ and $\psi'(x) \to \tau$ for $x \to \infty$. If $\int_0^\infty e^{bx}\,dG(x)$ is finite then* (2.1) *holds with* $A = \int e^{\tau x}\,dG(x)$.

Analogues of this result for distribution functions are given in [4].

We shall now formulate conditions which ensure that the convolution has a density on a neighbourhood of ∞ and that the asymptotic behaviour of this density is determined by the asymptotic behaviour of the densities of the summands in ∞.

PROPOSITION 2.2. *Let F_1 and F_2 be distributions on R with densities $f_i \sim e^{-\psi_i}$ on a neighbourhood of ∞ such that $\psi_i'(x) \to \infty$ for $x \to \infty$. Then $F_1 * F_2$ has a density on a neighbourhood of ∞ whose asymptotic behaviour depends only on the asymptotic behaviour of the functions f_i in ∞.*

Proof. Choose $c \geqslant 0$ so that the distributions F_i have a density on (c, ∞). Then the convolution $F_1 * F_2$ has a density f on $(2c, \infty)$ given by

$$f(t) = \int_c^{t-c} f_1(t-x) f_2(x)\, dx + \int_{-\infty}^{c} f_1(t-x)\, dF_2(x) + \int_{-\infty}^{c} f_2(t-x)\, dF_1(x).$$

By symmetry it suffices to show that

$$A = \int_{-\infty}^{c} e^{-\psi_1(t-x)}\, dF_2(x) = o\left(\int_c^{t-c} f_1(t-x) f_2(x)\, dx\right)$$

for $t \to \infty$. Choose c so large that $\alpha = \int_{c+1}^{c+2} f_2(x)\, dx > 0$. Observe that $A \leqslant e^{-\psi_1(t-c)}$ and

$$\int_{c+1}^{c+2} e^{-\psi_1(t-x)} f_2(x)\, dx \geqslant \alpha e^{-\psi_1(t-c-1)} = \alpha e^{-\psi_1(t-c)}\, e^{\psi_1'(t-\xi)}$$

with $|\xi| \leqslant c+1$ and hence $\psi_1'(t-\xi) \to \infty$.

Note that this result depends only on the first derivative of the functions ψ_i in the exponent. We still need a result in the case where the derivatives ψ_i' have a finite limit independent of the index i.

PROPOSITION 2.3. *Let F_1 and F_2 be distributions on R with densities $f_i \sim e^{-\psi_i}$ on a neighbourhood of ∞ with ψ_i convex and $\psi_i'(x) \to \tau < \infty$ for $x \to \infty$. Then $F_1 * F_2$ has a density on a neighbourhood of ∞ whose asymptotic behaviour depends only on the asymptotic behaviour of the functions f_i in ∞.*

Proof. First note that the derivatives ψ_i' are increasing, and hence $\tau > 0$. We may assume that the densities exist on $[c, \infty)$, where $c > 0$, and that the asymptotic equalities hold there within a factor 2. As above, $f(t)$ is the sum of three integrals and the sum over the finite interval $[c, t-c]$ dominates for $t \to \infty$. Let $\alpha_i = e^{-\psi_i(c)} > 0$ for $i = 1, 2$. We have the bounds $2e^{-\psi_i(t-c)}$ for the integral over the infinite intervals, whereas for $t > 2c$,

$$\int_c^{t-c} e^{-(\psi_1(t-x)+\psi_2(x))}\, dx \geqslant \tfrac{1}{2}(t-2c)(\alpha_2 e^{-\psi_1(t-c)} + \alpha_1 e^{-\psi_2(t-c)}),$$

by unimodality of the integrand on the interval $[c, t-c]$. Now let $t \to \infty$.

EXAMPLE. Let $f_i(x) \sim a_i x^{r_i - 1} e^{-x}$ for $i = 1, \ldots, d$ with a_i and r_i positive constants. Then

$$(f_1 * \ldots * f_d)(t) \sim a_1 \ldots a_d \frac{\Gamma(r_1) \ldots \Gamma(r_d)}{\Gamma(r)} t^{r-1} e^{-t}$$

with $r = r_1 + \ldots + r_d$. For $r_i \geqslant 1$ one can apply the result above; for $r_i < 1$ observe that the integral in the proof above divided by $f_1(t-c) \vee f_2(t-c)$ diverges for $t \to \infty$ even if we divide the integrand by the factor $x(t-x)$. Densities with gamma tails fall outside the scope of the present paper. The function

$$\psi(x) = x - (r-1)\log x$$

in the exponent is convex for $r > 1$, but $\sigma(x) = 1/\sqrt{\psi''(x)}$ is not self-neglecting. In geometrical terms this is the hyperbolic case, and we are concerned with the parabolic case.

3. *Self-neglecting functions*

In this section we collect some needed preliminaries about self-neglecting functions and the relation (1.4). (Cf. [**2**, **7**, **8**].) A function σ defined on a left neighbourhood of a point $t_\infty \leq \infty$ is *self-neglecting* (written $\sigma \in SN$) if it is strictly positive and satisfies

$$\frac{\sigma(t + x\sigma(t))}{\sigma(t)} \to 1 \quad \text{as } t \uparrow t_\infty \tag{3.1}$$

uniformly on bounded x-intervals. (In particular, this requires that for fixed $x > 0$ one has the inequality $t + x\sigma(t) < t_\infty$ for t sufficiently close to t_∞.) It helps to think of $\sigma(t)$ as the scale in a neighbourhood of t. If $t_\infty = \infty$ then sufficient for $\sigma \in SN$ is that σ have a density σ' satisfying $\sigma'(t) \to 0$ as $t \to \infty$. If $t_\infty < \infty$ and both σ and σ' vanish at t_∞, then $\sigma \in SN$. Membership in SN is preserved under asymptotic equivalence: if $\sigma \in SN$ and $\tau \sim \sigma$, then $\tau \in SN$.

Less obvious are the following facts. First, if σ is continuous, then pointwise convergence in (3.1) implies that convergence is locally uniform and hence $\sigma \in SN$. (See [**2**, p. 120] and [**6**, p. 49].) Second, if $\sigma \in SN$ then $\sigma \sim \tau$ where τ is C^1 and τ' vanishes at t_∞ (along with τ itself if t_∞ is finite). The proof of the first fact carries over to the function γ:

PROPOSITION 3.1. *If γ is continuous and satisfies* (1.4) *with σ self-neglecting, then* (1.4) *holds uniformly on bounded x-intervals.*

We give a short proof of the second fact for the case that $t_\infty = \infty$ since the basic argument is useful. The definition of a self-neglecting function supplies an s_0 such that σ is strictly positive on $[s_0, \infty)$ and

$$\tfrac{1}{2} < \sigma(t + x\sigma(t))/\sigma(t) < 2 \tag{3.2}$$

for $t \geq s_0$ and $|x| \leq 1$. Using s_0 as the base of a recursion, define the increasing sequence (s_n) by $s_{n+1} = s_n + \sigma(s_n)$. Then $s_n \to \infty$. (Otherwise, $s_\infty := \sup s_n < \infty$ and $\sigma(s_n) = s_{n+1} - s_n \to 0$. On the other hand, $\sigma(s_\infty) = c > 0$ and (3.2) evaluated at $t = s_\infty$ states that $\sigma > \frac{1}{2}c$ on the interval $(s_\infty - c, s_\infty + c)$.) Relation (3.1) now implies that $\sigma_n := \sigma(s_n) \sim \sigma_{n+1}$. Let τ be the piecewise linear continuous function which agrees with σ in the points s_n; that is, define

$$\tau(s) = \sigma_n + (\sigma_{n+1} - \sigma_n)(s - s_n)/(s_{n+1} - s_n) \quad \text{for } s_n \leq s \leq s_{n+1}.$$

Since $0 \leq (s - s_n)/(s_{n+1} - s_n) \leq 1$, the right-hand side above divided by σ_n converges to 1 and (3.1) gives $\sigma \sim \tau$. Furthermore, $\tau'(s) \to 0$.

