Technische Universität München
Zentrum Mathematik

Global consequences of local structure: Hamilton-based flow-lattices, logical limit-laws, and a measure on sign-matrices

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Vollständiger Abdruck der von der Fakultät für Mathematik der Technischen Universität München zur Erlangung des akademischen Grades eines

Doktors der Naturwissenschaften (Dr. rer. nat.)

genehmigten Dissertation.

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	2.	Univ.-Prof. Dr. Nina Gantert
	3.	Prof. Michael Krivelevich Tel Aviv University, Israel (schriftliche Beurteilung)

Die Dissertation wurde am 17. 6. 2014 bei der Technischen Universität München eingereicht und durch die Fakultät für Mathematik am 8. 9. 2014 angenommen.

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Wahrern und Mehrern weltweiten(den) Wissens

Preface

To preserve and extend the age-old text-based clockwork of formalised human knowledge, upholding a dichotomy into two states commonly called true and false, is a demiurgical pursuit of inestimable intrinsic and societal value, uniting human beings in a comparatively uncontroversial pursuit.

For this world-widening web of texts, and in the worlds that they chart, far less matters than what meets the eye: a binary truth value is most often decided by the presence or absence of much sparser substructures within a structure under consideration. With hierarchy in, and reduction of information becoming ever more pressing in view of the current deluge of automatically-created binary data, already the non-digital world forces us to reduce and deduce: this begins with the physical building blocks of the texts (to read Greek curves in the sand, or Arabic numerals scrawled as flaky soot-stripes on pocky paper, or Roman characters displayed as motley matrices of liquid-crystal-states, each time requires all of us to perform a massive parallel reduction of sensory data), continues with the texts themselves (how much padding there is, and of necessity, even in the most concise mathematical texts written by humans for humans!), and the observation extends to the ideal worlds opened up by those texts. There, too, much may be implied by little.

An important distinction, which often demarcates what one can from what one cannot prove, is whether the ambient structure is dense enough to allow for *premeditation*, i.e. putting one's mind to a pattern to look for, ahead of time. There are plausible truths (such as Conjecture 12 below) where it is probably still true that a much sparser substructure within a structure clinches the question, but where already the ambient structure is so sparse that the still sparser decisive substructure cannot be known in advance.

The present thesis contains new results which fit into this paradigm: (1) the existence of preselected, sparse, low-bandwidth, almost-two-colourable spanning subgraphs with Hamilton-based flow lattice can be used to prove that denser ambient graphs have Hamilton-based flow lattices as well, (2) the existence of preselected, constant-size 'universal models' as a subgraph of a large random graph on a surface implies that the entire graph is a model, allowing to compute probabilities, (3) the existence of circuits in an associated graph decides whether two measures on sign-matrices are equal or not.

There are two meta-properties ensuring that the appearance of a local structure can bestow a property upon an ambient structure:

(1) 'isolation' of the local structure from other local structures,
(2) 'monotonicity' of the global property (i.e. the global property holds as soon as some local substructure has it, intertwined as it may be with other substructures).

These are two different reasons for substructures not getting in the way of a global property. The arguments used in Chapter 3 to prove logical limit laws exemplify (1). The argument that we will use to prove flow lattices Hamilton-based in Chapter 2, and the characterisation of (non-)agreement of $\mathsf{P}_{\mathrm{chio}}$ and $\mathsf{P}_{\mathrm{lcf}}$ in Chapter 4 is an example of (2).

This thesis started with a working title 'Ganzzahlige Homologie zufälliger Simplizialkomplexe' ('Integral Homology of random simplicial complexes'), aiming at solving a problem posed in e.g. [112, Concluding Remarks] and [111], a talk that the author attended in Poznań, Poland: to prove (and explain more generally) the rarity of elements of large finite order in the homology of random simplicial complexes. The current state of the problem will not be discussed in this thesis. The author tries to develop a new approach to [112, Concluding Remarks] via *generalised inverses of incidence matrices*, taking a 'linearised' view by characterising the non-existence of finite-order elements in a quotient group as the existence of a point with integer coordinates in an associated affine subspace of a high-dimensional $\mathbb{Q}^d$; this is not ripe for publication and kept out of this thesis.

Examples of such quotient groups are the homology of a simplicial complex, or the quotient A of the flow lattice of a graph by the group generated by its Hamilton flows. There are analogies with the problem in [112, Concluding Remarks]: e.g., when granting the truth of Conjecture 12.(gnp.1), which is likely to be proved soon, then the content of the open Conjecture 12.(gnp.2) is that A a.a.s. does not contain non-zero elements of finite order; that is a problem which appears to be of comparable difficulty with the problem from [112, Concluding Remarks].

Acknowledgements. I remain grateful to my supervisor Anusch Taraz for advice, support and trust, that I tried to live and grow up to. I wholeheartedly thank Julia Böttcher, Mathias Schacht and Anusch Taraz for caring about epsilons and other small parameters, in particular for eking out as much as possible from the regularity method of Szemerédi and Komlós to publish their bandwidth theorem in the form [24, Theorem 2]; when studying their paper [24] I had the ideas from which Chapter 2 were developed.

I owe many thanks to Tobias Müller, Marc Noy, and Anusch Taraz for collaborating in the project which in parts is reported on in Chapter 3. Our project resulted in one more example of the cumulativeness, intersubjectivity and timeless consistency of mathematical knowledge, transcending accidents like time, space or nationality. Our project had breadth, depth, conceptual work, technical work, unexpected advances, and heated discussions on redactional and epistomological issues, eventually resulting in a paper condensing results from several decades into new quantitative statements about minor-closed classes of graphs. The project is rife with suggestions for further investigations; for example, it has drawn my attention to an apparently open construction problem in structural graph theory (*constructing critically 5-chromatic toroidal graphs with arbitrarily large edge-width*) which is motivated by, but itself does not involve, statistical considerations about 'almost all models'. The project also leads to an open proof-methodological problem (developing methods for *proving the existence of given nonconstantly large subgraphs in a random toroidal graph*). Moreover, our theorem about the explicit closure of the probability limits w.r.t. the set of planar graphs (see Theorem 99) makes statements which in the foreseeable future may be testable in the very practical sense of you sampling a thousand random planar graphs with a million vertices each, on a personal computer, checking properties and computing statistics. (This prediction is made in view of recent advances on Boltzmann samplers made by the French school of analytic combinatorics.) Hopefully, our project was only the end of a beginning.

I thank Nati Linial for giving my approach to the problem of proving rarity of finite-order elements in the homology of random complexes some consideration during two travels to Israel, and, on occasion of the same two travels, to Adam Chapman, Shira Chapman and Yuval Peled for their hospitality.

I thank Susanne Nieß and Tomasz Łuczak for collaboration on two projects, too complex and unfinished to appear in this thesis, and sincerely hope that we will bring both of them to fruition.

I did the research for this thesis as a graduate student of Technische Universität München, and thank this institution for the support and the freedom it accorded to me.

I gratefully acknowledge financial support via the Max Weber-Programm of the ENB, the DFG grant TA 309/2-2, and TUM Graduate School's Thematic Graduate Center TopMath, during some intervals of my studies. The latter, together with Institut Henri Poincaré, provided partial financial support for a trimester in Paris; in that context, I thank Jean-Marc Schlenker and the other organisers of the program 'Geometry and Analysis of Surface Group Representations' at IHP for allowing me to take part in this trimester, even though I do not have a published record of knowledge about arithmetic groups and their geometry. I thank Jean Raimbault for asking me to come to Jussieu campus and give a talk on the problem of rarity of large-finite-order-elements in the homology of random simplicial complexes.

Two mathematics teachers in high-school had a part in my getting interested in mathematics more substantial than the macerated regular fare of German secondary schools, and shall be thanked here: E. Christensen and K. Gruitrooy.

During my graduate studies I have appreciated learning more about life in (all too short!) conversations with many people, for example Hadi Afzali, Elad Aigner-Horev, Peter Allen, Andreas Alpers, Susanne Apel, Julia Böttcher, Steffen Borgwardt, Nathan Bowler, René Brandenberg, Max Butz, Johannes Carmesin, William Cavendish, Adam Chapman, Shira Chapman, Oliver Cooley, Valentina Disarlo, Nico Düvelmeyer, Julia Ehrenmüller, Joshua Erde, Holger Fink, Uriya First, Jan-Oliver Fröhlich, Jonah Gaster, Stefanie Gerke, Viviana Ghiglione, Roman Glebov, Nils Ole Glück, Bernhard Häupler, Georg Heise, Gottfried Herold, Melanie Herzog, Jonas Hirsch, Jan Hladký, Nan-Kuo Ho, Carlos Hoppen, Victoria Hoskins, Woojin Jeon, Thomas Kahle, Stefan Kahler, Florian Klöck, Christoph Koch, Stefan König, Tobias Köppl, Michael Kratzer, Robert Lang, Barbara Langfeld, Johannes Lengler, Anita Liebenau, Po-Shen Loh, Tamás Makai, Julien Marché, Anna Meine, Katarzyna Mieczkowska, Stefan Neukamm, Susanne Nieß, Andreas Noever, Yuval Peled, Yury Person, Florian Pfender, Oleg Pikhurko, Jean Raimbault, Wolfgang Riedl, Stephan Ritscher, Michael Ritter, Thomas Roche, Lucia Roth, Domingo Ruiz, Wojciech Samotij, Andy Sanders, Florent Schaffhauser, Marcus Schmidt, Tina Schmidt, Felix Schmiedl, Ole Schwen, Dmitriy Slutskiy, Piotr Śniady, Jan Spiess, Sina Straub, Tobias Strubel, Paul Stursberg, Ser Peow Tan, Louis Tehran, Jenna Thompson, Nicholas Vlamis, Graeme Wilkin, Andreas Würfl.

The price of freedom is unending vigilance. The unprecedented edifice of electronic technology, whose long-term expected value seems positive, is ever threatened by both wear and tear, and would be quick to collapse when left to itself. Sustainability is only possible through change, and this needs constant human stewardship. Everything around us that was made by living things, from the most simple to the most complex machines, depends for its survival on the transmission of information, which makes those who guard and preserve the requisite technology particularly important persons. I would like to thank the 'Rechnerbetriebsgruppe' of the mathematics department of TUM, for keeping the department's facilities up and running, in particular Michael Ritter for his maintenance of the technology at the department where I did much of the work for this thesis, and Andreas Dürmeier for quickly solving a problem related to non-delivery of email. I also thank Peter Baasch and Frank Bösch for help with technology while being employed at TUHH. Last but not least I thank the bookseller 'CŒNOBIUM. Libri antichi, stampe.' in Asti, Italy, for preserving old books, but also promptly dispatching them across the Alps, from Piedmont to piedmont.

Abstract

Chapter 2 among other things has as its main result a proof that for every $\gamma > 0$ there is $n_0 \in \mathbb{N}$ such that, whenever $n_0 \leqslant n \equiv 3 \pmod 8$, every n-vertex graph G with minimum-degree at least $(\frac{1}{2} + \gamma)n$ has the property that the free abelian group $\mathrm{Z}_1(G)$ generated by all the integer-valued circulations on G admits a *basis* consisting only of circulations whose support is a Hamilton-circuit of G. This would be a new result even with 'basis' replaced by 'generating set', but it adds to the interest of this statement that one can always make do with the smallest number of Hamilton-circuit-supported generators that is algebraically possible—that is, $\mathrm{rank}(\mathrm{Z}_1(G))$-many generators—in spite of the unwieldiness of these generators with their $\Omega(n)$-sized supports. As part of the proof of the above statement we provide the apparently first non-trivial examples at all for graphs whose group of circulations admits a basis of Hamilton-supported circulations (and these examples are sparse graphs). It is exceedingly likely that the above statement remains true when '$n \equiv 3 \pmod 8$' is replaced with the necessary condition '$n \equiv 1 \pmod 2$' but this is not proved in the thesis.

Chapter 3 among other things explicitly determines—as a set of closed intervals with explicitly defined endpoints—the closure of the set of the limits as $n \to \infty$ of the probability that a uniformly random n-vertex planar graph satisfies a given statement in MSO-logic. For *connected* n-vertex planar graphs we will prove that MSO-logic obeys a *zero-one law*. Another main result of Chapter 3 is a method to determine the closure of the set of probability limits of statements in MSO-logic w.r.t. to *any addable* class of graphs (of which the class of planar graphs is but one familiar one). Moreover, we prove a convergence law and a zero-one law for statements in FO-logic w.r.t. random graphs from the *non*-addable class of graphs embeddable on a fixed surface of arbitrary finite genus.

Chapter 4 makes a connection between $\{-1,+1\}$-matrices, $\{-1,0,+1\}$-matrices, and *signed graphs*. What can bind these objects together is the operation of *Chio condensation*. One of the motivations for Chapter 4 is to suggest a new approach to the old open problem of counting singular matrices with entries from $\{-1,+1\}$. The suggestion is to compare two measures, none of them uniform, but one of them closely related to it, the other asymptotically under control by results in the recent literature: we define a measure $\mathsf{P}_{\mathrm{chio}}$ on the set $\{-1,0,+1\}^{[n-1]^2}$ of all $(n-1)\times(n-1)$-matrices with entries from $\{-1,0,+1\}$ which (owing to a determinant identity published by M. F. Chio) is closely related to the uniform measure on $\{-1,+1\}^{[n]^2}$, yet is itself not a uniform measure, but rather resembles the so-called *lazy coin flip distribution* $\mathsf{P}_{\mathrm{lcf}}$ on $\{-1,0,+1\}^{[n-1]^2}$ on events defined by specifying a constant number of entries; this is interesting in view of a recent theorem of Bourgain, Vu and Wood which shows that if the entries of an $n \times n$ matrix whose $\{-1,0,+1\}$-entries are governed by $\mathsf{P}_{\mathrm{lcf}}$ and fully independent (they are not when governed by $\mathsf{P}_{\mathrm{chio}}$), then an optimal asymptotical bound on the singularity probability over $\mathbb{Z}$ *can* be proved. The two measures $\mathsf{P}_{\mathrm{chio}}$ and $\mathsf{P}_{\mathrm{lcf}}$ bear comparison, and the quality of $\mathsf{P}_{\mathrm{chio}}$ approximating $\mathsf{P}_{\mathrm{lcf}}$ is determined by the existence and number of *substructures* in an associated structure: roughly speaking, by the number of *circuits* in associated bipartite *signed graphs*. Regardless of whether this will lead to further quantitative advances, it seems a new and valuable contribution to the theory of signed graphs to point out how Chio condensation connects $\{\pm\}$-matrices to $\{0,\pm\}$-matrices, and that the sizes of the preimages under the Chio-map give rise to a distribution resembling the lazy coin flip measure, the approximation quality determined by the cyclomatic number of an associated graph.

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1 Introduction

1.1 Organisation of this thesis

> Un moment et un effort à peu près aussi petits qu'on veut peuvent suffire parfois pour jeter un livre d'une table, mélanger des papiers, tacher un vêtement, froisser du linge, brûler un champ de blé, [...] Il faut des efforts et du temps pour soulever le livre jusqu'à la table, mettre en ordre les papiers, nettoyer le vêtement, repasser le linge; un an de peine et de soins est nécessaire pour faire apparaître une autre moisson dans le champ; [...] On traduit ce principe par la fiction d'une grandeur qui, dans tout système où a lieu un changement, augmente toujours, sauf intervention de facteurs extérieurs; [...]
> Tel fut le couronnement de la science classique, qui devait dès lors se croire capable, par les calculs, les mesures, les équivalences numériques, de lire, à travers tous les phénomènes qui se produisent dans l'univers, de simples variations de l'énergie et de l'entropie [...] L'idée d'une telle réussite avait de quoi enivrer les esprits. [...] les choses juxtaposée dans l'étendue et qui changent d'instant en instant fournissent pourtant à l'homme une image de cette souveraineté perdue et interdite. [...] C'est à cause de cette image que l'univers, bien qu'impitoyable, mérite d'être aimé, même au moment où l'on souffre, comme une patrie et une cité.
>
> Simone Weil: Sur la science. Gallimard 1966.
> Chapter "La science et nous", pp. 128–140

This thesis adheres to the following design principles. Chapter 1 extensively introduces and contextualises (abridged versions of some of) the results of the thesis, without giving proofs. The Chapters 2–4 then prove the results. All definitions of auxiliary substructures used in Chapters 2–4 are gathered in the final Chapter 5. Also, all figures illustrating auxiliary substructures used for constructive purposes in the thesis are consigned to Chapter 5; this does not apply to figures which illustrate some point as we go along, such as Figures 2.2, 2.3, 2.8 and 2.9. Thus, Chapter 5 does more than merely state some basic notations and mathematical prerequisites; it is an essential part of the thesis. There is a list of symbols after Chapter 5, and at the very end an index of some technical terms used in the thesis. Each of Chapters 2–4 starts with an English verb as a motto.

Chapter 2 is not the result of collaboration. Some of the results of Chapter 2 have already been published in [82]. Some results in Chapter 2 about random graphs have been published by the author in [79]. Chapter 2 goes much further than [82] in that it extends some of the results on cycle spaces to flow lattices, i.e. *integral flows.* The proofs of the results on integral flows in Chapter 2, in particular the theorem about $n \equiv 3 \pmod 8$, have not been published before this thesis, but Conjecture 3 had already been announced by the author in [81].

Chapter 3 is the result of work done in collaboration. It presents joint work of the author with Tobias Müller, Marc Noy and Anusch Taraz. Results of Chapter 3 have already been published by the authors in [83], and have moreover been submitted to a journal.

Chapter 4 is not the result of collaboration, and has been published by the author in [80].

1.2 Flow lattices of graphs and the pursuit of constructibility

For as long as precise formal knowledge has been sought and kept, one of the main pursuits have been questions of *constructibility*: can a structure of a given type be built from other structures of another given type? Ancient examples are Archimedes' attempt at constructing a number system expressive enough to address each grain of sand, the question whether a square of area equal to a given circle can be constructed using only ruler and compass, the question whether every positive integer is uniquely a product of prime numbers, the question whether every even integer is the sum of at most two prime numbers, and the question whether every odd integer is the sum of at most three prime numbers (the latter was recently completely proved—with some reliance on machine computations—as explained in [84]).

More modern examples, getting progressively closer to the sense of 'construct' that Chapter 1 is concerned with, are the questions whether every ring of invariants of a finite group action on $\mathbb{C}[x_1, \ldots, x_n]$ is finitely-generated as a $\mathbb{C}$-algebra, the question whether the ring of integers of a number field admits a $\mathbb{Z}$-module-basis consisting of finitely-many powers of the same algebraic integer, and the question if and when a lattice (i.e., finitely-generated subgroup) in $\mathbb{R}^d$, assumed to be *generated* by the set of its shortest vectors, admits a *basis* of such vectors (cf. [122] [123] [124]). With the latter example we have arrived at a kind of construction close to the central topic of Chapter 2: a proof will be given for an essentially optimal vertex-degree-condition sufficient for the *flow lattice* of an n-vertex graph to admit a basis of *Hamilton-flows* (i.e., flows with magnitude-one-values and support equal to a Hamilton-circuit of the graph). The flow lattice of a graph is just the free abelian group generated by all the integer-valued circulations on this graph. A circulation is a flow without any sink or source; cf. [50, p. 140]. The flow lattice is the 1-dimensional cycle group of the graph, in the sense of standard simplicial homology. Figure 1.3 is an illustration of the group operation. The flow lattice of a graph contains more information than its more widely known mod-2-reduction, the *cycle space* of a graph; this lattice—when viewed in larger contexts—is the object of research to this day (cf. e.g. [9] and [155] for metric, or [68] for knot-theoretical, or [45] for category-theoretical aspects). Theorem 4 on p. 6 is a new result about the flow lattice.

In spite of homology over fields being determined by homology over $\mathbb{Z}$ via the universal coefficients theorem, if the homology groups are used as *ingredients inside other demands* made on a structure, then settings exist in which the use of field-coefficients can extract *more* information about an underlying space than the use of $\mathbb{Z}$-coefficients (cf. e.g. [77, p. 154–155]). Not so for our setting, and our demand of admitting Hamilton-supported bases of $\mathrm{Z}_1(G)$: since in a vector-space every generating-set contains a basis (for purely algebraic reasons, oblivious to the underlying set) it is only with *integer* coefficients that one can make the question about the generative power of the set of Hamilton-circuits more stringent. All graphs considered, requiring that a fixed set of $\|G\| - |G| + 1$ Hamilton-flow-generators is already sufficient to construct any flow makes a genuinely stronger demand on the underlying combinatorial setting than asking for *some* such set to suffice (cf. Proposition 47 on p. 45). In that sense, integer coefficients help to indirectly say something about the richness and flexibility of the set of Hamilton-circuits that one could *not* say when only using $\mathbb{F}_2$-coefficients: we can make do with the algebraically smallest-possible number of Hamilton-flows, despite the bulkiness of such generators.

Theorem 4 guarantees that every flow is constructible uniquely as a $\mathbb{Z}$-linear combination of a fixed rank-sized set of Hamilton-flows. The author conjectures more results of this type (see Conjecture 3). This thesis proves only part of Conjecture 3 (namely Theorems 4 and 5), but the thesis gives a technique to prove all the rest of it: a monotonicity argument that harks back to the title of this thesis. The global constructibility property (every flow can be constructed) of the graph will be deduced from a local reason: a preselected, highly structured, much sparser spanning subgraph, which already has the property itself. There is freedom in how to select that substructure, yet there are many restrictions, in particular it has to have properties which allow to use theorems guaranteeing a spanning embedding into the ambient graph. One such theorem is the *bandwidth theorem* of Böttcher, Schacht and Taraz (see Theorem 38 in Section 2.1.3 of Chapter 2), which, roughly speaking, provides an optimal sufficient vertex-degree-condition for the existence of a spanning substructure, the condition depending on the chromatic number of that substructure. In our proof of Theorem 4 below we will, in a sense, extract as much information as possible from the bandwidth theorem, in particular use its full form [24, Theorem 2] which allows the desired substructure to have the same chromatic number as the ambient graph, provided that it admits a colouring that has one of its colour classes of constant size.

