LÖWDIN'S APPROACH FOR ORTHOGONAL PULSES FOR UWB IMPULSE RADIO

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ABSTRACT

In this contribution we present a novel orthogonalization method for ultra–wideband (UWB) impulse radio transmission. Contrary to other work we utilize Löwdin's orthogonalization method which delivers a shift–orthogonal basis optimally close (in energy) to the initial pulse generating the shift–invariant space. We show that the shift–orthogonal basis can be well approximated using the Zak transform whenever the initial pulse fulfils certain conditions. This method can be efficiently implemented with the discrete Fourier transform. Furthermore we discuss the existence of compactly supported shift–orthogonal pulses, which are desirable for pulse position modulation.

Index Terms— Löwdin transformation, UWB impulse radio, PPM, Zak transform, shift–invariant spaces

1. INTRODUCTION

Ultra–wideband (UWB) transmission has been intensively investigated in recent years as a candidate for short–range indoor wireless communication. UWB aims for high data rates at short distances at a very low power density. A promising UWB technology is the impulse radio system that uses nanosecond pulses to transmit information (UWB-IR). Typical pulses in this setting are for example the Gaussian monocycle which can be realized at low cost. But for many applications also orthogonality of pulses is desirable, for example when multiple systems [1] or users will share the same spectrum. Moreover, the performance in bit error rate of M-ary orthogonal signal design increases with M for fixed energy per bit in a memoryless AWGN channel when assuming that all signals possess the same energy and support in time [2, Ch.4]. A shift–invariant structure for orthogonal and spectral efficient signaling is advisable in order to achieve also in this case simple and low–cost filter implementations.

Several methods for the orthogonalization of finite sets of pulse translates have been investigated in the context of UWB. A famous example is the Gram-Schmidt procedure which depends on a particular ordering of the pulses [3]. However, in such a way different pulse translates are distorted differently and the orthogonal pulses usually do not retain a shift–invariant structure, i.e. can not be used directly for pulse position modulation (PPM) in UWB systems. On the other hand there exists order–independent methods, where all given signals are handled simultaneously. The canonical and symmetric orthogonalization of Löwdin [4] are such methods. In the finite–dimensional setting these approaches are optimal in the L^2–sense, i.e. minimizing the overall sum of energy distortions [5]. For a certain class these ideas can be extended also to the infinite–dimensional case, i.e. orthogonal bases which are L^2–close to Riesz bases (known as Bari–bases). But for infinite–dimensional shift–invariant systems the objective diverges as for example already stated in [6].

However, we will show in this paper how Löwdin's construction can be used for approximating a shift–orthogonal sequence to implement an orthogonal overlapped PPM system for UWB-IR.

The paper is organized as follows: After introducing the system model we present in Section 2 the orthogonalization method, for which we show a simplified approximation. Afterwards we relate our construction to the frame theory and discuss the support properties. In Section 3 we apply our method to Gaussian monocycles, which is an UWB relevant setting.

Signal Model

PPM and pulse amplitude modulation are well–known techniques in UWB–IR transmission [7, 8]. The power spectrum density (PSD) of the radiated UWB signal u for such modulation strategies is governed by the power spectrum of a single pulse p, called the basic pulse. In this work, we will only consider the M-ary PPM method [9] together with a random polarity flip, to prevent discrete spectral lines in the PSD [10]. If we omit the time hopping code the PPM–transmitted waveform is:

$$u(t) = \sum_n \sqrt{\mathcal{E}} a_n \, p(t - nT_s - d_n T) \tag{1}$$

where the pulse p is normalized in energy such that $\mathcal{E}$ corresponds to the transmitted energy per symbol. To gain maximal SNR the shape of the basic pulse p is designed to exploit optimally the spectral mask required by the Federal Communication Commission (FCC) [1, 11]. The time shift T is the pulse position modulation factor. An i.i.d. zero-mean sequence $\{a_n\}$ models the random polarity flip and $\{d_n\}$ the M-ary i.i.d. information sequence where each random variable d_n corresponds to the nth data symbol. We assume d_n to be uniformly distributed taking integer values between 0 and $M-1$. T_s is the symbol duration, in our case the pulse repetition time, which is much greater then the length T_p of the support of p. To avoid intersymbol interference at the correlation receiver we would have to impose $T_s \geq (M-1)T + T_p$ with $T \geq T_p$. But then the data rate is limited to $(\log M)/T_s$.

