5.7 A stochastic approach to event-based control

5.7.1 Event-based control as a two-person team problem

This section considers the design of event-triggered controllers in the context of stochastic linear systems. The problem setup is posed as a two-person team problem, where the two agents are given by the controller and the event-trigger. Both agents aim at minimizing conjointly a common cost function, which comprises a quadratic control cost and a penalty that is paid whenever an event is generated. Triggering an event evokes the transmission of the current state information to the controller that adjusts the control inputs. The event generator, which is situated at the sensor, must therefore decide carefully, whenever it is worth to pay the penalty in order to update the controller. As the information available differs among the agents, the possibility of signalling between the agents is an integral part of the stochastic optimal control problem.

In the present system signaling can occur in two ways. By choosing specific control inputs, the controller may invoke the event generator to send another state update. On the other hand, by not sending information to the controller, the event generator implicitly signals also information to the controller depending on the choice of the event-triggering law.

Signalling can improve the system performance, but it is generally undesired in the design process, as it does not admit efficient numerical algorithms to be developed that solve the optimization problem. The resulting optimal policies are usually non-linear and depend on the complete observation history. In the time-triggered case, when transmission times are chosen beforehand, signaling is not possible and, therefore, the optimal design can be carried out in a straightforward manner. Interestingly, it turns out for the event-triggered transmission scheme that under some mild assumptions, signaling is not beneficial. These mild assumptions presume that the policies are deterministic and that the distributions of the noise process and the initial condition are symmetric.

The solution of the initial joint optimization problem reduces to an emulation-based approach: In the first step, the optimal controller is designed independently of the event generator and the communication penalty yielding a linear control law and a state estimator. In the second step, the event-triggering law is computed via stochastic dynamic programming, where events are triggered by the one-step ahead estimation error. Apart from the similar structure compared with the event-triggered controller in **[sec5-2]** , it is interesting to note that both designs use an emulation-based approach despite of the differing theoretical backgrounds.

This section is separated into three parts. In Section 5.7.2 the problem setup is introduced. This problem is solved for time-triggered transmission schemes in Section 5.7.3. The main results for the event-triggered case are given in Section 5.7.4.

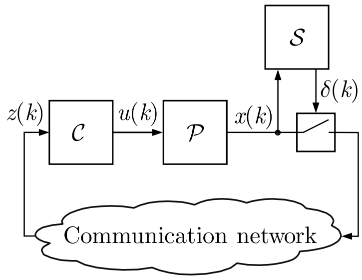

Fig. 5.1. Resource-constrained control system the with process $\mathcal{P}$, the sensor station $\mathcal{S}$ executing the event-triggering law, and the control station $\mathcal{C}$.

5.7.2 Problem formulation

The resource-constrained networked control system under consideration is illustrated in Fig. 5.1. It consists of a process $\mathcal{P}$ to be regulated by a controller $\mathcal{C}$, which is implemented at the actuator. The control station $\mathcal{C}$ receives measurements from a sensor station $\mathcal{S}$ that have to be transmitted over a resource-constrained communication network. Within the sensor station an event generator is implemented that decides upon current observations whether or not to send information to the controller. Subsequently, the individual functional blocks in Fig. 5.1 are defined.

The process $\mathcal{P}$ is described by the following discrete-time stochastic difference equation

$$\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}\boldsymbol{u}(k) + \boldsymbol{w}(k), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{5.1}$$

where $\boldsymbol{A} \in \mathrm{I\!R}^{n\times n}$, $\boldsymbol{B} \in \mathrm{I\!R}^{n\times d}$. The variables $\boldsymbol{x}(k)$ and $\boldsymbol{u}(k)$ denote the state and the control input and are taking values in $\mathrm{I\!R}^n$ and $\mathrm{I\!R}^d$, respectively. The initial state $\boldsymbol{x}_0$ is a random variable with finite mean and covariance $\boldsymbol{C}_{\boldsymbol{x}_0}$. The system noise process $\boldsymbol{w}(k)$ is i.i.d. (independent identically distributed), takes values in $\mathrm{I\!R}^n$, and is normal distributed with zero-mean and covariance matrix $\boldsymbol{C}_{\boldsymbol{w}}$. The random variables $\boldsymbol{x}_0$ and $\boldsymbol{w}(k)$ are statistically independent for each k. Let $(\Omega, \mathcal{F}, \mathrm{Prob})$ denote the probability space generated by the initial state $\boldsymbol{x}_0$ and noise sequence $\boldsymbol{W}^{N-1}$, where $\boldsymbol{W}^k = \{\boldsymbol{w}(0), \ldots, \boldsymbol{w}(k)\}$ denotes the truncated sequence up to time k. The variables $\boldsymbol{x}_0$ and $\boldsymbol{w}(k)$ are called the primitive random variables of the system. It is assumed that the statistics of the process $\mathcal{P}$ are known a-priori to both, the event generator and the controller.

