Oscillator Network Synchronization by Distributed Control

Y. Mozafari, A. Kiani, and S. Hirche[1]

Abstract— **The design and analysis of oscillator networks raise numbers of fundamental questions in systems and control. The stability and robustness of oscillator networks are the most significant challenges that must be addressed in control and communication design of such networks. In this paper we investigate the problem of synchronization in an oscillator network modeled by a non-uniform Kuramoto model. Conditions under which the network is synchronized, are studied and the procedure for designing distributed controllers, where information exchange between controllers occurs through communication network, is proposed. The guideline for designing communication topology for the overall distributed controllers is presented. Numerical studies are reported to validate the theoretical results and performance of distributed controllers.**

I. Introduction

Oscillator networks and their relevant key concept of synchronization provide a framework for describing many phenomena in biological and engineered systems, such as, flashing fireflies [1], cardiac pacemaker cells [2], laser arrays [3], and power grids [4]. Synchronization is achieved when all oscillators frequencies converge to a common frequency. More specifically, if the oscillators phases as well as their frequencies converge to a common value, phase synchronization is achieved. Synchronization is widely studied in literature, see e.g. [5], [6], [7], and [8]. One of the most successful models developed for describing the dynamic of oscillator networks is the Kuramoto model introduced in 1970s [9]. So far, many different extensions of the classic Kuramoto model are proposed, see e.g. [10].

Stability analysis of the Kuramoto model for arbitrary network topology and non-identical nodes, is carried out in [11]. The authors go beyond the all-to-all networks of identical oscillators and find some critical bounds on the coupling value. They find a necessary value, below which there is no totally synchronized state and show that there always exists a finite and large enough coupling value, beyond which synchronization is achieved [11]. In [12], a general Kuramoto model with heterogeneous time delays is investigated and it is proved that for some specific conditions, synchronization occurs even in the presence of delays. The necessary bound for the coupling value, accomplished in [11], was improved in [13]. The authors in [13] also find a sufficient condition for almost global exponential synchronization for finite N oscillators. In [14], the non-uniform Kuramoto model is proposed, which is derived based on power grid dynamics equations. The necessary as well as sufficient conditions for synchronization of the non-uniform Kuramoto model are proposed in [14].

In this paper we focus on engineered systems represented by an oscillator network and the controller design to achieve synchronization. Recently, the control design for synchronizing oscillator networks has received attention, such as, adaptive control [15], pinning control [16], and impulsive control [17]. In all these works a decentralized control law is developed, i.e. the local controller of an oscillator subsystem only uses local states for the control computation. In this paper, we concentrate on the design of distributed control laws, where the local controllers have access not only to their local subsystem states but also to the states of neighbor subsystems. Distributed control laws are known to provide better performance than decentralized laws for large-scale interconnected systems. Information exchange between the neighboring subsystems is realized through a communication network, i.e. the communication topology is an additional design parameter for the distributed control law. In [18], a topology-independent decentralized as well as distributed control law is designed for an infinite string of identical and linear subsystems. The design of a distributed control for local synchronization of dynamical networks is investigated in [19], where the constrained control design is formulated as a mixed-integer nonlinear optimization problem. The authors in [20], consider an interconnected system of heterogeneous, linear time-invariant subsystems and pose the problem of joint control gain and communication topology design and propose a two-layer architecture that is robust to communication link failure. First, a decentralized controller is designed to stabilize the overall system, and then the performance is improved by designing the distributed control law. The communication topology is also considered as a design parameter and the problem is formulated as a mixed-integer optimization problem.

The contribution of this paper is the design of a distributed control law for a finite and heterogeneous oscillator network, with arbitrary undirected physical network topology, modeled by non-uniform Kuramoto model. The only assumption on network topology is connectedness of the underlying physical graph. If the connectivity of the physical graph is small, which is expressed in terms of small algebraic connectivity [21], synchronization cannot be guaranteed. For this case we design a distributed control law to guarantee synchronization of the oscillator network. In the design procedure we also need to determine where to add the communication links between local controllers. The distributed control design procedure is proposed as an algorithm based on graph theory.

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This paper is organized as follows: in section II, the problem setup is presented and non-uniform Kuramoto model with a brief overview of its synchronization properties are discussed. In section III, the distributed controller design procedure is proposed. Simulation results are shown in section IV and finally concluding remarks are presented in section V.

