Stationary Consensus of Asynchronous Discrete-Time Second-Order Multi-Agent Systems under Switching Topology

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***Abstract*—This paper is concerned with the asynchronous consensus problem of discrete-time second-order multi-agent system under dynamically changing communication topology, in which the asynchrony means that each agent detects the neighbors' state information to update its state information by its own clock. It is not assumed that the agents' clocks are synchronized. Nor is it assumed that the time sequence over which each agent update its state information is evenly spaced. By using tools from graph theory and nonnegative matrix theory, particularly the product properties of row-stochastic matrices from an infinite set, we finally show that essentially the same result as that for the synchronous discrete-time system holds in the face of asynchronous setting. This generalizes the existing result to a very general case.**

***Index Terms*—Asynchronous consensus; Multi-agent systems; Second-order dynamics.**

I. Introduction

OVER the last decades, collective behaviors in networks of autonomous agents has received a huge amount of attention from different fields. One of the main reasons for that comes from the abundance of technological applications multi-agent systems (MAS): vehicle formations [10], [37], flocking and swarming [19], [26], scheduling of automated highway systems, sensor networks [13], [29], [38], microgrids [34], and power systems [27], just to name a few.

It is worth noting that to solve the load restoration problem for Microgrids, various soft computing algorithms have been applied, in which MAS is one of the most popular distributed solutions as opposed to those centralized control scheme which lacks adaptivity to the structure changes of the power networks, see, e.g., [8] and [27]. But these MAS-based methods apply only to certain power system of special structures. Another of its shortcoming is that they lack of rigorous stability or convergence analysis. Very recently, to address the problems arising in the existing solutions, a fully distributed load restoration algorithm based on MAS which applies to the network of any structure is proposed in [34], where mainly the consensus theory is used to perform the convergence and stability analysis. Such MAS consensus theory is further extended in [35] to solve the load shedding problem for power systems.

Consensus is well accepted as being a fundamental paradigm for coordination of multi-agent system. The consensus problem refers to the design of a consensus protocol through which all agents are coordinated in the sense that they all agree on some particular parameter of interest such as attitude, position, velocity and etc.. Starting with the agreement algorithm in [28], much progress has been made in studying the consensus problems from various perspectives [1], [9], [11], [14]–[16], [18], [25], [32], in which each agent dynamics is taken to be a first-order integrator. Recently, the second-order consensus problem in which each agent is governed by double-integrator dynamics has also spurred great interest partly due to its ability to model a broader class of complicated dynamical agents. For example, holonomic mobile robot dynamics can be feedback linearized as double integrators. Also, the unmanned aerial vehicles and underwater vehicles are adjusted for their desired motion directly by their accelerations rather than by their speeds. Progresses toward this direction can be founded in [21], [23], [24], [36], [39], [40], just to name a few. It is worth noting that synchronization of complex dynamical networks, such as small-world and scale-free networks, which is closely related to the multi-agent consensus problem, has also been widely studied (see, e.g., [4], [12], [17], [30]) from different perspectives.

Most of the aforementioned works are concerned with continuous-time dynamics. Considering in real applications the information transmission among agents may not be continuous due to the unreliability of communication channels or the limited sensing ability of agents, there have been a number of publications studying the discrete-time consensus problem [3], [7], [20], [22], where each agent synchronously receives its neighbors' information at discrete times, in which the synchrony means that all the agents update their states using latest information of its neighboring agents at the same time. However, considering that a central synchronizing clock may not be available and the communication topology is dynamically changing, it is of more practical interest to consider the asynchronous consensus, i.e., each agent's update action is independent of the others'.

There are several publications considering the information consensus of asynchronous first-order multi-agent systems [2], [5], [33]. To the best of our knowledge, few works have

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considered the asynchronous consensus for agents modeled by second-order dynamics with an exception in [6], in which asynchronous discrete-time system with communication delays is transformed into the convergence of a continuous system with time-varying delays, and the balanced and strongly connected assumption is imposed on the communication topologies at each time instant in order to help perform the convergence analysis. The balanced and strongly connected assumption is a rather restrictive condition on the communication topology as opposed to those investigating the asynchronous consensus of first-order multi-agent systems [2], [33], in which the communication topologies are relaxed to be repeatedly jointly rooted.

We consider in this paper the asynchronous consensus of discrete-time second-order multi-agent systems under dynamically changing communication topologies. The model to be investigated is closely related to our earlier work concerning the synchronous consensus of second-order discrete-time [22]. In contrast to the work in [22], we will be focusing on the case where the times at which the agent receives the state information from its neighbors are independent of the other agents, and the time sequence over which each agent update its state information is not necessarily evenly spaced. The differences between the current work and that in [22] lie mainly in the following three perspectives: (1) The asynchronous setting largely generalizes the synchronous case and further includes its synchronous counterpart as a very special case; (2) The technical issues brought by considering the asynchronous setting is much more challenging compared with the synchronous case, which require us to deal with the model considered by using different methods; (3) The analysis of the asynchronous systems is considerably more difficult than that of their synchronous counterparts. One of the reasons comes from the asynchronous setting itself. Another reason is purely from the technical part that the length of update intervals may take any value from an infinite set and that the set from which the possible weighting factors are chosen is also infinite, both of which makes the widely used method concerning product property of row-stochastic matrices from a finite set invalid in our work.

The remainder of the paper is organized as follows. Notation and definitions reside in the next section. We formulate the problem to be investigated in Section III and then state the main result in Section IV, while consensus analysis for the asynchronous discrete-time system is performed in Section V. In Section VI, application-inspired numerical example showing the effectiveness of the theoretical finding is simulated. Some concluding remarks are finally drawn in Section VII.

II. Notation and Definitions

Directed graphs (digraphs) will be used to model the communication topologies among the agents. Let $G = (\mathcal{V}, \mathscr{E}, A)$ be a weighted digraph of order N with a finite nonempty set of nodes $\mathcal{V} = \{1, 2, \ldots, N\}$, a set of edges $\mathscr{E} \subset \mathcal{V} \times \mathcal{V}$, and a weighted *adjacency matrix* $A = [a_{ij}] \in \mathbb{R}^{N\times N}$ with nonnegative adjacency elements a_{ij}. An edge of G is denoted by (i, j), meaning that there is a unidirectional exchange link from i to j. The adjacency elements associated with the edges are positive, i.e., $(j, i) \in \mathscr{E} \Leftrightarrow a_{ij} > 0$. Moreover, we assume $a_{ii} = 0$ for all $i \in \mathcal{V}$. Given a nonnegative matrix $S = [s_{ij}] \in \mathbb{R}^{n\times n}$, the associated digraph of S, denoted by $\Gamma(S)$, is the directed graph with the node set $\mathcal{V} = \{1, 2, \ldots, n\}$ such that there is an edge in $\Gamma(S)$ from j to i if and only if $s_{ij} > 0$.