Concerning our system model, it is needed to define the amount of information available at the control station at each time step k. The output signal $\delta(k)$ of the event generator takes values in $\{0, 1\}$ deciding whether information is transmitted at time k, i. e.,

$$\delta(k) = \begin{cases} 1, & \text{measurement } \boldsymbol{x}(k) \text{ is sent,} \\ 0, & \text{no measurement is transmitted.} \end{cases}$$

Therefore, the signal $\boldsymbol{z}(k)$ is defined as

$$\boldsymbol{z}(k) = \begin{cases} \boldsymbol{x}(k), & \delta(k) = 1, \\ \emptyset, & \delta(k) = 0. \end{cases} \tag{5.2}$$

As various steps of decisions are made within one time period k, a causal ordering is specified by the following sequence in which the events within the system occur:

$$\cdots \to \boldsymbol{x}(k) \to \delta(k) \to \boldsymbol{z}(k) \to \boldsymbol{u}(k) \to \boldsymbol{x}(k+1) \to \cdots$$

Note that the choice of $\delta(k)$ influences the amount of information at the controller station at time k.

We allow the control input $\boldsymbol{u}(k)$ and the event-trigger output $\delta(k)$ to depend on their complete past history. This implies in particular that they may have memory and are by themselves dynamical systems. Let the event-triggering law $\pi = \{\pi_0, \pi_1, \ldots, \pi_{N-1}\}$ and the control law $\gamma = \{\gamma_0, \gamma_1, \ldots \gamma_{N-1}\}$ denote admissible policies for the finite horizon N with

$$\delta(k) = \pi_k(\boldsymbol{X}^k), \qquad \boldsymbol{u}(k) = \gamma_k(\boldsymbol{Z}^k). \tag{5.3}$$

We assume that the mappings π_k and γ_k are measurable mappings of their available information $\boldsymbol{X}^k$ or $\boldsymbol{Z}^k$, respectively. Let us denote $\mathcal{U}_{\mathrm{ET}}$ to be the set of all admissible policy pairs (π, γ). As we will also consider time-triggered policies, we define the set $\mathcal{U}_{\mathrm{TT}}$ of admissible policy pairs, where the controller is given by (5.3) and π is either 0 or 1 at each time k, i. e.,

$$\mathcal{U}_{\mathrm{TT}} = \{(\pi, \gamma) \in \mathcal{U}_{\mathrm{ET}} | \pi_k \equiv c_k, c_k \in \{0, 1\}, k \in \{0, \ldots, N-1\}\}. \tag{5.4}$$

The communication channel takes the role of restricting or penalizing transmissions in the feedback loop. This will be reflected in the optimization problem. Let $J_{\mathcal{C}}$ be the control objective defined as

$$J_{\mathcal{C}} = \boldsymbol{x}^{\mathrm{T}}(N)\boldsymbol{Q}_N\boldsymbol{x}(N) + \sum_{k=0}^{N-1} \boldsymbol{x}(k)^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{x}(k) + \boldsymbol{u}(k)^{\mathrm{T}}\boldsymbol{R}\boldsymbol{u}(k), \tag{5.5}$$

where $\boldsymbol{Q}_N$ and $\boldsymbol{Q}$ are positive definite matrices in $\mathbb{R}^{n\times n}$and $\boldsymbol{R} \in \mathbb{R}^{d\times d}$ is positive semi-definite. Let r be the communication cost given by the number of transmissions, i. e.,

$$r = \sum_{k=0}^{N-1} \delta(k). \tag{5.6}$$

Problem 5.1. For a given $\lambda \geq 0$, find the optimal policies π^* and γ^*:

$$\inf_{(\pi,\gamma)\in\mathcal{U}_{\mathrm{ET}}} \mathrm{E}\left[J_{\mathcal{C}} + \lambda r\right].$$

The nonnegative value λ can be interpreted as the weight of penalizing transmissions over the communication channel. For notational convenience, we define the cost function $J(\pi, \gamma)$ for $(\pi, \gamma) \in \mathcal{U}_{\mathrm{ET}}$ to be

$$J(\pi, \gamma) = \mathrm{E}\left[J_{\mathcal{C}} + \lambda r\right]. \tag{5.7}$$

From the definition of $\mathcal{U}_{\mathrm{TT}}$ in (5.4), it can be observed that in our framework time-triggered controllers constitute a subset of event-triggered controllers, i. e., $\mathcal{U}_{\mathrm{TT}} \subset \mathcal{U}_{\mathrm{ET}}$. Therefore, we can state the following inequality

$$\inf_{(\pi,\gamma)\in\mathcal{U}_{\mathrm{ET}}} J(\pi, \gamma) \leq \inf_{(\pi,\gamma)\in\mathcal{U}_{\mathrm{TT}}} J(\pi, \gamma),$$

which says that the optimal event-based controller always outperforms the optimal time-triggered controller.

