Forward-backward systems for expected utility maximization

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December 28, 2013

Abstract

In this paper we deal with the utility maximization problem with general utility functions including power utility with liability. We derive a new approach in which we reduce the resulting control problem to the study of a system of fully-coupled Forward-Backward Stochastic Differential Equation (FBSDE) that promise to be accessible to numerical treatment.

AMS Subject Classification: Primary 60H10, 93E20
JEL Classification: C61, D52, D53

1 Introduction

One of the most commonly studied problems in mathematical finance and applied probability is related to maximizing expected terminal utility from trading in a financial market. In its version we consider here, the focus is more on an insurance issue: a small agent is interested in securitizing a random liability arising in his usual business by investing on a capital market. He therefore has two sources of income: his random liability, and the wealth obtained from trading on the capital market up to a terminal time with appropriate investment strategies. The agent's preferences are described by a utility function. So the *stochastic control problem* he faces results in the maximization of his terminal utility obtained from both sources of income with respect to all admissible strategies available to him. More formally, given his initial wealth $x > 0$, he aims at attaining the value function

$$V(0,x) := \sup_{\pi \in \mathcal{A}} \mathbb{E}[U(X_T^\pi + H)]. \tag{1.1}$$

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Here U is a general real-valued utility function with properties to be specified, $\mathcal{A}$ denotes the set of *admissible trading strategies*, $T < \infty$ the trading horizon, X_T^π is the agent's wealth obtained upon following a strategy $\pi \in \mathcal{A}$, and the random variable H describes a liability he must deliver at terminal time T. In the typical focus of mathematical interest of this problem we find questions related to existence and uniqueness of optimal solutions, as well as the characterization of optimal strategies and the *value function* V which is defined for times $0 \leq t \leq T$ and initial wealth $x > 0$ as

$$V(t,x) := \sup_{\pi \in \mathcal{A}} \mathbb{E}[U(X_{t,T}^\pi + H)|\mathcal{F}_t].$$

Here $X_{t,T}$ denotes the wealth the agent is able to obtain from trading in the capital market in the investment period $[t,T]$, and the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ describes the evolution of information.

The most important techniques to tackle the existence of optimal strategies π^* are based on concepts of *convex duality*. This tool first appears in Bismut [4], was further developed on Brownian bases in the works by Pliska [35], Karatzas and co-workers (see for instance Karatzas et al. [18], [19], [7]), with its modern and abstract form formulated in the setting of general semimartingales due to Kramkov and Schachermayer [21]. In this setting, growth conditions on U or related quantities such as the asymptotic elasticity, if U is defined on the half line, can be formulated. Together with mild regularity conditions on the liability and convexity assumptions on the set of admissible trading strategies (see e.g. [2] for details) they guarantee the existence of optimal investment strategies. Duality techniques are general and far-reaching, yet not constructive, especially from the perspective of numerical approximation: they are so far not amenable to computation or simulation of optimal strategies and value functions. They reveal yet another shortcoming arising if one is interested in working with non-convex constraints: convexity breaks down for instance if the strategies are supposed to take integer values.

A direct stochastic approach to simultaneously characterize optimal trading strategies and utilities is provided by an interpretation of the martingale optimality principle by the tools of (forward) backward stochastic differential equations (FBSDE). In the case of exponential utility it was discussed in El Karoui et al. [12], and Sekine [37], still with elements of convex analysis. Conceptually not linked to convex duality methods, in [14] it was seen to work in the setting of constraints that are just closed, not necessarily convex. If the filtration is generated by a standard Wiener process W, if either $U(x) := -\exp(-\alpha x)$ for some $\alpha > 0$ and $H \in L^2$, or $U(x) := \frac{x^\gamma}{\gamma}$ for $\gamma \in (0,1)$ or $U(x) = \ln x$ and $H = 0$, and if the selection of admissible strategies is restricted to a closed set, it has been shown by Hu et al. [14] that the control problem (1.1) can essentially be reduced to solving a BSDE of the form

$$Y_t = H - \int_t^T Z_s dW_s - \int_t^T f(s, Z_s) ds, \quad t \in [0,T], \tag{1.2}$$

where the *driver* $f(t,z)$ is a predictable process of quadratic growth in the z-variable. The method only works well in the cases of classical utility functions, i.e. exponential with general endowment, and power or logarithmic with zero endowment. In these cases the forward part, the portfolio process, and the backward part given by (1.2), are decoupled. This is due to a "separation of variables" property shared by the classical utility functions: their value function can be decomposed by $V(t,x) = g(x)V_t$ where g is a deterministic function and V is an adapted process. As a result, optimal future trading strategies are independent of current wealth levels. The stochastic approach has since been extended beyond the Brownian framework and to more general utility optimization problems with complete and incomplete information in numerous papers, of which we only quote [13], [29], [30], [31] and [27].

More generally, there has recently been an increasing interest in dynamic *translation invariant utility functions*. A utility function is called translation invariant if a cash amount added to a financial position increases the utility by that amount, and hence optimal trading strategies are wealth-independent[1]. Although the property of translation invariance renders the utility optimization problem mathematically tractable, independence of the trading strategies on wealth is rather unsatisfactory from an economic point of view. In [28] a verification theorem is derived for optimal trading strategies for more general utility functions in case $H = 0$. More precisely, given a general utility function U and assuming that there exists an optimal strategy regular enough such that the value function possesses some regularity in (t,x), it is shown that there exists a predictable random field $(\varphi(t,x))_{(t,x)\in[0,T]\times(0,\infty)}$ such that the pair (V,φ) solves a backward stochastic partial differential equation (BSPDE) of the form:

$$V(t,x) = U(x) - \int_t^T \varphi(s,x)dW_s - \int_t^T \frac{|\varphi_x(s,x)|^2}{V_{xx}(s,x)}ds, \quad t \in [0,T], \tag{1.3}$$

where φ_x denotes the partial derivative of φ with respect to x and V_{xx} the second partial derivative of V with respect to the same variable. The optimal strategy π^* can then be obtained from (V,φ). Unfortunately, the theory of BSPDE is still not well developed, and to the best of our knowledge the non-linearities arising in (1.3) cannot be handled except in the classical cases mentioned above where once again one benefits of the "separation of variables" (see [17]). Moreover, the utility function U only appears in the terminal condition which is not very handy. This corresponds to a general stochastic version of the Hamilton-Jacobi-Bellman equation in the Markovian setting.

In this paper we propose a new approach to solving the securitization problem with liability (1.1) for a larger class of utility functions and characterize the optimal strategy π^* in terms of a fully-coupled system of FBSDE. In contrast to the classical Markovian framework corresponding to the analytical HJB equations for which they are supposed to be functions of the terminal value of the forward process, the general terminal liabilities we work with usually create a system which is non-Markovian. It is therefore essentially more general than the system of FBSDE outlined in Peng [34] in the context of classical stochastic control problems. Coupled FBSDE have already been extensively studied, but essentially for Lipschitz coefficients. The treatment has focused on mainly three methods: one using contraction mappings ([1], [33]), one based on PDE ([24], [9]), and the method of continuation ([15], [38]). We refer to [25] for the investigation of FBSDE with Lipschitz coeffients.

[1]It has been shown by [10] that essentially all such utility functions can be represented in terms of a BSDE of the form 1.2.

The derivation of the FBSDE system appropriate for our purposes starts with a verification type observation. In case of utility functions defined on $\mathbb{R}$ (if they are defined on $\mathbb{R}_+$, a refinement of the argument will be applicable), given an optimal strategy π^* of the (forward) portfolio process X^{π^*}, to realize martingale optimality we postulate that $U'(X^{\pi^*} + Y)$ be a martingale, where (Y, Z) is the associated backward process. As a consequence, (Y, Z) is given by a certainty equivalent type expression for marginal utility $Y = (U')^{-1}(\mathbb{E}(U'(X_T^{\pi^*} + H)|\mathcal{F}_\cdot) - X^{\pi^*}$. This identification allows us to compute the driver of the BSDE related to (Y, Z). It is given in terms of the derivatives of U, involves the optimal forward process X^{π^*}, and provides the backward part of the FBSDE system. In a second step, we consider possible solution triples (X, Y, Z) of the FBSDE system obtained in the first step, not assuming that X corresponds to an optimal portfolio process. We then use the variational maximum principle in order to verify that under mild conditions on U the triple (X, Y, Z) solves the original optimization problem. This in particular means that X coincides with an optimal forward portfolio process X^{π^*}. In summary, under mild regularity conditions, solutions (X, Y, Z) of the FBSDE system provide solutions of the original securitization problem.

With this we also extend the fully stochastic approach of [14] of the optimization problem with terminal liability by means of a direct translation of martingale optimality into stochastic equations. Since it is not based on convex duality techniques, it should be able to cope with closed, non-convex constraints as well. Although it appears that we can write down systems of FBSDE with closed, non-convex constraints, we were not able to relate them to the corresponding utility optimization problem yet. However, it requires essentially more effort to solve a fully-coupled FBSDE system (which in general will fail to possess solutions) compared to the decoupled one of [14]. In classical cases in which decoupling techniques apply our FBSDE system possesses solutions. In a more general setting, solutions are constructed using a compactness criterium due to Delbaen and Schachermayer [11] for which so far we cannot avoid convexity assumptions.

Our approach provides in particular an FBSDE system for the case of power utility with general non-hedgeable liabilities. To the best of our knowledge ours is the first treatment that allows to characterize and calculate optimal strategies in this case.

The remainder of this paper is organized as follows. In Section 2 we introduce our financial market model. In Section 3 we first derive a verification theorem translated into a system of FBSDE for utilities defined on the real line. Its converse shows that a solution to the FBSDE system obtained allows to construct the optimal strategy. Section 4 is devoted to the discussion of analogous questions for utilities defined on the positive half line. In Section 5 we relate our approach to the stochastic maximum principle obtained by Peng [34] and the standard duality approach. We use the duality-BSDE link to construct a solution of the FBSDE associated with the problem of maximizing power utility with general positive endowment.

2 Preliminaries

We consider a financial market which consists of one bond S^0 with interest rate zero and of $d \geq 1$ stocks given by

$$d\tilde{S}_t^i := \tilde{S}_t^i dW_t^i + \tilde{S}_t^i \theta_t^i dt, \quad i \in \{1, \dots, d\}$$

where W is a standard Brownian motion on $\mathbb{R}^d$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, $(\mathcal{F}_t)_{t\in[0,T]}$ is the filtration generated by W, and $\theta := (\theta^1, \dots, \theta^d)$ is a predictable bounded process with values in $\mathbb{R}^d$. Let us remark at this place that a generalization of this model to one with a volatility matrix σ for which $\sigma\sigma^*$ is uniformly elliptic (as in Hu et al. [14]) is straightforward, and just adds notational complexity to the treatment. Since we assume the process θ to be bounded, Girsanov's theorem implies that the set of equivalent local martingale measures (i.e. probability measures under which $\tilde{S}$ is a local martingale) is not empty, and thus according to the classical literature (see e.g. [11]), arbitrage opportunities are excluded in our model. For simplicity we write throughout

$$dS_t^i := \frac{d\tilde{S}_t^i}{\tilde{S}_t^i}.$$

We denote by $\alpha \cdot \beta$ the inner product in $\mathbb{R}^d$ of vectors α and β and by $|\cdot|$ the usual associated L^2-norm on $\mathbb{R}^d$. In all the paper C will denote a generic constant which can differ from line to line. We also define the following spaces:

$$\mathbb{S}^2(\mathbb{R}^d) := \left\{ \beta : \Omega \times [0,T] \to \mathbb{R}^d, \text{ continuous and adapted }, \mathbb{E}[\sup_{t\in[0,T]} |\beta_t|^2] < \infty \right\},$$

$$\mathbb{H}^2(\mathbb{R}^d) := \left\{ \beta : \Omega \times [0,T] \to \mathbb{R}^d, \text{ predictable, } \mathbb{E}\left[\int_0^T |\beta_t|^2 dt\right] < \infty \right\}.$$

Since the market price of risk θ is assumed to be bounded, the stochastic process

$$\mathcal{E}(-\theta \cdot W)_t := \exp\left(-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t |\theta_s|^2 ds\right)$$

has finite moments of order p for any $p > 0$. We assume $d_1 + d_2 = d$ and that the agent can invest in the assets $\tilde{S}^1, \dots, \tilde{S}^{d_1}$ while the stocks $\tilde{S}^{d_1+1}, \dots, \tilde{S}^{d_2}$ are inaccessible to the agent. Denote $S^{\mathcal{H}} := (S^1, \dots, S^{d_1}, 0 \dots, 0)$, $W^{\mathcal{H}} := (W^1, \dots, W^{d_1}, 0 \dots, 0)$, $W^{\mathcal{O}} := (0, \dots, 0, W^{d_1+1}, \dots, W^{d_2})$, and $\theta^{\mathcal{H}} := (\theta^1, \dots, \theta^{d_1}, 0 \dots, 0)$ (the notation $\mathcal{H}$ refers to "hedgeable" and $\mathcal{O}$ to "orthogonal"). To any investment strategy π in $\mathbb{H}^2(\mathbb{R}^{d_1})$, we define X^π its associated wealth process defined as:

$$X_t^\pi := X_0^\pi + \int_0^t \pi_r dS_r^{\mathcal{H}} = X_0^\pi + \sum_{i=1}^{d_1} \int_0^t \pi_r^i dS_r^i, \quad t \in [0,T],$$

and for every $x > 0$ we denote by Π^x the set of trading strategies with initial capital x, that is:

$$\Pi^x := \left\{\pi \in \mathbb{H}^2(\mathbb{R}^{d_1}),\ X_0^\pi = x\right\}. \tag{2.1}$$

Every π in Π^x is extended to an $\mathbb{R}^d$-valued process by

$$\tilde{\pi} := (\pi^1, \ldots, \pi^{d_1}, 0, \ldots, 0).$$

In the following, we will always write π in place of $\tilde{\pi}$, i.e. π is an $\mathbb{R}^d$-valued process where the last d_2 components are zero. Moreover, we consider a utility function $U : I \to \mathbb{R}$ where I is an interval of $\mathbb{R}$ such that U is strictly increasing and strictly concave. We look for a strategy π^* in Π^x satisfying $\mathbb{E}[U(X_T^{\pi^*} + H)] < \infty$ such that

$$\pi^* = \mathrm{argmax}_{\pi \in \Pi^x,\ \mathbb{E}[|U(X_T^\pi + H)|] < \infty} \left\{\mathbb{E}[U(X_T^\pi + H)]\right\} \tag{2.2}$$

where H is a random variable in $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ such that the expression above makes sense. We concretize on sufficient conditions in the subsequent sections.

3 Utilities defined on the real line

In this section we consider a utility function $U : \mathbb{R} \to \mathbb{R}$ defined on the whole real line. We assume that U is strictly increasing and strictly concave and that the agent is endowed with a claim H. We introduce the following conditions.

(H1) $U : \mathbb{R} \to \mathbb{R}$ is three times differentiable

(H2) H is an element of $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$

(H3) We say that condition (H3) holds for an element π^* in Π^x, if $\mathbb{E}[|U'(X_T^{\pi^*} + H)|^2] < \infty$ and if for every bounded predictable process $h : [0, T] \times \Omega \to \mathbb{R}$, the family of random variables

$$\left(\int_0^T h_r dS_r^{\mathcal{H}} \int_0^1 U'\left(X_T^{\pi^*} + H + \varepsilon r \int_0^T h_u dS_u^{\mathcal{H}}\right) dr\right)_{\varepsilon \in (0,1)}$$

is uniformly integrable.