The set of neighbors of node i is denoted by $N_i = \{j \in \mathcal{V} : (j, i) \in \mathscr{E}\}$. Denote by $L = [l_{ij}]$ the graph Laplacian induced by weighted digraph $G = (\mathcal{V}, \mathscr{E}, A)$, which is defined by

$$l_{ij} = \begin{cases} \sum\limits_{k=1,k\neq i}^{N} a_{ik} & j = i \\ -a_{ij} & j \neq i \end{cases}.$$

The graph Laplacian L associated with an undirected graph is positive semi-definite, but the graph Laplacian associated with an digraph does not have this property. In both the undirected and directed cases, 0 is an eigenvalue of L with associated eigenvector $\mathbf{1}$, where $\mathbf{1}$ denotes the column vector of all ones with compatible dimension.

G is called a rooted graph or a graph has a directed spanning tree if there exists at least one node, called the root, which can be connected to each other node along a directed path within G. A matrix $M \in \mathbb{R}^{n\times n}$ is nonnegative (positive), denoted as $M \geq 0$ ($M > 0$), if all its entries are nonnegative (positive). Let N the square matrix with the same dimension as M, $M \geq N$ implies that $M - N \geq 0$. Note that for arbitrary nonnegative square matrices, say M and N, with the same dimension satisfying $M \geq \gamma N$, where $\gamma > 0$, if $\Gamma(N)$ is a rooted graph then $\Gamma(M)$ is also a rooted graph.

A nonnegative matrix M is said to be row stochastic if all its row sums are 1. A row-stochastic matrix M is called indecomposable and aperiodic (SIA) (or *ergodic*) if there exists a column vector $v \in \mathbb{R}^n$ such that $\lim_{k\to\infty} M^k = \mathbf{1}v^{\mathrm{T}}$. Let $\prod_{i=1}^{k} M_i = M_k M_{k-1} \cdots M_1$ denote the left product of the matrices $M_k, M_{k-1}, \cdots, M_1$. Given any row-stochastic matrix $P = [p_{ij}]$, define $\lambda(P) = 1 - \min_{i,j} \sum_k \min\{p_{ik}, p_{jk}\}$ [31]. $\lambda(P) = 0$ if and only if the rows of P are identical. Two nonnegative matrices M and N are said to be of the same type, denoted by $M \sim N$, if they have zero elements and positive elements in the same places. For example, given $A = \begin{bmatrix} 0.4 & 0 & 0.6 \\ 0 & 0.3 & 0.7 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0.2 & 0 & 0.8 \\ 0 & 0.9 & 0.1 \\ 0 & 0 & 0.3 \end{bmatrix}$, $C = \begin{bmatrix} 0.4 & 0 & 0.6 \\ 0.3 & 0 & 0.7 \\ 0 & 0 & 1 \end{bmatrix}$, then matrices A and B are of the same type but A and C are of different types.

III. Problem Description

The system to be considered consists of N autonomous agents, labeled 1 through N, all moving in the Euclidean space $\mathbb{R}^p$. Each agent is regarded as a node in a digraph G of order N.

In the continuous-time setting, each agent is modeled by second-order dynamics

$$\dot{x}_i = v_i, \quad \dot{v}_i = u_i, \quad i \in \mathcal{V}, \tag{1}$$

where $x_i \in \mathbb{R}^p$ and $v_i \in \mathbb{R}^p$ are, respectively, the position and velocity of the ith agent, $u_i \in \mathbb{R}^p$ is the distributed control input which uses only its own state information and the state information of its neighbors. A consensus algorithm for (1) that is investigated in [21], [23] and [36] is given by

$$u_i(t) = -\gamma v_i + \sum\nolimits_{j\in N_i(t)} \alpha_{ij}(t)(x_j(t) - x_i(t)), \quad (2)$$

where $\gamma > 0$ denotes the velocity damping gain and $N_i(t)$ denotes the set of neighbors of agent i at time t. The dependence of $N_i(t)$ on t means that the communication topology considered is dynamically changing.

In a real world application, information may not be transmitted continuously due to the unreliability of communication channels or the limited sensing ability of agents which results in discrete-time formulation in which each agent obtains its neighbors' state information only at discrete times.

A. Synchronous Discrete-time System

In the discrete-time case, using the forward difference approximation as that employed in [3] and [22], each agent in the network has an estimate of the second-order information state as follows:

$$\begin{cases} x_i(t_{k+1}) - x_i(t_k) = Tv_i(t_k) \\ v_i(t_{k+1}) - v_i(t_k) = Tu_i(t_k) \end{cases}, \quad (3)$$

where $x_i(t_k)$, $v_i(t_k)$ and

$$u_i(t_k) = -\gamma v_i(t_k) + \sum\nolimits_{j\in N_i(t_k)} \alpha_{ij}(t_k)(x_j(t_k) - x_i(t_k)), \quad (4)$$

are, respectively, the position, velocity and control input of the ith agent at time t_k, and $T > 0$ is the step-size (or sampling time). Obviously, the times at which each agent gets the state information from its neighbors are synchronized and the step sizes are same for each updating process. These relatively restrictive conditions in real application naturally motivates the investigation of the asynchronous system which is elaborated in the following subsection.

B. Asynchronous System

In contrast to that specified in the above subsection, we consider in this paper the asynchronous consensus in which each agent independently detects its neighbors' state information at times determined by its own clock. Also, the event times at which each agent update its states are are not necessarily evenly spaced. More specifically, we assume that each agent $i \in \{1, 2, \ldots, N\}$ receives or detects its neighbors' states at times $t_0^i, t_1^i, \ldots, t_k^i, \ldots$, which is denoted by a real number sequence $\{t_k^i\}$ for simplicity. We further assume for each $i \in \{1, 2, \ldots, N\}$ that $\{t_k^i\}$ satisfies the following constraints:

$$T_u \le t_{k+1}^i - t_k^i \le \bar{T}_u, \quad k \in \mathbb{N}, \quad (5)$$

where $\mathbb{N}$ denotes the set of nonnegative integers, $t_0^i = 0$ and T_u and $\bar{T}_u$ are positive numbers.

Definition 1: We say that discrete-time system (3), (4) is asynchronous if the times at which the agent receives its neighbors' states are independent of each other, i.e., $\{t_k^i\}$ is independent of $\{t_k^j\}$, $\forall i, j \in \{1, 2, \ldots, N\}$, $i \neq j$.

Similar to that in [2], let us merge all the N time sequences $\{t_k^i\}$, $i = 1, \ldots, N$, into a single ordered sequence $\mathcal{T}$. Relabel the elements of $\mathcal{T}$ as $t_0, t_1, t_2, \ldots$ in such a way so that $t_0 = 0$ and $t_k < t_{k+1}, k \in \mathbb{N}$. Let $\tau_k = t_{k+1} - t_k$, $k \in \mathbb{N}$. Note that the independence of sequences $\{t_k^i\}$, $i = 1, \ldots, N$, does not preclude the arbitrary closeness of such sequences from different agents. Thus τ_k could be any positive number in $(0, \bar{T}_u]$, i.e. $\tau_k \in (0, \bar{T}_u]$, $k \in \mathbb{N}$.