In particular, for a part of Conjecture 3, left unproved in this thesis, the method which is outlined in ($\mathbb{Z}$-St.1)–($\mathbb{Z}$-St.3) of Section 2.2 applies, too, and the thesis already provides suitable sparsest-possible auxiliary substructures for carrying out the steps (namely $\mathrm{M}_r^{\boxbslash}$ in Definition 216 in Chapter 5). The proof of Conjecture 3.(I.2), and the proof of the auxiliary Conjecture 17 about $\mathrm{M}_r^{\boxbslash}$ in particular, will not be carried out in this thesis, partly for aesthetic and proof-economical reasons: the execution of the proof once again requires arbitrary choices during the technical, linear algebraic work, and the author could not yet make these choices judiciously enough to result in calculations up to the standards of simplicity set by the complete proof of Conjecture 3.(I.1) in Chapter 2. In the author's opinion, the main technical achievement of that proof is to have constructed a general argument in which the matrix for the change of bases contains only entries of absolute value at most *two* (see Definition 68 in Chapter 2, which describes this change of bases). Many alternatives for the arbitrary choices made in the proof lead to sets of Hamilton-flows which, while also constituting a Hamilton-flow-basis for the flow lattice, result in an unmanageably complicated matrix for the change of bases.

1.2.1 Generating sets for $\mathrm{Z}_1(G)$ not containing any basis

In an abelian group, unlike for vector spaces, a basis cannot in general be obtained from a generating set by removing generators. The simplest example: the generating set $\{2, 3\}$ of $\mathbb{Z}$ as a $\mathbb{Z}$-module. More generally, it can happen that *extraneous structural demands* on the generators of an abelian group lead to the following situation:

(1) there exist generating sets meeting the demands,
(2) there does not exist any basis meeting the demands.

In the above example the extraneous structural demand (making use of the extraneous structure $<$ on $\mathbb{Z}$) is 'having generators with magnitude > 1'). Then $\{2, 3\} \subseteq \mathbb{Z}$ is a *generating set* of $\mathbb{Z}$ meeting the extraneous demands, but among the two $\mathbb{Z}$-bases of $\mathbb{Z}$, none does.

There are trivial examples of (1) and (2) in the case of abelian groups of the form $\mathrm{Z}_1(G)$ with G some graph (i.e., flow lattices): just extend the basic example $\{2, 3\} \subseteq \mathbb{Z}$ coefficient-wise to two copies of one and the same element of $\mathrm{Z}_1(G)$ (i.e., take $G = C$ to be a circuit, let $\vec{C}$ denote one of its two orientations, and consider the generating-set $\{2\vec{C}, 3\vec{C}\}$ of $\mathrm{Z}_1(G)$). However, there are non-trivial examples, too (cf. Proposition 1).

A priori, it is not an implausible conjecture that for every graph G, the abelian group $\mathrm{Z}_1(G)$ might have a property forcing every generating set consisting only of *simple* flows (i.e., having values of magnitude at most one on each edge, ruling out the above trivial example) to contain a basis. However, this is not true.[1] The smallest example the author could find (Proposition 1) of a graph G admitting a generating set of $\mathrm{Z}_1(G)$ consisting of simple flows and not containing any basis is $G = \mathrm{W}_6$, the wheel with six spokes. The wheel with 5-spokes appears *not* to provide any such example—there, every generating set appears to contain a basis (this is not proved in this thesis). While hard to believe, this elementary phenomenon (which should be mentioned in every textbook introducing the group of circulations) has apparently not been described before. The author does not know of any such example in the literature[2] and thinks it not superfluous to present it here,

[1] Nor is it true if in the above conjecture 'generating set' is replaced with 'Hamilton-supported generating set' (cf. Proposition 47 in Chapter 2).

[2] For *other lattices* than $\mathrm{Z}_1(G)$, though, this *is* a well-documented topic: in the classical geometric setting which allows *arbitrary* vectors of $\mathbb{R}^d$ as elements of the lattice, it is a well-studied question as to whether (and when)

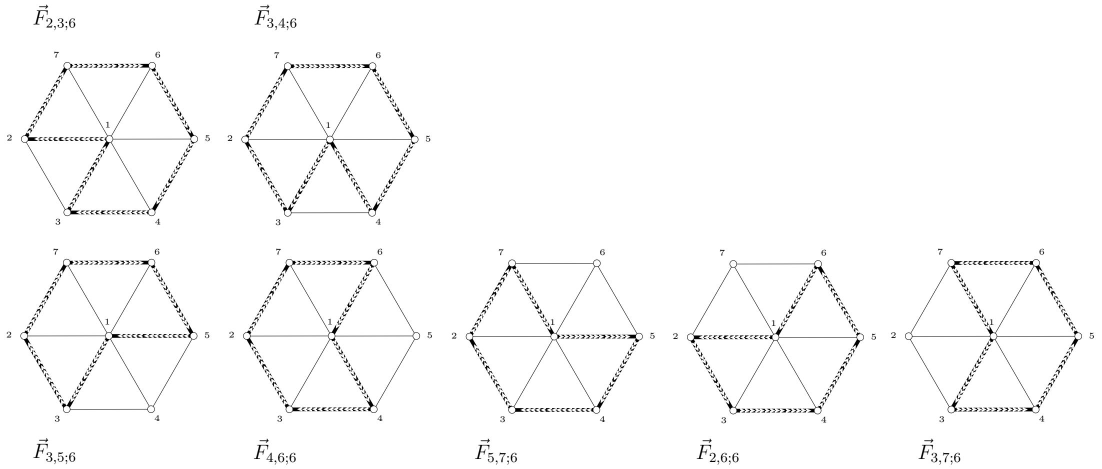


Figure 1.1: The wheel with six spokes is an example of a graph G admitting a generating set of the flow lattice $\mathrm{Z}_1(G)$ not containing any basis. The author could not find a smaller example, and e.g. the 5-wheel does not provide such an example (this is not proved in the present thesis). Figure 1.1 gives an example of a generating set not containing a basis. The incidence matrix of the simple flows (the indices in the name of a flow indicate the end-vertices of the two spokes in the flow's support, followed by the number of spokes of the wheel after the semi-colon; $\vec{F}_{i,j;s}$ is a convenient notation for hub-containing simple flows in a general s-spoke wheel) in the generating set is given in (5.2). The rank of the flow lattice of this graph is $12 - 7 + 1 = 6$, while $\{\vec{F}_{2,3;6}, \vec{F}_{3,4;6}, \vec{F}_{3,5;6}, \vec{F}_{4,6;6}, \vec{F}_{5,7;6}, \vec{F}_{2,6;6}, \vec{F}_{3,7;6}\}$ is a 7-element generating set. Leaving out any of its elements leaves a set not generating the flow lattice anymore. Note that the last five flows are not Hamilton-flows, and in fact, wheel graphs cannot yield examples such as the more complicated one given in Section 2.2.1 of Chapter 2, for the simple reason that a wheel with r spokes contains only r Hamilton-circuits, while the rank of its flow lattice is $r + 1$. This is some justification for giving the complicated example in Section 2.2.1. Moreover, note that none of the flows has as its support an induced circuit. The author conjectures that for every 3-connected graph G, if $\mathcal{S}$ is a generating set of $\mathrm{Z}_1(G)$ consisting only of flows whose support is an induced non-separating circuit, then $\mathcal{S}$ contains a basis for $\mathrm{Z}_1(G)$.

before we move to Hamilton-supported flows. This explicit example of a generating-set of $\mathrm{Z}_1(\mathrm{W}_6)$ not containing a basis is shown in Figure 1.1.

Proposition 1. *The set* $\mathcal{G}(\mathrm{W}_6) := \{ \vec{F}_{2,3;6}, \vec{F}_{3,4;6}, \vec{F}_{3,5;6}, \vec{F}_{4,6;6}, \vec{F}_{5,7;6}, \vec{F}_{2,6;6}, \vec{F}_{3,7;6} \}$ *from Definition 220 and shown in Figure 1.1 is a generating set of* $\mathrm{Z}_1(W_6)$*, and each of the* $7 = \binom{7}{6} = \binom{7}{\mathrm{rk}(\mathrm{Z}_1(\mathrm{W}_6))}$ *rank-sized subsets is* not *a generating set of* $\mathrm{Z}_1(W_6)$.

To prove Proposition 1, it suffices to do a few computations. The elementary divisors of the incidence matrix in (5.2) on p. 204 are $(1^{\times 6})$, which, by standard theory (cf. Section 5.2 in Chapter 5), proves $\mathcal{G}(\mathrm{W}_6)$ to be a generating set of $\mathrm{Z}_1(\mathrm{W}_6)$. By the well-known formula for the Betti-number (a reference in the context of the flow lattice is e.g. [9, Lemma 2]), we have $\mathrm{rk}_{\mathbb{Z}}(\mathrm{Z}_1(\mathrm{W}_6)) = \|\mathrm{W}_6\| - |\mathrm{W}_6| + 1 = 12 - 7 + 1 = 6$. Since $|\mathcal{S}| = 7 > 6 = \mathrm{rk}_{\mathbb{Z}}(\mathrm{Z}_1(\mathrm{W}_6))$, the generating set $\mathcal{B}$ is not a basis. Leaving out any single generator leaves a rank-sized set which, while $\mathbb{Z}$-linearly independent, is not generating anymore, i.e. the subgroup it generates has index > 1 in $\mathrm{Z}_1(\mathrm{W}_6)$. Specifically, leaving out only the $\vec{F}_{2,3;6}$-indexed row results in an incidence matrix with elementary divisors $(1^{\times 5}, 2^{\times 1})$; leaving out only the $\vec{F}_{3,4;6}$-indexed row results in an incidence matrix with elementary

there is a basis consisting only of minimum-length vectors in the lattice, under the assumption that there is *some* (possibly larger than rank-sized) generating set of minimum-length vectors; see e.g. [122] [123] [124]. But for lattices *of the special form* $\mathrm{Z}_1(G)$ with G a graph, the entire literature on flows, circulations, cycle spaces and so on seems never to moot the topic of generating sets not containing a basis.

divisors $(1^{\times 5}, 2^{\times 1})$; leaving out only $\vec{F}_{3,5;6}$ gives $(1^{\times 5}, 3^{\times 1})$; leaving out only $\vec{F}_{4,6;6}$ gives $(1^{\times 5}, 2^{\times 1})$; leaving out only $\vec{F}_{5,7;6}$ gives $(1^{\times 5}, 3^{\times 1})$; leaving out only $\vec{F}_{2,6;6}$ gives $(1^{\times 5}, 2^{\times 1})$; and leaving out only $\vec{F}_{3,7;6}$ gives $(1^{\times 5}, 3^{\times 1})$. Thus, in each case leaving out a single generator results in a set generating a full sublattice with index > 1 in $\mathrm{Z}_1(\mathrm{W}_6)$.

1.2.2 Sufficient conditions for the flow lattice to be Hamilton-based

The following definition is contained[3] in the more technical and condensed Definition 204 in Chapter 5. The notation from Definition 204 will be used for the proofs in Chapter 2, and from a strictly logical point of view it is redundant to give Definition 2. For the purposes of this introduction though, it seems advisable to give this more leisurely version:

Definition 2. *Let us say that a graph G with n vertices and e edges*

(D.1) *has* Hamilton-generated cycle space $\mathrm{Z}_1(G;\mathbb{F}_2)$ *if and only if there exists a set*[4] $\mathcal{S}$ *of Hamilton-circuits of G such that every cycle in G can be constructed as the symmetric difference of (the edge-sets of) elements of $\mathcal{S}$,*
(D.2) *has* almost-Hamilton-generated cycle space $\mathrm{Z}_1(G;\mathbb{F}_2)$ *if and only if in G there exists a circuit C^- of length $n-1$ and some set*[5] $\mathcal{S}$ *of Hamilton-circuits such that every cycle in G can be constructed as the symmetric difference of (the edge-sets of) elements of $\mathcal{S} \sqcup \{C^-\}$,*
(D.3) *has* Hamilton-generated flow lattice $\mathrm{Z}_1(G)$ *if and only if there exists a generating set of the abelian group $\mathrm{Z}_1(G)$ consisting only of Hamilton-flows,*
(D.4) *has* almost-Hamilton-generated flow lattice $\mathrm{Z}_1(G)$ *if and only if there exists a simple flow z^- of length $n-1$ and some set $\mathcal{S}$ of Hamilton-flows such that $\{z^-\} \cup \mathcal{S}$ is a generating set of $\mathrm{Z}_1(G)$,*
(D.5) *has* Hamilton-based flow lattice $\mathrm{Z}_1(G)$ *if and only if G has a Hamilton-generated flow lattice (cf.* (D.3)*), with the generating set having $e-n+1$ elements only,*
(D.6) *has* almost-Hamilton-based flow lattice $\mathrm{Z}_1(G)$ *if and only if it has almost-Hamilton-generated flow lattice (cf.* (D.4)*), with the generating set having $e-n+1$ elements only.*

It seems very likely that the following is true:

Conjecture 3 (sufficient conditions for the flow lattice to be Hamilton-based; announced in [81, (C.1)–(C.3)]). *For every $\gamma > 0$ there is $n_0 \in \mathbb{N}$ such that for every graph G with at least $n \geqslant n_0$ vertices, e edges and minimum-degree δ,*

(I.1) if $\delta \geqslant (\frac{1}{2}+\gamma)n$ and n is odd, then G has Hamilton-based flow lattice,
(I.2) if $\delta \geqslant (\frac{1}{2}+\gamma)n$ and n is even, then G has almost-Hamilton-based flow lattice,
(I.3) if $\delta \geqslant (\frac{1}{4}+\gamma)n$ and G is balanced bipartite, then G has Hamilton-based flow lattice,
(I.4) if in (I.1) and (I.2) the condition '$\delta \geqslant (\frac{1}{2}+\gamma)n$' is replaced with '$\delta \geqslant \frac{2}{3}n$', then without further change to (I.1) or (I.2), it suffices to take $n_0 := 2 \cdot 10^8$.

(I.1) becomes false if '$\delta \geqslant (\frac{1}{2}+\gamma)n$' is replaced with '$\delta \geqslant \lfloor \frac{1}{2}n \rfloor$ and G Hamilton-connected'.

Chapter 2 of this thesis gives

(1) a proof of Conjecture 3.(I.1) restricted to the infinite set
$\{n \in \mathbb{N} \colon n \geqslant n_0(\gamma), \quad n \equiv 3 \pmod 8\}$,
with $n_0(\gamma)$ some constant depending only on γ, cf. Theorem 4,

[3] Property (D.1) is property $\mathrm{Cd}_0\mathcal{C}_{|\cdot|}$ in Definition 204.(5); property (D.2) is property $\mathrm{Cd}_0\mathcal{C}_{|\cdot|}{}^-$ in Definition 204.(6); property (D.3) is property $\mathrm{Quo}_{\{0\}}\mathcal{C}_{|\cdot|}$ in Definition 204.(7); property (D.4) is property $\mathrm{Quo}_A\mathcal{C}_{|\cdot|}{}^-$ in Definition 204.(8); property (D.5) is property $\mathrm{Bas}\mathcal{C}_{|\cdot|}$ in Definition 204.(15); property (D.6) is property $\mathrm{Bas}\mathcal{C}_{|\cdot|}{}^-$ in Definition 204.(16).

[4] By linear algebra alone, oblivious to the underlying combinatorics, it would be equivalent to write 'a set of $e-n+1$ Hamilton-circuits'.

[5] Again by linear algebra alone, it would be equivalent to write 'a set of $e-n$ Hamilton-circuits'.

(2) a proof of Conjecture 3.(I.4) restricted to the infinite set
$\{n \in \mathbb{N} \colon n \geqslant n_0(\gamma), \quad n \equiv 3 \pmod 8\}$,
with $n_0(\gamma)$ some constant depending only on γ, cf. Theorem 5,
(3) a proof of the version of Conjecture 3 which is obtained by reading all statements modulo 2, see Theorem 6 below,
(4) auxiliary subgraphs to prove Conjecture 3.(I.2), via the same techniques as were sufficient for (1) and (2): the graphs $\mathrm{M}_r^{\boxslash}$ from Definition 216; that proof is not carried out, though.

For further reference, let us state (1), (2) and (3) separately, as Theorems 4, 5 and 6:

Theorem 4. *For every $\gamma > 0$ there is $n_0 \in \mathbb{N}$ such that for every graph G with $n_0 \leqslant |G| \equiv 3 \pmod 8$ and minimum degree $\delta(G) \geqslant (\frac{1}{2} + \gamma)|G|$, its flow lattice $\mathrm{Z}_1(G)$ is Hamilton-based.*

Theorem 5. *If $2 \cdot 10^8 \leqslant |G| \equiv 3 \pmod 8$ and $\delta(G) \geqslant \frac{2}{3}|G|$, then $\mathrm{Z}_1(G)$ is Hamilton-based.*

Theorems 4 and 5 would even then be new results if 'Hamilton-based' were replaced by 'Hamilton-*generated*'. It is the main concern of Chapter 1, though, to conclude as much as possible from the strong structural information provided by the *bandwidth theorem* of Böttcher, Schacht and Taraz [24, Theorem 2]. Then, asking for *bases* is the right question. Bases consisting of Hamilton-flows, as opposed to mere generating sets, are where the bandwidth theorem can fully weigh in with its spanning substructures: the theorem allows one to rather freely preselect auxiliary spanning substructures, and this makes it possible to select rank-many flows in advance and then prove, by explicit calculations, that they constitute a basis. In doing so, one has to judiciously make rather arbitrary choices, both of the substructure itself and, another free choice (which can easily turn out to be wrong): a spanning-tree-basis on the one, and an explicit Hamilton-circuit-basis on the other hand, in such a way that the resulting change-of-basis-matrix is of manageable complexity. If one uses the bandwidth theorem, much of the information it offers would be squandered when merely concluding the existence of a Hamilton-supported *generating set*.

Theorems 4 and 5 will be proved in Section 2.2 of Chapter 2. It is exceedingly likely that both Theorem 4 and Theorem 5 are true with '$|G| \equiv 3 \pmod 8$' replaced by the (necessary) condition '$|G| \equiv 1 \pmod 2$', and that instead of '$\delta(G) \geqslant (\frac{1}{2} + \gamma)|G|$' the hypothesis '$\delta(G) \geqslant \frac{1}{2}|G| + c$' with a small constant c is sufficient (and provable in practice), but this is not proved in this thesis—analogous techniques will suffice, but the decision to employ sparsest-possible auxiliary substructures incurs dependencies on divisibility-properties of n more complicated than being odd.

Now to the mod-2-version mentioned in (3) (see Definition 204 for the $\mathcal{M}$-notation):

Theorem 6 (sufficient conditions for a cycle space generated by Hamilton-circuits [82, Theorem 1]; the bipartite case (I3) had already been announced in [22])**.** *For every $\gamma > 0$ there exists $n_0 \in \mathbb{N}$ such that for every graph G with $|G| \geqslant n_0$, the following is true:*

(I1) if $\delta(G) \geqslant (\frac{1}{2} + \gamma)|G|$ and $|G|$ is odd, then $G \in \mathcal{M}_{|\cdot|,0}$,
(I2) if $\delta(G) \geqslant (\frac{1}{2} + \gamma)|G|$ and $|G|$ is even, then $G \in \mathcal{M}_{|\cdot|,1}$ and $G \in \mathcal{M}_{\{|\cdot|-1,|\cdot|\},0}$,
(I3) if $\delta(G) \geqslant (\frac{1}{4} + \gamma)|G|$ and G is balanced bipartite, then $G \in \mathcal{M}_{|\cdot|,0}$,
(I4) if in (I1) and (I2) the condition '$\delta(G) \geqslant (\frac{1}{2} + \gamma)|G|$' is replaced by '$\delta(G) \geqslant \frac{2}{3}|G|$', then without further change to (I1) or (I2) it suffices to take $n_0 := 2 \cdot 10^8$.

Implication (I1) becomes false if ' $(\frac{1}{2} + \gamma)|G|$ ' is replaced by ' $\lfloor \frac{|G|}{2} \rfloor$ and G Hamilton-connected'.

Theorem 6 will be proved in Section 2.1 of Chapter 2.

In (I1), the hypothesis of odd $|G|$ is necessary: as a consequence of Mantel's theorem [120], every graph G with $\delta(G) \geqslant \lfloor |G|/2 \rfloor + 1$ contains a triangle T. If $|G|$ is even, the vector with support T cannot be an $\mathbb{F}_2$-linear combination of the (even-length) Hamilton-circuits.

A purely combinatorial way of phrasing (I1) and (I3) is to say that 'every circuit in G can be realised as a symmetric difference of some Hamilton-circuits of G'. In this variant phrasing, talking about graph-theoretical circuits does not lose any generality since for any graph G and

Aspects of Hamilton-circuits	Literature
efficient algorithms for finding a copy	[21, Section 4], [148]
number of all copies	[149], [42], [41]
number of mutually edge-disjoint copies	[140] [141] [106]
host graph random	[17] [100] [12] [99] [110] [105]
linear algebraic properties, recombining them into shorter circuits	this thesis

Table 1.1: Some aspects of Hamilton-circuits in graphs with high minimum degree.

any cycle $c \in \mathrm{Z}_1(G;\mathbb{F}_2)$, the support $\mathrm{Supp}(c)$ is an edge-disjoint union of graph-theoretical circuits [50, Proposition 1.9.2]. Let us note in passing that this generalises to locally-finite *infinite* graphs [51, Theorem 7.2, equivalence (i) $\Leftrightarrow$ (iii)], and that it has been given a precise sense for arbitrary compact metric spaces [62]. *Linear-algebraic* properties of Hamilton cycles *in infinite graphs*—i.e. the role of infinite Hamilton circles vis-à-vis the cycle space (in the sense of [48] [49] [51] [52])—present an unexplored research topic (the search for conditions sufficient for the mere existence of one Hamilton circle was begun in [30]).

Not only as a result of Theorem 4 (which of course yields infinitely-many such examples), but already *during the proof* of Theorem 4 will we construct what are apparently the first non-trivial examples of graphs having a flow lattice admitting a *basis* of Hamilton-flows (namely, the auxiliary graphs $\mathrm{C}_n^{2^-}$ from Definition 214). While examples (certain Cayley graphs) of flow lattices admitting a *generating set* of Hamilton-flows have been known ever since [117] and [137], the question of whether it is possible to use the smallest number of Hamilton-flow-generators that is possible for algebraical reasons alone (i.e., the rank of the flow lattice) seems never to have been mooted before. In the literature, the only other result about *bases* of the flow lattice other than the very well documented spanning-tree-bases appear to be *ear-decomposition-bases* (cf. [118, Theorem 2.2]), these bases, however, somewhat resemble spanning-tree-basis in that they typically contain many flows with rather short circuits as their supports, and also in that they have the linear independence of the basis elements more or less built into the inductive definition (which is completely different in the case of Hamilton-flows, where to prove linear independence can be problematic).