Before presenting the first main result of this section, we prove that condition $(H3)$ is satisfied for every strategy π^* such that $\mathbb{E}[|U'(X_T^{\pi^*} + H)|] < \infty$ when one has an exponential growth condition on the marginal utility of the form:

$$U'(x + y) \leq C\left(1 + U'(x)\right)(1 + \exp(\alpha y)) \quad \text{for some } \alpha \in \mathbb{R}.$$

Indeed, let $G := \int_0^T h_r dS_r^{\mathcal{H}}$ and $d > 0$. We will show that the quantity

$$q(d) := \sup_{\varepsilon \in (0,1)} \mathbb{E}\left[\left|G \int_0^1 U'(X_T^{\pi^*} + H + \varepsilon r G) dr\right| \mathbf{1}_{\left|G \int_0^1 U'(X_T^{\pi^*} + H + \varepsilon r G) dr\right| > d}\right]$$

vanishes as d tends to infinity. For simplicity we write $\delta_{\varepsilon,d} := \mathbf{1}_{\left|G\int_0^1 U'(X_T^{\pi^*}+H+\varepsilon rG)dr\right|>d}$. By the Cauchy-Schwarz inequality

$$
\begin{aligned}
q(d) &\leq \sup_{\varepsilon\in(0,1)} \mathbb{E}\left[(1+U'(X_T^{\pi^*}+H))\left|G(1+\int_0^1 \exp(\alpha\varepsilon rG))dr\right|\delta_{\varepsilon,d}\right] \\
&\leq C\mathbb{E}\left[|U'(X_T^{\pi^*}+H)|^2\right]^{1/2} \sup_{\varepsilon\in(0,1)} \mathbb{E}\left[\left|G\int_0^1 \exp(\alpha\varepsilon rG)dr\right|^2 \delta_{\varepsilon,d}\right]^{1/2}.
\end{aligned}
$$

Since $\mathbb{E}\left[|U'(X_T^{\pi^*}+H)|^2\right]$ is assumed to be finite we deduce from the inequality

$$
\exp(\alpha\zeta x) \leq 1+\exp(\alpha x) \quad \text{for all } x\in\mathbb{R},\ 0<\zeta<1
$$

that

$$
q(d) \leq C \sup_{\varepsilon\in(0,1)} \mathbb{E}\left[|G(2+\exp(\alpha G))|^2\,\delta_{\varepsilon,d}\right]^{1/2}.
$$

Applying successively the Cauchy-Schwarz inequality and the Markov inequality, we deduce

$$
\begin{aligned}
q(d) &\leq C\mathbb{E}\left[|G(2+\exp(\alpha G))|^4\right]^{1/4} \sup_{\varepsilon\in(0,1)} \mathbb{E}[\delta_{\varepsilon,d}]^{1/4} \\
&\leq C\mathbb{E}\left[|G(2+\exp(\alpha G))|^4\right]^{1/4} d^{-1/4} \sup_{\varepsilon\in(0,1)} \mathbb{E}\left[|G|\int_0^1 U'(X_T^{\pi^*}+H+\varepsilon rG)dr\right]^{1/4} \\
&\leq C\mathbb{E}\left[|G(2+\exp(\alpha G))|^4\right]^{1/4} d^{-1/4}\, \mathbb{E}\left[|G(2+\exp(\alpha G))|^2\right]^{1/8}.
\end{aligned}
$$

Let $p\geq 2$. Since h and θ are bounded it is clear that $\mathbb{E}\left[|G|^{2p}\right]<\infty$ and

$$
\begin{aligned}
&\mathbb{E}\left[|G(2+\exp(\alpha G))|^p\right] \\
&\leq \mathbb{E}\left[|G|^{2p}\right]^{1/2} \mathbb{E}\left[|2+\exp(\alpha G)|^{2p}\right]^{1/2} \\
&\leq C\left(2+\mathbb{E}\left[|\exp(\alpha G)|^{2p}\right]\right)^{1/2} \\
&= C\left(2+\mathbb{E}\left[\exp\left(\int_0^T 2p\alpha h_r dW_r^{\mathcal{H}} - \frac{1}{2}\int_0^T |2p\alpha h_r|^2 dr\right)\right.\right. \\
&\qquad \left.\left.\exp\left(\frac{1}{2}\int_0^T |2p\alpha h_r|^2 + 2p\alpha h_r\cdot\theta_r dr\right)\right]\right)^{1/2} \\
&\leq C.
\end{aligned}
$$

Hence $\lim_{d\to\infty} q(d)=0$ which proves the assertion.

3.1 Characterization and verification: incomplete markets

We are now ready to state and prove the first main results of this paper namely a characterization of the optimal strategy (*cf.* Theorem 3.1) and a verification theorem presented in Theorem 3.5.

Theorem 3.1. *Assume that $(H1)-(H2)$ hold. Let $\pi^* \in \Pi^x$ be an optimal solution to the problem (2.2) such that $(H3)$ is satisfied. Then there exists a continuous adapted process Y with $Y_T = H$ such that $U'(X^{\pi^*}+Y)$ is a square integrable martingale and the optimal strategy allows for the representation*

$$\pi_t^{*^i} = -\theta_t^i \frac{U'(X_t^{\pi^*}+Y_t)}{U''(X_t^{\pi^*}+Y_t)} - Z_t^i, \quad t \in [0,T], \quad i = 1, \dots, d_1$$

where $Z_t := \frac{d\langle Y,W\rangle_t}{dt} := \left(\frac{d\langle Y,W^i\rangle_t}{dt}, \dots, \frac{d\langle Y,W^d\rangle_t}{dt}\right)$.

Proof. We first prove the existence of Y. Since $\mathbb{E}[|U'(X_T^{\pi^*}+H)|^2] < \infty$, the stochastic process α defined as $\alpha_t := \mathbb{E}[U'(X_T^{\pi^*}+H)|\mathcal{F}_t]$, for t in $[0,T]$ is a square integrable martingale. Define $Y_t := (U')^{-1}(\alpha_t) - X_t^{\pi^*}$. Then Y is $(\mathcal{F}_t)_{t\in[0,T]}$-predictable. Now Itô's formula yields

$$Y_t + X_t^{\pi^*} = Y_T + X_T^{\pi^*} - \int_t^T \frac{1}{U''(U'^{-1}(\alpha_s))} d\alpha_s + \frac{1}{2}\int_t^T \frac{U^{(3)}(U'^{-1}(\alpha_s))}{(U''(U'^{-1}(\alpha_s)))^3} d\langle \alpha, \alpha\rangle_s. \quad (3.1)$$

By definition, α is the unique solution of the zero driver BSDE

$$\alpha_t = U'(X_T^{\pi^*}+Y_T) - \int_t^T \beta_s dW_s, \quad t \in [0,T], \quad (3.2)$$

where β is a square integrable predictable process with respect to $dt \otimes d\mathbb{P}$ with values in $\mathbb{R}^d$. Plugging (3.2) into (3.1) yields

$$Y_t + X_t^{\pi^*} = X_T^{\pi^*} + H - \int_t^T \frac{1}{U''(X_s^{\pi^*}+Y_s))}\beta_s dW_s + \frac{1}{2}\int_t^T \frac{U^{(3)}(X_s^{\pi^*}+Y_s)}{(U''(X_s^{\pi^*}+Y_s))^3}|\beta_s|^2 ds.$$

Setting $\tilde{Z} := \frac{1}{U''(X^{\pi^*}+Y))}\beta$, we have

$$Y_t + X_t^{\pi^*} = X_T^{\pi^*} + H - \int_t^T \tilde{Z}_s dW_s + \frac{1}{2}\int_t^T \frac{U^{(3)}}{U''}(X_s^{\pi^*}+Y_s)|\tilde{Z}_s|^2 ds.$$

Now by putting $Z^i := \tilde{Z}^i - \pi^{*i}$, $i = 1, \dots, d$, we have shown that Y is a solution to the BSDE

$$Y_t = H - \int_t^T Z_s dW_s - \int_t^T f(s, X_s^{\pi^*}, Y_s, Z_s) ds, \quad t \in [0,T], \quad (3.3)$$

where f is given by

$$f(s, X_s^{\pi^*}, Y_s, Z_s) = -\frac{1}{2}\frac{U^{(3)}}{U''}(X_s^{\pi^*}+Y_s)|\pi_s^* + Z_s|^2 - \pi_s^* \cdot \theta_s. \quad (3.4)$$

Finally, by construction we have $U'(X_t^{\pi^*}+Y_t) = \alpha_t$, which thus is a martingale.

Now we deal with the characterization of the optimal strategy. To this end, let $h : [0,T] \times \Omega \to \mathbb{R}^{d_1}$ be a bounded predictable process. We extend h into $\mathbb{R}^d$ by concatenating zeros via $\tilde{h} := (h^1, \dots, h^{d_1}, 0, \dots, 0)$ and by abuse of notation denote $\tilde{h}$ again by h. Thus for every

ε in $(0,1)$ the perturbed strategy $\pi^*+\varepsilon h$ belongs to Π^x. Since π^* is optimal it is clear that for every such h we have

$$l(h) := \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}\left[U(x+\int_0^T(\pi_r^*+\varepsilon h_r)dS_r^{\mathcal{H}}+Y_T) - U(x+\int_0^T \pi_r^* dS_r^{\mathcal{H}}+Y_T)\right] \leq 0. \tag{3.5}$$

Moreover

$$\begin{aligned}
&\frac{1}{\varepsilon}\left(U(x+\int_0^T(\pi_r^*+\varepsilon h_r)dS_r^{\mathcal{H}}+Y_T) - U(x+\int_0^T \pi_r^* dS_r^{\mathcal{H}}+Y_T)\right)\\
&= \int_0^T h_r dS_r^{\mathcal{H}} \int_0^1 U'\left(X_T^{\pi^*}+Y_T+\theta\varepsilon\int_0^T h_r dS_r^{\mathcal{H}}\right)d\theta.
\end{aligned}$$

Now by (H3), Lebesgue's dominated convergence theorem implies that (3.5) can be rewritten as

$$\mathbb{E}\left[U'(X_T^{\pi^*}+Y_T)\int_0^T h_r dS_r^{\mathcal{H}}\right] \leq 0 \tag{3.6}$$

for every bounded predictable process h. Applying integration by parts to $U'(X_s^{\pi^*}+Y_s)_{s\in[0,T]}$ and $\left(\int_0^s h_r dS_r^{\mathcal{H}}\right)_{s\in[0,T]}$, we get

$$\begin{aligned}
&U'(X_T^{\pi^*}+Y_T)\int_0^T h_r dS_r^{\mathcal{H}}\\
&= U'(x+Y_0)\times 0 + \int_0^T U'(X_s^{\pi^*}+Y_s)h_s dS_s^{\mathcal{H}}\\
&\quad + \int_0^T\int_0^s h_r dS_r^{\mathcal{H}}\, U''(X_s^{\pi^*}+Y_s)\left[(\pi_s^*+Z_s)dW_s^{\mathcal{H}} + (\pi_s^*\cdot\theta_s + f(s,X_s^{\pi^*},Y_s,Z_s))ds\right]\\
&\quad + \frac{1}{2}\int_0^T\int_0^s h_r dS_r^{\mathcal{H}}\, U^{(3)}(X_s^{\pi^*}+Y_s)|\pi_s^*+Z_s|^2 ds\\
&\quad + \int_0^T U''(X_s^{\pi^*}+Y_s)h_s\cdot(\pi_s^*+Z_s)ds.
\end{aligned}$$

By definition of the driver f, the previous expression reduces to

$$\begin{aligned}
&U'(X_T^{\pi^*}+Y_T)\int_0^T h_r dS_r^{\mathcal{H}}\\
&= \int_0^T\left(U'(X_s^{\pi^*}+Y_s)\theta_s + U''(X_s^{\pi^*}+Y_s)(\pi_s^*+Z_s)\right)\cdot h_s ds\\
&\quad + \int_0^T\int_0^s h_r dS_r^{\mathcal{H}}\, U''(X_s^{\pi^*}+Y_s)(\pi_s^*+Z_s)dW_s^{\mathcal{H}} + \int_0^T U'(X_s^{\pi^*}+Y_s)h_s dW_s^{\mathcal{H}}.
\end{aligned} \tag{3.7}$$

The next step would be to apply the conditional expectations in (3.7). However the two terms on the second line of the right hand side are a priori only local martingales. We start by showing that the first one is a uniformly integrable martingale. Indeed, from the computations which have led to (3.3) we have that

$$U''(X^{\pi^*}+Y)(\pi^*+Z) = \beta,$$

where we recall that β is the square integrable process appearing in (3.2). Using the BDG inequality we get

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[0,T]}\left|\int_0^s\int_0^r h_u dS_u^{\mathcal{H}}\, U''(X_r^{\pi^*}+Y_r)(\pi_r^*+Z_r)dW_r^{\mathcal{H}}\right|\right]\\
&\leq C\mathbb{E}\left[\left|\int_0^T\left|\int_0^s h_r dS_r^{\mathcal{H}}\right|^2|\beta_s|^2 ds\right|^{1/2}\right]\\
&\leq C\mathbb{E}\left[\left(\sup_{s\in[0,T]}\left|\int_0^s h_r dS_r^{\mathcal{H}}\right|^2\right)^{1/2}\left(\int_0^T|\beta_s|^2 ds\right)^{1/2}\right].
\end{aligned}$$

Young's inequality furthermore yields

$$\begin{aligned}
&\mathbb{E}\left[\left(\sup_{s\in[0,T]}\left|\int_0^s h_r dS_r^{\mathcal{H}}\right|^2\right)^{1/2}\left(\int_0^T|\beta_s|^2 ds\right)^{1/2}\right]\\
&\leq C\mathbb{E}\left[\sup_{s\in[0,T]}\left|\int_0^s h_r dS_r^{\mathcal{H}}\right|^2\right]+C\mathbb{E}\left[\int_0^T|\beta_s|^2 ds\right]\\
&\leq C\left(1+\mathbb{E}\left[\sup_{s\in[0,T]}\left|\int_0^s h_r dW_r^{\mathcal{H}}\right|^2\right]\right),
\end{aligned}$$

where we have used that h and θ are bounded. Applying once again the BDG inequality, we obtain

$$\mathbb{E}\left[\sup_{s\in[0,T]}\left|\int_0^s h_r dW_r^{\mathcal{H}}\right|^2\right]\leq 4\mathbb{E}\left[\int_0^T|h_r|^2 dr\right]<\infty.$$

Putting together the previous steps, we have that

$$\mathbb{E}\left[\sup_{s\in[0,T]}\left|\int_0^s\int_0^r h_u dS_u^{\mathcal{H}}\, U''(X_r^{\pi^*}+Y_r)(\pi_r^*+Z_r)dW_r^{\mathcal{H}}\right|\right]<\infty,$$

and thus we get

$$\mathbb{E}\left[\int_0^T\int_0^s h_r dS_r^{\mathcal{H}}\, U''(X_s^{\pi^*}+Y_s)(\pi_s^*+Z_s)dW_s^{\mathcal{H}}\right]=0.$$

Note that $\left(\int_0^t U'(X_s^{\pi^*}+Y_s)h_s dW_s^{\mathcal{H}}\right)_{t\in[0,T]}$ is a square integrable martingale. Indeed $U'(X^{\pi^*}+Y)=\alpha$ is a square integrable martingale and thus

$$\mathbb{E}\left[\int_0^T\left|U'(X_s^{\pi^*}+Y_s)h_s\right|^2 ds\right]<\infty.$$

Similarly,

$$\mathbb{E}\left[\left|U'(X_T^{\pi^*}+Y_T)\int_t^T h_r dS_r^{\mathcal{H}}\right|\right]<\infty.$$

Taking expectation in (3.7) we obtain for every $n \geq 1$ that

$$\begin{aligned}
&\mathbb{E}\left[U'(X_T^{\pi^*} + Y_T)\int_0^T h_r dS_r^{\mathcal{H}}\right] \\
&= \mathbb{E}\left[\int_0^T \left(U'(X_s^{\pi^*} + Y_s)\theta_s + U''(X_s^{\pi^*} + Y_s)(\pi_s^* + Z_s)\right) \cdot h_s ds\right],
\end{aligned} \tag{3.8}$$

which in conjunction with (3.6) leads to

$$\mathbb{E}\left[\int_0^T \left(U'(X_s^{\pi^*} + Y_s)\theta_s + U''(X_s^{\pi^*} + Y_s)(\pi_s^* + Z_s)\right) \cdot h_s ds\right] \leq 0$$

for every bounded predictable process h. Replacing h by $-h$, we get

$$\mathbb{E}\left[\int_0^T \left(U'(X_s^{\pi^*} + Y_s)\theta_s + U''(X_s^{\pi^*} + Y_s)(\pi_s^* + Z_s)\right) \cdot h_s ds\right] = 0. \tag{3.9}$$

Now fix i in $\{1, \dots, d_1\}$. Let $A_s^i := U'(X_s^{\pi^*} + Y_s)\theta_s^i + U''(X_s^{\pi^*} + Y_s)(\pi_s^{*^i} + Z_s^i)$ and $h_s := (0, \dots, 0, \mathbf{1}_{\{A_s^i > 0\}}, 0, \dots, 0)$ where the non-vanishing component is the i-th component. From (3.9) we get

$$\mathbb{E}\left[\int_0^T \mathbf{1}_{\{A_s^i > 0\}}[U'(X_s^{\pi^*} + Y_s)\theta_s^i + U''(X_s^{\pi^*} + Y_s)(\pi_s^{*^i} + Z_s^i)]ds\right] = 0.$$

Hence, $A^i \leq 0$, $d\mathbb{P} \otimes dt - a.e.$. Similarly by choosing $h_s = (0, \dots, 0, \mathbf{1}_{\{A_s^i < 0\}}, 0, \dots, 0)$ we deduce that

$$U'(X_t^{\pi^*} + Y_t)\theta_t^i + U''(X_t^{\pi^*} + Y_t)(\pi_t^{*^i} + Z_t^i) = 0, \quad d\mathbb{P} \otimes dt - a.e.$$

This concludes the proof since $i \in \{1, \dots, d_1\}$ is arbitrary. □

The characterization of the optimal strategy above can also be given in terms of a fully-coupled Forward-Backward system.

