

# Analysis, Interpretation und Generalization of a Strictly Dissipative State Space Formulation of Second Order Systems

Heiko K. F. Panzer\*, Benjamin Kleinherne, Boris Lohmann  
Lehrstuhl für Regelungstechnik, Technische Universität München  
Boltzmannstr. 15, D-85748 Garching  
\*E-Mail: panzer@tum.de

*Abstract.* We discuss a novel arrangement of second order systems in convenient state space representations fulfilling  $\mathbf{E} = \mathbf{E}^T > \mathbf{0}$  and  $\mathbf{A} + \mathbf{A}^T < \mathbf{0}$ , which has first been presented in [1]. In this contribution, the technique is motivated, analyzed from a system theoretical point of view, and extensively illustrated at the example of a single mass oscillator. In the second part of the article, the prerequisites for symmetric positive definite mass, damping, and stiffness matrices are weakened in several ways. We firstly consider the presence of gyroscopic forces resulting in a skew-symmetric component in the damping matrix. Secondly, algebraic constraints in the form of a singular mass matrix are treated for two relevant scenarios. It is shown how under each of these conditions a state space formulation with the above properties can be derived.

## 1 Introduction and Motivation

For the modeling of complex technical systems, finite element methods (FEM) are commonly used to convert partial differential equations into systems of ordinary differential equations (ODE) by spatial discretization on a given geometry. Typically—for instance in the context of structural mechanics—this results in linear time invariant (LTI) systems of second order,

$$\mathbf{G}(s) : \begin{cases} \mathbf{M} \ddot{\mathbf{z}}(t) + \mathbf{D} \dot{\mathbf{z}}(t) + \mathbf{K} \mathbf{z}(t) = \mathbf{F} \mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{S} \mathbf{z}(t), \end{cases} \quad (1)$$

where  $\mathbf{z}(t) \in \mathbb{R}^n$ ,  $\mathbf{u}(t) \in \mathbb{R}^m$ , and  $\mathbf{y}(t) \in \mathbb{R}^p$  contain the  $n$  displacement variables,  $m$  inputs, and  $p$  outputs of the system, respectively.  $\mathbf{F} \in \mathbb{R}^{n \times m}$  and  $\mathbf{S} \in \mathbb{R}^{p \times n}$  denote the input and output matrix, respectively.<sup>1</sup>  $\mathbf{M}, \mathbf{D}, \mathbf{K} \in \mathbb{R}^{n \times n}$  are named mass, damping, and stiffness matrix and are often symmetric positive definite:

$$\mathbf{M} = \mathbf{M}^T > \mathbf{0}, \quad \mathbf{K} = \mathbf{K}^T > \mathbf{0}, \quad \mathbf{D} = \mathbf{D}^T > \mathbf{0}. \quad (2)$$

With increasing demands on the accuracy of the model, the dimension  $n$  can grow dramatically, as the number of degrees of freedom rises with finer spatial discretization. Thereby, the increasing computational complexity can easily hinder the use of the model for the purpose of simulation, control, or optimization of the technical system.

Model order reduction (MOR) offers one possible remedy by approximating the transfer behavior of the high fidelity model (1) with a reduced order model (ROM) of much smaller complexity. To this end, two lines of action are basically viable. Either, the second order system is firstly linearized, i. e. transformed into a state space model of the form

$$\mathbf{G}(s) : \begin{cases} \mathbf{E} \dot{\mathbf{x}}(t) = \mathbf{A} \mathbf{x}(t) + \mathbf{B} \mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{C} \mathbf{x}(t), \end{cases} \quad (3)$$

with  $\mathbf{x} \in \mathbb{R}^N$ ,  $\mathbf{A}, \mathbf{E} \in \mathbb{R}^{N \times N}$ ,  $\mathbf{B} \in \mathbb{R}^{N \times m}$ , and  $\mathbf{C} \in \mathbb{R}^{p \times N}$ ; then, one can apply standard MOR techniques like modal or balanced truncation, or KRYLOV subspace methods (cf. Section 2.2).

<sup>1</sup>Sometimes,  $\mathbf{y}(t)$  also contains a summand  $\mathbf{S}_v \dot{\mathbf{z}}(t)$  which accounts for velocity-based output variables. Though all results of this article carry over to the general case, we assume  $\mathbf{S}_v = \mathbf{0}$  for ease of presentation.

Or alternatively, the second order system is reduced directly by means of adaptations of the given MOR methods—see, for instance, [2, 3, 4, 5, 6, 7], and references therein. In addition to numerical simplifications, this procedure has the advantage that preservation of stability can easily be guaranteed by one-sided projection.

The workaround using a state space model, however, is particularly advisable when—for simulation purposes, the design of a controller, or similar—a first order system of ODEs is required anyway after the reduction process. Also, the aforementioned preservation of stability can be assured as well, if the linearization of the second order system into a state space model is accomplished judiciously. It has recently been shown how second order systems can even be transformed into a strictly dissipative state space formulation [1], i. e. with the definiteness properties

$$\mathbf{E} = \mathbf{E}^T > \mathbf{0} \quad \text{und} \quad \mathbf{A} + \mathbf{A}^T < \mathbf{0}. \quad (4)$$

Not only does this formulation guarantee preservation of asymptotic stability in one-sided reduction, but also it enables the application of novel error bounds for reduction using KRYLOV subspace methods (cf. [8] and Section 2.2).

In this contribution, we first revise preliminaries related to state space models with the above properties (4). In Section 3, the procedure in [1] is picked up, analyzed and illustrated in detail. In Sections 4 and 5 we demonstrate how the prerequisites (2) can be relaxed to even transform certain classes of systems with algebraic constraints—so-called differential-algebraic equations (DAE)—into strictly dissipative state space models. Conclusions and an outlook are given at the end of the article.

## 2 Preliminaries

In the following, we recall important preliminaries on systems in strictly dissipative state space formulation and on their importance for MOR.

### 2.1 Logarithmic norm and the speed of contraction

Consider an asymptotically stable LTI state space model (3) with positive definite  $\mathbf{E} = \mathbf{E}^T > \mathbf{0}$ , possibly identity. Then,  $\mathbf{E}$  induces a norm in which the state vector  $\mathbf{x}(t)$  can be measured. Define

$$f(t) := \|\mathbf{x}(t)\|_{\mathbf{E}} = \sqrt{\mathbf{x}^T \mathbf{E} \mathbf{x}} \geq 0.$$

Although the system is known to be asymptotically stable, i. e.  $\lim_{t \rightarrow \infty} f(t) = 0$  for  $\mathbf{u}(t) \equiv 0$ , we know little on the evolution of the norm in time. If no input is applied, its derivative is<sup>2</sup>

$$\frac{d}{dt} f(t) = \frac{\mathbf{x}^T \mathbf{E} \dot{\mathbf{x}} + \dot{\mathbf{x}}^T \mathbf{E} \mathbf{x}}{2 \|\mathbf{x}\|_{\mathbf{E}}} = \frac{\mathbf{x}^T \mathbf{A} \mathbf{x} + \mathbf{x}^T \mathbf{A}^T \mathbf{x}}{2 \|\mathbf{x}\|_{\mathbf{E}}} = \frac{\mathbf{x}^T \frac{\mathbf{A} + \mathbf{A}^T}{2} \mathbf{x}}{\sqrt{\mathbf{x}^T \mathbf{E} \mathbf{x}}} = \frac{\mathbf{x}^T \frac{\mathbf{A} + \mathbf{A}^T}{2} \mathbf{x}}{\mathbf{x}^T \mathbf{E} \mathbf{x}} \cdot f(t).$$

Obviously, the *symmetric part* of  $\mathbf{A}$ ,  $\text{sym } \mathbf{A} := \frac{\mathbf{A} + \mathbf{A}^T}{2}$ , has an important influence on the way the norm decreases. If  $\mathbf{A} + \mathbf{A}^T$  is negative definite,  $\frac{d}{dt} f(t) < 0$  holds where  $f(t) > 0$ , and the norm decays strictly monotonically. If  $\text{sym } \mathbf{A}$  has at least one positive eigenvalue, this means that the norm can temporarily increase before it eventually tends towards zero.

---

<sup>2</sup>For better readability, we sometimes omit the time argument ( $t$ ) in the following.

In fact, a worst-case bound on the decay rate of  $\|\mathbf{x}(t)\|_{\mathbf{E}}$  can be given in terms of the right-most generalized eigenvalue of  $\text{sym } \mathbf{A}$ . Using the Cholesky decomposition of  $\mathbf{E} = \mathbf{L}^T \mathbf{L}$ , we can reformulate the RAYLEIGH quotient [9] to derive

$$\begin{aligned} \frac{d}{dt} f(t) &= \frac{\mathbf{x}^T \mathbf{L}^T \mathbf{L}^{-T} \frac{\mathbf{A} + \mathbf{A}^T}{2} \mathbf{L}^{-1} \mathbf{L} \mathbf{x}}{\mathbf{x}^T \mathbf{L}^T \mathbf{L} \mathbf{x}} \cdot f(t) \stackrel{\mathbf{y} = \mathbf{L} \mathbf{x}}{=} \frac{\mathbf{y}^T \mathbf{L}^{-T} \frac{\mathbf{A} + \mathbf{A}^T}{2} \mathbf{L}^{-1} \mathbf{y}}{\mathbf{y}^T \mathbf{y}} \cdot f(t) \\ &\leq \lambda_{\max} \left( \mathbf{L}^{-T} \frac{\mathbf{A} + \mathbf{A}^T}{2} \mathbf{L}^{-1} \right) \cdot f(t) = \lambda_{\max} \left( \frac{\mathbf{A} + \mathbf{A}^T}{2} \mathbf{E}^{-1} \right) \cdot f(t). \end{aligned}$$

Hence, the right-most solution of the generalized eigenvalue problem above, also known as the (generalized) numerical abscissa or (generalized) logarithmic norm [1, 10]

$$\mu := \mu_{\mathbf{E}}(\mathbf{A}) := \lambda_{\max} \left( \frac{\mathbf{A} + \mathbf{A}^T}{2}, \mathbf{E} \right),$$

gives us some upper bound on the slope of  $f(t)$  over  $f(t)$ :  $\frac{\dot{f}(t)}{f(t)} \leq \mu$ . In other words:  $\mu$  is a worst-case estimate on the speed of contraction the system exhibits. As it was shown in [11] and [12], the norm of  $\mathbf{x}$  decays at least as fast as an exponential function and can therefore be upper bounded by

$$f(t) \leq f(0) \cdot e^{\mu t} \quad \text{for } t \geq 0. \quad (5)$$

Accordingly,  $\mu < 0$  implies strictly monotonic decay of  $\|\mathbf{x}(t)\|_{\mathbf{E}}$  and asymptotic stability. If  $\mu = 0$ , the decay is only monotonic, i. e. the norm can not increase, at least. If  $\mu > 0$ , however, a phenomenon called "the hump", which is typical for non-normal system operators, can occur in the norm signal [13].

