A Strictly Dissipative State Space Representation of Second Order Systems

Eine strikt dissipative Zustandsraumdarstellung von Systemen zweiter Ordnung

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Summary This article introduces a novel way of transforming linear second order systems to first order state space models in strictly dissipative realization. This property extends the potential for the analysis of large-scale systems in many ways: it allows for the application of efficient methods for the solution of Lyapunov equations and the guaranteed preservation of asymptotic stability during model order reduction. The transformation is easy to implement and applicable to models of very high order due to its negligible computational effort. ►►► **Zusammenfassung** Dieser Beitrag stellt eine Transformation linearer Systeme zweiter Ordnung auf strikt dissipative Zustandsraummodelle vor. Strikte Dissipativität erweitert die Möglichkeiten zur Analyse sehr großer Modelle in mehrfacher Hinsicht: Beispielsweise erlaubt sie den Einsatz effizienter Methoden zur Lösung von Ljapunow-Gleichungen oder die garantierte Erhaltung von asymptotischer Stabilität bei der Modellordnungsreduktion. Aufgrund ihres vernachlässigbaren numerischen Aufwands ist die einfach zu implementierende Transformation auch auf größte Modelle anwendbar.

Keywords Dissipativity, second order systems, model order reduction ►►► **Schlagwörter** Dissipativität, Systeme zweiter Ordnung, Modellordnungsreduktion

1 Introduction

Mathematical modeling of complex dynamic systems, like mechanical structures or integrated circuits, often leads to linear time-invariant (LTI) systems $\mathbf{G}(s)$ of the form

$$\begin{aligned} \mathbf{E}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\,, \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t)\,, \end{aligned} \tag{1}$$

with $m \in \mathbb{N}$ inputs, $p \in \mathbb{N}$ outputs and $n \in \mathbb{N}$ state variables, where n is called the order of the system. $\mathbf{E}$, $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times p}$ and $\mathbf{C} \in \mathbb{R}^{m\times n}$ are matrices with constant coefficients; in this paper, we assume $\mathbf{E}$ to be regular. $\mathbf{u}(t) \in \mathbb{R}^p$, $\mathbf{y}(t) \in \mathbb{R}^m$ and $\mathbf{x}(t) \in \mathbb{R}^n$ are, respectively, the input, output and state vectors.

With increasing demands on the accuracy of the model, its order n typically grows dramatically, for instance due to finer meshing in a finite element method (FEM). The direct analysis or simulation of such a system is then numerically expensive or not even feasible [1].

Instead, one aims to approximate $\mathbf{G}(s)$ by a system of far smaller order $q \ll n$ by means of so-called model order reduction (MOR). Unfortunately, most MOR techniques, like proper orthogonal decomposition, approximate balanced truncation or Krylov subspace methods do not guarantee preservation of stability during the reduction process, in general.

One remedy to this issue is the theory of the logarithmic norm, which was introduced independently by Dahlquist and Lozinskii in the 1950's and defined as

$$\mu(\mathbf{A}) = \lim_{h\downarrow 0} \frac{||\mathbf{I} + h\,\mathbf{A}|| - 1}{h}\,. \tag{2}$$

Originally, it was intended to bound the error growth of solutions to ODE systems; one central finding was the bound on the matrix exponential [3; 9]:

$$||e^{\mathbf{A}t}|| \leq e^{\mu(\mathbf{A})\,t} \quad \forall\, t \geq 0\,. \tag{3}$$

Meanwhile, however, the theory of the logarithmic norm has been used for various applications, among which we recall the following sufficient conditions for preservation of stability during the reduction of $\mathbf{G}(s)$: when the logarithmic norm of $\mathbf{A}$ is non-positive while $\mathbf{E}$ is positive definite (e. g. identity), stability of the reduced model is assured in a one-sided (GALERKIN-type) projection [4]. A *strictly* negative value of $\mu(\mathbf{A})$ can even guarantee *asymptotic* stability of the reduced order model.

The same prerequisite also holds in other domains, including the solution of LYAPUNOV equations: the recently introduced KPIK algorithm [11], for instance, exhibits remarkable speed-up compared to standard solvers but requires $\mu(\mathbf{A})$ to be strictly negative.