To further increase the data rate we need to allow $T < T_p$, resulting in overlapping pulses. But with orthogonality, further performance gains could be achieved then by operating at a higher data rate (see for example [2, (4.95)]) or more users could be supported in a multi–user setting [8]. Thus, in the view application for PPM, we are essentially confronted with the problem of adequate orthogonalization of a set of pulse translates.

2. ORTHOGONALIZATION OF PULSE-TRANSLATES

Let be $p \in L^2(\mathbb{R}) = L^2$ a finite–energy pulse ($\|\cdot\|_2$ will denote the usual L^2–norm). For simplicity we rescale the time–axis in (1) such that $T = 1$ and consider the first frame ($n = 0$). The set of all semi–discrete convolutions of p with sequences $\mathbf{c} = \{c_m\} \in \ell^2(\mathbb{Z}) = \ell^2$ (the Hilbert space of square–summable sequences)

$$V(p) := \left\{ \sum_{m\in\mathbb{Z}} c_m p(\cdot - m) \;\middle|\; \mathbf{c} \in \ell^2 \right\} \tag{2}$$

defines a *shift-invariant space*, i.e. for each $f \in V(p)$ we have also $\{f(\cdot - m)\}_{m\in\mathbb{Z}} \subset V(p)$. We call then p a *generator* for $V(p)$ [12]. For any $K \in \mathbb{N}$ we denote finite sets of translates with $\{p(\cdot - m)\}_{m=-K}^{K}$ and with $\{p(\cdot - m)\}$ the infinite set of all translates $m \in \mathbb{Z}$.

Löwdin Orthogonalization

Our goal is to find for the space $V(p)$ a new pulse $p^\circ \in V(p)$ such that its integer translates $\{p^\circ(\cdot - n)\}$ constitute an orthonormal basis (ONB) for $V(p)$. For the existence of such a countable ONB it is necessary and sufficient that $V(p)$ is a separable Hilbert space, in particular a closed subspace of L^2. This can be ensured for a $p \in L^2$ if there exists two constants $0 < A \le B < \infty$ such that

$$A \le \Phi_p(\nu) := \sum_{k\in\mathbb{Z}} |\hat{p}(\nu + k)|^2 \le B \tag{3}$$

holds for almost every frequency $\nu \in \mathbb{R}$ ($\hat{p}$ denotes here the Fourier transform of p). In this case $\{p(\cdot - m)\}$ is a Riesz basis for $V(p)$, which is then a closed subspace of L^2 [13, 14]. If $A = B = 1$ the pulse p generates an ONB [15, Th. 7.2.3] and (3) is also known as the Nyquist condition. Since any Riesz basis can be transformed by a bounded bijective operator $L : V(p) \to V(p)$ to an ONB, the main contribution of the present work is to derive the expansion coefficients of the ONB elements in the Riesz basis together with a stable approximation method. For this we use the Gram matrix of $\{p(\cdot - m)\}$, which is given element wise by the L^2–scalar product :

$$[\mathbf{G}]_{nm} := \int_{-\infty}^{\infty} p(t - m)\overline{p(t - n)}dt, \tag{4}$$

where the bar denotes complex conjugation. This defines by (3) a positive, bounded and invertible operator from ℓ^2 into itself having a bounded inverse. For any $K \in \mathbb{N}$ the first K translates in both directions $\{p(\cdot - m)\}_{m=-K}^{K}$ define the $(2K + 1) \times (2K + 1)$ dimensional Gram matrix $\mathbf{G}_K$, which is a submatrix of $\mathbf{G}$. An orthonormal pulse set $\{p_n^{\circ,K}\}_{n=-K}^{K}$ is then obtained by the Löwdin transformation [16, 17, 18]:

$$p_n^{\circ,K} := \sum_{m=-K}^{K} [\mathbf{G}_K^{-\frac{1}{2}}]_{nm} p(\cdot - m). \tag{5}$$