5.7.3 Optimal time-triggered transmission

In the following we are concerned with the optimal design of the time-triggered controller. This means we want to minimize J within the admissible time-triggered control policies given by $\mathcal{U}_{\mathrm{TT}}$. First, we fix an arbitrary triggering sequence π^{TT} and investigate the corresponding optimal control law γ^* stated by the following problem:

Problem 5.2.

$$\inf_{\gamma} J(\pi^{\mathrm{TT}}, \gamma), \quad \text{s.t. } (\pi^{\mathrm{TT}}, \gamma) \in \mathcal{U}_{\mathrm{TT}}.$$

It can be observed that the communication cost r is constant and can, therefore, be omitted from the optimization for a fixed triggering sequence π^{TT}. What remains is the expected quadratic cost term $\mathrm{E}[J_{\mathcal{C}}]$. Second, the equation can be written as a linear time-varying measurement equation

$$\boldsymbol{z}(k) = \boldsymbol{C}(k)\boldsymbol{x}(k), \quad \boldsymbol{C}(k) = \begin{cases} \boldsymbol{I}_n, & \delta(k) = 1, \\ \boldsymbol{0}_n, & \delta(k) = 0. \end{cases} \tag{5.8}$$

Therefore, we can conclude that the problem of finding the optimal control law in Problem 5.2 reduces to a standard LQG problem without measurement noise, as the process evolves according to a linear difference equation given by (5.1) and the measurement equation given by (5.8) is also linear in the state $\boldsymbol{x}(k)$ and the costs are quadratic in the state and the control input. Hence, the separation principle of stochastic control is applicable and the solution is given by a certainty-equivalence controller consisting of a linear gain and a Kalman estimator. A certainty-equivalence controller is given by solving a related deterministic control problem, where all primitive random variables are set to their means, and by replacing the state variable by its least-squares estimate within the deterministic solution.

This result is summarized in the following theorem.

Theorem 5.1. *Let an arbitrary time-triggered transmission sequence π^{TT} be fixed. Then, the solution of Problem 5.2 is given by the certainty-equivalence controller*

$$\boldsymbol{u}(k) = \gamma_k^*(\boldsymbol{Z}^k) = -\boldsymbol{L}(k)\mathrm{E}[\boldsymbol{x}(k)|\boldsymbol{Z}^k], \quad k \in \{0, \dots, N-1\} \tag{5.9}$$

with

$$\begin{aligned}
\boldsymbol{L}(k) &= \left(\boldsymbol{R} + \boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}(k+1)\boldsymbol{B}\right)^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}(k+1)\boldsymbol{A},\\
\boldsymbol{P}(k) &= \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}(k+1)\boldsymbol{A} + \boldsymbol{Q}\\
&\quad - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}(k+1)\boldsymbol{B}\left(\boldsymbol{R} + \boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}(k+1)\boldsymbol{B}\right)^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}(k+1)\boldsymbol{A},
\end{aligned} \tag{5.10}$$

where $\boldsymbol{P}(N) = \boldsymbol{Q}_N$ and $\boldsymbol{P}(k) \in \mathbb{R}^{n\times n}$ is non-negative definite for $k \in \{0, \dots, N\}$. The estimator $\mathrm{E}[\boldsymbol{x}(k)|\boldsymbol{Z}^k]$ is given by the following recursive form

$$\mathrm{E}[\boldsymbol{x}(k)|\boldsymbol{Z}^k] = \begin{cases} \boldsymbol{x}(k), & \delta(k) = 1,\\ (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{L}(k))\mathrm{E}[\boldsymbol{x}(k-1)|\boldsymbol{Z}^{k-1}], & \delta(k) = 0. \end{cases} \tag{5.11}$$

Having obtained the optimal controller for a given transmission sequence, we focus now on the calculation of the optimal time-triggered transmission scheme $\pi^{\mathrm{TT},*}$. For that reason, let $\boldsymbol{e}(k)$ be the estimation error at time k defined as

$$\boldsymbol{e}(k) = \boldsymbol{x}(k) - \mathrm{E}[\boldsymbol{x}(k)|\boldsymbol{Z}^k].$$