For any agent $i \in \{1, 2, \ldots, N\}$ and $k \in \mathbb{N}$, there exists $s \in \mathbb{N}$ such hat $t_s^i \le t_k < t_{k+1} \le t_{s+1}^i$. Then the dynamics of asynchronous discrete-time systems can be written as follows.

$$\begin{cases} x_i(t_{k+1}) - x_i(t_k) = \tau_k v_i(t_k) \\ v_i(t_{k+1}) - v_i(t_k) = \tau_k u_i(t_k) \end{cases}, \quad (6)$$

where

$$u_i(t_k) = -\gamma v_i(t_k) + \sum\nolimits_{j\in N_i(t_k)} \alpha_{ij}(t_k)(x_j(t_s^i) - x_i(t_k)). \quad (7)$$

Different from the assumption in [22], [36] that the set from which all the weighting factors are chosen is finite, it is assumed in this paper that all the nonzero and thus positive weighting factors are uniformly and upper bounded, i.e. $\alpha_{ij}(t_k) \in [\underline{\alpha}, \bar{\alpha}]$ whenever $j \in N_i(t_k)$, where $0 < \underline{\alpha} < \bar{\alpha}$. It is worth pointing out that in (7), it is the state information $x_j(t_s^i)$ instead of $x_j(t_k)$ that is received by agent i in updating the state information. This is because agent i only receives the state information of agent j at time t_s^i.

Remark 1: By slightly modifying algorithm (7), we can get the following algorithm

$$\begin{aligned} &u_i(t_k) \\ =& -\gamma v_i(t_k) + \sum_{j\in N_i(t_k)} \alpha_{ij}(t_k)\left([x_j(t_s^i) - x_i(t_k)] - [\delta_j - \delta_i]\right), \end{aligned}$$

which can be used to guarantee the differences of the agents' position states converge to the desired values, i.e. $x_j(t_k) - x_i(t_k) \to \Delta_{ij} = \delta_i - \delta_j$, where $\delta_i \in \mathbb{R}^p$, $i = 1, \ldots, N$, are constant vectors. However, the consensus analysis for these two algorithms are essentially the same. To see this, one can use $x_j - \delta_j$ to replace x_j perform the consensus analysis.

For simplicity, we assume in what follows $p = 1$. However, all results still hold for any positive integer p by introducing the notation of the Kronecker product '$\bigotimes$'.

We say that consensus is reached for asynchronous system (6), (7) if for any initial position and velocity states,

$$\lim\nolimits_{k\to\infty} x_i(t_k) = \lim\nolimits_{k\to\infty} x_j(t_k)$$

and

$$\lim\nolimits_{k\to\infty} v_i(t_k) = 0, \ i, j \in \mathcal{V}.$$

It is assumed that $x_i(t_k) = x_i(0)$ and $v_i(t_k) = v_i(0)$ for any $k < 0$ and $i, j \in \mathcal{V}$.

IV. Main Result

To state the main result, we need to introduce a few definitions. Denote $\bar{G}$ as the set of all possible communication topologies for all the N agents[1]. The *union of a group of digraphs* $\{G_{i_1}, \ldots, G_{i_k}\} \subset \bar{G}$ is a digraph with the same

[1] $\bar{G}$ is infinite since the set consisting of the weighting factors is infinite. However, there are only finite different topological structures in $\bar{G}$.

node set and the edge set given by the union of the edge sets of G_{i_j}, $j = 1, \ldots, k$.

A finite sequence of digraphs with the same node set is said to be jointly rooted if the union of such finite sequence of digraphs is a rooted graph, while an infinite sequence of digraphs $G_0, G_1, G_2, \ldots,$ with the same node set is called *repeatedly jointly rooted* if there exists an infinite sequence of contiguous, nonempty, uniformly bounded time intervals $[k_j, k_{j+1})$, $j = 1, 2, \ldots,$ starting at $k_1 = 0$, for which each finite set of digraphs $G_{k_j}, G_{k_j+1}, \ldots, G_{k_{j+1}-1}$ is jointly rooted [9], [25].

Remark 2: The definition of repeatedly jointly rooted digraphs used here is slightly different from that in [2], in which all the time intervals are with the same length, i.e. all $k_{j+1} - k_j$'s, $j = 1, 2, \ldots,$ are the same as each other.

Each edge (i, j) in digraph $G(t_k)$ corresponds to a unidirectional information link from i to j at time t_k, where $G(t_k)$ denotes the communication topology at time t_k, $k \in \mathbb{N}$.

In the face of synchronous setting, the result concerning the consensus of discrete-time second-order agents under switching topology is rephrased as follows.

Theorem 1: (Theorem 3, [22]) Assume that the velocity damping gain γ satisfies $2\sqrt{d_{max}} \leq \gamma < \frac{2}{T}$, where $d_{max} = (N-1)\bar{\alpha}$. Then consensus is reached for the synchronous discrete-time system (3), (4) if the infinite sequence of digraphs $G(t_1), G(t_2), \ldots$ is repeatedly jointly rooted.

The final aim of this paper is to prove that essentially the same result holds in the face of asynchronous setting.

Theorem 2: Assume that γ satisfies $2\sqrt{d_{max}} \leq \gamma < \frac{2}{\bar{T}_u}$, where $d_{max} = (N-1)\bar{\alpha}$. Then consensus is reached for asynchronous system (6), (7) if the infinite sequence of digraphs $G(t_1)$, $G(t_2)$, …is repeatedly jointly rooted.

V. Consensus Analysis

This section aims to give a Proof of Theorem 2. The analysis is motivated by the work in [2] and [33], the time sequences at which each agent detects its neighbors' state information are merged into a single ordered sequence $\mathcal{T}$ and then asynchronous discrete-time system is casted into an equivalent augmented synchronous discrete-time which evolves over time sequence $\mathcal{T}$. Finally, mixed tools from graph theory and nonnegative matrix theory, particularly the infinite product properties of row-stochastic matrices from an infinite set, is employed to prove that essentially the same result as that for the synchronous case holds in the face of asynchronous setting.

Follow the above proof guidelines, we first perform the following model transformation, which helps us deal with the asynchronous consensus problem for an equivalent transformed synchronous discrete-time system evolving on the index set of $\mathcal{T}$. Denote $y(t_k) = \frac{2}{\gamma} v(t_k) + x(t_k)$ and $r(t_k) = \left[x^{\mathrm{T}}(t_k), y^{\mathrm{T}}(t_k)\right]^{\mathrm{T}}$, where $x(t_k) = [x_1(t_k), \cdots, r_N(t_k)]^{\mathrm{T}}$ and $v(t_k) = [v_1(t_k), \cdots, v_N(t_k)]^{\mathrm{T}}$.

Denote by $L(t_k)$ the graph Laplacian induced by graph $G(t_k)$. Further, let $\check{m}$ denote the upper bound for the number of elements in set $\{t_j : t_j \in [t_k^i, t_{k+1}^i),\ j \in \mathbb{N}\}$ for any $i = 1, \ldots, N$, and $k = 0, 1, \ldots$. The following result is from [33] (see Lemma 1 therein).

Lemma 1: Let $\check{m}$ be the integer as defined above, $\check{m} = (\lfloor \bar{T}_u / T_u \rfloor + 1)(N-1) + 1$, where $\lfloor \bar{T}_u / T_u \rfloor$ is the maximum integer not greater than $\bar{T}_u / T_u$.