1.2.2.1 The conditions for Hamilton-based flow lattice in a larger context

We now give more context for the central notion of Chapter 2: longest-possible circuit-supported generators, i.e. Hamilton-circuit-supported generators for cycle spaces and flow lattices. Some $\mathbb{F}_2$-specific context is furthermore to be found at the beginning of Chapter 2.

For the systematising efforts in the present section we in particular use the graph properties $\mathcal{M}_{|\cdot|}^{\mathbb{Z}\mathrm{Bas}}$ (resp. $\mathcal{M}_{|\cdot|,\langle 0\rangle}^{\mathbb{Z}}$, resp. $\mathcal{M}_{|\cdot|,0}$) from Definition 204.(19) (resp. 204.(14), resp. 204.(12)) in Chapter 5, i.e., the set of all graphs with Hamilton-based flow lattice (resp. all graphs with Hamilton-generated flow lattice, resp. all graphs with Hamilton-generated cycle space).

For the graph $\mathrm{X}_{\neg\mathrm{hb}}^{\mathrm{hg}}$ from Definition 221 in Chapter 5 we have $\mathrm{X}_{\neg\mathrm{hb}}^{\mathrm{hg}} \in \mathcal{M}_{|\cdot|,\langle 0\rangle}^{\mathbb{Z}}$ yet $\mathrm{X}_{\neg\mathrm{hb}}^{\mathrm{hg}} \notin \mathcal{M}_{|\cdot|}^{\mathbb{Z}\mathrm{Bas}}$, by Proposition 47. For the graph Pr_6 from Definition 206 we have $\mathrm{Pr}_6 \in \mathcal{M}_{|\cdot|,0}$ by Lemma 37.(a10), yet $\mathrm{Pr}_6 \notin \mathcal{M}_{|\cdot|,\langle 0\rangle}^{\mathbb{Z}}$ (see Figure 2.3). These two examples prove

$$\mathcal{M}_{|\cdot|}^{\mathbb{Z}\mathrm{Bas}} \subsetneq \mathcal{M}_{|\cdot|,\langle 0\rangle}^{\mathbb{Z}} \subsetneq \mathcal{M}_{|\cdot|,0} \ . \tag{1.1}$$

In principle, (1.1) leaves the possibility that the probability thresholds w.r.t. $G(n,p)$ of the three graph properties $\mathcal{M}_{|\cdot|}^{\mathbb{Z}\mathrm{Bas}}$, $\mathcal{M}_{|\cdot|,\langle 0\rangle}^{\mathbb{Z}}$ and $\mathcal{M}_{|\cdot|,0}$, each of which is monotone by (7), (5) and (1) in Lemma 43, might *differ*. It seems more plausible, however, that w.r.t. the measure of $G(n,p)$ these three sets are asymptotically almost surely indistinguishable and that the threshold for all three of them coicides with the threshold for minimum-degree three, i.e. $(\log n + 2\log\log n)/n$; by

Theorem 11 alone, the threshold for each of these three monotone properties must be at least that large. Each of the three thresholds is still not settled, only for $\mathcal{M}_{|\cdot|,0}$ it soon might be (see the remarks after Conjecture 12 on p. 12).

There is much reason to believe that all statements in Conjecture 3 on p. 5 are true. Nevertheless, the proof of each of the statements in Conjecture 3 is more difficult than its corresponding mod-2-version in Theorem 6 above. In view of time- and space-constraints, it was decided to prove Conjecture 3.(I.1) restricted to sufficiently large $n \equiv 3 \pmod 8$, but this completely. The other cases of Conjecture 3 to all appearances yield to the same method, but they seem to necessitate the use of slightly different auxiliary graphs, and then there seems no getting around redoing some technical parts of the proof.

The divisibility-condition is a side-effect of (and justified by) the author's decision to use sparsest-possible auxiliary graphs, with a view towards proving Conjecture 79 on p. 114 and an unconditional strengthening of Dirac's theorem on Hamilton-circuits (cf. [50, Theorem 10.1.1]). The author is working on a proof of Conjecture 3 for *all* sufficiently large odd n, sparsest-possible auxiliary substructures and all cases up to the standards set by the proof in Section 2.2 of Chapter 2.

Non-constantly-long generators for the cycle spaces can be a mathematical necessity. For example in the *prism* graph $\mathrm{P}_r = \mathrm{K}^2 \,\square\, \mathrm{C}^r$ over a circuit of length r, also called a *r-rung cyclic ladder graph* (delete the vertex z from $\mathrm{P}_6^{\boxtimes}$ in Figure 5.1 of Chapter 5 to obtain an illustration of that graph), *any* generating set of $\mathrm{Z}_1(\mathrm{P}_r;\mathbb{F}_2)$ must contain a cycle with support of size at least $r = \frac{1}{2}|\mathrm{P}_r|$, the reason being that the obvious (extension to $\mathrm{Z}_1(\mathrm{P}_r;\mathbb{F}_2)$ of the) setwise-rung-fixing involution $\varphi \in \mathrm{Aut}(\mathrm{P}_r)$ fixes each of those elements of $\mathrm{Z}_1(\mathrm{P}_r;\mathbb{F}_2)$ which have as their support a union of any of the (to use self-explanatory language left undefined) 'rectangular circuits' of P_r; but there do exist elements of $\mathrm{Z}_1(\mathrm{P}_r;\mathbb{F}_2)$ which are *not* fixed by φ, hence no generating set can consist only of cycles with supports consisting of unions of rectangular circuits, and each cycle not having such a support must have length at least r.

The above example is about generators of $\mathrm{Z}_1(G;\mathbb{F}_2)$ with length $\frac{1}{2}|G|$. Beyond this length, in particular with regard to Hamilton-circuit-supported generators, there are three ways of viewing the results of Chapter 2 as a continuation of existing lines of investigation:

(ct.1) strengthening results by Locke, Morris, Moulton and Witte about Hamilton-flows in Cayley graphs,
(ct.2) approaching a conjecture of Bondy on the generative power of long circuits in sufficiently dense and connected graphs,
(ct.3) strengthening the conclusion of sufficient criteria for the existence of a Hamilton-circuit under any of the aspects of
  (i) total number,
  (ii) relative position,
  (iii) generative power of the set of all Hamilton-circuits
  w.r.t. *combining them* in some algebraic sense.

As to (ct.1), non-trivial examples of flow lattices of graphs admitting *generating sets* consisting entirely of Hamilton-supported flows have been known since [117]. The MR-review of [137] says:

> Brian Alspach asked which flows can be written as a sum of Hamiltonian cycles. This paper gives an answer to this question for non-cubic Cayley graphs on abelian groups of even order.

While it is true that [137] gives some answer to Alspach's question, one might add that Alspach could have asked for *more*: which flows can be generated from one fixed *basis* (i.e., $\mathrm{rank}(\mathrm{Z}_1(G))$-sized set) of Hamilton-flows? This appears to be a natural open question, even for Cayley-graphs on abelian groups, and it is a question which seems a natural target for the method of preselecting specific spanning substructures for which to find the basis first. In particular, there is the following open question (bases are never mentioned in any of [135] [136] [117] or [137], and the method of proof in [137] seems not to offer enough control on the total number of Hamilton-flow-generators used to answer Question 7):

Question 7. *Do all those statements in [117] and [137] which guarantee a* generating set *for the lattice of integer flows consisting of Hamilton-flows remain true if strengthened*[6] *to guaranteeing a* basis *of Hamilton-flows?*

In particular, and more specifically, the following question appears to be an open, yet natural and approachable research problem:

Question 8. *Is the flow lattice* $\mathrm{Z}_1(G)$ *of every connected Cayley-graph* G *on a finite, odd-order abelian group Hamilton*-based*?*

For the special Cayley graph $G = \mathrm{C}_n^2 \cong \mathrm{Cay}(\mathbb{Z}/n, \{1, 2, n-2, n-1\})$ and $n \equiv 3 \pmod 8$, Question 8 has an affirmative answer, as a consequence (cf. Corollary 70) of the stronger statement Proposition 69 in Chapter 2. A proof via a monotonicity argument analogous to ($\mathbb{Z}$-St.1)–($\mathbb{Z}$-St.3) in Section 2.2.2 of Chapter 2 might already be enough to settle Question 8, the difference being that instead of the bandwidth theorem one would have to use a dedicated embedding lemma for Cayley-graphs. Thus, analogously to the problem (1.12) in the context of random graphs on surfaces, there arises an associated open methodological problem:

Develop techniques for embedding spanning subgraphs into Cayley-graphs, techniques which in some sense are general enough to avoid having to develop a new proof for each desired subgraph. (1.2)

As to (ct.2), another body of context for Theorem 6 is provided by literature on the following open conjecture of Bondy:

Conjecture 9 (Bondy 1979; [76, p. 246] [114, Conjecture 1] [115, p. 256] [116, Conjecture 1] [11, Conjecture A] [2, p. 21] [3, p. 12])**.** *If* $d \in \mathbb{Z}$, *in every vertex-*3*-connected graph* G *with* $|G| \geqslant 2d$ *and* $\delta(G) \geqslant d$, *the set of all circuits of length at least* $2d-1$ *is an* $\mathbb{F}_2$*-generating set of* $\mathrm{Z}_1(G; \mathbb{F}_2)$.

Over thirty years ago, Locke proved [114, Theorem 2 and Corollary 4] that Bondy's conjecture is true under the *additional* assumption of 'G non-hamiltonian or $|G| \geqslant 4d-5$'.

Theorem 6 gives an asymptotic answer for two special cases of Conjecture 9: if $\gamma > 0$, $|G|$ is sufficiently large, and '$\delta(G) \geqslant d$' is replaced by '$\delta(G) \geqslant (1+\gamma)d$', then (I2) in Theorem 6 says that if '$|G| \geqslant 2d$' holds as '$|G| = 2d$', Bondy's conclusion is true, and if '$|G| \geqslant 2d$' holds as '$|G| = 2d+1$', then (I1) in Theorem 6 says that of the three lengths $|G|-2$, $|G|-1$ and $|G|$ which Bondy allows as lengths of generating circuits, $|G|$ alone is enough. It seems likely that with the techniques used in this thesis, i.e., embedding a preselected auxiliary seed graph, it will be possible to make further inroads towards Conjecture 9.

More context for Theorem 6 can be found in Section 2.1 of Chapter 2. The formulation of Theorem 6 invites improvements (e.g. eliminating the lower bound on $|G|$, proving non-asymptotic results, or finding an infinite set of counter-examples disproving the weakened implications for *every* $|\cdot|$, instead of only for $|G| = 7$ and $|G| = 12$ as is done in Section 2.1.3.3). In particular, the following questions[7] are still open:

(Q1) Does (I1) remain true when $(\frac{1}{2}+\gamma)|G|$ is lowered to the Dirac threshold $\frac{1}{2}|G|$?
(Q2) Does (I3) remain true when $(\frac{1}{4}+\gamma)|G|$ is lowered to $\delta(G) \geqslant \frac{1}{4}|G|+1$?

The road we took to (I1) and (I3) suggests the following open questions about spanning subgraphs:

(Q3) Let G be a graph with $|G| \equiv 3 \pmod 4$ and $\delta(G) \geqslant \frac{1}{2}|G|$. Does it follow that there is a graph embedding of $\mathrm{C}_{|G|}^{2-}$ into G? (For $\mathrm{C}_{|G|}^{2-}$ cf. Definition 214 on p. 199 with $n := |G|$.)
(Q4) Let G be balanced bipartite with $\delta(G) \geqslant \frac{1}{4}|G|+1$. Does it follow that $\mathrm{CL}_{\frac{1}{2}|G|} \hookrightarrow G$?

[6] And this would indeed be a not only formally but materially stronger statement, as proved by Proposition 47.
[7] Note that in (Q1), because of the necessary hypothesis that $|G|$ be odd, the threshold $\frac{1}{2}|G|$ equals $\lfloor\frac{1}{2}|G|\rfloor + 1$.

An affirmative answer to (Q3) implies an affirmative answer to (Q1) in the case $|G| \equiv 3 \pmod 4$. An affirmative answer to (Q4) implies an affirmative answer to (Q2).

The two latter implications hold because of the argument summarised in ($\mathbb{F}_2$-St.2)—($\mathbb{F}_2$-St.3) above. The graphs $\mathrm{Pr}_r^{\boxtimes}$ and $\mathrm{M}_r^{\boxtimes}$ from (Q3) are visualised in Figure 5.1.

As to (Q4), it should be noted that a theorem of Czygrinow and Kierstead [43, Theorem 1] comes tantalizingly close: if G is a sufficiently large balanced bipartite graph, then $\delta(G) \geqslant \frac{1}{4}|G|+1$ implies that G contains a spanning copy of the *non*-cyclic ladder NCL_r (defined as CL_r with the two edges $\{a_{r-1}, b_0\}$ and $\{a_0, b_{r-1}\}$ removed). Alas, this small defect is enough to render this spanning subgraph unsuitable for serving as an auxiliary substructure in the same way CL_r did above: while the non-cyclic ladder still is Hamilton-laceable, the loss of the two edges causes a drastic drop in the dimension of $\langle \mathcal{H}(\cdot) \rangle_{\mathbb{F}_2}$: to use the technical notation from Definition 204 in Chapter 5, whereas $\mathrm{CL}_r \in \mathrm{Cd}_0\mathcal{C}_{|\cdot|}$ by Lemma 37.(a15) in Chapter 2, it can be checked that NCL_r contains only *one* Hamilton-circuit, hence $\mathrm{NCL}_r \in \mathrm{Cd}_{\beta_1(\mathrm{NCL}_r)-1}\mathcal{C}_{|\cdot|}$.

We now survey the literature relevant to (Q1). In the pursuit of Question (Q1), an affirmative answer to which would give a nice strengthening of Dirac's theorem, one should simultaneously keep in mind the following two facts:

(1) any graph G with $|G|$ odd and $\delta(G) \geqslant \frac{1}{2}|G|$ is Hamilton-connected,
(2) Hamilton-connectedness alone does not imply a Hamilton-generated cycle space.

Here, (1) is an immediate corollary of a theorem of O. Ore (owing to oddness of $|G| =: n$, it follows from $\delta(G) \geqslant n/2$ that in [145, Theorem 3.1] we have $\rho(u) + \rho(v) \geqslant n+1$ for any two non-adjacent vertices u and v). Moreover, (2) is proved by the example $\mathrm{CE}_{(\mathrm{I1})}$ from Definition 212.

Question (Q1) seems not to have been explicitly asked in the literature. There is, however, the aforementioned Conjecture 9, which according to [114, Reference 1] [116, Reference 3] dates back to 1979 and apparently is still open. For $n := |G| = 2d$, Conjecture 9 asks for a generating set consisting of Hamilton-circuits together with all circuits shorter by one. For the case of even $n = 2d$, these additional circuits are clearly necessary, but the point of Question (Q1) is that for odd $n := 2d+1$ it seems quite possible to make do solely with Hamilton-circuits (instead of the three lengths $2d-1$, $2d$ and $2d+1 = |G|$ allowed by Bondy's conjecture), all the more so as Theorem 6 gives an asymptotic affirmative answer to (Q1). The only papers explicitly addressing Bondy's conjecture apparently are [76] [114] [115] [116] [11] [2] [3]. We will briefly consider each of them. In [76, p. 246], Conjecture 9 is merely mentioned at the end as a related open conjecture. In [114, Theorem 2 and Corollary 4] it is proved that for every $d \in \mathbb{Z}$, if G is a 3-connected graph with $\delta(G) \geqslant d$ which is either non-hamiltonian or has $|G| \geqslant 4d-5$, then $\mathrm{Z}_1(G; \mathbb{F}_2)$ is generated by its circuits of length at least $2d-1$ (note that if $|G| \geqslant 4d-5$, the conclusion in Bondy's conjecture is far from generatedness by Hamilton-circuits). Furthermore, [115] does not have the cycle space as its main concern but announces the results of [114] at the very end. Moreover, [116] and [3] study the question if and when $\mathcal{CO}_{\mathfrak{L}'} \subseteq \mathrm{Cd}_0\mathcal{C}_{\mathfrak{L}''}$ for different sets of lengths $\mathfrak{L}'$ and $\mathfrak{L}''$, but these papers do not deal with with minimum-degree conditions and Conjecture 9 is merely mentioned in passing [116, p. 77] [3, p. 12]. As to [11], it can be proved that that paper does not answer (Q1) either:

Theorem 10 (Barovich–Locke [11, Theorem 2.2])**.** *Let $d \in \mathbb{Z}$, let G be a finite hamiltonian graph, let G be 3-connected, $\delta(G) \geqslant d$ and $|G| \geqslant 2d-1$. If $|G| \in \{9, \dots, 4d-8\}$, and if there exists at least one $v \in \mathrm{V}(G)$ such that $G - v$ is* not *hamiltonian, and if another condition holds (which is irrelevant here), then $\mathrm{Z}_1(G; \mathbb{F}_2)$ is generated by the set of all circuits of length at least $2d-1$.*

The point to be made is that if $|G|$ is odd and $\delta(G) \geqslant \lceil \frac{|G|}{2} \rceil$, and if the theorem of Barovich–Locke is to yield generatedness by Hamilton-circuits, then necessarily we must set $2d-1 = |G|$. While this automatically makes the hypothesis $|G| \in \{9, \dots, 4d-8\}$ true, and while $\delta(G) \geqslant \lceil \frac{|G|}{2} \rceil$ ensures that G is hamiltonian and also that G is 3-connected, the remaining hypothesis of Theorem 10 above *cannot possibly be true* in the setting of Question (Q1): for every $v \in \mathrm{V}(G)$ we have $\delta(G-v) \geqslant \delta(G) - 1$ $\geqslant$ (since $\delta(G)$ is an integer) $\geqslant \lceil \frac{1}{2}|G| \rceil - 1 = \frac{|G|}{2} - \frac{1}{2} = \frac{1}{2}|G-v|$, hence $G - v$ is still hamiltonian by Dirac's theorem. Hence Theorem 10, as it stands, does not answer Question (Q1). Furthermore, in

[2] the sentence "in the presence of a long cycle every k-path-connected graph is $(k+1)$-generated" [2, Introduction, last paragraph] cannot be construed so as to answer Question (Q1): each of the slightly different ways in which this phrase is made precise by the authors (cf. [2, Corollary 5, Lemmas 9 and 10]) involves additional assumptions one of which always is that there exists a circuit of length $2k-2$ or $2k-3$. The existence of such a circuit implies that '$(k+1)$-generated' is far from meaning 'generated by Hamilton-circuits'.

As to (ct.3), let us start by mentioning that, regrettably, at present there appears not to be any precise mathematical synergy between the known results on (ct.3).(i), (ct.3).(ii) and (ct.3).(iii); these are but variations on a theme: that the set of Hamilton-circuits in a graph satisfying a sufficient condition for hamiltonicity is rich and flexible. Implications between results pertaining to any of (ct.3).(i)–(ct.3).(iii) seem not be found in the literature. However, the present thesis gives one precise negative conjecture (Conjecture 22) and one precise negative result (Theorem 11) on hypothetical implications of the type (i) $\Rightarrow$ (iii). In a binomial random graph, the edge-probability $p(n) = \frac{\log n + \log\log n + \omega(1)}{n}$ gives the threshold for the a.a.s. existence of at least one Hamilton-circuit. By results of Glebov and Krivelevich [64] this already a.a.s. guarantees a superexponential number of Hamilton-circuits (an example of a result of type (ct.3).(i)). In particular, it already guarantees far more Hamilton-circuits than the dimension of the cycle space of such a random graph. However, in Section 2.3 of Chapter 2 of the present thesis we will prove that nevertheless $p(n) = \frac{\log n + \log\log n + \omega(1)}{n}$ is *not yet sufficient* to a.a.s. guarantee that all these many Hamilton-circuits generate the cycle space! Although in $G_{n,p}$ the property of a.a.s. hamiltonicity necessarily comes with extras (e.g., [39] [64]), in particular with far more Hamilton-circuits than the dimension of the cycle space, the property of the Hamilton-circuits generating the cycle space is *not* guaranteed right from the start of a.a.s. hamiltonicity. This gives a sense in which a result of type (ct.3).(iii) (Hamilton-circuit-generated cycle space) is *not* implied by quite a strong example (superexponentially-many Hamilton-circuits) of a result of type (ct.3).(i).

Let us repeat that (cf. e.g. [103, Theorem 1]) a.a.s. hamiltonicity already begins at $p \geqslant (\log n + 1 \cdot \log\log n + \omega(1))/n$. In the following Theorem 11, we do not require in the hypotheses that p imply a.a.s. Hamilton-connectedness, but it is part of the meaning of Theorem 11 that for $G_{n,p}$ to a.a.s. and non-trivially have its cycle space generated by Hamilton-circuits, p necessarily must be at least as large as the threshold for Hamilton-connectedness. (This threshold is known since at least [119, Theorem 1 with $k=2$].)

One important remark on Theorem 11 is that it is an example of a non-implication w.r.t. the set of all graphs becoming an a.a.s. implication when restricted to random graphs as models: obviously, the cycle space $\mathrm{Z}_1(G;\mathbb{F}_2)$ being generated by Hamilton-circuits implies that any two adjacent vertices are connected by a Hamilton-path; it is not clear, however, that it also implies that any two *non*-adjacent vertices are connected by a Hamilton-path. And, indeed, all graphs considered, this is false, see the example briefly described on p. 43 below the non-inclusion (2.14). To sum up, being Hamilton-generated does not imply being Hamilton-connected, even restricted to 3-connected graphs; but, in view of Theorem 11 and [119, Theorem 1], the implication *does* hold in an a.a.s. sense when restricted to graphs $G_{n,p}$ as models:

Theorem 11 (a Hamilton-generated cycle space does not appear from the onset of hamiltonicity of $G_{n,p}$; cf. [79])**.** *If $p \in [0,1]^{\mathbb{N}}$ is such that with $G \sim G_{n,p}$ a.a.s. for odd n*

(1) *G is neither a forest nor a circuit,*
(2) *every cycle in G is a symmetric difference of Hamilton-circuits,*

then there exists an infinite sequence $(n_k)_{k\in\mathbb{N}}$ of odd numbers and an infinite sequence $(\omega_{n_k})_{k\in\mathbb{N}}$ with $\omega_{n_k} \xrightarrow{k\to\infty} \infty$, such that $p_{n_k} > \frac{\log n_k + 2\log\log n_k + \omega_{n_k}}{n_k}$ for all sufficiently large $k \in \mathbb{N}$.

Theorem 11 is proved on p. 113 in Section 2.3 of Chapter 2.