By pushing this construction further, one can make $\tau \in C^\infty$ such that $\tau \sim \sigma$ and such that the derivatives of τ satisfy $\sigma^k \tau^{(k+1)} \to 0$. To do this one changes the definition of τ by interpolating not with a linear function but rather with an increasing C^∞-function from $[0, 1]$ onto $[0, 1]$ all of whose derivatives vanish in the endpoints 0 and 1. If this C^∞ function is denoted by f then we define τ on $[s_n, s_{n+1}]$ by

$$\tau(s) = \sigma_n + f((s - s_n)/(s_{n+1} - s_n))(\sigma_{n+1} - \sigma_n).$$

If we start the sequence (s_n) with a point s_0 determined by the function γ and define the function β by

$$\beta(s) = \gamma(s_n) + f((s - s_n)/(s_{n+1} - s_n))(\gamma(s_{n+1}) - \gamma(s_n))$$

for $s \in [s_n, s_{n+1})$, then the construction above yields the following result:

PROPOSITION 3.2. *Suppose σ is self-neglecting and γ satisfies* (1.4) *uniformly on bounded x-intervals. Then there exists a C^∞-function $\beta \sim \gamma$ such that $\sigma^k \beta^{(k)}/\beta$ vanishes at t_∞ for $k = 1, 2, \ldots$.*

The following closure result for SN will be needed.

PROPOSITION 3.3. *If σ and τ are self-neglecting with the same upper endpoint t_∞, then so is ρ where ρ is defined by*

$$\rho^{-2} = \sigma^{-2} + \tau^{-2}.$$

Proof. If σ and τ are C^1 with derivatives which vanish in t_∞, then ρ is C^1 and

$$\frac{\rho'}{\rho^3} = \frac{\sigma'}{\sigma^3} + \frac{\tau'}{\tau^3} \quad \Rightarrow \quad \rho' = \left(\frac{\rho}{\sigma}\right)^3 \sigma' + \left(\frac{\rho}{\tau}\right)^3 \tau' \to 0$$

since $\rho \leq \sigma$ and $\rho \leq \tau$. In the general case, choose $\tilde{\sigma} \sim \sigma$ and $\tilde{\tau} \sim \tau$ so that $\tilde{\sigma}'$ and $\tilde{\tau}'$ vanish in t_∞. Then $\tilde{\sigma}^{-2} \sim \sigma^{-2}$ and $\tilde{\tau}^{-2} \sim \tau^{-2}$, whence $\tilde{\rho} \sim \rho$.

4. *The conjugate transform*

We need some simple facts about the conjugate (Legendre) transform of a convex function **[2, 3, 9, 11]**.

A convex function ψ is defined on the whole real line R with values in $(-\infty, \infty]$. Its *domain* is the set on which it is finite. This is a connected set, which we shall assume to contain at least two points. The function ψ is continuous on the interior of the domain. For $\xi \in R$ we define

$$\psi^*(\xi) = \sup_x (\xi x - \psi(x)). \tag{4.1}$$

Note that ψ^* is convex and lower semi-continuous since it is the supremum of affine functions. For all x and ξ one has the inequality

$$\xi x \leq \psi(x) + \psi^*(\xi). \tag{4.2}$$

The function ψ^* is called the *convex conjugate* of ψ. It is well known that $\psi^{**} = (\psi^*)^*$ is the convex hull of ψ:

$$\psi^{**} = \sup\{\xi \mid \xi \text{ is affine and } \xi \leq \psi\},$$

where the inequality and supremum are defined pointwise in the previous display.

Now assume ψ is C^1 and strictly convex on the open interval D. Then ψ' is continuous and strictly increasing on D. The image $\Delta = \psi'(D)$ is an open interval and ψ^* is finite on Δ. Each $\xi \in \Delta$ is the slope of the tangent line at a unique point $x = q(\xi) = \xi^T \in D$. Also $q = \psi'^{\leftarrow} = \psi^{*\prime}$. The symmetry in this situation is reflected in the symmetric relation

$$\xi x = \psi(x) + \psi^*(\xi) \tag{4.3}$$

for $\xi \in \Delta$, $x \in D$ related as above. Note that (4.3) is a local definition of ψ^* which does not depend on the behaviour of ψ outside the open set D, whereas (4.1) apparently is a global definition. These results also hold for convex functions of several variables:

PROPOSITION 4.1. *Let ψ be a convex function on R^d with convex conjugate ψ^* defined on the dual space R^{d*} of all linear functionals on R^d by* (4.1). *Let $D \subset R^d$ be an open set. Assume that ψ is finite and C^2 on D, and that $\psi''(x)$ is positive definite in each point $x \in D$. Then there exist an open set $\Delta \subset R^{d*}$ and a homeomorphism q: $\Delta \to D$ with inverse θ: $D \to \Delta$ such that if x and ξ satisfy* (4.3), *then*

$$q(\xi) = x \quad \Leftrightarrow \quad \theta(x) = \xi. \tag{4.4}$$

Proof. The function θ: $x \to \psi'(x)$ is C^1 with non-singular derivative $\psi''(x)$ on D. The inverse function theorem implies that $\Delta = \theta(D)$ is open and that θ locally is a C^1 diffeomorphism. Convexity ensures that the function is injective.

The relation between the variables x and ξ in (4.4) is symmetric. We shall express this symmetry in the notation

$$x = \xi^T, \quad \xi = x^T, \tag{4.5}$$

which states that x is the unique point in D where the tangent plane to ψ has slope ξ.

The formula for ψ_0 in the statement of Theorem 1.1 has an elegant interpretation in terms of conjugate transforms. The *infimal convolution* of two convex functions ψ_1 and ψ_2 is defined in [9] as

$$\psi_1 \square \psi_2(x) = \inf_y \{\psi_1(x-y) + \psi_2(y)\}$$

and it is shown there (Theorem 16.4) that

$$\psi_1 \square \psi_2 = (\psi_1^* + \psi_2^*)^*. \tag{4.6}$$

THEOREM 4.2. *Let the conditions of Theorem* 1.1 *hold. Then*

$$\psi_0 = \psi_1 \square \ldots \square \psi_d \tag{4.7}$$

on a left neighbourhood of $t_\infty = t_{1\infty} + \ldots + t_{d\infty}$.