Theorem 3.2. *Under the assumptions of Theorem 3.1, the optimal strategy π^* for (2.2) is given by*

$$\pi_t^{*^i} = -\theta_t^i \frac{U'(X_t + Y_t)}{U''(X_t + Y_t)} - Z_t^i, \quad t \in [0, T], \quad i = 1, \dots, d_1,$$

where (X, Y, Z) is a triple of adapted processes which solves the FBSDE

$$\begin{cases}
X_t &= x - \int_0^t \left(\theta_s \frac{U'(X_s+Y_s)}{U''(X_s+Y_s)} + Z_s\right) dW_s^{\mathcal{H}} - \int_0^t \left(\theta_s \frac{U'(X_s+Y_s)}{U''(X_s+Y_s)} + Z_s\right) \cdot \theta_s^{\mathcal{H}} ds \\
Y_t &= H - \int_t^T Z_s dW_s - \int_t^T \left[-\frac{1}{2}|\theta_s^{\mathcal{H}}|^2 \frac{U^{(3)}(X_s+Y_s)|U'(X_s+Y_s)|^2}{(U''(X_s+Y_s))^3} \right. \\
&\qquad \left. + |\theta_s^{\mathcal{H}}|^2 \frac{U'(X_s+Y_s)}{U''(X_s+Y_s)} + Z_s \cdot \theta_s^{\mathcal{H}} - \frac{1}{2}|Z_s^{\mathcal{O}}|^2 \frac{U^{(3)}}{U''}(X_s + Y_s) \right] ds,
\end{cases} \tag{3.10}$$

with the notation $Z = (\underbrace{Z^1, \dots, Z^{d_1}}_{=:Z^{\mathcal{H}}}, \underbrace{Z^{d_1+1}, \dots, Z^d}_{=:Z^{\mathcal{O}}})$. In addition, the optimal wealth process X^{π^*} is equal to X.

Proof. From Theorem 3.1 we know that the optimal strategy is given by

$$\pi_t^{*^i} = -\theta_t^i \frac{U'(X_t^{\pi^*} + Y_t)}{U''(X_t^{\pi^*} + Y_t)} - Z_t^i, \quad t \in [0,T], \quad i \in \{1, \dots, d_1\},$$

where (Y, Z) is a solution of the BSDE (3.3) with driver f as in (3.4). Now plugging the expression of π^* into equation (3.4) yields

$$\begin{cases} X_t^{\pi^*} &= x - \int_0^t \left(\theta_s \frac{U'(X_s^{\pi^*}+Y_s)}{U''(X_s^{\pi^*}+Y_s)} + Z_s \right) dW_s^{\mathcal{H}} - \int_0^t \left(\theta_s \frac{U'(X_s^{\pi^*}+Y_s)}{U''(X_s^{\pi^*}+Y_s)} + Z_s \right) \cdot \theta_s^{\mathcal{H}} ds \\ Y_t &= H - \int_t^T Z_s dW_s - \int_t^T \Big[-\frac{1}{2} |\theta_s^{\mathcal{H}}|^2 \frac{U^{(3)}(X_s^{\pi^*}+Y_s)|U'(X_s^{\pi^*}+Y_s)|^2}{(U''(X_s^{\pi^*}+Y_s))^3} \\ & \quad + |\theta_s^{\mathcal{H}}|^2 \frac{U'(X_s^{\pi^*}+Y_s)}{U''(X_s^{\pi^*}+Y_s)} + Z_s \cdot \theta_s^{\mathcal{H}} - \frac{1}{2} |Z_s^{\mathcal{O}}|^2 \frac{U^{(3)}}{U''}(X_s^{\pi^*} + Y_s) \Big] ds. \end{cases} \tag{3.11}$$

Recalling that $X^\pi := x + \int_0^\cdot \pi_s (dW_s^{\mathcal{H}} + \theta_s^{\mathcal{H}} ds)$ for any admissible strategy π, we get the forward part of the FBSDE. □

Remark 3.3. *Using Itô's formula and the FBSDE* (3.10)*, we have*

$$U'(X+Y) = U'(x+Y_0) + \int_0^\cdot -\theta_s^{\mathcal{H}} U'(X_s+Y_s) dW_s^{\mathcal{H}} + \int_0^\cdot U''(X_s+Y_s) Z_s^{\mathcal{O}} dW_s^{\mathcal{O}}.$$

Remark 3.4. *Note that using the system* (3.10)*, for* $\alpha := U'(X^{\pi^*} + Y)$*, integration by parts yields for every* t *in* $[0,T]$

$$\begin{aligned} & U'(X_t^{\pi^*} + Y_t)(X_t^\pi - X_t^{\pi^*}) \\ &= \int_0^t (X_s^\pi - X_s^{\pi^*}) d\alpha_s + \int_0^t \alpha_s (\pi_s - \pi_s^*) dW_s^{\mathcal{H}} \\ &\quad + \int_0^t \left(\alpha_s \theta_s^{\mathcal{H}} + U''(X_s^{\pi^*} + Y_s)(Z_s^{\mathcal{H}} + \pi_s^*) \right) \cdot (\pi_s - \pi_s^*) ds \\ &= \int_0^t (X_s^\pi - X_s^{\pi^*}) d\alpha_s + \int_0^t \alpha_s (\pi_s - \pi_s^*) dW_s^{\mathcal{H}}. \end{aligned}$$

This shows that $U'(X^{\pi^*} + Y)(X^\pi - X^{\pi^*})$ *is a local martingale for every* π *in* Π^x*.*

The converse implication of Theorems 3.1 and 3.2 constitutes the second main result.

Theorem 3.5. *Let* $(H1)$ *and (H2) be satisfied. Let* (X, Y, Z) *be a triple of stochastic processes which solves the FBSDE* (3.10) *satisfying:* Z *is in* $\mathbb{H}^2(\mathbb{R}^d)$*,* $\mathbb{E}[|U(X_T + H)|] < \infty$*,* $\mathbb{E}[|U'(X_T + H)|^2] < \infty$*, and* $U'(X+Y)$ *is a positive martingale. Moreover, assume that there exists a constant* $\kappa > 0$ *such that*

$$-\frac{U'(x)}{U''(x)} \leq \kappa$$

for all $x \in \mathbb{R}$*. Then*

$$\pi_t^{*^i} := -\frac{U'(X_t + Y_t)}{U''(X_t + Y_t)} \theta_t^i - Z_t^i, \quad t \in [0,T], \quad i \in \{1, \dots, d_1\},$$

is an optimal solution of the optimization problem (2.2)*.*

Proof. Note first that by definition of π^*, $X = X^{\pi^*}$. Since the risk tolerance $-\frac{U'(x)}{U''(x)}$ is bounded and since $Z \in \mathbb{H}^2(\mathbb{R}^d)$, we immediately get $\mathbb{E}\left[\int_0^T |\pi_s^*|^2 ds\right] < \infty$, thus, $\pi \in \Pi^x$. By assumption, $U'(X+Y)$ is a positive continuous martingale, hence there exists a continuous local martingale L such that $U'(X+Y) = \mathcal{E}(L)$. And we know from Remark 3.3 that

$$L = \log(U'(x+Y_0)) + \int_0^\cdot -\theta_s^{\mathcal{H}} dW_s^{\mathcal{H}} + \int_0^\cdot \frac{U''(X_s+Y_s)}{U'(X_s+Y_s)} Z_s^{\mathcal{O}} dW_s^{\mathcal{O}}.$$

Define the probability measure $\mathbb{Q} \sim \mathbb{P}$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := \frac{U'(X_T+H)}{\mathbb{E}[U'(X_T+H)]}.$$

Girsanov's theorem implies that $\tilde{W} := \tilde{W}^{\mathcal{H}} + \tilde{W}^{\mathcal{O}} = (W^1 + \int_0^\cdot \theta_t^1 dt, \ldots, W^{d_1} + \int_0^\cdot \theta_t^{d_1} dt, W^{d_1+1} - \int_0^\cdot \frac{U''(X_t+Y_t)}{U'(X_t+Y_t)} Z_t^{d_1+1} dt, \ldots, W^{d_2} - \int_0^\cdot \frac{U''(X_t+Y_t)}{U'(X_t+Y_t)} Z_t^{d_2} dt)$ is a standard Brownian motion under $\mathbb{Q}$. Thus X^π is a local martingale under $\mathbb{Q}$ for every π in Π^x. Now fix π in Π^x with $\mathbb{E}[|U(X_T^\pi + H)|] < \infty$. Let $(\tau_n)_n$ be a localizing sequence for the local martingale $X^\pi - X^{\pi^*}$. Since U is a concave, we have

$$U(X_T^\pi + H) - U(X_T^{\pi^*} + H) \leq U'(X_T^{\pi^*} + H)(X_T^\pi - X_T^{\pi^*}). \tag{3.12}$$

Taking expectations in (3.12) we get

$$\begin{aligned}
\frac{\mathbb{E}[U(X_T^\pi + H) - U(X_T^{\pi^*} + H)]}{\mathbb{E}[U'(X_T+H)]} &\leq \mathbb{E}_{\mathbb{Q}}[X_T^\pi - X_T^{\pi^*}] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\lim_{n\to\infty} \int_0^{T\wedge\tau_n} (\pi_s - \pi_s^*) d\tilde{W}_s^{\mathcal{H}}\right] \\
&= \lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}}\left[\int_0^{T\wedge\tau_n} (\pi_s - \pi_s^*) d\tilde{W}_s^{\mathcal{H}}\right] = 0,
\end{aligned}$$

which eventually follows as a consequence of Lebesgue's dominated convergence theorem. To this end we prove that

$$\mathbb{E}_{\mathbb{Q}}\left[\sup_{t\in[0,T]} \left|\int_0^t (\pi_s - \pi_s^*) d\tilde{W}_s^{\mathcal{H}}\right|\right] < \infty.$$

Indeed the BDG inequality and the Cauchy-Schwarz inequality imply that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[\sup_{t\in[0,T]} \left|\int_0^t (\pi_s - \pi_s^*) d\tilde{W}_s^{\mathcal{H}}\right|\right] \\
&\leq C\mathbb{E}_{\mathbb{Q}}\left[\left(\int_0^T |\pi_s - \pi_s^*|^2 ds\right)^{\frac{1}{2}}\right] \\
&= C\mathbb{E}\left[\frac{U'(X_T+H)}{\mathbb{E}[U'(X_T+H)]} \left(\int_0^T |\pi_s - \pi_s^*|^2 ds\right)^{\frac{1}{2}}\right]
\end{aligned}$$

$$\leq C\mathbb{E}\left[\left|\frac{U'(X_T+H)}{\mathbb{E}[U'(X_T+H)]}\right|^2\right]^{\frac{1}{2}} \mathbb{E}\left[\int_0^T |\pi_s-\pi_s^*|^2 ds\right]^{\frac{1}{2}} < \infty.$$

□

We have proved in Theorem 3.2 that if (2.2) exhibits an optimal strategy $\pi^* \in \Pi^x$, then there exists an adapted solution to the FBSDE (3.10). As a byproduct we showed that the optimization procedure singles out a "pricing measure" under which the asset prices and marginal utilities are martingales. In this sense, the process Y captures the impact of future trading gains on the agent's marginal utilities. If we assume additional conditions on the utility function U, we get the following regularity properties of the solution (X,Y,Z).

Proposition 3.6. *Assume that $H \in L^\infty(\Omega,\mathcal{F}_T,\mathbb{P})$ and that the FBSDE (3.10) admits an adapted solution (X,Y,Z) such that Y is bounded. Let*

$$\varphi_1(x) := \frac{U'(x)}{U''(x)}, \quad \varphi_2(x) := \frac{U^{(3)}(x)|U'(x)|^2}{(U''(x))^3}, \quad \varphi_3(x) := \frac{U^{(3)}(x)}{U''(x)}, \quad x \in \mathbb{R}.$$

Assume that U is such that φ_i, $i=1,2,3$ are bounded and Lipschitz continuous functions. Then (X,Y,Z) is the unique solution of (3.10) in $\mathbb{S}^2(\mathbb{R})\times\mathbb{S}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R}^d)$. In addition, $Z\cdot W$ is a BMO-martingale.

Proof. Let (X,Y,Z) be a solution to (3.10) such that Y is bounded. Then, using the usual theory on quadratic growth BSDEs (see for example [30, Theorem 2.5 and Lemma 3.1]) we have only from the backward part of the FBSDE that Z is in $\mathbb{H}^2(\mathbb{R}^d)$ and that $Z\cdot W$ is a BMO-martingale. In addition there exists a unique bounded solution to the backward component for a given process X. Now the previous regularity properties of the processes (Y,Z) imply that X is in $\mathbb{S}^2(\mathbb{R})$. We turn to the uniqueness of the process X. Assume that there exists another solution (X',Y',Z') of (3.10). Hence, Theorem 3.5 implies that $\pi^{*\prime} := -\frac{U'(X'+Y')}{U''(X'+Y')}\theta^i + Z'^i, \quad i\in\{1,\dots,d_1\}$ is an optimal solution to our original problem (2.2) and X' is the optimal wealth process. However, by strict concavity of U and by convexity of Π^x the optimal strategy has to be unique. So X and X' are the wealth processes of the same optimal strategy, thus, they have to coincide (for instance $X_T = X'_T$, $\mathbb{P}-a.s.$) which implies $(Y',Z')=(Y,Z)$. □

In the complete case we are able to construct the solution (X,Y,Z). This is the subject of the following Section.