Intuitively, the result that the flow lattice can be constructed from $\mathrm{rank}(\mathrm{Z}_1(G))$-many generators conflicts with the relative unwieldiness of such generators (compared to, say, non-separating induced circuits). Results about the cycle space tend to require generators to have smallness-properties like being an induced or non-separating circuit; Hamilton-generators can be seen as diametrically opposed to non-separating induced circuits: Hamilton-circuits *do* separate and almost always are *non*-induced. Given this unwieldiness, it would have been a plausible outcome that as soon as δ implies being Hamilton-generated over $\mathbb{F}_2$, it also implies being Hamilton-generated over $\mathbb{Z}$ but *not yet* being Hamilton-*based* over $\mathbb{Z}$, i.e., that there was a minimum-degree-regime (resp. an edge-probability-regime), where one can generate the flow lattice by Hamilton-flows, yet needs more than rank-many generators. Theorem 4 shows (resp. Conjecture 12 postulates) that this is not the case: while the property of being Hamilton-generated in general does not imply being Hamilton-based (even in the presence of Hamilton-connectedness, cf. Proposition 47), seeing them through the lens of minimum-degree-hypotheses (resp. binomial random graphs) blurs that distinction.

Conjecture 12. *With the notions from Definition 2, and with* $\mathbb{N}_{\mathrm{odd}}$ *denoting the odd natural numbers, and for every* $\omega \in [0,1]^{\mathbb{N}}$ *with* $\omega_n \xrightarrow{n\to\infty} \infty$ *and every* $p \in [0,1]^{\mathbb{N}_{\mathrm{odd}}}$,

(gnp.1) *if* $p_n = (\log n + 2\log\log n + \omega_n)/n$ *and* $G \sim G(n, p_n)$,
then P [$\mathrm{Z}_1(G;\mathbb{F}_2)$ *is Hamilton-generated (equivalently, Hamilton-based)*] $\xrightarrow{\mathbb{N}_{\mathrm{odd}}\ni n\to\infty} 1$,

(gnp.2) *if* $p_n = (\log n + 2\log\log n + \omega_n)/n$ *and* $G \sim G(n, p_n)$,
then P [$\mathrm{Z}_1(G)$ *is Hamilton-generated*] $\xrightarrow{\mathbb{N}_{\mathrm{odd}}\ni n\to\infty} 1$,

(gnp.3) *if* $p_n = (\log n + 2\log\log n + \omega_n)/n$ *and* $G \sim G(n, p_n)$,
then P [$\mathrm{Z}_1(G)$ *is Hamilton-based*] $\xrightarrow{\mathbb{N}_{\mathrm{odd}}\ni n\to\infty} 1$.

In Section 2.3 of Chapter 2, an upper bound of $n^{-1/2+\varepsilon}$ (see Theorem 74) on the smallest p sufficient for a Hamilton-generated cycle space, much larger than the one conjectured in (gnp.1), will be deduced from a result of Kühn and Osthus.

Since [$\mathrm{Z}_1(G)$ is Hamilton-based] $\Rightarrow$ [$\mathrm{Z}_1(G)$ is Hamilton-generated] $\Rightarrow$ [$\mathrm{Z}_1(G;\mathbb{F}_2)$ is generated by Hamilton-circuits], for trivial reasons we have (gnp.3) $\Rightarrow$ (gnp.2) $\Rightarrow$ (gnp.1). All graphs considered, the converses of the first two implications in the preceding sentence are false. *If* (gnp.3) is true, though, then with Theorem 11 we can show that the converses of these implications *do* hold restricted to random graphs as models: if $G \sim G_{n,p}$ and satisfies [$\mathrm{Z}_1(G;\mathbb{F}_2)$ is generated by Hamilton-circuits], then $p(n) \geqslant \frac{\log n + 2\log\log n + \omega(1)}{n}$ by Theorem 11, hence, almost surely, [$\mathrm{Z}_1(G)$ is Hamilton-based] by (gnp.3), whose truth we have just assumed. (Since (gnp.2) then follows for trivial reasons, this shows that if (gnp.3) is true, then both converses hold a.a.s.)

Proof-methodological gaps seem to yawn between (gnp.1) and (gnp.2), and between (gnp.2) and (gnp.3). An argument (not described in this thesis), which the author learned from T. Łuczak, appears to settle (gnp.1), but this argument uses $-1 = 1$ in $\mathbb{F}_2$ and appears not to be salvageable in the setting of $\mathbb{Z}$-coefficients. And even if some modification of Łuczak's argument can settle (gnp.2), it appears likely that such a modification will still be of a local and 'element-chasing' kind, expressing a given cycle by *some* Hamilton-flow-generators, not giving enough control on the total number of such generators used, so even then (gnp.3) seems likely to remain out of reach. Currently, none of (gnp.1), (gnp.2) or (gnp.3) is completely proved; while (gnp.1) seems settled by the above-mentioned argument of T. Łuczak, for (gnp.2), and in particular for (gnp.3), even a strategy is missing. The conjecture in (gnp.2) may be of similar difficulty as the problem (cf. [112, Concluding Remarks]) about the rarity of elements of finite order in the one-dimensional homology of a random simplicial complex, and the author has some ideas for treating both that problem and Conjecture 12.(gnp.2) by a uniform, more general method; not so for (gnp.3) though, which seems significantly more difficult.

1.2.2.2 A positive instance of Question (Q1)

We will now analyse a small yet relevant example which is a positive instance for Question (Q1). This example illustrates how a minimum degree just barely satisfying the Dirac threshold can endow a *non*-Cayley graph with the property of having its cycle space generated by its Hamilton-circuits: the graph G from Definition 223 in Chapter 5 satisfies the hypotheses in Question (Q1) (barely

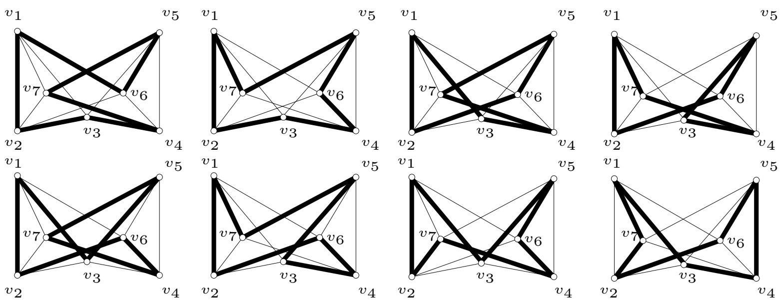


Figure 1.2: An example of an $\mathbb{F}_2$-basis for $\mathrm{Z}_1(\mathrm{G};\mathbb{F}_2)$ consisting only of Hamilton-circuits in a situation where the underlying graph G is not a Cayley graph and presumably owes its being Hamilton-generated to the Dirac condition (which it satisfies just barely). This graph is moreover Hamilton-based, so it is a positive instance of Conjecture 23. Examples illustrating the operation of adding integral flows in this graphs are shown in Figure 1.3. Incidentally, the $\mathbb{F}_2$-basis for $\mathrm{Z}_1(\mathrm{G};\mathbb{F}_2)$ shown in the present figure is *not* the set of supports of a *basis* for $\mathrm{Z}_1(\mathrm{G})$.

so), and $\dim_{\mathbb{F}_2}(\mathrm{G};\mathbb{F}_2) = \beta_1(\mathrm{G}) = \|\mathrm{G}\| - |\mathrm{G}| + 1 = 14 - 7 + 1 = 8$. Furthermore, because of the following fact we cannot prove that G is a positive instance for Question (Q1) just by appealing to Theorem 36.(2) from Section 2.1.2 of Chapter 2:

Proposition 13. *The graph* G *from Definition 223 is not a Cayley graph on any group.*

Proof. While provable elementarily, let us give a high-context proof of this: the order $|\mathrm{G}| = 7$ being prime, the only possible underlying group is $\mathbb{Z}/7$ with addition. Suppose that G were a Cayley graph on $\mathbb{Z}/7$. Since the spectrum of the adjacency matrix of G is $(4, 1, -1, -1, 0, 0, -3)$, the graph G would then be a quartic connected Cayley graph on an abelian group having only integer adjacency-eigenvalues. But this would contradict a classification theorem due to Abdollahi and Vatandoost [1, Theorem 1.1] according to which the set of all orders of such graphs is a finite set which does not contain 7. $\square$

Proposition 14 (G is Hamilton-generated). $\langle \mathcal{H}(\mathrm{G}) \rangle_{\mathbb{F}_2} = \mathrm{Z}_1(\mathrm{G};\mathbb{F}_2)$.

Proof. We give an $\mathbb{F}_2$-basis (shown in Figure 1.2) for $\mathrm{Z}_1(\mathrm{G};\mathbb{F}_2)$ consisting of Hamilton-circuits only. Let $C_1^{\mathrm{G}} := v_1v_2v_3v_4v_7v_5v_6v_1$, $C_2^{\mathrm{G}} := v_1v_2v_3v_4v_6v_5v_7v_1$, $C_3^{\mathrm{G}} := v_1v_2v_6v_5v_7v_4v_3v_1$, $C_4^{\mathrm{G}} := v_1v_2v_6v_5v_3v_4v_7v_1$, $C_5^{\mathrm{G}} := v_1v_2v_6v_4v_7v_5v_3v_1$, $C_6^{\mathrm{G}} := v_1v_2v_6v_4v_3v_5v_7v_1$, $C_7^{\mathrm{G}} := v_1v_2v_7v_4v_6v_5v_3v_1$, $C_8^{\mathrm{G}} := v_1v_7v_2v_6v_5v_4v_3v_1$. w.r.t. the standard basis of $\mathrm{C}_1(\mathrm{G};\mathbb{F}_2)$ the circuits $C_1^{\mathrm{G}}, \ldots, C_8^{\mathrm{G}}$ give rise to the matrix shown in (1.3), which has $\mathbb{F}_2$-rank equal to $8 = \dim_{\mathbb{F}_2}(\mathrm{Z}_1(\mathrm{G};\mathbb{F}_2))$.

$$\begin{array}{c|cccccccc}
 & C_1^{\mathrm{G}} & C_2^{\mathrm{G}} & C_3^{\mathrm{G}} & C_4^{\mathrm{G}} & C_5^{\mathrm{G}} & C_6^{\mathrm{G}} & C_7^{\mathrm{G}} & C_8^{\mathrm{G}} \\
v_1v_2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
v_1v_3 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
v_1v_6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_1v_7 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
v_2v_3 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_2v_6 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
v_2v_7 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
v_3v_4 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\
v_3v_5 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
v_4v_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
v_4v_6 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
v_4v_7 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
v_5v_6 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
v_5v_7 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0
\end{array} \tag{1.3}$$

Therefore the $\mathbb{F}_2$-span of $C_1^{\mathrm{G}}, \ldots, C_8^{\mathrm{G}}$ is an 8-dimensional subspace of the 8-dimensional $\mathbb{F}_2$-vector space $\mathrm{Z}_1(\mathrm{G};\mathbb{F}_2)$, hence is *equal* to $\mathrm{Z}_1(\mathrm{G};\mathbb{F}_2)$, completing the proof of Proposition 14. $\square$

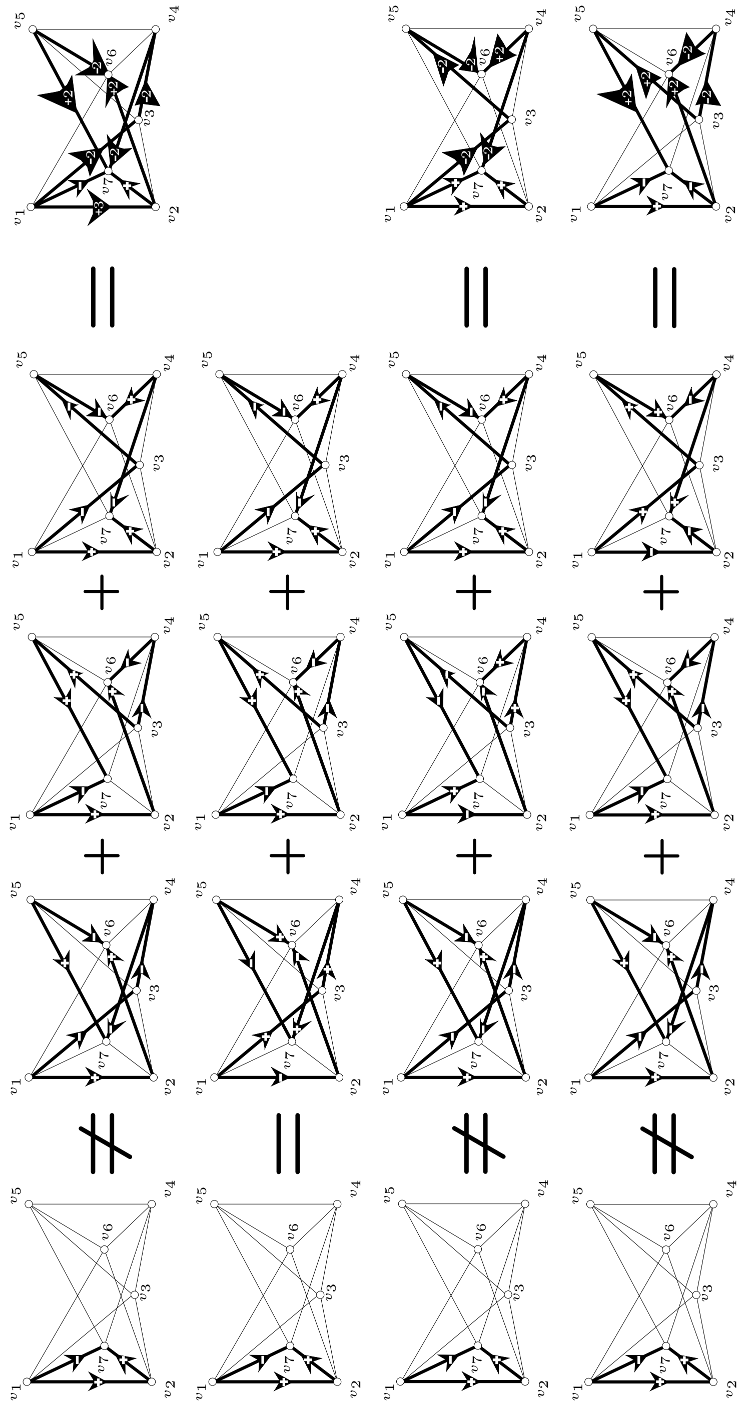


Figure 1.3: Examples illustrating the addition of Hamilton-flows on the non-Cayley-graph from Definition 223. The arrowheads do not have anything to do with the various flows but they indicate the arbitrary orientation selected for each edge (which one *has* to select in order to compute with incidence matrices). The signs in the arrowheads give the coefficient of the respective flow at that oriented edge. Two of the four calculations show that the support of a sum of flows need not be a Eulerian graph.

Incidentally, while G actually *has* Hamilton-based $\mathrm{Z}_1(G)$, the $\mathbb{F}_2$-generating set used to prove Proposition 14 is not a $\mathbb{Z}$-generating set of $\mathrm{Z}_1(\mathrm{G})$. Orienting the circuits in the proof of Proposition 14 yields the following incidence matrix:

$$
\begin{array}{c|cccccccc}
 & \vec{C}_1^{\mathrm{G}} & \vec{C}_2^{\mathrm{G}} & \vec{C}_3^{\mathrm{G}} & \vec{C}_4^{\mathrm{G}} & \vec{C}_5^{\mathrm{G}} & \vec{C}_6^{\mathrm{G}} & \vec{C}_7^{\mathrm{G}} & \vec{C}_8^{\mathrm{G}} \\
\hline
v_1v_2 & + & + & + & + & + & + & + & 0 \\
v_1v_3 & 0 & 0 & - & 0 & - & 0 & - & + \\
v_1v_6 & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_1v_7 & 0 & - & 0 & - & 0 & - & 0 & - \\
v_2v_3 & + & + & 0 & 0 & 0 & 0 & 0 & 0 \\
v_2v_6 & 0 & 0 & + & + & + & + & 0 & - \\
v_2v_7 & 0 & 0 & 0 & 0 & 0 & 0 & + & + \\
v_3v_4 & + & + & - & + & 0 & - & 0 & + \\
v_3v_5 & 0 & 0 & 0 & - & - & + & - & 0 \\
v_4v_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + \\
v_4v_6 & 0 & + & 0 & 0 & - & - & + & 0 \\
v_4v_7 & + & 0 & - & + & + & 0 & - & 0 \\
v_5v_6 & + & - & - & - & 0 & 0 & - & + \\
v_5v_7 & - & + & + & 0 & - & + & 0 & 0
\end{array}
\tag{1.4}
$$

The vector of elementary divisors (cf. Section 5.2 in Chapter 5) of the rank-8-matrix in (1.4) is $(1^{\times 7}, 3^{\times 1})$, hence $\mathrm{Z}_1(\mathrm{G}) \,/\, \langle\, \vec{C}_1^{\mathrm{G}}\ \vec{C}_2^{\mathrm{G}}, \vec{C}_3^{\mathrm{G}}, \vec{C}_4^{\mathrm{G}}, \vec{C}_5^{\mathrm{G}}, \vec{C}_6^{\mathrm{G}}, \vec{C}_7^{\mathrm{G}}, \vec{C}_8^{\mathrm{G}} \,\rangle \cong \mathbb{Z}/3$.

1.2.3 Some more open problems and conjectures related to flow lattices

Problems are inevitable, interesting problems are soluble; this section gives a few more of them. Perhaps the most accessible open problem in this section is:

Conjecture 15. *For every $\gamma > 0$ there is $n_0 \in \mathbb{N}$ such that for every odd $n \geqslant n_0$, in every n-vertex graph G with $\delta(G) \geqslant (\frac{1}{2} + \gamma)n$ the flow lattice $\mathrm{Z}_1(G)$ admits generating sets not containing a basis.*

There is much reason to believe Conjecture 15: by Lemma 43, the property $\mathcal{M}^{\beta_0=1}_{\mathrm{B}\nsubseteq\mathrm{G}}$ of being connected and admitting generating sets not containing any basis is a monotone graph property. So if there is only one infinite family of graphs in it whose existence as a spanning subgraph in G is implied by $\delta(G) \geqslant (\frac{1}{2} + \gamma)n$, then Conjecture 15 holds. Specifically, it would suffice to construct some example of a generating set not containing any basis within each of the seed graphs $\mathrm{C}_n^{2^-}$ from Definition 214. This is probably not too difficult, but the author did not try.

A more difficult conjecture, which by its quantitative conclusion would make more of the rather strong minimum-degree-assumption, is this:

Conjecture 16. *For every $m \in \mathbb{N}$ and every $\gamma > 0$ there is $n_0 \in \mathbb{N}$ such that for every odd $n \geqslant n_0$, in every n-vertex graph G with $\delta(G) \geqslant (\frac{1}{2} + \gamma)n$ the flow lattice $\mathrm{Z}_1(G)$ admits* minimal *generating sets with $m \cdot \mathrm{rank}_{\mathbb{Z}}\mathrm{Z}_1(G)$ elements.*

The following can hardly be called an open problem, since to all appearances it can be treated analogously to Lemma 66. See Figure 5.7 in Chapter 5 for an illustration of Conjecture 17.(3); moreover, the low bandwidth conjectured in (5) is witnessed by the set $\{0, r, \frac{1}{2}(r-1), \frac{1}{2}(3r-1)\} \subseteq \mathrm{V}(\mathrm{M}_r^{\boxbslash})$, which is a $(4, \frac{1}{2})$-separator (in the sense of [23]).

Conjecture 17 (properties of $\mathrm{M}_r^{\boxbslash}$)**.** *For every odd $r \geqslant 5$, and with $\mathrm{M}_r^{\boxbslash}$ as in Definition 216,*

(1) $\mathrm{M}_r^{\boxbslash}$ *has* $|\mathrm{M}_r^{\boxbslash}| - 2$ *vertices of degree three and* 2 *vertices of degree four,*
(2) $|\mathrm{M}_r^{\boxbslash}| = 2r \equiv 2 \pmod 4$ *and* $\|\mathrm{M}_r^{\boxbslash}\| = 3r + 1 = \frac{3}{2}|\mathrm{M}_r^{\boxbslash}| + 1$,
(3) $\mathrm{M}_r^{\boxbslash}$ *is not bipartite but it admits a* 3*-colouring which uses the third colour exactly twice,*
(4) $\mathrm{M}_r^{\boxbslash}$ *is Hamilton-connected,*
(5) $\mathrm{M}_r^{\boxbslash}$ *has sublinear bandwidth in the sense that, for every $\beta > 0$ there exists $n_0 = n_0(\beta)$ such that* $\mathrm{bw}(\mathrm{M}_r^{\boxbslash}) \leqslant \beta \cdot n$ *whenever* $2r \geqslant n_0$,
(6) $\mathrm{Z}_1(\mathrm{M}_r^{\boxbslash})$ *is almost-Hamilton-based.*

Let us now briefly mention some complexity-theoretic questions: let us recall that having the flow lattice Hamilton-generated (resp. -based) is a property of ordinary undirected graphs (just as having elements of finite order in the homology is a property of an abstract unoriented hypergraph);

integers and orientations are props used in defining the property. It is thus natural to restrict to the language of finite graphs only (as is often done, e.g. in the study of the complexity of the Hamiltonian problem). Then the complexity-theoretic status of the graph property $\mathrm{Bas}\mathcal{C}_{|\cdot|}$ (cf. Definition 204.(15)) is an open problem. In particular, while the language of Hamiltonian graph is a standard example for an NP-language, the following seems an open but approachable problem (it probably can be solved by making use of existing specialised literature on the size of solutions to linear equations over the integers; the author has not tried):

Conjecture 18. *W.r.t. to the language of finite graphs:*

(1) *the language* $\mathrm{Bas}\mathcal{C}_{|\cdot|}$ *of all graphs with* Hamilton-based *flow lattice is in NP ,*
(2) *the language* $\mathrm{Quo}_{\langle 0\rangle}\mathcal{C}_{|\cdot|}$ *of all graphs with* Hamilton-generated *flow lattice is in NP .*

The difficulty of Conjecture 18 stems from the restriction to the language of finite graphs (w.r.t. a much richer language, and a much looser measure of certificate-size, both (1) and (2) are of course answered many times over in this thesis).