II. POBLEM SETUP

A. Preliminaries

For better understanding of issues addressed in this paper, we first present a brief introduction to graph theory. A graph $G(V, E)$ consists of a set of N vertices and a set of M edges. We denote each vertex by $v_i \in V$ for $i = \{1, \cdots, N\}$ and each edge by $e_k \in E$, $k = \{1, 2, \cdots, M\}$. If $v_i, v_j \in V$ and $(v_i, v_j) \in E$, then v_i and v_j are neighbors and they are connected through edge e_l as $e_l \sim (v_i, v_j)$. The set $(E)^c$ is the complement of E, which includes $\frac{N(N-1)}{2} - M$ edges and $E \bigcup (E)^c$ is the set of edges of a complete graph with N nodes. All graphs in this paper are considered undirected. A graph G is said to be connected if there is a path between any two vertices in the graph G. In a completed graph all distinct vertices are connected by an edge. The adjacency matrix $A(G) = [A_{ij}]$ of an undirected graph is an $N \times N$ symmetric matrix, such that $A_{ij} = 1$ if v_i and v_j are neighbors, and $A_{ij} = 0$ otherwise. A graph is weighted if we assign a positive number W_k to each edge, and $W_{M \times M} := diag(W_k)$ is the corresponding weight matrix. Elements of the adjacency matrix of a weighted graph are the weights of the edges. The incidence matrix $B(G) = [B_{ij}]$ of an oriented graph G^σ (with σ being some assigned orientation) is an $N \times M$ matrix such that $B_{ij} = 1$ if edge j enters vertex i, $B_{ij} = -1$ if edge j leaves vertex i, and zero otherwise. The symmetric $N \times N$ matrix defined as $L = BB^T$ is called the Laplacian of G and is independent of choice of orientation. The Laplacian has several important properties: (1) L is always positive semidefinite with a zero eigenvalue; (2) The algebraic multiplicity of its zero eigenvalue is equal to the number of connected components in the graph; (3) The first non-zero eigenvalue $\lambda_2(L)$, called algebraic connectivity, gives a measure of connectedness of the graph. The corresponding eigenvector is called Fiedler vector. Laplacian of a weighted graph is defined as

$$L_{ij} = \begin{cases} -W_l, & e_l \sim (v_i, v_j) \\ \sum_{e_l \sim (v_i, v_k)} W_l, & i = j \\ 0, & otherwise \end{cases} \quad (1)$$

In this paper, $L(A)$ stands for the Laplacian matrix of a graph induced by adjacency matrix A. Furthermore, we define two different graphs for the oscillator network: the physical graph $G_p(V, E_p)$ and the communication graph $G_c(V, E_c)$ which represent physical interconnections between the oscillators and the communication links between local controllers, respectively. $G_T(V, E_T)$ represents the whole system graph with $E_T = E_p \cup E_c$. The operator $A \circ B$ that is called "Hadamard product" is element by element matrix multiplication.

B. Non-uniform Kuramoto Model

In 1975, Kuramoto proposed the classical Kuramoto model [9], which is one of the most successful models for synchronization analysis of oscillator networks. The classic Kuramoto model for describing dynamics of a set of N oscillators is given by:

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{N} \frac{K}{N} sin(\theta_i - \theta_j), i = \{1, \cdots, N\} \quad (2)$$

where θ_i is phase of the i^{th} oscillator, K is the coupling value which influences the synchronization properties, and ω_i is the natural frequency of i^{th} oscillator.

Depending on the specific application, various extensions of the Kuramoto models are proposed in literature. For a survey of Kuramoto models, see e.g. [10]. One of the extensions proposed in [14], inspired by power grid dynamics, is the *non-uniform Kuramoto model*

$$D_i \dot{\theta}_i = \omega_i - \sum_{j=1}^{N} P_{ij} sin(\theta_i - \theta_j + \varphi_{ij}), i = \{1, \cdots, N\} \quad (3)$$

where $P_{ij}, i = \{1, \cdots, N\}$ are the coupling values, $\varphi_{ij} \in [0, \pi/2), i, j = \{1, \cdots, N\}$ are phase shift parameters, and $D_i > 0, i = \{1, \cdots, N\}$ are time constants. In fact, matrix $P = [P_{ij}]$ is the adjacency matrix for the oscillator network's physical graph $G_p(V, E_p)$ and for the rest of the paper we assume connectedness of physical graph G_p.

The differences between equations (2) and (3) are as follows: the coupling values P_{ij} are heterogeneous, the phase shifts φ_{ij} are added to the model, and heterogeneous time constants D_i are considered. This model can obviously be simplified to the classic Kuramoto model by setting $P_{ij} = \frac{K}{N}$, $\varphi_{ij} = 0$ and $D_i = 1$. We choose the non-uniform Kuramoto model for further investigations, because it can be extended to capture the dynamics of power grid and furthermore provide more general overview than the classical Kuramoto model.

C. Synchronization

Consider the set $\Delta(\gamma)$ for $\gamma \in [0, \pi]$ as an open set of phase angles of the non-uniform Kuramoto model in (3) as $(\theta_1, \cdots, \theta_N)$ such that there exists an arc of length γ containing all $(\theta_1, \cdots, \theta_N)$ in its interior. Therefore an array of angles $\theta(t) = (\theta_1, \cdots, \theta_N) \in \Delta(\gamma)$ satisfies $\max_{i,j \in [1, \cdots, N]} |\theta_i - \theta_j| < \gamma$. For $\gamma \in [0, \pi]$ we also define $\bar{\Delta}(\gamma)$ to be the closure of the open set $\Delta(\gamma)$.