3.2 Characterization and verification: complete markets

In this section we consider the benchmark case of a complete market. We assume $d=1$ for simplicity. H denotes a square integrable random variable measurable with respect to the Brownian motion W.

In the complete case we can give sufficient conditions for the existence of a solution to the system (3.10). Our construction relies on the following remark.

Remark 3.7. *Using* (3.10) *the martingale* $U'(X^{\pi^*}+Y)$ *becomes more explicit, because Itô's formula applied to* $U'(X^{\pi^*}+Y)$ *yields*

$$\begin{aligned}U'(X_t^{\pi^*}+Y_t) &= U'(x+Y_0)+\int_0^t U''(X_s^{\pi^*}+Y_s)(\pi_s^*+Z_s)dW_s\\ &= U'(x+Y_0)-\int_0^t U'(X_s^{\pi^*}+Y_s)\theta_s dW_s,\end{aligned}$$

where we have replaced π^* *by its characterization in terms of* (X,Y,Z) *from Theorem 3.1. Hence,*

$$U'(X_t^{\pi^*}+Y_t)=U'(x+Y_0)\mathcal{E}(-\theta\cdot W)_t,\quad t\in[0,T]. \tag{3.13}$$

This remark will allow us to prove existence of a solution to the system (3.10) under a condition on the risk aversion coefficient $-\frac{U''}{U'}$ of U. To this end, we give a sufficient condition on U for the system (3.10) to exhibit a solution. We have the following remark.

Remark 3.8. *If* (X,Y,Z) *is an adapted solution to the system* (3.10)*, then* $P:=X+Y$ *is solution of the forward SDE*

$$P_t=x+Y_0-\int_0^t\theta_s\frac{U'(P_s)}{U''(P_s)}dW_s-\int_0^t\frac{1}{2}|\theta_s|^2\frac{U^{(3)}(P_s)|U^{'}(P_s)|^2}{(U''(P_s))^3}ds,\quad t\in[0,T]. \tag{3.14}$$

In addition, if (X,Y,Z) *is in* $\mathbb{S}^2(\mathbb{R})\times\mathbb{S}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R}^d)$*, then* $P\in\mathbb{S}^2(\mathbb{R})$*. Thus a necessary condition for the FBSDE* (3.10) *to have a solution is that the SDE* (3.14) *admits a solution.*

We are now going to state an existence result for the FBSDE system (3.10) that characterizes optimal trading strategies in terms of the functions $\varphi_1(x)=\frac{U'(x)}{U''(x)}$ and $\varphi_2(x)=\frac{U^{(3)}(x)|U^{'}(x)|^2}{(U''(x))^3}$.

Proposition 3.9. *Assume that the functions* φ_1 *and* φ_2 *are bounded and Lipschitz continuous. Then the FBSDE*

$$\begin{cases} X_t=x-\int_0^t\left(\theta_s\frac{U'(X_s+Y_s)}{U''(X_s+Y_s)}+Z_s\right)dW_s-\int_0^t\left(\theta_s\frac{U'(X_s+Y_s)}{U''(X_s+Y_s)}+Z_s\right)\cdot\theta_s ds\\ \\ Y_t=H-\int_t^T Z_s dW_s-\int_t^T\Big(-\frac{1}{2}|\theta_s|^2\frac{U^{(3)}(X_s+Y_s)|U^{'}(X_s+Y_s)|^2}{(U''(X_s+Y_s))^3}+|\theta_s|^2\frac{U^{'}(X_s+Y_s)}{U''(X_s+Y_s)}\\ \qquad\qquad +Z_s\cdot\theta_s\Big)ds\end{cases} \tag{3.15}$$

admits a solution (X,Y,Z) *in* $\mathbb{S}^2(\mathbb{R})\times\mathbb{S}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R}^d)$ *such that* $\mathbb{E}[|U(X_T+H)|]<\infty$ *and* $\mathbb{E}[|U'(X_T+H)|^2]<\infty$.

Proof. Let m in $\mathbb{R}$. Consider the following SDE

$$P_t^m=x+m-\int_0^t\theta_s\varphi_1(P_s^m)dW_s-\int_0^t\frac{1}{2}|\theta_s|^2\varphi_2(P_s^m)ds,\quad t\in[0,T].$$

Since this SDE has Lipschitz coefficients the existence and uniqueness of a solution in $\mathbb{S}^2(\mathbb{R})$ is guaranteed (see for example [36, V.3. Lemma 1]). Next, consider the BSDE

$$Y_t^m=H-\int_t^T Z_s^m dW_s-\int_t^T\left(-\frac{1}{2}|\theta_s|^2\varphi_2(P_s^m)+|\theta_s|^2\varphi_1(P_s^m)+Z_s^m\cdot\theta_s\right)ds. \tag{3.16}$$

We denote its driver by $f(s,p,z) := -\frac{1}{2}|\theta_s|^2\varphi_2(p) + |\theta_s|^2\varphi_1(p) + z\cdot\theta_s$. Using the regularity properties of φ_1 and φ_2 and the fact that θ is bounded, we see that there exists a constant $K>0$ such that

$$|f(s,p,z)| \leq K(1+|z|)$$

and the constant K depends only on α_1,α_2 and on $\|\theta\|_\infty$, thus in particular K does not depend on m. Since the driver f is Lipschitz continuous in z, there exists a unique pair of adapted processes (Y^m, Z^m) in $\mathbb{S}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R}^d)$ which solves (3.16). In addition, $|Y_t^m|\leq K$ holds $\mathbb{P}$-a.s. for all t in $[0,T]$. We recall that this constant K does not depend on m, thus $|Y_0^m|\leq K$. Using standard arguments we can show that the map $m\mapsto Y_0^m$ is continuous. Even if this procedure is straightforward, we reprove this fact here to make the paper self-contained. Fix m,m' in $\mathbb{R}$ with $m\neq m'$. We set $\delta Y_t := Y_t^m - Y_t^{m'}$, $\delta Z_t := Z_t^m - Z_t^{m'}$. By (3.16) it follows that $(\delta Y,\delta Z)$ is solution to the Lipschitz BSDE:

$$\delta Y_t = 0 - \int_t^T \delta Z_s dW_s - \int_t^T (\theta_s\delta Z_s + h(s))ds$$

with $h(s) := \frac{1}{2}|\theta_s|^2(\varphi_2(P_s^m) - \varphi_2(P_s^{m'})) + |\theta_s|^2(\varphi_1(P_s^m)-\varphi_1(P_s^{m'}))$. Using classical a priori estimates for Lipschitz growth BSDEs (see for example [26, Lemma 2.2]) we get

$$|\delta Y_0|^2 \leq \mathbb{E}[\sup_{t\in[0,T]}|\delta Y_t|^2] \leq C\mathbb{E}\left[\int_0^T |h(s)|^2 ds\right].$$

The boundedness of θ and the Lipschitz assumption on φ_1 and on φ_2 immediately imply that

$$\mathbb{E}\left[\int_0^T |h(s)|^2 ds\right] \leq C\mathbb{E}\left[\int_0^T |P_s^m - P_s^{m'}|^2 ds\right] \leq C\mathbb{E}\left[\sup_{t\in[0,T]}|P_t^m - P_t^{m'}|^2\right].$$

Combining the inequalities above with classical estimates on Lipschitz SDEs (see for example [36, Estimate (***) in the proof of Theorem V.7.37]) we finally get that

$$|\delta Y_0|^2 \leq C|m-m'|^2$$

which concludes the proof by letting m' tend to m. This in conjunction with $m\mapsto Y_0^m$ being bounded yields that there exists an element $m^*\in\mathbb{R}$ such that $Y_0^{m^*} = m^*$. Setting

$$X_t^{m^*} := P_t^{m^*} - Y_t^{m^*}, \quad t\in[0,T],$$

it is straightforward to check that $(X^{m^*}, Y^{m^*}, Z^{m^*})$ satisfies (3.15). Moreover, we have $X^{m^*}\in\mathbb{S}^2(\mathbb{R})$ since Y^{m^*} is bounded and since $P^{m^*}\in\mathbb{S}^2(\mathbb{R})$. Next, note that $\mathbb{E}[|U'(X_T + Y_T)|^2]<\infty$ since $U'(X_T+Y_T) = U'(x+m)\mathcal{E}(-\theta\cdot W)$. Now using the concavity of U, we see

$$U(x)\leq U'(0)x + U(0), \quad -U(x)\leq -U'(x)x - U(0), \quad \forall x\in\mathbb{R}.$$

Consequently, we have

$$\mathbb{E}[|U(X_T+H)|] \leq \mathbb{E}[|U'(0)|\ |X_T+H| + |U(0)|] + \mathbb{E}[|U'(X_T+H)(X_T+H)| + |U(0)|] < \infty,$$

which concludes the proof. □

4 Utility functions on the positive half-line

In this section we study utility functions $U : \mathbb{R}^+ \to \mathbb{R}$ defined on the positive half-line. Again, we assume that U is strictly increasing and strictly concave.

In the previous section we have derived a FBSDE characterization of the optimal strategy for the utility maximization problem (2.2). The key observation was that there exists a stochastic process Y such that $U'(X^{\pi^*} + Y)$ is a martingale. However if U is only defined on the positive half-line, it is not clear *a priori* that the expression $U'(X^{\pi^*} + Y)$ makes sense. We could generalize this approach by looking for a function Φ such that $\Phi(X_t^{\pi^*}, Y_t)$ is a martingale and such that $\Phi(X_T^{\pi^*}, Y_T) = U'(X_T^{\pi^*} + H)$. If $H = 0$, it turns out that a good choice for Φ is given by $\Phi(x, y) := U'(x)\exp(y)$, since the system we obtain coincides (up to a non-linear transformation) with the one obtained by Peng in [34, Section 4] using the maximum principle. Note that the system of Peng is not formulated as a FBSDE but rather as a system of equations: one for the wealth process whose dynamics depend on the strategy and one adjoint equation, but a reformulation of this system of equation allows to get a FBSDE (details are given in Section 5.1).

In the previous section, π denoted the total amount of money invested into the stock (the number of shares being $\pi/\tilde{S}$). Now we denote by π^i the proportion of wealth invested in the i-th stock S^i. Once again we denote by Π^x the set of admissible strategies with initial capital x which is now defined by

$$\Pi^x := \left\{ \pi \in \mathbb{H}^2(\mathbb{R}^{d_1}),\ X_0^\pi = x, \right\}. \tag{4.1}$$

where X^π stands for the associated wealth process given by

$$X_t^\pi := X_0^\pi + \int_0^t \pi_s X_s^\pi dS_s^{\mathcal{H}}, \quad t \in [0, T].$$

Again, we extend π to $\mathbb{R}^d$ via $\tilde{\pi} := (\pi^1, \ldots, \pi^{d_1}, 0, \ldots, 0)$ and make the convention that we write π instead of $\tilde{\pi}$. Thus, we have

$$X_t^\pi = x\mathcal{E}\left(\int_0^\cdot \pi_r dS_r^{\mathcal{H}}\right)_t, \quad t \in [0, T].$$

We need to impose the following assumptions.

(H3) $U : \mathbb{R}^+ \to \mathbb{R}$ is three times differentiable, strictly increasing and concave

(H4) H is a positive $\mathcal{F}_T$-measurable random variable in $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$

(H5) We say that assumption (H5) holds for an element π^* in Π^x, if

(i) $\mathbb{E}[|X_T^{\pi^*} U'(X_T^{\pi^*} + H)|^2] < \infty$;

(ii) the sequence of random variables

$$\left(\frac{1}{\varepsilon}(X_T^{\pi^*+\varepsilon\rho} - X_T^{\pi^*}) \int_0^1 U'(X_T^{\pi^*} + H + r(X_T^{\pi^*+\varepsilon\rho} - X_T^{\pi^*}))dr\right)_{\varepsilon\in(0,1)}$$

is uniformly integrable;

(iii)

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T]} \mathbb{E}\left[\left|\frac{1}{\varepsilon}(X_t^{\pi^*+\varepsilon\rho} - X_t^{\pi^*}) - \xi_t\right|^2\right] = 0,$$

where $d\xi_t = \pi_t^* \xi_t dS_t^{\mathcal{H}} + \rho_t X_t^{\pi^*} dS_t^{\mathcal{H}}, \quad t \in [0,T]$, and $\sup_{t\in[0,T]} \mathbb{E}[|\xi_t|^2] < \infty$.

(H6) There exists a constant $c > 0$ such that $\frac{-U'(x)}{xU''(x)} \leq c$ for all $x \in \mathbb{R}^+$.

4.1 Characterization and verification: incomplete markets

Note that in condition (H5), if $U'(0) < \infty$ or if $H \geq a > 0$ is satisfied, then (iii) implies (ii).

Theorem 4.1. *Assume that (H3) and (H4) hold. Let π^* be an optimal solution to* (2.2) *satisfying $\mathbb{E}[|U(X_T^{\pi^*} + H)|] < \infty$ and assumption (H5). Then there exists a predictable process Y with $Y_T = \log(U'(X_T^{\pi^*} + H)) - \log(U'(X_T^{\pi^*}))$ such that $X^{\pi^*} U'(X^{\pi^*}) \exp(Y)$ is a martingale and*

$$\pi_s^{*i} = -\frac{U'(X_s^{\pi^*})}{X_s^{\pi^*} U''(X_s^{\pi^*})}(Z_s^i + \theta_s^i), \quad s \in [0,T], \quad i = 1, \ldots, d_1,$$

where $Z_t := \left(\frac{d\langle Y, W^1\rangle_t}{dt}, \ldots, \frac{d\langle Y, W^d\rangle_t}{dt}\right)$.