Here we denote the inverse square root of $\mathbf{G}_K$ by $\mathbf{G}_K^{-\frac{1}{2}}$. Note that this is in general not a submatrix of $\mathbf{G}^{-\frac{1}{2}}$.

Since the sum in (5) is finite, this is also pointwise well-defined for each $t \in \mathbb{R}$. But to construct an orthogonal generator for the shift-invariant space $V(p)$ we have to take the limit in (5). To guarantee hence a pointwise convergence we need at least continuity and a smooth decay at infinity. If we further assume that p and $\hat{p}$ are in the Wiener space $W(\mathbb{R})$, i.e. essentially bounded functions with:

$$\|p\|_W := \sum_{m\in\mathbb{Z}} \operatorname{ess\,sup}_{t\in[0,1]} |p(t - m)| < \infty \tag{6}$$

then the condition (3) holds pointwise (see for example [19, p.105] and [12]). Let us denote by $W_0(\mathbb{R})$ the space of all continuous Wiener functions. Note that by the Riemann–Lebesgue Lemma $p, \hat{p} \in W(\mathbb{R})$ already implies continuity. The spectral function Φ_p in (3) can also be expressed in terms of the Zak transform [20], defined as:

$$(\mathbf{Z}f)(t,\nu) := \sum_{k\in\mathbb{Z}} f(t - k)e^{2\pi i k t \nu} \quad \text{for} \quad \nu, t \in \mathbb{R} \tag{7}$$

for any continuous function f. Let $r_p := p * \overline{p}$ be the auto–correlation of p. The spectral function is then:

$$\Phi_p(\nu) = \mathbf{Z}(p * \overline{p}_-)(0,\nu), \tag{8}$$

where the function $p_-(t) := p(-t)$ denotes here the time reversal of p. The integer samples of the auto–correlation r_p define also the Gram matrix elements:

$$\begin{aligned} r_p(n - m) = (p * \overline{p}_-)(n - m) &= \int_{-\infty}^{\infty} p(n - m - t)\overline{p(-t)}dt \\ &= \int_{-\infty}^{\infty} p(t - m)\overline{p(t - n)}dt = [\mathbf{G}]_{nm}. \end{aligned} \tag{9}$$

Note that if the pulse fulfills the symmetry $\overline{p}(-t) = p(t)$, the auto-correlation is the auto–convolution.

Stability and Approximation

In the limit $K \to \infty$ the initial pulse sequence $\{p(\cdot - n)\}_{n=-K}^{K}$ will be shift–invariant, hence the Riesz and Nyquist conditions are expressed by (3) in terms of $\hat{p}$. This suggests together with (8) a straightforward construction of an orthogonalization procedure with the discrete Fourier transform (DFT). However, for any finite K, shift–invariance can only be achieved by cyclic extension and then it is not clear, whether such an approach is also stable.

Theorem (Stability of Löwdin Orthogonalization). *Let $M \in \mathbb{N}$ and $p, \hat{p} \in W_0(\mathbb{R})$ such that it holds:*

(i) $\operatorname{supp}(p) \subset [-\frac{M}{2}, \frac{M}{2}]$ *and*

(ii) $\{p(\cdot - k)\}$ *is a Riesz basis for $V(p)$, i.e. satisfy* (3).