Due to the importance of the property  $\mu < 0$ , we make the following definitions.

**Definition 1.** A quadratic matrix  $\mathbf{A}$  is called *dissipative* if  $\mathbf{A} + \mathbf{A}^T \leq \mathbf{0}$ . It is called *strictly dissipative*, if  $\mathbf{A} + \mathbf{A}^T < \mathbf{0}$ .

**Definition 2.** A state space model (3) is called *dissipative*, if  $\mathbf{E} = \mathbf{E}^T > \mathbf{0}$  and  $\mathbf{A} + \mathbf{A}^T \leq \mathbf{0}$ , i. e. if  $\mu_{\mathbf{E}}(\mathbf{A}) = 0$ .

It is called *strictly dissipative*, if  $\mathbf{E} = \mathbf{E}^T > \mathbf{0}$  and  $\mathbf{A} + \mathbf{A}^T < \mathbf{0}$ , i. e. if  $\mu_{\mathbf{E}}(\mathbf{A}) < 0$ .

Please note that these definitions are not directly related to the concept of dissipativity/passivity in the sense of WILLEMS [14, 15], but constitute a realization-dependent property of a state space model (cf. Section 2.3).

## 2.2 Order reduction of models in strictly dissipative state space formulation

Projection-based MOR in state space, as introduced above, typically involves the choice of two matrices  $\mathbf{V}, \mathbf{W} \in \mathbb{R}^{N \times q}$  with  $q \ll N$  to obtain a reduced order model  $\mathbf{G}_r(s)$  of the form<sup>3</sup>

$$\mathbf{G}_r(s) : \begin{cases} \overbrace{\mathbf{W}^T \mathbf{E} \mathbf{V}}^{\mathbf{E}_r} \dot{\mathbf{x}}_r(t) = \overbrace{\mathbf{W}^T \mathbf{A} \mathbf{V}}^{\mathbf{A}_r} \mathbf{x}_r(t) + \overbrace{\mathbf{W}^T \mathbf{B}}^{\mathbf{B}_r} \mathbf{u}(t), \\ \mathbf{y}_r(t) = \underbrace{\mathbf{C} \mathbf{V}}_{\mathbf{C}_r} \mathbf{x}_r(t). \end{cases} \quad (6)$$

<sup>3</sup>With the common abuse of notation, in this article we refer by  $\mathbf{G}(s)$  to the transfer function in the LAPLACE domain and to the corresponding dynamic system itself.

$\mathbf{V}$  and  $\mathbf{W}$  can be computed by MOR methods like modal truncation, balanced truncation or rational KRYLOV subspaces. The special case  $\mathbf{V} = \mathbf{W}$  is referred to as one-sided reduction. For details, please refer to [16] and the references therein.

**Lemma 1** (cf. [17, 18, 19, 20]). *One-sided reduction of a strictly dissipative state space model (3) including a projection matrix  $\mathbf{V}$  of full column rank delivers an asymptotically stable ROM.*

*Proof.*  $\mathbf{E}_r = \mathbf{V}^T \mathbf{E} \mathbf{V}$  is symmetric positive definite and  $\mathbf{A}_r = \mathbf{V}^T \mathbf{A} \mathbf{V}$  fulfills  $\mu_{\mathbf{E}_r}(\mathbf{A}_r) < 0$ , as

$$\mathbf{A}_r + \mathbf{A}_r^T = \mathbf{V}^T \mathbf{A} \mathbf{V} + \mathbf{V}^T \mathbf{A}^T \mathbf{V} = \mathbf{V}^T \underbrace{(\mathbf{A} + \mathbf{A}^T)}_{< \mathbf{0}} \mathbf{V} < \mathbf{0}.$$

From our considerations above, this implies asymptotic stability of  $\mathbf{G}_r(s)$ .  $\square$

A second advantage of strict dissipativity has recently been presented in [8]: when a strictly dissipative state space model is reduced by rational KRYLOV subspace methods, upper bounds on the  $\mathcal{H}_2$  and  $\mathcal{H}_\infty$  norm of the resulting error can be derived. As the bounds only apply for strictly negative  $\mu_{\mathbf{E}}(\mathbf{A}) < 0$ , this is another motivation to model systems in such a form.

### 2.3 The influence of the system representation

It is important to note that  $\mu = \mu_{\mathbf{E}}(\mathbf{A})$  is not a system invariant, but depends on the state space representation. In particular, the transition towards  $\mathbf{A} \rightarrow \mathbf{E}^{-1} \mathbf{A}$  and  $\mathbf{E} \rightarrow \mathbf{I}$  typically has a strong—but not the desired—effect on  $\mu$ . The modal canonical form of an asymptotically stable system with distinct poles, on the other hand, always fulfills  $\mu < 0$ . But can any stable LTI system also be transformed into a strictly dissipative realization when an eigen decomposition is not feasible?

To answer this question, consider an asymptotically stable LTI system (3) with symmetric positive definite  $\mathbf{E}$  and arbitrary  $\mathbf{A}$ . In general, there are two ways to influence  $\mu$ : After a state transformation  $\mathbf{x} = \mathbf{T} \mathbf{z}$ , on the one hand, the coordinate basis is changed and the state vector  $\mathbf{z}(t)$  is measured instead of  $\mathbf{x}(t)$ . If, on the other hand, the ODE system is multiplied from the left by a matrix  $\mathbf{T}$ , then the state vector  $\mathbf{x}(t)$  remains unaffected but is measured in a different norm  $\|\cdot\|_{\mathbf{T}\mathbf{E}}$ <sup>4</sup>.

In the following, we focus on the second scenario, as it is generally desirable not to change the coordinate basis. One way to find a strictly dissipative realization is to choose a suitable symmetric positive definite matrix  $\mathbf{P}$  and set  $\mathbf{T} := \mathbf{E}^T \mathbf{P}$ :

$$\begin{aligned} \mathbf{E}^T \mathbf{P} \mathbf{E} \dot{\mathbf{x}}(t) &= \mathbf{E}^T \mathbf{P} \mathbf{A} \mathbf{x}(t) + \mathbf{E}^T \mathbf{P} \mathbf{B} \mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C} \mathbf{x}(t), \end{aligned} \tag{7}$$

The new matrix  $\widehat{\mathbf{E}} = \mathbf{E}^T \mathbf{P} \mathbf{E}$  obtained is symmetric positive definite—even if  $\mathbf{E}$  is only regular, but not positive definite—, so the first condition (4) for a strictly dissipative realization is fulfilled for all admissible  $\mathbf{P}$ . In addition, however,

$$\mathbf{E}^T \mathbf{P} \mathbf{A} + (\mathbf{E}^T \mathbf{P} \mathbf{A})^T = \mathbf{E}^T \mathbf{P} \mathbf{A} + \mathbf{A}^T \mathbf{P} \mathbf{E} \stackrel{!}{<} \mathbf{0}$$

must hold. This condition is a generalized LYAPUNOV inequality, which can always be solved for an asymptotically stable model [16, 21], but requires tremendous numerical effort for high order  $N$ . Accordingly, though a strictly dissipative system representation can always be found in theory, the computational complexity is generally not affordable in practice.

<sup>4</sup>Please note that either way, the resulting matrix  $\mathbf{E}^T \mathbf{T}$  or  $\mathbf{T} \mathbf{E}$ , respectively, must be positive definite for the logarithmic norm theory to apply.

### 3 Second order systems in state space

It turns out, however, that the structure inherent in the positive definiteness of the matrices  $\mathbf{M}$ ,  $\mathbf{D}$ , and  $\mathbf{K}$  enables the inexpensive derivation of a strictly dissipative state space formulation.

#### 3.1 Classical linearization

It has already been noted in [3], that for any regular matrix  $\mathbf{R} \in \mathbb{R}^{n \times n}$ , equivalent realizations of the second order system (1) were given by the  $N = 2n$  dimensional state space model

$$\begin{aligned} \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{z}}(t) \\ \ddot{\mathbf{z}}(t) \end{bmatrix} &= \begin{bmatrix} \mathbf{0} & \mathbf{R} \\ -\mathbf{K} & -\mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{z}(t) \\ \dot{\mathbf{z}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{F} \end{bmatrix} \mathbf{u}(t), \\ \mathbf{y}(t) &= \begin{bmatrix} \mathbf{S} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{z}(t) \\ \dot{\mathbf{z}}(t) \end{bmatrix}. \end{aligned} \quad (8)$$

Traditionally,  $\mathbf{R} = \mathbf{I}$  was a common choice in view of the little storage requirements introduced by the identity matrix. In the light of the above results, however, we will consider the choice  $\mathbf{R} = \mathbf{K}$  [3]. It is easy to see that the resulting matrix  $\mathbf{E}$  is positive definite while the symmetric part of  $\mathbf{A}$  is negative semidefinite, which characterizes a (not strictly) dissipative realization.

#### 3.2 The novel approach

We now review the novel approach first presented in [1]. One starts from the realization (8) and multiplies the state equation in (8) from the left by a matrix

$$\mathbf{T} := \begin{bmatrix} \mathbf{I} & \alpha \mathbf{I} \\ \alpha \mathbf{M} \mathbf{K}^{-1} & \mathbf{I} \end{bmatrix} \in \mathbb{R}^{2n \times 2n} \quad (9)$$

which depends on the real positive scalar  $\alpha \in \mathbb{R}^+$ . The result is

$$\begin{aligned} \underbrace{\tilde{\mathbf{E}}}_{\mathbf{T}\mathbf{E}} \dot{\mathbf{x}}(t) &= \underbrace{\tilde{\mathbf{A}}}_{\mathbf{T}\mathbf{A}} \mathbf{x}(t) + \underbrace{\tilde{\mathbf{B}}}_{\mathbf{T}\mathbf{B}} \mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C} \mathbf{x}(t), \end{aligned} \quad (10)$$

with

$$\tilde{\mathbf{A}} = \begin{bmatrix} -\alpha \mathbf{K} & \mathbf{K} - \alpha \mathbf{D} \\ -\mathbf{K} & -\mathbf{D} + \alpha \mathbf{M} \end{bmatrix}, \quad \tilde{\mathbf{E}} = \begin{bmatrix} \mathbf{K} & \alpha \mathbf{M} \\ \alpha \mathbf{M} & \mathbf{M} \end{bmatrix}, \quad \tilde{\mathbf{B}} = \begin{bmatrix} \alpha \mathbf{F} \\ \mathbf{F} \end{bmatrix}. \quad (11)$$

As was shown in [1], this state space representation is strictly dissipative if

$$0 < \alpha < \alpha^* := \lambda_{\min} \left[ \mathbf{D} \left( \mathbf{M} + \frac{1}{4} \mathbf{D} \mathbf{K}^{-1} \mathbf{D} \right)^{-1} \right] \quad (12)$$

is fulfilled, where  $\alpha^*$  is the smallest solution to the eigenvalue problem

$$\mathbf{D} \mathbf{v} = \lambda \cdot \left( \mathbf{M} + \frac{1}{4} \mathbf{D} \mathbf{K}^{-1} \mathbf{D} \right) \mathbf{v}, \quad \lambda \in \mathbb{R}, \mathbf{v} \in \mathbb{R}^n \setminus \{\mathbf{0}\}.$$

Its computation does not require much numerical effort. In MATLAB, for instance, it can be performed by the following code:

```
[L_K,~,P_K] = chol(sparse(K));
alpha_fun = @(x) (M*x + D*(P_K*(L_K\((L_K'\(P_K'*(D*x/4))))));
opts = struct('issym', true, 'isreal', true);
alpha_max = 1/eigs(alpha_fun, size(D,1), sparse(D), 1, 'LA', opts);
```

For the benchmark model of a butterfly gyroscope [22] with  $n = 17361$  degrees of freedom, this computation lasts 1.6 seconds<sup>5</sup>.