Proof. As in the proof of Theorem 3.1, we prove the existence of Y such that $X^{\pi^*} U'(X^{\pi^*}) \exp(Y)$ is a martingale with $Y_T = \log(U'(X_T^{\pi^*} + H)) - \log(U'(X_T^{\pi^*}))$. Consequently, $U'(X_T^{\pi^*} + H) = U'(X_T^{\pi^*}) \exp(Y_T)$. By (H5), the process

$$\alpha_t := \mathbb{E}[X_T^{\pi^*} U'(X_T^{\pi^*} + H) | \mathcal{F}_t]$$

is a square integrable martingale. In addition it is the unique solution to the BSDE

$$\alpha_t = X_T^{\pi^*} U'(X_T^{\pi^*} + H) - \int_t^T \beta_s dW_s, \quad t \in [0,T],$$

where β is a square integrable predictable process with values in $\mathbb{R}^d$. We set $Y := \log(\alpha) - \log(U'(X^{\pi^*})) - \log(X^{\pi^*})$. As in the proof of Theorem 3.1, Itô's formula implies that

$$Y_t = Y_T - \int_t^T \left[\frac{\beta_s}{\alpha_s} - \frac{U''(X_s^{\pi^*})}{U'(X_s^{\pi^*})} X_s^{\pi^*} \pi_s^* - \pi_s^*\right] dW_s$$

$$-\int_t^T \Big[-\frac{1}{2}\frac{|\beta_s|^2}{|\alpha_s|^2} - \Big(\frac{U''(X_s^{\pi^*})}{U'(X_s^{\pi^*})} X_s^{\pi^*}\pi_s^* + \pi_s^* \Big) \cdot \theta_s^{\mathcal{H}}$$
$$+ \frac{|X_s^{\pi^*}\pi_s^*|^2}{2} \left(\left| \frac{U''(X_s^{\pi^*})}{U'(X_s^{\pi^*})} \right|^2 - \frac{U^{(3)}(X_s^{\pi^*})}{U'(X_s^{\pi^*})} \right) + \frac{|\pi_s^*|^2}{2} \Big] ds.$$

Setting

$$Z_t^i = \frac{\beta_t^i}{\alpha_t} - \frac{\pi_t^*}{U'(X_t^{\pi^*})} (X_t^{\pi^*} U''(X_t^{\pi^*}) + U'(X_t^{\pi^*})), \quad i = 1, \ldots, d, \tag{4.2}$$

we get

$$\begin{aligned} Y_t &= Y_T - \int_t^T Z_s dW_s - \int_t^T \Big[-\frac{1}{2}\frac{U^{(3)}(X_s^{\pi^*})}{U'(X_s^{\pi^*})} |X_s^{\pi^*}\pi_s^*|^2 - (Z_s^{\mathcal{H}} + \theta_s^{\mathcal{H}}) \cdot \Big(\frac{U''(X_s^{\pi^*})}{U'(X_s^{\pi^*})} X_s^{\pi^*}\pi_s^* + \pi_s^* \Big) \\ &\quad - \frac{U''(X_s^{\pi^*})}{U'(X_s^{\pi^*})} X_s^{\pi^*} |\pi_s^*|^2 - \frac{1}{2}|Z_s|^2 \Big] ds, \quad t \in [0,T]. \end{aligned}$$

We now derive the characterization of π^* in terms of U' and Y and Z. We employ an argument put forth in [34] and then substitute the Hamiltonian by a BSDE. Fix $\pi \in \Pi^x$. Since the latter is a convex set, for $\rho := \pi - \pi^*$, $\pi^* + \varepsilon\rho$ remains an admissible strategy for every $\varepsilon \in (0,1)$. We have

$$\frac{1}{\varepsilon}(U(X_T^{\pi^*+\varepsilon\rho} + H) - U(X_T^{\pi^*} + H)) =$$
$$\frac{1}{\varepsilon}(X_T^{\pi^*+\varepsilon\rho} - X_T^{\pi^*}) \int_0^1 U'(X_T^{\pi^*} + H + r(X_T^{\pi^*+\varepsilon\rho} - X_T^{\pi^*}))dr.$$

Since π^* is optimal we find

$$\mathbb{E}\left[\frac{1}{\varepsilon}(X_T^{\pi^*+\varepsilon\rho} - X_T^{\pi^*}) \int_0^1 U'(X_T^{\pi^*} + H + r(X_T^{\pi^*+\varepsilon\rho} - X_T^{\pi^*}))dr \right] \le 0, \quad \forall \varepsilon > 0. \tag{4.3}$$

Now let ξ be defined by

$$d\xi_t = (\pi_t^* \xi_t + \rho_t X_t^{\pi^*}) dS_t^{\mathcal{H}}, \quad t \in [0,T].$$

By (H5), we can apply Lebesgue's dominated convergence theorem in inequality (4.3) which, possibly passing to a subsequence, yields

$$\mathbb{E}[\xi_T U'(X_T^{\pi^*} + H)] = \lim_{\varepsilon \to 0} \mathbb{E}\left[\frac{1}{\varepsilon}(X_T^{\pi^*+\varepsilon\rho} - X_T^{\pi^*}) \int_0^1 U'(X_T^{\pi^*} + H + r(X_T^{\pi^*+\varepsilon\rho} - X_T^{\pi^*}))dr \right].$$

Combined with (4.3), this leads to

$$\mathbb{E}[\xi_T (X_T^{\pi^*})^{-1} U'(X_T^{\pi^*}) X_T^{\pi^*} \exp(Y_T)] = \mathbb{E}[\xi_T U'(X_T^{\pi^*} + H)] \le 0, \quad \forall \pi \in \Pi^x. \tag{4.4}$$

We now restrict our attention to a particular class of processes π. We choose ρ to be a bounded predictable process and define $\pi := \rho + \pi^*$ which is an admissible strategy since it is square integrable. The integration by parts formula for continuous semimartingales implies that

$$\xi_t(X_t^{\pi^*})^{-1} = \int_0^t \rho_s dW_s^{\mathcal{H}} + \int_0^t [\rho_s \cdot \theta_s^{\mathcal{H}} - \rho_s \cdot \pi_s^*] ds, \quad t \in [0,T].$$

Another application of integration by parts to $\alpha = U'(X^{\pi^*})X^{\pi^*}\exp(Y)$ and $\xi(X^{\pi^*})^{-1}$ yields

$$\begin{aligned}
\xi_T U'(X_T^{\pi^*} + Y_T) &= \xi_T (X_T^{\pi^*})^{-1} U'(X_T^{\pi^*}) X_T^{\pi^*} \exp(Y_T) \\
&= \int_0^T \xi_t (X_t^{\pi^*})^{-1} d\alpha_t + \int_0^T \alpha_t \rho_t dW_t^{\mathcal{H}} \\
&\quad + \int_0^T \rho_t \exp(Y_t) X_t^{\pi^*} \cdot (U'(X_t^{\pi^*})(Z_t^{\mathcal{H}} + \theta_t^{\mathcal{H}}) + U''(X_t^{\pi^*}) X_t^{\pi^*} \pi_t^*) dt. \quad (4.5)
\end{aligned}$$

We now intend to take the expectation in the above relation. To this end, we need the following moment estimates. Using that ρ is bounded, we obtain

$$\begin{aligned}
\mathbb{E}[\sup_{t\in[0,T]} |\xi_t(X_t^{\pi^*})^{-1}|^2] &= \mathbb{E}\left[\sup_{t\in[0,T]} \left| \int_0^t \rho_s dW_s^{\mathcal{H}} + \int_0^t (\rho_s \cdot \theta_s^{\mathcal{H}} - \rho_s \cdot \pi_s^*) ds \right|^2\right] \\
&\le C\mathbb{E}\left[\sup_{t\in[0,T]} \left| \int_0^t \rho_s dW_s^{\mathcal{H}} \right|^2\right] + \mathbb{E}\left[\sup_{t\in[0,T]} \left| \int_0^t |\rho_s \cdot \theta_s^{\mathcal{H}} - \rho_s \cdot \pi_s^*| ds \right|^2\right] \\
&\le C\left(\mathbb{E}\left[\int_0^T |\rho_s|^2 ds\right] + \mathbb{E}\left[\left|\int_0^T \rho_s \cdot \theta_s^{\mathcal{H}} ds\right|^2\right] + \mathbb{E}\left[\left|\int_0^T \rho_s \cdot \pi_s^* ds\right|^2\right]\right) \\
&\le C\left(1 + \mathbb{E}\left[\int_0^T |\pi_s^*|^2 ds\right]\right) < \infty, \quad (4.6)
\end{aligned}$$

where we have used Doob's inequality. Consequently, we get

$$\mathbb{E}[|\xi_T(X_T^{\pi^*})^{-1}\alpha_T|] \le \mathbb{E}[|\alpha_T|^2]^{1/2}\mathbb{E}[||\xi_T(X_T^{\pi^*})^{-1}|^2]^{1/2} < \infty,$$

which follows from the Cauchy-Schwarz inequality. With ρ being bounded, we get for some generic constant $C > 0$

$$\mathbb{E}\left[\int_0^T |\alpha_s \rho_s|^2 ds\right] \le C\mathbb{E}\left[\int_0^T |\alpha_s|^2 ds\right] < \infty.$$

Hence $\int_0^\cdot \alpha_t \rho_t dW_t^{\mathcal{H}}$ is a square integrable martingale. Next, let $(\tau_n)_{n\ge1}$ be a localizing sequence for the local martingale $\int_0^\cdot \xi_t(X_t^{\pi^*})^{-1} d\alpha_t$. Then we have

$$\left|\int_0^{\tau_n} \xi_t(X_t^{\pi^*})^{-1} d\alpha_t\right| \le \sup_{t\in[0,T]} \left|\int_0^t \xi_t(X_t^{\pi^*})^{-1} d\alpha_t\right|.$$

To apply Lebesgue's dominated convergence theorem and show that $\mathbb{E}\left[\int_0^T \xi_t(X_t^{\pi^*})^{-1} d\alpha_t\right] = 0$, we need to prove $\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t \xi_t(X_t^{\pi^*})^{-1} d\alpha_t\right|\right] < \infty$. In fact,

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|\int_0^t \xi_t(X_t^{\pi^*})^{-1} d\alpha_t\right|\right] \le C\mathbb{E}\left[\left|\int_0^T |\xi_t|^2 |(X_t^{\pi^*})^{-1}|^2 d\langle\alpha\rangle_t\right|^{1/2}\right]$$

$$\leq C\mathbb{E}\left[\sup_{t\in[0,T]}|\xi_t|^2|(X_t^{\pi^*})^{-1}|^2\right]^{1/2}\mathbb{E}\left[\langle\alpha\rangle_T\right]^{1/2}$$
$$<\infty,$$

where we have used the estimate (4.6). Thus, (4.5) entails

$$\mathbb{E}\left[\left|\int_0^T \rho_t \exp(Y_t)X_t^{\pi^*}\cdot(U'(X_t^{\pi^*})(Z_t^{\mathcal{H}}+\theta_t^{\mathcal{H}})+U''(X_t^{\pi^*})X_t^{\pi^*}\pi_t^*)dt\right|\right]<\infty,$$

and from (4.4), it holds that for every π in Π^x such that ρ is bounded, we get

$$\mathbb{E}\left[\int_0^T \rho_t \exp(Y_t)X_t^{\pi^*}\cdot(U'(X_t^{\pi^*})(Z_t^{\mathcal{H}}+\theta_t^{\mathcal{H}})+U''(X_t^{\pi^*})X_t^{\pi^*}\pi_t^*)dt\right]\leq 0.$$

Substituting ρ with $-\rho$ in the previous inequality, we obtain for every ρ

$$\mathbb{E}\left[\int_0^T \rho_t \exp(Y_t)X_t^{\pi^*}\cdot(U'(X_t^{\pi^*})(Z_t^{\mathcal{H}}+\theta_t^{\mathcal{H}})+U''(X_t^{\pi^*})X_t^{\pi^*}\pi_t^*)dt\right]=0. \tag{4.7}$$

Now let $A_t := U'(X_t^{\pi^*})(Z_t^{\mathcal{H}}+\theta_t^{\mathcal{H}})+U''(X_t^{\pi^*})X_t^{\pi^*}\pi_t^*$ and let $\rho_t(\omega):=\mathbf{1}_{\{A_t(\omega)>0\}}$. Recall that we have $d\mathbb{P}\otimes dt$-a.s. $\exp(Y_t)X_t^{\pi^*}>0$. Plugging ρ into (4.7) yields

$$A_t(\omega)\leq 0,\ d\mathbb{P}\otimes dt - a.e.$$

Similarly choosing $\rho_t(\omega):=\mathbf{1}_{\{A_t(\omega)<0\}}$, we find

$$A_t(\omega)=0,\ d\mathbb{P}\otimes dt - a.e.$$

Thus, we achieve

$$\pi_t^{*^i}=-\frac{U'(X_t^{\pi^*})}{X_t^{\pi^*}U''(X_t^{\pi^*})}(Z_t^i+\theta_t^i),\quad \forall t\in[0,T],\quad i=1,\ldots,d_1.$$

□

Let us now deal with the converse implication.

Theorem 4.2. *Assume (H3)-(H4) and (H6). Let (X,Y,Z) be an adapted solution of the FBSDE*

$$\begin{cases} X_t = x-\int_0^t \frac{U'(X_s)}{U''(X_s)}(Z_s^{\mathcal{H}}+\theta_s^{\mathcal{H}})dW_s^{\mathcal{H}}-\int_0^t \frac{U'(X_s)}{U''(X_s)}(Z_s^{\mathcal{H}}+\theta_s^{\mathcal{H}})\theta_s ds,\\ Y_t=\log\left(\frac{U'(X_T+H)}{U'(X_T)}\right)-\int_t^T\left[(|Z_s^{\mathcal{H}}+\theta_s^{\mathcal{H}}|^2)\left(1-\frac{1}{2}\frac{U^{(3)}(X_s)U'(X_s)}{|U''(X_s)|^2}\right)-\frac{1}{2}|Z_s|^2\right]ds\\ \qquad\qquad -\int_t^T Z_s dW_s, \end{cases} \tag{4.8}$$

such that $\mathbb{E}[|U(X_T^{\pi^*}+H)|]<\infty$, Z is an element of $\mathbb{H}^2(\mathbb{R}^d)$ and the positive local martingale $XU'(X)\exp(Y)$ is a true martingale. Then

$$\pi_t^{*^i}:=-\frac{U'(X_s)}{X_sU''(X_s)}(Z_s^i+\theta_s^i),\quad s\in[0,T],\quad i=1,\ldots,d_1$$

is a solution to the optimization problem (2.2).

Proof. We first note that $\pi^* \in \Pi^x$ since by the fact that Z is in $\mathbb{H}^2(\mathbb{R}^d)$, there is a constant $C > 0$ such that

$$\mathbb{E}\left[\int_0^T |\pi_t^*|^2 dt\right] \leq C\ \mathbb{E}\left[\int_0^T |Z_t^{\mathcal{H}} + \theta_t^{\mathcal{H}}|^2 dt\right] < \infty.$$

Now let π be an element of Π^x. Let $D := U'(X)\exp(Y)$. Applying Itô's formula and inserting the expression for π^*, we find

$$dD_t = D_t(-\theta_t dW_t^{\mathcal{H}} + Z_t dW_t^{\mathcal{O}}), \quad D_0 = U'(x)\exp(Y_0).$$

Hence

$$D_t = U'(x)\exp(Y_0)\mathcal{E}\left(-\int_0^\cdot \theta_s dW_s^{\mathcal{H}} + \int_0^\cdot Z_s dW_s^{\mathcal{O}}\right)_t, \quad t \in [0,T], \tag{4.9}$$

which is a positive local martingale. Now fix π in Π^x. By definition of X^π and of D, the product formula implies that $X^\pi D$ satisfies

$$DX^\pi = xD_0\mathcal{E}((\pi - \theta)\cdot W^{\mathcal{H}} + Z \cdot W^{\mathcal{O}}).$$

Hence, $X^\pi D$ is a supermartingale and so $\mathbb{E}[D_T X_T^\pi] \leq D_0 x$. By assumption, $X^{\pi^*}D = XU'(X)\exp(Y)$ is a true martingale, so $\mathbb{E}[D_T X_T^{\pi^*}] = D_0 x$. Finally combining the facts above, recalling that $D_T = U'(X_T^{\pi^*} + H)$ and using the concavity of U, we obtain

$$\mathbb{E}[U(X_T^\pi + H) - U(X_T^{\pi^*} + H)] \leq \mathbb{E}[U'(X_T^{\pi^*} + H)(X_T^\pi - X_T^{\pi^*})] \leq 0. \tag{4.10}$$

□

Remark 4.3. *In the previous proof, if we apply the integration by parts formula to $D = U'(X)\exp(Y)$ and $X^\pi - X^{\pi^*}$, we get*

$$U'(X^\pi)\exp(Y)(X^\pi - X^{\pi^*}) = \int_0^\cdot (X_t^\pi - X_t^{\pi^*})dD_t + \int_0^\cdot D_t(\pi_t X_t^\pi - \pi_t^* X_t^{\pi^*})dW_t^{\mathcal{H}}.$$

Thus $U'(X^\pi)\exp(Y)(X^\pi - X^{\pi^*})$ is a local martingale for every admissible strategy π.

Remark 4.4. *Note that using the regularity assumptions of the FBSDE* (4.8)*, we derived that $D := U'(X^{\pi^*})\exp(Y)$ is a true martingale*

$$D_t = U'(x)\exp(Y_0)\mathcal{E}\left(-\theta \cdot W^{\mathcal{H}} + Z^{\mathcal{O}} \cdot W^{\mathcal{O}}\right).$$

4.2 Characterization and verification: complete markets

We adopt the setting and notations of Section 4 with $d_1 = d = 1$ and $H = 0$. In the complete case we can give sufficient conditions for the existence of a solution to the system (4.8). To this end, take note of the following remark.

