Recent developments in mixed integer linear programming formulations for the resource-constrained project scheduling problem

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Outline

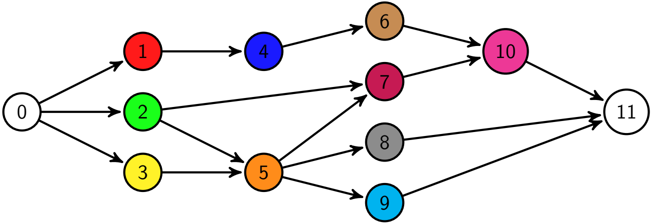
1. RCPSP
2. MILP for RCPSP
3. Standard and novel MILP formulations
   - Pseudo-polynomial time-indexed formulations
   - Extended time-indexed formulations and valid inequalities
   - Compact sequencing and natural date variable formulations
   - Compact event-based formulations
4. Synthesis of theoretical and experimental results
5. Perspectives
6. References

The Resource-Constrained Project Scheduling Problem (RCPSP)

- A central problem in many industrial applications
  - Project management, manufacturing, process industry, parallel processor architectures
- The "standard" RCPSP : An NP-hard problem posing a computational challenge since the the eighties
  - Benchmark instances [Patterson 1984], [Alvarez-Valdes and Tamarit 1989], [Kolisch, Sprecher and Drexl 1995,1997] (**PSPLIB**), [Baptiste and Le Pape 2000], [Carlier and Néron 2003].
  - 686 citations on PSPLIP (Google Scholar) 1/1/2014
  - 48 (out of 480) still open instances with 60 activities and 4 resources from PSPLIB

The RCPSP : data

- R set of resources, limited constant availability $B_k \geq 0$,
- A set of activities, duration $p_i \geq 0$, resource requirement $b_{ik} \geq 0$ on each resource k,
- E set of precedence constraints (i, j), $i, j \in A$, $i < j$
- $\mathcal{T}$ time interval (scheduling horizon)



$|R| = 1, B = 4, \mathcal{T} = [0, 30)$

i	p_i	b_i
1	3	2
2	5	3
3	1	3
4	3	1
5	2	1
6	4	2
7	5	3
8	6	1
9	4	1
10	4	1

The RCPSP : variables, objective and constraints

- $S_i \geq 0$ start time of activity i
- $C_{\max}$ makespan or total project duration

RCPSP (conceptual formulation)

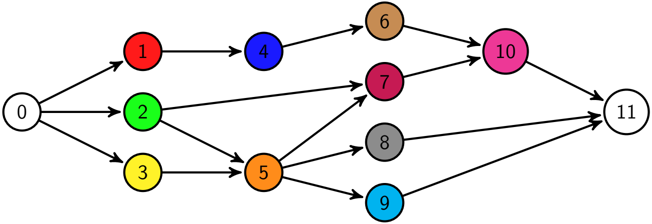
$$\min C_{\max} = \max_{i \in A} S_i + p_i$$

$$\text{s.t.} \begin{cases} S_j \geq S_i + p_i & (i,j) \in E & \textit{Precedence constraints} \\ \sum_{i \in A(t)} b_{ik} \leq B_k & t \in \mathcal{T}, k \in R & \textit{Resource constraints} \\ S_j \geq 0 & i \in A & \end{cases}$$

where $A(t) = \{j \in A | t \in [S_j, S_j + p_j)\}, \forall t \in \mathcal{T}$

The RCPSP : solution example

$|R| = 1, B = 4, \mathcal{T} = [0, 30)$



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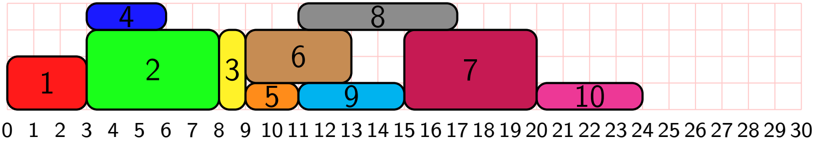
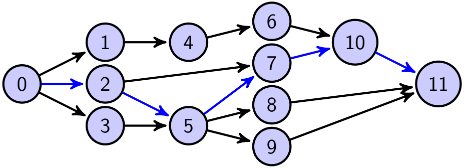


The RCPSP : complexity, variants and methods

- Strongly NP-hard
- Generalizes single/parallel machine, X-shop problems
- Many relevant variants
  - Other objectives : $\min \sum_{i \in A} w_i(S_i + p_i)$
  - Generalized precedence constraints $S_j \geq S_i + l_{ij}$
  - Setup times, multiple modes, non renewable resources, ...
  - Uncertainty $p_i \in [p_i^{\min}, p_i^{\max}]$, $p_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$
- Exact and heuristic Methods
  - Heuristics and metaheuristics
  - Dedicated branch and bound methods
  - Specific lower bounds
  - Constraint programming (CP) or hybrid SAT/CP
  - Mixed Integer Linear Programming (MILP)

The RCPSP : pre-processing and trivial bounds

- Upper bounds $|T|$: parallel or serial list scheduling heuristics
- CPM lower bound : longest 0–$n+1$ path (16)
- Resource lower bound $\max_{k\in R}\sum_{i\in A} b_{ik} * p_i/B_k$ ($16.5 \rightarrow 17$)
- Reduce time windows $[ES_i, LS_i]$ by constraint propagation :



- Temporal constraint propagation TW
- Temporal + Resource constraint propagation TW^+

$UB = 24$ (parallel SGS / Min LFT rule)

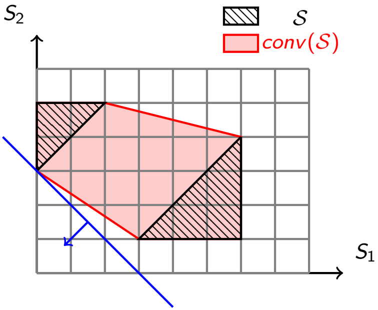
i	p_i	b_i	TW	TW^+
1	3	2	$[0, 10]$	$[0, 10]$
2	5	3	$[0, 8]$	$[0, 6]$
3	1	3	$[0, 12]$	$[0, 12]$
4	3	1	$[3, 13]$	$[3, 13]$
5	2	1	$[5, 13]$	$[6, 13]$
6	4	2	$[6, 16]$	$[8, 16]$
7	5	3	$[7, 15]$	$[9, 15]$
8	6	1	$[7, 18]$	$[8, 18]$
9	4	1	$[7, 20]$	$[8, 20]$
10	4	1	$[12, 20]$	$[18, 20]$
11	0	0	$[16, 24]$	$[22, 24]$

Outline

The scheduling polyhedron

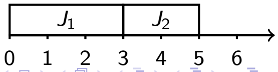
Example (release dates r_i, deadlines $\tilde{d}_i$)