Definition 1: The solution $\theta(t)$ of the Kuramoto model achieves exponential synchronization, if there exists a length $\gamma \in [0, \pi)$ such that $\theta(t) \in \bar{\Delta}(\gamma)$ for all $t \geq 0$ and all frequencies $\dot{\theta}_i(t)$ converge exponentially to a common frequency $\dot{\theta}_\infty$.

In other words, in the synchronized state all frequency differences $|\dot{\theta}_i - \dot{\theta}_j| \to 0$, as $t \to \infty$.

III. DISTRIBUTED CONTROL DESIGN

A. Distributed control structure

The goal of this section is to design a distributed controller to guarantee synchronization for an oscillator network, which consists of N heterogeneous oscillators with natural frequencies $\omega_i, i = \{1, \cdots, N\}$ and nonlinear couplings modeled in (3). In the controller design procedure two major issues need be taken into account. The first one is the appropriate structure of the controller to guarantee the overall synchronization, and the second is an appropriate communication design, i.e. the placement of the new communication link to maintain and enhance the synchronization. Consider the non-uniform Kuramoto model in (3) with the control u_i

$$D_i\dot{\theta}_i = \omega_i - \sum_{j=1}^{N} P_{ij} sin(\theta_i - \theta_j + \varphi_{ij}) + u_i, i = \{1, \cdots, N\} \tag{4}$$

where $u \in R^N$ is vector of control signals as

$$u_i = -\sum_{j=1}^{N} K_{ij} Cij sin(\theta_i - \theta_j), i = \{1, 2, \cdots, N\} \tag{5}$$

where $C_{ij} \in \{0, 1\}$ indicates communication links between the local controllers and the $K_{ij} \geq 0$ are the controller gains. In fact matrix C is adjacency matrix for communication graph. Let us further assume that

$$\varphi_{ij} = 0, (i, j) \in E_c \tag{6}$$

$$P_{ij}.C_{ij} = 0, i, j = \{1, 2, \cdots, N\} \tag{7}$$

where assumption (6) implies that $\varphi_{ij} = 0$ for edges which belong to the communication graph, and in (7) we assume that the physical topology is given and we only improve the whole system's graph connectivity by inserting communication links, and consequently when we have physical connections between oscillators, we do not add new communication link. From equations (4) and (5) we have

$$D_i\dot{\theta}_i = \omega_i - \sum_{j=1}^{N} T_{ij} sin(\theta_i - \theta_j + \varphi_{ij}) \tag{8}$$

where $T \in R^{N \times N}$ is adjacency matrix of the whole system's graph $G_T(V, E_T)$ and is defined as

$$T_{ij} = P_{ij} + K_{ij} C_{ij}, i, j = \{1, 2, \cdots, N\} \tag{9}$$

In the following theorem, the conditions for synchronization and the region of attraction for the oscillator network in (4) controlled by a distributed control law (5) will be characterized. Before stating the result, we introduce the following variable notation

- $u = \max_{i \neq j} D_i D_j, i, j = \{1, 2, \cdots, N\}$;
- $l = \min_{i \neq j}\{D_i D_j\}, i, j = \{1, 2, \cdots, N\}$;
- $\kappa = \sum_{k=1}^{N} D_k$;
- $\alpha = \sqrt{l/u}$;
- $\tilde{X} = \sqrt{N} \left\| \left[\sum_{j=1}^{N} \frac{T_{1j}}{D_1} sin(\varphi_{1j}), \cdots, \sum_{j=1}^{N} \frac{T_{Nj}}{D_N} sin(\varphi_{Nj})\right] \right\|_2$;
- $H \in R^{N(N-1)/2 \times N}$ is the incidence matrix for complete graph with N nodes;
- $D = diag(D_i)$;
- $\lambda_c = \frac{uN(\|HD^{-1}\omega\|_2 + \tilde{X})}{\kappa\alpha}$;
- $\mathcal{T} \in R^{N \times N}$, $\mathcal{T}_{ij} = T_{ij} \cos(\varphi_{ij}), i, j = \{1, 2, \cdots, N\}$.

Theorem 1: Consider the oscillator network (4). The distributed control law of the form

$$u_i = -\sum_{j=1}^{N} K^*_{ij} C^*_{ij} sin(\theta_i - \theta_j) \tag{10}$$

with K^*_{ij} and C^*_{ij} satisfying

$$\lambda_2 \left(L\left(\mathcal{T}^*\right)\right) > \lambda_c, i, j = \{1, 2, \cdots, N\} \tag{11}$$

where $\mathcal{T}^*_{ij} = \left(P_{ij} + K^*_{ij} C^*_{ij}\right) \cos(\varphi_{ij}), i, j = \{1, 2, \cdots, N\}$, then for all initial conditions $\theta(0) \in \{\theta(0) \in \Delta(\pi) : \|H\theta(0)\|_2 < \alpha\rho\}$, the solution of oscillator networks dynamics in (8) is bounded and all frequencies $\dot{\theta}_i$ converge to $\dot{\theta}_\infty$.