Remark 4.5. *Similarly to Remark 4.4, we can use* (4.8) *to characterize further the martingale $U'(X^{\pi^*})\exp(Y)$. Applying Itô's formula to $U'(X^{\pi^*})\exp(Y)$ gives rise to*

$$U'(X_t^{\pi^*})\exp(Y_t) = U'(x)\exp(Y_0) - \int_0^t U'(X_s)\exp(Y_s)\theta_s dW_s.$$

Hence we have

$$U'(X_t^{\pi^*})\exp(Y_t) = U'(x)\exp(Y_0)\mathcal{E}(-\theta \cdot W)_t, \quad t \in [0,T]. \tag{4.11}$$

This observation allows to prove the existence of (4.8) under a condition on the risk aversion coefficient $-\frac{U''}{U'}$. Let $\varphi_1(x) := \frac{U'(x)}{U''(x)}$ and $\varphi_2(x) := 1 - \frac{1}{2}\frac{U^{(3)}(x)U'(x)}{|U''(x)|^2}$. We will now give a sufficient condition for the system (4.8) to exhibit a solution. We begin with the following remark.

Remark 4.6. *Note that if φ_2 is constant then the system above decouples. An elementary analysis shows that this happens if and only is U is the exponential, power, logarithmic or quadratic (mean-variance hedging) function. If $U(x) = -\exp(-\alpha_1 x) - \exp(-\alpha_2 x)$ then φ_2 is bounded and Lipschitz and if $U(x) := \frac{x^{\gamma_1}}{\gamma_1} + \frac{x^{\gamma_2}}{\gamma_2}$ then φ_2 is a bounded function.*

Theorem 4.7. *Assume that φ_2 is a continuous bounded function. Then there exists an adapted solution (X,Y,Z) in $\mathbb{S}^2(\mathbb{R}) \times \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R})$ of the FBSDE*

$$\begin{cases} X_t = x - \int_0^t \frac{U'(X_s)}{U''(X_s)}(Z_s+\theta_s)dW_s - \int_0^t \frac{U'(X_s)}{U''(X_s)}(Z_s+\theta_s)\theta_s ds \\ Y_t = 0 - \int_t^T Z_s dW_s - \int_t^T \left[|Z_s+\theta_s|^2\left(1-\frac{1}{2}\frac{U^{(3)}(X_s)U'(X_s)}{|U''(X_s)|^2}\right) - \frac{1}{2}|Z_s|^2\right] ds. \end{cases} \tag{4.12}$$

Moreover, $\mathbb{E}[|U(X_T)|] < \infty$ and $\mathbb{E}[|U'(X_T)|^2] < \infty$.

Proof. Fix $m > 0$ and consider the BSDE

$$Y_t^m = 0 - \int_t^T \left[|Z_s^m+\theta_s|^2 \varphi_2\left((U')^{-1}(U'(x)\exp(m)\mathcal{E}(-\theta\cdot W)_t \exp(-Y_t^m))\right) - \frac{1}{2}|Z_s^m|^2 ds\right] \\ - \int_t^T Z_s^m dW_s.$$

Since φ_2 is bounded, the driver of the BSDE above in (Y^m, Z^m) can be bounded uniformly in m. Hence [20] yields a solution pair $(Y^m, Z^m) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R})$ of this equation with $|Y^m| \le C$, where C does not depend on m and $Z\cdot W$ is a BMO-martingale. In addition (once again using standard arguments as in the proof of Proposition 3.9) we may state that $m \mapsto Y_0^m$ is continuous. Thus there exists an element $m^* > 0$ such that $Y_0^{m^*} = m^*$. Now applying Itô's formula to

$$X^{m^*} := (U')^{-1}(U'(x)\exp(m^*)\mathcal{E}(-\theta\cdot W)\exp(-Y^m)),$$

we check that $(X^{m^*}, Y^{m^*}, Z^{m^*})$ satisfies (4.12). It remains to show that $\mathbb{E}[|U(X_T)|] < \infty$. From the concavity of U we have

$$\mathbb{E}[|U(X_T)|] \le |U'(0)|\mathbb{E}[|X_T|] + |U(0)| + \mathbb{E}[|U'(X_T)X_T|] + |U(0)|.$$

Since $X = x\mathcal{E}(-\frac{U'(X)}{XU''(X)}(Z+\theta)\cdot W)$, $-\frac{U'(x)}{xU''(x)} \le \kappa$ for $x \in \mathbb{R}$ and $(Z+\theta)\cdot W$ is a BMO-martingale, X is a true martingale, and thus $\mathbb{E}[X_T] = x$. Similarly we have that $X_T U'(X_T) = X_T U'(X_T)\exp(Y_T) = xU'(x)\exp(Y_0)\mathcal{E}((-\frac{U'(X)}{XU''(X)}(Z+\theta)-\theta)\cdot W)$ and so $XU'(X)\exp(Y)$ is a true martingale. This proves $\mathbb{E}[|X_T U'(X_T)|] < \infty$. □

5 Links to other approaches

In this section we link our approach to characterizing optimal investment strategies to two other approaches based on the stochastic maximum principle and duality theory, respectively.

5.1 Stochastic maximum principle

This section links our approach in the complete market setting to the approach using the stochastic maximum principle. As this section is solely of illustrative character, we will only give a formal derivation. In particular, we assume here that U and U^{-1} are sufficiently smooth functions with bounded and continuous derivatives. Moreover, we confine the consideration to the complete market case with $d_1 = d = 1$ and $H = 0$ and recall that in this setting, the wealth process is given by

$$X_t^\pi = x + \int_0^t \pi_s dW_s + \int_0^t \pi_s \theta_s ds, \quad t \in [0,T].$$

We consider $J(\pi) := \mathbb{E}[U(X_T^\pi)]$ and set $\tilde{X}_t^\pi := U(X_t^\pi)$. Itô's formula yields

$$d\tilde{X}_t^\pi = U'(U^{-1}(\tilde{X}_t^\pi))\pi_t dW_t + \Big[U'(U^{-1}(\tilde{X}_t^\pi))\pi_t\theta_t + \frac{1}{2}U''(U^{-1}(\tilde{X}_t^\pi))|\pi_t|^2\Big]dt$$

and $J(\pi) = \mathbb{E}[\tilde{X}_T^\pi]$. Applying the maximum principle technique described in [5] (see also [34, Section 4]), we introduce the adjoint equation to get

$$\begin{cases} d\tilde{X}_t^\pi = U'(U^{-1}(\tilde{X}_t^\pi))\pi_t dW_t + \Big[U'(U^{-1}(\tilde{X}_t^\pi))\pi_t\theta_t + \frac{1}{2}U''(U^{-1}(\tilde{X}_t^\pi))|\pi_t|^2\Big]dt,\ \tilde{X}_0^\pi = U(x), \\ -dp_t = \Big[\Big(\frac{U''}{U'}(U^{-1}(\tilde{X}_t^\pi))\theta_t\pi_t + \frac{1}{2}\frac{U^{(3)}}{U''}(U^{-1}(\tilde{X}_t^\pi))|\pi_t|^2\Big)p_t + k_t\frac{U''}{U'}(U^{-1}(\tilde{X}_t^\pi))\pi_t\Big]dt + k_t dW_t,\ p_T = 1. \end{cases} \tag{5.1}$$

We now introduce the corresponding Hamiltonian, defined as

$$H(t,p,k,\pi) := p[U'(U^{-1}(\tilde{X}_t^\pi))\pi_t\theta_t + \frac{1}{2}U''(U^{-1}(\tilde{X}_t^\pi))|\pi_t|^2] + kU'(U^{-1}(\tilde{X}_t^\pi))\pi_t.$$

A formal maximization gives

$$\pi_t^* := -\frac{U'}{U''}(U^{-1}(\tilde{X}_t^\pi))\left[\frac{k_t}{p_t} + \theta_t\right].$$

Plugging this into (5.1) yields

$$\begin{cases} d\tilde{X}_t^\pi = -\frac{|U'|^2}{U''}(U^{-1}(\tilde{X}_t^\pi))\left(\frac{k_t}{p_t} + \theta_t\right)\left[dW_t - \frac{1}{2}\left(\frac{k_t}{p_t} - \theta_t\right)dt\right],\ \tilde{X}_0^\pi = U(x), \\ dp_t = -\left(\frac{k_t}{p_t} + \theta_t\right)^2 p_t\left[-1 + \frac{1}{2}\frac{U^{(3)}U'}{|U''|^2}(U^{-1}(\tilde{X}_t^\pi))\right]dt + k_t dW_t,\ p_T = 1 \end{cases} \tag{5.2}$$

We now relate this system with (4.12) using a Cole-Hopf type transformation. First we plug π^* into (5.2) and obtain

$$\begin{cases} dX_t^{\pi^*} = -\frac{U'}{U''}(X_t^{\pi^*})\left[\frac{k_t}{p_t} + \theta_t\right](dW_t + \theta dt),\ X_0^{\pi^*} = x, \\ dp_t = -\left(\frac{k_t}{p_t} + \theta_t\right)^2 p_t\left[-1 + \frac{1}{2}\frac{U^{(3)}U'}{|U''|^2}(X_t^{\pi^*})\right]dt + k_t dW_t,\ p_T = 1. \end{cases} \tag{5.3}$$

Next consider the system

$$\begin{cases} dX_t^{\pi^*} = -\frac{U'}{U''}(X_t^{\pi^*})(Z_t+\theta_t)(dW_t+\theta dt),\ X_0^{\pi^*}=x, \\ dY_t = \left[(Z_t+\theta_t)^2\big(1-\frac{1}{2}\frac{U^{(3)}(X_t^{\pi^*})U'(X_t^{\pi^*})}{|U''|^2(X_t^{\pi^*})}\big)-\frac{1}{2}|Z_t|^2\right]dt + Z_t dW_t,\ Y_T=0. \end{cases} \tag{5.4}$$

Setting $\tilde{p}_T := \exp(Y_t)$, $\tilde{k} := Z\tilde{p}$ and $\tilde{X} := X$, we see that Itô's formula implies that $(\tilde{p},\tilde{k})$ is a solution of (5.3).

5.2 FBSDE solution via convex duality methods

Let us now turn to a very important link of our approach with convex duality theory. We have seen in Sections 3 and 4 that our approach relies on choosing a process Y such that the quantities $U'(X^{\pi^*}+Y)$ and $X^{\pi^*}U'(X^{\pi^*})\exp(Y)$, respectively, are martingales. In fact, these martingales are not any martingales. For instance in case of a utility function on the whole real line, $U'(X^{\pi^*}+Y)$ is exactly $U'(x+Y_0)\mathcal{E}(-\theta\cdot W^{\mathcal{H}}+\frac{U''}{U'}(X^{\pi^*}+Y)Z^{\mathcal{O}}\cdot W^{\mathcal{O}})$. So in the complete case it is exactly the martingale under which the price is itself a martingale. For utility functions defined on the positive half line this leads directly to duality theory, since it is known from [18] and [21] that (under some growth-type condition on U) the optimal wealth process X^{π^*} and the stochastic process Y^* that solve the dual problem are such that the stochastic process $X^{\pi^*}Y^*$ is a martingale. In addition, in our notation, it is known from the dual approach that Y^* has the form $Y^*=Y_0^*\mathcal{E}(-\theta\cdot W^{\mathcal{H}}+M)$ where M is a martingale orthogonal to $W^{\mathcal{H}}$. Recall that in our case $X^{\pi^*}U'(X^{\pi^*})\exp(Y)$ is a martingale and from (4.9), we have proved that $D:=U'(X^{\pi^*})\exp(Y)$ is exactly of the form $D_0\mathcal{E}(-\theta\cdot W^{\mathcal{H}}+Z^{\mathcal{O}}\cdot W^{\mathcal{O}})$. In other words $Y^*=D$ and the $Z^{\mathcal{O}}$ component appearing in the solution of our FBSDE exactly represents the orthogonal part of the dual optimizer in the language of the convex dual approach. Obviously, this needs to be derived more formally.

Utility functions defined on the real line

The aim of this section is to employ convex duality results to obtain a solution to the forward-backward system (3.10) that has been derived for the case of utility functions defined on the entire real line. To this end, we adopt the convex duality results from [32] and summarize in the following their framework. We also remark that a more general setting is considered in [3]. The utility function $U:\mathbb{R}\to\mathbb{R}$ is assumed to satisfy the Inada conditions $\lim_{x\to-\infty}U'(x)=\infty$ and $\lim_{x\to\infty}U'(x)=0$ as well as the reasonable asymptotic elasticity conditions

$$\liminf_{x\to-\infty}\frac{xU'(x)}{U(x)}>1,\quad \limsup_{x\to\infty}\frac{xU'(x)}{U(x)}<1.$$

The Fenchel-Legendre transform of U is given by

$$V(y):=\sup_{x\in\mathbb{R}}\big\{U(x)-xy\big\},\ y\in\mathbb{R}.$$

Rather than tackling the primal problem (2.2), the dual approach attacks the convex optimization problem

$$v:=\inf_{\mu_T\in\mathcal{C}}\mathbb{E}\Big[V(\mu_T)+\mu_T H\Big], \tag{5.5}$$

where $\mathcal{C}$ denotes the set of all measure densities $y\nu_t := y\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t}$ where $y \geq 0$, $\mathbb{Q} \ll \mathbb{P}$ is a probability measure such that $S^{\mathcal{H}}$ becomes a $\mathbb{Q}$-local martingale and which has finite entropy $\mathbb{E}\big[V\big(\frac{d\mathbb{Q}}{d\mathbb{P}}\big)\big] < \infty$. Two conditions are essential in [32]:

(A1) the set $\mathcal{C}$ is non-empty;

(A2) there exists constants $C_1, C_2 \in \mathbb{R}$ and $\varphi \in \mathbb{H}^2_{loc}(\mathbb{R}^{d_1})$ such that $\int_0^T \varphi_s dS^{\mathcal{H}}_s$ is bounded from below such that the endowment H satisfies a.s.

$$C_1 \leq H \leq C_2 + \int_0^T \varphi_s dS^{\mathcal{H}}_s,$$

where for any integer k, $\mathbb{H}^2_{loc}(\mathbb{R}^k)$ denotes the set of stochastic processes[2] X for which there exists a sequence of stopping times $(\tau_n)_n$ increasing to T such that for every n, $X\mathbf{1}_{[0,\tau_n]}$ is an element of $\mathbb{H}^2(\mathbb{R}^k)$. The key results Theorem 1.1 and Proposition 4.1 from [32] then state that the dual problem (5.5) admits a unique solution $\mu^*_T \in \mathcal{C}$ that satisfies

$$\mu^*_T = U'(X^*_T + H),$$

where $X^*_t := X^{\pi^*}_t = x + \int_0^t \pi^*_s dS^{\mathcal{H}}_s$ is the solution to the primal problem (2.2) and that $X^*\mu^*$ is a true martingale. The following lemma is an easy observation on the structure of the dual optimizer μ^*.

Lemma 5.1. *Under Assumptions* (A1) *and* (A2), *there exists a process* $\nu \in \mathbb{H}^2_{loc}(\mathbb{R}^{d_1})$ *such that the dual optimizer* μ^* *yields the representation*

$$\mu^*_t = \mu^*_0 \mathcal{E}\big(-\theta^{\mathcal{H}} \cdot W^{\mathcal{H}} + \nu \cdot W^{\mathcal{O}}\big)_t, \quad 0 \leq t \leq T.$$

Proof. Let us assume w.l.o.g. that μ^* is normalized, hence the density of a probability measure that is absolutely continuous with respect to $\mathbb{P}$. Thus, there exist $\kappa, \nu \in \mathbb{H}^2_{loc}(\mathbb{R}^{d_1})$ such that

$$\mu^*_t = \mathcal{E}\big(\kappa \cdot W^{\mathcal{H}} + \nu \cdot W^{\mathcal{O}}\big)_t, \quad 0 \leq t \leq T.$$

By Itô's formula, we obtain

$$d(X^*_t\mu^*_t) = \big(X^*_t\mu^*_t\kappa_t + \mu^*_t\pi^*_t\big)dW^{\mathcal{H}}_t + X^*_t\mu^*_t\nu_t dW^{\mathcal{O}}_t + \mu^*_t\pi^*_t\Big(\theta^{\mathcal{H}}_t + \kappa_t\Big)dt.$$

Due to the martingale property of $X^*\mu^*$, the drift must vanish leading to $\kappa = -\theta^{\mathcal{H}}$. □

We are now in the position to construct a solution to the coupled FBSDE (3.10) by making use of the characterization of the dual optimizer from the previous lemma.