$|A| = 2$, $|R| = 1$, $b_1 = b_2 = B = 1$
$p_1 = 3$, $p_2 = 2$, $r_1 = 0$, $r_2 = 1$, $\tilde{d}_1 = 9$, $\tilde{d}_2 = 7$).
Objective function $f(S) = S_1 + S_2 + p_1 + p_2$.



$$(P) \min S_1 + S_2 + 5$$
$$S_1 \geq 0$$
$$S_2 \geq 1$$
$$S_1 \leq 6$$
$$S_2 \leq 5$$
$$S_2 \geq S_1 + 3 \vee S_1 \geq S_2 + 2$$

(P) can be solved by LP on $conv(\mathcal{S})$



MILP for RCPSP : principle

- Let $\mathbf{S}$, $\mathbf{c}\,\mathbf{S}$ and $\mathcal{S}$ denote the start time vector,the linear objective and the feasible set of the RCPSP.
- Let $\mathbf{x}$ denote a vector of additional p binary variables.
- The MILP $\min_{\mathbf{S},\mathbf{x}}\{\mathbf{c}\,\mathbf{S}|\mathbf{M}\,\mathbf{S}+\mathbf{N}\,\mathbf{x}\leq \mathbf{q},\mathbf{S}\geq \mathbf{0},\mathbf{x}\in\{0,1\}^p\}$ is a correct formulation for the RCPSP if we have

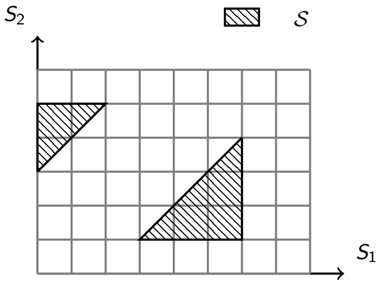
$$\mathcal{S}=\{\mathbf{S}\geq\mathbf{0}|\exists\mathbf{x}\in\{0,1\}^p,\mathbf{M}\,\mathbf{S}+\mathbf{N}\,\mathbf{x}\leq\mathbf{q}\}$$

- $\mathcal{S}$ can be searched by branch and bound (and cut)
  - Branching : tree search on $\mathbf{x}$
  - Bounding : solve at each node the LP relaxation by considering unfixed $x_q\in[0,1]$ (and possibly incorporating valid inequalities)

The bound is tight if the relaxed set $\tilde{\mathcal{S}}=\{\mathbf{S}\geq\mathbf{0}|\exists\mathbf{x}\in[0,1]^p,\mathbf{M}\,\mathbf{S}+\mathbf{N}\,\mathbf{x}\leq\mathbf{q}\}$ is close to $conv(\mathcal{S})$.

MILP for RCPSP : example and issues

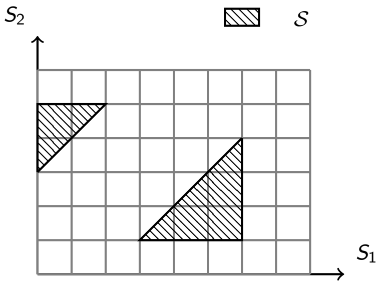
- Design a MIP formulation for the scheduling problem
- Solve by branch-and-bound



MILP for RCPSP : example and issues

- Design a MIP formulation for the scheduling problem
- Solve by branch-and-bound

$$
\begin{aligned}
(P)\ \min S_1 + S_2 + 5 & \\
S_1 &\geq 0 \\
S_2 &\geq 1 \\
S_1 &\leq 6 \\
S_2 &\leq 5 \\
S_2 - S_1 + 8x &\geq 3 \\
S_1 - S_2 + 7(1-x) &\geq 2 \\
x &\in \{0,1\}
\end{aligned}
$$



The projection of the MILP feasible set on **S** maps $\mathcal{S}$

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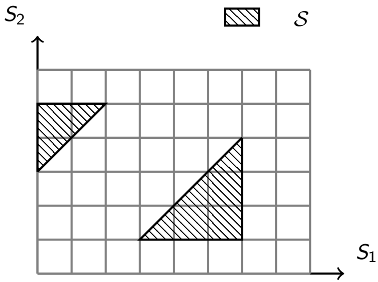

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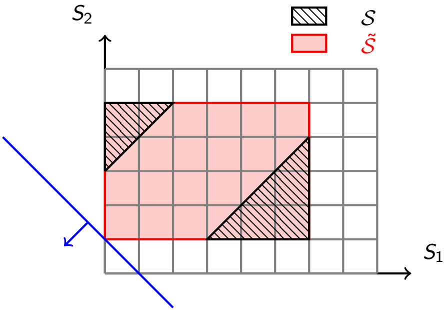

Root node LB=6

issue $x = 0.5$ always feasible

MILP for RCPSP : example and issues

- Design a MIP formulation for the scheduling problem
- Solve by branch-and-bound

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\begin{aligned}
(P) \min S_1 + S_2 + 5 \\
S_1 &\geq 0 \\
S_2 &\geq 1 \\
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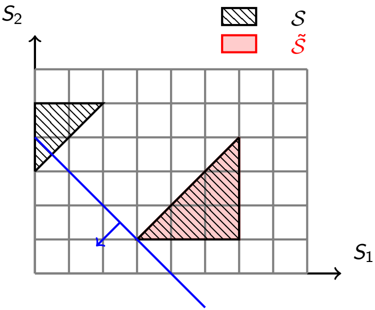
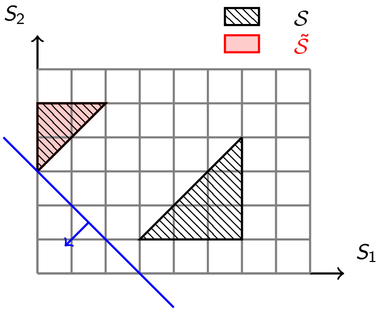


Left node $x = 1$, obj=9

MILP for RCPSP : example and issues

- Design a MIP formulation for the scheduling problem
- Solve by branch-and-bound

$$
\begin{aligned}
(P)\ \min S_1 + S_2 + 5 & \\
S_1 &\geq 0 \\
S_2 &\geq 1 \\
S_1 &\leq 6 \\
S_2 &\leq 5 \\
S_2 - S_1 + 8x &\geq 3 \\
S_1 - S_2 + 7(1-x) &\geq 2 \\
x &\in \{0,1\}
\end{aligned}
$$



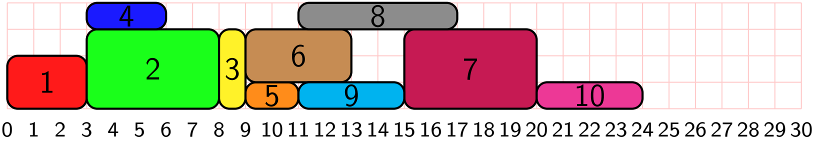
Right node $x = 0$, obj=8

MILP for RCPSP : tradeoffs

- Designing pseudo-polynomial or extended formulations
  - Pros : obtain better LP relaxations, early node pruning in the search tree
  - Cons : increase of the MILP size (number of binary variables, constraints) towards pseudo-polynomial and even exponential sizes (need of column and cut generation techniques)
- Design compact formulations (polynomial size)
  - Pros : fast node evaluation, mode nodes explored
  - Cons : need to generate cuts

MILP for RCPSP : families of formulations

[Queyranne and Schulz 1994] classify the scheduling MILP for scheduling according to the type of decision variables, each yielding different families of valid inequalities.