Proof: See Appendix. ■

In Theorem 1, we show that after applying controller (5), if condition (11) is satisfied, the oscillator network will synchronize. In the next subsection we will present an algorithm for designing the communication topology for the distributed controller represented in (5).

Remark 1: Note that according to condition (11) the control design procedure is a central task which needs the information from the whole system's graph in order to compute λ_2. However, the control operation is distributed and the local controllers act based on their own and their neighbor's information.

B. Communication topology design

In this section we consider the design of communication topology as well as controller gains. The goal of the design is to find a controller which satisfies condition (11) and guarantees synchronization. For this purpose we assume that communication links are added once at a time, thereafter algebraic connectivity of the whole system is computed and condition (11) is checked. This is repeated until the sufficient condition (11) is satisfied. The problem of adding each communication link to the system can be formulated as the following optimization problem

$$\max \lambda_2 \left(L(\mathcal{T})\right) \tag{12}$$

$$s.t. \quad \mathbf{1}^T C \mathbf{1} = 2 \tag{13}$$

$$C_{ij} \in \{0, 1\}, i, j \in (E)^c \tag{14}$$

$$C_{ij} = 0, i, j \in E \tag{15}$$

$$0 \leq K_{ij} \leq K_{ij,max} \tag{16}$$

where the objective function is to maximize algebraic connectivity by adding each of the communication links. In constraint (13), $\mathbf{1}$ is the vector of all ones and it models the fact that we want to add one link in each run. Constraints (14) and (15) are corresponding to assumption (7) that indicate the feasible set for adding communication links, in which C_{ij} can take the value zero or one. The last constraint (16) is the feasible interval for controller gains which is specified according to physical limitations of the system.

Lemma 1 (*Monotonicity Theorem*) [22]: If A and B are N by N Hermitian Matrices, and B is positive semidefinite, then $\lambda_i(A) \leq \lambda_i(A+B)$ for all $i = \{1, \cdots, N\}$.

Lemma 2: Consider a weighted graph with adjacency matrix A. Increasing the weight corresponding to one edge will result in increase of algebraic connectivity of the graph.

Proof: The adjacency matrix of the graph with increased weight of an edge $e_{l^*} \sim (v_{i^*}, v_{j^*})$ can be represented by $A + \Delta A$, where

$$\Delta A_{ij} = \begin{cases} \delta a, & (i,j) = (i^*, j^*) \text{ or } (i,j) = (j^*, i^*) \\ 0, & \text{otherwise} \end{cases} \tag{17}$$

The Laplacian matrix $L(A)$ of a weighted matrix is defined as (1). When weight of one edge is increased, the new Laplacian will become $L + \Delta L$ and we have

$$\Delta L = \left[\begin{array}{ccccccccccc} & & & 0 & & & & 0 & & & \\ & \mathbf{0} & & \vdots & & \mathbf{0} & & \vdots & & \mathbf{0} & \\ & & & 0 & & & & 0 & & & \\ 0 & \cdots & 0 & \delta l_{i^*i^*} & 0 & \cdots & 0 & \delta l_{i^*j^*} & 0 & \cdots & 0 \\ & & & 0 & & & & 0 & & & \\ & \mathbf{0} & & \vdots & & \mathbf{0} & & \vdots & & \mathbf{0} & \\ & & & 0 & & & & 0 & & & \\ 0 & \cdots & 0 & \delta l_{j^*i^*} & 0 & \cdots & 0 & \delta l_{j^*j^*} & 0 & \cdots & 0 \\ & & & 0 & & & & 0 & & & \\ & \mathbf{0} & & \vdots & & \mathbf{0} & & \vdots & & \mathbf{0} & \\ & & & 0 & & & & 0 & & & \end{array}\right] \tag{18}$$

where $\delta l_{i^*i^*} = \delta l_{j^*j^*} = \delta a$ and $\delta l_{i^*j^*} = \delta l_{j^*i^*} = -\delta a$. Clearly $rank(\Delta L) = 1$ and we know that $\sum_{k=1}^{N} \lambda_k^{(\Delta L)} = tr(\Delta L)$, therefore ΔL has only one non-zero eigenvalue which equals to $2\delta a$. Since ΔL is a symmetric matrix, it implies that ΔL is a positive semi-definite matrix. Using Lemma 1, it follows that the algebraic connectivity of the graph induced by adjacency matrix $A + \Delta A$ is larger than algebraic connectivity of the graph induced by adjacency matrix A. This in turn completes the proof of Lemma 2. ■

According to Lemma 2, we can see that objective function (12) is monotonically increasing in the set (16) and the optimal solution will be with $K_{ij,max}$. Therefore the constraint (16) can be rewritten as $K_{ij} = K_{ij,max}$. The obtained optimization problem can be solved by the heuristic method proposed in [23], which yields an almost optimal but very fast converging solution, that is specially important for the large-scale systems. In the proposed method, Fiedler vector is computed and used to locate a new link in order to maximize algebraic connectivity. The new communication link is added between two nodes i and j with maximum $(\nu_{2i} - \nu_{2j})^2$, where ν_{2i} is the i^{th} element of Fiedler vector ν_2. The algorithm for adding communication links is summarized in Table I.