Proof. Set $\psi(x) = \psi_1(x_1) + \ldots + \psi_d(x_d)$, define ψ_0 as in Theorem 1.1, and set

$$\chi(t) = \inf\{\psi(x) |\ x_1 + \ldots + x_d = t\}.$$

We shall construct $t_0 < t_\infty$ such that $\chi \equiv \psi_0$ on the interval (t_0, t_∞).

The functions ψ_i' are continuous and strictly increasing on a left neighbourhood of the upper endpoints $t_{i\infty}$ with limit τ_∞ in $t_{i\infty}$. Hence the d inverse functions $q_i = (\psi_i')^{\leftarrow}$ are defined, continuous and strictly increasing on an interval $[\tau_0, \tau_\infty)$. Set $t_{i0} = q_i(\tau_0)$. Then $t = q_1 + \dots + q_d$ is a continuous strictly increasing function from the interval (τ_0, τ_∞) onto (t_0, t_∞) where $t_0 = t_{10} + \dots + t_{d0}$.

Let $\tau \in (\tau_0, \tau_\infty)$. Then $\psi_0(t) = \psi(q)$ where $q = (q_1(\tau), \dots, q_d(\tau))$ and $t = q_1 + \dots + q_d$. The function $\varphi(x) = \psi(x) - (x_1 + \dots + x_d)\tau$ is convex and the derivative φ' vanishes in $q = (\tau, \dots, \tau)^T$. Since the matrix $\varphi''(q) = \psi''(q)$ is positive definite, the function φ has a unique minimum in q. *A fortiori* the function φ restricted to the set $S_t = \{x_1 + \dots + x_d = t\}$ achieves its minimum in the point $q \in S_t$, and so does ψ since the linear function $x_1 + \dots + x_d$ is constant on S_t. This proves that $\chi(t) = \psi(q)$.

5. *Asymptotically parabolic functions*

In § 4 we have studied the convexity of ψ. We shall now investigate the implications of the extra condition (1.3) that $\sigma = 1/\sqrt{\psi''}$ is self-neglecting. The class of asymptotically parabolic (AP) functions defined below is natural for our problem. It possesses a number of pleasing and useful closure properties which we explore before describing the asymptotic behaviour of the functions ψ in this class. These closure properties further illuminate the statement of Theorem 1.1.

DEFINITION. A function ψ is *asymptotically parabolic* (AP) if it is a convex C^2-function which is defined on an open interval D such that ψ'' is strictly positive on D and such that $\sigma := 1/\sqrt{\psi''}$ is self-neglecting in the upper endpoint $t_\infty \leq \infty$ of the open interval D. We call σ the *auxiliary* or *scale function*. Sometimes we shall write $AP(t_\infty)$ for the set of asymptotically parabolic functions with common endpoint t_∞.

EXAMPLES. Asymptotically parabolic functions:

$\psi(t)$	parameter	t_∞	$\sigma(t)$	
t^α	$\alpha > 1$	∞	$ct^{1-\alpha/2}$	$c = 1/\sqrt{(\alpha(\alpha-1))}$
e^t		∞	$e^{-t/2}$	
$t - t^\alpha$	$0 < \alpha < 1$	∞	$ct^{1-\alpha/2}$	$c = 1/\sqrt{(\alpha(1-\alpha))}$
$t \log t$		∞	$\sqrt{t}$	
$(-t)^{-\alpha}$	$\alpha > 0$	0	$c\,\lvert t\rvert^{1+\alpha/2}$	$c = 1/\sqrt{(\alpha(1+\alpha))}$

The functions $\psi(t) = t - \log t$ and $\psi(t) = 1/t$ are not asymptotically parabolic.

We begin with some closure properties of the class AP. The first proposition has an obvious proof and the second is a simple consequence of Lemma 3.3.

PROPOSITION 5.1. *The set of AP functions is closed for addition of constants and linear functions: if ψ is AP then so is $\psi(t) + \rho + \lambda t$ for any ρ and λ in R.*

PROPOSITION 5.2. *The set of AP functions is a convex cone:*

(a) *if ψ is AP with scale function σ then for any $c > 0$ the function $c\psi$ is AP with scale function $\sigma/\sqrt{c}$;*

(b) *suppose $\psi_1, \psi_2 \in AP(t_\infty)$ with scale functions σ_1, σ_2; then $\psi_1 + \psi_2 \in AP(t_\infty)$ with scale function $\sigma = (\sigma_1^{-2} + \sigma_2^{-2})^{-\frac{1}{2}}$.*

THEOREM 5.3. *The class of asymptotically parabolic functions is closed under conjugation in the following sense: if ψ is AP with domain D and endpoint t_∞ and scale function σ, then the restriction of ψ^* to $\Delta = \psi'(D)$ is AP with endpoint $\tau_\infty = \psi'(t_\infty)$ and scale function $s(\tau) = 1/\sigma(q)$ with $q = \psi^{*\prime}(\tau)$.*

Proof. The relation $q = (\psi')^{\leftarrow}(\tau) = (\psi^*)'(\tau)$ determines a diffeomorphism $q \leftrightarrow \tau$ between points $q \in D$ and points $\tau \in \Delta = \psi'(D)$ (see § 4). On the one hand,

$$q'(\tau) = (\psi^*)''(\tau) = 1/(s(\tau))^2,$$

and on the other,

$$q'(\tau) = ((\psi')^{\leftarrow}(\tau))' = 1/\psi''((\psi')^{\leftarrow}(\tau)) = \sigma^2(q).$$

This shows that the asserted relationship between σ and s holds. We still have to prove that the function s is self-neglecting. Observe that for $|\theta| \leq 1$ we have

$$\begin{aligned}\psi'(q + u\sigma(q)) &= \psi'(q) + u\sigma(q)\psi''(q + \theta u\sigma(q))\\ &= \psi'(q) + (u/\sigma(q))(\sigma^2(q)/\sigma^2(q + \theta u\sigma(q)))\\ &= \psi'(q) + (1 + o(1))u/\sigma(q)\end{aligned}$$

uniformly on bounded u-sets, by the self-neglecting property of σ. So given $u = u(q) \to u_0$, we may find $v(q) \sim u(q)$ such that

$$\psi'(q) + u(q)/\sigma(q) = \psi'(q + v(q)\sigma(q)). \tag{5.1}$$

This relation implies that $s \in SN$ as follows:

$$\begin{aligned}s(\tau + us(\tau)) &= 1/\sigma((\psi')^{\leftarrow}(\tau + us(\tau)))\\ &= 1/\sigma((\psi')^{\leftarrow}(\psi'(q) + u/\sigma(q)))\\ &= 1/\sigma((\psi')^{\leftarrow}(\psi'(q + v\sigma(q))))\\ &= 1/\sigma(q + v\sigma(q))\\ &\sim 1/\sigma(q) = s(\tau).\end{aligned}$$

We shall now derive some useful asymptotic properties of asymptotically parabolic functions.