Whenever this is not the case, it has been shown that an asymptotically stable system can be transformed into an equivalent realization that fulfills $\mu(\mathbf{A}) < 0$ and $\mathbf{E} = \mathbf{I}$ [6; 7]. But the computation of a suitable state transformation requires the solution of a LYAPUNOV inequality, which is not feasable for large n.

The modeling of mechanical systems or electric circuits, on the other hand, often leads to ODE systems of second order form:

$$\begin{aligned} &\mathbf{M}\ddot{\mathbf{z}}(t) + \mathbf{D}\dot{\mathbf{z}}(t) + \mathbf{K}\mathbf{z}(t) = \mathbf{F}\mathbf{u}(t)\,, \\ &\mathbf{y}(t) = \mathbf{S}\mathbf{z}(t)\,, \end{aligned} \tag{4}$$

where $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{K}$ are called mass, damping and stiffness matrices. For a large class of finite element models $\mathbf{M} = \mathbf{M}^T > \mathbf{0}$, $\mathbf{K} = \mathbf{K}^T > \mathbf{0}$ and $\mathbf{D} = \mathbf{D}^T > \mathbf{0}$ can be assumed. The special structure of these systems can be used to reformulate them equivalently in a non-strictly dissipative state space model, i. e. such that $\mathbf{E} > \mathbf{0}$ and $\mu(\mathbf{A}) = 0$ hold; see [10] and Sect. 2.3.

In this work, it is shown how second order systems with the above described properties can be arranged differently in state space such that $\mu(\mathbf{A})$ is *strictly* negative and the logarithmic norm theory applies to its full extent.

The rest of this paper is organized as follows: in Sect. 2, related preliminary knowledge is given; the new modeling is described in Sect. 3; numerical results are given in Sect. 4, followed by conclusions.

2 Preliminaries and Problem Formulation

In this section, preliminaries towards the logarithmic norm in LTI systems are presented. We first introduce the logarithmic 2-norm and its application to state space and descriptor systems, respectively. We then show how second order systems can be transformed to (non-strictly) dissipative first order form.

2.1 The Logarithmic Norm Induced by the Euclidian Norm

We will focus on the logarithmic norm subordinate to the Euclidian norm. For this case, $\mu_2(\mathbf{A})$ can be expressed as

$$\mu_2(\mathbf{A}) = \lambda_{max}(\operatorname{sym}\mathbf{A}) = \lambda_{max}\left(\frac{\mathbf{A}+\mathbf{A}^T}{2}\right), \tag{5}$$

which is the right-most eigenvalue of the symmetric part of $\mathbf{A}$ [3], also called the *numerical abscissa* of $\mathbf{A}$.

The matrix $\mathbf{A}$ is called (strictly) dissipative if $\mu_2(\mathbf{A}) \leq 0$ ($\mu_2(\mathbf{A}) < 0$) [12].

2.2 The Logarithmic Norm in Descriptor Systems

It is well known, that systems (1) with regular $\mathbf{E} \neq \mathbf{I}$ can be transformed to an explicit representation or realization, respectively, e. g. by premultiplication of the ODE system by $\mathbf{E}^{-1}$ or by a change of basis $\mathbf{z} = \mathbf{E}\mathbf{x}$.

Inspired by the form of (5), a third possibility is often favorable if $\mathbf{E}$ is positive definite: using the CHOLESKY factorization $\mathbf{E} = \mathbf{L}\mathbf{L}^T$, system (1) can be transformed in the following way:

$$\begin{aligned} \overbrace{\mathbf{L}^{-1}\mathbf{E}\mathbf{L}^{-T}}^{\mathbf{I}}\,\dot{\widehat{\mathbf{x}}}(t) &= \overbrace{\mathbf{L}^{-1}\mathbf{A}\mathbf{L}^{-T}}^{\widehat{\mathbf{A}}}\,\widehat{\mathbf{x}}(t) + \overbrace{\mathbf{L}^{-1}\mathbf{B}}^{\widehat{\mathbf{B}}}\,\mathbf{u}(t)\,, \\ \mathbf{y}(t) &= \underbrace{\mathbf{C}\mathbf{L}^{-T}}_{\widehat{\mathbf{C}}}\,\widehat{\mathbf{x}}(t)\,. \end{aligned} \tag{6}$$