An obvious attempt at Conjecture 18.(1) is this: let an arbitrary n-vertex graph $G \in \mathrm{Bas}\mathcal{C}_{|\cdot|}$ be given. Let $r := \|G\| - |G| + 1 = \mathrm{rank}(\mathrm{Z}_1(G))$. We now describe a certificate *in the language of finite graphs* for G's being contained in $\mathrm{Bas}\mathcal{C}_{|\cdot|}$. Let $\vec{H}_1, \ldots, \vec{H}_r$ denote any Hamilton-flow-basis for $\mathrm{Z}_1(G)$. Let $H_1, \ldots, H_r$ denote the underlying Hamilton-circuits. Consider any spanning tree T of G. Let $\vec{F}_1, \ldots, \vec{F}_r$ denote the (unique up to the r arbitrary choices of orientations) fundamental-circuit-basis of $\mathrm{Z}_1(G)$ w.r.t. T. For every $i \in [r]$ let $(\lambda_i) \in \mathbb{Z}^{[r]}$ be the unique vector of integers with $\vec{F}_i = \sum_{1 \leqslant j \leqslant r} \lambda_{i,j} \vec{H}_j$. These linear relations are a certificate for $\mathrm{Z}_1(G)$ being Hamilton-based, and they can easily be encoded in the language of finite graphs: first list each of $\vec{H}_1, \ldots, \vec{H}_r$, by listing $r \cdot n$ adjacencies. Then for each $i \in [r]$ write down the orientation of $\vec{H}_i$ encoded as (say) an r-vector of triples (i.e. having length $3r$), each triple consisting of the vertex $1 \in V$, the smaller vertex $v \in V$ of the two neighbours of 1 in H_i, and then either 1 or v again, depending on the orientation of the edge $\{1, v\} \in \mathrm{E}(H_i)$ in $\vec{H}_i$. Then, continuing the certificate string, encode—in the language of finite graphs, according to some encoding rule—the integers $\lambda_{1,1}, \ldots, \lambda_{r,r}$, which can be done in space $O(\sum_{(i,j) \in [r]^2} |\lambda_{i,j}|)$.

The resulting certificate string, which to check a Turing machine can easily be programmed, has three coding 'portions' and size bounded by $r \cdot n + 3 \cdot r + \sum_{(i,j) \in [r]^2} |\lambda_{i,j}|$. Since $r \leqslant n^2 - n + 1 \in O(n^2)$, this is $O(n^3 + \sum_{(i,j) \in [r]^2} |\lambda_{i,j}|)$. What is now needed is an argument that $\sum_{(i,j) \in [r]^2} |\lambda_{i,j}|$ is polynomial in n (for at least one choice of spanning-tree and Hamilton-circuit-basis) for every n-vertex $G \in \mathrm{Bas}\mathcal{C}_{|\cdot|}$. There exists literature on the size of solutions to linear equations over the integers, which might already provide tools sufficient to provide such an argument.

Conjecture 18.(2) seems a little harder; in brief, here the polynomial size of the number of generators is not granted from the beginning, so with regard to the above argument, there is more to be done than proving the sum of magnitudes of coefficients to be polynomial.

Let us mention that, of course, if $\mathcal{B}$ is any Hamilton-supported basis for the free abelian group $\mathrm{Z}_1(G)$, then its *incidence matrix* $\mathrm{A}_{\mathcal{B}}$, together with *any pair of unimodular matrices* P and Q such that PAQ is the Smith Normal Form of A, constitutes something of a polynomial-sized certificate for $\mathcal{B}$'s being a basis of $\mathrm{Z}_1(G)$. The triple $(\mathrm{A}_{\mathcal{B}}, P, Q)$ has size polynomial in n, provided that one uses a *unit cost per number* model. The multiplying out of PAQ and checking whether this rectangular-diagonal matrix contains rank-many ones takes time polynomial in n in the unit cost model, too. However, all of this uses a language much richer than the language of finite graphs, and the unit cost model for the entries just dodges the difficulty of the question.

Moreover, it seems a harder open question whether the set of all graphs which do *not* have a Hamilton-based flow lattice is in NP, equivalently, whether $\mathrm{Bas}\mathcal{C}_{|\cdot|}$ is in coNP. I.e., is it possible to efficiently certify that a given n-vertex graph G does not admit a Hamilton-supported basis for $\mathrm{Z}_1(G)$, for *every* graph which does not admit one? The question is whether one can do significantly better than the exponential-time algorithm of encoding all the Smith Normal Forms (and their certificates), for each rank-sized subset of the typically exponentially-many Hamilton-supported flows. It seems unlikely that non-existence of a Hamilton-basis for Z_1 can in general be efficiently certified:

Conjecture 19. *W.r.t. the language of finite graphs,* $\mathrm{Bas}\mathcal{C}_{|\cdot|}$ *is* not *in coNP.*

In view of inexpressibility theorems about monadic NP (cf. [152]), the following is plausible:

Conjecture 20. *W.r.t. the language of finite graphs,* $\mathrm{Bas}\mathcal{C}_{|\cdot|}$ *is* not *in monadic NP.*

Another open problem is the following:

Conjecture 21. *Deciding whether a given graph has Hamilton-based flow lattice is NP-hard.*

To deduce Conjecture 21 from the known NP-hardness of deciding the existence of *some* Hamilton-circuit, it would suffice to give a construction which, given a graph G, in time polynomial in n constructs a graph $f(G)$ such that G contains a Hamilton-circuit if and only if $f(G)$ is Hamilton-based. The author could not find any such reduction but thinks that proving Conjecture 21 might be doable for someone well-versed in the construction of polynomial reductions.

The following is an open problem. Conjecture 22 is true for $m = 2$, see Figure 2.3, and only interesting because of the two conditions of both odd $|G|$ and $\delta(G) \geqslant 3$. (When dropping *only* the odd-order-condition, then Conjecture 22 is true for trivial reasons, e.g. for even-order complete graphs; when dropping *only* the condition $\delta(G) \geqslant 3$, then Conjecture 22 is true in view of the example of the odd-order graphs consisting of a degree-two vertex joined to an even order complete graph):

Conjecture 22. *For every constant $m \in \mathbb{N}$ there exists an odd-order graph G with $\delta(G) \geqslant 3$ and containing at least $m \cdot \mathrm{rank}(\mathrm{Z}_1(G))$ Hamilton-circuits, yet the flow lattice $\mathrm{Z}_1(G)$ is not generated by Hamilton-flows.*

The following unconditional strengthening of Dirac's theorem on Hamilton-circuits is still not proved but will probably yield to the argument via embedding pre-selected spanning subgraphs (unlike for random graphs, for minimum-degree conditions that argumentation seems to be enough to prove a best-possible result); this is one motivation for the effort that the author invested in constructing sparsest-possible seed graphs:

Conjecture 23. *For every graph G with $|G|$ odd and minimum-degree at least $\lceil \frac{1}{2}|G| \rceil$ the abelian group $\mathrm{Z}_1(G)$ admits a basis of Hamilton-flows.*

1.3 Logical limit laws

For any infinite set $\mathcal{S}$ of finite structures one may ask for the behaviour as $n \to \infty$ of the ratio obtained by dividing the number of all structures of size n modelling a fixed logical formula, by the number of all the structures of size n. In the case of finite graphs, a classic example of such an investigation is a theorem of Glebskii, Kogan, Liogon'kiĭ and Talanov [65] that the set of all finite relational structures of a given finite arity obeys a zero-one-law w.r.t. sentences in first-order logic (FO for short; a brief explanation of the special case of FO-logic in the language of finite graphs can be found on p. 193 in Chapter 5). In the wake of that result, a wealth of analogous results about more restricted classes of structures has been obtained.

The pursuit of selecting a structure $\mathcal{S}$, and then investigating the (existence of) limits $\lim_{n\to\infty} |\{S \in \mathcal{S}_n : S \models \varphi\}| / |\mathcal{S}_n|$ for a given logical statement φ, can be seen as a line of study avoiding the danger of taking too close a look, of overly focusing on contingent details of a set. Density limits can be seen as condensed global pieces of information about a set $\mathcal{S}$. In the present thesis, this point of view will be taken w.r.t. two important classes of graphs, forests and planar graphs, and the (closure of) the set of probability-limits will be explicitly determined, not only for FO-logic, but also for the *monadic second order logic of graphs* (MSO for short). In that language, over and above the variables signifying vertices, there is a second type of variable, whose semantics are that they signify sets, and there is a second binary relation (other than the adjacency relation) whose semantics are that it signifies being an element of a set. For the class of graphs on a fixed surface, which lacks a property called 'addability' (and therefore makes it harder to find 'well-separated' substructures deciding about the truth of a statement) we will be able to at least characterise the probability limits of statements in FO-logic.

MSO-logic is a considerably more expressive language than FO-logic of graphs, and it can define a large part of the notions within the purview of contemporary graph theory: for example, being connected, being acyclic, and being c-colorable for fixed $c \in \mathbb{N}$, each are definable in MSO, but not in FO, the latter not even for $c = 2$. Regarding the central notion of Chapter 2, let us note that even MSO cannot define the set of graphs containing a Hamilton-circuit (cf. [161] [54, Corollary 6.3.5]).

Let us mention in passing that there is a dual aspect to such structure-concerned investigations: instead of fixing the universe, i.e. the structure, and then studying the asymptotic probabilities of various logical statements, one may dually fix the logic, let the structures roam freely and study what abstract universes have what explanatory qualities for assigning probabilities to statements. If you will, that is the essence of scientific thought: freely pursuing structural explanations for logical propositions, by constructing abstract worlds in which they can be true (and with what ease). We will not pursue this point of view; one point of departure to the literature is [61].

1.3.1 Logical limit laws for graphs from an addable class

Taking a point of view more concerned about structures than logic, we will prove that the structures called *addable minor-closed classes of graphs*, i.e. infinite sets of labelled finite graphs, closed under applying graph-isomorphisms, and closed under each of the operations of (1) deleting an edge, (2) contracting an edge and removing multiple edges, (3) adding a new edge in between two connected components, (4) deleting an entire connected component (5) adding an element of the class as a new connected component, can always can be studied from a limit-density point of view:

Theorem 24 (joint work with T. Müller, M. Noy and A. Taraz; [83])**.** *If $\mathcal{A}$ is any addable, minor-closed class of graphs, and φ any sentence in MSO-logic, then* $\lim_{n\to\infty} \frac{|\{G\in\mathcal{A}_n\colon G\models\varphi\}|}{|\mathcal{A}_n|}$ *exists.*

Theorem 24 will be proved in the more explicit version of Theorem 87 in Section 3.1.1 of Chapter 3, a version which will also enable us to answer a natural follow-up question: the question of *realisability*, i.e., of what numbers $p \in [0,1]$ can be realised by probability limits as in Theorem 24. Since negating gives another MSO-sentence about graphs, the set of all such limits is symmetric about $\frac{1}{2}$, so this reduces to numbers $p \in [0, \frac{1}{2}]$. Moreover, there are two versions of this question: (1) fixing an addable minor-closed class $\mathcal{A}$ and letting only the MSO-sentences vary, (2) letting both $\mathcal{A}$ and φ vary. As to (1), certainly not *every* number in $p \in [0, \frac{1}{2}]$ is a probability limit of an MSO sentence, for the strong reason that there are only countably-many MSO-sentences about graphs, hence the set of their probability limits w.r.t. some fixed class is again countable. A natural weakening of that question is whether the set of probability limits is at least *dense* in $[0,1]$. The answer is that this cannot happen with *any* addable minor-closed class as the class of allowed models, however large, neither for variant (1) nor variant (2) of the realisability question:

Proposition 25 (joint work with T. Müller, M. Noy and A. Taraz; [83])**.** *Suppose $\mathcal{G}$ is an addable, minor-closed class of graphs. Then for every $\varphi \in$ MSO either* $\lim_{n\to\infty}|\{G \in \mathcal{G}_n \colon G \models \varphi\}|/|\mathcal{G}_n| \leqslant 1 - \mathrm{e}^{-1/2} < 0.394$ *or* $\lim_{n\to\infty}|\{G \in \mathcal{G}_n \colon G \models \varphi\}|/|\mathcal{G}_n| \geqslant \mathrm{e}^{-1/2} > 0.606$.

Proposition 25 will be proved in a more detailed version (Theorem 91) in Chapter 3.

We will moreover prove that the set of probability limits of FO-sentences is dense in the set of such limits of MSO-sentences (Lemma 90), and completely describe the closure of these sets:

Theorem 26 (joint work with T. Müller, M. Noy and A. Taraz)**.** *If $\mathcal{A}$ denotes any addable, minor-closed class of graphs,* FO *(resp.* MSO*) the set of all FO-sentences (resp. MSO-sentences) about* *graphs, and* cl *closure of subsets of* $\mathbb{R}$ *w.r.t. the usual metric topology, then we have the following equality, and the set is equal to a union of finitely-many intervals:*

$$\mathrm{cl}\left(\left\{\lim_{n\to\infty}\frac{|\{G\in\mathcal{A}_n : G\models\varphi\}|}{|\mathcal{A}_n|} : \varphi\in\mathrm{FO}\right\}\right)=\mathrm{cl}\left(\left\{\lim_{n\to\infty}\frac{|\{G\in\mathcal{A}_n : G\models\varphi\}|}{|\mathcal{A}_n|} : \varphi\in\mathrm{MSO}\right\}\right)\ . \quad (1.5)$$

Theorem 26 will be proved in a more detailed form (Theorem 92) in Chapter 3.

A key to our proofs are graphs that one might call 'universal models for $\equiv_r^{\mathrm{MSO}}$-equivalence classes'. Here, $\equiv_r^{\mathrm{MSO}}$ denotes a certain equivalence relation on MSO-formulas that is a standard tool in finite model theory; roughly speaking, two structures being equivalent w.r.t. $\equiv_r^{\mathrm{MSO}}$ means that for every MSO-formula with quantifiers nesting to at most a depth of r, either both structures satisfy the formula, or both do not. We describe a construction of such models (cf. Lemma 226 in Chapter 5) which is applicable to any addable minor-closed set of graphs. This construction is inspired by work of G. L. McColm [126, Theorem 2.1]. McColm considered trees sampled uniformly at random from all labelled trees on n vertices and proved that they obey a zero-one law w.r.t. statements in the monadic second order logic of graphs. In our proofs we also use the fact that a.a.s. there is a so-called giant component of size $n-O(1)$ (i.e., there is a positive probability of the giant component to be smaller by a given *constant* than the total number n of vertices), and the asymptotic distribution of the isomorphism-type of the non-giant components is known, thanks to work of C. McDiarmid.

In this thesis we also treat one non-addable class of graphs: graphs embeddable on a fixed surface of genus larger than zero (this class, while non-addable, retains some properties of addable classes, as proved by McDiarmid, cf. Theorem 134).

Random graphs from other non-addable classes have been investigated by several authors. For example, the results of [27] show how differently they may behave; this remains visible from the point of view of probability limits, too (cf. [83, Section 5]).

1.3.2 Logical limit laws for forests and planar graphs

In Chapter 3 we will prove the following two theorems quantifying how limited the scope for rating logical sentence is when only forests are considered as 'possible worlds', and how the spectrum for rating logical sentences widens when all planar graphs are allowed as models:

Theorem 27 (joint work with T. Müller, M. Noy and A. Taraz [83])**.** *If* $\mathcal{F}_n$ *denotes the set of all* n*-vertex labelled forests,* MSO *the set of all MSO-sentences about graphs,* cl *the closure of subsets of* $\mathbb{R}$ *w.r.t. the usual metric topology, then the set*

$$\mathrm{cl}\left(\left\{\lim_{n\to\infty}\frac{|\{G\in\mathcal{F}_n : G\models\varphi\}|}{|\mathcal{F}_n|} : \varphi\in\mathrm{MSO}\right\}\right)\ , \quad (1.6)$$

equals the union of 4 *disjoint intervals with the same length (which is of order* 10^{-1}*). The endpoints of these intervals are (except for* 0 *and* 1*) irrational numbers of the form* e^x *with rational* x*.*

Theorem 27 will be proved in a more explicit version (Theorem 97) in Section 3.2 of Chapter 3.

Theorem 28 (joint work with T. Müller, M. Noy and A. Taraz; [83])**.** *If* $\mathcal{P}_n$ *denotes the set of all* n*-vertex labelled planar graphs,* MSO *the set of all MSO-sentences about graphs,* cl *the closure of subsets of* $\mathbb{R}$ *w.r.t. the usual metric topology, then the set*

$$\mathrm{cl}\left(\left\{\lim_{n\to\infty}\frac{|\{G\in\mathcal{P}_n : G\models\varphi\}|}{|\mathcal{P}_n|} : \varphi\in\mathrm{MSO}\right\}\right)\ , \quad (1.7)$$

equals the union of 108 *disjoint intervals, all having the same length (which is of order* 10^{-6}*). The endpoints of these intervals can be defined explicitly as rational functions in the two quantities* ρ *and* $G(\rho)$*, with* $G(z)$ *the exponential generating function of finite planar graphs, and* ρ *is its radius of convergence.*

Theorem 28 will be proved in a more explicit version (Theorem 99) in Section 3.3 of Chapter 3, after first having illustrated the general method of proof for the simpler class of forests. An indispensable tool used for our proof of Theorem 28 are the precise asymptotics for labelled planar graphs recently found by O. Giménez and M. Noy in [63]: in particular, the number 108 sensitively depends on the numerical values of two constants precisely determined in [63] (as solutions to implicit, non-algebraic equations). More precisely, the number of independent indices a, b, c, d, e within the union in Theorem 99 depends on such approximations; in *that* sense, quantitative errors on the decimal order of about 10^{-5} can result in a millionfold magnification into what one might dare call a qualitative error: an erroneous number of intervals.

A (perhaps overly) simplified sample of these results is this: whatever you say in MSO-logic of graphs, the probability that your statement is true for a large finite random planar graph is either at most 3.7% or at least 96.3%. In particular, it will not be possible for you to say something in MSO logic that is true for roughly *half* of all large finite planar graphs. And if you do say something that is true for at most 3.7% of large random planar graphs, then in fact it will either be true for less than 1.36%, or it will be true for more than 3.5% of such graphs; it is impossible that your statement is true for (say) 2% of them. Moreover, if only *connected* planar graphs are admitted as models of your statement, then whatever your statement may be, it is either true for *almost no* or *almost every* large model.

1.3.3 Logical limit laws for graphs on a surface

A natural path to take when enlarging the possible worlds in which to evaluate MSO-sentences is to proceed from forests, to planar graphs and then to graphs embeddable on an arbitrary fixed surface. However, the latter class of graphs lacks the property of being *addable* (see Definition 203.(3) in Chapter 5) and this makes some of our otherwise valid arguments impossible. While Theorem 28 might have analogues for graphs on any fixed surface, currently no proof of this is known. Restricting the logic, though, opens up new ways: for FO-logic of graphs, using 'local normal forms' provided by a theorem of H. Gaifman, we could prove that random graphs drawn uniformly at random from the set of all graphs embeddable on a fixed surface obey a convergence law:

Theorem 29 (joint work with T. Müller, M. Noy and A. Taraz; [83])**.** *If S denotes some fixed surface (whether orientable or not) and $\mathcal{G}_S$ the class of all graphs embeddable on S, then for every statement φ in FO-logic of graphs,* $\lim_{n\to\infty} \frac{|\{G\in(\mathcal{G}_S)_n\colon G\models\varphi\}|}{|(\mathcal{G}_S)_n|}$ *exists.*

Theorem 29 will be proved in the more detailed version of Theorem 138 in Chapter 3.

Theorem 30 (connected graphs a fixed surface obey a zero-one-law, and the same as do obey connected planar graphs; joint work with T. Müller, M. Noy and A. Taraz [83])**.** *If S denotes some fixed surface (whether orientable or not) and $\mathcal{G}_S$ the class of all graphs embeddable on S, then for every statement φ in FO-logic of graphs,* $\frac{|\{G\in(\mathcal{G}_S)_n\colon G \text{ connected, } G\models\varphi\}|}{|(\mathcal{G}_S)_n|} \xrightarrow{n\to\infty} \{0,1\}$. *Moreover, for fixed φ, for any choice of the surface S the same limit results (hence, the same as when S is taken to be the sphere).*

Theorem 30 will be proved in in Section 3.4 of Chapter 3. One way to phrase the last statement of Theorem 30 is to say 'the almost-sure-FO-theory for connected graphs on a fixed surface is the same as the almost-sure-FO-theory of connected planar graphs'.

A natural follow-up question arising from our Theorem 30 is whether the statement remains true when instead of FO-logic we permit *existential monadic second-order logic* (or *EMSO*-logic for short, a synonym is *monadic NP*); this is a logic somewhere between FO- and MSO-logic in which universal quantification over sets is not allowed, and in which for example c-colourability is still definable while being connected is not. With $\mathcal{G}_S$ not addable, and EMSO-logic not[8] admitting

[8] In view of the fact that c-colourability *is* definable in EMSO-logic, together with the existence of graphs with arbitrarily large chromatic number and arbitrarily large girth [50, Theorem 5.2.5], i.e. graphs which *locally* are 2-colourable.

of local normal forms, two properties which are keys in our proofs of Theorem 24 and Theorem 29 now simultaneously fail to hold true. In particular, the following is still not proved:

Conjecture 31. *If S denotes some fixed surface (whether orientable or not) and $\mathcal{G}_S$ the class of all graphs embeddable on S, then for every statement φ in EMSO-logic of graphs, the limit* $\lim_{n\to\infty} \frac{|\{G\in(\mathcal{G}_S)_n :\, G\models\varphi\}|}{|(\mathcal{G}_S)_n|}$ *exists, and* $\lim_{n\to\infty} \frac{|\{G\in(\mathcal{G}_S)_n :\, G \text{ connected, } G\models\varphi\}|}{|(\mathcal{G}_S)_n|} \xrightarrow{n\to\infty} \{0,1\}$.

If Conjecture 31 is true, then it might moreover be true that the almost-sure-EMSO-theory of graphs on a fixed surface is *still* independent of the choice of the surface S. The author does not know whether the property of containing a given graph H as a minor is (known to be) definable in EMSO-logic (but this indeed seems to be a problematic question, and to depend on H). If it were, then the almost-sure-EMSO-theories would depend on the surface.

Being connected, while easily definable in MSO, is *not* definable in EMSO (by a theorem of T. Schwentick, not even if there is—as an additional structure to the adjacency relation—a total order on the vertices, cf. [152, Theorem 17]). Therefore, when trying to define containment of a minor in EMSO, one cannot just do what one can with *MSO*-logic, namely formalise the well-known characterisation that H is a minor of G if and only if there exists a $\mathrm{V}(H)$-indexed family $\mathcal{V}$ of mutually disjoint vertex-sets (called *branch-sets*), each of them inducing a connected graph, such that for every edge of H there is at least one edge in between the corresponding sets in $\mathcal{V}$: the connectedness of the graphs induced by the branch-sets cannot be expressed in EMSO-logic.