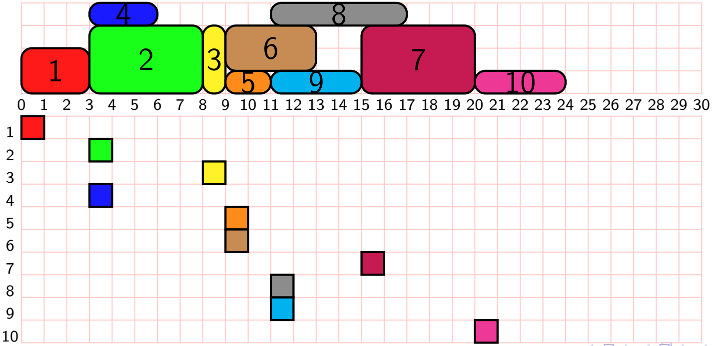
1. Time-indexed variables
2. Linear-ordering variables → Strict-order or sequencing variables
3. Positional dates and assignment variables → Event-based formulations

Outline

Outline

Time-indexed pulse variables

- For integer data, $\mathcal{S}$ can be restricted to its integer vectors $\mathcal{S}^{\text{int}}$.
- "Pulse" binary variable $x_{it} = 1 \Leftrightarrow S_i = t$, for $t \in T = \mathcal{T} \cap \mathbb{N}$
- Pseudo-polynomial number of variables $|A||T|$



The aggregated time-indexed formulation

- $S_i = \sum_{t \in T} t\, x_{it}$
- $A(t) = \{i \in A | \exists \tau \in \{t - p_i + 1, \ldots, t\}, x_{i\tau} = 1\}$

$$
\begin{aligned}
(DT)\ \text{Min.}\ & \sum_{t \in T} t x_{n+1,t} \\
\text{s.t.}\ & \sum_{t \in T} t x_{jt} - \sum_{t \in H} t x_{it} \geq p_i \quad (i,j) \in E \\
& \sum_{i \in V} \sum_{\tau = t - p_i + 1}^{t} b_{ik} x_{i\tau} \leq B_k \quad t \in T;\ k \in \mathcal{R} \\
& \sum_{t \in T} x_{it} = 1 \quad i \in A \\
& x_{it} \in \{0,1\} \quad i \in A
\end{aligned}
$$

[Pritsker et al. 1969]

Back to the small example : a better relaxation...

$$
\begin{aligned}
&(P) \min S_1 + S_2 + 5\\
&S_1 = x_{1,1} + 2x_{1,2} + 3x_{1,3} + 4x_{1,4} + 5x_{1,5} + 6x_{1,6}\\
&S_2 = x_{2,1} + 2x_{2,2} + 3x_{2,3} + 4x_{2,4} + 5x_{2,5}\\
&x_{1,0} + x_{1,1} + x_{1,2} + x_{1,3} + x_{1,4} + x_{1,5} + x_{1,6} = 1\\
&x_{2,1} + x_{2,2} + x_{2,3} + x_{2,4} + x_{2,5} = 1\\
&x_{1,0} + x_{1,1} + x_{2,1} \leq 1\\
&x_{2,1} + x_{2,2} + x_{1,0} + x_{1,1} + x_{1,2} \leq 1\\
&x_{2,2} + x_{2,3} + x_{1,1} + x_{1,2} + x_{1,3} \leq 1\\
&x_{2,3} + x_{2,4} + x_{1,2} + x_{1,3} + x_{1,4} \leq 1\\
&x_{2,4} + x_{2,5} + x_{1,3} + x_{1,4} + x_{1,5} \leq 1\\
&x_{2,5} + x_{1,4} + x_{1,5} + x_{1,6} \leq 1\\
&x_{1,t} \in \{0,1\} \quad t \in \{0,\ldots,6\}\\
&x_{2,t} \in \{0,1\} \quad t \in \{1,\ldots,5\}
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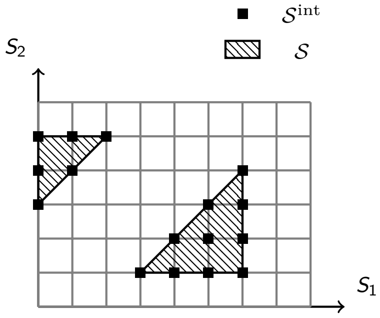
$$



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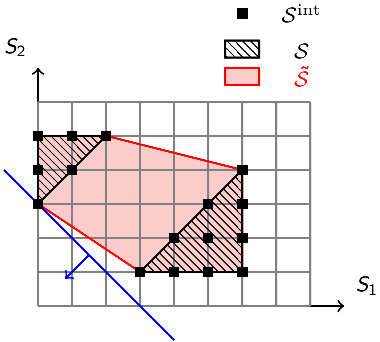

In this example $\tilde{\mathcal{S}} = conv(\mathcal{S})$ and the relaxation is tight...

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x_{2,1} + x_{2,2} + x_{1,0} + x_{1,1} + x_{1,2} &\leq 1 \\
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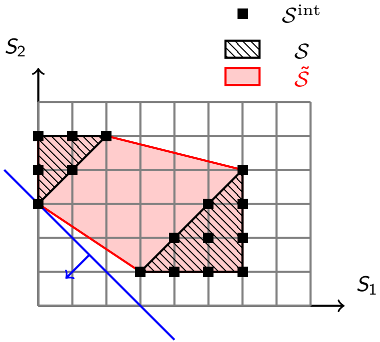
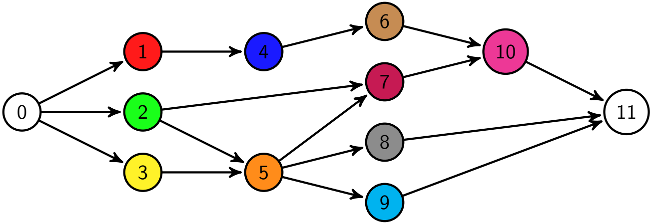