Theorem 5.2. *Under the assumptions* (A1) *and* (A2), *the FBSDE* (3.10) *admits a solution* (X, Y, Z) *which satisfies the identity* $\mu^* = U'(X + Y)$.

[2] note that here we identify the set of $\frac{dS^{\mathcal{H}}}{S}$-integrable processes with $\mathbb{H}^2_{loc}$

Proof. Given the optimal wealth process $X^*_t = x+\int_0^t \pi^*_s dS^{\mathcal{H}}_s$, let us define $Y := (U')^{-1}(\mu^*)-X^*$. Note that the Inada conditions imply that $(U')^{-1}$ is continuous. This obviously induces the terminal condition $Y_T = H$ and Itô's formula yields

$$\begin{aligned} dY_t = &-\Big(\theta^{\mathcal{H}}_t \frac{\mu^*_t}{U''\big((U')^{-1}(\mu^*_t)\big)} + \pi^*_t\Big) dW^{\mathcal{H}}_t + \frac{\mu^*_t}{U''\big((U')^{-1}(\mu^*_t)\big)}\nu_t dW^{\mathcal{O}}_t \\ &- \left(\frac{1}{2}\frac{U^{(3)}\big((U')^{-1}(\mu^*_t)\big)|\mu^*_t|^2}{U''\big((U')^{-1}(\mu^*_t)\big)^3}\big(|\theta^{\mathcal{H}}_t|^2 + |\nu_t|^2\big) + \pi^*_t\theta^{\mathcal{H}}_t\right) dt, \end{aligned}$$

which by setting

$$Z^{\mathcal{H}}_t = -\theta^{\mathcal{H}}_t \frac{\mu^*_t}{U''\big((U')^{-1}(\mu^*_t)\big)} - \pi^*_t,\ Z^{\mathcal{O}}_t = \frac{\mu^*_t}{U''\big((U')^{-1}(\mu^*_t)\big)}\nu_t, \tag{5.6}$$

and using the identity $\mu^*_t = U'(X^*_t + Y_t)$ becomes

$$\begin{aligned} dY_t = &\left(-\frac{1}{2}|\theta^{\mathcal{H}}_t|^2 \frac{U^{(3)}(X^*_t+Y_t)|U'(X^*_t+Y_t)|^2}{(U''(X^*_t+Y_t))^3} + |\theta^{\mathcal{H}}_t|^2\frac{U'(X^*_t+Y_t)}{U''(X^*_t+Y_t)} + Z_t\cdot\theta^{\mathcal{H}}_t \right.\\ &\left. -\frac{1}{2}|Z^{\mathcal{O}}_t|^2\frac{U^{(3)}}{U''}(X^*_t+Y_t)\right) dt + Z^{\mathcal{H}}_t dW^{\mathcal{H}}_t + Z^{\mathcal{O}}_t dW^{\mathcal{O}}_t. \end{aligned}$$

This is the backward equation from the system (3.10). However we obtain from (5.6) $\pi^*_t = -\theta^{\mathcal{H}}_t \frac{U'(X^*_t+Y_t)}{U''(X^*_s+Y_s)} - Z^{\mathcal{H}}_t$ which gives rise to

$$X^*_t = x - \int_0^t \big(\theta^{\mathcal{H}}_t \frac{U'(X^*_t+Y_t)}{U''(X^*_s+Y_s)} + Z^{\mathcal{H}}_t\big) dS^{\mathcal{H}}_s, \quad 0 \le t \le T.$$

Hence, putting $X := X^*$ we finish the proof. □

Using the framework of duality theory, we recall below as Proposition 5.3 and Corollary 5.4 an alternative verification theorem to Theorem 3.5 proposed in Section 3 under a different set of assumptions. The difference between these assumptions lies in the fact that the set of admissible strategy consists of predictable and integrable (with respect to $\frac{dS^{\mathcal{H}}}{S}$) processes π the associated wealth process of which is a supermartingale with respect to any probability measure whose density is in $\mathcal{C}$ (with $y = 1$). This set of strategies will be denoted as $\mathscr{H}^{Perm}$ according to [32, Definition 1.1]. We provide a proof of Proposition 5.3 in order to make this paper self-contained.

Proposition 5.3. *Assume conditions (A1)-(A2) are in force. If there exist a process π^* in $\mathscr{H}^{Perm}$, and a triple (μ^*, ν, X^*) solving*

$$\begin{cases} X^*_t = x + \int_0^t \pi^*_s (dW^{\mathcal{H}}_s + \theta^{\mathcal{H}}_s ds) \\ \mu^*_t = U'(H + X^*_T) + \int_t^T \theta^{\mathcal{H}}_s \mu^*_s dW^{\mathcal{H}}_s - \int_t^T \mu^*_s \nu_s dW^{\mathcal{O}}_s, \end{cases} \tag{5.7}$$

such that $X^*\mu^*$ is a true martingale and $\mathbb{E}[\mu^*_T] < \infty$, then π^* is an optimal strategy for

$$\sup_{\pi\in\mathscr{H}^{Perm}} \mathbb{E}[U(X^\pi_T + H)].$$

Proof. Indeed, let π be an admissible strategy in $\mathscr{H}^{Perm}$. Hence X^π is a supermartingale under the probability measure $\mathbb{Q}^*$ defined by $\frac{\mu_T^*}{\mu_0^*}$. Using the convexity of U, we have

$$\begin{aligned}\mathbb{E}[U(X_T^\pi + H)] - \mathbb{E}[U(X_T^* + H)] &\leq \mathbb{E}[U'(X_T^* + H)(X_T^\pi - X_T^*)] \\ &= \mu_0^* \, \mathbb{E}_{\mathbb{Q}^*}[X_T^\pi] - \mathbb{E}[\mu_T^* X_T^*] \leq 0.\end{aligned}$$

Here we have used the supermartingale property of X^π under $\mathbb{Q}^*$ and the martingale property of $\mu^* X^*$. □

Corollary 5.4. *Assume Conditions (A1)-(A2) are in force. If there exists a process triple of adapted processes* (X, Y, Z) *solution to the system* (3.10) *such that* $XU'(X^{\pi^*} + Y)$ *is a true martingale and* $\pi^* := -\theta \frac{U'(X+Y)}{U''(X+Y)} - Z$ *belongs to* $\mathscr{H}^{Perm}$*, then* π^* *is an optimal strategy for*

$$\sup_{\pi \in \mathscr{H}^{Perm}} \mathbb{E}[U(X_T^\pi + H)].$$

Proof. If (X, Y, Z) is solution to (3.10), then by Remark 3.3, (X, Y, Z) is solution to the system (5.7) with $\nu := \frac{U''}{U'}(X + Y)Z^{\mathcal{O}}$. The conclusion follows from Proposition 5.3. □

Utility functions defined on the positive half-line

The aim of this section is to employ convex duality results to obtain a solution to the forward-backward system (4.12) that has been derived for the case of utility functions defined on the positive half-line. We denote by Π^1 the set of admissible strategies with initial capital given by one unit of currency. In the case of zero endowment $H = 0$, the solution to the concave optimization problem (2.2) is achieved by formulating and solving the following dual problem. Denote the convex conjugate of the concave function U by

$$V(y) := \sup_{x>0} \big\{U(x) - xy\big\}, \quad y > 0,$$

and consider wealth processes given by $dX_t^\pi = X_t^\pi \pi_t \frac{d\tilde{S}_t}{\tilde{S}_t}$, $X_0^\pi = x > 0$. Define a family of nonnegative semimartingales via

$$\mathcal{D} := \big\{D \geq 0 : D_0 = 1,\ X^\pi D \text{ is a supermartingale for every } \pi \in \Pi^1\big\}.$$

Then the primal problem (2.2) is solved by finding a solution to the convex dual optimization problem

$$v(y) = \inf_{D_T \in \mathcal{D}} \mathbb{E}\big[V(yD_T)\big], \quad y > 0. \tag{5.8}$$

If this dual problem admits a unique solution $D_T^* \in \mathcal{D}$, then the primal problem (2.2) with $H = 0$ also yields a unique solution

$$\begin{aligned}X_T^{\pi^*} &= x + \int_0^T X_s^{\pi^*} \pi_s^* \frac{d\tilde{S}_s}{\tilde{S}_s} \\ &= x + \int_0^T \alpha_s^* dS_s\end{aligned}$$

$$= I(yY_T^*),$$

with the corresponding optimal control $\pi^* = \frac{\alpha^* \tilde{S}}{X^{\pi^*}}$. Here we have $I = (U')^{-1}$ and $x = -v'(y)$.[3] The case of bounded terminal endowment H is dealt with in [8], where instead of (5.8) a dual problem of the type

$$v(y) = \inf_{D_T \in \mathcal{D}} \mathbb{E}\big[V(yD_T) + yD_T H\big], \quad y > 0,$$

is considered with a different choice of the domain $\mathcal{D}$. The case of general integrable H has been studied in [16], using the original dual problem (5.8) but a modification of the domain $\mathcal{D}$ for the case $H = 0$. A ubiquitous property of the convex duality method is that once the primal and the dual optimizers are obtained, their product $X^{\pi^*} D^*$ is a nonnegative true martingale (hence uniformly integrable), see [21] for an economic interpretation. In the context of utility maximization with bounded random endowments, this martingale property of $X^{\pi^*} D^*$ is pointed out in [8, Remark 4.6]. This martingale property of $X^{\pi^*} D^*$ constitutes the first main ingredient for deriving a solution to the forward-backward equation (4.12). A second main ingredient is constituted by the characterization of the dual domain $\mathcal{D}$. Note that in the setting of continuous processes, $\mathcal{D}$ is the family of all non-negative supermartingales (see e.g. [21, 16]). According to a well known result, every nonnegative càdlàg supermartingale $D \in \mathcal{D}$ admits a unique multiplicative decomposition

$$D = AM$$

where A is a predictable, non-increasing process such that $A_0 = 1$ and M is càdlàg local martingale. However, [23] characterizes the elements of $D \in \mathcal{D}$ by the multiplicative decomposition

$$D = A\mathcal{E}(-\theta^{\mathcal{H}} \cdot W^{\mathcal{H}} + K \cdot W^{\mathcal{O}}), \tag{5.9}$$

where A is a predictable non-increasing process such that $A_0 = 1$ and $K \in \mathbb{H}^2_{loc}(\mathbb{R}^{d_2})$ (see [23, Proposition 3.2]). Using that the Fenchel-Legendre transform V is strictly decreasing, [23, Corollary 3.3] shows that the dual optimizer is a (continuous) local martingale and admits the representation

$$D^* = \mathcal{E}\big(-\theta^{\mathcal{H}} \cdot W^{\mathcal{H}} + K^* \cdot W^{\mathcal{O}}\big) \tag{5.10}$$

for a uniquely determined $K^* \in \mathbb{H}^2_{loc}(\mathbb{R}^{d_2})$. If $v(y) = \mathbb{E}\Big[V(yD_T^*)\Big] < \infty$, then we can check that the optimal K^* actually belongs to $\mathbb{H}^2(\mathbb{R}^{d_2})$. This is done in the following lemma whose proof is in the same spirit as the one of [22, Lemma 3.2].

Lemma 5.5. *If for some $y > 0$ we have*

$$v(y) = \inf_{\nu \in \mathbb{H}^2_{loc}(\mathbb{R}^{d_2})} \mathbb{E}\Big[V\Big(y\mathcal{E}\big(-\theta^{\mathcal{H}} \cdot W^{\mathcal{H}} + \nu \cdot W^{\mathcal{O}}\big)\Big)\Big] < \infty,$$

[3]This is equivalent to $u'(x) = y$ where $u(x) = \sup_\pi \mathbb{E}\big[U(X_T^\pi + H)\big]$. The differentiability of both $v(y)$ and $u(x)$ are shown in [8].

we obtain

$$v(y) = \inf_{\nu \in \mathbb{H}^2(\mathbb{R}^{d_2})} \mathbb{E}\Big[V\Big(y\mathcal{E}\big(-\theta^{\mathcal{H}} \cdot W^{\mathcal{H}} + \nu \cdot W^{\mathcal{O}}\big)\Big)\Big],$$

i.e. the optimal K^* minimizing $v(y)$ can be assumed to belong to $\mathbb{H}^2(\mathbb{R}^{d_2})$.

Proof. We introduce the family of stopping times

$$\tau^n := \inf\big\{t > 0 : \int_0^t \big(|\theta_s^{\mathcal{H}}|^2 + |K_s^*|^2\big)ds \geq n\big\},\ n \in \mathbb{N}.$$

Let $y > 0$. Then we have

$$\begin{aligned} v(y) &= \mathbb{E}\Big[V\Big(y\mathcal{E}_T\big(-\theta^{\mathcal{H}} \cdot W^{\mathcal{H}} + K^* \cdot W^{\mathcal{O}}\big)\Big)\Big] \\ &= \mathbb{E}\Big[\mathbb{E}\Big[V\Big(y\mathcal{E}_T\big(-\theta^{\mathcal{H}} \cdot W^{\mathcal{H}} + K^* \cdot W^{\mathcal{O}}\big)\Big)\big|\mathcal{F}_{\tau^n}\Big]\Big] \\ &\geq \mathbb{E}\Big[V\Big(y\mathcal{E}_{\tau^n}\big(-\theta^{\mathcal{H}} \cdot W^{\mathcal{H}} + K^* \cdot W^{\mathcal{O}}\big)\Big)\Big], \end{aligned}$$

where the last line follows by Jensen's inequality. Continuing the last line and recalling that V is a strictly convex function, we have

$$\begin{aligned} v(y) &\geq \mathbb{E}\Big[V\Big(y\exp\big(\int_0^{\tau^n}\big(-\theta_s^{\mathcal{H}}dW_s^{\mathcal{H}} + K_s^*dW_s^{\mathcal{O}}\big)\big)\exp\big(-\frac{1}{2}\int_0^{\tau^n}\big(|\theta_s^{\mathcal{H}}|^2 + |K_s^*|^2\big)ds\big)\Big)\Big] \\ &\geq V\left(\mathbb{E}\Big[y\exp\big(\int_0^{\tau^n}\big(-\theta_s^{\mathcal{H}}dW_s^{\mathcal{H}} + K_s^*dW_s^{\mathcal{O}}\big)\big)\exp\big(-\frac{1}{2}\int_0^{\tau^n}\big(|\theta_s^{\mathcal{H}}|^2 + |K_s^*|^2\big)ds\big)\Big]\right) \\ &\geq V\left(y\exp\left(\mathbb{E}\Big[-\frac{1}{2}\int_0^{\tau^n}\big(|\theta_s^{\mathcal{H}}|^2 + |K_s^*|^2\big)ds\Big]\right)\right), \end{aligned}$$

where Jensen's inequality has been used twice. By continuity of V and of the exponential function, it follows from the monotone convergence theorem that

$$\begin{aligned} v(y) &\geq \lim_{n\to\infty} V\Big(\exp\Big(-\frac{1}{2}\mathbb{E}\Big[\int_0^{\tau^n}\big(|\theta_s^{\mathcal{H}}|^2 + |K_s^*|^2\big)ds\Big]\Big)\Big) \\ &= V\Big(\exp\Big(-\frac{1}{2}\mathbb{E}\Big[\int_0^{T}\big(|\theta_s^{\mathcal{H}}|^2 + |K_s^*|^2\big)ds\Big]\Big)\Big). \end{aligned}$$

Since $v(y) < \infty$ and $V\big(\exp(-\infty)\big) = V(0) = U(\infty) = \infty$, it follows that

$$\mathbb{E}\Big[\int_0^T\big(|\theta_s^{\mathcal{H}}|^2 + |K_s^*|^2\big)ds\Big] < \infty.$$

We deduce that $K^* \in \mathbb{H}^2(\mathbb{R}^{d_2})$. $\square$

Now using that $X^{\pi^*}Y^*$ is a true martingale and that the dual optimizer Y^* is a local martingale satisfying (5.10), we get the following result.