THEOREM 5.4. *Suppose ψ is AP with endpoint t_∞. Then $\psi(t_\infty)$ is infinite.*

Proof. This is obvious if $t_\infty = \infty$ and $\tau_\infty = \sup \psi'(t) \neq 0$. We are left with two cases: t_∞ is finite, and $\tau_\infty = 0$.

First assume that t_∞ is finite. Since $\sigma = 1/\sqrt{\psi''}$ is self-neglecting with finite endpoint t_∞, it follows from the parenthetical remark after (3.1) that $\sigma(t) = o(t_\infty - t)$. Hence for any $M > 1$, eventually we have $\psi''(t) > M/(t_\infty - t)^2$ and $\psi'(t) - \psi'(t_0) > M_1/(t_\infty - t)$. This implies that $\psi' \to \infty$ and in fact

$$(t_\infty - t)\psi'(t) \to \infty \quad \text{as } t \uparrow t_\infty. \tag{5.2}$$

Next assume that $t_\infty = \infty$ and $\tau_\infty = 0$. Then $\sigma(t) = o(t)$ implies that for any $M > 0$ eventually $\psi''(t) > M/t^2$, $\psi'(t) < -M/t$ and hence we obtain the limit relation

$$(\tau_\infty - \psi'(t))t \to \infty \quad \text{as } t \to \infty. \tag{5.3}$$

PROPOSITION 5.5. *Suppose ψ is asymptotically parabolic with endpoint t_∞. If t_∞ is finite then $\tau_\infty = \sup \psi'(t) = \infty$ and* (5.2) *holds. If τ_∞ is finite then* (5.3) *holds.*

Proof. See the proof of Theorem 5.4 above.

Theorem 5.4 can be regarded as a regularity condition of the convex function ψ in its right endpoint, and has an important consequence for the domain of the conjugate function ψ^*. By definition, the domain D of an AP function is an open interval. The domain of the conjugate function is a connected subset of R and contains the open interval $\Delta = \psi'(D)$. Now assume $\tau_\infty = \sup \Delta$ is finite. Then $\psi^*(\tau_\infty) = -\lim(\psi(t) - \tau_\infty t)$ for $t \to t_\infty$. This limit is infinite by Theorem 5.4 and hence τ_∞ is not an element of the domain of ψ^*. (If the limit were finite then τ_∞ would lie in the domain of ψ^*, and it would even be an interior point if t_∞ were finite too.) The domain of ψ^* need not be equal to $\Delta = \psi'(D)$, but for any $\tau \in \Delta$ the two sets agree on the half line $[\tau, \infty)$, and since we are only concerned with the asymptotics in the upper endpoint, this is exactly what we need.

Suppose ψ is AP with scale function σ. A positive function γ defined on the domain of ψ is *flat* (for ψ) if (1.4) holds uniformly on bounded x-intervals.

EXAMPLE. If ψ is asymptotically parabolic then ψ'' is flat and so is $\sigma = 1/\sqrt{\psi''}$. Any continuous positive function γ which satisfies (1.4) is flat. A product of flat functions is flat. Since flatness is an asymptotic property, we shall often relax the positivity condition: the function should be strictly positive on a left neighbourhood of the endpoint t_∞ of ψ.

PROPOSITION 5.6. *Suppose ψ is AP. Let γ be a positive function defined on the domain D of ψ. Define γ^T on $\Delta = \psi'(D)$ by*

$$\gamma^T(\xi) = \gamma(\xi^T). \tag{5.4}$$

Then γ^T is flat for ψ^* if γ is flat for ψ.

Proof. Let $\tau = q^T$. Then $\tau + us(\tau) = (q + v\sigma(q))^T$ by (5.1) with $v \sim u$. This gives

$$\gamma^T(\tau + us(\tau)) = \gamma(q + v\sigma(q)) \sim \gamma(q) = \gamma^T(\tau).$$

The next result explains the terminology 'asymptotically parabolic' and 'flat'.

THEOREM 5.7. *Suppose ψ is asymptotically parabolic with endpoint t_∞ and scale function σ, and γ is flat. Given $t_0 < t_\infty$, $M > 1$ and $\varepsilon > 0$ there exists $t_1 \in (t_0, t_\infty)$ such that for any $t \in (t_1, t_\infty)$,*

$$J_t = [t - M\sigma(t), t + M\sigma(t)] \subset (t_0, t_\infty),$$

$$1 - \varepsilon < \gamma(t + u\sigma(t))/\gamma(t) < 1 + \varepsilon, \quad |u| \leq M,$$

$$|\varphi(u) - \tfrac{1}{2}u^2| \leq \tfrac{1}{2}\varepsilon u^2, \quad |u| \leq M,$$

where φ is the normalized function

$$\varphi(u) = \psi(t + u\sigma(t)) - \psi(t) - u\sigma(t)\psi'(t).$$

Proof. The first relation is a consequence of the assumption that $\sigma \in SN$; the second follows from the definition of 'flat'. It holds in particular for the flat function ψ''. This implies that $|\varphi''-1|<\varepsilon$ on $[-M, M]$. Repeated integration gives the third relation.

The next two results quantify the behaviour of ψ and its first two derivatives at the upper endpoint.

PROPOSITION 5.8. *Suppose ψ is AP with scale function σ and right endpoint t_∞. Then ψ' is flat and*

$$\lim_{t \uparrow t_\infty} \psi'(t)\sigma(t) = \infty. \tag{5.5}$$

Proof. Note that ψ' is increasing. By Theorem 5.4, $\psi(t_\infty)=\infty$ and hence the derivative ψ' is eventually strictly positive. Let $x = x(t) \to x_0 \in R$. Then $\sigma \in SN$ gives

$$\begin{aligned}\psi'(t + x\sigma(t)) - \psi'(t) &= \int_0^x \psi''(t + y\sigma(t))\sigma(t)\, dy \\ &= \int_0^x (\sigma(t)/\sigma^2(t + y\sigma(t)))\, dy \\ &\sim \int_0^x (\sigma(t)/\sigma^2(t))\, dy = x/\sigma(t) \quad \text{as } t \to t_\infty.\end{aligned}$$

Thus we have

$$\psi'(t + x\sigma(t))/\psi'(t) - 1 \sim x/(\sigma(t)\psi'(t)),$$

and it only remains to show that $\sigma(t)\psi'(t) \to \infty$. For any $M > 1$,

$$\begin{aligned}\psi'(t)\sigma(t) &\geq (\psi'(t) - \psi'(t - M\sigma(t)))\sigma(t) \\ &= \sigma(t)\int_{t-M\sigma(t)}^{t} \psi''(u)\, du \\ &= \sigma^2(t)\int_{-M}^{0} (1/\sigma^2(t + u\sigma(t)))\, du \to M.\end{aligned}$$

So $\liminf_{t\uparrow t_\infty} \psi'(t)\sigma(t) \geq M$ for any M and this suffices to establish (5.5).