Let $\xi(k) = [r(t_k)^{\mathrm{T}}, r(t_{k-1})^{\mathrm{T}}, \ldots, r(t_{k-\check{m}+1})^{\mathrm{T}}]^{\mathrm{T}}$, where $k \geq \check{m} - 1$. Given any square matrix $A = [a_{ij}]$, let $\mathrm{diag}\{A\}$ denote the diagonal matrix with the ith diagonal element equals to a_{ii}. By observing the expression of system (6), (7), there exists a state matrix, denoted by $M(\gamma, \tau_k, \Xi_1(\gamma, \tau_k, t_k), \ldots, \Xi_{\check{m}}(\gamma, \tau_k, t_k))$, which is defined as that in (8),

where

$$\Xi_1(\gamma, \tau_k, t_k) = \begin{bmatrix} (1 - \frac{\gamma\tau_k}{2}) I_n & \frac{\gamma\tau_k}{2} I_n \\ \frac{\gamma\tau_k}{2} I_n - \frac{2\tau_k}{\gamma}(\mathrm{diag}\{L(t_k)\} - A_1(t_k)) & (1 - \frac{\gamma\tau_k}{2}) I_n \end{bmatrix}$$

and

$$\Xi_\ell(\gamma, \tau_k, t_k) = \begin{bmatrix} 0_{N,N} & 0_{N,N} \\ \frac{2\tau_k}{\gamma} A_\ell(t_k) & 0_{N,N} \end{bmatrix},\ \ell = 2, 3, \ldots, \check{m},$$

such that

$$\xi[k+1] = M(\gamma, \tau_k, \Xi_1(\tau_k, t_k, t_k), \ldots, \Xi_{\check{m}}(\tau_k, t_k, t_k))\xi[k]. \tag{9}$$

Note that $A_1(t_k), \ldots, A_{\check{m}}(t_k)$ are nonnegative matrices satisfying $A_1(t_k) + A_2(t_k) + \cdots + A_{\check{m}}(t_k) = A(t_k)$, where $A(t_k) = \mathrm{diag}\{L(t_k)\} - L(t_k)$ is the adjacency matrix associated with digraph $G(t_k)$ and if $t_s^i = t_{k-k'}$, where $k' \in \{0, 1, \ldots, \check{m} - 1\}$, then the ith row of matrix $A_{k'+1}(t_k)$ is equal to the ith row of matrix $A(t_k)$, while the ith rows of all the other matrices in $A_1(t_k), \ldots, A_{\check{m}}(t_k)$ are equal to zeros.

We begin our analysis with the following observation.

Lemma 2: Let $d_{max} = (N-1)\bar{\alpha}$, then $\Xi(\gamma, \tau_k, t_k) = \Xi_1(\gamma, \tau_k, t_k) + \Xi_2(\gamma, \tau_k, t_k) + \cdots + \Xi_{\check{m}}(\gamma, \tau_k, t_k)$, $k \in \mathbb{N}$, is a row-stochastic matrix with positive diagonal elements if γ satisfies

$$2\sqrt{d_{max}} \leq \gamma < \frac{2}{\bar{T}_u}. \tag{10}$$

Proof. The result follows directly by observing the fact that

$$\begin{aligned} & \Xi(\gamma, \tau_k, t_k) \\ = \ & \Xi_1(\gamma, \tau_k, t_k) + \Xi_2(\gamma, \tau_k, t_k) + \cdots + \Xi_{\check{m}}(\gamma, \tau_k, t_k) \\ = \ & \begin{bmatrix} (1 - \frac{\gamma\tau_k}{2}) I_n & \frac{\gamma\tau_k}{2} I_n \\ \frac{\gamma\tau_k}{2} I_n - \frac{2\tau_k}{\gamma} L(t_k) & (1 - \frac{\gamma\tau_k}{2}) I_n \end{bmatrix}. \end{aligned}$$

and the constraints on γ. ■

Based on the above lemma, now we have the following result.

Lemma 3: Suppose that γ satisfies the inequality in (10). Let $\{z_1, z_2, \ldots, z_q\}$ be any finite subset of $\mathbb{N}$ for which the sequence of digraphs $G(t_{z_1})$, $G(t_{z_2})$, $\ldots, G(t_{z_q})$ is jointly rooted. Then the sequence of digraphs $\Gamma(\Xi(\gamma, \tau_{z_1}, t_{z_1})), \Gamma(\Xi(\gamma, t_{z_2}, t_{z_2})), \ldots, \Gamma(\Xi(\gamma, t_{z_q}, t_{z_q}))$ is also jointly rooted.

Proof. According to the definition of the union of a group of digraphs, the union of digraphs $\Gamma(\Xi(\gamma, \tau_{z_1}, t_{z_1}))$,

$$M(\gamma, \tau_k, \Xi_1(\gamma, \tau_k, t_k), \Xi_2(\gamma, \tau_k, t_k), \ldots, \Xi_{\check{m}}(\gamma, \tau_k, t_k)) = \begin{bmatrix} \Xi_1(\tau_k, t_k, t_k) & \Xi_2(\tau_k, t_k, t_k) & \cdots & \Xi_{\check{m}-1}(\tau_k, t_k, t_k) & \Xi_{\check{m}}(\tau_k, t_k, t_k) \\ I_{2N} & 0 & \cdots & 0 & 0 \\ 0 & I_{2N} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_{2N} & 0 \end{bmatrix} \in \mathbb{R}^{2N\check{m} \times 2N\check{m}}, \quad (8)$$

$\Gamma(\Xi(\gamma, \tau_{z_2}, t_{z_2})), \ldots, \Gamma(\Xi(\gamma, \tau_{z_q}, t_{z_q}))$ is exactly the digraph $\Gamma(\sum_{\ell=1}^{q} \Xi(\gamma, \tau_{z_\ell}, t_{z_\ell}))$. Because γ satisfies the inequality in (10), it follows from Lemma 2 that $\Xi(\gamma, \tau_{z_\ell}, t_{z_\ell})$, $\ell = 1, 2, \ldots, q$, is a row-stochastic matrix with positive diagonal entries. Let $\delta = \min\{\tau_{z_1}, \tau_{z_2}, \ldots, \tau_{z_q}\}$, by observing the form that $\Xi(\gamma, \tau_{z_\ell}, t_{z_\ell})$ takes in, one can get that

$$\begin{aligned} \sum\nolimits_{\ell=1}^{q} \Xi(\gamma, \tau_{z_\ell}, t_{z_\ell}) &\geq \begin{bmatrix} 0 & \frac{\gamma}{2} q\delta I_n \\ \frac{2}{\gamma}\delta(\sum_{\ell=1}^{q} A(t_{z_\ell})) & 0 \end{bmatrix} \\ &\geq \bar{\delta} \begin{bmatrix} 0 & I_n \\ \sum_{\ell=1}^{q} A(t_{z_\ell}) & 0 \end{bmatrix} \\ &= \bar{\delta} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \end{aligned}$$

where $\bar{\delta} = \min\{\frac{2}{\gamma}\delta, \frac{\gamma}{2} q\delta\} > 0$, $M_{11} = 0_{N\times N}$, $M_{12} = I_N$, $M_{21} = \sum_{\ell=1}^{q} A(t_{z_\ell})$, and $M_{22} = 0_{N\times N}$. With the above inequality which implies a close relation between digraph $\Gamma(\sum_{\ell=1}^{q} \Xi(\gamma, \tau_{z_\ell}, t_{z_\ell}))$ and $\Gamma(\sum_{\ell=1}^{q} A(t_{z_\ell}))$, we can find a root of digraph $\Gamma(\sum_{\ell=1}^{q} \Xi(\gamma, \tau_{z_\ell}, t_{z_\ell}))$ based on $\Gamma(\sum_{\ell=1}^{q} A(t_{z_\ell}))$, the union of digraphs $G(t_{z_1}), G(t_{z_2}), \ldots, G(t_{z_q})$, which is a rooted graph according to the given condition.