By using Lemma 6.1 in Chapter 8 of [1] and a couple of straightforward reformulations, the cost function can be rewritten in the following form.

$$\begin{aligned}
J =& \lambda\mathrm{E}\left[\sum_{k=0}^{N}\delta(k)\right] + \mathrm{E}[\boldsymbol{x}^{\mathrm{T}}(0)\boldsymbol{P}(0)\boldsymbol{x}(0)] + \mathrm{E}\left[\sum_{k=0}^{N-1}\boldsymbol{w}(k)^{\mathrm{T}}\boldsymbol{P}_{k+1}\boldsymbol{w}(k)\right]\\
&+ \mathrm{E}\left[\sum_{k=0}^{N-1}\boldsymbol{e}(k)^{\mathrm{T}}\boldsymbol{L}(k)^{\mathrm{T}}\boldsymbol{\Gamma}(k)\boldsymbol{L}(k)\boldsymbol{e}(k)\right]\\
&+ \mathrm{E}\left[\sum_{k=0}^{N-1}(\boldsymbol{u}(k) + \boldsymbol{L}(k)\mathrm{E}[\boldsymbol{x}(k)|\boldsymbol{Z}^k])^{\mathrm{T}}\boldsymbol{\Gamma}(k)(\boldsymbol{u}(k) + \boldsymbol{L}(k)\mathrm{E}[\boldsymbol{x}(k)|\boldsymbol{Z}^k)\right],
\end{aligned} \tag{5.12}$$

where $\boldsymbol{\Gamma}(k)$ is defined as

$$\boldsymbol{\Gamma}(k) = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}(k+1)\boldsymbol{B} + \boldsymbol{R}, \quad k \in \{0, \dots, N-1\}.$$

Obviously, the second and the third term on the right-hand side of (5.12) are constant. For a fixed time-triggered transmission sequence, we observe that the first term is constant. In addition, it can be shown that $\boldsymbol{e}(k)$ is a random variable that is independent of the policy γ. This gives the optimal control law γ^* in (5.9) for a fixed transmission sequence as already stated in Theorem 5.1. On the other hand, only the first and the forth term are varying with different transmission sequences when assuming that the control law is given by (5.9). In order to calculate the optimal time-triggered transmission sequence, we define the one-step ahead estimation error $\boldsymbol{e}_1(k)$ by

$$\boldsymbol{e}_1(k) = \boldsymbol{x}(k) - \mathrm{E}[\boldsymbol{x}(k)|\boldsymbol{Z}^{k-1}]. \tag{5.13}$$

From this definition, we have the following connection to the estimation error.

$$\boldsymbol{e}(k) = \begin{cases} 0, & \delta(k) = 1, \\ \boldsymbol{e}_1(k), & \delta(k) = 0. \end{cases}$$

The evolution of $\boldsymbol{e}_1(k)$ can be derived by

$$\begin{aligned} \boldsymbol{e}_1(k+1) &= \boldsymbol{x}(k+1) - \mathrm{E}[\boldsymbol{x}(k+1)|\boldsymbol{Z}^k] \\ &= \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}\boldsymbol{u}(k) + \boldsymbol{w}(k) - \mathrm{E}[\boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}\boldsymbol{u}(k) + \boldsymbol{w}(k)|\boldsymbol{Z}^k] \\ &= \boldsymbol{A}(\boldsymbol{x}(k) - \mathrm{E}[\boldsymbol{x}(k)|\boldsymbol{Z}^k]) + \boldsymbol{w}(k) \\ &= (1-\delta_k)\boldsymbol{A}\boldsymbol{e}_1(k) + \boldsymbol{w}(k). \end{aligned}$$

The remaining optimization problem has then the following form

$$\begin{aligned} \pi^{\mathrm{TT},*} = \arg \inf_{\delta_0,\ldots,\delta_{N-1}} \mathrm{E}\left[\sum_{k=0}^{N-1} (1-\delta(k))\boldsymbol{e}_1(k)^{\mathrm{T}}\boldsymbol{L}(k)^{\mathrm{T}}\boldsymbol{\Gamma}(k)\boldsymbol{L}(k)\boldsymbol{e}_1(k) + \lambda\delta(k)\right] \\ \text{s.t.} \quad \boldsymbol{e}_1(k+1) = (1-\delta(k))\boldsymbol{A}\boldsymbol{e}_1(k) + \boldsymbol{w}(k) \end{aligned} \tag{5.14}$$