<sup>5</sup>All numerical computations for this article have been performed on an Intel Core2 Duo CPU running at 3 GHz.

### 3.3 Illustrating Example: Single Mass Oscillator

The simple example of a single mass harmonic oscillator can ideally assist to understand the effect of the change of realization presented in the previous section.

The initial conditions  $z(0)$  and  $\dot{z}(0)$  and the second order ODE

$$m \ddot{z}(t) + d \dot{z}(t) + k z(t) = 0.$$

define an initial value problem to describe the position and velocity of the mass over time. In our example, let the constants be given by  $m = 1, d = 1, k = 2$ , and assume  $z(0) = \dot{z}(0) = 0.5$ . We will now compare two different linearizations of this model. The first one is the classical formulation (8) with  $\mathbf{R} = \mathbf{K}$ . Secondly, we follow the procedure from Section 3.2 and obtain  $\alpha^* = 0.889$ , so for simplicity we choose  $\alpha := 0.5$  (cf. Figure 3 in Section 3.4). In both cases, the state vector of the state space model is given by  $\mathbf{x} = [\dot{z} \ z]^T$ .

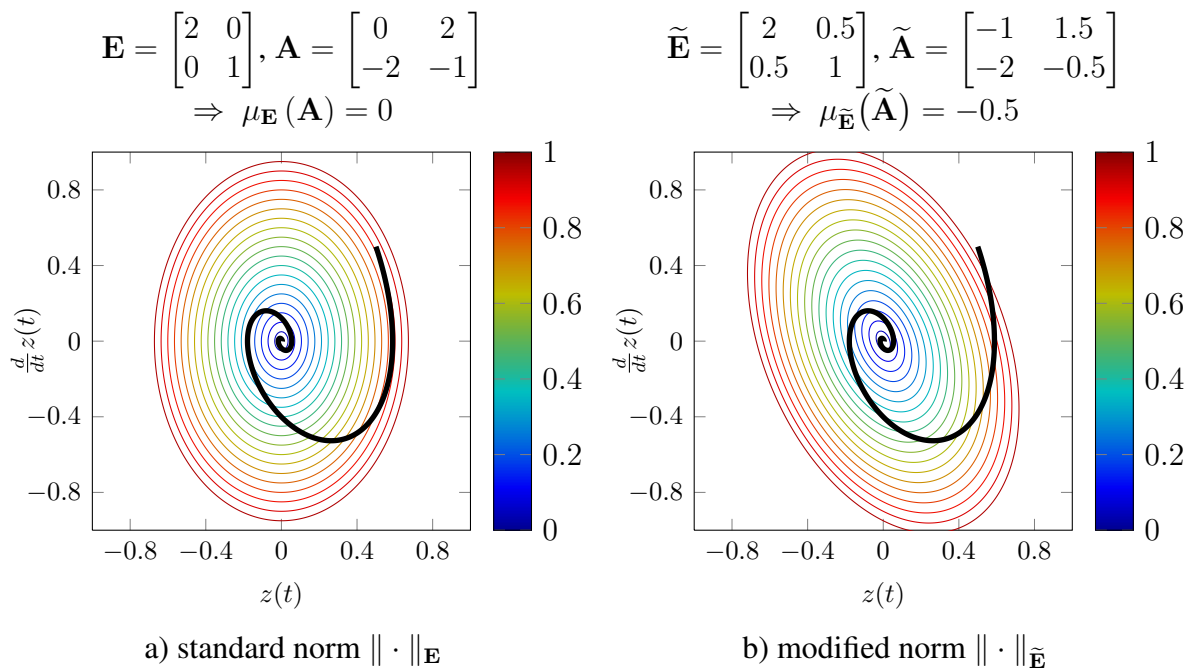


Figure 1: Dissipative vs. strictly dissipative state space realization

Figure 1 shows a phase portrait of the state vector  $\mathbf{x}(t)$ —which is identical for both cases (cf. Section 2.3)—overlaid by contour plots indicating sets of constant values w. r. t. the respective norm. In either case, the trajectories contract monotonically, yet in case a), one can see how for  $\dot{z}(t) = 0$  the decline is not strictly monotonic, but the norm can stand still for an instant. This makes sense from a physical point of view, as

$$\|\mathbf{x}\|_{\mathbf{E}} = \sqrt{\mathbf{z}^T \mathbf{K} \mathbf{z} + \dot{\mathbf{z}}^T \mathbf{M} \dot{\mathbf{z}}} = \sqrt{k z^2 + m \dot{z}^2} \quad (13)$$

is closely related to the physical energy which is only dissipated by damping and hence constant for any time  $t$  where  $\dot{z}(t) = 0$ . The modified norm, on the other hand, defines an energy pot which is reshaped in such a manner that the trajectory never touches contour lines parallelly, but always crosses them. In fact, the minimal (worst-case) angle between the trajectory and any of the contour lines is related to the logarithmic norm  $\mu$ : the more negative  $\mu$ , the steeper is the slowest decay rate of the trajectory.

The effect of deforming the energy pot can be seen even better in the 3D-plots in Figure 2. They result from the plots in Figure 1 by assigning the value of the norm to the vertical axes  $x_3$  of the plots and looking at the resulting solid figures from the sides, i. e. from the directions of the  $x_1 = z$  axis and the  $x_2 = \dot{z}$  axis. The trajectory  $\mathbf{x}(t)$  as it were is projected along the vertical axis  $x_3$  from the state space plane ( $x_3 = 0$ ) onto the respective energy pot.

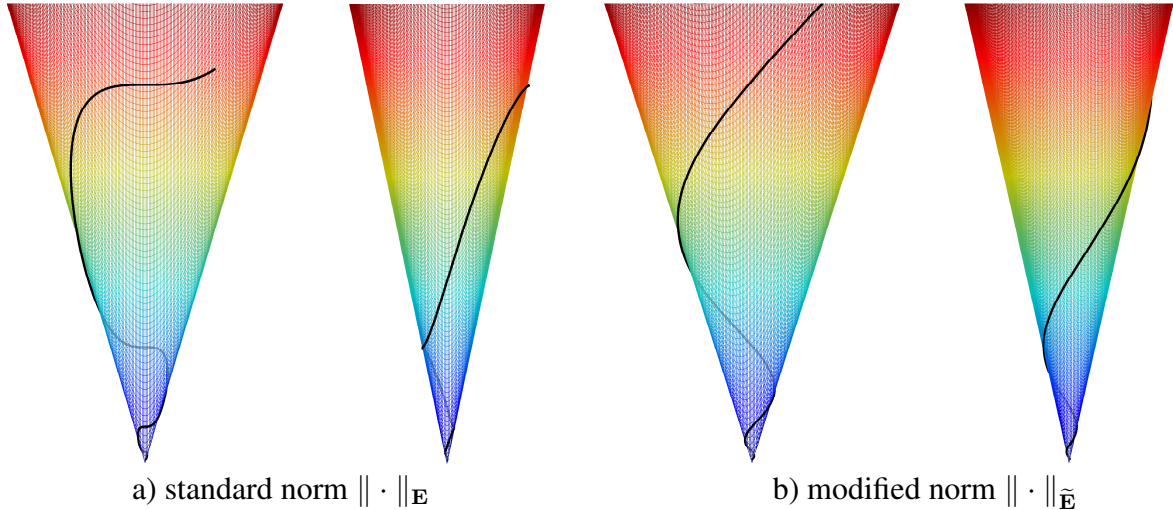


Figure 2: 3D view of standard vs. modified energy pot

Although the energy pots look similar at first sight, it is obvious that the transformation (9) introduces a deformation which avoids the plateaus of the standard norm, and at the same time balances and harmonizes the projected 3D curve of the trajectory in general.

Please note, finally, that the standard procedure of multiplying the classical ODE system from the left by  $\mathbf{E}^{-1}$  would have delivered a non-dissipative system matrix with  $\mu = +\frac{\sqrt{2}-1}{2}$ .

### 3.4 Optimal choice of $\alpha$ and the influence of damping

So far we have analyzed the effect of the transformation towards the modified state space representation (10); we have also derived an easy way to determine the valid interval of  $\alpha$  in which the system representation (10) is strictly dissipative. But the question remains open how exactly  $\alpha$  should be chosen within this interval.

From our above considerations we can conclude that the minimization of the logarithmic norm  $\mu_{\tilde{\mathbf{A}}}(\tilde{\mathbf{A}})$  is one meaningful objective: the smaller  $\mu$  becomes, the tighter is the exponential bound (5) and the more smoothly the system contracts. Although an analytic expression for the optimal value of  $\alpha$  minimizing  $\mu$  could not yet be found, we will show how  $\alpha$  can still be suitably chosen to obtain small  $\mu$ . To this end, we once more investigate the above example of a single mass oscillator. Functions  $\mu$  over  $\alpha$  for various damping values  $d$  can be seen in Figure 3. As it was predicted in Section 3.2, they all change their sign at  $\alpha = 0$  and at the respective values of  $\alpha^*$ , which have been computed according to (12) and marked by an orange spot.

It is interesting to observe that—although the exact relationship is more complicated—the graphs resemble shifted absolute value functions, all the more, the smaller  $d$  gets. To highlight this, Figure 3 also shows corresponding functions  $|\alpha - \frac{\alpha^*}{2}| - \frac{\alpha^*}{2}$  in dashed orange. This very behavior has also been observed for high-dimensional models, e. g. the benchmark model of a butterfly gyroscope [22] whose damping matrix is given by  $\mathbf{D} := 10^{-6}\mathbf{K}$ . The course of the function  $\mu(\alpha)$  is visually indistinguishable from the shifted absolute value function.