The logarithmic norm of the transformed dynamic matrix $\widehat{\mathbf{A}}$ is then equivalent to

$$\begin{aligned} \mu_2(\widehat{\mathbf{A}}) &= \mu_2\left(\mathbf{L}^{-1}\mathbf{A}\mathbf{L}^{-T}\right) = \\ &= \frac{1}{2}\lambda_{max}\left[\mathbf{L}^{-1}\mathbf{A}\mathbf{L}^{-T} + \left(\mathbf{L}^{-1}\mathbf{A}\mathbf{L}^{-T}\right)^T\right] = \\ &= \frac{1}{2}\lambda_{max}\left[\mathbf{L}^{-1}\left(\mathbf{A}+\mathbf{A}^T\right)\mathbf{L}^{-T}\right]. \end{aligned} \tag{7}$$

One can see from Eq. (7) that the eigenvalues λ_i of $\operatorname{sym}\widehat{\mathbf{A}}$ are identical to the solutions of the generalized eigenvalue problem $\det(\operatorname{sym}\mathbf{A} - \lambda_i\mathbf{E}) = 0$. For that reason, the resulting value is also referred to as [5]

$$\mu_{\mathbf{E}}(\mathbf{A}) := \mu_2(\mathbf{E},\mathbf{A}) := \mu_2(\widehat{\mathbf{A}}) \tag{8}$$

In order to compute $\mu_{\mathbf{E}}(\mathbf{A})$, the explicit calculation of $\widehat{\mathbf{A}} = \mathbf{L}^{-1}\mathbf{A}\mathbf{L}^{-T}$ can therefore be avoided by finding the right-most generalized eigenvalue instead. A possible Matlab implementation is

```
mu = eigs(A+A', E, 1, 'LA')/2;
```

Please note that if $\mu_2(\mathbf{A}) < 0$, then $\mu_2(\widehat{\mathbf{A}})$ can be shown to be negative as well, since $\mathbf{L}$ is of full rank:

$$\begin{aligned} \mu_2(\mathbf{A}) < 0 &\Longleftrightarrow \mathbf{A}+\mathbf{A}^T < \mathbf{0} \\ &\Longleftrightarrow \mathbf{x}^T(\mathbf{A}+\mathbf{A}^T)\mathbf{x} < 0 \qquad \forall\,\mathbf{x} \\ &\Longleftrightarrow \underbrace{\mathbf{x}^T\mathbf{L}}_{\mathbf{y}^T}\,\mathbf{L}^{-1}(\mathbf{A}+\mathbf{A}^T)\mathbf{L}^{-T}\,\underbrace{\mathbf{L}^T\mathbf{x}}_{\mathbf{y}} < 0\,\forall\,\mathbf{x} \\ &\Longleftrightarrow \mathbf{y}^T\mathbf{L}^{-1}(\mathbf{A}+\mathbf{A}^T)\mathbf{L}^{-T}\mathbf{y} < 0 \qquad \forall\,\mathbf{y} \\ &\Longleftrightarrow \widehat{\mathbf{A}}+\widehat{\mathbf{A}}^T < \mathbf{0} \\ &\Longleftrightarrow \mu_2(\widehat{\mathbf{A}}) < 0\,. \end{aligned} \tag{9}$$

Accordingly, every system with $\mu_2(\mathbf{A})<0$ and $\mathbf{E}>\mathbf{0}$ can be transformed in a way such that $\widehat{\mathbf{E}}=\mathbf{I}$ and $\widehat{\mathbf{A}}$ preserves the dissipativity of $\mathbf{A}$.