Checking the entries in matrix M_{21} one can find if (i, j) is an edge in digraph $\Gamma(\sum_{\ell=1}^{q} A(t_{z_\ell}))$ then $(i, n+j)$ is an edge in digraph $\Gamma(\sum_{\ell=1}^{q} \Xi(\gamma, \tau_{z_\ell}, t_{z_\ell}))$ for any $i, j = 1, \ldots, N$, $i \neq j$, and checking the entries in matrix M_{12} one can get that edge $(n+i, i) \in \Gamma(\sum_{\ell=1}^{q} \Xi(\gamma, \tau_{z_\ell}, t_{z_\ell}))$ for any $i = 1, \ldots, N$. In what follows, we will specify how to find a root of digraph $\Gamma(\sum_{\ell=1}^{q} \Xi(\gamma, \tau_{z_\ell}, t_{z_\ell}))$. Assume ℓ is the root of digraph $\Gamma(\sum_{\ell=1}^{q} A(t_{z_\ell}))$ and without loss of generality denote by $(\ell \to k)$ the directed path connecting node ℓ to node k within digraph $\Gamma(\sum_{\ell=1}^{q} A(t_{z_\ell}))$, $k \in \{1, \ldots, N\}$, $k \neq \ell$. By splitting each edge, say (i, j) in directed path $(\ell \to k)$, into edges $(i, N+j), (N+j, j)$ and adding edge $(N+\ell, \ell)$ to node ℓ one can get a directed path in digraph $\Gamma(\sum_{\ell=1}^{q} \Xi(\gamma, \tau_{z_\ell}, t_{z_\ell}))$ connecting node $N+\ell$ to node k, $\forall k \in \{1, \ldots, N\}$, $k \neq \ell$. Obviously, this procedure simultaneously results in a directed path connecting node $N+\ell$ to node $N+k$, $\forall k \in \{1, \ldots, N\}$, $k \neq \ell$. Combing the above arguments implies that node $N+\ell$ can be connected to any other node in digraph $\Gamma(\sum_{\ell=1}^{q} \Xi(\gamma, \tau_{z_\ell}, t_{z_\ell}))$ and thus it is a root of digraph $\Gamma(\sum_{\ell=1}^{q} \Xi(\gamma, \tau_{z_\ell}, t_{z_\ell}))$. ■

Assume, in the sequel, that γ satisfies $2\sqrt{d_{max}} \leq \gamma < \frac{2}{\bar{T}_u}$. Then, by Lemma 2, all possible $\Xi(\gamma, \tau_k, t_k)$ must be nonnegative with positive diagonal elements. In addition, for each fixed γ, denote by $\tilde{D}(\gamma)$ the set consisting of all possible matrices $\Xi(\gamma, \tau_k, t_k), k \in \mathbb{N}$. To proceed further we need to exploit the compactness of $\tilde{D}(\gamma)$ (see Remark 4 for the reason why we need to do so), this is reasonable since the set of all $2N\check{m} \times 2N\check{m}$ matrices can be viewed as the metric space $\mathbb{R}^{(2N\check{m})^2}$. Unfortunately, $\tilde{D}(\gamma)$ itself is not compact, this is because $(0, \bar{T}_u]$, the interval from which τ_k is chosen, is not a compact set. Instead of investigate $\tilde{D}(\gamma)$ directly, we consider the following set

$$\begin{aligned} &D(\gamma) \\ &= \left\{ \Xi(\gamma, \tau) \middle| \Xi(\gamma, \tau) = \begin{bmatrix} (1 - \frac{\gamma\tau}{2}) I_n & \frac{\gamma\tau}{2} I_n \\ \frac{\gamma\tau}{2} I_n - \frac{2\tau}{\gamma} L & (1 - \frac{\gamma\tau}{2}) I_n \end{bmatrix}, \text{where} \right. \\ &\quad L = [l_{ij}] \text{ is a graph Laplacian}, l_{i,j} \in \{0\} \cup [\underline{\alpha}, \bar{\alpha}], \\ &\quad \left. \forall i, j \in \{1, \ldots, N\}, i \neq j, \text{ and } \tau \in [0, \bar{T}_u] \right\}. \end{aligned}$$

Clearly, $\tilde{D}(\gamma)$ is a subset of $D(\gamma)$. However, different from $\tilde{D}(\gamma)$ which is not compact, $D(\gamma)$ is a compact set. This argument is stated and carefully proved as follows.

Lemma 4: For each γ satisfying $2\sqrt{d_{max}} \leq \gamma < \frac{2}{\bar{T}_u}$, $D(\gamma)$ is a compact set.

Proof. Note that the set of all $2N \times 2N$ matrices can be viewed as the metric space $\mathbb{R}^{4N^2}$. Each $\Xi = [\Xi_{i,j}]$ in $D(\gamma)$ can be viewed as a vector $[\Xi_{1,1}, \ldots, \Xi_{1,2N}, \Xi_{2,1}, \ldots, \Xi_{2,2N}, \Xi_{2N,1}, \ldots, \Xi_{2N,2N}]$ in $\mathbb{R}^{4N^2}$. Denote by $D(\gamma, \tau)$ the set consisting of the elements in $D(\gamma)$ for each fixed $\tau \in [0, \bar{T}_u]$, it is then clear that $D(\gamma)$ is compact if each $D(\gamma, \tau)$ is compact since $D(\gamma) = \bigcup_{\tau \in [0, \bar{T}_u]} D(\gamma, \tau)$ and $[0, \bar{T}_u]$ is a compact set. Note that when $\tau = 0$, $D(\gamma, \tau)$ is compact since it is a set consisting of only one point in $\mathbb{R}^{4N^2}$. In what follows, we consider the case that $\tau \in (0, \bar{T}_u]$. Let

$$\begin{aligned} &S_i \\ &= \left\{ [\Xi_{i,1}, \ldots, \Xi_{i,2n}] \middle| [\Xi_{i,1}, \ldots, \Xi_{i,2N}] \text{ is the vector taken from} \right. \\ &\quad \left. \text{the } i\text{-th row of } \Xi, \Xi \in D(\gamma, \tau) \right\}, \; i = 1, \ldots, 2N, \tau \in (0, \bar{T}_u]. \end{aligned}$$

Then, $D(\gamma, \tau) = S_1 \times S_2 \times \cdots \times S_{2N}$ and $D(\gamma, \tau)$ is compact if each S_i $(i = 1, 2, \ldots, 2N)$ is compact. Considering that each S_i, $i = 1, 2, \ldots, N$, is a set with only one element in $\mathbb{R}^{4N^2}$, the compactness of which follows directly, and thus we will only prove in the sequel that S_{N+1} is compact, but the proof of the compactness for the other S_{N+k}'s, $k = 2, \ldots, N$ can be obtained in exactly the same way.