Moreover, if Conjecture 31 is strengthened by replacing 'EMSO' with 'MSO', then we still arrive at a statement that we could not refute, but about *this* statement we can at least say something negative: even if convergence holds, the almost-sure-MSO-theories then *would* depend on the surface: it is known that for each fixed surface S the set $\mathcal{G}_S$ equals the set of all graphs not containing some finite set of graphs as a minor (cf. [147] [134, Theorem 7.0.1] [50, Corollary 12.5.2]), and MSO-logic of graphs *can* express the existence of a fixed minor. Therefore, being embeddable into a given surface can be defined by a single MSO-sentence in the language of graphs. Thus, there would be MSO-sentences which are in the almost-sure-theory of $\mathcal{G}_S$ for some S, but not in that theory for other surfaces S.

Specialising Conjecture 31 by taking S to be the torus-surface and φ any fixed EMSO-sentence equivalent to 4-colourability of a graph, we are still faced with a problem that we could not solve:

$$\text{Is a uniformly random toroidal graph a.a.s. 4-colourable?} \tag{1.8}$$

In view of McDiarmid's Theorem 135, an affirmative answer to the statistical question (1.8) would imply the full deterministic four-colour-theorem (if there were even one non-four-colourable planar graph, it would a.a.s. appear in a large random graph on the torus, making that large graph non-four-colourable and contradicting the affirmative answer to (1.8)). Therefore, one probably should not hope for a relatively short proof of an affirmative answer to (1.8). What one nevertheless *can* reasonably hope for is (1.8) to have a negative answer in the strong sense of:

$$\text{Conjecture: a uniformly random toroidal graph is a.a.s. not 4-colourable.} \tag{1.9}$$

If (1.9) is true, then it is true with 'non-4-colourable' replaced by '5-chromatic', since a large uniformly random graph on the torus is a.a.s. 5-colourable (the reason being that [158] completely characterises 5-colourability in terms of local substructures: a toroidal graph is 5-colourable if and only if it does not contain any of four explicitly known small non-planar graphs *as a subgraph*, and by local planarity a large random toroidal graph a.a.s. does not contain *any* constant-sized non-planar graph). Some reason to believe in (1.9) is [159, Theorem 3.3]. There, Thomassen proved:

$$\text{There are infinitely-many nonisomorphic critically-5-chromatic toroidal graphs.} \tag{1.10}$$

Here, 'critically-5-chromatic' is taken to mean '5-colourable, non-4-colourable, but deleting any *edge* leaves a 4-colourable graph'; this is the sense of '5-critical' in [134, p. 232] and is different from the sense of 'critically 5-chromatic' in [50, p. 134] which requires deleting vertices. In a sense, critically 5-chromatic toroidal graphs as subgraphs are the most economical structural reason for a large toroidal graph to be 5-chromatic, and by Thomassen's theorem, there are infinitely many of them. What distinguishes them further is that Thomassen also proved that the set of all isomorphism types of critically 6-chromatic toroidal graphs is already *finite*; according to [134, Theorem 8.4.5] there are only four of them, and the largest has eleven vertices. It follows from results of Chapuy, Fusy, Giménez, Mohar and Noy that such small non-planar graphs a.a.s. do not exist in a large random toroidal graph. In that sense, critically 5-chromatic toroidal graphs as subgraphs are the most one can hope for when trying to prove (1.10) by showing some substructure to a.a.s. exist.

There would be stronger intuitive reasons to believe in the open conjecture (1.9), though, if (1.10) could be strengthened as follows:

Conjecture: there exist critically-5-chromatic toroidal graphs of arbitrarily large edge-width. (1.11)

Here, the rather non-descriptive term 'edge-width w' (cf. [134, p. 129]) of a toroidal graph G means that there exists an embedding of G into the torus such that the shortest length among those graph-theoretical circuits of G that are not contractible within the torus (and for that embedding) is w. Only if (1.11) is true do there exist infinitely-many structural reasons for 5-chromaticity *of a kind that can be expected* in a large random toroidal graph: because of its a.a.s. local planarity, a large random toroidal graph a.a.s. does not contain any non-planar graphs with bounded edge-width.

Conjecture (1.11) is a deterministic, apparently open, and hopefully constructively solvable problem in structural graph theory. No constructions proving (1.11) seem to be known to date. At least since [56] it *is* known that there exist 5-*chromatic* (no 'critically' here) toroidal graphs with arbitrarily large edge-width, but these examples of Fisk, essentially triangulations, are rather dense and non-critically-5-chromatic, hence not suitable to serve as a structural explanation for 5-chromaticity of a large random toroidal graph. In view of the results of Thomassen and Fisk, we seem to face the following situation: separate constructions are known for

(1) 5-chromatic toroidal graphs with arbitrarily large edge-width,
(2) critically-5-chromatic graphs with arbitrarily large number of vertices,

but a construction of toroidal graphs which would *simultaneously* offer both large edge-width and critical 5-chromaticity seems not to have been found.

For projective graphs however, such a construction *is* known by now (cf. [133]), while for toroidal graphs such a construction still seems not to have been found, despite enduring interest in the topic (cf. e.g. [146] [166]). Intuitively, if one views the size of the set of forbidden minors as some sort of measure of the difficulty in constructing something on a given surface, then the apparently greater difficulty in constructing critically-5-chromatic graphs with large edge-width on the torus compared to constructing them on the projective plane is in keeping with the *much* larger set of forbidden minors for toroidal graphs compared to projective graphs: while a graph is projective if and only if it does not contain any graph from an explicitly known set of 103 graphs as a topological minor (cf. [134, Theorem 6.5.1]), there exist thousands of forbidden topological minors for toroidal graphs (cf. see [134, p. 202]), and the exact number of those forbidden minors, known to be finite as a consequence of the graph minor theorem, still appears to be unknown (cf. [70, Chapter 1, p. 8]).

Let us close this section by reiterating that our interest in the construction problem (1.11) stems from the statistical problem (1.8), but that there is an associated open methodological problem, harking back to (1.2):

Develop methods for proving the existence of non-constantly-sized subgraphs in a large random graph embeddable on a fixed surface. (1.12)

Even for random planar graphs, only requests for the existence of *constantly*-sized subgraphs can be met at the current state of the art.

1.4 Sign matrices

For a commutative ring R and a finite subset $U \subseteq R$ one may ask how many among the $|U|^{n^2}$ matrices $A \in U^{[n]^2}$ have $\det(A) = 0$. Much is known precisely when R is the finite field $\mathbb{F}_q$ (q power of a prime) and $U = R$. For instance, it follows from elementary linear algebra that the number of singular $n \times n$ matrices with entries from $\mathbb{F}_q$ is precisely $q^{n^2} - \prod_{0 \leqslant i \leqslant n-1}(q^n - q^i)$. As an advanced example, precise statements can be proved for matrices over finite fields even if the entries are i.i.d. according to (quite) arbitrary distributions (cf. the work of Kahn–Komlós [93], and Maples [121]).

By contrast, if $R = \mathbb{Z}$ and $U = \{-1, +1\}$, the correct order of decay of the density of singular matrices is still unknown, yet there is an old and plausible conjecture of uncertain origin, which has been studied at least since [101], and about which the last two decades have brought new knowledge:

Conjecture 32. $\frac{|\{A \in \{-1,+1\}^{[n]^2} :\ A \ \text{singular over } \mathbb{Z}\}|}{|\{-1,+1\}^{[n]^2}|} \sim (\frac{1}{2} + o(1))^n$ for $n \to \infty$.

In Chapter 4 we define a measure P_{chio} on the set $\{-1, 0, +1\}^{[n-1]^2}$ (see Definition 144 in Chapter 4) which can be used to give a more constrained and relative, yet equivalent version: Conjecture 32 is equivalent (cf. (S3) on p. 188) to an inequality between two summations over singular $\{0, \pm\}$-matrices, weighted by the two measures P_{chio} and P_{lcf} (cf. p. 153–154 for their definitions):

Conjecture 33. *For $n \to \infty$,*

$$\sum_{B' \in \{0,\pm\}^{[n-1]^2}:\ B' \ \text{singular}} \mathsf{P}_{\text{chio}}[B'] \leqslant \left(\frac{1}{2} + o(1)\right) \cdot \sum_{B'' \in \{0,\pm\}^{[n-1]^2}:\ B'' \ \text{singular}} \mathsf{P}_{\text{lcf}}\big[B''\big] \ . \tag{1.13}$$

The inequality (1.13) suggests experimenting with grouping the matrices indexing the sums on either side of the conjectural inequality differently.

The measure P_{chio} behaves similarly (in a sense that Chapter 4 will make precise) to the *lazy-coin-flip* measure P_{lcf} on $\{-1, 0, +1\}^{[n-1]^2}$ (see Definition 139). The latter is interesting since w.r.t. P_{lcf} an optimal result on the asymptotics of the measure of singular matrices from $\{-1, 0, +1\}^{[n-1]^2}$ has recently been obtained [26]. In Chapter 4 we will prove statements to the effect that special events, called *entry-specification-events*, are approximately k-wise-independent w.r.t. P_{chio} (see Theorem 187), for constants k not depending on n. Moreover, we will provide a graph-theoretical characterisation (see Theorem 167) in the language of signed graphs (cf. [167]), and use it to prove that, given $B \in \{-1, 0, +1\}^{[n-1]^2}$, deciding whether $\mathsf{P}_{\text{chio}}[B] = \mathsf{P}_{\text{lcf}}[B]$ is equivalent to *deciding an evasive property of bipartite graphs* (cf. [165]), hence this decision necessarily takes time $\Omega(n^2)$:

Theorem 34 (complexity of deciding whether P_{chio} and P_{lcf} agree)**.** *For every $B \in \{0, \pm\}^{[n-1]^2}$, the answer to the decision problem of whether $\mathsf{P}_{\text{chio}}[B] = \mathsf{P}_{\text{lcf}}[B]$ cannot be computed in time $o(n^2)$. I.e., there does not exist a fixed algorithm (with entry-wise access to B as its only source of information) which decides that question on arbitrary instances $B \in \{0, \pm\}^{[n-1]^2}$ in time $o(n^2)$.*

Theorem 34 will be proved in the more explicit version of Theorem 173 in Section 4.2 of Chapter 4.

So there does not exist any algorithm deciding whether $\mathsf{P}_{\text{lcf}}[B] = \mathsf{P}_{\text{chio}}[B]$, for given $B \in \{0, \pm\}^{[n-1]^2}$, which would run faster than the obvious $\Theta(n^2)$-algorithm which examines every entry of the given B, separately computes $\mathsf{P}_{\text{lcf}}[B]$ and $\mathsf{P}_{\text{chio}}[B]$, and then compares. This seems not to be obvious, and the present author does not know how to prove it without the above-mentioned graph-theoretical characterisation. Of course, each of the two separate tasks of computing $\mathsf{P}_{\text{chio}}[B]$ or $\mathsf{P}_{\text{lcf}}[B]$, for given $B \in \{0, \pm\}^{[n-1]^2}$, needs time $\Theta(n^2)$, for the trivial reason that the last unknown entry to be read after reading all others can still make a difference for the values of the separate measures. However, only asking the yes-no-question $\mathsf{P}_{\text{chio}}[B] = \mathsf{P}_{\text{lcf}}[B]$, without asking to be told the value, is a much weaker demand, and a priori it is not clear whether there might be some fixed algorithm which, for any $B \in \{0, \pm\}^{[n-1]^2}$, needs to only examine some $o(n^2)$-proportion of the entries of B to make this decision. The graph-theoretical characterisation will allow us to prove that this is not the case.

2 Hamilton-based flow lattices of graphs

> **construct** ▸ **verb** [...] [with obj.] build or make (something, typically a building, road or machine): [...] ORIGIN late Middle English: from Latin *construct*-'heaped together, built', from the verb *construere*, from *con*- 'together' + *struere* 'pile, build'.
>
> Oxford Dictionary of English. Second Edition 2003. ISBN 0-19-8613474. p. 371

This chapter contains proofs for the results introduced in Section 1.2 of Chapter 1.

These proofs proceed by an argument (outlined in the steps ($\mathbb{F}_2$-St.1)–($\mathbb{F}_2$-St.3) resp. ($\mathbb{Z}$-St.1)–($\mathbb{Z}$-St.3) below) which one may call a 'monotonicity-argument': a preselected substructure is shown to *spanningly embed* into (i.e., to exist as a spanning subgraph of) a denser graph (satisfying some condition sufficient for such an embedding). Moreover, it is shown that the substructure itself has some desired property; one then concludes that the ambient graph must have the property, too.

For $\mathbb{F}_2$-coefficients at least (there are additional complications for $\mathbb{Z}$-coefficients), where equality of dimensions implies equality of spaces, one can approach this argument by contrasting it with a graph process, involving a 'race' between two numbers, which we call β_1 and h_1: we keep adding edges, one after another, into an initially empty n-vertex graph. We call our current graph $\widetilde{G}$ and consider the two integers $\beta_1 := \beta_1(\widetilde{G}) = \dim_{\mathbb{F}_2} \mathrm{Z}_1(\widetilde{G}; \mathbb{F}_2) = \|\widetilde{G}\| - n + 1$ and $\mathrm{h}_1 := \mathrm{h}_1(\widetilde{G}) := \dim_{\mathbb{F}_2}(\langle \mathcal{H}(\widetilde{G}) \rangle_{\mathbb{F}_2})$. In the very early stages of the process, β_1 is negative but keeps incrementing with each single edge added, while h_1 remains (there not being any Hamilton-circuit) zero, biding its time. When $\|\widetilde{G}\| = n$ edges have been added, a circuit must exist and β_1 has reached value 1, while h_1 typically will still stay at 0 for a while. When for the first time Hamilton-circuits appear, a *catching-up-phase* begins, at the end of which h_1 will have caught up with β_1. Some time into this phase, a non-edge is typically connected by many Hamilton-paths, so adding it creates several new Hamilton-circuits linearly-independent to both one another, and to the span of those already there. Because of this, the adding-in of new edges in this phase now often causes large *jumps* of h_1. It seems that the catching-up of h_1 with β_1 happens during a relatively short epoch of the process. From then on, the numbers of course proceed equally, by algebraic reasons alone. This process, in particular the size of jumps in h_1 and the duration of the catching-up period, have not yet been analysed mathematically; they are mentioned here as suggestions for further research, not as part of the proofs in Chapter 2.

Graphs which are Hamilton-connected, yet do not have their cycle space generated by Hamilton-circuits, seem to be rather rare among Hamilton-connected graphs, and have not been systematically studied from the viewpoint of structural graph theory. The graph $\mathrm{CE}_{(\mathrm{I1})}$ underlying Figure 2.1 is one example. In the notation of the technical Definition 204 in Chapter 5, we have $\mathrm{CE}_{(\mathrm{I1})} \in \mathcal{CO}_{|\cdot|-1}$ and $\mathrm{CE}_{(\mathrm{I1})} \in \mathrm{Cd}_1\mathcal{C}_{|\cdot|}$, hence $\mathrm{CE}_{(\mathrm{I1})} \in \mathcal{M}_{|\cdot|,1}$ but $\mathrm{CE}_{(\mathrm{I1})} \notin \mathcal{M}_{|\cdot|,0}$.

Moreover, whichever of the missing edges of $\mathrm{CE}_{(\mathrm{I1})}$ is added, the dimension of the cycle space (henceforward β_1) steps up by one (as for any graph and any edge) while the dimension of the $\mathbb{F}_2$-span of the Hamilton-circuits jumps by *two*, catching up with the dimension of the cycle space. From then on, when the remaining missing edges are added in, β_1 and h_1 will proceed, as equal numbers, all the way to $\binom{n}{2} - n + 1 = \dim_{\mathbb{F}_2}(\mathrm{Z}_1(\mathrm{K}^n)) = \beta_1(\mathrm{K}^n)$.

It is possible that a Hamilton-connected graph has relatively few Hamilton-circuits. Apparently (they are not given in this thesis), the author has found infinitely-many Hamilton-connected graphs each having only four distinct Hamilton-circuits. This shows that even when restricted to Hamilton-connected graphs G, the dimension h_1 of the $\mathbb{F}_2$-span of all Hamilton-circuits can be arbitrarily smaller than $\beta_1 = \dim_{\mathbb{F}_2} \mathrm{Z}_1(G)$, so even for Hamilton-connected graphs the difference that the number h_1 has to catch up with can be arbitrarily large. Intuitively speaking, these examples are highly non-random (in particular, they are planar). A formal treatment of these examples is not finished, and it was decided to keep them out of this thesis. If correct, then, using the notation of Definition 204 in Chapter 5, these examples show

$$\mathcal{CO}_{|\cdot|-1} \cap \mathrm{Cd}_d\mathcal{C}_{|\cdot|} \neq \varnothing \text{ for every } d \in \mathbb{N}. \tag{2.1}$$

The example $\mathrm{CE}_{(\mathrm{I1})}$ from Definition 212 proves (2.1) only for $d = 1$.

If one could get quantitative control of the short catching-up phase in the process just sketched, then this might provide another (random) source of graphs in $\mathcal{CO}_{|\cdot|-1} \cap \mathrm{Cd}_d\mathcal{C}_{|\cdot|} \neq \varnothing$ with large d.

The essence of the embedding technique underlying Chapter 2 is that it allows one to *step into the midst of the above-mentioned process*, when the catching up is already over, by pre-selecting a specific seed graph, for which one can carry out a proof of $\mathrm{h}_1 = \beta_1$; the linear-independence arguments used to prove (a20)–(a29) are the heart of the matter: these arguments show that h_1 has caught up with β_1.

Not every graph suitable as a seed graph in the context of $\mathbb{F}_2$-coefficients is suitable as a seed graph when $\mathbb{Z}$-coefficients are used. I.e., it will not be sufficient for Section 2.2.2 to recycle the auxiliary graphs used in [82]. For example, the graphs $\mathrm{M}_r^{\boxtimes}$ and $\mathrm{M}_r^{\boxminus}$ *are* suitable seed graphs for the $\mathbb{F}_2$-results, but unfortunately not for the results about integral flows (cf. e.g. Figure 2.3 on p. 53). This is the reason for the use of $\mathrm{C}_n^{2^-}$ from Definition 214, which apparently are the first published non-trivial examples of graphs with Hamilton-*based* flow lattice.

2.1 Cycles modulo two ($\mathbb{F}_2$-coefficients)

There exist investigations in which the set underlying a finite-dimensional vector space is not forgotten, but made to play a central part. One such investigation was begun thirty years ago by Hartman: when does the cycle space $\mathrm{Z}_1(G;\mathbb{F}_2)$ of a graph G admit an $\mathbb{F}_2$-basis consisting of *long* graph-theoretical circuits only? In [76, Theorem 1] Hartman proved that—barring the sole exception of G being a complete graph with an even number of vertices—for every 2-connected finite graph G, the set of all circuits of length *at least* $\delta(G) + 1$ generates $\mathrm{Z}_1(G;\mathbb{F}_2)$.

The lower the minimum degree $\delta(G)$, the larger the set of cycle-lengths one has to allow in order to be guaranteed a generating set by Hartman's theorem. In particular, statements guaranteeing a generating set consisting entirely of *Hamilton-circuits* remain almost inaccessible via this theorem: one has to set $\delta(G) := |G| - 1$, hence $G \cong \mathrm{K}^{|G|}$, and what remains of Hartman's general theorem is a rather special (albeit still non-obvious) statement about the complete graph. The property of $\mathrm{Z}_1(G;\mathbb{F}_2)$ being generated by the Hamilton-circuits of G seems to have been first studied by Alspach, Locke and Witte (see Theorem 36.(2) in Section 2.1.2). They proved that G has the property if G is a connected Cayley graph on a finite abelian group and is either bipartite or has odd order (these hypotheses being mutually exclusive for connected Cayley graphs on finite abelian groups). In Section 2.1 we will prove Theorem 6 from Chapter 1, which is the mod-2-version of Conjecture 3 from Chapter 1.

Theorem 6 is an addition to the growing corpus of knowledge about the following phenomenon: when studying the set of Hamilton-circuits as a function of the minimum degree $\delta(G)$, it pours if it rains—slightly below a sufficient threshold there still exist graphs which do not have *any* Hamilton-circuit, slightly above the threshold suddenly every graph contains not merely one but rather a plethora of Hamilton-circuits satisfying many *additional requirements*. This line of investigation appears to begin with Nash-Williams' proof [140, Theorem 2] [141, Theorem 3] that for every graph G with $\delta(G) \geqslant \frac{1}{2}|G|$ there exists not only one (Dirac's theorem [53, Theorem 3] [50, Theorem 10.1.1]) but at least $\lfloor \frac{5}{224} n \rfloor$ edge-disjoint Hamilton-circuits. For sufficiently large graphs G with $\delta(G)$ a little larger than $\frac{1}{2}|G|$, Nash-Williams' theorem was improved by Christofides, Kühn and Osthus [38, Theorem 2] to the guarantee that there are at least $\frac{1}{8}n$ edge-disjoint Hamilton circuits—this being an asymptotically best-possible result in view of examples [140, p. 818] which show that in graphs

G with $\delta(G) \geqslant \frac{1}{2}|G|$ and having a slightly irregular degree sequence, the number of edge-disjoint Hamilton-circuits is bounded by $\frac{1}{8}n$. More can be achieved if besides a high minimum-degree, additional requirements are imposed on the host graph. Two aspects of this are (1) a regular degree sequence, (2) a random host graph.

As to (1), if the host graph is required to be regular in advance, a still unsettled conjecture of Jackson [89, p. 13, l. 17] posits that a d-regular graph with $d \geqslant \frac{|G|-1}{2}$ actually realises the obvious upper bound $\lfloor \frac{1}{2}d \rfloor$ for the number of edge-disjoint Hamilton-circuits. Christofides, Kühn and Osthus proved a theorem which in a sense comes arbitrarily close to the conjecture [38, Theorem 5]. This has recently been further improved by Kühn and Osthus [109, Theorem 1.3].

As to (2), Frieze and Krivelevich conjectured [59, p. 222] that for *any* function $0 \leqslant p_n \leqslant 1$ a binomial random graph $G(n, p_n)$ asymptotically almost-surely attains the a priori maximum of $\lfloor \delta/2 \rfloor$ edge-disjoint Hamilton-circuits, which they proved [59, Theorem 1] for $p_n \leqslant (1 + o(1))\frac{\log n}{n}$. In [99, Theorem 2] Knox, Kühn and Osthus proved the conjecture for a class of functions p_n that sweeps a huge portion of the range $\frac{\log n}{n} \ll p_n \ll 1$. A remaining gap (starting at $\frac{\log n}{n}$) in the probability range heretofore covered was closed by Krivelevich and Samotij [105]. As a consequence of a recent breakthrough of Kühn and Osthus [108], using their notion of *robust outexpanders*, Krivelevich and Frieze's conjecture has now been completely proved [109, Theorem 1.10 and Section 5.2]. The question about the number of edge-disjoint copies of Hamilton-circuits has been refined by means of a function related to spanning subgraphs [109, Theorem 1.5] [106, Theorem 3].