2.3 Transforming Second Order Systems to First Order

Second order systems (4) can be transformed into an equivalent state space formulation [10]:

$$\begin{bmatrix}\mathbf{R} & \mathbf{0}\\ \mathbf{0} & \mathbf{M}\end{bmatrix}\begin{bmatrix}\dot{\mathbf{z}}(t)\\ \ddot{\mathbf{z}}(t)\end{bmatrix}=\begin{bmatrix}\mathbf{0} & \mathbf{R}\\ -\mathbf{K} & -\mathbf{D}\end{bmatrix}\begin{bmatrix}\mathbf{z}(t)\\ \dot{\mathbf{z}}(t)\end{bmatrix}+\begin{bmatrix}\mathbf{0}\\ \mathbf{F}\end{bmatrix}\mathbf{u}(t)\,,$$
$$\mathbf{y}(t)=\begin{bmatrix}\mathbf{S} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{z}(t)\\ \dot{\mathbf{z}}(t)\end{bmatrix}. \qquad (10)$$

As the first line only serves to constitute equality of $\dot{\mathbf{z}}(t)=\dot{\mathbf{z}}(t)$, it can be equivalently premultiplied by any regular matrix $\mathbf{R}$ of appropriate size. Setting $\mathbf{R}=\mathbf{K}$, as it was suggested in [10], can be directly seen to deliver $\mathbf{E}>\mathbf{0}$ and $\mu_2(\mathbf{A})=0$, as the symmetric part of $\mathbf{A}$ only contains $\mathbf{D}$ in the bottom-right entry, which makes it negative semi-definite. This is beneficial in many respects but not sufficient in others, as we have seen.

3 Strictly Dissipative State Space Formulation

In this section, the main contribution of the article is presented: It is shown how second order systems (4) can be transformed to a state space representation with positive definite $\mathbf{E}$ and *strictly* dissipative $\mathbf{A}$.

3.1 A New Arrangement in State Space

The main idea is to premultiply (10) by a matrix

$$\mathbf{T}:=\begin{bmatrix}\mathbf{I} & \alpha\mathbf{I}\\ \alpha\mathbf{M}\mathbf{K}^{-1} & \mathbf{I}\end{bmatrix} \qquad (11)$$

with $\alpha\in\mathbb{R}^+$. This changes the representation of (1) towards

$$\overbrace{\mathbf{TE}}^{\widetilde{\mathbf{E}}}\;\dot{\mathbf{x}}(t)=\overbrace{\mathbf{TA}}^{\widetilde{\mathbf{A}}}\;\mathbf{x}(t)+\overbrace{\mathbf{TB}}^{\widetilde{\mathbf{B}}}\;\mathbf{u}(t)\,,$$
$$\mathbf{y}(t)=\mathbf{C}\;\mathbf{x}(t)\,, \qquad (12)$$

with

$$\widetilde{\mathbf{A}}=\begin{bmatrix}-\alpha\mathbf{K} & \mathbf{K}-\alpha\mathbf{D}\\ -\mathbf{K} & -\mathbf{D}+\alpha\mathbf{M}\end{bmatrix},$$
$$\widetilde{\mathbf{E}}=\begin{bmatrix}\mathbf{K} & \alpha\mathbf{M}\\ \alpha\mathbf{M} & \mathbf{M}\end{bmatrix},\quad \widetilde{\mathbf{B}}=\begin{bmatrix}\alpha\mathbf{F}\\ \mathbf{F}\end{bmatrix}. \qquad (13)$$

It is stressed that neither the state vector $\mathbf{x}(t)$ nor the output $\mathbf{y}(t)$ are affected by the transformation; besides, $\widetilde{\mathbf{E}}$ remains symmetric. It will be shown in the following, though, that the choice of α has a strong effect on the eigenvalues of $\widetilde{\mathbf{E}}$ and the logarithmic norm of $\widetilde{\mathbf{A}}$. This dependency will be used to find a strictly dissipative system representation.