Since the triggering variable δ_k is chosen before execution, i. e., it is independent of $\boldsymbol{e}_1(k)$, it is possible to rewrite above optimization problem in order to apply dynamic programming. For that reason, we define the error covariance

$$\boldsymbol{\Phi}(k) = \mathrm{E}[\boldsymbol{e}_1(k)\boldsymbol{e}_1^{\mathrm{T}}(k)].$$

The evolution of $\boldsymbol{\Phi}(k)$ is given by

$$\boldsymbol{\Phi}(k+1) = (1-\delta(k))\boldsymbol{A}\boldsymbol{\Phi}(k)\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{C}_{\boldsymbol{w}}, \quad \boldsymbol{\Phi}(0) = \boldsymbol{C}_{\boldsymbol{x}_0}.$$

Then, the optimization problem in (5.14) can be written as

$$\begin{aligned} \pi^{\mathrm{TT},*} = \arg \inf_{\delta_0,\ldots,\delta_{N-1}} \sum_{k=0}^{N-1} (1-\delta(k))\mathrm{tr}[\boldsymbol{\Phi}(k)\boldsymbol{L}(k)^{\mathrm{T}}\boldsymbol{\Gamma}(k)\boldsymbol{L}(k)] + \lambda\delta(k) \\ \text{s.t.} \quad \boldsymbol{\Phi}(k+1) = (1-\delta(k))\boldsymbol{A}\boldsymbol{\Phi}(k)\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{C}_{\boldsymbol{w}}, \quad \boldsymbol{\Phi}(0) = \boldsymbol{C}_{\boldsymbol{x}_0} \end{aligned} \tag{5.15}$$

We observe that the initially stochastic optimization problem reduces to a deterministic optimal control problem with state variable $\boldsymbol{\Phi}(k)$. This implies that the calculation of the optimal time-triggered transmission sequence $\pi^{\mathrm{TT},*}$ can be performed by deterministic dynamic programming. In summary, the optimal time-triggered controller within the set $\mathcal{U}^{\mathrm{TT}}$ can be calculated in two steps:

1. Obtain the optimal control gain $L(k)$ from the discrete-time Riccati equation in (5.10)
2. Solve optimization problem (5.15) that yields the optimal transmission timings.

Inspired by this design approach, the more challenging problem of event-triggered transmission strategies is studied in the next section.

5.7.4 Optimal event-triggered transmission

What makes the derivation in the previous section appealing relies on the fact that the cost function J is completely separable with respect to the control law and the transmission times. This becomes evident when regarding (5.12) that results from reordering the cost terms of J. When allowing the transmissions to be triggered by events rather than by a-priori fixed timings, the separation does not hold in the way as for the time-triggered mechanism. This is due to the fact that the estimation error $\boldsymbol{e}_k$ is generally not independent of the control law anymore when assuming a fixed event-triggering law π. In other words, the controller is able to signal through the plant to the event-trigger that it may want to receive another state update.

Such signalling is called the dual effect of control and refers to the dual role of control: (i) influencing the state evolution and (ii) decreasing the estimation error. When the second phenomenon is not present, which is also referred to as the absence of the dual effect, then the optimal control law is given by (5.9). On the other hand, in our case the dual effect is present in general, which implies that the optimal control law will be a nonlinear function of the complete history $\boldsymbol{Z}^k$, which highly depends on the choice of the fixed event generator.

Another approach that might be taken is the direct optimization of both the control law and the event-triggering law at the same time. But as the information available at the controller and at the event generator differ, the optimization problem has a non-classical information pattern, whose solution is very hard to find and no systematic algorithms are available, even for simple cases. In fact, the joint optimization problem under consideration falls into the category of sequential stochastic control problems, for which a dynamic programming formulation is possible. But the value function must be parameterized by the distribution of the state, which implies an infinite dimensional state space, and the minimum is taken over all control laws rather than over the inputs. Obviously, this formulation does not allow efficient numerical algorithms and also restrains us from getting new insights.

The aforementioned arguments suggest that only little can be said about the optimal event-triggered controller that solves Problem 5.1. It is, therefore, even more surprising that the actual optimal event-triggered controller is a certainty equivalence controller given by (5.9), i. e., it takes the same form as in the time-triggered case.

The crucial property that is exploited to show the above statement is the nestedness property of the information pattern. The information pattern is nested because the information available at the controller represented by the sigma algebra of $\boldsymbol{Z}^k$ is a subset of the information available at the event-trigger given by the sigma algebra $\boldsymbol{X}^k$ for any k, i. e.,

$$\sigma(\boldsymbol{Z}^k) \subset \sigma(\boldsymbol{X}^k) \subset \mathcal{F}, \quad k \in \{0, \dots, N-1\}.$$

The key idea to show that certainty-equivalence controller are optimal bases on the following common concept in optimal control.