By observing the form that each Ξ in $D(\gamma, \tau)$ takes in, we have

$$S_{N+1} = \Big\{ \Xi^{(N+1)} = [\Xi_{N+1,1}, \ldots, \Xi_{N+1,N}, 1 - \frac{\gamma\tau}{2}, 0, \ldots, 0] \Big| \; \Xi^{(N+1)} \text{ is the vector taken from the } (N+1)\text{−th row of } \Xi, \; \Xi \in \Upsilon(\alpha, T) \Big\}.$$

Denote C_1 as

$$C_1 = \Big\{ [x_1, x_2 \ldots, x_N, 1 - \frac{\gamma\tau}{2}, 0, \ldots, 0] \Big| \frac{\gamma\tau}{2} - \frac{2\tau d_{max}}{\gamma} \le x_1 \le \frac{2\tau}{\gamma}, \text{and } x_j \in \{0\} \cup [\frac{2\tau}{\gamma}\underline{\alpha}, \frac{2\tau}{\gamma}\bar{\alpha}], j = 2, \ldots, n \Big\}.$$

C_1 is compact since it is the product space of $2N$ compact spaces in $\mathbb{R}^1$. Moreover, $S_{N+1} \subset C_1$, but $S_{N+1} \neq C_1$. This is because there is an extra constraint imposed on S_{N+1}, i.e., the sum of all the elements in each vector of S_{N+1} is 1. To illustrate accurately the relation between S_{N+1} and C_1, we introduce the following continuous multivariate function:

$$f: \;\; \mathbb{R}^{2N} \to \mathbb{R},$$

$$f(x) := \sum_{i=1}^{2N} x_i, \; \forall x = [x_1, x_2, \ldots, x_{2N}] \in \mathbb{R}^{2N}.$$

Since f is continuous and $\{1\}$ is a compact set, $f^{-1}(\{1\})$ is compact. This, together with the fact that $S_1 = f^{-1}(\{1\}) \cap C_1$ and C_1 is compact, implies that S_1 is compact, thereby completing the proof. ■

Let $\Pi(\gamma)$ denote the set of matrices

$$\begin{bmatrix} \Xi_1 & \Xi_2 & \cdots & \Xi_{\check{m}-1} & \Xi_{\check{m}} \\ I_{2N} & 0 & \cdots & 0 & 0 \\ 0 & I_{2N} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_{2N} & 0 \end{bmatrix},$$

such that $\Xi_1, \Xi_2, \ldots, \Xi_{\check{m}} \in \Lambda(\Xi(\gamma, \tau))$ and $\Xi_1 + \Xi_2 + \ldots + \Xi_{\check{m}} = \Xi(\gamma, \tau)$, where $\Xi(\gamma, \tau) \in D(\gamma)$, $\tau \in [0, \bar{T}_u]$, and $\Lambda(\Xi(\gamma, \tau)) = \{\Xi = [\Xi_{ij}] : \Xi_{ij} = \Xi_{ij}(\gamma, \tau) \text{ or } \Xi_{ij} = 0, \; i, j = 1, 2, \ldots, 2N\}$. Clearly, the set $\Pi(\gamma)$ include all possible state matrices of system (9). $\Pi(\gamma)$ is a compact set which can be obtained by observing the the fact that given any $\Xi(\gamma, \tau) \in D(\gamma)$, all the possible choices of $\Xi_1, \cdots, \Xi_{\check{m}}$ are finite and then using the similar proof as that for Lemma 4.

Given any positive integer K, define

$$\Pi(\gamma, \mathrm{K}) = \Big\{ \prod_{i=1}^{\epsilon} M(\gamma, \tau_i, \Xi_{i1}, \ldots, \Xi_{i\check{m}}) : M(\gamma, \tau_i, \cdot) \in \Pi(\gamma) \text{ and there exists an integer } \epsilon, 1 \le \epsilon \le \mathrm{K} \text{ such that the sequence of digraphs } \Gamma(\sum_{j=1}^{\check{m}} \Xi_{ij}), \; i = 1, \ldots, \epsilon, \text{ is jointly rooted} \Big\}.$$

$\Pi(\gamma, \mathrm{K})$ is also a compact set, which can be derived by noticing the following facts: 1) $\Pi(\gamma)$ is a compact set; 2) all possible choices of ϵ are finite since ϵ is bounded by K; 3) all possible choices of the root node are finite; 4) given ϵ, a root and a directed spanning tree that is incurred by this root, the following set

$$\Big\{ \prod_{i=1}^{\epsilon} M(\gamma, \tau_i, \Xi_{i1}, \ldots, \Xi_{i\check{m}}) : M(\gamma, \tau_i, \cdot) \in \Pi(\gamma) \text{ and the union of the digraphs } \Gamma(\sum_{j=1}^{\check{m}} \Xi_{ij}), \; i = 1, \ldots, \epsilon, \text{ contains the specified directed spanning tree} \Big\}$$

is compact (this can be proved by following the similar proof of Lemma 10 in [33]). Note that the set $\Pi(\gamma, \mathrm{K})$ includes all possible products of ϵ, $\epsilon \le \mathrm{K}$, consecutive state matrices of system (9).

To derive the main result, we need the classical results concerning the infinite product properties of row-stochastic matrices. Before that, we first introduce some useful notation from [31]. Let $\mathbb{A} = \{A_1, \ldots, A_k\}$ ($\mathbb{A}$ can be an infinite set) be a set of square matrices which are of the same order. By a word (in the A's, $A \in \mathbb{A}$) of length m we mean the product of m A's (repetitions permitted). The main results in [31] is rephrased as follows.

Lemma 5: *([31])* Let $\mathbb{M} = \{M_1, M_2, \ldots, M_q\}$ be a finite set of $n \times n$ SIA matrices with the property that for each sequence $M_{i_1}, M_{i_2}, \ldots, M_{i_j}$ of positive length, the matrix product $M_{i_j} M_{i_{j-1}} \cdots M_{i_1}$ is SIA. Then, for each infinite sequence $M_{i_1}, M_{i_2}, \ldots$ there exists a column vector $\mathbf{c} \in \mathbb{R}^n$ such that

$$\lim_{j\to\infty} M_{i_j} M_{i_{j-1}} \cdots M_{i_1} = \mathbf{1}\mathbf{c}^{\mathrm{T}}. \tag{11}$$

In addition, if there are infinite many elements in $\mathbb{M}$, i.e. $\mathbb{M}$ is an infinite set, let $\varphi(n)$ (which may depend on n) denote the number of different types of all $n \times n$ SIA matrices, then $\lambda(W) < 1$ for any word W in the M's, $M \in \mathbb{M}$, of length $\varphi(n) + 1$. Furthermore, if there there exists a constant $0 \le d < 1$ satisfying $\lambda(W) \le d$ for all the words in the M's of length $\varphi(n) + 1$, then (11) still holds.