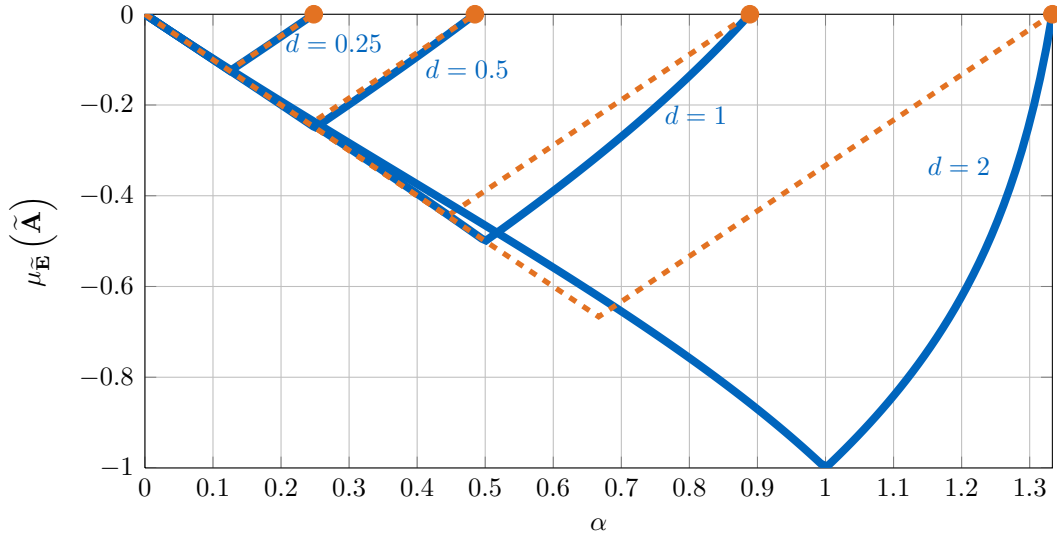


Figure 3: Logarithmic norm over  $\alpha$  for one mass oscillator with various damping

Accordingly, for models in structural mechanics, which typically exhibit very little damping, the choice  $\alpha := \frac{\alpha^*}{2}$  is recommended. In all test cases treated so far, it delivered excellent approximations of the true optimum, so the lack of an analytic expression to find the minimum can be considered a minor drawback of the method. In fact, slight differences from the optimum hardly change the resulting logarithmic norm anyway, therefore it is not of crucial importance to find the optimum as accurately as possible. Besides, the initial choice  $\alpha := \frac{\alpha^*}{2}$  can of course be improved by means of optimization.

To compute the logarithmic norm, finally, the `eigs` command in MATLAB can be used. Its standard convergence criteria, however, are quite strict and sometimes cannot be fulfilled in practice. In such a situation, two measures can be taken: Either the convergence condition can be relaxed by setting the tolerance to a larger number than the default (`eps`). Or the number of LANCZOS vectors can be increased. The following lines present a possible implementation, whose execution lasted 4.7 seconds for the butterfly gyroscope [22] with  $N = 34722$ .

```
alpha = alpha_max/2;

E = [K, alpha*M; alpha*M, M];
symA = [-alpha*K, -alpha/2*D; -alpha/2*D, -D+alpha*M];

p = 20; % number of Lanczos vectors
tol = 1e-10; % convergence tolerance
opts = struct('issym', true, 'p', p, 'tol', tol);
mu = eigs(symA, E, 1, 0, opts);
```

Please note that this code must only be used if  $\mu < 0$  is guaranteed as it is the case here. In general, the code will deliver the eigenvalue closest to zero and not the largest (right-most) one. For detailed information on large eigenvalue problems, please refer to [23].

The above procedure enables the approximate minimization of  $\mu$ . One can, however, also think of other optimization objectives: When the strictly dissipative state space model is reduced by means of KRYLOV subspace methods, for instance, the rigorous upper bounds on the  $\mathcal{H}_2$  and  $\mathcal{H}_\infty$  error norms derived in [8] apply. To keep the bounds as tight as possible, one might also choose  $\alpha$  such that one or both of the bounds become minimal instead of  $\mu$ , but this is out of the scope of this article.



### 3.5 Physical vs. modified energy norm

We have seen in Section 2.1 that a state space model in strictly dissipative form can allow for an upper bound (5) on the state vector. Consequently, for the representation derived above,

$$\|\mathbf{x}(t)\|_{\tilde{\mathbf{E}}} \leq \|\mathbf{x}(0)\|_{\tilde{\mathbf{E}}} \cdot e^{\mu_{\tilde{\mathbf{E}}}(\tilde{\mathbf{A}})t} \quad \forall t \geq 0 \quad (14)$$

holds. As noted before, the  $\tilde{\mathbf{E}}$ -norm is not directly related to a physical quantity, but amounts to

$$\|\mathbf{x}\|_{\tilde{\mathbf{E}}} = \sqrt{\mathbf{z}^T \mathbf{K} \mathbf{z} + 2\alpha \mathbf{z}^T \mathbf{M} \dot{\mathbf{z}} + \dot{\mathbf{z}}^T \mathbf{M} \dot{\mathbf{z}}}.$$

In order to analyze  $\mathbf{x}(t)$  in the physical  $\mathbf{E}$ -norm, one can perform the following estimation:

$$\begin{aligned} \|\mathbf{x}(t)\|_{\mathbf{E}} &= \sqrt{\mathbf{x}^T \mathbf{E} \mathbf{x}} = \sqrt{\mathbf{x}^T \tilde{\mathbf{E}} \mathbf{x}} \cdot \sqrt{\frac{\mathbf{x}^T \mathbf{E} \mathbf{x}}{\mathbf{x}^T \tilde{\mathbf{E}} \mathbf{x}}} \\ &\leq \sqrt{\mathbf{x}^T \tilde{\mathbf{E}} \mathbf{x}} \cdot \sqrt{\lambda_{\max}(\mathbf{E}, \tilde{\mathbf{E}})} = \sqrt{\mathbf{x}^T \tilde{\mathbf{E}} \mathbf{x}} \cdot \sqrt{\lambda_{\max}(\mathbf{E} \tilde{\mathbf{E}}^{-1})} \\ &\leq \|\mathbf{x}(0)\|_{\tilde{\mathbf{E}}} \cdot e^{\mu_{\tilde{\mathbf{E}}}(\tilde{\mathbf{A}})t} \cdot \underbrace{\sqrt{\lambda_{\max}(\mathbf{T}^{-1})}}_{\eta} \end{aligned} \quad (15)$$

Accordingly, the bound (14) in the modified norm carries over to the standard norm (13) when multiplied with the constant  $\eta$ . To compute the eigenvalue of  $\mathbf{T}^{-1}$  from (9), one can avoid the unfavorable term  $\mathbf{M}\mathbf{K}^{-1}$  in  $\mathbf{T}$  by formulating a generalized eigenvalue problem

$$\eta^2 = \lambda_{\max}(\mathbf{T}^{-1}) = \lambda_{\max} \left( \begin{bmatrix} \mathbf{I} & \alpha \mathbf{I} \\ \alpha \mathbf{M} \mathbf{K}^{-1} & \mathbf{I} \end{bmatrix}^{-1} \right) = \lambda_{\max} \left( \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{K} & \alpha \mathbf{I} \\ \alpha \mathbf{M} & \mathbf{I} \end{bmatrix}^{-1} \right) \quad (16)$$

and using an implementation like

```
I = speye(size(M));
O = sparse(size(M,1), size(M,2));
eta = sqrt(eigs([K, O; O, I], [K, alpha*I; alpha*M, I], 1));
```

whose execution lasted 2.6 seconds for the butterfly gyroscope [22]. Results for the one mass oscillator and the benchmark model of a cantilever beam [24] with ten finite elements and length  $L = 1$  are given in Figure 4. The bounds are less tight for systems of higher complexity, but one can see their potential and the basic effect of the transformation. Please note that the bound (5) directly evaluated in the standard norm would have delivered  $\|\mathbf{x}(t)\|_{\mathbf{E}} \leq \|\mathbf{x}(0)\|_{\mathbf{E}} (= const.)$ .

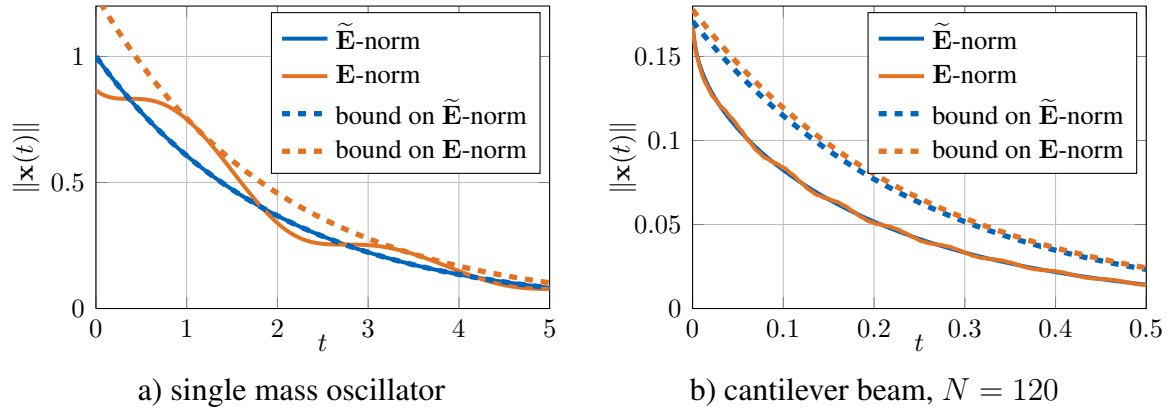


Figure 4: Upper bounds on the norms of  $\mathbf{x}(t)$

## 4 Generalization to systems with gyroscopic forces

So far we have assumed  $\mathbf{M}$ ,  $\mathbf{D}$ , and  $\mathbf{K}$  to be symmetric and positive definite. In practice, these conditions can be quite restrictive and will therefore be relaxed in the following two sections. Firstly, we will concentrate on the case that  $\mathbf{M}$  and  $\mathbf{K}$  remain symmetric positive definite while  $\mathbf{D}$  contains not only a positive definite damping, but also a skew-symmetric component as it is typical for systems with gyroscopic forces. For that reason, in this section we assume

$$\mathbf{D} \neq \mathbf{D}^T \quad \text{but} \quad \mathbf{x}^T \mathbf{D} \mathbf{x} > 0 \quad \forall \mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}. \quad (17)$$

**Theorem 1.** *When  $\mathbf{D}$  exhibits a skew-symmetric component, the state space representation (10) with matrices as in (11) is strictly dissipative if  $\alpha$  fulfills*

$$\alpha < \alpha^* := \lambda_{\min} \left[ \text{sym } \mathbf{D} \cdot \left( \mathbf{M} + \frac{1}{4} \mathbf{D}^T \mathbf{K}^{-1} \mathbf{D} \right)^{-1} \right]. \quad (18)$$

*Proof.* In the state space representation (10), the symmetric part of the matrix  $\tilde{\mathbf{A}}$  according to (11) now has to be expressed by

$$\text{sym } \tilde{\mathbf{A}} = \begin{bmatrix} -\alpha \mathbf{K} & -\frac{\alpha}{2} \mathbf{D} \\ -\frac{\alpha}{2} \mathbf{D}^T & -\text{sym } \mathbf{D} + \alpha \mathbf{M} \end{bmatrix}.$$

Formula (12) for  $\alpha^*$  is then no longer valid. It turns out, however, that the approach in [1] can be adopted in a similar way to proof the theorem.