IV. SIMULATION RESULTS

We now numerically evaluate our proposed controller for a network consisting of $N = 10$ oscillators. Elements of

TABLE I

ALGORITHM FOR ADDING COMMUNICATION LINKS

1.	Initialization $C = 0$; Input: matrices K, P, $\cos(\varphi_{ij})$for$i, j = \{0, 1, \cdots, N\}$, λ_c;
2.	Compute $T = P + K \circ C$;
3.	Calculate eigenvalue $\lambda_2(L(\mathcal{T}))$ and corresponding eigenvector ν_2;
4.	If $\lambda_2(L(\mathcal{T})) > \lambda_c$ in equation (11), then go to step 6; else go to step 5;
5.	Find $\max_{i,j \in (E)^c} (\nu_{2i} - \nu_{2j})^2$, set $C_{ij} = C_{ji} = 1$, go to step 2;
6.	Output C.

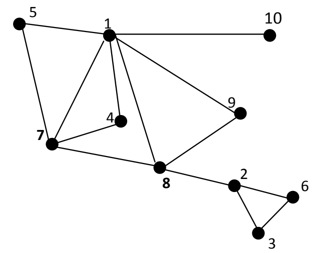


Fig. 1. Physical graph topology

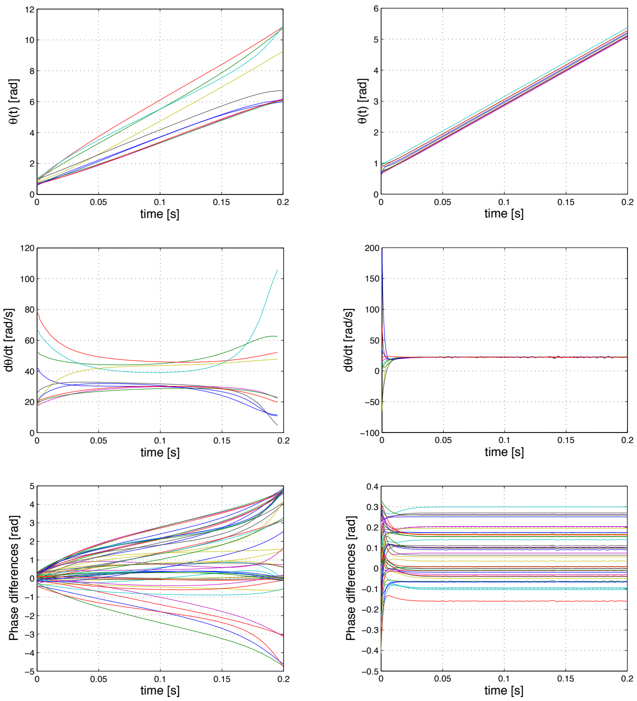
physical graph's adjacency matrix P, time constants D_i and natural frequencies ω_i are chosen randomly from a uniform distribution over $[0.7, 1.27], [0.05, 0.08]$ and $[0, 5]$ respectively. For a sake of simplicity, we ignore phase shift, i.e. $\varphi_{ij} = 0$. Matrix P is randomly generated as follows

$$P = \begin{bmatrix} 0 & 0 & 0 & 1.01 & 0.7 & 0 & 0.89 & 0.79 & 1.03 & 0.83 \\ 0 & 0 & 0.75 & 0 & 0 & 0.73 & 0 & 0.84 & 0 & 0 \\ 0 & 0.75 & 0 & 0 & 0 & 1.14 & 0 & 0 & 0 & 0 \\ 1.01 & 0 & 0 & 0 & 0 & 0 & 1.03 & 0 & 0 & 0 \\ 0.7 & 0 & 0 & 0 & 0 & 0 & 0.93 & 0 & 0 & 0 \\ 0 & 0.73 & 1.14 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.89 & 0 & 0 & 1.03 & 0.93 & 0 & 0 & 0.82 & 0 & 0 \\ 0.79 & 0.84 & 0 & 0 & 0 & 0 & 0.82 & 0 & 0 & 0.78 \\ 1.03 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.83 & 0 & 0 & 0 & 0 & 0 & 0 & 0.78 & 0 & 0 \end{bmatrix}$$