One way to look at these results is as providing 'extremely orthogonal' (i.e. no additive cancellation is involved in the vanishing of the standard bilinear form) sets of Hamilton-circuits. As they stand, these theorems are far from providing 'orthogonal' Hamilton-circuit-bases for $\mathrm{Z}_1(G; \mathbb{F}_2)$: at the relevant minimum degrees, the dimension of $\mathrm{Z}_1(G; \mathbb{F}_2)$ is much higher than $\delta(G)/2$ (roughly, one has $\dim_{\mathbb{F}_2} \mathrm{Z}_1(G; \mathbb{F}_2) \in \Theta_{|G| \to \infty}(\delta(G)^2)$), so the sets of mutually disjoint Hamilton-circuits are—while 'very' orthogonal—far from being generating sets of $\mathrm{Z}_1(G; \mathbb{F}_2)$. Yet it does not seem unlikely that the above-mentioned theorems can be extended in a more algebraic vein by devising generalizations of 'edge-disjoint' (e.g. 'size of the intersection of the supports even') and thus be made to resonate with results like Theorem 6.

2.1.1 Plan of the proof of Theorem 6 from Chapter 1

The proof of Theorem 6 from Section 1.2.2 in Chapter 1 will be structured into the following steps. The strategy is the same for (I1)–(I4), but the auxiliary spanning subgraphs used are different:

($\mathbb{F}_2$-St.1) Find an infinite set of graphs H which are simultaneously Hamilton-connected[1], have their cycle space generated by Hamilton-circuits (resp., themselves have one of the other properties mentioned in Theorem 6), *and* can be guaranteed as spanning subgraphs in step ($\mathbb{F}_2$-St.2). These graphs H serve as 'rebar' during steps ($\mathbb{F}_2$-St.2) and ($\mathbb{F}_2$-St.3); they are used to confer the desired properties to the host graph G.

($\mathbb{F}_2$-St.2) For the graphs H from ($\mathbb{F}_2$-St.1), prove the existence of embeddings $H \hookrightarrow G$; for (I1) and (I2) by using Theorem 38, for (I3) by using Theorem 39, and for (I4) by using Theorem 40 below.

($\mathbb{F}_2$-St.3) By using Lemma 41 with $R := \mathbb{F}_2$, argue that the properties proved in ($\mathbb{F}_2$-St.1) transfer from H to the host graph G, completing the proof of Theorem 6.

[1] While the argument in [82] treats the issue correctly, on p. 507 of [82] there unfortunately remained a misleading footnote saying that graphs with the weaker property defined by requiring only *non-adjacent* vertices to be connected by a Hamilton path were sufficient for the monotonicity argument. This is false of course: the requirement that a seed-substructure *itself* has to have its cycle space generated by Hamilton-circuits implies that any two *adjacent* vertices are connected by at least one Hamilton-circuit, while for the monotonicity argument to work, any two *non*-adjacent vertices must be so connected. Taken together, it follows that any suitable seed graph must be Hamilton-connected (hence in particular have minimum-degree 3). The mistake is confined to that sole careless footnote alone; [82] correctly uses seed-substructures which *are* Hamilton-connected.

2.1.2 Details on step ($\mathbb{F}_2$-St.1)

If $\mathcal{A}$ is a finite abelian group in additive notation, and $0 \notin S \subseteq \mathcal{A}$ has the property that $-S := \{-s \colon s \in S\} = S$, then we write $\langle S \rangle := \sum_{s \in S} \mathbb{Z}s$ for the abelian group generated by S and define a graph $G := \mathrm{Cay}(\langle S \rangle; S)$ by $\mathrm{V}(G) := \langle S \rangle$ and $\{a, b\} \in \mathrm{E}(G) :\Leftrightarrow a - b \in S$, called the *Cayley graph* associated to $\mathcal{A}$ and S. The following theorem of Chen and Quimpo has proved to be fertile for the theory of Cayley graphs on finite abelian groups:

Theorem 35 (Chen–Quimpo; [36, Theorem 4] gives the non-bipartite case[2])**.** *For every finite abelian group $\mathcal{A}$ and every $S \subseteq \mathcal{A}$ with $-S = S$ and $|S| \geqslant 3$ the graph $G = \mathrm{Cay}(\langle S \rangle; S)$ is Hamilton-connected in case G is not bipartite, and Hamilton-laceable in case G is bipartite.* $\square$

We will use the following theorem of Alspach, Locke and Witte which appears to be the first result in the literature dealing with linear algebraic properties of Hamilton-circuits (as to terminology, a graph G is called a *prism over the graph H* if and only if $G \cong H \,\square\, \mathrm{P}_1$, where P_1 is an edge):

Theorem 36 (Alspach–Locke–Witte [7, Theorem 2.1 and Corollary 2.3])**.** *For every finite abelian group $\mathcal{A}$ and every $0 \notin S \subseteq \mathcal{A}$ with $-S = S$, the graph $G := \mathrm{Cay}(\langle S \rangle; S)$ has the following properties:*

(1) *if G is bipartite, then $\mathcal{H}(G)$ generates $\mathrm{Z}_1(G; \mathbb{F}_2)$,*
(2) *if $|G| = |S|$ is odd, then $\mathcal{H}(G)$ generates $\mathrm{Z}_1(G; \mathbb{F}_2)$,*
(3) *if $|G| = |S|$ is even and G is not bipartite and not a prism over any circuit of odd length, then $\dim_{\mathbb{F}_2}\big(\mathrm{Z}_1(G; \mathbb{F}_2)/\langle \mathcal{H}(G) \rangle_{\mathbb{F}_2}\big) = 1$.* $\square$

Lemma 37 (properties of the auxiliary structures)**.** *For every $n \geqslant 5$ and every $r \in \mathbb{Z}_{\geqslant 4}$, and with the various graph properties as in Definition 204,*

(a1) $\mathrm{C}_n^2 \cong \mathrm{Cay}(\mathbb{Z}/n; \{1, 2, n-2, n-1\})$,
(a2) C_n^2 *is not a prism over a graph (i.e. there does not exist H with $\mathrm{C}_n^2 \cong H \,\square\, \mathrm{P}_1$)* ,
(a3) *if n is even, then $\mathrm{C}_n^2 \in \mathcal{M}_{\{|\cdot|\},1}$* , (a4) *if n is odd, then $\mathrm{C}_n^2 \in \mathcal{M}_{\{|\cdot|\},0}$* ,
(a5) $\mathrm{C}_n^2 \in \mathcal{M}_{\{|\cdot|\},0}{}^{-}$ *for every $n \geqslant 5$, in the sense of Definition 204.*(13) ,
(a6) $\mathrm{Pr}_r \cong \mathrm{Cay}(\mathbb{F}_2 \oplus \mathbb{Z}/r; \{(1,0), (0,1), (0, r-1)\})$, (a7) $\mathrm{M}_r \cong \mathrm{Cay}(\mathbb{Z}/(2r); \{1, r, 2r-1\})$,
(a8) *if r is even, then $\mathrm{Pr}_r \in \mathcal{LA}_{|\cdot|-1}$* , (a9) *if r is odd, then $\mathrm{M}_r \in \mathcal{LA}_{|\cdot|-1}$* ,
(a10) *if r is even, then $\mathrm{Pr}_r \in \mathrm{b}\mathcal{M}_{\{|\cdot|\},0}$* , (a11) *if r is odd, then $\mathrm{M}_r \in \mathrm{b}\mathcal{M}_{\{|\cdot|\},0}$* ,
(a12) *if r is even, then $\mathrm{CL}_r \cong \mathrm{Pr}_r$* , (a13) *if r is odd, then $\mathrm{CL}_r \cong \mathrm{M}_r$* ,
(a14) $\mathrm{CL}_r \in \mathcal{LA}_{|\cdot|-1}$, (a15) $\mathrm{CL}_r \in \mathrm{b}\mathcal{M}_{\{|\cdot|\},0}$,
(a16) *if r is even, then $\mathrm{Pr}_r^{\boxtimes} \in \mathcal{CO}_{\{|\cdot|-1\}}$* , (a17) *if r is odd, then $\mathrm{M}_r^{\boxtimes} \in \mathcal{CO}_{\{|\cdot|-1\}}$* ,
(a18) *if r is even, then $\mathrm{Pr}_r^{\boxminus} \in \mathcal{CO}_{\{|\cdot|-1\}}$* , (a19) *if r is odd, then $\mathrm{M}_r^{\boxminus} \in \mathcal{CO}_{\{|\cdot|-1\}}$* ,
(a20) *concerning $\mathrm{Pr}_r^{\boxtimes}$ and $\mathrm{Pr}_r^{\boxminus}$ for even r, and concerning $\mathrm{M}_r^{\boxtimes}$ and $\mathrm{M}_r^{\boxminus}$ for odd r, the set $\{\mathrm{c}_C \colon C \in \mathcal{CB}_G^{(1)}\}$ is a linearly independent subset of $\mathrm{Z}_1(G; \mathbb{F}_2)$ for all $G \in \{\mathrm{Pr}_r^{\boxtimes}, \mathrm{Pr}_r^{\boxminus}, \mathrm{M}_r^{\boxtimes}, \mathrm{M}_r^{\boxminus}\}$* ,
(a21) *concerning $\mathrm{Pr}_r^{\boxtimes}$ and $\mathrm{Pr}_r^{\boxminus}$ for even r, and concerning $\mathrm{M}_r^{\boxtimes}$ and $\mathrm{M}_r^{\boxminus}$ for odd r, the set $\{\mathrm{c}_C \colon C \in \mathcal{CB}_G^{(2)}\}$ is a linearly independent subset of $\mathrm{Z}_1(G; \mathbb{F}_2)$ for all $G \in \{\mathrm{Pr}_r^{\boxtimes}, \mathrm{Pr}_r^{\boxminus}, \mathrm{M}_r^{\boxtimes}, \mathrm{M}_r^{\boxminus}\}$* ,
(a22) *concerning $\mathrm{Pr}_r^{\boxtimes}$ and $\mathrm{Pr}_r^{\boxminus}$ for even $r \geqslant 4$, and concerning $\mathrm{M}_r^{\boxtimes}$ and $\mathrm{M}_r^{\boxminus}$ for odd $r \geqslant 5$, the sum $\Big\langle \mathcal{CB}_G^{(1)} \Big\rangle_{\mathbb{F}_2} + \Big\langle \mathcal{CB}_G^{(2)} \Big\rangle_{\mathbb{F}_2} \subseteq \mathrm{C}_1(G; \mathbb{F}_2)$ is direct for all $G \in \{\mathrm{Pr}_r^{\boxtimes}, \mathrm{Pr}_r^{\boxminus}, \mathrm{M}_r^{\boxtimes}, \mathrm{M}_r^{\boxminus}\}$* ,
(a23) *concerning $\mathrm{Pr}_r^{\boxtimes}$ and $\mathrm{Pr}_r^{\boxminus}$ for even r, and concerning $\mathrm{M}_r^{\boxtimes}$ and $\mathrm{M}_r^{\boxminus}$ for odd r* ,
($\boxtimes$.(0)) $\langle \mathcal{H}(\mathrm{Pr}_r^{\boxtimes}) \rangle_{\mathbb{F}_2} = \mathrm{Z}_1(\mathrm{Pr}_r^{\boxtimes}; \mathbb{F}_2)$, ($\boxtimes$.(1)) $\langle \mathcal{H}(\mathrm{M}_r^{\boxtimes}) \rangle_{\mathbb{F}_2} = \mathrm{Z}_1(\mathrm{M}_r^{\boxtimes}; \mathbb{F}_2)$,
($\boxminus$.(0)) $\dim_{\mathbb{F}_2}\big(\mathrm{Z}_1(\mathrm{Pr}_r^{\boxminus}; \mathbb{F}_2)/\langle \mathcal{H}(\mathrm{Pr}_r^{\boxminus}) \rangle_{\mathbb{F}_2}\big) = 1$, ($\boxminus$.(1)) $\dim_{\mathbb{F}_2}\big(\mathrm{Z}_1(\mathrm{M}_r^{\boxminus}; \mathbb{F}_2)/\langle \mathcal{H}(\mathrm{M}_r^{\boxminus}) \rangle_{\mathbb{F}_2}\big) = 1$,

($\boxminus$.$|\cdot| - 1$.(0)) $\langle \mathcal{C}_{\{|\cdot|-1, |\cdot|\}}(\mathrm{Pr}_r^{\boxminus}) \rangle_{\mathbb{F}_2} = \mathrm{Z}_1(\mathrm{Pr}_r^{\boxminus}; \mathbb{F}_2)$,
($\boxminus$.$|\cdot| - 1$.(1)) $\langle \mathcal{C}_{\{|\cdot|-1, |\cdot|\}}(\mathrm{M}_r^{\boxminus}) \rangle_{\mathbb{F}_2} = \mathrm{Z}_1(\mathrm{M}_r^{\boxminus}; \mathbb{F}_2)$,

(a24) *if r is even, then $\mathrm{Pr}_r^{\boxtimes} \in \mathcal{M}_{\{|\cdot|\},0}$* , (a25) *if r is odd, then $\mathrm{M}_r^{\boxtimes} \in \mathcal{M}_{\{|\cdot|\},0}$* ,
(a26) *if r is even, then $\mathrm{Pr}_r^{\boxminus} \in \mathcal{M}_{\{|\cdot|\},1}$* , (a27) *if r is odd, then $\mathrm{M}_r^{\boxminus} \in \mathcal{M}_{\{|\cdot|\},1}$* ,
(a28) *if r is even, then $\mathrm{Pr}_r^{\boxminus} \in \mathcal{M}_{\{|\cdot|-1, |\cdot|\},0}$* , (a29) *if r is odd, then $\mathrm{M}_r^{\boxminus} \in \mathcal{M}_{\{|\cdot|-1, |\cdot|\},0}$* ,

[2] The bipartite case appears to be susceptible to analogous arguments as in [36]. The author does not know of any published proof of the bipartite case. Nevertheless, it is mentioned in [8, Theorem 1.4], [6, Theorem 1.7], [132, Introductory Remarks and Proposition 2.1] and [131, Proposition 3]. Moreover, what little we need of the general bipartite case, namely Lemma 37.(a14), is easy to show directly.

(a30) *for every $\beta > 0$ there exists $n_0 = n_0(\beta)$ such that—in case of* $\mathrm{Pr}_r^{\boxtimes}$ *and* $\mathrm{Pr}_r^{\boxminus}$ *for even r while in case of* $\mathrm{M}_r^{\boxtimes}$ *and* $\mathrm{M}_r^{\boxminus}$ *for odd r—if $H \in \{C_n^2, \mathrm{CL}_r, \mathrm{Pr}_r^{\boxtimes}, \mathrm{Pr}_r^{\boxminus}, \mathrm{M}_r^{\boxtimes}, \mathrm{M}_r^{\boxminus}\}$ and $|H| \geqslant n_0$, the following is true: the bandwidth satisfies* $\mathrm{bw}(H) \leqslant \beta \cdot |H|$, *and moreover for each $H \in \{\mathrm{Pr}_r^{\boxtimes}, \mathrm{Pr}_r^{\boxminus}, \mathrm{M}_r^{\boxtimes}, \mathrm{M}_r^{\boxminus}\}$ there exists a bijection $\mathrm{b}_H \colon \mathrm{V}(H) \to \{1, \ldots, |H|\}$ and a map $\mathrm{h}_H \colon \mathrm{V}(H) \to \{0, 1, 2\}$ such that b_H is a bandwidth-$\beta|H|$-labelling and h_H a 3-colouring of H, and h_H has $|\mathrm{h}_H^{-1}(0)| \leqslant \beta|H|$ and is $\big(8 \cdot 2 \cdot \beta \cdot |H|, 4 \cdot 2 \cdot \beta \cdot |H|\big)$-zero-free w.r.t. b_H* .

There are arbitrary choices to be made when proving Lemma 37. Let us especially mention that there are three different feasible strategies for proving (a15):

(A1) Realise CL_r as a Cayley graph on a finite abelian group. Then cite a theorem of Alspach, Locke and Witte which implies that $\mathrm{Z}_1(\mathrm{CL}_r; \mathbb{F}_2)$ is generated by Hamilton-circuits.

(A2) Determine the full set of non-separating induced circuits of CL_r, then realise every single such circuit as a $\mathbb{F}_2$-sum of Hamilton-circuits of CL_r and then appeal to a theorem of Tutte ([162, Statement (2.5)] [50, Theorem 3.2.3]) which states that in a 3-connected graph G, the cycle space $\mathrm{Z}_1(G; \mathbb{F}_2)$ is generated by the set of all non-separating induced circuits.

(A3) Exhibit sufficiently many explicit Hamilton-circuits of CL_r so that after choosing some basis the matrix of these circuits has $\mathbb{F}_2$-rank equal to $\dim_{\mathbb{F}_2} \mathrm{Z}_1(\mathrm{CL}_r; \mathbb{F}_2)$. It then follows that $\mathrm{Z}_1(\mathrm{CL}_r; \mathbb{F}_2) = \langle \mathcal{H}(\mathrm{CL}_r) \rangle_{\mathbb{F}_2}$, since in a vector space, a maximal linearly independent subset is a generating set.

Each of (A1)–(A3) demands attention to the parity of r, for despite a superficial similarity, the sets of circuits in CL_r for odd and even r turn out to be quite different. A positive way to look at this is as helping to decide which of (A1)–(A3) to choose. While each argument can be used for each parity of r, there are some reasons to choose (A2) for even r. The reason is a *trade-off between being a circulant graph (i.e. a Cayley graph on a finite cyclic group) and being a planar graph:* if r is even, then it can be shown that CL_r is not isomorphic to any Cayley graph on a cyclic group, whereas when r is odd, CL_r *is* a circulant graph. In return, CL_r is *planar* if and only if r is even, and this facilitates (A2): when it comes to proving that no non-separating induced circuits of CL_r have been overlooked, the planarity of CL_r for even r opens up a shortcut via a theorem of Kelmans [98, p. 264] (a hypothetical overlooked non-separating induced circuit implies an edge contained in more than two such circuits, contradicting Kelmans' theorem). For odd r, however, the non-planarity of CL_r (easy to prove via Kuratowski's theorem, cf. [74, p. 494]), makes this shortcut disappear. For these reasons, (A2) takes considerably more work when r is odd than when r is even, and we will not make any use of it. In the proofs in Section 2.1.2 we will opt for the shortest route, i.e. (A1). Argument (A3), the most arbitrary of all three (usually there is no overriding justification for choosing a particular set of linearly-independent Hamilton-circuits except that it works) will be used for proving (a23), i.e. for dealing with the rather ad-hoc auxiliary structures $\mathrm{Pr}_r^{\boxtimes}$, $\mathrm{Pr}_r^{\boxminus}$, $\mathrm{M}_r^{\boxtimes}$ and $\mathrm{M}_r^{\boxminus}$.

Justifying that CL_r is indeed one of the subgraphs guaranteed by Theorem 39 will pose no difficulty and can be done uniformly for every $r \in \mathbb{Z}_{\geqslant 3}$. Matters are being complicated by parity issues when it comes to step ($\mathbb{F}_2$-St.1). We will later make essential use of the following sets (for each of the circuits C in these sets, the reader may use Figure 5.1 to visualise C):

Proof of Lemma 37. As to (a1), an easy verification shows that the map $\{v_0, \ldots, v_{n-1}\} \to \mathbb{Z}/n$, $v_i \mapsto i$ is a graph isomorphism $\mathrm{C}_n^2 \to \mathrm{Cay}(\mathbb{Z}/n; \{1, 2, n-2, n-1\})$. (Both for this, and for the verifications required in (a6), (a7), (a12), (a13), it is recommendable to use an obvious and known [85, Section 1.5, first paragraph] characterisation of graph isomorphisms: *every injective graph homomorphism between two graphs with equal f-vectors is a graph isomorphism.* This relieves one of the responsibility to explicitly show that non-edges are mapped to non-edges.)

As to (a2), the definition of $\square$ implies that for every graph G, every vertex of the graph $G \,\square\, \mathrm{P}_1$ has odd degree. But for every $n \geqslant 5$ the graph C_n^2 is regular with vertex-degree four.

As to (a3) and (a4), first note that C_n^2 is non-bipartite, for both parities of n, and therefore (a1) and Theorem 35 combined imply that $\mathrm{C}_n^2 \in \mathcal{CO}_{\{|\cdot|-1\}}$, for every n. It remains to justify that

$\mathrm{C}_n^2 \in \mathrm{Cd}_1\mathcal{C}_{\{|\cdot|\}}$ for even n, resp. $\mathrm{C}_n^2 \in \mathrm{Cd}_0\mathcal{C}_{\{|\cdot|\}}$ for odd n. Both these statements follow from combining (a1) and (a2) with Theorem 36.(2) and Theorem 36.(3).

As to (a5), we first note that if n is odd, then $\mathrm{C}_n^2 \in \mathcal{M}_{|\cdot|,0}$ by (a4), which proves (a5) since directly from the definitions we have $\mathcal{M}_{|\cdot|,0} \subseteq \mathcal{M}_{|\cdot|,0}{}^{-}$. So we may assume that n is even. Then we choose any of the $n = |\mathrm{C}_n^2|$ circuits C of length $n-1 = |\mathrm{C}_n^2| - 1$ in C_n^2 and (in the notation of Definition 204.(6)) set $z^- := C$ (identifying C with the element of $\mathrm{Z}_1(G;\mathbb{F}_2)$ defined by it). Since n is even, C has odd length, so $C \notin \langle \mathcal{H}(\mathrm{C}_n^2)\rangle_{\mathbb{F}_2}$. Moreover, $\dim_{\mathbb{F}_2}\langle \mathcal{H}(\mathrm{C}_n^2)\rangle_{\mathbb{F}_2} = \dim_{\mathbb{F}_2} \mathrm{Z}_1(\mathrm{C}_n^2;\mathbb{F}_2) - 1$ by (a3), hence $\dim_{\mathbb{F}_2}\langle \{C\} \sqcup \mathcal{H}(\mathrm{C}_n^2)\rangle_{\mathbb{F}_2} \geqslant \dim_{\mathbb{F}_2} \mathrm{Z}_1(\mathrm{C}_n^2;\mathbb{F}_2)$. Due to $\langle \{C\} \sqcup \mathcal{H}(\mathrm{C}_n^2)\rangle_{\mathbb{F}_2}$ being an $\mathbb{F}_2$-linear subspace of $\mathrm{Z}_1(\mathrm{C}_n^2;\mathbb{F}_2)$, this must hold with equality, proving (a5).