Then the limit $\{p_k^\circ\}$ of the Löwdin orthonormalization in (5) *can be approximated by the sequence $\{\tilde{p}_k^{\circ,K}\}_{k=-K}^{K}$, which is represented pointwise for $K \ge M$ and each $k \in [-K, K]$ by:*

$$\tilde{p}_k^{\circ,K}(t) := \begin{cases} \sum_{l=0}^{2K} \dfrac{e^{-\frac{2\pi i\cdot kl}{2K+1}} (\mathbf{Z}p)(t,\frac{l}{2K+1})}{\sqrt{(\mathbf{Z}r_p)(0,\frac{l}{2K+1})}} & ,|t| \le K + \frac{M}{2} \\ 0 & ,\textit{else} \end{cases}, \tag{10}$$

such that for each $k \in \mathbb{Z}$:

$$p_k^\circ(t) = p^\circ(t - k) = \lim_{K\to\infty} \tilde{p}_k^{\circ,K}(t) \tag{11}$$

converges for each $t \in \mathbb{R}$ (i.e. pointwise) and defines a shift-orthonormal basis for $V(p)$.

Sketch of proof. Before we can start with an approximation of the limit in (5) we have to establish the existence of the limit in a pointwise sense. In general, this follows by requiring continuity and a certain polynomial decay for the generator p [11, Lemma 1]. However, for compact and bounded pulses (i.e. with polynomial decay) (5) is well-defined in the limit (it is a finite sum). But if one relaxes the condition from compactly supported to a certain decay condition, one needs the argumentation by the so called Schur class. From Jaffards Theorem on matrix operators with polynomial diagonal-off decay [21] it follows that the Löwdin pulses p_k° are continuous.

The main part for the approximation uses the *finite section method* for the Gram matrix. In [11] we apply an approximation result by Christensen and Strohmer in [22] (given for frame operators) to the Gram matrix $\mathbf{G}$. Let us denote by $\mathbf{P}_K$ the matrix of the orthogonal projection of $\mathbf{c} \in \ell^2$ to $\mathbf{c}^K = \{c_k\}_{k=-K}^K$. Then we can use a sequence of the inverse-square roots of the "Strang circulant preconditioner" $\tilde{\mathbf{G}}_K$ of the matrix $\mathbf{G}_K = \mathbf{P}_K\mathbf{G}\mathbf{P}_K$, which are cyclic matrices defined by the first row:

$$c_m^K = \begin{cases} r_p(m) & m \in \{0,\dots,M\} \\ 0 & \text{else} \\ r_p(2K+1-m) & m \in \{2K+1-M,\dots,2K\} \end{cases}, \quad (12)$$

as an applicable approximation method for $\mathbf{G}^{-\frac{1}{2}}$. This is possible, since $\left\|\tilde{\mathbf{G}}_K - \mathbf{G}_K\right\|_{\ell^2\to\ell^2} \to 0$ as K goes to infinity, which follows from a result of Gray in [23]. For sufficiently large values of K and for all sequences $\mathbf{c} \in \ell^2$ we have then $\tilde{\mathbf{G}}_K^{-1}\mathbf{P}_K\mathbf{c} \to \mathbf{G}^{-1}\mathbf{c}$ as $K \to \infty$ (strong convergence as operators $\ell^2 \to \ell^2$). If $\tilde{\mathbf{G}}_K$ is also positive, than we can deduce the strong convergence for its inverse square root. Since $\mathbf{c}_t = \{p(t-l)\} \in \ell^2$ for all $t \in \mathbb{R}$ we have:

$$\tilde{p}_k^{\circ,K}(t) = \sum_{l=-K}^{K} [\tilde{\mathbf{G}}_K^{-\frac{1}{2}}]_{kl}\, p(t-l) \quad (13)$$

for all $K \in \mathbb{N}$ such that $p_k^\circ(t) = \lim_{K\to\infty} \tilde{p}_k^{\circ,K}(t)$ for all $k \in \mathbb{Z}$.