In this example $\tilde{\mathcal{S}} = conv(\mathcal{S})$ and the relaxation is tight...
... but we need 11 binary variables for a 2-task example

... but not so good in general

$|R| = 1, B = 4, \mathcal{T} = [0, 30)$



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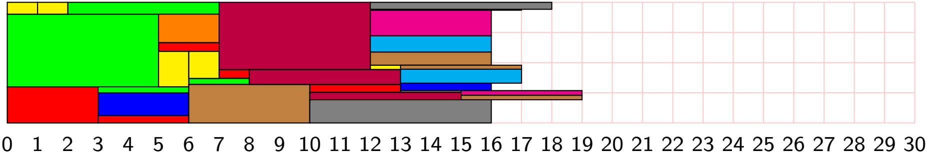


Bound = 16.46 (17) (not better than trivial Res. Bount)

The disaggregated time-indexed formulation (DDT)

The model can be reinforced by disaggregation of the precedence constraints, i.e. replacing precedence constraints by

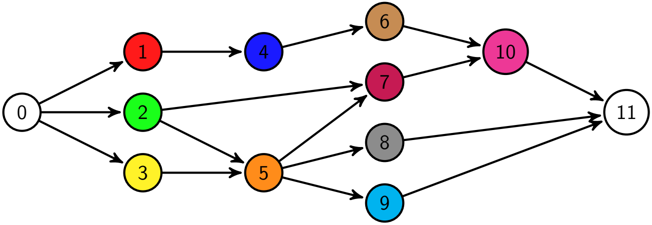
$$\sum_{\tau=0}^{t-p_i} x_{i\tau} - \sum_{\tau=0}^{t} x_{j\tau} \geq 0 \quad (i,j) \in E;\ t \in T$$

[Christofides *et al.* 1997]

- Modeling the logical relation : $S_j \leq t \Rightarrow S_i \leq t - p_i$
- The constraint matrix without resource constraints is totally unimodular.
- Total unimodularity preserved by lagrangean relaxation of the resource constraints Also efficiently computable by a max flow algorithm [Möhring *et al.* 2003]

DDT : relaxation quality

$|R| = 1, B = 4, \mathcal{T} = [0, 30)$



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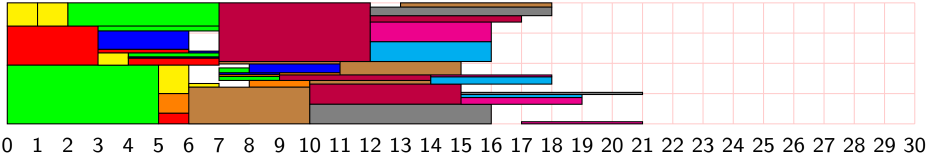
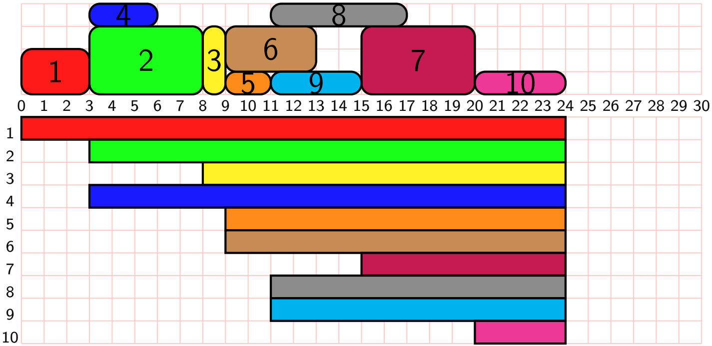


Bound = 17.14 (18) Strictly better than trivial bounds

Time-indexed step variables

- "Step" binary variable $\xi_{it} = 1 \Leftrightarrow S_i \leq t$, for $t \in T$
- Introduced by [Pritsker and Watters 1968] rediscovered several times... `[citations removed]`



Time-indexed formulations with step variables

- The time-indexed formulation with step variable (SDDT) can be obtained by (DDT) by the following transformation :

$$\xi_{it} = \sum_{\tau=0}^{t} x_{it}$$

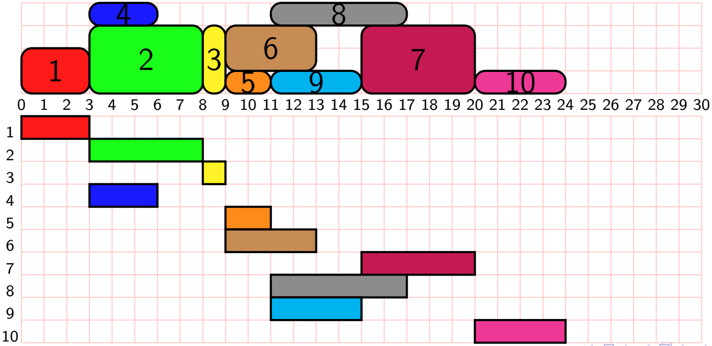
- Conversely, $x_{it} = \xi_{it} - \xi_{it-1}$
- This is a non-singular transformation (NST)
- Formulations that can be obtained from each other by a NST are strictly equivalent. They have the same $\tilde{S}$ and the same relaxation value.
- [Bianco and Caramia 2013] present a variant of the step formulation based on variables $\xi'_{it} = 1 \Leftrightarrow S_i + p_i \leq t$. We can shown that it is equivalent to (SDDT) by NST [A. 2013].

On/off time-indexed step variables

- "On/off" binary variable

$$\mu_{it} = 1 \Leftrightarrow t \in [S_i, S_i + p_i[$$

- Introduced by [Lawler 1964, Kaplan 1998] for preemptive problems and [Klein, 2000] for the RCPSP



Time-indexed formulations with on/off variables

Consider the following non singular transformation :

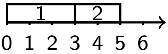
- $\mu_{it} = \sum_{\tau=t-p_i+1}^{t} x_{i\tau}$
- $x_{it} = \sum_{k=0}^{\lfloor t/p_i \rfloor} \mu_{i,t-kp_i} - \sum_{k=0}^{\lfloor (t-1)/p_i \rfloor} \mu_{i,t-kp_i-1}$
- [A. 2013] Applying the transformation yields a time-indexed formulations with on/off variables OODDT equivalent to DDT and tighter than that of [Klein 2000].
- Many "new" formulations presented in the litterature are in fact weaker than or equivalent to DDT.
- Need to be distinguished from actual cutting planes or extended formulations

Outline

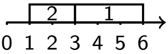
Extended formulations

- Formulation having better relaxations...
- ... with an exponential number of constraints and/or variables
- Need to use cut and/or column generation techniques

Small example again. $\mathcal{S}^E$ dominant set of earliest schedules Let $x_s = 1$ iff schedule $S^s = \mathcal{S}^E$ is selected. $S_i = \sum_{s \in \mathcal{S}^E} S_i^s x_s$