Looking at $\widetilde{\mathbf{E}}$, it is clear that due to continuity the positive definiteness of $\mathbf{K}$ and $\mathbf{M}$ will be carried over to $\widetilde{\mathbf{E}}$ for small values of α. To investigate the logarithmic norm of $\widetilde{\mathbf{A}}$, on the other hand, we consider the symmetric part of $\widetilde{\mathbf{A}}$:

$$\operatorname{sym}\widetilde{\mathbf{A}}=\frac{\widetilde{\mathbf{A}}+\widetilde{\mathbf{A}}^T}{2}=\begin{bmatrix}-\alpha\mathbf{K} & -\frac{\alpha}{2}\mathbf{D}\\ -\frac{\alpha}{2}\mathbf{D} & -\mathbf{D}+\alpha\mathbf{M}\end{bmatrix} \qquad (14)$$

Different from the symmetric part of $\mathbf{A}$ in (10), the upper left diagonal block is now negative definite; the eigenvalues of $-\alpha\mathbf{K}$ drop as α increases. This leads to an evident trade-off, because at the same time the eigenvalues of the bottom right block, $-\mathbf{D}+\alpha\mathbf{M}$, climb too far unless α is chosen sufficiently small. Therefore, it seems plausible that the matrix can be negative definite in a certain range of α – which leads to the desired negative logarithmic norm of $\widehat{\mathbf{A}}$ – although the influence of the off-diagonal blocks is not clear a priori.

3.2 How Large May α Be?

We shall clearify and quantify this idea in the following.

Theorem 1. *Let $\widetilde{\mathbf{E}},\widetilde{\mathbf{A}}$ be defined as in (13). Then the following conditions hold for α:*

a) $\alpha<\alpha^*_{\mathbf{E}}:=\sqrt{\lambda_{min}(\mathbf{K}\mathbf{M}^{-1})}\quad\Rightarrow\quad\widetilde{\mathbf{E}}>\mathbf{0}$.

b) $\alpha<\alpha^*_{\mathbf{A}}:=\lambda_{min}\left[\mathbf{D}(\mathbf{M}+\frac{1}{4}\mathbf{D}\mathbf{K}^{-1}\mathbf{D})^{-1}\right]$
$\Rightarrow\quad\mu_2(\widetilde{\mathbf{A}})<0$.

Proof: Both parts follow directly from Schur's Lemma on positive definite matrices [13].

a) $\widetilde{\mathbf{E}}$ is positive definite iff
i) $\mathbf{M}>\mathbf{0}$, which is true by assumption,
ii) $\mathbf{K}-\alpha\mathbf{M}\mathbf{M}^{-1}\alpha\mathbf{M}>\mathbf{0}\iff\mathbf{K}-\alpha^2\mathbf{M}>\mathbf{0}$,
which is equivalent to $\alpha<\alpha^*_{\mathbf{E}}$.

b) $\operatorname{sym}\widetilde{\mathbf{A}}$ is negative definite iff
i) $-\alpha\mathbf{K}<\mathbf{0}$, which is true by assumption,
ii) $-\mathbf{D}+\alpha\mathbf{M}-\frac{\alpha}{2}\mathbf{D}(-\alpha\mathbf{K})^{-1}\frac{\alpha}{2}\mathbf{D}<\mathbf{0}\iff$
$-\mathbf{D}+\alpha(\mathbf{M}+\frac{1}{4}\mathbf{D}\mathbf{K}^{-1}\mathbf{D})<\mathbf{0}$,
which is equivalent to $\alpha<\alpha^*_{\mathbf{A}}$. □

In order to obtain a system with positive definite $\widetilde{\mathbf{E}}$ and strictly dissipative $\widetilde{\mathbf{A}}$, both conditions must be true. Accordingly, when $\alpha^*_{\mathbf{E}}$ and $\alpha^*_{\mathbf{A}}$ are computed separately, any positive value of α which is smaller than $\min\{\alpha^*_{\mathbf{E}},\alpha^*_{\mathbf{A}}\}$ fulfills the required properties.

It turns out, however, that $\alpha^*_{\mathbf{E}}$ is always greater or equal to $\alpha^*_{\mathbf{A}}$, which is shown in the following:

Theorem 2. $\alpha^*_{\mathbf{E}}\geq\alpha^*_{\mathbf{A}}$.