Remark 3: Apparently, the product of row-stochastic matrices from an infinite set is much more complicated than that of the finite case. Lemma 5 shows for the infinite case, the existence of such d as that defined in Lemma 5 is of key role in establishing equation (11).

The following result is key in establishing the convergence analysis, which paves the way for us to use the result in Lemma 5.

Lemma 6: For any $\Phi_1, \ldots, \Phi_k \in \Pi(\gamma, \mathrm{K})$, where $k = \varphi(2N\check{m}) + 1$, there exists a constant $0 \le d < 1$ such that $\lambda(\prod_{i=1}^{k} \Phi_i) \le d$.

Proof. We first prove that for any $\Phi \in \Pi(\gamma, \mathrm{K})$, Φ is a SIA matrix. According to the definition of $\Pi(\gamma, \mathrm{K})$, there exist positive integer ϵ ($1 \le \epsilon \le \mathrm{K}$), $M(\gamma, \tau_i, \Xi_{i1}, \ldots, \Xi_{i\check{m}}) \in \Pi(\gamma)$, $i = 1, \ldots, \epsilon$, such that $\Phi = \prod_{i=1}^{\epsilon} M(\gamma, \tau_i, \Xi_{i1}, \ldots, \Xi_{i\check{m}})$ and the sequence of digraphs $\Gamma(\sum_{j=1}^{\check{m}} \Xi_{ij}), i = 1, \ldots, \epsilon$ is jointly rooted.

Since $M(\gamma, \tau_i, \Xi_{i1}, \ldots, \Xi_{i\check{m}}) \in \Pi(\gamma)$, $\sum_{j=1}^{\check{m}} \Xi_{ij}$ must be nonnegative matrices with positive diagonal elements. Furthermore, by observing the form that $\sum_{j=1}^{\check{m}} \Xi_{ij}$ takes in, one can get that there exists a positive number $\mu = \min\{1, 1 - \frac{\gamma \bar{T}_u}{2}\} < 1$ such that $diag(\sum_{j=1}^{\check{m}} \Xi_{ij}) \geq \mu I_{2N}$, for any $M(\gamma, \tau_i, \Xi_{i1}, \ldots, \Xi_{i\check{m}}) \in \Pi(\gamma)$. Combining this with the condition that the sequence of digraphs $\Gamma(\sum_{j=1}^{\check{m}} \Xi_{ij}), i = 1, \ldots, \epsilon$, is jointly rooted, we can prove that matrix Φ is SIA by following the proof of Lemma 7 in [32]. Let

$$d = \max_{\Phi_i \in \Pi(\gamma, \mathrm{K})} \lambda(\prod_{i=1}^{k} \Phi_i).$$

Recall that each matrix in $\Pi(\gamma, \mathrm{K})$ is of order $2N\check{m}$. Since $\prod_{i=1}^{k} \Phi_i$ is a word in the Φ's ($\Phi \in \Pi(\gamma, \mathrm{K})$) is of length $k = \varphi(2N\check{m}) + 1$, it follows from Lemma 5 that $\lambda(\prod_{i=1}^{k} \Phi_i) < 1$. This, together with the fact that $\Pi(\gamma, \mathrm{K})$ is a compact set and $\lambda(\cdot)$ is continuous, implies that there exists a positive number d and $0 \leq d < 1$ such that $\lambda(\prod_{i=1}^{k} \Phi_i) \leq d$ for any $\Phi_1, \ldots, \Phi_k \in \Pi(\gamma, \mathrm{K})$. ■

Remark 4: It can be seen from the proof of Lemma 6 that the compactness of set $\Pi(\gamma, \mathrm{K})$ is key to prove the existence of such d that satisfies $0 \leq d < 1$. This is also the reason why we define $\Pi(\gamma)$ and $\Pi(\gamma, \mathrm{K})$ based on $D(\gamma)$ rather than $\tilde{D}(\gamma)$.

We will denote $M(\gamma, \tau_k, \Xi_1(t_k), \ldots, \Xi_{\check{m}}(t_k))$, the state matrix of system (9), by $M(t_k)$ for simplicity if it is clear from the context. Recall that $\Xi(\gamma, \tau_k, t_k) = \Xi_1(\gamma, \tau_k, t_k) + \Xi_2(\gamma, \tau_k, t_k) + \cdots + \Xi_{\check{m}}(\gamma, \tau_k, t_k)$. With the above preparations, we are now finally in a position to prove the main result.

Proof of Theorem 2: We first prove that consensus can be reached for system (9). Let $\Phi(t_k, t_k) = I_{2N\check{m}}$, $k \geq 0$, and $\Phi(t_k, t_l) = M(t_{k-1}) \cdots M(t_{l+1}) M(t_l)$, $k > l \geq 0$.

Since the infinite sequence of graphs $G(t_0), G(t_1), \ldots$ is repeatedly jointly rooted, there exists an infinite sequence of contiguous, nonempty, uniformly bounded time intervals $[k_j, k_{j+1}), j = 1, 2, \ldots$, starting at $k_1 = 0$, for which each finite sequence of graphs $G(t_{k_j}), G(t_{k_j+1}), \ldots, G(t_{k_{j+1}-1})$ is jointly rooted. Assume, without loss of generality, that the lengths of all the time intervals $[k_j, k_{j+1}), j = 1, 2, \ldots$, are bounded by K. It follows from Lemma 3 and the condition that the sequence of digraphs $G(t_{k_j})$, $G(t_{k_j+1}), \ldots$, $G(t_{k_{j+1}-1})$ is jointly rooted that the sequence of digraphs $\Gamma(\Xi(\gamma, \tau_{k_j}, t_{k_j}))$, $\Gamma(\Xi(\gamma, \tau_{k_j+1}, t_{k_j+1}))$, $\ldots$, $\Gamma(\Xi(\gamma, \tau_{k_{j+1}-1}, t_{k_{j+1}-1}))$ is also jointly rooted for each $j \in \mathbb{N}$, which, together with the proof of Lemma 6, implies that $\Phi(t_{k_{j+1}, t_{k_j}}) = \prod_{k=k_j}^{k_{j+1}-1} M(t_k) \in \Pi(\gamma, \mathrm{K})$. Since $\Phi(t_{k_j}, 0) = \Phi(t_{k_j}, t_{k_{j-1}}) \Phi(t_{k_{j-1}}, t_{k_{j-2}}) \cdots \Phi(t_{k_2}, t_{k_1})$, it then follows from Lemma 5 and Lemma 6 that

$$\lim_{j \to \infty} \Phi(k_j, 0) = \mathbf{1}_{2N\check{m}} \mathbf{w}^{\mathrm{T}}, \quad (12)$$

where $\mathbf{w} \in \mathbb{R}^{2N\check{m}}$ and $\mathbf{w} \geq 0$.