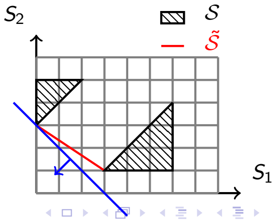
S^1 

$\sum C_i = 8$

S^2 

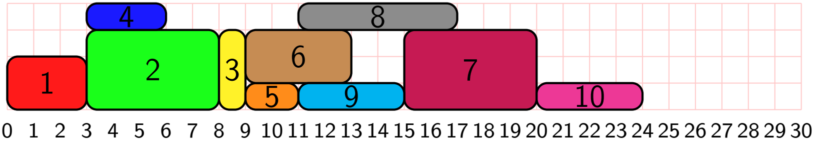
$\sum C_i = 9$

$$\min S_1 + S_2 + 5$$
$$S_1 = 3x_2$$
$$S_2 = 3x_1 + x_2$$
$$x_1 + x_2 = 1$$
$$x_1, x_2 \in \{0, 1\}$$



Forbidden sets

- Minimal forbidden set (MFS) F : a minimal set of activities that cannot be scheduled in parallel :
  $\sum_{i \in F} b_{ik} > B_k$ and $\forall j \in C, \sum_{i \in F \setminus \{j\}} b_{ik} \leq B_k$



  $\mathcal{F} = \{\{1,2\},\{1,3\},\{2,3\},\ldots,\{7,8,9\},\ldots\}$

- There is in general an exponential number of MFS.
- Can be reduced by excluding MFS having two activities with a precedence relation or non intersecting time windows.

Valid inequalities

- Forbidden set-based valid inequalities [Hardin *et al* 2008]
  - Basic inequality : $\sum_{i \in A} \sum_{s=t-p_i+1}^{t} x_{is} \leq |F| - 1, \quad \forall F \in \mathcal{F}$
    The resource constraints can be replaced by this set of inequalities → extended formulation
  - A more general family of inequalities : extension to an interval of length v

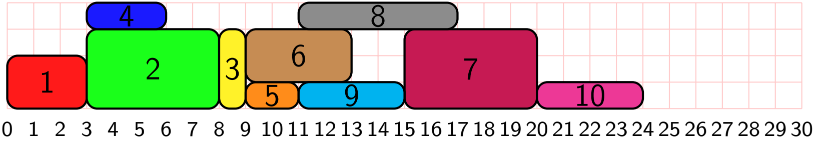
$$\sum_{i \in F \setminus \{j\}} \sum_{s=t-p_i+1+v}^{t} x_{is} + \sum_{s=t-p_j+1}^{t+v} x_{js} \leq |F| - 1 \quad \forall F \in \mathcal{F}$$

  - Lifting procedure and separation heuristic

- other valid inequalities [Christofides et al. 1987, de Sousa and Wolsey 1997, Cavalcante et al. 2001, Baptiste and Demassey 2004, Demassey *et al* 2005]

Feasible subsets

- Feasible subset P : a set of activities that can be scheduled in parallel :
  $\sum_{i \in P} b_{ik} \leq B_k$ and $(i,j) \notin TA$ and
  $[ES_i, LS_i + p_i] \cap [ES_j, LS_j + p_j] \neq \emptyset$



  $\mathcal{P} = \{\{1\}, \{2\}, ..., \{10\}, \{1,5\}, \{2,4\}, \ldots, \}$

- There is in general an exponential number of FS.
- a schedule : an assignment of feasible subset to each time period
  1–2 : $\{1\}$; 3–5 : $\{2,4\}$; 6,7 : $\{2\}$; 8 : $\{3\}$; 9,10 : $\{5,6\}$; . . .

The feasible subset-based formulation (FS)

- obtained from (DDT) by replacing the resource constraints by

$$\text{s.t.} \sum_{P \in \mathcal{P}_i} \sum_{t \in T} y_{Pt} = p_i \quad i \in A,\ p_i \geq 1$$

$$\sum_{P \in \overline{\mathcal{P}}} y_{Pt} \leq 1 \quad t \in T$$

$$x_i^t - \sum_{P \in \mathcal{P}_i} y_{Pt} - \sum_{P \in \mathcal{P}_i} y_{P,t-1} \geq 0 \quad i \in A;\ t \in T$$

$$y_{At} \in \{0,1\} \quad P \in \mathcal{P};\ t \in \cap_{i \in P}\{ES_i, \ldots, LS_i\}$$

where $\mathcal{P}_i \subseteq \mathcal{P}$ is the set of all feasible subsets that contain activity i.

[Mingozzi *et al* 1998]

Lower bounds based on the feasible subset-based formulation

- Weighted Node packing combinatorial bound issued from the dual of the preemptive relaxation [Mingozzi *et al.* 1998]
- Destructive preemptive relaxation solved by constraint propagation and column generation or lagrangian relaxation [Brucker and Knust 2000, Demassey *et al* 2004, Baptiste and Demassey 2004]
- Preemptive FS solved by branch and price. [Moukrim *et al.* 2013]

Limits of time-indexed formulations

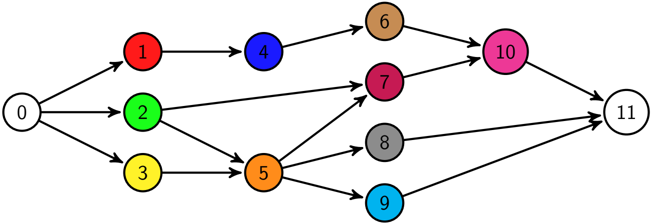
1. Equivalent relaxations does not mean equivalent behaviour of the MILP solver for obtaining solutions
   - [Bianco and Caramia 2013] show that the ξ'_{it} formulation outperforms others in terms of integer solving
2. Even weaker relaxations may yield better integer solutions
   - Well-known that (DT) formulation may also perform better than (DDT) formulation for integer solving.
3. Time-indexed formulation cannot be used for problems where large horizons are needed
   - Some examples with 15 activities are out of reach of time-indexed formulaiton [Kone *et al.* 2011]

Need of compact and/or hybrid formulations

Outline

Sequencing or strict ordering variable

- Principle : adding precedence constraints such that all resource conflicts are resolved
- Any schedule satisfying these new precedence constraints is feasible
- Sequencing variable $z_{ij} = 1 \Leftrightarrow S_j \geq S_i + p_i$



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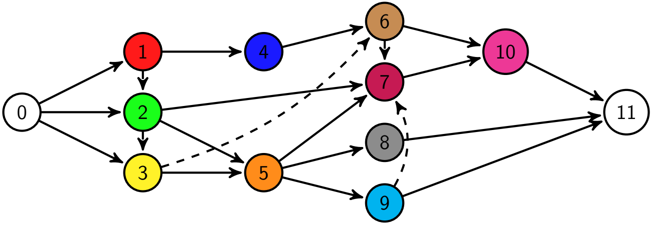