Theorem 5.6. *Let H be a non-negative bounded random endowment and assume that the coefficient of relative risk aversion $-\frac{xU''(x)}{U'(x)}$ satisfies*

$$\limsup_{x\to\infty}\left(-\frac{xU''(x)}{U'(x)}\right)<\infty. \tag{5.11}$$

Then there exists $x_0>0$ such that for all $x>x_0$ the coupled FBSDE (4.8) has a solution (X,Y,Z) such that $X_0=x$. In addition, X is the optimal wealth of the problem (2.2) and the dual optimizer Y^* associated with it is given by $Y^*=U'(X)\exp(Y)$ (so that $yY_T^*=U'(X_T+H)$).

Proof. The existence of $x_0>0$ such that for every $x>x_0$ the quantity

$$u(x)=\sup_{\pi\in\Pi^x}\mathbb{E}\big[U\big(X_T^\pi+H\big)\big]=\mathbb{E}\big[U\big(X_T^{\pi^*}+H\big)\big]$$

is finite has been shown in [8]. We set $X^*:=X^{\pi^*}$. Also recall that we have $y=u'(x)>0$ for $x>x_0$ and that we have

$$\mathbb{E}\Big[yX_T^*Y_T^*\Big]=xy.$$

Moreover, $yY_T^*=U'(X_T^*+H)$. We define the true martingale $\alpha:=yX^*Y^*$. We set $Y:=\log(\alpha)-\log(X^*)-\log(U'(X^*))$. We observe

$$\begin{aligned}Y_t&=\log\left(\frac{\alpha_t}{X_t^*U'(X_t^*)}\right)\\&=\log\left(\frac{yY_t^*}{U'(X_t^*)}\right)\\&=\log(y)+\log(Y_t^*)-\log(U'(X_t^*).\end{aligned}$$

Recall that by definition of X^* and Y^* we have

$$dY_t^*=Y_t^*\left(-\theta_t^{\mathcal{H}}dW_t^{\mathcal{H}}+K_t^*dW_t^{\mathcal{O}}\right)$$

and

$$dX_t^*=X_t^*\left(\pi_t^*dW_t^{\mathcal{H}}+\pi_t^*\theta_t^{\mathcal{H}}dt\right).$$

Hence

$$\begin{aligned}dY_t=&-\theta_t^{\mathcal{H}}dW_t^{\mathcal{H}}+K_t^*dW_t^{\mathcal{O}}-\frac{1}{2}(|\theta_t^{\mathcal{H}}|^2+|K_t^*|^2)dt\\&-\frac{U''(X_t^*)}{U'(X_t^*)}(\pi_t^*X_t^*dW_t^{\mathcal{H}}+\pi_t^*X_t^*\theta_t^{\mathcal{H}}dt)\\&-\frac{1}{2}\frac{U^{(3)}(X_t^*)U'(X_t^*)-(U''(X_t^*))^2}{(U'(X_t^*))^2}|\pi_t^*X_t^*|^2dt.\end{aligned}$$

We define

$$Z_t^{\mathcal{H}}:=-\theta_t^{\mathcal{H}}-\frac{U''(X_t^*)}{U'(X_t^*)}\pi_t^*X_t^*,$$

so that $\pi_t^* X_t^* = -(Z_t^{\mathcal{H}} + \theta_t^{\mathcal{H}}) \frac{U'(X_t^*)}{U''(X_t^*)}$, and

$$Z_t^{\mathcal{O}} := K_t^*.$$

Then

$$\begin{aligned}
dY_t &= Z_t^{\mathcal{H}} dW_t^{\mathcal{H}} + Z_t^{\mathcal{O}} dW_t^{\mathcal{O}} - \frac{1}{2}(|\theta_t^{\mathcal{H}}|^2 + |K_t^*|^2)dt \\
&\quad + \left[\theta_t^{\mathcal{H}}(Z_t^{\mathcal{H}} + \theta_t^{\mathcal{H}}) - \frac{1}{2} \frac{U^{(3)}(X_t^*)U'(X_t^*) - (U''(X_t^*))^2}{(U'(X_t^*))^2} \frac{|Z_t^{\mathcal{H}} + \theta_t^{\mathcal{H}}|^2 |U'(X_t^*)|^2}{|U''(X_t^*)|^2} \right] dt \\
&= Z_t^{\mathcal{H}} dW_t^{\mathcal{H}} + Z_t^{\mathcal{O}} dW_t^{\mathcal{O}} + \left[|Z_t^{\mathcal{H}} + \theta_t^{\mathcal{H}}|^2 \left(1 - \frac{1}{2} \frac{U^{(3)}(X_t^*)U'(X_t^*)}{|U''(X_t^*)|^2} \right) - \frac{1}{2}|Z_t^{\mathcal{H}}|^2 \right] dt.
\end{aligned}$$

Finally note that by construction $Y_T = \log\left(\frac{U'(X_T^*+H)}{U'(X_{T^*})}\right)$. Hence, $(X, Y, Z) = (X^*, Y, Z)$ is a solution to (4.8) and

$$yY^* = U'(X)\exp(Y).$$

□

Let us recall that the *absolute risk aversion* of U is defined by $ARA(x) := -\frac{U''(x)}{U'(x)}$ and the *risk tolerance* as $\frac{1}{ARA(x)}$. We say that U has *hyperbolic absolute risk aversion (HARA)* if and only if its risk tolerance $\frac{1}{ARA(x)}$ is linear in x. More precisely, it can be shown that a utility function U is *HARA* if and only if

$$U(x) = \frac{1-\gamma}{\gamma}\Big(\frac{ax}{1-\gamma} + b\Big)^{\gamma}, \quad \frac{ax}{1-\gamma} + b > 0,$$

for given real numbers $\gamma, a, b \in \mathbb{R}$.

Corollary 5.7. *Assume that U is HARA. Then there exists a constant $\kappa \in \mathbb{R}$ such that the backward equation from* (4.8) *can be written as*

$$\begin{aligned}
Y_t &= \log\Big(\frac{U'(X_T^* + H)}{U'(X_T^*)}\Big) - \int_t^T Z_s dW_s - \int_t^T \Big(-\frac{1}{2}|Z_s|^2 + \kappa |Z_s^{\mathcal{H}} + \theta_s^{\mathcal{H}}|^2 \Big) ds \qquad (5.12) \\
&= \log\Big(\frac{U'(X_T^* + H)}{U'(X_T^*)}\Big) - \int_t^T Z_s dW_s - \int_t^T g(s, Z_s) ds.
\end{aligned}$$

Proof. Notice that for the risk tolerance

$$f(x) := \frac{1}{ARA(x)} = -\frac{U'(x)}{U''(x)}$$

the equation

$$f'(x) = -1 + \frac{U'(x)U^{(3)}(x)}{|U''(x)|^2}$$

holds. U being *HARA* implies that f is linear in x. It follows that there exist constants $c, d \in \mathbb{R}$ such that $f^{'}(x) = cx + d$. Hence the BSDE from (4.8) can also be written as

$$\begin{aligned}
Y_t &= \log\Big(\frac{U^{'}(X_T^* + H)}{U^{'}(X_T^*)}\Big) - \int_t^T Z_s dW_s - \int_t^T \Big(-\frac{1}{2}|Z_s|^2 + (\frac{1}{2} - \frac{1}{2}f^{'}(X_s^*))|Z_s^{\mathcal{H}} + \theta_s^{\mathcal{H}}|^2\Big) ds \\
&= \log\Big(\frac{U^{'}(X_T^* + H)}{U^{'}(X_T^*)}\Big) - \int_t^T Z_s dW_s - \int_t^T \Big(-\frac{1}{2}|Z_s|^2 + \kappa|Z_s^{\mathcal{H}} + \theta_s^{\mathcal{H}}|^2\Big) ds,
\end{aligned}$$

for $\kappa = \frac{1}{2} - \frac{1}{2}c$. □

Obviously the driver of the BSDE (5.12), $g(s, z)$, satisfies the quadratic growth condition

$$|g(s, z)| \leq \alpha + \frac{\gamma}{2}|z|^2$$

for suitably chosen real numbers $\alpha, \gamma > 0$. In this setting [6, Theorem 2] yields the following result.

Corollary 5.8. *If* $\xi = \log\Big(\frac{U^{'}(X_T^*+H)}{U^{'}(X_T^*)}\Big)$ *satisfies* $\mathbb{E}\Big[e^{\gamma|\xi|}\Big] < \infty$, *then the BSDE* (5.12) *admits a solution* (Y, Z) *such that* Y *is continuous and* $Z \in \mathbb{H}^2_{loc}(\mathbb{R}^d)$.

5.3 The case of power utility with general endowment

We finally show that our FBSDE approach yields a constructive solution to a central problem in utility optimization: the case of power utility with general endowment. We know from duality theory that an optimal solution exists. We aim at describing the solution more explicitly, this way proving for instance that the optimal strategy is square integrable (whereas convex duality theory usually only ensures measurability), and characterizing it in terms of the solution to an FBSDE which is potentially amenable to numerical computation. We will use definitions and notations of Section 4. Let $U(x) := \frac{x^\gamma}{\gamma}$ with γ a fixed parameter in $(0, 1)$. Let H be a positive bounded $\mathcal{F}_T$-measurable random variable, where we recall that $(\mathcal{F}_t)_{t\in[0,T]}$ is the filtration generated by $W = (W^{\mathcal{H}}, W^{\mathcal{O}})$. We recall that we denote by Π^x the set of admissible strategies with initial capital x which is now defined by

$$\Pi^x := \Big\{\pi \in \mathbb{H}^2(\mathbb{R}^{d_1}),\ X_0^\pi = x, \Big\}. \tag{5.13}$$

where X^π stands for the associated wealth process given by

$$X_t^\pi := X_0^\pi + \int_0^t \pi_s X_s^\pi dS_s^{\mathcal{H}}, \quad t \in [0, T].$$

Here $\pi^i, i = 1, \ldots, d_1$ denotes the proportion of wealth invested in the stock. Again, we extend π to $\mathbb{R}^d$ via $\tilde{\pi} := (\pi^1, \ldots, \pi^{d_1}, 0, \ldots, 0)$ and make the convention that we write π instead of $\tilde{\pi}$. Thus, we have

$$X_t^\pi = x\mathcal{E}\left(\int_0^\cdot \pi_r dS_r^{\mathcal{H}}\right)_t, \quad t \in [0, T].$$

Note that this setting covers the case of a purely orthogonal endowment of the form $H := \phi(S_T^{\mathcal{O}})$ where ϕ is positive. Now we can proceed to the analysis of the problem

$$\sup_{\pi \in \Pi^x} \mathbb{E}\left[\frac{(X_T^\pi + H)^\gamma}{\gamma}\right]. \tag{5.14}$$

It is known in this case is that an optimal strategy exists (see [16]) which is located in a much larger space than Π^x. In particular it is not proved that the optimal strategy is square integrable. For the characterization of this optimal strategy we can write the Hamilton-Jacobi-Bellman PDE in the Markovian case. Yet, it is not clear how it can be solved. We shall show by a combination of duality theory and BSDE concepts that the optimal strategy belongs to the space Π^x, and that it can be characterized as a solution of a system of FBSDE. Let us be more precise.

Theorem 5.9. *There exists $x_0 > 0$ such that for every $x > x_0$, the system*

$$\begin{cases} X_t = x + \int_0^t \frac{X_s(Z_s^{\mathcal{H}} + \theta_s^{\mathcal{H}})}{1-\gamma} dW_s^{\mathcal{H}} + \int_0^t \theta_s^{\mathcal{H}} \frac{X_s(Z_s^{\mathcal{H}} + \theta_s^{\mathcal{H}})}{1-\gamma} ds \\ Y_t = (\gamma - 1)\log\left(1 + \frac{H}{X_T}\right) - \int_t^T Z_s dW_s - \int_t^T \left(\frac{\gamma}{2(\gamma-1)}|Z_s^{\mathcal{H}} + \theta_s^{\mathcal{H}}|^2 - \frac{1}{2}|Z_s|^2\right) ds \end{cases} \tag{5.15}$$

admits an adapted solution (X, Y, Z). If in addition $Z^{\mathcal{H}} = (Z^1, \ldots, Z^{d_1})$ is in $\mathbb{H}^2(\mathbb{R}^{d_1})$, then

$$\pi^{*i} := \frac{1}{1-\gamma}(Z^i + \theta^i), \; i = 1, \ldots, d_1 \tag{5.16}$$

is the optimal solution to the maximization problem (5.14).

Proof. First note that the system (5.15) is exactly the system (4.8) with $U(x) = \frac{x^\gamma}{\gamma}$. Hence due to Theorem 5.6 there exists $x_0 > 0$ such that the system (5.15) admits a solution (X, Y, Z) if $x > x_0$. We fix $x > x_0$ and consider the associated solution (X, Y, Z) (that is $X_0 = x$). In addition, we know from Theorem 5.6 that $X = X^*$. Hence π^* is given by (5.16). It just remains to prove that π^* is in Π^x, which is a direct consequence of the fact that Z is in $\mathbb{H}^2(\mathbb{R}^{d_1})$. Indeed, this fact plus the fact that θ is bounded implies that

$$\mathbb{E}\left[\int_0^T |\pi_s|^2 ds\right] = \frac{1}{(\gamma-1)^2}\mathbb{E}\left[\int_0^T |Z_s^{\mathcal{H}} + \theta_s|^2 ds\right] < \infty. \tag{5.17}$$

$\square$

Remark 5.10. *Note that since we know that the dual optimizer Y^* is given by $Y^* = U'(X)\exp(Y)$, it is clear that $XU'(X)\exp(Y)$ is a true martingale. Hence the square integrability of Z implies that the condition of Theorem 4.2 is satisfied: $\mathbb{E}[(X_T + H)^\gamma] < \infty$. Finally notice that $Z^{\mathcal{O}}$ is in $\mathbb{H}^2(\mathbb{R}^{d_2})$ by Lemma 5.5.*

So the only element missing in the proof is to establish that $Z^{\mathcal{H}}$ is in $\mathbb{H}^2(\mathbb{R}^{d_1})$ which would imply that π^* given by (5.16) is in $\mathbb{H}^2(\mathbb{R}^{d_1})$ as proved above in (5.17). This question requires a deeper analysis of the system and is currently investigated by the authors.

Acknowledgments

The authors are very grateful to an anonymous Referee for illuminating comments and for indicating us References [3] and [32]. Horst acknowledges financial support through the SFB 649 *Economic Risk*. Imkeller and Réveillac are grateful to the *DFG Research Center MATHEON, Project E2*. Hu is partially supported by the Marie Curie ITN Project *Deterministic and Stochastic Controlled Systems and Applications*, call: F97-PEOPLE-2007-1-1-ITN, n. 213841-2. Zhang acknowledges support by *DFG IRTG 1339 SMCP*.

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