Definition 5.1 (Dominating policies). *A set of policies $\mathcal{U}'_{\mathrm{ET}} \subset \mathcal{U}_{\mathrm{ET}}$ is called a dominating class of policies for Problem 5.1 if for any feasible $(\pi, \gamma) \in \mathcal{U}_{\mathrm{ET}}$, there exists a feasible $(\pi', \gamma') \in \mathcal{U}'_{\mathrm{ET}}$, such that*

$$J(\pi', \gamma') \leq J(\pi, \gamma),$$

where J is the cost function defined by (5.7) for the corresponding problem.

Once a dominating class of policies is found, the above definition implies that we can restrict the solutions of the optimization problem to such policies. In the following, we show that the set of policy pairs where the controller is a certainty-equivalence controller denoted by γ^* is a dominating class of policies. Therefore, the remaining goal is to prove that for any pair (π, γ), we can find a pair (π', γ^*) whose costs are at most that of (π, γ).

In order to achieve this, we introduce a suitable reparametrization of the triggering law. Given a policy (π, γ), we define another policy (ρ, γ) where $\rho = \{\rho_0, \dots, \rho_{N-1}\}$ is the triggering law and ρ_k is a function of $\{\boldsymbol{x}_0, \boldsymbol{W}^{k-1}\}$, such that

$$\rho_k(\boldsymbol{x}_0, \boldsymbol{W}^{k-1}) = \pi_k(\boldsymbol{X}^k), \quad k \in \{0, \dots, N-1\}, \omega \in \Omega, \tag{5.16}$$

when both systems use the control law γ. As the control inputs $\boldsymbol{U}^{k-1}$ are known at the event time k by the law γ due to $\sigma(\boldsymbol{Z}^k) \subset \sigma(\boldsymbol{X}^k)k$, the variables $\{\boldsymbol{x}_0, \boldsymbol{W}^{k-1}\}$ can be fully recovered by the state sequence $\boldsymbol{X}^k$ and vice versa. Therefore, the triggering law ρ satisfying (5.16) always exists. On the other hand, this also implies that given (ρ, γ), there is always a (π, γ) satisfying (5.16).

The next auxiliary result gives a statement on the optimal control law for fixed ρ.

Lemma 5.1. *Let the triggering law ρ be a function of primitive variables given by*

$$\delta_k = \rho_k(\boldsymbol{x}_0, \boldsymbol{W}^{k-1}), \quad k \in \{0, \dots, N-1\}. \tag{5.17}$$

If the triggering law ρ is fixed, then the optimal control law γ^* minimizing $J(\rho, \gamma)$ is a certainty-equivalence controller defined in (5.9)*.*

Proof. The proof can be outlined as follows. First, it is shown that the estimation error $\boldsymbol{e}_k$ is a random variable independent of the control law chosen for a fixed event-triggering law ρ. Inspecting Eq. (5.12), which is also valid for event-triggering law ρ, it can be seen that unlike the last expression all others are constants. Hence, it can be concluded that the certainty equivalence controller defined by (5.9) is optimal. The details of the proof can be found in [7]. □

Lemma 5.1 enables us to show that the certainty-equivalence controller is optimal for Problem 5.1, which is stated in the following theorem.

Theorem 5.2. *Let the system be given by (5.1) and (5.2). The class of policies $\mathcal{U}_{\mathrm{CE}} \subset \mathcal{U}_{\mathrm{ET}}$ defined by*

$$\mathcal{U}_{\mathrm{CE}} = \{(\pi, \gamma^*) \in \mathcal{U}_{\mathrm{ET}} \,|\, \gamma^* = -\boldsymbol{L}(k)\mathrm{E}[\boldsymbol{x}(k)|\boldsymbol{Z}^k],\ \boldsymbol{L}(k) \text{ given by (5.10)}\}$$

is a dominating class of policies for Problem 5.1.

Proof. According to Def. 5.1, it suffices to show that for any feasible pair $(\pi, \gamma) \in \mathcal{U}_{\mathrm{ET}}$, there is a feasible policy $(\pi', \gamma^*) \in \mathcal{U}_{\mathrm{CE}}$ whose costs are at most that of (π, γ).