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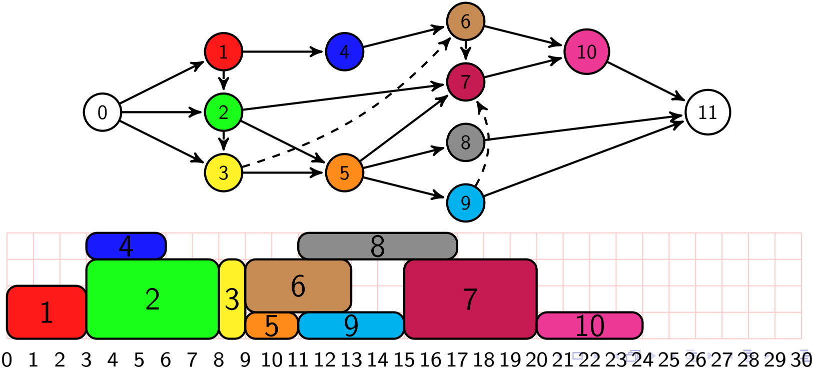


A first formulation based on forbidden sets

The set of additional precedence constraints has to "destroy" all forbidden sets.

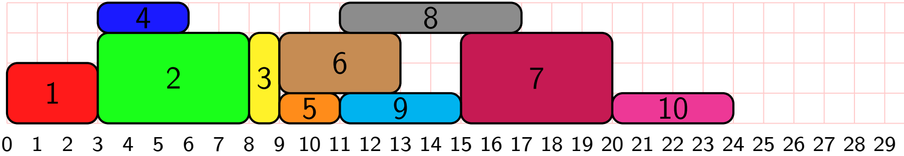
$$
\begin{aligned}
\text{Min. } & S_{n+1} \\
\text{s. t. } & z_{ij} + z_{ji} \leq 1 \quad i,j \in V,\ i < j \\
& z_{ij} + z_{jh} - z_{ih} \leq 1 \quad i,j,h \in V,\ i \neq j \neq h) \\
& z_{ij} = 1 \quad (i,j) \in E \\
& S_j - S_i + (1 - M_{ij}) z_{ij} \geq p_i \quad i,j \in V,\ i \neq j \\
& \sum_{i,j \in F, i \neq j} z_{ij} \geq 1 \quad F \in \mathcal{F} \\
& z_{ij} \in \{0,1\} \quad i,j \in V,\ i \neq j
\end{aligned}
$$

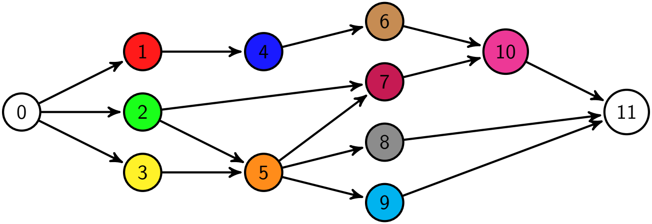
[Alvarez-Valdés and Tamarit 1993]

Extension of the disjunctive formulation for the job-shop problem [Balas 1985] with an exponential number of constraints

Resource flow variables

$\phi_{ij}^k \geq 0$: numbers of units of resource k transferred from i to j





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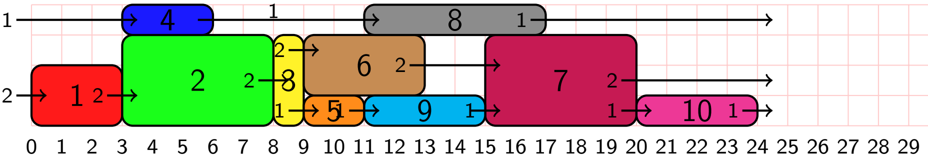

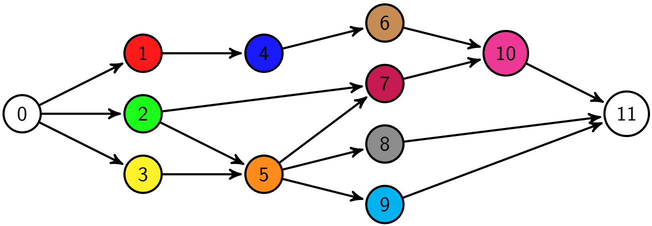

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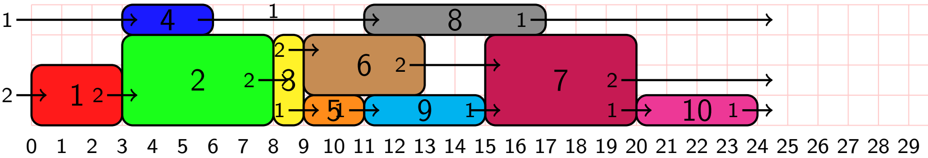
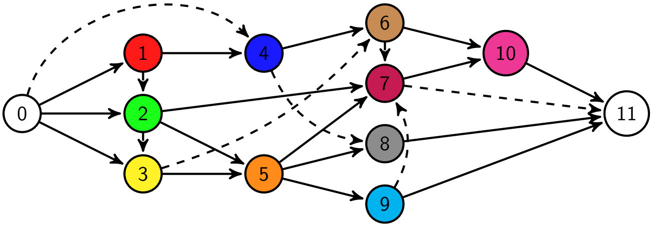


Enforcing sequencing variables to be compatible with the flow

$\phi_{ij}^k > 0 \Rightarrow z_{ij} = 1$



A formulation based on resource flows

- Replace the forbidden set constraints by the following flow constraints

$$\phi_{ij}^k - \min(\tilde{r}_{ik}, \tilde{r}_{jk}) z_{ij} \leq 0 \quad (i, j \in V,\ i \neq j,\ \forall k \in \mathcal{R})$$

$$\sum_{j \in V \setminus \{i\}} \phi_{ij}^k = \tilde{r}_{ik} \quad (i \in V \setminus \{n+1\})$$

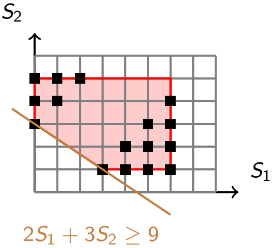
$$\sum_{i \in V \setminus \{j\}} \phi_{ij}^k = \tilde{r}_{jk} \quad (j \in V \setminus \{0\})$$

$$0 \leq \phi_{ij}^k \leq \min(\tilde{r}_{ik}, \tilde{r}_{jk}) \quad (i, j \in V,\ i \neq n+1,\ j \neq 0,\ i \neq j;\ k \in \mathcal{R})$$

- $O(|A|^2 R)$ additional continuous variables
- FB : A compact formulation. [A. *et al* 2003]

Valid inequalities for sequencing formulations

- Relaxation of poor quality, need to generate valid inequalities
- Example 1 : Extension of valid inequalities by [Balas 85,Applegate & Cook 1991,Dyer & Wolsey 1990] for the disjunctive formulation of the job-shop (half-cuts, late job cuts...)