The first goal is to find the maximal value  $\alpha^*$  for which the matrix  $\text{sym } \tilde{\mathbf{A}}$  is still strictly dissipative. SCHUR's Lemma [25] splits the definiteness condition into two matrix inequalities:

$$\text{sym } \tilde{\mathbf{A}} < \mathbf{0} \quad \Leftrightarrow \quad \begin{cases} i) & -\alpha \mathbf{K} < \mathbf{0} \\ ii) & -\text{sym } \mathbf{D} + \alpha \mathbf{M} - \left( -\frac{\alpha}{2} \mathbf{D}^T \right) \left( -\alpha \mathbf{K} \right)^{-1} \left( -\frac{\alpha}{2} \mathbf{D} \right) < \mathbf{0} \end{cases}$$

Condition *i)* is true by assumption. The second requirement leads to

$$\alpha \left( \mathbf{M} + \frac{1}{4} \mathbf{D}^T \mathbf{K}^{-1} \mathbf{D} \right) < \text{sym } \mathbf{D}$$

which is equivalent to (18). It remains to show that  $\tilde{\mathbf{E}}$  is positive definite in the derived interval of  $\alpha$ . To this end, we define the matrix

$$\mathbf{W} := \begin{bmatrix} \frac{1}{\sqrt{\alpha}} \mathbf{I} & \mathbf{0} \\ 2\sqrt{\alpha} \mathbf{M} \mathbf{K}^{-1} & -2\sqrt{\alpha} \mathbf{M} \mathbf{D}^{-T} \end{bmatrix}. \quad (19)$$

We know that  $\mathbf{Y} := \mathbf{W} \left( -\text{sym } \tilde{\mathbf{A}} \right) \mathbf{W}^T > \mathbf{0}$  and want to show that  $\tilde{\mathbf{E}} \geq \mathbf{Y}$ . Consider

$$\mathbf{Y} = \begin{bmatrix} \mathbf{K} & \alpha \mathbf{M} \\ \alpha \mathbf{M} & 4\alpha \mathbf{M} \mathbf{D}^{-T} (\text{sym } \mathbf{D} - \alpha \mathbf{M}) \mathbf{D}^{-1} \mathbf{M} \end{bmatrix}. \quad (20)$$

Only the lower right block differs from  $\tilde{\mathbf{E}}$ , so we have to make sure that the following holds:

$$\begin{aligned} & \mathbf{D}^T \mathbf{M}^{-1} \cdot \left| \begin{array}{c} \mathbf{M} \geq 4\alpha \mathbf{M} \mathbf{D}^{-T} (\text{sym } \mathbf{D} - \alpha \mathbf{M}) \mathbf{D}^{-1} \mathbf{M} \\ \mathbf{D}^T \mathbf{M}^{-1} \mathbf{D} - 4\alpha (\text{sym } \mathbf{D} - \alpha \mathbf{M}) \geq \mathbf{0} \\ \mathbf{D}^T \mathbf{M}^{-1} \mathbf{D} - 2\alpha \mathbf{D} - 2\alpha \mathbf{D}^T + 4\alpha^2 \mathbf{M} \geq \mathbf{0} \\ \underbrace{(\mathbf{D}^T - 2\alpha \mathbf{M})}_{=: \mathbf{X}^T} \mathbf{M}^{-1} \underbrace{(\mathbf{D} - 2\alpha \mathbf{M})}_{=: \mathbf{X}} \geq \mathbf{0} \end{array} \right| \cdot \mathbf{M}^{-1} \mathbf{D} \\ \Leftrightarrow & \\ \Leftrightarrow & \\ \Leftrightarrow & \end{aligned}$$

The last line is true, so  $\tilde{\mathbf{E}} \geq \mathbf{Y}$  holds, which is sufficient for  $\tilde{\mathbf{E}} > \mathbf{0}$ .  $\square$

**Remark 1.** *Please note that (18) is a real generalization of the primary formula (12).*

## 5 Generalization to systems with algebraic constraints

In this section we concentrate on systems with positive definite stiffness and positive semidefinite mass matrices

$$\mathbf{M} = \mathbf{M}^T \geq \mathbf{0}.$$

For the damping, we consider two important cases: in Section 5.2, we assume  $\mathbf{D}$  to be singular and symmetric positive semidefinite and to share the null space of  $\mathbf{M}$ , and in Section 5.3, we assume  $\mathbf{D}$  to be regular and symmetric positive definite—as in the first part of this paper.

These two special cases have been chosen due to their importance in the context of constrained FEM models, which are firstly discussed in the following.

### 5.1 Modeling of algebraic constraints

In most applications, a FEM model is subjected to given constraints, for instance when certain degrees of freedom are connected through massless elements or forced to zero. In either case, the constraints can be formulated with the help of a rectangular selector matrix  $\mathbf{H} \in \mathbb{R}^{n \times r}$ , where  $r$  denotes the number of constraints. Oftentimes,  $\mathbf{H}$  is composed of unity vectors; in the following, we assume w. l. o. g. that  $\mathbf{H}$  is orthogonal so that  $\mathbf{H}^T \mathbf{H} = \mathbf{I}_{r \times r}$  holds.

The constrained model is then given by

$$\mathbf{M}_U \ddot{\mathbf{z}}(t) + \mathbf{D}_U \dot{\mathbf{z}}(t) + \mathbf{K}_U \mathbf{z}(t) + \mathbf{H} \boldsymbol{\lambda}(t) = \mathbf{F}_U \mathbf{u}(t) \quad (21)$$

$$\mathbf{H}^T \mathbf{z}(t) = \mathbf{0}, \quad (22)$$

where (22) describes the algebraic constraints, while the term  $\mathbf{H} \boldsymbol{\lambda}(t)$  in (21) accounts for the reactive forces that provide for the abidance by the constraints with the LAGRANGE multiplier  $\boldsymbol{\lambda}$ .  $\mathbf{M}_U$ ,  $\mathbf{D}_U$ ,  $\mathbf{K}_U$ , and  $\mathbf{F}_U$  are the matrices of the unconstrained second order system.

Let  $\mathbf{H}_\perp \in \mathbb{R}^{n \times (n-r)}$  be a basis of the orthogonal complement of  $\mathbf{H}$ , so that

$$\underbrace{[\mathbf{H}_\perp \ \mathbf{H}]^T}_{\mathbf{Q}^T} \cdot \underbrace{[\mathbf{H}_\perp \ \mathbf{H}]}_{\mathbf{Q}} = \underbrace{[\mathbf{H}_\perp \ \mathbf{H}]}_{\mathbf{Q}} \cdot \underbrace{[\mathbf{H}_\perp \ \mathbf{H}]^T}_{\mathbf{Q}^T} = \mathbf{I} \quad (23)$$

holds. Then, the solution  $\mathbf{z}(t)$  can be transformed and partitioned into

$$\widehat{\mathbf{z}}(t) := \mathbf{Q}^T \mathbf{z}(t) = \begin{bmatrix} \mathbf{H}_\perp^T \mathbf{z}(t) \\ \mathbf{H}^T \mathbf{z}(t) \end{bmatrix} = \begin{bmatrix} \widehat{\mathbf{z}}_1(t) \\ \widehat{\mathbf{z}}_2(t) \end{bmatrix}. \quad (24)$$

Due to the algebraic constraints (22),

$$\widehat{\mathbf{z}}_2(t) \equiv \mathbf{0} \quad (25)$$

holds and

$$\mathbf{z}(t) = \mathbf{Q} \widehat{\mathbf{z}}(t) = [\mathbf{H}_\perp \ \mathbf{H}] \widehat{\mathbf{z}}(t) = \mathbf{H}_\perp \widehat{\mathbf{z}}_1(t) \quad (26)$$

lives in the subspace  $\text{span } \mathbf{H}_\perp$ . Inserting (26) into (21) and multiplying from the left by  $\mathbf{H}_\perp^T$  reduces the system to  $n - r$  independent purely differential equations (cf. [5]):

$$\underbrace{\mathbf{H}_\perp^T \mathbf{M}_U \mathbf{H}_\perp}_{=: \widehat{\mathbf{M}}_{11}} \ddot{\widehat{\mathbf{z}}}_1(t) + \underbrace{\mathbf{H}_\perp^T \mathbf{D}_U \mathbf{H}_\perp}_{=: \widehat{\mathbf{D}}_{11}} \dot{\widehat{\mathbf{z}}}_1(t) + \underbrace{\mathbf{H}_\perp^T \mathbf{K}_U \mathbf{H}_\perp}_{=: \widehat{\mathbf{K}}_{11}} \widehat{\mathbf{z}}_1(t) + \underbrace{\mathbf{H}_\perp^T \mathbf{H}}_{\mathbf{0}} \boldsymbol{\lambda}(t) = \underbrace{\mathbf{H}_\perp^T \mathbf{F}_U}_{=: \widehat{\mathbf{F}}_1} \mathbf{u}(t). \quad (27)$$

The constraints (25) can then again be incorporated to create a complete set of  $n$  DAEs:

$$\begin{bmatrix} \widehat{\mathbf{M}}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \ddot{\widehat{\mathbf{z}}}(t) + \begin{bmatrix} \widehat{\mathbf{D}}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \dot{\widehat{\mathbf{z}}}(t) + \begin{bmatrix} \widehat{\mathbf{K}}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_r \end{bmatrix} \widehat{\mathbf{z}}(t) = \begin{bmatrix} \widehat{\mathbf{F}}_1 \\ \mathbf{0} \end{bmatrix} \mathbf{u}(t). \quad (28)$$

Substituting  $\widehat{\mathbf{z}}(t)$  by  $\mathbf{Q}^T \mathbf{z}(t)$  and multiplying the equation from the left by  $\mathbf{Q}$  yields a DAE model (1) in the original basis  $\mathbf{z}$ . The respective matrices read

$$\mathbf{M} = \mathbf{Q} \begin{bmatrix} \widehat{\mathbf{M}}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{Q}^T, \quad \mathbf{D} = \mathbf{Q} \begin{bmatrix} \widehat{\mathbf{D}}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{Q}^T, \quad \mathbf{K} = \mathbf{Q} \begin{bmatrix} \widehat{\mathbf{K}}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_r \end{bmatrix} \mathbf{Q}^T, \quad \mathbf{F} = \mathbf{Q} \begin{bmatrix} \widehat{\mathbf{F}}_1 \\ \mathbf{0} \end{bmatrix}. \quad (29)$$

$\mathbf{M}$  and  $\mathbf{D}$  are symmetric positive semidefinite and share the same null space, while  $\mathbf{K}$  is symmetric positive definite ( $\rightarrow$ Sec. 5.2).