The circulant matrix $\tilde{\mathbf{G}}_K = \mathbf{F}_K\tilde{\mathbf{D}}_K\mathbf{F}_K^*$ can now be diagonalized by the $(2K+1)\times(2K+1)$ DFT matrix $\mathbf{F}_K$ where the eigenvalues of the diagonal matrix $\tilde{\mathbf{D}}_K$ are given by [23, Theorem 7]:

$$\tilde{\lambda}_j^K = \sum_{n=0}^{2K} c_n^K e^{-2\pi i \frac{nj}{2K+1}} \quad (14)$$

for $j \in \{0,\dots,2K\}$. Next we use (9) to rewrite the eigenvalues in terms of r_p by definition (12):

$$\tilde{\lambda}_j^K = \sum_{n=-M}^{M} r_p(n) e^{-2\pi i j \frac{n}{2K+1}} = (\mathbf{Z}r_p)(0, \frac{j}{2K+1}). \quad (15)$$

The second equality follows from the support $[-M, M]$ of r_p, i.e. it is the Zak transform (7) of r_p evaluated at time $t = 0$ and frequency $\nu = \frac{j}{2K+1}$. Note that r_p is independent of K. Thus, all eigenvalues are given as samples of the spectral function $\Phi_p(\nu) := \sum_k |\hat{p}(\nu+k)|^2$ which is bounded by A and B almost everywhere (the Riesz condition in (3)). But, since $p, \hat{p} \in W(\mathbb{R})$ they are continuous and the above is true for all ν and hence for all eigenvalues of any $\tilde{\mathbf{G}}_K$. With this we have for each $t \in \mathbb{R}$:

$$\tilde{p}_k^{\circ,K}(t) = \sum_{l=-K}^{K} [\mathbf{F}_K\tilde{\mathbf{D}}_K^{-\frac{1}{2}}\mathbf{F}_K^*]_{kl}\, p(t-l) \quad (16)$$

$$= \frac{1}{2K+1}\sum_{j=0}^{2K} e^{-\frac{2\pi i\cdot jk}{2K+1}} \frac{\sum_{l=-K}^{K} p(t-l) e^{\frac{2\pi i\cdot lj}{2K+1}}}{\sqrt{\tilde{\lambda}_j^K}} \quad (17)$$

For each t this is nothing else than the DFT of the sample sequence $\mathbf{c}_t^K = \{p(t-l)\}_{l=-K}^K$, where each j'th sample of the DFT is divided by the square root of the j'th eigenvalue. Moreover, if we restrict the numerator to the time-domain $[-K - \frac{M}{2}, K + \frac{M}{2}]$, this agrees with the Zak transform $(\mathbf{Z}p)(t, \frac{j}{2K+1})$, since we can replace the sum by an infinite sum. Together with (15) this yields (10). The Zak expression in (10) is for each t a continuous function in ν, since it consists of finite sums of continuous functions. Hence, in the limit we can express the partial sum as a Riemann integral:

$$p_k^\circ(t) = \int_0^1 \frac{(\mathbf{Z}p)(t-k,\nu)}{\sqrt{(\mathbf{Z}r_p)(0,\nu)}} d\nu. \quad (18)$$

The quotient of two continuous functions is continuous if the nominator is not zero at some point, which is granted by the positivity of Φ_p due to the Riesz condition. The last step in the numerator comes from the periodic property in time of the Zak transform (see [20, eq. (2.18) + (2.19)]). This shows immediately that $\{p_k^\circ\}$ is a shift–sequence, i.e. $p_k^\circ = p_0^\circ(\cdot - k) =: p^\circ(\cdot - k)$ for all $k \in \mathbb{Z}$. So it remains to show the orthonormality. If we write (18) in the Zak domain, multiplying both sides by $e^{-2\pi i\nu t}$ and integrating over the time t, we get for every $\nu \in \mathbb{R}$:

$$\int_0^1 e^{-2\pi i\nu t}(\mathbf{Z}p^\circ)(t,\nu)dt = \frac{\int_0^1 (\mathbf{Z}\hat{p})(\nu,-t)dt}{\sqrt{(\mathbf{Z}r_p)(0,\nu)}}$$

by the commutation relation between Zak and Fourier transform and the periodicity in frequency (see for example [20, (2.14)+(2.19)]). Finally, by using the inversion formula and the spectral function Φ_p as representation for the eigenvalues, we get from [20, (2.30)] the orthogonalization in the frequency domain:

$$\hat{p}^\circ(\nu) = \frac{\hat{p}(\nu)}{\sqrt{\sum_m |\hat{p}(\nu+m)|^2}}. \quad (19)$$

With this relation we can easily show that p° fulfils the Nyquist condition with $A = B = 1$ in (3). In fact the orthogonalization (19) in the frequency domain is a well know result (see for example [15, Prop. 7.3.9]).