The examples at the beginning of this section show that the long term behaviour of asymptotically parabolic functions ψ may be far from parabolic.

Consider the derivative. The derivative ψ' is continuous and strictly increasing from D onto Δ. One can construct an increasing sequence of points $p_n = (t_n, \lambda_n)$ on the graph of ψ' with $t_n \uparrow t_\infty$ such that the area of the rectangle $[p_n, p_{n+1}]$ tends to 1. Then $t_{n+1} - t_n =: \sigma_n \sim \sigma(t_n)$, and hence $\lambda_{n+1} - \lambda_n \sim 1/\sigma_n$. (Indeed if we choose $t_{n+1} - t_n \sim \sigma(t_n)$ then $\psi'(t_{n+1}) - \psi'(t_n) \sim \sigma(t_n)\psi''(t_n) = 1/\sigma(t_n)$ since $\sigma(t) = 1/\sqrt{(\psi''(t))}$ is self-neglecting. Compare the construction of the sequence (s_n) in § 3.) This gives a graphic illustration of the symmetry between the function ψ (with derivative ψ') and the conjugate function ψ^* (with derivative $\psi^{*\prime} = (\psi')^{\leftarrow}$).

Now let $m \geq 1$ be a fixed integer. The area of the rectangle $[p_n, p_{n+m}]$ will tend to m^2. Since ψ'' is flat, it is asymptotically constant over the interval $[t_n, t_{n+m}]$ and ψ' is asymptotically linear on $[t_n, t_{n+m}]$. Hence the area of the rectangle below the curve $y = \psi'(x)$ will tend to $\frac{1}{2}m^2$. This yields the following useful inequality.

PROPOSITION 5.9. *Let ψ be AP and let t_n be as above. For any $M > 1$ there exists an index n_0 such that*

$$\psi(t_{n+k}) - \psi(t_n) \geq (t_{n+k} - t_n)\psi'(t_n) + kM$$

for $n \geq n_0$, $k > 2M$.

Proof. Set $R(n, k) = \psi(t_{n+k}) - \psi(t_n) - (t_{n+k} - t_n)\psi'(t_n)$. For fixed $k \geq 1$ we have seen that $R(n, k) \to \frac{1}{2}k^2$. Let $m = [2M + 1]$ denote the integer part of $2M + 1$. Then $R(n, m) > mM$ eventually. The area of the oblong rectangle $[t_n, t_{n+1}] \times [\lambda_{n-m}, \lambda_n]$ tends to m and hence exceeds M eventually.

We now return to Theorem 1.1.

Let ψ be AP with endpoint t_∞ and scale function σ and let $f(t) \sim \gamma(t)e^{-\psi(t)}$, for $t \to t_\infty$, be a probability density. We assume that γ is continuous and flat. In § 3 we have shown that we may then assume that γ is C^2 and that $\sigma^k(t)\gamma^{(k)}(t)/\gamma(t)$ vanishes in t_∞ for $k = 1, 2$. This implies that $\varphi = \psi - \log \gamma$ is AP with scale function $1/\sqrt{(\varphi''(t))} \sim \sigma(t)$. Integrability of f implies that $\varphi(t_\infty) = \infty$ (by Theorem 5.4 above) and hence that $\sup \varphi'(t)$ is strictly positive (by convexity). The limit relation (5.5), $\varphi'(t)\sigma(t) \to \infty$, together with the relation $\sigma(t)\gamma'(t)/\gamma(t) \to 0$, implies that $(\log \gamma(t))' = o(\varphi'(t))$ and hence $\tau_\infty = \psi'(t_\infty) = \varphi'(t_\infty) > 0$. The relations (5.2) and (5.3) on the derivatives ψ_i' give superlogarithmic increase for the functions ψ_i and hence determine the tail behaviour of the densities f_i. We combine these results in the following proposition.

PROPOSITION 5.10. *Let the conditions of Theorem 1.1 hold. Then $\tau_\infty > 0$, and $\psi_i(t_{i\infty}) = \infty$ for $i = 1, \ldots, d$. If $t_{i\infty}$ is finite then $f_i(t) = o(t_{i\infty} - t)^{-n}$ for all n; if τ_∞ is finite then $t^{-n}e^{\tau_\infty t}f_i(t) \to \infty$ for $t \to \infty$, for all n.*

There exist flat functions $\beta_i \sim \gamma_i$ such that $\varphi_i = \psi_i - \log \beta_i$ is AP with upper endpoint $t_{i\infty}$ and such that $1/\varphi_i''(t) \sim \sigma_i^2(t)$ for $t \to t_{i\infty}$ and $\sup_t \varphi_i'(t) = \tau_\infty$, and such that $e^{-\varphi_i} = \beta_i e^{-\psi_i}$ is a probability density.

In view of the results of § 2, it suffices to prove Theorem 1.1 in the case where $f_i = \gamma_i e^{-\psi_i} = e^{-\varphi_i}$ where the functions γ_i are flat and the functions ψ_i and φ_i are asymptotically parabolic.

Our last result shows that the domain of the conjugate transform ψ^* and of the cumulant generating function of $f \sim \gamma e^{-\psi}$ coincide, at least on $[0, \infty)$.

PROPOSITION 5.11. *Let X have density $f \sim \gamma e^{-\psi}$ where ψ is AP and the function γ is flat. Let $\tau_\infty = \sup \psi'(t)$. Then $C(\tau) = Ee^{\tau X}$ is finite for $0 \leq \tau < \tau_\infty$. If τ_∞ is finite then $Ee^{\tau X} = \infty$ for $\tau = \tau_\infty$.*

Proof. First assume that $\tau < \tau_\infty$. Set $\varphi = \psi - \log \gamma$. If $\psi'(t) \to \tau_\infty < \infty$, then by Proposition 5.8, $\sigma(t) \to \infty$ and thus by Proposition 3.2 we have $(\log \gamma)'(t) \to 0$.

Hence $\varphi(t)-\tau t>\varepsilon t$ eventually for some $\varepsilon>0$ and $\int_0^\infty e^{\tau t-\varphi(t)}\,dt$ converges. If $\psi'(t)\to\infty$, then $\varphi'(t)=\psi'(t)-(\log\gamma)'(t)\to\infty$ and the result follows similarly. If $\tau=\tau_\infty<\infty$ then $\tau t-\varphi(t)\to\infty$ by Theorem 5.4 and $Ee^{\tau X}=\infty$. This completes the proof.

6. *Asymptotic normality of the exponential family*

Before proceeding with the proof of Theorem 1.1 we make a brief digression and discuss the asymptotic normality of the exponential family f_λ, for $\lambda\in\Lambda$, generated by a d-dimensional density f. We begin by defining precisely what we mean by asymptotic normality with exponential tails.