Sometimes, when damping cannot be easily modeled by means of finite elements, one uses RAYLEIGH damping by assigning to  $\mathbf{D}$  a linear combination of  $\mathbf{M}$  and  $\mathbf{K}$ . This is often denoted by  $\mathbf{D} = \alpha \mathbf{M} + \beta \mathbf{K}$ , but please note that this  $\alpha$  is *not* related to the variable  $\alpha$  as it is defined in the rest of this article. When the constraints are added after computing the RAYLEIGH damping, then the final matrix  $\mathbf{D}$  is semidefinite as in (28), as we have just seen. When RAYLEIGH damping is, however, defined *after* introducing the constraints,  $\mathbf{D}$  is usually positive definite as it contains a scalar multiple of the regular stiffness matrix ( $\rightarrow$ Sec. 5.3).

Those two cases are therefore treated in the following subsections. We will formulate strictly dissipative state space formulations of models with the respective properties.

## 5.2 Singular, symmetric, positive semidefinite damping $\mathbf{D}$ with the same null space as $\mathbf{M}$

Let us first assume  $\mathbf{D} = \mathbf{D}^T \geq \mathbf{0}$  in our second order system (1) is positive semidefinite and fulfills

$$\ker \mathbf{D} = \ker \mathbf{M} \iff \mathbf{D}\mathbf{x} = \mathbf{0} \quad \forall \mathbf{x} \text{ with } \mathbf{M}\mathbf{x} = \mathbf{0}. \quad (30)$$

Then there exists an orthogonal matrix  $\mathbf{Q} \in \mathbb{R}^{n \times n}$  with

$$\widehat{\mathbf{M}} := \mathbf{Q}^T \mathbf{M} \mathbf{Q} = \begin{bmatrix} \widehat{\mathbf{M}}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{r \times r} \end{bmatrix} \quad \text{and} \quad \widehat{\mathbf{D}} := \mathbf{Q}^T \mathbf{D} \mathbf{Q} = \begin{bmatrix} \widehat{\mathbf{D}}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{r \times r} \end{bmatrix}, \quad (31)$$

where  $\widehat{\mathbf{M}}_{11}, \widehat{\mathbf{D}}_{11} \in \mathbb{R}^{(n-r) \times (n-r)}$ ;  $r$  denotes the column rank of the null space  $\ker \mathbf{M}$ . Define

$$\widehat{\mathbf{K}} := \mathbf{Q}^T \mathbf{K} \mathbf{Q} = \begin{bmatrix} \widehat{\mathbf{K}}_{11} & \widehat{\mathbf{K}}_{12} \\ \widehat{\mathbf{K}}_{12}^T & \widehat{\mathbf{K}}_{22} \end{bmatrix}, \quad \widehat{\mathbf{F}} := \mathbf{Q}^T \mathbf{F} = \begin{bmatrix} \widehat{\mathbf{F}}_1 \\ \widehat{\mathbf{F}}_2 \end{bmatrix}, \quad \text{and} \quad \widehat{\mathbf{S}} := \mathbf{Q} \mathbf{S} = [\widehat{\mathbf{S}}_1 \quad \widehat{\mathbf{S}}_2]. \quad (32)$$

Please note that the block-diagonal structure of  $\widehat{\mathbf{K}}$  in (29) is a special case of (32), which is not necessarily implied by our conditions (31) on  $\mathbf{Q}$ . We therefore use this general formulation.

**Theorem 2.** *The original DAE model is equivalent to the purely ODE system*

$$\underbrace{\widehat{\mathbf{M}}_{11}}_{\widehat{\mathbf{M}}_C} \ddot{\widehat{\mathbf{z}}}_1(t) + \underbrace{\widehat{\mathbf{D}}_{11}}_{\widehat{\mathbf{D}}_C} \dot{\widehat{\mathbf{z}}}_1(t) + \underbrace{\left( \widehat{\mathbf{K}}_{11} - \widehat{\mathbf{K}}_{12} \widehat{\mathbf{K}}_{22}^{-1} \widehat{\mathbf{K}}_{12}^T \right)}_{\widehat{\mathbf{K}}_C} \widehat{\mathbf{z}}_1(t) = \underbrace{\left( \widehat{\mathbf{F}}_1 - \widehat{\mathbf{K}}_{12} \widehat{\mathbf{K}}_{22}^{-1} \widehat{\mathbf{F}}_2 \right)}_{\widehat{\mathbf{F}}_C} \mathbf{u}(t) \quad (33)$$

with the output equation

$$\mathbf{y}(t) = \underbrace{\left( \widehat{\mathbf{S}}_1 + \widehat{\mathbf{S}}_2 \widehat{\mathbf{K}}_{22}^{-1} \widehat{\mathbf{K}}_{12}^T \right)}_{\widehat{\mathbf{S}}_C} \widehat{\mathbf{z}}_1(t) + \left( \widehat{\mathbf{S}}_2 \widehat{\mathbf{K}}_{22}^{-1} \widehat{\mathbf{F}}_2 \right) \mathbf{u}(t), \quad (34)$$

whose constrained mass, damping, and stiffness matrices are symmetric positive definite. The system can therefore be transformed into a strictly dissipative state space model according to Section 3; the resulting order is  $N = 2(n - r)$ .

*Proof.* An equivalent formulation of the second order system is given by

$$\begin{bmatrix} \widehat{\mathbf{M}}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{z}}_1(t) \\ \widehat{\mathbf{z}}_2(t) \end{bmatrix} + \begin{bmatrix} \widehat{\mathbf{D}}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \widehat{\dot{\mathbf{z}}}_1(t) \\ \widehat{\dot{\mathbf{z}}}_2(t) \end{bmatrix} + \begin{bmatrix} \widehat{\mathbf{K}}_{11} & \widehat{\mathbf{K}}_{12} \\ \widehat{\mathbf{K}}_{12}^T & \widehat{\mathbf{K}}_{22} \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{z}}_1(t) \\ \widehat{\mathbf{z}}_2(t) \end{bmatrix} = \begin{bmatrix} \widehat{\mathbf{F}}_1 \\ \widehat{\mathbf{F}}_2 \end{bmatrix} \mathbf{u}(t), \quad (35)$$

$$\mathbf{y}(t) = \begin{bmatrix} \widehat{\mathbf{S}}_1 & \widehat{\mathbf{S}}_2 \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{z}}_1(t) \\ \widehat{\mathbf{z}}_2(t) \end{bmatrix}.$$

The algebraic constraints are

$$\widehat{\mathbf{K}}_{12}^T \widehat{\mathbf{z}}_1(t) + \widehat{\mathbf{K}}_{22} \widehat{\mathbf{z}}_2(t) = \widehat{\mathbf{F}}_2 \mathbf{u}(t) \iff \widehat{\mathbf{z}}_2(t) = -\widehat{\mathbf{K}}_{22}^{-1} \widehat{\mathbf{K}}_{12}^T \widehat{\mathbf{z}}_1(t) + \widehat{\mathbf{K}}_{22}^{-1} \widehat{\mathbf{F}}_2 \mathbf{u}(t). \quad (36)$$

The inverse exists because  $\widehat{\mathbf{K}}_{22}$  is positive definite since  $\widehat{\mathbf{K}} = \mathbf{Q}^T \mathbf{K} \mathbf{Q}$  is positive definite. Eq. (36) can be inserted into (35) and yields the purely differential ODE system (33) with the output equation (34)—which can also contain a feedthrough term.  $\widehat{\mathbf{K}}_C$  is symmetric positive definite due to SCHUR's Lemma.

Accordingly, this new formulation fulfills all requirements of Section 3.2 and can hence be rewritten in a strictly dissipative state space formulation.  $\square$

We complete this subsection with two remarks on numerical properties of the procedure.

**Remark 2.**  $\mathbf{Q}$  might for instance result from an eigen- or SCHUR decomposition in theory, but from a numerical point of view one must note that such transformations are no longer feasible when  $n$  is very large. However, in many practical cases the null space of  $\mathbf{M}$  and  $\mathbf{D}$  is composed of unity vectors, in which case  $\mathbf{Q}$  can be chosen as a permutation matrix and the transformation (31) can be easily performed.

**Remark 3.** Even though the sparsity of  $\mathbf{K}$  is often retrieved by  $\widehat{\mathbf{K}}$  due to the previous remark, the matrix  $\widehat{\mathbf{K}}_C$  is typically large and dense. However, judicious implementation can partially avoid the related drawbacks; in fact,  $\widehat{\mathbf{K}}_C$  does not have to be computed explicitly. Multiplication with a vector, for instance, can be performed using the distributive and associative property

$$\widehat{\mathbf{K}}_C \mathbf{x} = \widehat{\mathbf{K}}_{11} \mathbf{x} - \widehat{\mathbf{K}}_{12} \left( \widehat{\mathbf{K}}_{22}^{-1} \left( \widehat{\mathbf{K}}_{12}^T \mathbf{x} \right) \right). \quad (37)$$

For division by  $\widehat{\mathbf{K}}_C$ , one can use WOODBURY's identity [26]

$$\widehat{\mathbf{K}}_C^{-1} = \left( \widehat{\mathbf{K}}_{11} - \widehat{\mathbf{K}}_{12} \widehat{\mathbf{K}}_{22}^{-1} \widehat{\mathbf{K}}_{12}^T \right)^{-1} = \widehat{\mathbf{K}}_{11}^{-1} - \widehat{\mathbf{K}}_{11}^{-1} \widehat{\mathbf{K}}_{12} \left( \widehat{\mathbf{K}}_{22}^{-1} + \widehat{\mathbf{K}}_{12}^T \widehat{\mathbf{K}}_{11}^{-1} \widehat{\mathbf{K}}_{12} \right)^{-1} \widehat{\mathbf{K}}_{12}^T \widehat{\mathbf{K}}_{11}^{-1}. \quad (38)$$

### 5.3 Symmetric positive definite damping matrix D

Let us now consider the case that  $\mathbf{D}$  and  $\mathbf{K}$  are symmetric positive definite and  $\mathbf{M}$  is symmetric positive semidefinite. As above, we use a suitable orthogonal matrix  $\mathbf{Q} \in \mathbb{R}^{n \times n}$  to transform the system to a representation where

$$\widehat{\mathbf{M}} = \begin{bmatrix} \widehat{\mathbf{M}}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{r \times r} \end{bmatrix} \quad (39)$$

holds and  $\widehat{\mathbf{K}}$ ,  $\widehat{\mathbf{F}}$ , and  $\widehat{\mathbf{S}}$  read as in (32). The transformed damping matrix, however, is now given by

$$\widehat{\mathbf{D}} = \mathbf{Q}^T \mathbf{D} \mathbf{Q} = \begin{bmatrix} \widehat{\mathbf{D}}_{11} & \widehat{\mathbf{D}}_{12} \\ \widehat{\mathbf{D}}_{12}^T & \widehat{\mathbf{D}}_{22} \end{bmatrix} > \mathbf{0}. \quad (40)$$

This time, we directly arrange the matrices in a state space model according to (11). In fact, the resulting matrix  $\tilde{\mathbf{A}}$  is then strictly dissipative for all  $\alpha$  in the interval defined by (12), because  $\widehat{\mathbf{M}}$  only affects the lower right entry of  $\tilde{\mathbf{A}}$  and the considerations from Section 3.2 carry over. The eigenvalue problem (12), for instance, is still well-defined, as

$$\widehat{\mathbf{M}} + \frac{1}{4} \widehat{\mathbf{D}} \widehat{\mathbf{K}}^{-1} \widehat{\mathbf{D}}$$

is positive definite and hence regular even though  $\widehat{\mathbf{M}}$  loses rank. For that reason,  $\tilde{\mathbf{A}} + \tilde{\mathbf{A}}^T < \mathbf{0}$  can still be fulfilled under the weakened condition of a semidefinite mass matrix.