Proof: We will show that $\widetilde{\mathbf{E}}>\mathbf{0}$ is always true when $\operatorname{sym}\widetilde{\mathbf{A}}<\mathbf{0}$ is fulfilled. To this end, we define

$$\mathbf{W}:=\begin{bmatrix}\frac{1}{\sqrt{\alpha}}\mathbf{I} & \mathbf{0}\\ 2\sqrt{\alpha}\mathbf{M}\mathbf{K}^{-1} & -2\sqrt{\alpha}\mathbf{M}\mathbf{D}^{-1}\end{bmatrix}. \qquad (15)$$

Obviously, $\operatorname{sym}\widetilde{\mathbf{A}}<\mathbf{0}$ is equivalent to

$$\mathbf{0}<\mathbf{X}:=\mathbf{W}(-\operatorname{sym}\widetilde{\mathbf{A}})\mathbf{W}^T=\begin{bmatrix}\mathbf{K} & \alpha\mathbf{M}\\ \alpha\mathbf{M} & 4\alpha\mathbf{M}\mathbf{D}^{-1}(\mathbf{D}-\alpha\mathbf{M})\mathbf{D}^{-1}\mathbf{M}\end{bmatrix}.$$

We want to show that $\widetilde{\mathbf{E}} \geq \mathbf{X}$. It is sufficient to compare the bottom-right entry:

$$\begin{aligned}
&\widetilde{\mathbf{E}} \geq \mathbf{X} = \mathbf{W}(-\mathrm{sym}\,\widetilde{\mathbf{A}})\mathbf{W}^T \\
\Longleftrightarrow\ &\widetilde{\mathbf{E}} - \mathbf{W}(-\mathrm{sym}\,\widetilde{\mathbf{A}})\mathbf{W}^T \geq \mathbf{0} \\
\Longleftrightarrow\ &\mathbf{M} - 4\alpha\mathbf{M}\mathbf{D}^{-1}(\mathbf{D}-\alpha\mathbf{M})\mathbf{D}^{-1}\mathbf{M} \geq \mathbf{0} \\
\Longleftrightarrow\ &\mathbf{D}\mathbf{M}^{-1}\mathbf{D} - 4\alpha(\mathbf{D}-\alpha\mathbf{M}) \geq \mathbf{0}
\end{aligned}$$

Using the Cholesky decomposition $\mathbf{M} = \mathbf{R}\mathbf{R}^T$:

$$\begin{aligned}
\Longleftrightarrow\ &4\alpha^2\mathbf{R}\mathbf{R}^T - 4\alpha\mathbf{D} + \mathbf{D}(\mathbf{R}\mathbf{R}^T)^{-1}\mathbf{D} \geq \mathbf{0} \\
\Longleftrightarrow\ &4\alpha^2\mathbf{I} - 4\alpha\mathbf{R}^{-1}\mathbf{D}\mathbf{R}^{-T} + \mathbf{R}^{-1}\mathbf{D}\mathbf{R}^{-T}\mathbf{R}^{-1}\mathbf{D}\mathbf{R}^{-T} \geq \mathbf{0} \\
\Longleftrightarrow\ &(2\alpha\mathbf{I} - \mathbf{R}^{-1}\mathbf{D}\mathbf{R}^{-T})^T(2\alpha\mathbf{I} - \mathbf{R}^{-1}\mathbf{D}\mathbf{R}^{-T}) \geq \mathbf{0}
\end{aligned}$$

The last line is always true; accordingly, $\widetilde{\mathbf{E}} \geq \mathbf{X}$ holds. Therefore, $\mu_2(\widetilde{\mathbf{A}}) < 0$ induces $\widetilde{\mathbf{E}} \geq \mathbf{X} > \mathbf{0}$, which completes the proof. □

From the above considerations it follows that α should neither be too small nor too large in order to make $\mu_{\widetilde{\mathbf{E}}}(\widehat{\mathbf{A}})$ as negative as possible. Without claim of optimality, it is therefore suggested to choose

$$\alpha := \frac{\alpha^*_{\mathbf{A}}}{2}. \tag{16}$$

3.3 Numerical Considerations

Please note that both $\alpha^*_{\mathbf{E}}$ (which is not actually required, as we have seen above) and $\alpha^*_{\mathbf{A}}$ can be computed as solutions to generalized eigenvalue problems involving only symmetric matrices:

$$a)\ \mathbf{K}\mathbf{v}_i = \lambda_i^2 \cdot \mathbf{M}\mathbf{v}_i$$

$$b)\ \mathbf{D}\mathbf{v}_i = \lambda_i \cdot \left(\mathbf{M} + \frac{1}{4}\mathbf{D}\mathbf{K}^{-1}\mathbf{D}\right)\mathbf{v}_i$$

with eigenvalues λ_i and corresponding eigenvectors $\mathbf{v}_i$.