Given an admissible policy (π, γ), there exists an admissible policy (ρ, γ) with ρ_k being a function of primitive variables that satisfies (5.16). Condition (5.16) implies that for (π, γ) and (ρ, γ), we have identical random variables $\boldsymbol{u}_k$ and δ_k for $k \in \{0, \dots, N-1\}$ and, therefore, identical costs. In the same way for the pair (ρ, γ^*), we find a triggering law π' being a function of $\boldsymbol{X}^k$, such that both (ρ, γ^*) and (π', γ^*) output identical random variables $\boldsymbol{u}_k$ and δ_k for $k \in \{0, \dots, N-1\}$. Due to Lemma 5.1, we obtain

$$J(\pi, \gamma) = J(\rho, \gamma) \geq \min_{\gamma} J(\rho, \gamma) = J(\rho, \gamma^*) = J(\pi', \gamma^*).$$

This concludes the proof. □

Theorem 5.2 implies that we can characterize optimal control policies to be certainty-equivalent control laws given by (5.9). The remaining problem is to design the optimal event generator π^*. Opposed to the time-triggered case, special care needs to be taken, as the estimator $\mathrm{E}[\boldsymbol{x}(k)|\boldsymbol{Z}^k]$ depends on the choice of the event-triggering policy π. This is because not sending update information to the controller may still be useful information for the state estimate $\mathrm{E}[\boldsymbol{x}(k)|\boldsymbol{Z}^k]$ and constitutes another type of signalling between the event generator and the controller. Throughout the subsequent paragraph, suppose that a particular event-triggering law π has been selected. We define $\tau(k)$ to be the last time an update was sent, i. e.

$$\tau(k) = \max\{\ell|\delta_\ell, \ell < k\}$$

with $\tau(k) = -1$, if there were no transmissions before k. Obviously, for $\delta(k) = 1$, we get $\mathrm{E}[\boldsymbol{x}(k)|\boldsymbol{Z}^k] = \boldsymbol{x}_k$. However, for $\delta(k) = 0$, we have

$$\mathrm{E}[\boldsymbol{x}(k)|\boldsymbol{Z}^k] = \boldsymbol{A}^{k-\tau(k)}\boldsymbol{x}(\tau(k)) + \sum_{\ell=\tau(k)}^{k-1} \boldsymbol{A}^{k-\ell-1}\boldsymbol{B}\boldsymbol{u}_\ell + \mathrm{E}\left[\sum_{\ell=\tau(k)}^{k-1} \boldsymbol{A}^{k-\ell-1}\boldsymbol{w}(\ell)|\boldsymbol{Z}^k\right] \tag{5.18}$$

In the time-triggered case, the noise process $\boldsymbol{w}(\ell)$ for $\ell \in \{\tau(k), \ldots, k-1\}$ is statistically independent of $\boldsymbol{Z}^k$, as no information is gathered after $\tau(k)$ at the controller. Therefore, the last term vanishes, because $\boldsymbol{w}(k)$ is zero-mean. On the other hand, by not sending information, when having an event-triggering law π_k that depends on $\boldsymbol{X}^k$, it also depends on $\boldsymbol{w}_{k-1}$. Therefore, the last term may not be zero in the event-triggering case due to the statistical dependence between $\boldsymbol{Z}^k$ and $\boldsymbol{w}(\ell)$ for $\ell \in \{\tau(k), \ldots, k-1\}$. The additional bias represented by the last term in (5.18) is constant for fixed $\tau(k)$ at any time $k \in \{0, \ldots, N-1\}$. Therefore, the estimator eventually takes the following form

$$\mathrm{E}[\boldsymbol{x}(k)|\boldsymbol{Z}^k] = \begin{cases} \boldsymbol{x}(k), & \delta(k) = 1 \\ (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{L}(k))\mathrm{E}\left[\boldsymbol{x}(k-1)|\boldsymbol{Z}^{k-1}\right] + \boldsymbol{\alpha}(\tau(k), k), & \delta(k) = 0, \end{cases} \tag{5.19}$$

which is similar to the least-squares estimator for the time-triggered case given by (5.11), but differs by the additional bias term $\boldsymbol{\alpha}(\tau(k), k)$.

This formulation of the optimal estimator allows us to recover a similar optimization as for the optimal time-triggered transmission sequence given by (5.14).

$$\begin{aligned} \pi^* = \arg\inf_\pi \mathrm{E}\left[\sum_{k=0}^{N-1} (1-\delta(k))(\boldsymbol{e}_1(k) - \boldsymbol{\alpha}(\tau(k), k))^{\mathrm{T}} \boldsymbol{L}(k)^{\mathrm{T}} \boldsymbol{\Gamma}(k) \boldsymbol{L}(k)\right] \times \\ \times (\boldsymbol{e}_1(k) - \boldsymbol{\alpha}(\tau(k), k)) + \lambda\delta(k)] \\ \text{s.t.} \quad \boldsymbol{e}_1(k+1) = (1-\delta(k))\boldsymbol{A}\boldsymbol{e}_1(k) + \boldsymbol{w}(k). \end{aligned} \tag{5.20}$$