However, our approach provides now a stable approximation of this limit by finite constructions. This is achieved by demanding fast and smooth decay in time and frequency (e.g. in using a continuous compact supported pulse) which implies continuity. The results has therefore the important advantage of having a *pointwise* meaning. □

2.1. Discussion

In this section we state some properties of our presented orthogonalization construction.

Canonical Tight Frame

For (regular) shift-invariant spaces $V(p)$ the Löwdin orthogonalization method corresponds to the canonical tight frame construction. Thus, if $Sf := \sum_l (p(\cdot - l), f)\, p(\cdot - l)$ defines the frame operator S on $V(p)$ for the Riesz–sequence $\{p(\cdot - l)\} \subset V(p)$, then a straightforward calculation gives:

Lemma ([24] eq. (3.3)). *Let the sequence $\{p(\cdot - k)\}$ be a Riesz basis for its closed span, then the Löwdin orthogonalization:*

$$p^\circ(\cdot - k) = S^{-\frac{1}{2}}p(\cdot - k) = \sum_l [\mathbf{G}^{-\frac{1}{2}}]_{kl}\, p(\cdot - l) \quad , \quad k \in \mathbb{Z} \quad (20)$$

yields the canonical tight frame.

This was already stated (without proof) by Meyer in [24, (3.3)]. In the case of Gabor frames for L^2 Janssen and Strohmer [6] have shown by duality that this minimize $\|p - \tilde{p}^\circ\|_2$ over all orthogonal Gabor bases $\tilde{p}^\circ$ for a closed subspace of L^2. For a proof of this Lemma and further details see [11].

Compactly Supported Orthogonal Generators

For PPM transmission the existence of compactly supported orthogonal generators is of central interest, since in this case the transmission and receiving time is finite and hence realizable without further modifications.

Since the setting, i.e. the condition on the generator pulse p in the theorem, is very strict, we yield a small shift–invariant space. In fact we have $V(p) = \mathcal{S}(p)$ where $\mathcal{S}(p)$ is called a principal shift–invariant subspace of L^2 which is defined as the L^2-closure of all finite linear combinations of translates of p. This result was established in general L^q spaces by Jia in [25, Thm. 2]. In case of compactly supported generators the generated principal shift–invariant subspaces were characterized in detail by de Boor et al. in [26]. Since we demand for p the Wiener condition (6), we have pointwise boundedness below and above of Φ_p in (3). De Boor et al. could show in their work that this implies stability for p. If additionally all shifts of p are *linearly independent* in the sense of [26], then this generator is unique up to shifts and scalar multiples. Finally they show in this case a negative result, which excludes the existence of a compactly supported orthogonal generator if p itself is not already orthogonal. But if p would be already orthogonal, then p is unique up to shifts and scalar multiples and the Löwdin construction becomes a scaled identity (normalizing p). Moreover, the uniqueness then rules out the existence of a filter construction $\mathbf{c} \in \ell^2$ on p such that $\sum_k c_k p(\cdot - k)$ becomes a compactly supported orthogonal generator for $\mathcal{S}(p)$. The linearly independent property of a compactly supported generator p is equivalent to:

$$\{(\mathscr{L}p)(z-n)\}_{n\in\mathbb{Z}} \neq 0 \quad \text{for all} \quad z \in \mathbb{C} \tag{21}$$

where $(\mathscr{L}p)$ denotes the Laplace transform of p [26, p.53]. This means that $(\mathscr{L}p)$ is required to not have periodic zero points. See Cor. 2.17, Cor. 2.27, Prop. 2.25 and Thm. 2.29 in [26] for more details.