The task of finding the smallest solution λ_{min} of such symmetric problems has extensively been studied and can typically be performed without tremendous numerical effort [8]. In particular, one can avoid the explicit calculation of matrix inverses (e. g. $\mathbf{D}\mathbf{K}^{-1}\mathbf{D}$) by adept implementation. In Matlab, for instance, the generalized eigenvalue problem *b)* can be solved by passing an anonymous function to the `eigs` command:

```
mfun = @(x) (M*x + D*(K\(D*x))/4);
n = size(D, 1);
alpha = 1/eigs(mfun, n, D, 1, 'LR');
```

4 Numerical Example: Butterfly Gyroscope

In order to demonstrate the procedure derived above, we use the model of a Butterfly Gyroscope [2]. The dynamics is described by a second order system of 17361 ODEs; damping is modeled artificially as $\mathbf{D} = 10^{-6}\mathbf{K}$.

For this system, the computation of α has been performed according to Sect. 3, lasted below 5 sec and yielded $\alpha^*_{\mathbf{A}} = 113.06$ (for completeness, $\alpha^*_{\mathbf{E}} = 10\,633$ was computed, too). According to Eq (16), we choose $\alpha = 56.529$. In fact, the resulting system fulfills all desired properties: $\widetilde{\mathbf{E}} > \mathbf{0}$ and $\mathrm{sym}\,\widetilde{\mathbf{A}} < \mathbf{0}$ hold; the logarithmic norm of $\widehat{\mathbf{A}}$ is $\mu_2(\widehat{\mathbf{A}}) = \mu_{\widetilde{\mathbf{E}}}(\widetilde{\mathbf{A}}) = -56.528$.

Additionally, the transformed system (12) has been analyzed for multiple values of α; results can be seen in Fig. 1. Figure 1a shows the smallest eigenvalue of $\widetilde{\mathbf{E}}$ over α, Fig. 1b and c give the logarithmic 2-norm of $\widetilde{\mathbf{A}}$ and $\widehat{\mathbf{A}}$, respectively.

One can see that the smallest eigenvalue of $\widetilde{\mathbf{E}}$ merely decays (although the graph is slightly disturbed by numerical noise); positive definiteness is successfully preserved. This is due to the fact that $\alpha^*_{\mathbf{E}}$ is about 100 times larger than the visible section of α. The right-