Assuming that the distributions of the primitive random variables are symmetric, as in our case, it can be shown for first-order linear systems that

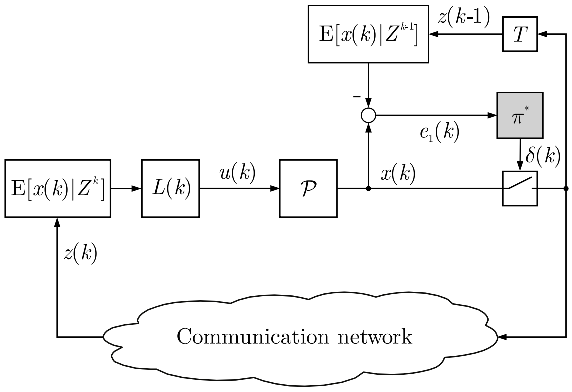

Fig. 5.2. Complete structure of the optimal event based control system with event-trigger π^* and certainty-equivalence controller represented by $\boldsymbol{L}(k)$. System block T denotes a 1-step delay element.

the optimal event generator is a symmetric threshold function and the state estimator is given by the optimal least-squares estimator for the time-triggered case, i. e., $\boldsymbol{\alpha} \equiv \boldsymbol{0}$. Symmetry means that the event-triggering mapping π_k is an even function in $\boldsymbol{e}_1(k)$. The question whether symmetric policies are also optimal for higher-order systems remains an open problem. In the following, we assume that also for higher dimensional systems symmetric even-triggering policies are optimal. Then, the optimization problem takes the standard form of a optimal stochastic control problem with state $\boldsymbol{e}_1(k)$, which can be solved by means of dynamic programming. As the terminal and the running costs are radially increasing functions and the noise distributions are symmetric, it can be concluded for the scalar case that the event-triggering law will be a time-varying threshold function of the one-step ahead estimation error $\boldsymbol{e}_1(k)$.

In summary, it has been shown that the initial Problem 5.1, which was supposed to be hard to solve, turns out to be numerically tractable, as the optimal solution can be constrained to a certain structure without loosing optimality. Fig. 5.2 illustrates the structure of the optimal event-triggered controller. The steps involved in the computation, which can be performed offline, can be summarized as follows.

1. Obtain the optimal control gain $\boldsymbol{L}(k)$ from the discrete-time Riccati equation in (5.10).
2. Solve optimization problem (5.15) that yields the optimal event-trigger as a function of $\boldsymbol{e}_1(k)$ by stochastic dynamic programming.

In terms of numerical complexity, the first step can be solved for higher dimensional processes, as it has polynomial complexity with respect to the state dimension n. Therefore, this step does not put severe requirements on the computations. On the other hand, it is well known that the computational complexity of the dynamic programming algorithm grows exponentially with the state dimension when the cost-to-go function has no closed form solution, which is also the case for (5.20) with $\boldsymbol{\alpha} \equiv \mathbf{0}$. However, near-optimal solutions, which can be solved in polynomial-time, can be obtained by using approximate dynamic programming.

Bibliographical notes

A proof for the time-triggered case in Section 5.7.3 that shows that the estimation error is a random variable that is independent of the control policy can be found in section 5.2 of [2].

The fact that stochastic optimal control problems with non-classical information pattern are hard to solve is shown in [9] for a very elementary problem setting and is also supported by the complexity theoretic viewpoint taken in [8]. The dynamic programming formulation for problems with a certain non-classical information pattern has been discussed in [10].

The optimality of the certainty equivalence controller for event-triggered systems has been discussed in [7], where also problem settings with transmission constraints rather than communication penalties have been analyzed.

The problem of event-triggered estimation for scalar systems has been extensively studied in [4, 6]. They both show that the optimal event-trigger is a symmetric threshold function, when the distributions of the primitive random variables are symmetric. While this fact is the consequence of results in majorization theory and the Riesz rearrangement inequality in [4], the work in [6] bases on global convergence properties of the proposed iterative algorithm and uses Lyapunov methods. In [6], the bias parameter $\boldsymbol{\alpha}$ is considered as a free optimization parameter and an iterative method is developed that alternates between optimizing the event-triggering policy while fixing the estimator and vice versa.

The suboptimal design of the event generator is studied in [3] by means of approximate dynamic programming.

The obtained results can be used to extend the design of event-triggered controllers for multiple feedback loops over a common communication medium analyzed in Section **[sec7-3]** beyond the consideration of integrator subsystems. This is demonstrated in [5] through a bi-level design approach, where the communication penalty functions as a Lagrange multiplier constraining the average number of transmissions.

□

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