- Example 2 : constraint propagation-based cutting planes [Demassey *et al* 2005]
  - Compute conditional distances $d_{ij}^{k \prec l}$, $d_{ij}^{l \prec k}$ and $d_{ij}^{k||l}$ by CP
  - Lifted distance inequalities

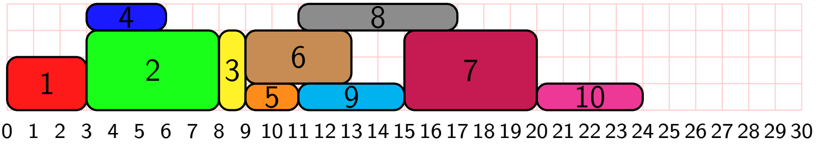
$$S_j - S_i \geq d_{ij}^{h||l} + (d_{ij}^{h \prec l} - d_{ij}^{h||l})z_{hl} + (d_{ij}^{l \prec h} - d_{ij}^{h||l})z_{lh}$$

Outline

1. RCPSP
2. MILP for RCPSP
3. Standard and novel MILP formulations
   - Pseudo-polynomial time-indexed formulations
   - Extended time-indexed formulations and valid inequalities
   - Compact sequencing and natural date variable formulations
   - Compact event-based formulations
4. Synthesis of theoretical and experimental results
5. Perspectives
6. References

Start and End Event variables

- $\mathcal{E}$: set of remarkable events.
- $t_e \geq 0$: event date : representing the start and end of at least one activity
- Start binary assignment variables $a^-_{ie} = 1 \leftrightarrow S_i = t_e$
- End binary assignment variables $a^+_{ie} = 1 \leftrightarrow S_i + p_i = t_e$
- Maximum $n+1$ events $\Longrightarrow$ $2(n+1)|\mathcal{E}|$ binary variables.



Extension of models proposed for machine scheduling [Lasserre and Queyranne 1994,Dauzère-Pérès and Lasserre 1995], widely used also in the process scheduling industry [Pinto and Grossmann 1995, Zapata *et al* 2008].

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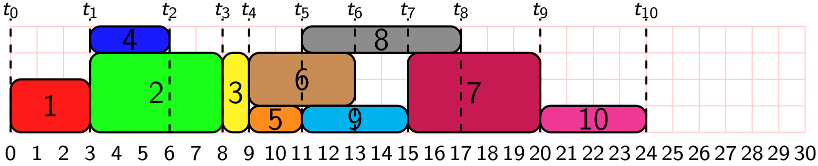

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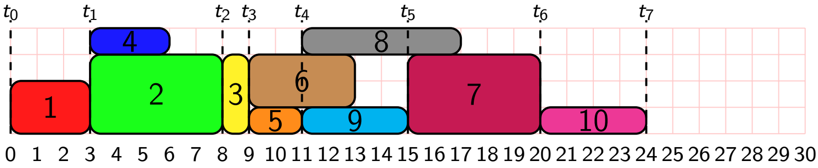
On/Off Event variables

- $\mathcal{E}$: set of remarkable events.
- $t_e \geq 0$: event date : representing the start of at least one activity
- On/off binary variable $aie = 1 \Leftrightarrow [S_i, S_i + p_i] \cap [t_e, t_e + 1] \neq \emptyset$
- Each activity such that $a_{ie} = 1$ can be assumed of length $[t_e, t_e + 1]$
- $n|\mathcal{E}|$ binary variables



(OOE) Min. $C_{\max}$

s. t. $C_{\max} \geq t_e + (\overline{a}_{ie} - \overline{a}_{i(e-1)})p_i \quad (e \in \mathcal{E};\ i \in A)$

$$t_0 = 0$$

$$t_{e+1} \geq t_e \quad (e \neq n-1 \in \mathcal{E})$$

$$t_f \geq t_e + (\overline{a}_{ie} - \overline{a}_{i,e-1} - \overline{a}_{if} + \overline{a}_{i,f-1} - 1)p_i \quad ((e,f,i) \in \mathcal{E}^2 \times A,\ f > e \neq 0)$$

$$\sum_{e'=0}^{e-1} \overline{a}_{ie'} \geq e(1 - \overline{a}_{ie} + \overline{a}_{i,e-1})) \quad (i \in A; e \neq 0 \in \mathcal{E})$$

$$\sum_{e'=e}^{n-1} \overline{a}_{ie'} \geq e(1 + \overline{a}_{ie} - \overline{a}_{i,e-1}) \quad (i \in A; e \neq 0 \in \mathcal{E})$$

$$\sum_{e \in \mathcal{E}} \overline{a}_{ie} \geq 1 \quad (i \in A)$$

$$\overline{a}_{ie} + \sum_{e'=0}^{e} \overline{a}_{je'} \leq 1 + (1 - \overline{a}_{ie})e \quad (e \in \mathcal{E};\ (i,j) \in E)$$

$$\sum_{i=0}^{n-1} r_{ik}\overline{a}_{ie} \leq R_k \quad (e \in \mathcal{E};\ k \in \mathcal{R})$$

$$t_e \geq 0 \quad (e \in \mathcal{E})$$

$\overline{a}_{ie} \in \{0,1\} \quad (i \in A;\ e \in \mathcal{E})$ [Koné *et al.* 2011]

Valid inequalities for event-based formulations

- Wanted ! !

Done for the one machine problem in [Della croce *et al* 2014]

Outline

Comparison of formulations : LB

instance	LCG12	%RDDT	%DDT(1h)	PFS(3h)
j609_1	85	17.65%	2.35%	
j609_3	99	17.17%	9.09%	
j609_5	81	14.81%	3.70%	
j609_6	105	11.43%	4.76%	
j609_7	105	18.10%	2.86%	
j609_8	95	18.95%	7.37%	
j609_9	99	12.12%	7.07%	
j609_10	90	15.56%	3.33%	
j6013_1	105	16.19%	1.90%	-1.90%
j6013_2	103	20.39%	1.94%	
j6013_3	84	19.05%	1.19%	
j6013_4	98	20.41%	3.06%	
j6013_5	92	21.74%	1.09%	
j6013_6	91	16.48%	1.10%	
j6013_7	83	19.28%	3.61%	
j6013_8	115	20.00%	3.48%	
j6013_9	97	16.49%	2.06%	
j6013_10	114	24.56%	0.88%	
j6025_2	95	14.74%	5.26%	
j6025_4	106	18.87%	8.49%	
j6025_6	105	14.29%	4.76%	
j6025_7	88	15.91%	6.82%	
j6025_8	95	22.11%	5.26%	
j6025_10	107	15.89%	6.54%	