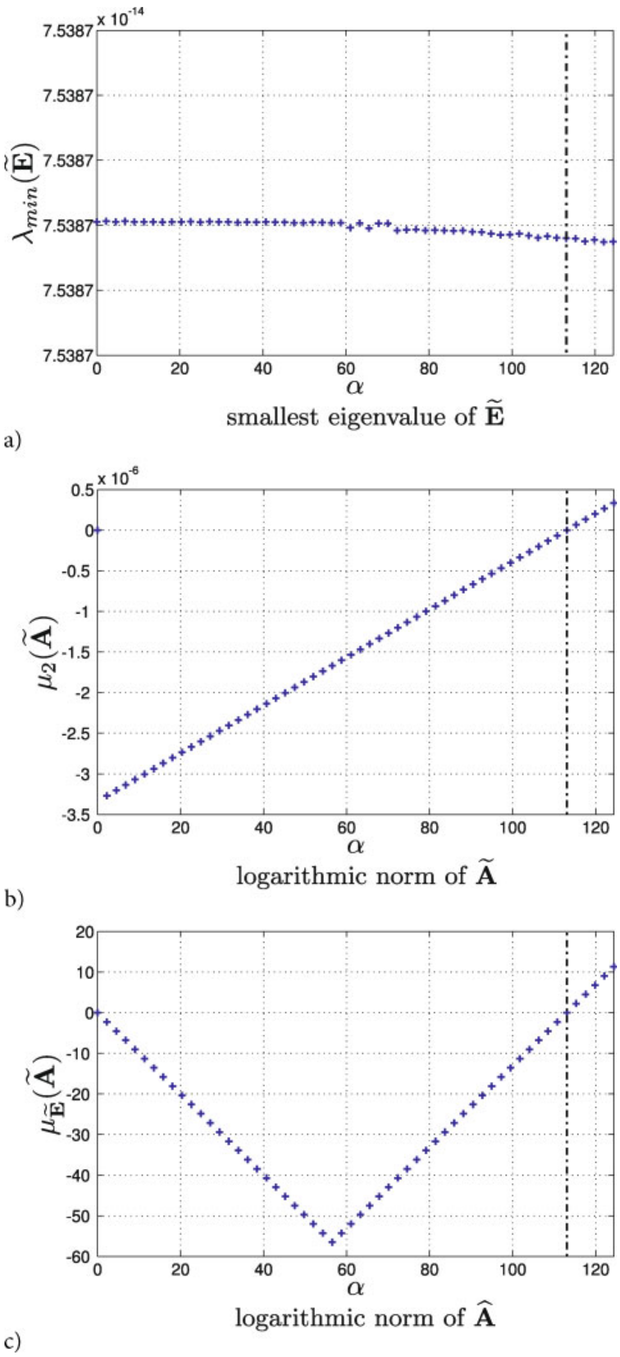

Figure 1 Simulation results for the Butterfly Gyroscope.

most eigenvalue of $\mathrm{sym}\,\widetilde{\mathbf{A}}$, on the other hand, drops abruptly to about -3.3×10^{-6} even for very small values of α – although it is clearly zero for $\alpha = 0$. With increasing α it rises towards zero again and changes sign at $\alpha = \alpha^*_{\mathbf{A}} = 113.06$, which confirms the results from above. Looking at the evolution of the generalized logarithmic norm $\mu_{\widetilde{\mathbf{E}}}(\widetilde{\mathbf{A}})$, finally, one can perfectly see a minimum which is indeed almost caught by our choice of $\alpha = 56.529$.

In fact, the graph of $\mu_{\widetilde{\mathbf{E}}}(\widetilde{\mathbf{A}})$ resembles a kind of shifted absolute value function. This observation has been confirmed also for other models with moderate damping. The exact relationship between α and $\mu_{\widetilde{\mathbf{E}}}(\widetilde{\mathbf{A}})$ is, however, more complicated.

Please note, finally, that among $\mu_2(\widetilde{\mathbf{A}})$ and $\mu_{\widetilde{\mathbf{E}}}(\widetilde{\mathbf{A}})$, the latter is the quantity whose absolute value matters for applications like error bounds or the analysis of the transient system behavior, as $\mu_2(\widetilde{\mathbf{A}})$ disregards the influence of the matrix $\widetilde{\mathbf{E}}$ and therefore uses a non-physical norm.

5 Conclusions and Outlook

We have shown how large scale second order systems with standard definiteness properties can be transformed into state space models with positive definite $\mathbf{E}$ and strictly dissipative $\mathbf{A}$. This is achieved by a novel arrangement of the matrices which depends on a scalar parameter α.

Although no optimal choice w.r.t. the resulting logarithmic norm of $\mathbf{A}$ has been derived so far, a straightforward way to compute a valid interval for α has been presented. Numerical examples supported that in practice it is satisfactory to choose α in the middle of this interval.

Future work is aimed at various generalizations of the restrictive case $\mathbf{M}, \mathbf{D}, \mathbf{K} > \mathbf{0}$ presented in this article. First studies indicate that definiteness is not generally necessary to allow for a strictly dissipative state space realization. For instance, gyroscopic (i. e. skew-symmetric) components in $\mathbf{D}$ hardly affect the above results, if the symmetric part of $\mathbf{D}$ remains positive definite.

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