Corresponding physical graph is shown in Fig. 1. Using the condition (11) in Theorem 1, we can see that for the generated graph (without a controller) $\lambda_2(L(P)) = 0.24$, and $\lambda_c = 21.3$. This in turn indicates that for the current physical parameters, oscillators may not synchronize. As can be seen in Fig. 2(a) the synchronization can not be achieved and the oscillator's phases as well as their frequencies diverge. Now consider the control input $u_i = -\sum_{j=1}^{10} K_{ij} C_{ij} \sin(\theta_i - \theta_j)$, where for the sake of simplicity we consider identical controller gain $K_{ij,max} = K = 10$. Using the iterative algorithm proposed in Table I, we design proper communication topology to fulfill the sufficient condition (11). The desired communication graph matrix C which corresponds to the communication topology is driven as follows

$$C = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Simulation results after applying controller is shown in Fig. 2(b). As can be seen in this figure, the slope of all θ_i are



(a) (b)

Fig. 2. Simulation results for a network of 10 oscillators, before (a) and after (b) applying controller

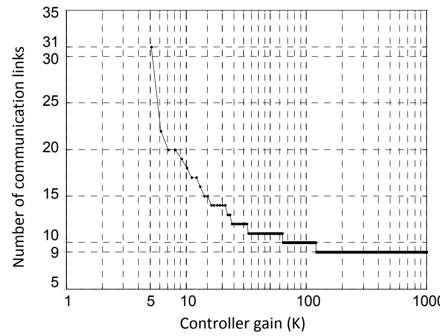


Fig. 3. Number of communication links needed to guarantee synchronization decreases as the controller gains increase

TABLE II

NUMBER OF COMMUNICATION LINKS FOR DIFFERENT GAINS

Gain (K)	Number of communication links	Comments
1	-	No solution
5.1285	31	Maximum possible communication links
5.5	26	-
6	23	-
10	18	-
20	14	-
30	12	-
100	10	-

converging to one constant value, all frequencies converge to a common value and moreover, phase differences remain constant which corresponds to synchronization as well as phase lock. It can be seen in Table II, number of communication links decreases as the largest possible controller gains increases, where we have considered same $K_{ij,max}$ for all controllers ($K_{ij,max} = K, i, j = \{1, 2, \cdots, N\}$). Fig. 3 shows the variation in number of communication links for different control gain K in a logarithmic scale.

Remark 2: In Table II we see that for the controller gains $K = 10$ the number of required communication links in order to satisfy the condition (11), equals 18. However, simulation results show that this system can be synchronized with only 7 communication links, which is less than 18 communication links. Also the achieved algebraic connectivity with 7 communication links is $\lambda_2 = 3.0725 < \lambda_c = 21.3$. This result indicates that the sufficient condition (11) is a very conservative condition.

V. CONCLUSION

In this paper, we consider a network of N heterogeneous oscillators modeled by non-uniform Kuramoto model, and propose the distributed control to synchronize the overall network. In particular, a sufficient condition for synchronization of the networked control system is established. We assume that all physical parameters such as graph topology, coupling values, and time constants are given and the design ultimate goal is to find a proper distributed control law to satisfy the sufficient condition of synchronization. It is shown that with the presented distributed control, the optimal control gains are the maximum possible gains, which are defined by the physical system limitations. The communication topology design is then considered and the communication links are added once at a time, between two nodes i and j with maximum $(\nu_{2i}-\nu_{2j})^2$, where ν_{2i} is the i^{th} element of Fiedler vector ν_2, that is equal to adding each communication link between the most connected node and the most isolated node in the whole system's graph.

Numerical results validate the effectiveness of the distributed control design. The simulation results show that our proposed distributed controller can improve the connectivity of the overall network by adding communication links such that sufficient condition for synchronization is satisfied.

REFERENCES

[1] J. Buck, "Synchronous rhythmic flashing of fireflies," *The Quarterly Review of Biology*, vol. 63, no. 3, pp. 256–289, September 1988.

[2] R. E. Mirollo and S. H. Strogatz, "Synchronization of pulse-coupled biological oscillators," *SIAM Journal on Applied Mathematics*, vol. 50, no. 6, pp. 1645–1662, 1990.

[3] G. Kozyreff, A. Vladimirov, and P. Mandel, "Global coupling with time delay in an array of semiconductor lasers," *Physical Review Letters*, vol. 85, no. 18, pp. 3809–3812, October 2000.

[4] P. Kundur, *Power System Stability and Control*. McGraw-Hill, 1994.

[5] X. Li, "Sync in complex dynamical networks: stability, evolution, control, and application," *International Journal of Computational Cognition*, vol. 3, no. 4, pp. 16–26, 2005.

[6] S. Strogatz, "Exploring complex networks," *Nature*, vol. 410, pp. 268–276, 2001.

[7] X. Wang, "Complex networks: topology, dynamics and synchronization," *International Journal of Bifurcation and Chaos*, vol. 12, no. 5, pp. 885–916, 2002.