From this discussion follows that if the initial pulse p fulfils (21) we can not expect to construct strictly interference-free orthogonal PPM systems by FIR filtering with time shift $T < T_p$. This means for practical implementations, that K has to be chosen such that the matched filter output for wrong decision is reasonably below the noise level. The detailed interference level depending on K has to be further investigated.

3. ALGORITHMIC IMPLEMENTATION

The most common pulse for UWB-IR transmission in (1) is a truncated Gaussian monocycle: $q(t') \simeq t' \cdot \exp(-t'^2/\sigma^2)$ where σ is chosen such that the maximum of $|\hat{q}(f)|^2$ is reached at the center $f_c = 6.85$GHz of the passband [27]. Since we need continuity, we mask q with a unit triangle window Λ of width $T_p = 4.95\sigma = 0.1626$ns such that 99.99% of the energy of q is concentrated in the window $[-T_p/2, T_p/2]$. Let us scale the time $t = t'T_p/M$ such that the support is $[-M/2, M/2]$ and the time shift T is one. If we fix M and p, then $\tilde{p}_0^{\circ,K}$ is the K–approximation to the limit pulse $p^\circ := p_0^\circ$ which is an orthogonal generator for $V(p)$, hence we call them approximated Löwdin orthogonal (ALO). The question is, how well $\{\tilde{p}_k^{\circ,K}\}$ approximates a shift–orthogonal set for finite K?

For this we compute with MATLAB the ALO pulses $\tilde{p}_0^{\circ,K}$ in (17) by using the DFT and plot the result in Fig. 1 for the parameters $M = 4$ and $K = 12$ together with the initial pulse p and the Löwdin orthogonal (LO) pulse $p_0^{\circ,K}$. It can be observed that $\tilde{p}_0^{\circ,K}$ and $p_0^{\circ,K}$ are quite similar but have increased support as compared to p. Due to the circulant construction $p_0^{\circ,K}$ has slight more concentration at the boundaries which vanishes with increasing K.

In Fig. 2 we plot the correlation functions of $\tilde{p}_k^{\circ,K}$, given as:

$$\tilde{\xi}_{kl}^K(\tau) = \int \tilde{p}_k^{\circ,K}(t)\overline{\tilde{p}_l^{\circ,K}(t-\tau)}dt \quad , \quad \tau \in \mathbb{R},$$

and respectively for $p_k^{\circ,K}$ as ξ_{kl}^K, for $k = l = 0$ (auto–correlation). Furthermore, Fig. 2 contains the auto-correlation r_p of p. As one has to expect, the orthogonalized pulses have zeros at almost each shift–instant T reinforcing its shift–orthogonal character. Again – the cyclic extension in the ALO pulses results in slightly increased correlation at the boundaries.

In Fig. 3 we plot the condition numbers of the Strang matrix $\tilde{\mathbf{G}}_K$ for $K = 4$ to $K = 100$. The minimum eigenvalue is for each K almost constant, whereby variation in the condition number is caused only by the variation of the maximum eigenvalue.

In Fig. 4 we derive numerically the Riesz bounds in (3) for various M. We computed the minimal and maximal eigenvalue λ_{min} resp. λ_{max} of $\mathbf{G}$ ($\|p\| = 1$) with a semi-definite program (SDP) from [28, p.59], as well as the minimal and maximal eigenvalue $\tilde{\lambda}_{min}^K$ resp. $\tilde{\lambda}_{max}^K$ of $\tilde{\mathbf{G}}_K$. One can see that for finite K the minimal and maximal eigenvalue of $\mathbf{G}$ are almost attained, see also Fig. 3. For comparison we plot the eigenvalues λ_Π of a rectangle masked Gaussian monocycle. Increasing M results in more overlap and hence in a degradation of the Riesz bounds.