instance	LCG12	%RDDT	%DDT(1h)	PFS13(3h)
j6029_1	98	19.39%	3.06%	
j6029_2	123	17.89%	7.32%	-3.25%
j6029_3	114	19.30%	1.75%	-3.51%
j6029_4	126	15.87%	7.14%	-3.17%
j6029_5	102	12.75%	3.92%	-2.94%
j6029_6	144	17.36%	9.03%	-1.39%
j6029_7	117	19.66%	4.27%	
j6029_8	98	13.27%	2.04%	-9.18%
j6029_9	105	18.10%	4.76%	
j6029_10	111	20.72%	1.80%	
j6030_2	69	4.35%	1.45%	
j6041_3	90	16.67%	4.44%	
j6041_5	109	20.18%	7.34%	
j6041_10	108	12.04%	2.78%	
j6045_1	90	12.22%	4.44%	-1.11%
j6045_2	134	20.90%	11.94%	-2.99%
j6045_3	133	13.53%	6.02%	-3.76%
j6045_4	101	15.84%	4.95%	-1.98%
j6045_5	99	21.21%	3.03%	-2.02%
j6045_6	132	21.97%	21.21%	-3.79%
j6045_7	113	19.47%	5.31%	-3.54%
j6045_8	119	15.13%	5.04%	-3.36%
j6045_9	114	16.67%	5.26%	-4.39%
j6045_10	102	16.67%	3.92%	-4.90%

LCG12 : [Schutt *et al* 2013] (hybrid CP/SAT method : Lazy clause generation)

PFS13 : [Moukrim *et al* 2013] Preemptive feasible subset formulation solved by B&P

Comparison of formulations : exact solving

Instances	Formulations	%Integer	%Opt	%Gap	%ΔCPM	Time Opt (s)
KSD30	DDT	91	82	0.47	8.91	10.45
	DT	86	78	0.55	6.74	12.76
	FCT	67	62	0.16	3.76	22.66
	OOE_Prec	46	30	1.69	13.65	52.31
	OOE	33	24	1.22	7.00	112.62
	SEE	3.1	2.9	0.24	0.61	123.62
	MCS	–	**97**	0.00	11.48	7.39
PACK	DDT	95	**76**	1.08	199.02	63.39
	DT	85	55	0.49	203.58	48.24
	OOE_Prec	55	5	3.25	227.19	18.92
	OOE	49	9	2.89	231.29	61.78
	FCT	2	0	1.28	14.49	–
	SEE	0	0	–	–	–
	MCS	–	25	0.00	149.81	115.88
BL	DDT	100	100	0.00	32.40	13.68
	DT	100	100	0.00	32.40	37.93
	OOE_Prec	54	0	7.26	40.30	–
	OOE	49	0	7.90	41.65	–
	FCT	21	3	6.14	30.64	310.58
	SEE	8	0	12.81	29.96	–
	MCS	–	**100**	0.00	32.40	3.29
KSD15_d	OEE_Prec	99.8	86	0.00	10.02	6.49
	FCT	99	94	0.02	9.02	12.06
	OEE	99	83	0.01	10.14	4.68
	SEE	92	76	0.15	9.86	13.04
	DT	55	54	0.23	4.31	12.10
	DDT	1	1	0.00	2.63	3.34
	MCS	–	**100**	0.00	10.18	0.07
PACK_d	OEE	60	18	1.26	120.13	75.58
	OOE_Prec	60	14	1.62	117.56	54.35
	FCT	7	7	0.00	0.00	60.88
	SEE	4	4	0.00	0.00	215.08
	DT	0	0		–	–
	DDT	0	0		–	–
	MCS	–	**38**	0.00	50.59	72.34

- MCS [Laborie 2005] (MFS-based CP)
- LCG [Schutt *et al* 2013]

	KSD30	PACK	BL	KSD15_d	PACK_d
LCG	100	70.91	100	100	67.27
MCS	82	25	100	100	38
MIP	97	76	100	94	18
	(DDT)	(DDT)	(DDT)	(FB)	(OOE)

- KSD30 “highly disjunctive” instances
- PACK,BL “highly cumulative” instances
- KSD15_d : first 15 activities of KSD30 with modified durations
- PACK_d : PACK instances with modified durations

Synthesis of theoretical and experimental results

- Time indexed formulations have the best LP relaxations with FS$\succ$DDT$\succ$DT
- Compact formulations have poor relaxation but can be the only alternative for large scheduling horizons
  - Highly disjunctive instances : flow-based models
  - Highly cumulative instances : event-based models
  - Valid inequalities stricly necessary
- MILP vs Lazy Clause Generation
  - MILP outperformed by LCG for exact solving disjunctive instances
  - Competitive with LCG for lower bounds based on preemptive exact solving of FS through B&P.
  - Competitive with LCG for exact highly cumulative instances

Outline

- Time aggregation / energetic reasoning / dual feasible functions [Carlier and Néron 2000, Kooli 2012]
- Mixed continuous/discrete models [Haït and A. 2012]
- Preprocessing [Baptiste *et al* 2010]
- B&P for the non-preemptive feasible set formulations
- CG for chain decomposition models [Kimms 2001,Van den Akker *et al.* 2005
- Matheuristics [Palpant *et al.* 2004,Della croce *et al* 2014]
- Hybrid SAT/CP/MILP

Find x

s.t.

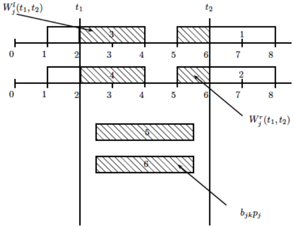
$$\sum_{l\in I^j} x_{jl} = p_j, \quad \forall j \in \mathcal{A}$$

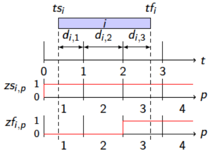
$$x_{jl} \le \Delta_l, \quad \forall j \in \mathcal{A}, \forall l \in \mathcal{I}^j$$

$$\sum_{j\in \mathcal{A}^l} b_{jk} x_{jl} \le B_k \Delta_l, \quad \forall k \in \mathcal{R}, \forall l \in \mathcal{L}$$

$$\sum_{s\in \mathcal{I}^i/s\le l} \frac{x_{is}}{p_i} \ge \sum_{s=\tau/t_\tau=r_j}^{l} \frac{x_{js}}{p_j}, \quad \forall (i,j) \in A, \forall l \in \mathcal{I}_i^j$$

$$x_{jl} \ge 0, \quad \forall j \in \mathcal{A}, \ \forall l \in \mathcal{I}^j$$





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