DEFINITION. A sequence of random vectors X_n in R^d is *asymptotically normal* (AN) if there exist affine transformations α_n on R^d, $\alpha_n(x)=A_n^{-1}(x-b_n)$ with A_n an invertible linear transformation on R^d, such that $\alpha_n(X_n)\xrightarrow{d} U$ where the random vector U is standard normal. The sequence (X_n) is *asymptotically normal with exponential tails* (ANET) if the vectors $\alpha_n(X_n)$ have densities g_n which satisfy the condition: for any $\varepsilon>0$ there exists an index n_0 such that for $n\geq n_0$,

$$|g_n(x)-\nu_d(x)|<\varepsilon e^{-\|x\|/\varepsilon}\quad\text{for } x\in R^d, \tag{6.1}$$

where ν_d is the standard normal density on R^d and $\|x\|$ denotes the Euclidean norm. We shall sometimes say that the sequence of densities (g_n) is ANET.

The reason for the name 'with exponential tails' is contained in the following result, which once stated hardly needs proof.

PROPOSITION 6.1. *The sequence (g_n) is ANET if and only if*

$$g_n(x)\to\nu_d(x) \tag{6.2}$$

locally uniformly in x, and given $\varepsilon>0$ there exist an index n_0 and constants $M>1$ and $C>1$ such that

$$g_n(x)<Ce^{-\|x\|/\varepsilon},\quad \|x\|\geq M,\ n\geq n_0. \tag{6.3}$$

One may take $C=1$ in (6.3) without loss of generality since $Ce^{-t/\varepsilon}\leq e^{-t/2\varepsilon}$ for $t\geq 2\varepsilon\log C$.

REMARK 1. If (X_n) is asymptotically normal under two different affine transformations α_n and β_n, and if the sequence is ANET with respect to one set of scaling transformations, then it is also ANET with respect to the other set.

REMARK 2. If a norm $\|x\|_1$ on R^d other than the Euclidean norm is chosen then the right-hand side of (6.1) may be replaced by $\varepsilon e^{-\|x\|_1/\delta}$ with $\delta=c\varepsilon$ where the constant c depends on the new norm $\|\cdot\|_1$. In particular, if we take the norm $\|x\|_1=|x_1|+\ldots+|x_d|$ then there is a nice relation between the bounds in d dimensions and in $d-1$ dimensions: if g_n satisfies (6.1), then the marginal density h_n on R^{d-1} defined by $h_n(y)=\int_R g_n(y,u)\,du$, $y\in R^{d-1}$, satisfies

$$|h_n(y)-\nu_{d-1}(y)|<\varepsilon\int e^{-\|y\|_1/\delta-|u|/\delta}\,du=2\varepsilon\delta e^{-\|y\|_1/\delta}.$$

This proves that the property of ANET is perserved under affine surjections. If the random vectors (X_n) are ANET and if (γ_n) is a sequence of affine surjections from R^d to R^m with $m \leqslant d$ then the sequence of random vectors $\gamma_n(X_n)$ in R^m is ANET.

THEOREM 6.2. *Suppose that the random vectors X_n in R^d are ANET. Let γ_n: $R^d \to R^m$ with $m \leqslant d$ be affine surjections such that $W_n := \gamma_n(X_n) \xrightarrow{d} W$ where W is an m-dimensional normal vector with non-singular covariance matrix. Then the sequence (W_n) is ANET. Moreover, the conditional densities $f_{n|w}$ of X_n given $W_n = w$ are ANET and the inequality* (6.1) *holds uniformly on bounded w-sets.*

Proof. We may assume that $W \in R^m$ has a standard normal density, that $X_n \xrightarrow{d} U \in R^d$ where U has a standard normal density on R^d, and that γ_n is the projection on the first m coordinates. Let X_n have density f_n so that

$$f_{n|w}(y) = f_n(w, y) \Big/ \int_{R^{d-m}} f_n(w, y)\,dy \tag{6.4}$$

for $w \in R^m$ and $y \in R^{d-m}$. Fix $r > 0$ and restrict w to the closed ball with radius r around the origin in R^m. The marginal in the denominator of (6.4) is ANET by Remark 2 above, and with w restricted to the ball, the denominator is bounded away from zero. Also $f_n(w, y)$ converges to $\nu_d(w, y)$, the standard normal density on R^d, locally uniformly and thus $f_{n|w}$ satisfies (6.2). To prove the property of exponential tails it is necessary to verify (6.3). Since the denominator of (6.4) is bounded away from zero, it suffices to show for given $\varepsilon > 0$ that there exists a constant $r' > 0$ such that, for large n,

$$f_n(w, y) \leqslant e^C e^{-\|y\|/\varepsilon}$$

for $\|w\| \leqslant r$, $\|y\| \geqslant r'$. This follows from the fact that f_n is ANET and

$$\{\|w\| \leqslant r,\ \|y\| \geqslant r'\} \subset \{\|(w, y)\| \geqslant r''\}.$$

The next result discusses the behaviour of moment generating functions and moments when there is convergence to normality with exponential tails.

PROPOSITION 6.3. *Suppose X_n in R^d is ANET and $Y_n := A_n(X_n - b_n) \xrightarrow{d} U$ where U has a standard normal density on R^d. Then the moment generating function of Y_n converges to that of the standard normal density. For each $t \in R^d$,*

$$Ee^{(t, Y_n)} \to e^{\|t\|^2/2}.$$

Proof. Take $1/\varepsilon \geqslant 1 + \|t\|$. Eventually the density g_n of Y_n satisfies

$$e^{(t,y)} g_n(y) \leqslant e^{\|t\|\,\|y\|} K e^{-\|y\|/\varepsilon} = K e^{-\|y\|}.$$

Now use dominated convergence.

COROLLARY. *The moments of Y_n converge to those of U. If μ_n denotes the expectation of X_n and Σ_n the covariance, then $A_n(\mu_n - b_n) \to 0$ and $A_n' \Sigma_n A_n \to I$ where I is the identity matrix.*

The next result shows that ANET is precisely the convergence concept we need.

THEOREM 6.4. *Let $f_n = e^{-\psi_n}$ be probability densities on R^d with ψ_n convex and C^2. If $\psi_n'(0)\to 0$ and if $\psi_n''(x)\to I$ uniformly on bounded x-sets, where I is the identity matrix, then the sequence (f_n) is ANET.*

Proof. This follows from the next more technical result by a Taylor expansion.