4. CONCLUSIONS

We have presented a novel orthogonalization method for UWB impulse radio transmission. Contrary to prior work our method has the advantage of being order–independent and optimal in the L^2–sense. We have shown that the shift–orthogonal basis can be well and stable approximated. Thus, our method provide a simple and realizable procedure of orthogonalizing UWB pulses. However, it seems not to be clear if there exist a stable construction method out of p for a compactly supported orthogonal generator. This question will be further studied in [11]. Since our construction is applicable to a wide class of compactly supported pulses, one can also start with an initial pulse which is optimized to the FCC mask constraints, [27, 11].

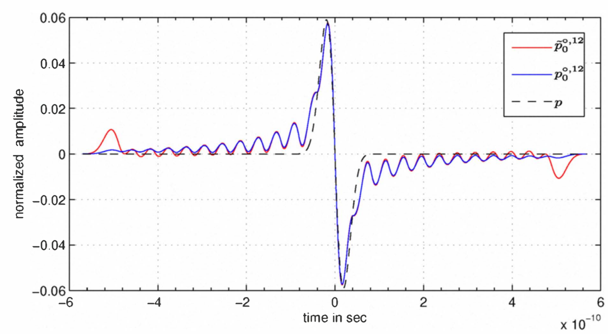

Fig. 1. Pulse shapes in time for $K = 12$ and $M = 4$

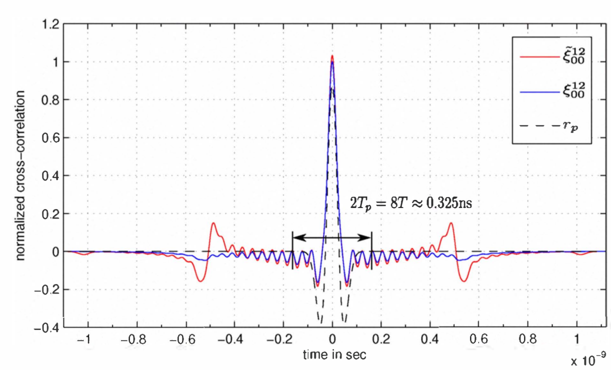

Fig. 2. Auto-correlation of pulses $K = 12$ and $M = 4$.

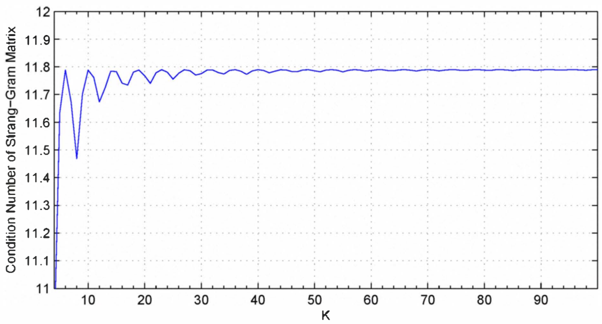

Fig. 3. Convergence of the condition–number of $\tilde{\mathbf{G}}_K$ for $M = 4$.

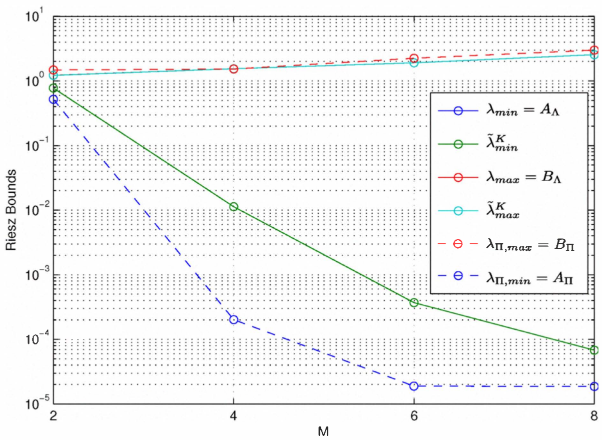

Fig. 4. Calculated Riesz Bounds A, B for various M and $K = 3M$.

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