However, the resulting matrix  $\tilde{\mathbf{E}}$  is singular and does not satisfy the second requirement for a strictly dissipative state space representation. It has rank  $2n - r$  and the following structure:

$$\tilde{\mathbf{E}} = \begin{bmatrix} \widehat{\mathbf{K}} & \alpha \widehat{\mathbf{M}} \\ \alpha \widehat{\mathbf{M}} & \widehat{\mathbf{M}} \end{bmatrix} = \begin{bmatrix} \widehat{\mathbf{K}}_{11} & \widehat{\mathbf{K}}_{12} & \alpha \widehat{\mathbf{M}}_{11} & \mathbf{0} \\ \widehat{\mathbf{K}}_{12}^T & \widehat{\mathbf{K}}_{22} & \mathbf{0} & \mathbf{0} \\ \alpha \widehat{\mathbf{M}}_{11} & \mathbf{0} & \widehat{\mathbf{M}}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{E}}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Similarly to the procedure in the previous subsection, we can therefore partition the state vector of the first order state space model

$$\begin{bmatrix} \tilde{\mathbf{E}}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_1(t) \\ \dot{\mathbf{x}}_2(t) \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{A}}_{11} & \tilde{\mathbf{A}}_{12} \\ \tilde{\mathbf{A}}_{21} & \tilde{\mathbf{A}}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \end{bmatrix} + \begin{bmatrix} \tilde{\mathbf{B}}_1 \\ \tilde{\mathbf{B}}_2 \end{bmatrix} \mathbf{u}(t), \quad (41)$$

$$\mathbf{y}(t) = \begin{bmatrix} \tilde{\mathbf{C}}_1 & \tilde{\mathbf{C}}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \end{bmatrix}$$

with  $\mathbf{x}_1 \in \mathbb{R}^{2n-r}$ ,  $\mathbf{x}_2 \in \mathbb{R}^r$ . The matrices  $\tilde{\mathbf{A}}$ ,  $\tilde{\mathbf{B}}$ , and  $\tilde{\mathbf{C}}$  are partitioned appropriately, for instance

$$\tilde{\mathbf{A}} = \begin{bmatrix} -\alpha \widehat{\mathbf{K}}_{11} & -\alpha \widehat{\mathbf{K}}_{12} & \widehat{\mathbf{K}}_{11} - \alpha \widehat{\mathbf{D}}_{11} & \widehat{\mathbf{K}}_{12} - \alpha \widehat{\mathbf{D}}_{12} \\ -\alpha \widehat{\mathbf{K}}_{12}^T & -\alpha \widehat{\mathbf{K}}_{22} & \widehat{\mathbf{K}}_{12}^T - \alpha \widehat{\mathbf{D}}_{12}^T & \widehat{\mathbf{K}}_{22} - \alpha \widehat{\mathbf{D}}_{22} \\ -\widehat{\mathbf{K}}_{11} & -\widehat{\mathbf{K}}_{12} & -\widehat{\mathbf{D}}_{11} + \alpha \widehat{\mathbf{M}}_{11} & -\widehat{\mathbf{D}}_{12} \\ -\widehat{\mathbf{K}}_{12}^T & -\widehat{\mathbf{K}}_{22} & -\widehat{\mathbf{D}}_{12}^T & -\widehat{\mathbf{D}}_{22} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{A}}_{11} & \tilde{\mathbf{A}}_{12} \\ \tilde{\mathbf{A}}_{21} & \tilde{\mathbf{A}}_{22} \end{bmatrix}.$$

Then the following holds.

**Theorem 3.** *The given second order system is equivalently expressed by the state space model*

$$\underbrace{\tilde{\mathbf{E}}_{11}}_{\tilde{\mathbf{E}}_C} \dot{\mathbf{x}}_1(t) = \underbrace{\left( \tilde{\mathbf{A}}_{11} - \tilde{\mathbf{A}}_{12} \tilde{\mathbf{A}}_{22}^{-1} \tilde{\mathbf{A}}_{21} \right)}_{\tilde{\mathbf{A}}_C} \mathbf{x}_1(t) + \underbrace{\left( \tilde{\mathbf{B}}_1 - \tilde{\mathbf{A}}_{12} \tilde{\mathbf{A}}_{22}^{-1} \tilde{\mathbf{B}}_2 \right)}_{\tilde{\mathbf{B}}_C} \mathbf{u}(t), \quad (42)$$

$$\mathbf{y}(t) = \underbrace{\left( \tilde{\mathbf{C}}_1 - \tilde{\mathbf{C}}_2 \tilde{\mathbf{A}}_{22}^{-1} \tilde{\mathbf{A}}_{21} \right)}_{\tilde{\mathbf{C}}_C} \mathbf{x}_1(t) + \left( \tilde{\mathbf{C}}_2 \tilde{\mathbf{A}}_{22}^{-1} \tilde{\mathbf{B}}_2 \right) \mathbf{u}(t), \quad (43)$$

which is strictly dissipative for all  $\alpha$  fulfilling (12) and whose order is  $N = 2n - r$ .

*Proof.* We first show the equivalence of this realization to the state space model (41), and its existence. To this end, we identify the algebraic conditions

$$\tilde{\mathbf{A}}_{21} \mathbf{x}_1(t) + \tilde{\mathbf{A}}_{22} \mathbf{x}_2(t) = \tilde{\mathbf{B}}_2 \mathbf{u}(t) \iff \mathbf{x}_2(t) = -\tilde{\mathbf{A}}_{22}^{-1} \tilde{\mathbf{A}}_{21} \mathbf{x}_1(t) + \tilde{\mathbf{A}}_{22}^{-1} \tilde{\mathbf{B}}_2 \mathbf{u}(t). \quad (44)$$

Reinserting (44) into (41) yields the purely differential system (42), (43). The inverse  $\tilde{\mathbf{A}}_{22}^{-1}$  exists because  $\tilde{\mathbf{A}}$  is strictly dissipative—therefore,  $\tilde{\mathbf{A}}_{22}$  is strictly dissipative, too, which implies that all eigenvalues of  $\tilde{\mathbf{A}}_{22}$  have strictly negative real part and are hence different from zero.

To show the strict dissipativity of the state space model, we must firstly prove that the strict dissipativity of  $\tilde{\mathbf{A}}$  carries over to its SCHUR complement  $\tilde{\mathbf{A}}_C$ . Secondly, we must show that  $\tilde{\mathbf{E}}_C$  is positive definite in the relevant range of  $\alpha$ .

We admit values of  $\alpha$  for which  $\tilde{\mathbf{A}} + \tilde{\mathbf{A}}^T < \mathbf{0}$  holds. For the partitioned matrix  $\tilde{\mathbf{A}}$ , this means

$$\text{sym } \tilde{\mathbf{A}} = \text{sym} \begin{bmatrix} \tilde{\mathbf{A}}_{11} & \tilde{\mathbf{A}}_{12} \\ \tilde{\mathbf{A}}_{21} & \tilde{\mathbf{A}}_{22} \end{bmatrix} = \begin{bmatrix} \text{sym } \tilde{\mathbf{A}}_{11} & \frac{1}{2} (\tilde{\mathbf{A}}_{12} + \tilde{\mathbf{A}}_{21}^T) \\ \frac{1}{2} (\tilde{\mathbf{A}}_{12}^T + \tilde{\mathbf{A}}_{21}) & \text{sym } \tilde{\mathbf{A}}_{22} \end{bmatrix} < \mathbf{0}.$$

SCHUR's Lemma yields

$$\begin{aligned} & \text{sym } \tilde{\mathbf{A}}_{11} - \frac{1}{2} (\tilde{\mathbf{A}}_{12} + \tilde{\mathbf{A}}_{21}^T) \underbrace{(\text{sym } \tilde{\mathbf{A}}_{22})^{-1}}_{=\tilde{\mathbf{A}}_{22}^{-1}} \frac{1}{2} (\tilde{\mathbf{A}}_{12}^T + \tilde{\mathbf{A}}_{21}) < \mathbf{0} \\ \implies & \frac{1}{2} \tilde{\mathbf{A}}_{11} - \frac{1}{4} \tilde{\mathbf{A}}_{12} \tilde{\mathbf{A}}_{22}^{-1} \tilde{\mathbf{A}}_{21} + \frac{1}{2} \tilde{\mathbf{A}}_{11}^T - \frac{1}{4} \tilde{\mathbf{A}}_{21}^T \tilde{\mathbf{A}}_{22}^{-1} \tilde{\mathbf{A}}_{12}^T < \frac{1}{4} \tilde{\mathbf{A}}_{12} \tilde{\mathbf{A}}_{22}^{-1} \tilde{\mathbf{A}}_{12}^T + \frac{1}{4} \tilde{\mathbf{A}}_{21}^T \tilde{\mathbf{A}}_{22}^{-1} \tilde{\mathbf{A}}_{21} \\ \implies & \text{sym } \tilde{\mathbf{A}}_S = \text{sym} (\tilde{\mathbf{A}}_{11} - \tilde{\mathbf{A}}_{12} \tilde{\mathbf{A}}_{22}^{-1} \tilde{\mathbf{A}}_{21}) < \frac{1}{4} (\tilde{\mathbf{A}}_{12} - \tilde{\mathbf{A}}_{21}^T) \underbrace{\tilde{\mathbf{A}}_{22}^{-1}}_{< \mathbf{0}} (\tilde{\mathbf{A}}_{12} - \tilde{\mathbf{A}}_{21}^T)^T < \mathbf{0}. \end{aligned}$$