PROPOSITION 6.5. *Let f_n be probability densities on R^d such that $f_n(x) \sim c_n e^{-\|x\|^2/2}$ uniformly on bounded x-sets in R^d for $n\to\infty$. If $f_n \leq e^{-\psi_n}$ on R^d with ψ_n convex, and if there exists a constant $M>1$ such that for each $x\in R^d$ eventually $f_n(x) > e^{-\psi_n(x)}/M$, then $c_n \to (2\pi)^{-d/2}$ and the sequence (f_n) is ANET.*

Proof. Integration over a ball with radius 1 around the origin shows that the sequence (c_n) is bounded by a constant c. For fixed $x\in R^d$ eventually

$$\tfrac{1}{2}\|x\|^2 - 1 - \log M < \psi_n(x) + \log c_n < \tfrac{1}{2}\|x\|^2 + 1.$$

The second inequality eventually holds uniformly on any closed ball

$$B = \{\|x\| \leq r\}$$

by assumption, and so does the first by convexity of the functions ψ_n; see [**9**, Corollary 10.8.1]. The convex function $\varphi_n = \psi_n + \log c_n + 1 + \log M$ lies between the two paraboloids $\frac{1}{2}\|x\|^2$ and $\frac{1}{2}\|x\|^2 + A$ on the ball $\{\|x\|\leq r\}$, with $A = 4 + \log M$. We may assume that $r \geq 2\sqrt{A}$. Let $\|y\| > r$. Since $\varphi_n \geq \frac{1}{2}\|y\|^2$, we get

$$\varphi_n(y) \geq (\|y\|/r)(\tfrac{1}{2}r^2 - A) \geq \tfrac{1}{4}\|y\|\, r,$$
$$f_n(y) \leq ce^{-\|y\|^2/2} \leq ce^{A}e^{-\varphi_n(y)} \leq cMe^4 e^{-r\|y\|/4}, \qquad \|y\| > r.$$

The integral of f_n over the complement of a large ball is small, and hence the integral over the ball is close to 1. This implies that c_n is close to $(2\pi)^{-d/2}$. Thus f_n converges to the normal density uniformly on bounded sets. Since the constant cM in the inequality above is fixed, and r is arbitrary, we have asymptotic normality with exponential tails.

Now consider the exponential family generated by a vector X in R^d with density f. It is well known that the set

$$\Lambda = \{\lambda \in R^d |\ C(\lambda) = Ee^{(\lambda, X)} < \infty\}$$

is convex and that the cumulant generating function $\lambda \mapsto \log C(\lambda)$ is a convex function on Λ. With the density f we associate the exponential family of densities f_λ, for $\lambda\in\Lambda$, by defining

$$f_\lambda(x) = e^{(\lambda, x)} f(x)/C(\lambda), \quad \text{for } x \in R^d. \tag{6.5}$$

For the proof of Theorem 1.1 we need consider only the one-dimensional case.

THEOREM 6.6. *Let the random variable X have bounded density $f \sim \gamma e^{-\psi}$ where ψ is AP with endpoint t_∞ and scale function σ, and γ is flat. Let $\tau_\infty = \sup \psi'$. Define f_τ, with $\tau\in\Lambda$, by (6.5). Then $\Lambda \cap [0,\infty) = [0, \tau_\infty)$ and $q = \tau^T \to t_\infty$ is equivalent with $\tau = q^T \to \tau_\infty$. The normalized densities $g_\tau(u) = \sigma f_\tau(q + \sigma u)$ are ANET. Note that $q = \tau^T = (\psi')^{\leftarrow}(\tau)$ depends on τ and that $\sigma = \sigma(q)$ depends on q. The moment generating function $C(\tau)$ of the density f and the conjugate transform ψ^* of the function ψ in the exponent of the density f are related by the asymptotic equality*

$$C(\tau) \sim \gamma(q)\sigma(q)(\sqrt{2\pi})e^{\psi^*(\tau)}. \tag{6.6}$$

Proof. Let $0<\tau<\tau_\infty$ and assume $\tau\to\tau_\infty$. Proposition 5.11 gives $\Lambda\cap[0,\infty)=[0,\tau_\infty)$. Define $q=\tau^T$, so that $\psi'(q)=\tau$. Then $q\to t_\infty$ if and only if $\tau\to\tau_\infty$ by strict monotonicity of ψ' on D and by definition of $\tau_\infty=\sup\psi'(D)$. Define normalized densities $g_\tau(u)=\sigma f_\tau(q+\sigma u)$ with $\sigma=\sigma(q)$, $q=\tau^T$. For $x=q+\sigma u$,

$$\begin{aligned}\tau x-\psi(x)&=\tau q+\tau\sigma u-\psi(q)-\psi'(q)\sigma u-\tfrac{1}{2}\psi''(q)\sigma^2u^2+r_\tau(u)\\&=\psi^*(\tau)-\tfrac{1}{2}u^2+r_\tau(u).\end{aligned}$$

The remainder term vanishes uniformly on bounded intervals for $\tau\to\tau_\infty$ by Theorem 5.7. By the same theorem

$$g_\tau(u)\sim c(\tau)e^{-u^2/2}\quad\text{as } \tau\to\tau_\infty$$

uniformly on bounded u-sets where $c(\tau)=\gamma(q)\sigma(q)e^{\psi^*(\tau)}/C(\tau)$.

If $-\log f$ is convex then the functions $-\log g_\tau$ are convex and g_τ is ANET by Proposition 6.5 for $\tau\to\tau_\infty$. In particular, $c(\tau)\to 1/\sqrt{(2\pi)}$ which gives (6.6).

In the general case we only have asymptotic equality. We then enclose $-\log g_\tau$ between two convex functions. This is done as follows: $f\sim e^{-\varphi}$ where φ is asymptotically parabolic, by Proposition 5.10. We may assume that t_∞ is positive and that $\frac{1}{2}\leqslant fe^{\varphi}\leqslant 2$ on a left neighbourhood of t_∞, say on $[0,t_\infty)$ for convenience. Choose $M>1-\varphi(0)$ so that $f\leqslant e^M$ on R, and let φ_0 be the convex hull of the function φ_1 which is identically $-M$ on $(-\infty,0]$ and $\varphi-1$ on $(0,t_\infty)$. Then $\varphi_0\equiv\varphi-1$ on an interval (c,t_∞). (The function φ^* is AP. This implies by Theorem 5.4 that the tangent line to $\varphi-1$ in t intersects the y-axis in a point $y_t\to-\infty$ for $t\to t_\infty$.) Now apply Proposition 5 above: $g_\tau\leqslant e^{-\varphi_\tau}$ with $\varphi_\tau(u)=\varphi(q+u\sigma(q))-\log\sigma(q)-1$ convex. For any $r>1$ we have the inequality $g_\tau\geqslant e^{-\varphi_\tau-2}$ on $[-r,r]$ if we choose τ so large that $[q-r\sigma(q),q+r\sigma(q)]\subset(c,t_\infty)$. This is possible by Theorem 5.7.