[8] C. W. Wu, *Synchronization in complex networks of nonlinear dynamical systems*. World Scientific, 2007.

[9] Y. Kuramoto, In International Symposium on Mathematical Problems in Theoretical Physics, volume 39 of Lecture Nom in Physics, page 420. Springer, New York, 1975.

[10] J. Acebron, L. Bonilla, C. P. Vicente, F. Ritort, and R. Spigler, "The kuramoto model: A simple paradigm for synchronization phenomena," *Reviews of Modern Physics*, vol. 77, no. 1, pp. 137–185, 2005.

[11] A. Jadbabaie, N. Motee, and M. Barahona, "On the stability of the kuramoto model of coupled nonlinear oscillators," in *In Proceedings of the American Control Conference*, vol. 5, 2004, pp. 4296–4301.

[12] A. papachristodoulou and A. Jadbabaie, "Synchronization in oscillator networks: switching topologies and non-homogeneous delays," in *In Proceedings of the Conference on Desicion and Control*, 2005, pp. 5692–5697.

[13] N. Chopra and M. Spong, "On exponential synchronization of kuramoto oscillators," *IEEE transactions on Automatic Control*, vol. 54, no. 2, pp. 353–357, February 2009.

[14] F. Dorfler and F. Bullo, "Synchronization and transient stability in power networks and non-uniform kuramoto oscillators," in *In Proceedings of the American Control Conference*, 2010.

[15] J. Zhou, J. an Lu, and J. Lu, "Adaptive synchronization of an uncertain complex dynamical network," *IEEE Transactions on Automatic Control*, vol. 51, no. 4, pp. 652–656, April 2006.

[16] X. Li, X. Wang, and G. Chen, "Pinning a complex dynamical network to its equilibrium," *IEEE Transactions on Circuits and Systems I*, vol. 51, no. 10, pp. 2074–2087, October 2004.

[17] B. Liu, X. Z. Liu, G. Chen, and H. Wang, "Robust impulsive synchronization of uncertain dynamical networks," *IEEE Transactions on Circuits and Systems I*, vol. 52, no. 7, pp. 1431–1441, July 2005.

[18] R. Cogill and S. Lall, "Control design for topology-independent stability of interconnected systems," in *In Proceedings of the American Control Conference*, 2004.

[19] T. Liu, D. Hill, and J. Zhao, "Synchronization of dynamical networks by network control," in *In Proceedings of the Conference on Desicion and Control*, 2009.

[20] A. Gusrialdi and S. Hirche, "Performance-oriented communication topology design for large-scale interconnected systems," in *In Proceedings of the Conference on Desicion and Control*, 2010.

[21] R. Grone, R. Merris, and V. Sunder, "The laplacian spectrum of a graph," *SIAM Journal on Matrix Analysis and Applications*, vol. 11, no. 2, pp. 218–238, 1990.

[22] R. A. Horn and C. R. Johnson, *Matrix Analysis Matrix Analysis Matrix analysis*. Cambridge University Press, 1990.

[23] A. Ghosh and S. Boyd, "Growing well-connected graphs," *In Proceedings of the Conference on Desicion and Control*, 2006.

[24] F. Dorfler and F. Bullo, "Synchronization and transient stability in power networks and non-uniform kuramoto oscillators," *SIAM Journal on Control and Optimization*, vol. 50, no. 3, pp. 1616–1642, 2012.

[25] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Prentice Hall, 2002.

VI. Appendix: Proof of Theorem 1

The oscillator network (3) with the distributed controller (5), for all $i,j=\{1,2,\cdots,N\}$ that $i\neq j$, can be represented in the matrix form as

$$\frac{d}{dt}H\theta = HD^{-1}\omega - HD^{-1}H^{T}diag(T_{ij}\cos(\varphi_{ij}))\sin(H\theta) - HX \tag{19}$$

A continuous differentiable function of the phase differences for all $i,j=\{1,2,\cdots,N\}$ that $i\neq j$ is considered as $V(H\theta)=\frac{1}{2}(H\theta)^{T}P(H\theta)$ where $P=diag(D_iD_j), i,j=\{1,2,\cdots,N\}$ and $V(H\theta)\in\omega$, $\omega=\{x\in R^{N}|\alpha_1(\|x\|)\leq V(x)\leq\alpha_2(\|x\|)\}$, where α_1 and α_2 are class K functions for positive $D_i, i=\{1,2,\cdots,N\}$ as $l\|H\theta\|_2^2\leq 2V(H\theta)\leq u\|H\theta\|_2^2$. Differentiating the Lyapunov function $V(H\theta)$ along solutions of (19) yields

$$\dot{V}(H\theta)=(H\theta)^{T}PHD^{-1}\omega-(H\theta)^{T}PHX- (H\theta)^{T}PHD^{-1}H^{T}diag(T_{ij}\cos(\varphi_{ij}))\sin(H\theta) \tag{20}$$