Accordingly,  $\tilde{\mathbf{A}}_S + \tilde{\mathbf{A}}_S^T < \mathbf{0}$  is fulfilled. It remains to prove that  $\tilde{\mathbf{E}}_{11} > \mathbf{0}$  is also true. We proceed similarly to the proof of Theorem 1. Let a truncation matrix  $\mathbf{Z}$  be defined as

$$\mathbf{Z} = \begin{bmatrix} \mathbf{I}_{n-r} \\ \mathbf{0}_{r \times n} \end{bmatrix} \in \mathbb{R}^{n \times (n-r)},$$

such that  $\hat{\mathbf{M}} = \mathbf{Z} \hat{\mathbf{M}}_{11} \mathbf{Z}^T$  and  $\hat{\mathbf{M}}_{11} = \mathbf{Z}^T \hat{\mathbf{M}} \mathbf{Z}$  holds. Similarly to above, define

$$\mathbf{W} := \begin{bmatrix} \frac{1}{\sqrt{\alpha}} \mathbf{I} & \mathbf{0} \\ 2\sqrt{\alpha} \mathbf{Z}^T \hat{\mathbf{M}} \hat{\mathbf{K}}^{-1} & -2\sqrt{\alpha} \mathbf{Z}^T \hat{\mathbf{M}} \hat{\mathbf{D}}^{-T} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{\alpha}} \mathbf{I} & \mathbf{0} \\ 2\sqrt{\alpha} \hat{\mathbf{M}}_{11} \mathbf{Z} \hat{\mathbf{K}}^{-1} & -2\sqrt{\alpha} \hat{\mathbf{M}}_{11} \mathbf{Z} \hat{\mathbf{D}}^{-T} \end{bmatrix} \quad (45)$$

and

$$\mathbf{Y} := \mathbf{W} \left( -\text{sym } \tilde{\mathbf{A}} \right) \mathbf{W}^T = \begin{bmatrix} \mathbf{K} & \alpha \mathbf{Z} \hat{\mathbf{M}}_{11} \\ \alpha \hat{\mathbf{M}}_{11} \mathbf{Z}^T & 4\alpha \hat{\mathbf{M}}_{11} \mathbf{Z}^T \hat{\mathbf{D}}^{-1} (\hat{\mathbf{D}} - \alpha \hat{\mathbf{M}}) \hat{\mathbf{D}}^{-1} \mathbf{Z} \hat{\mathbf{M}}_{11} \end{bmatrix} > \mathbf{0}. \quad (46)$$

Again,  $\mathbf{Y}$  equals  $\tilde{\mathbf{E}}_{11}$  except for the lower right entry, and we need to show

$$\begin{aligned} & \hat{\mathbf{M}}_{11} \geq 4\alpha \hat{\mathbf{M}}_{11} \mathbf{Z}^T \hat{\mathbf{D}}^{-1} (\hat{\mathbf{D}} - \alpha \hat{\mathbf{M}}) \hat{\mathbf{D}}^{-1} \mathbf{Z} \hat{\mathbf{M}}_{11} \\ \iff & \hat{\mathbf{M}}_{11}^{-1} \geq 4\alpha \mathbf{Z}^T \hat{\mathbf{D}}^{-1} \mathbf{Z} - 4\alpha^2 \mathbf{Z}^T \hat{\mathbf{D}}^{-1} \hat{\mathbf{M}} \hat{\mathbf{D}}^{-1} \mathbf{Z} \\ \iff & \hat{\mathbf{M}}_{11}^{-1} - 4\alpha \mathbf{Z}^T \hat{\mathbf{D}}^{-1} \mathbf{Z} + 4\alpha^2 \mathbf{Z}^T \hat{\mathbf{D}}^{-1} \mathbf{Z} \hat{\mathbf{M}}_{11} \mathbf{Z}^T \hat{\mathbf{D}}^{-1} \mathbf{Z} \geq \mathbf{0} \\ \iff & (\mathbf{I} - 2\alpha \mathbf{Z}^T \hat{\mathbf{D}}^{-1} \mathbf{Z} \hat{\mathbf{M}}_{11}) \hat{\mathbf{M}}_{11}^{-1} (\mathbf{I} - 2\alpha \mathbf{Z}^T \hat{\mathbf{D}}^{-1} \mathbf{Z} \hat{\mathbf{M}}_{11})^T \geq \mathbf{0} \end{aligned}$$

which is true. So,  $\tilde{\mathbf{E}}_{11}$  is positive definite for all  $\alpha$  in the valid range.  $\square$

## 6 Conclusions and Outlook

We have shown how a second order system can be formulated in a strictly dissipative state space representation. Although its state vector is identical to the one of classical state space formulations, a modified norm is introduced in which the state norm decays steadily when no input is applied. This formulation enables stability preservation in model order reduction as well as error bounds when KRYLOV subspace methods are employed. The original requirements in [1] have also been weakened to admit systems with gyroscopic forces and algebraic constraints.

The case of semidefinite damping, however, which often occurs in the modeling of electrical circuits, remains unsolved. The presented approach can in fact be shown to be generally unsuited for solving this problem; and to date, no alternative approach has yet been found. Another open question is the effect of the choice of  $\alpha$  on the aforementioned error bounds in KRYLOV-based MOR.

## Acknowledgments

The first author of this article was partially supported by the Cusanuswerk Scholarship Award Programme, which is gratefully acknowledged. We also thank DR. JENS SAAK from the MPI Magdeburg for his inspiring comments on inner products in non-standard state space models and PROF. TIMO REIS for his advice on DAE systems.

## References

- [1] H. Panzer, T. Wolf, and B. Lohmann. “A Strictly Dissipative State Space Representation of Second Order Systems”. In: *at–Automatisierungstechnik* 60 (2012), pp. 392–396.
- [2] Y. Chahlaoui, K. A. Gallivan, A. Vandendorpe, and P. V. Dooren. “Model reduction of second order systems”. In: *Lecture Notes in Computational Science and Engineering* 45 (2005), pp. 149–172.
- [3] B. Salimbahrami. “Structure Preserving Order Reduction of Large Scale Second Order Models”. PhD thesis. Technische Universität München, 2005.
- [4] B. Salimbahrami and B. Lohmann. “Order Reduction of Large Scale Second Order Systems Using Krylov Subspace Methods”. In: *Linear Algebra and its Applications* 415.23 (2006), pp. 385–405.
- [5] M. Lehner. “Modellreduktion in elastischen Mehrkörpersystemen”. PhD thesis. Universität Stuttgart, 2007.
- [6] T. Reis and T. Stykel. “Balanced Truncation Model Reduction of Second-Order Systems”. In: *Mathematical and Computer Modelling of Dynamical Systems* 14.5 (2008), pp. 391–406.
- [7] C. Nowakowski, P. Kürschner, P. Eberhard, and P. Benner. “Model reduction of an elastic crankshaft for elastic multibody simulations”. In: *ZAMM – Zeitschrift für Angewandte Mathematik und Mechanik* 93.4 (2013), pp. 198–216.
- [8] H. Panzer, T. Wolf, and B. Lohmann. “ $\mathcal{H}_2$  and  $\mathcal{H}_\infty$  Error Bounds for Model Order Reduction of Second Order Systems by Krylov Subspace Methods”. In: *Proceedings of the European Control Conference*. 2013.



- [9] F. R. Gantmacher. *The Theory of Matrices Vol. 1*. Chelsea Publishing Company, 1959.
- [10] I. Higuera and B. García-Celayeta. “Logarithmic Norms for Matrix Pencils”. In: *SIAM J. Matrix Anal. Appl.* 20 (3 1999), pp. 646–666.
- [11] G. Dahlquist. *Stability and error bounds in the numerical integration of ordinary differential equations*. Almqvist & Wiksells, Uppsala, 1959.
- [12] S. M. Lozinskii. “Error estimate for numerical integration of ordinary differential equations”. In: *Izvestiya Vysshikh Uchebnykh Zavedenii Matematika* 5 (1958), pp. 52–90.
- [13] L. Trefethen and M. Embree. *Spectra and pseudospectra: the behavior of nonnormal matrices and operators*. Princeton Univ Pr, 2005.
- [14] J. C. Willems. “Dissipative dynamical systems Part I: General theory”. In: *Archive for Rational Mechanics and Analysis* 45.5 (1972), pp. 321–351.
- [15] J. C. Willems. “Dissipative dynamical systems Part II: Linear systems with quadratic supply rates”. In: *Archive for Rational Mechanics and Analysis* 45.5 (1972), pp. 352–393.
- [16] A. C. Antoulas. *Approximation of Large-Scale Dynamical Systems*. SIAM, 2005.
- [17] L. M. Silveira, M. Kamon, I. Elfadel, and J. White. “A coordinate-transformed Arnoldi algorithm for generating guaranteed stable reduced-order models of RLC circuits”. In: *Proceedings of the 1996 IEEE/ACM International Conference on Computer-Aided Design*. 1997, pp. 288–294.
- [18] A. Odabasioglu, M. Celik, and L. T. Pileggi. “PRIMA: Passive Reduced-order Interconnect Macromodeling Algorithm”. In: *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems* 17.8 (1998), pp. 645–654.
- [19] A. Yousefi. “Preserving Stability in Model and Controller Reduction with Application to Embedded Systems”. PhD thesis. Technische Universität München, 2006.
- [20] R. Castañé Selga, B. Lohmann, and R. Eid. “Stability Preservation in Projection-based Model Order Reduction of Large Scale Systems”. In: *European Journal of Control* 18.2 (2012), pp. 122–132.
- [21] P. Benner and E. S. Quintana-Ortí. “Solving stable generalized Lyapunov equations with the matrix sign function”. In: *Numerical Algorithms* 20.1 (1999), pp. 75–100.
- [22] D. Billger. *The Butterfly Gyro*. [http://portal.uni-freiburg.de/imteksimulation/downloads/benchmark/ The Butterfly Gyro \(35889\)](http://portal.uni-freiburg.de/imteksimulation/downloads/benchmark/The%20Butterfly%20Gyro%20(35889).pdf).
- [23] Y. Saad. *Numerical methods for large eigenvalue problems*. Vol. 158. SIAM, 1992.
- [24] H. Panzer, J. Hubele, R. Eid, and B. Lohmann. *Generating a Parametric Finite Element Model of a 3D Cantilever Timoshenko Beam Using MATLAB*. Technical Reports on Automatic Control. Lehrstuhl für Regelungstechnik, Technische Universität München, 2009.
- [25] F. Zhang. *The Schur complement and its applications*. Springer, 2005.
- [26] K. B. Petersen and M. S. Pedersen. *The matrix cookbook*. Tech. rep. Technical University of Denmark, 2005.