The remaining part then can be completed by mimicking an argument similar to the proof of Theorem 2 in [9]. That is, for each $m > 0$, let k_l be the largest nonnegative integer such that $k_l \leq m$. Note that matrix $\Phi(t_m, t_{k_l})$ is row stochastic, thus we have

$$\begin{aligned} \Phi(t_m, 0) - \mathbf{1}\mathbf{w}^{\mathrm{T}} &= \Phi(t_m, t_{k_l}) \Phi(t_{k_l}, 0) - \Phi(t_m, t_{k_l}) \mathbf{1}\mathbf{w}^{\mathrm{T}} \\ &= \Phi(t_m, t_{k_l})(\Phi(t_{k_l}, 0) - \mathbf{1}\mathbf{w}^{\mathrm{T}}). \end{aligned}$$

The matrix $\Phi(t_m, t_{k_l})$ is bounded because it is the product of finite matrices which come from a bounded set $\Pi(\gamma)$. By using (12), we immediately have $\lim_{m \to \infty} \Phi(t_m, 0) = \mathbf{1}_{2N\check{m}} \mathbf{w}^{\mathrm{T}}$. Combining this with the fact that $\xi(t_m) = \Phi(t_m, 0)\xi(0)$ yields $\lim_{m \to \infty} \xi(t_m) = (w^{\mathrm{T}} \xi(0)) \mathbf{1}_{2N\check{m}}$, which in turn, together with the fact that $y(t_m) = \frac{2}{\gamma} v(t_m) + x(t_m)$, implies $\lim_{m \to \infty} x(m) = \lim_{m \to \infty} y(m) = (w^{\mathrm{T}} \xi(0)) \mathbf{1}_N$ and $\lim_{k \to \infty} v(m) = 0$, and therefore completing the proof. ■

VI. Numerical Example

In this section, we present an example to demonstrate the effectiveness of our result.

Consider a group of four autonomous ground robots which are shown in Figure 1. Assume that each robot is governed by double-integrator dynamics (1), and further that each robot's update intervals are uniform random variables over $[0.2, 0.5]$. Assume that the weighting factor of each edge of the communication topology is 1. We further assume the system evolves in the following asynchronous way

(1) robot 2 can receive the state information of robot 1 at update times t_{4k+1}^2, $k \in \mathbb{N}$;
(2) robot 3 can receive the state information of robot 4 at update times t_{4k+2}^3, and receive the state information of robot 1 at update times t_{4k+3}^3, $k \in \mathbb{N}$;
(3) robot 4 can receive the sate information of robot 2 at update times t_{4k+2}^4, and receive the state information of robot 3 at update times t_{4k}^4, $k \in \mathbb{N}$.

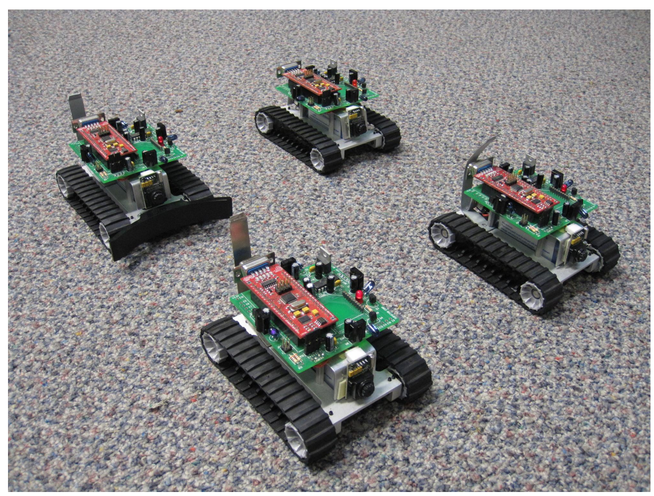

Fig. 1. Four ground robots (photo taken at the Australian National University).

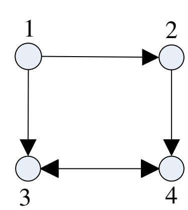

Fig. 2. A rooted graph in which node 1 is the root.

We further assume that $\gamma = 2$ which satisfies the inequality in (10) since $d_{max} = 1$ and $\bar{T}_u = 0.5$. Obviously, the infinite sequence of digraphs $G(t_1), G(t_2), \ldots$ is repeatedly jointly

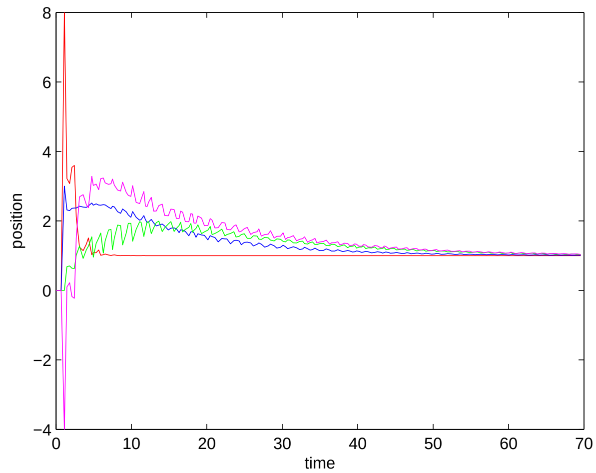

Fig. 3. Position trajectories for the robots evolving according to asynchronous system (6), (7).

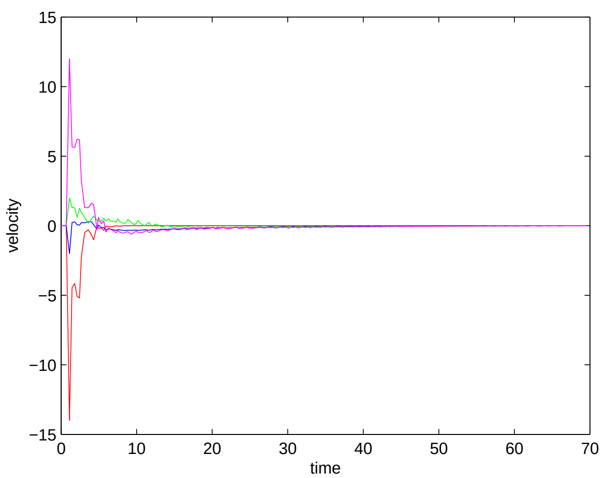

Fig. 4. Velocity trajectories for the robots evolving according to asynchronous system (6), (7).

rooted (see Figure 2 for the union of the communication topologies across a sufficiently long but bounded time interval, in which node 1 is the root). It can be seen from Figure 3 and Figure 4, which show respectively the position and velocity trajectories of the four robots that consensus is finally reached for the asynchronous system (6), (7).

VII. Conclusions

We have investigated in this paper the asynchronous consensus problem of discrete-time second-order multi-agent system under dynamically changing communication topology, in which a very general setting as opposed to the synchronous case has been considered. By merging the time sequences at which each agents detects its neighbors' state information into a single ordered sequence $\mathcal{T}$ and then casting the asynchronous discrete-time system into an equivalent augmented synchronous discrete-time system which evolves over time sequence $\mathcal{T}$, it has been shown by rigorous analysis that based on some conditions on the velocity damping gain, consensus can be reached if the infinite sequence of communication topologies over the time sequence $\mathcal{T}$ is repeatedly jointly rooted, which is essentially the same condition for guaranteeing the synchronous consensus and thus extending the existing results to a very generalized case.

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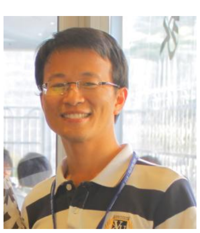

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