7. *The proof*

Let f denote the density of the random vector $X=(X_1,\ldots,X_d)$ with independent components X_i having density $f_i\sim\gamma_ie^{-\psi_i}$ as in the statement of Theorem 1.1. The exponential family of densities f_λ is defined in (6.5). These densities are products. Choose $\lambda=(\tau,\ldots,\tau)$ with $\tau\to\tau_\infty$, and apply Theorem 6.6 to the components X_i to find that the normalized densities $g_{i\tau}(u_i)=\sigma_if_{i\tau}(q_i+\sigma_iu_i)$ are ANET for $\tau\to\tau_\infty$ (equivalently for $q_i\to t_{i\infty}$ where $q=\lambda^T$, that is, $\psi_i'(q_i)=\tau$ for $i=1,\ldots,d$). Hence the multivariate density $\prod_{i=1}^d g_{i\tau}$ is ANET and

$$\prod_{i=1}^d g_{i\tau}(u_i)=\prod_{i=1}^d\sigma_if_{i\tau}(q_i+\sigma_iu_i)\to\prod_{i=1}^d e^{-u_i^2/2}/\sqrt{(2\pi)}. \tag{7.1}$$

If the $X_{i\tau}$ are independent variables with densities $f_{i\tau}$, then the corollary to Proposition 6.3 gives

$$\operatorname{Var}(X_{i\tau})\sim\sigma_i^2,\quad (EX_{i\tau}-q_i)/\sigma_i\to 0 \tag{7.2}$$

for $\tau\to\tau_\infty$. The sum $X_{0\tau}=X_{1\tau}+\ldots+X_{d\tau}$ with density $f_{0\tau}$, say, is ANET for $\tau\to\tau_\infty$ by Theorem 6.2; hence

$$\sigma_0f_{0\tau}(t+\sigma_0u)\to e^{-u^2/2}/\sqrt{(2\pi)}\quad\text{as } \tau\to\tau_\infty \tag{7.3}$$

where, by (7.2) and the convergence-of-types theorem, we may choose $\sigma_0^2=\sigma_1^2+\ldots+\sigma_d^2$ and $t=q_1+\ldots+q_d$ (see the corollary to Proposition 6.3).

Setting $u_i = 0$, $i = 1, \ldots, d$ in (7.1) and (7.3) we find that

$$f_\lambda(q) = \prod_{i=1}^{d} f_{i\tau}(q_i) = e^{(\lambda, q)} f(q)/C(\lambda) \sim \prod_{i=1}^{d} 1/\sigma_i \surd(2\pi), \tag{7.4}$$

and

$$f_{0\tau}(t) = e^{\tau t} f_0(t)/C_0(\tau) \sim 1/\sigma_0 \surd(2\pi). \tag{7.5}$$

Observe that $(\lambda, q) = \tau t$ and $C(\lambda) = Ee^{(\lambda, X)} = Ee^{\tau X_0} = C_0(\tau)$. The relations (7.4) and (7.5) now give

$$f_0(t) \sim \frac{\prod_{i=1}^d (\sigma_i \surd(2\pi))}{\sigma_0 \surd(2\pi)} f(q).$$

Substitute $f(q) = \prod_{i=1}^d \gamma_i(q_i) e^{-\psi_i(q_i)}$ to obtain

$$f_0(t) \sim \frac{\prod_{i=1}^d ((\surd 2\pi)\sigma_i(q_i)\gamma_i(q_i))}{\sigma_0 \surd(2\pi)} \exp\left\{-\sum_{i=1}^{d} \psi_i(q_i)\right\}.$$

If we define ψ_0 and γ_0 by (1.6) and (1.8), that is,

$$\psi_0(t) = \sum_{i=1}^{d} \psi_i(q_i), \quad \gamma_0(t) = \frac{\prod_{i=1}^d (\sigma_i(q_i)\gamma_i(q_i)\surd(2\pi))}{\sigma_0(t)\surd(2\pi)},$$

then we obtain the desired asymptotic identity

$$f_0(t) \sim \gamma_0(t) e^{-\psi_0(t)}.$$

We still have to show that ψ_0 is asymptotically parabolic and that γ_0 is flat.

The first statement holds since the functions ψ_i^* are asymptotically parabolic with common endpoint τ_∞ by Theorem 5.3. This implies that $\psi_0^* = \psi_1^* + \ldots + \psi_d^*$ (see (4.6) and Theorem 4.2) is AP by Proposition 5.2, and hence also $\psi_0 = \psi_0^{**}$ by Theorem 5.3.

Define

$$\delta_i(q_i) = (\surd 2\pi)\sigma_i(q_i)\gamma_i(q_i) \quad \text{for } i = 1, \ldots, d,$$

$$\delta_0(t) = \prod \delta_i(q_i) = (\surd 2\pi)\sigma_0(t)\gamma_0(t).$$

The function δ_i is flat for ψ_i since the factors σ_i and γ_i are flat by assumption (see Proposition 3.1). The function δ_i^T is flat for ψ_i^* (see Proposition 5.6) and hence for $\psi_0^* = \sum \psi_i^*$. (Note that $1/s_0^2 = \psi_0^{*\prime\prime} = \sum \psi_i^{*\prime\prime} = \sum 1/s_i^2$ implies $s_0 < \min s_i$; compare Proposition 3.3.) The product $d_0 = \delta_1^T \ldots \delta_d^T$ is flat for ψ_0^* and hence d_0^T is flat for ψ_0. Now observe that

$$d_0^T(t) = d_0(\tau) = \delta_1^T(\tau) \ldots \delta_d^T(\tau) = \delta_1(q_1) \ldots \delta_d(q_d) = \delta_0(t),$$

which shows that δ_0 is flat for ψ_0, and hence also $\gamma_0 = \delta_0/\sigma_0 \surd(2\pi)$.

References

1. O. Barndorff-Nielsen and C. Klüppelberg, 'A note on the tail accuracy of the univariate saddlepoint approximation', Research Report 238, Department of Theoretical Statistics, University of Aarhus, 1991; *Ann. Toulouse* to appear.
2. N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular variation*, Encyclopedia of Mathematics and its Applications 27 (Cambridge University Press, 1987).

3. L. BROWN, *Fundamentals of statistical exponential families with applications in statistical decision theory*, IMS Lecture Notes—Monographic Series 9 (Institute of Mathematical Statistics, Hayward, California, 1986).
4. D. CLINE, 'Convolution tails, product tails and domains of attraction', *Probab. Th. Rel. Fields* 72 (1986) 529–557.
5. P. FEIGIN and E. YASHCHIN, 'On a strong Tauberian result', *Z. Wahrsch. Verw. Gebiete* 65 (1980) 35–48.
6. J. L. GELUK and L. DE HAAN, *Regular variation, extensions and Tauberian theorems*, CWI Tract 40 (Centrum voor Wiskunde en Informatica, Amsterdam, 1987).
7. L. DE HAAN, *On regular variation and its application to the weak convergence of sample extremes*, Mathematical Centre Tract 32 (Mathematics Centre, Amsterdam, 1970).
8. S. RESNICK, *Extreme values, regular variation and point processes* (Springer, New York, 1987).
9. R. T. ROCKAFELLAR, *Convex analysis* (Princeton University Press, 1970).
10. H. ROOTZEN, 'A ratio limit theorem for the tails of weighted sums', *Ann. Probab.* 15 (1987) 728–747.
11. J. VAN TIEL, *Convex analysis* (Wiley, New York, 1984).

University of Amsterdam
FWI
Plantage Muidergracht 24
1018 *TV Amsterdam*
Holland

Department of Mathematics
ETH
Zurich
Switzerland

School of Operations Research
and Industrial Engineering
ETC Building
Cornell University
Ithaca
New York 14853
U.S.A.