Simplifying (20) we get

$$\dot{V}(H\theta)=(H\theta)^{T}PHD^{-1}\omega-(H\theta)^{T}PHX- \kappa(H\theta)^{T}diag(T_{ij}\cos(\varphi_{ij}))\sin(H\theta) \tag{21}$$

where $\kappa=\sum_{k=1}^{N}D_k$. Assuming that all initial conditions $\theta(0)$ belong to $\theta(0)\in(0,\pi)$, i.e. $\theta(0)\in\Delta(\pi)$, and for some $\rho\in(0,\pi)$, $\|H\theta(0)\|_2<\rho$. Since $\|H\theta(0)\|_\infty < H\theta(0)\|_2$, it follows that $H\theta(0)\in\bar{\Delta}(\rho)$ and $sinc(\rho)\leq sinc(\theta_i-\theta_j)\leq 1$. Thus, from (21) we have

$$\dot{V}(H\theta)\leq(H\theta)^{T}PHD^{-1}\omega-(H\theta)^{T}PHX- \kappa sinc(\rho)(H\theta)^{T}diag(T_{ij}\cos(\varphi_{ij}))H\theta \tag{22}$$

Using lemma V.9 in [24], we have

$$\dot{V}(H\theta)\leq\|H\theta\|_2 u(\|HD^{-1}\omega\|_2+\tilde{X})- \left(\tfrac{\kappa}{N}\right)sinc(\rho)\lambda_2\left(L\left(T_{ij}\cos(\varphi_{ij})\right)\right)\|H\theta\|_2^2 \tag{23}$$

The right-hand side of (23) is negative for all $i,j=\{1,2,\cdots,N\}$ that $i\neq j$ if

$$\|H\theta\|_2>\mu_c:=\frac{uN(\|HD^{-1}\omega\|_2+\tilde{X})}{\kappa sinc(\rho)\lambda_2\left(L\left(T_{ij}\cos(\varphi_{ij})\right)\right)} \tag{24}$$

Choosing $\mu\in(0,\rho)$ such that $\mu>\mu_c$, we have

$$\dot{V}(H\theta)\leq -W(H\theta),\forall\|H\theta\|_2\geq\mu>0 \tag{25}$$

where for all $i,j=\{1,2,\cdots,N\}$ that $i\neq j$, $W=\left(1-\frac{\mu_c}{\mu}\right)\cdot\left(\frac{\kappa}{N}\right)sinc(\rho)\lambda_2\left(L\left(T_{ij}\cos(\varphi_{ij})\right)\right)\|H\theta\|_2^2$ is a continuous positive definite function. Applying theorem 4.18 in [25], we conclude that if $\mu<\alpha_2^{-1}(\alpha_1(\rho))$ then for every initial condition $\|H\theta(0)\|_2<\alpha_2^{-1}(\alpha_1(\rho))$, there is $T\geq 0$ such that the solution of (11) is bounded as $\|H\theta(t)\|_2\leq\alpha_1^{-1}(\alpha_2(\mu)), t\geq T$ where $\alpha_2^{-1}(\alpha_1(\rho))=\alpha\rho$, and $\alpha_1^{-1}(\alpha_2(\mu))=\mu/\alpha$. Now we have to choose ρ and μ to define region of attraction and solution bound. We note that it is desirable to choose ρ as large as possible, since it determines region of attraction. However, as mentioned before, these conditions must also be satisfied by μ and ρ

$$\mu<\alpha\rho \tag{26}$$

$$\mu>\frac{uN(\|HD^{-1}\omega\|_2+\tilde{X})}{\kappa sinc(\rho)\lambda_2\left(L\left(T_{ij}\cos(\varphi_{ij})\right)\right)}=\frac{\alpha\lambda_c}{sinc(\rho)} \tag{27}$$

From (26) and (27), it follows that $\lambda_c<\rho sinc(\rho)$. Since we assumed that $\rho\in(0,\pi)$, we achieve the following condition

$$\lambda_2\left(L\left(T_{ij}\cos(\varphi_{ij})\right)\right)>\lambda_c=\frac{uN(\|HD^{-1}\omega\|_2+\tilde{X})}{\kappa\alpha} \tag{28}$$

Therefore, we can conclude that if condition (28) is satisfied for an oscillator network, then for every $\|H\theta(0)\|_2<\alpha\rho$,there is $T\geq 0$ such that $\|H\theta(t)\|_2\leq\mu/\alpha$, which means the ultimate solution of the oscillator network (4), $H\theta(t)$ for $t\geq T$, with the distributed control law of the form (5) is bounded. Now using Theorem V.1 in [24], the frequencies of the oscillator network with the proposed controller in (5) that satisfies the sufficient condition (28) will exponentially synchronize to the common frequency $\dot{\theta}_\infty$.

■