As to (a6), an easy verification shows that the map $\{x_0,\dots,x_{r-1},\ y_0,\dots,y_{r-1}\} \to \mathbb{F}_2 \oplus \mathbb{Z}/r$, $x_i \mapsto (0,i)$, $y_i \mapsto (1,i)$ is a graph isomorphism $\mathrm{Pr}_r \to \mathrm{Cay}(\mathbb{F}_2 \oplus \mathbb{Z}/r; \{(1,0),(0,1),(0,r-1)\})$.

As to (a7), an easy verification shows that the map $\mathrm{V}(\mathrm{M}_r) = \{x_0,\dots,x_{r-1},\ y_0,\dots,y_{r-1}\} \to \mathbb{Z}/(2r)$, $x_i \mapsto i$, $y_i \mapsto i+r$ is a graph isomorphism $\mathrm{M}_r \to \mathrm{Cay}\,(\mathbb{Z}/(2r); \{1, r, 2r-1\})$.

As to (a8), it is easy to check that r being even implies that Pr_r is bipartite. Therefore (a8) follows from (a6) combined with Theorem 35. Moreover, (a8) is straightforward to prove directly.

As to (a9), it is easy to check that r being odd implies that M_r is bipartite. Therefore (a9) follows from (a7) combined with Theorem 35. Moreover, (a9) is straightforward to prove directly.

As to (a10), it is easy to check that r being even implies that Pr_r is bipartite. Therefore, combining (a6) with Theorem 35 yields that $\mathrm{Pr}_r \in \mathcal{LA}_{\{|\cdot|-1\}}$, and combining (a6) with Theorem 36.(1) yields $\mathrm{Pr}_r \in \mathrm{Cd}_0\mathcal{C}_{\{|\cdot|\}}$, completing the proof of (a10).

As to (a11), it is easy to check that r being odd implies that M_r is bipartite. Therefore, combining (a7) with Theorem 35 yields that $\mathrm{M}_r \in \mathcal{LA}_{\{|\cdot|-1\}}$, and combining (a7) with Theorem 36.(1) yields $\mathrm{M}_r \in \mathrm{Cd}_0\mathcal{C}_{\{|\cdot|\}}$, completing the proof of (a11).

As to (a12) and (a13), an easy verification shows that the map $\mathrm{V}(\mathrm{CL}_r) \to \mathrm{V}(\mathrm{Pr}_r) = \mathrm{V}(\mathrm{M}_r)$ defined by $a_i \mapsto x_i$ for every even $0 \leqslant i \leqslant r-1$, $a_i \mapsto y_i$ for every odd $0 \leqslant i \leqslant r-1$, $b_i \mapsto y_i$ for every even $0 \leqslant i \leqslant r-1$, $b_i \mapsto x_i$ for every odd $0 \leqslant i \leqslant r-1$, is a graph isomorphism $\mathrm{CL}_r \to \mathrm{Pr}_r$ for every even $r \geqslant 4$ and a graph isomorphism $\mathrm{CL}_r \to \mathrm{M}_r$ for every odd $r \geqslant 4$.

As to (a14), this follows by combining (a8) and (a9) with (a12) and (a13).

As to (a15), this follows by combining (a10) and (a11) with (a12) and (a13).

As to (a16) and (a17), the literature apparently does not contain a sufficient criterion for Hamilton-connectedness which would apply to either $\mathrm{Pr}_r^{\boxtimes}$ or $\mathrm{M}_r^{\boxtimes}$. Therefore a direct proof by distinguishing cases and providing explicit Hamilton paths appears to be unavoidable. Let $\{v,w\} \subseteq \mathrm{V}(\mathrm{M}_r^{\boxtimes}) = \mathrm{V}(\mathrm{Pr}_r^{\boxtimes})$ be arbitrary distinct vertices.

We will repeatedly reduce the work to be done by making use of symmetries. The automorphism group of both $\mathrm{Pr}_r^{\boxtimes}$ and $\mathrm{M}_r^{\boxtimes}$ is the group generated by the two unique homomorphic extensions of the maps $\left(\substack{\{z,x_0,y_0,x_1,y_1\}\to\{z,x_0,y_0,x_1,y_1\} \\ z\mapsto z,\ x_0\leftrightarrow y_0,\ x_1\leftrightarrow y_1}\right)$ and $\left(\substack{\{z,x_0,y_0,x_1,y_1\}\to\{z,x_0,y_0,x_1,y_1\} \\ z\mapsto z,\ x_0\leftrightarrow x_1,\ y_0\leftrightarrow y_1}\right)$ to all of $\mathrm{V}(\mathrm{Pr}_r^{\boxtimes}) = \mathrm{V}(\mathrm{M}_r^{\boxtimes})$ (thus both $\mathrm{Aut}(\mathrm{Pr}_r^{\boxtimes})$ and $\mathrm{Aut}(\mathrm{M}_r^{\boxtimes})$ are isomorphic to the Klein four-group $\mathbb{F}_2 \oplus \mathbb{F}_2$). These extensions are involutions on $\mathrm{V}(\mathrm{Pr}_r^{\boxtimes}) = \mathrm{V}(\mathrm{M}_r^{\boxtimes})$ and will be denoted by Ψ_{xy} (the map $z \mapsto z$ and $x_i \leftrightarrow y_i$ for every $0 \leqslant i \leqslant r-1$) and Ψ_{xx} (the map $z \mapsto z$ and, for $u \in \{x,y\}$, by $u_1 \leftrightarrow u_0$, $u_2 \leftrightarrow u_{r-1}$, $u_3 \leftrightarrow u_{r-2}$, $\dots$, $u_{\lfloor \frac{r+1}{2} \rfloor} \leftrightarrow u_{\lceil \frac{r+1}{2} \rceil}$). Both Ψ_{xy} and Ψ_{xx} are automorphisms of both $\mathrm{M}_r^{\boxtimes}$ (for every $r \geqslant 5$) and $\mathrm{Pr}_r^{\boxtimes}$ (for every $r \geqslant 4$).

Case 1. $z \in \{v,w\}$. In the absence of information distinguishing v from w we may assume $z = v$.

Case 1.1. $w \in \{x_0,y_0,x_1,y_1\}$. Since $\mathrm{Aut}(\mathrm{Pr}_r^{\boxtimes})$ acts transitively on the set $\{x_0,y_0,x_1,y_1\}$ while keeping z fixed, we may assume that $w = x_0$. Then $x_0x_1\dots x_{r-1}y_{r-1}y_{r-2}\dots y_1y_0z$ in both $\mathrm{Pr}_r^{\boxtimes}$ and $\mathrm{M}_r^{\boxtimes}$ is Hamilton path linking v and w. This proves both (a16) and (a17) in the Case 1.1.

Case 1.2. $w \notin \{x_0,y_0,x_1,y_1\}$. Due to Ψ_{xy} we may assume that $w = x_i$ with $2 \leqslant i \leqslant r-1$. Now consider the expressions:

$$\begin{array}{ll} (\mathrm{Pr}.1.2.(0)) & x_iy_iy_{i+1}x_{i+1}x_{i+2}y_{i+2}\cdots y_{r-2}y_{r-1}x_{r-1}x_0x_1x_2\cdots x_{i-1}y_{i-1}y_{i-2}y_{i-3}\cdots y_0z \quad , \\ (\mathrm{Pr}.1.2.(1)) & x_iy_iy_{i+1}x_{i+1}x_{i+2}y_{i+2}\cdots x_{r-2}x_{r-1}y_{r-1}y_0y_1y_2\cdots y_{i-1}x_{i-1}x_{i-2}x_{i-3}\cdots x_0z \quad , \end{array}$$

$$\begin{array}{ll} (\mathrm{M}.1.2.(0)) & x_iy_iy_{i+1}x_{i+1}x_{i+2}y_{i+2}\cdots x_{r-2}x_{r-1}y_{r-1}x_0x_1x_2\cdots x_{i-1}y_{i-1}y_{i-2}y_{i-3}\cdots y_0z \quad , \\ (\mathrm{M}.1.2.(1)) & x_iy_iy_{i+1}x_{i+1}x_{i+2}y_{i+2}\cdots y_{r-2}y_{r-1}x_{r-1}y_0y_1y_2\cdots y_{i-1}x_{i-1}x_{i-2}x_{i-3}\cdots x_0z \quad . \end{array}$$

If i is even, then (Pr.1.2.(0)), and if i is odd then (Pr.1.2.(1)) is a Hamilton path of Pr_r linking v and w, for every even $r \geqslant 4$. If i is even, then (M.1.2.(0)), and if i is odd then (M.1.2.(1)) is a Hamilton path of M_r linking v and w, for every odd $r \geqslant 5$. This proves both (a16) and (a17) in the Case 1.2.

Case 2. $z \notin \{v, w\}$.

Case 2.1. $\{v, w\} \subseteq \{x_0, \ldots, x_{r-1}\}$ or $\{v, w\} \subseteq \{y_0, \ldots, y_{r-1}\}$. In view of Φ_{xy} we may assume that $\{v, w\} \subseteq \{x_0, \ldots, x_{r-1}\}$.

Case 2.1.1. $\{v, w\} \cap \{x_0, x_1\} \neq \varnothing$. In the absence of information distinguishing v from w we may assume that $v \in \{x_0, x_1\}$. In view of the transitivity of both $\mathrm{Aut}(\mathrm{Pr}_r^{\boxtimes})$ and $\mathrm{Aut}(\mathrm{M}_r^{\boxtimes})$ on $\{x_0, x_1, y_0, y_1\}$ we may further assume that $v = x_0$. Then $w = x_i$ for some $i \in [1, r-1]$. We can now reduce the claim we are currently proving to claims about a cartesian product of the form $\mathrm{P}_1 \,\square\, \mathrm{P}_l$ (for some l) which is obtained after deleting certain vertices. The reduction is made possible by making—depending on the parity of the i in x_i—the right choice of a 3-path or a 4-path within the graph induced by $\{z, x_0, x_1, y_0, y_1\}$.

If i is even (hence in particular $i \geqslant 2$), then starting out with the 4-path $x_0 y_0 z x_1 y_1$ leaves us facing the task of connecting y_2 with x_i (which lies in the opposite colour class compared to y_2) via a Hamilton path of the graph remaining after deletion of $\{x_0, y_0, x_1, y_1, z\}$. This remaining graph is—regardless of whether we are currently speaking about $\mathrm{M}_r^{\boxtimes}$ or $\mathrm{Pr}_r^{\boxtimes}$—isomorphic to the cartesian product $\mathrm{P}_2 \,\square\, \mathrm{P}_{r-3}$, of which the vertex y_2 is a 'corner vertex' in the sense of [36, Section 2]. Therefore this task *can* be accomplished according to [36, Lemma 1].

If on the contrary i is odd, then starting out with the 3-path $x_0 z y_0 y_1$ leaves us facing the task of connecting y_1 with x_i (which lies in the opposite colour class compared to y_1) by a Hamilton path of the graph remaining after deletion of $\{x_0, y_0, z\}$. This remaining graph is—regardless of whether we are currently speaking about $\mathrm{M}_r^{\boxtimes}$ or $\mathrm{Pr}_r^{\boxtimes}$—isomorphic to the cartesian product $\mathrm{P}_2 \,\square\, \mathrm{P}_{r-2}$, of which the vertex '$y_1$' is a corner vertex. Therefore this task, too, *can* be accomplished according to [36, Lemma 1]. This proves both (a16) and (a17) in the Case 2.1.1.

Case 2.1.2. $\{v, w\} \cap \{x_0, x_1\} = \varnothing$. Then $v = x_i$ and $w = x_j$ for some $\{i, j\} \in \binom{\{2,3,\ldots,r-1\}}{2}$. In the absence of information distinguishing v from w we may assume that $2 \leqslant i < j \leqslant r-1$.

Now consider the expressions

$$
\begin{array}{ll}
(\mathrm{Pr}.2.1.2.(1)) & x_i x_{i+1} \ldots x_{j-1} y_{j-1} y_{j-2} \ldots y_{i-1} x_{i-1} x_{i-2} y_{i-2} y_{i-3} \ldots x_2 y_2 y_1 x_1 z y_0 x_0 x_{r-1} y_{r-1} y_{r-2} \ldots x_{j+1} y_{j+1} y_j x_j \quad , \\
(\mathrm{Pr}.2.1.2.(2)) & x_i x_{i+1} \ldots x_{j-1} y_{j-1} y_{j-2} \ldots y_{i-1} x_{i-1} x_{i-2} y_{i-2} y_{i-3} \ldots x_2 y_2 y_1 x_1 z x_0 y_0 y_{r-1} x_{r-1} x_{r-2} \ldots x_{j+1} y_{j+1} y_j x_j \quad , \\
(\mathrm{Pr}.2.1.2.(3)) & x_i x_{i+1} \ldots x_{j-1} y_{j-1} y_{j-2} \ldots y_{i-1} x_{i-1} x_{i-2} y_{i-2} y_{i-3} \ldots y_2 x_2 x_1 y_1 z y_0 x_0 x_{r-1} y_{r-1} y_{r-2} \ldots x_{j+1} y_{j+1} y_j x_j \quad , \\
(\mathrm{Pr}.2.1.2.(4)) & x_i x_{i+1} \ldots x_{j-1} y_{j-1} y_{j-2} \ldots y_{i-1} x_{i-1} x_{i-2} y_{i-2} y_{i-3} \ldots y_2 x_2 x_1 y_1 z x_0 y_0 y_{r-1} x_{r-1} x_{r-2} \ldots x_{j+1} y_{j+1} y_j x_j \quad .
\end{array}
$$

and

$$
\begin{array}{ll}
(\mathrm{M}.2.1.2.(1)) & x_i x_{i+1} \ldots x_{j-1} y_{j-1} y_{j-2} \ldots y_{i-1} x_{i-1} x_{i-2} y_{i-2} y_{i-3} \ldots x_2 y_2 y_1 x_1 z y_0 x_0 y_{r-1} x_{r-1} x_{r-2} \ldots x_{j+1} y_{j+1} y_j x_j \quad , \\
(\mathrm{M}.2.1.2.(2)) & x_i x_{i+1} \ldots x_{j-1} y_{j-1} y_{j-2} \ldots y_{i-1} x_{i-1} x_{i-2} y_{i-2} y_{i-3} \ldots x_2 y_2 y_1 x_1 z x_0 y_0 x_{r-1} y_{r-1} y_{r-2} \ldots x_{j+1} y_{j+1} y_j x_j \quad , \\
(\mathrm{M}.2.1.2.(3)) & x_i x_{i+1} \ldots x_{j-1} y_{j-1} y_{j-2} \ldots y_{i-1} x_{i-1} x_{i-2} y_{i-2} y_{i-3} \ldots y_2 x_2 x_1 y_1 z y_0 x_0 y_{r-1} x_{r-1} x_{r-2} \ldots x_{j+1} y_{j+1} y_j x_j \quad , \\
(\mathrm{M}.2.1.2.(4)) & x_i x_{i+1} \ldots x_{j-1} y_{j-1} y_{j-2} \ldots y_{i-1} x_{i-1} x_{i-2} y_{i-2} y_{i-3} \ldots y_2 x_2 x_1 y_1 z x_0 y_0 x_{r-1} y_{r-1} y_{r-2} \ldots x_{j+1} y_{j+1} y_j x_j \quad .
\end{array}
$$

If i is even and j is even, then (Pr.2.1.2.(1)) for even r is a Hamilton path of $\mathrm{Pr}_r^{\boxtimes}$ linking v and w and (M.2.1.2.(1)) for odd r is one of $\mathrm{M}_r^{\boxtimes}$, while if i is even and j is odd, then (Pr.2.1.2.(2)) for even r is a Hamilton path of $\mathrm{Pr}_r^{\boxtimes}$ linking v and w and (M.2.1.2.(2)) for odd r is one of $\mathrm{M}_r^{\boxtimes}$, while if i is odd and j is even, then (Pr.2.1.2.(3)) for even r is a Hamilton path of $\mathrm{Pr}_r^{\boxtimes}$ linking v and w and (M.2.1.2.(3)) for odd r is one of $\mathrm{M}_r^{\boxtimes}$, while if i is odd and j is odd, then (Pr.2.1.2.(4)) for even r is a Hamilton path of $\mathrm{Pr}_r^{\boxtimes}$ linking v and w and (M.2.1.2.(4)) for odd r is one of $\mathrm{M}_r^{\boxtimes}$. This proves both (a16) and (a17) in the Case 2.1.2.

Case 2.2. $\{v, w\} \cap \{x_0, \ldots, x_{r-1}\} \neq \varnothing$ and $\{v, w\} \cap \{y_0, \ldots, y_{r-1}\} \neq \varnothing$. Being within Case 2, we know $\{v, w\} \subseteq \{x_0, \ldots, x_{r-1}\} \sqcup \{y_0, \ldots, y_{r-1}\}$. Therefore the statement defining Case 2.2 is the negation of the one defining Case 2.1. Due to Φ_{xy} we may assume $v = x_i$ with $0 \leqslant i \leqslant r-1$ and $w = y_j$ with $0 \leqslant j \leqslant r-1$. Due to Φ_{xx} we may further assume that $i \leqslant j$.

Case 2.2.1. $i \in \{0, 1\}$. Not only do both $\mathrm{Aut}(\mathrm{Pr}_r^{\boxtimes})$ and $\mathrm{Aut}(\mathrm{M}_r^{\boxtimes})$ act transitively on $\{x_0, x_1, y_0, y_1\}$, but it is possible to use this symmetry while still preserving the assumption $i \leqslant j$ that we already made: namely, if $i = 1$, hence $v = x_1$ and $w = y_j$ with $1 = i \leqslant j$, then $\Psi_{xx}(v) = x_0$ and $\Psi_{xx}(w) = y_{r+1-i}$ (with $y_r := y_0$) and still $0 = i \leqslant j = r + 1 - i$. Therefore we may further assume that $i = 0$, i.e. $v = x_0$. Now consider the expressions

(Pr.2.2.1.(0)) $x_0zx_1x_2\dots x_{j+1}y_{j+1}y_{j+2}x_{j+2}\dots x_{r-2}x_{r-1}y_{r-1}y_0\dots y_{j-1}y_j$,
(Pr.2.2.1.(1)) $x_0x_{r-1}x_{r-2}\dots x_{j+1}y_{j+1}y_{j+2}\dots y_{r-1}y_0zx_1y_1y_2x_2\dots x_{j-1}x_jy_j$.

(M.2.2.1.(0)) $x_0zx_1x_2\dots x_{j+1}y_{j+1}y_{j+2}x_{j+2}\dots y_{r-2}y_{r-1}x_{r-1}y_0y_1\dots y_j$,
(M.2.2.1.(1)) $x_0y_{r-1}x_{r-1}x_{r-2}y_{r-2}\dots x_jx_{j+1}\dots x_1zy_0y_1\dots y_j$.

If j is even, then (Pr.2.2.1.(0)), and if j is odd then (Pr.2.2.1.(1)) is a Hamilton path of $\mathrm{Pr}_r^{\boxtimes}$ linking v and w, for every even $r \geqslant 4$. If j is even, then (M.2.2.1.(0)), and if j is odd then (M.2.2.1.(1)) is a Hamilton path of $\mathrm{M}_r^{\boxtimes}$ linking v and w, for every odd $r \geqslant 4$. This proves (a16) in the Case 2.2.1.

Case 2.2.2. $i \notin \{0,1\}$. Now consider the expressions

(Pr.2.2.2.(0)) $x_ix_{i+1}\dots x_{j+1}y_{j+1}y_{j+2}x_{j+2}\dots x_{r-2}x_{r-1}y_{r-1}y_0x_0zx_1y_1y_2x_2x_3y_3\dots x_{i-2}x_{i-1}y_{i-1}y_iy_{i+1}\dots y_j$,
(Pr.2.2.2.(1)) $x_ix_{i+1}\dots x_{j+1}y_{j+1}y_{j+2}x_{j+2}\dots y_{r-2}y_{r-1}x_{r-1}x_0y_0zx_1y_1y_2x_2x_3y_3\dots x_{i-2}x_{i-1}y_{i-1}y_iy_{i+1}\dots y_j$.

(M.2.2.2.(0)) $x_ix_{i+1}\dots x_{j+1}y_{j+1}y_{j+2}x_{j+2}\dots y_{r-2}y_{r-1}x_{r-1}y_0x_0zx_1y_1y_2x_2x_3y_3\dots x_{i-2}x_{i-1}y_{i-1}y_iy_{i+1}\dots y_j$,
(M.2.2.2.(1)) $x_ix_{i+1}\dots x_{j+1}y_{j+1}y_{j+2}x_{j+2}\dots x_{r-2}x_{r-1}y_{r-1}x_0y_0zx_1y_1y_2x_2x_3y_3\dots x_{i-2}x_{i-1}y_{i-1}y_iy_{i+1}\dots y_j$.

Since the automorphism Ψ_{xx} changes the parity of the index of an x_i, and since (as explained in Case 2.2.1) the relation $i \leqslant j$ is preserved by Ψ_{xx}, we may assume that i is even.

If j is even, (Pr.2.2.2.(0)), and if j is odd, (Pr.2.2.2.(1)) is a Hamilton path of $\mathrm{Pr}_r^{\boxtimes}$ linking v and w, for every even $r \geqslant 4$. If j is even, then (M.2.2.2.(0)), and if j is odd then (M.2.2.2.(1)) is a Hamilton path of $\mathrm{M}_r^{\boxtimes}$ linking v and w, for every odd $r \geqslant 5$, completing the Case 2.2.2.

Since at each level of the case distinction the property defining the preceding level was partitioned into mutually exclusive properties, both (a16) and (a17) have now been proved.

As to (a18) and (a19), let $\{v,w\} \subseteq \mathrm{V}(\mathrm{Pr}_r^{\boxminus})$ be arbitrary distinct vertices. For most of the instances of the property of being Hamilton-connected it is possible to deduce the Hamilton-connectedness of $\mathrm{Pr}_r^{\boxminus}$ and $\mathrm{M}_r^{\boxminus}$ from (the proof of) (a16) in Lemma 37: if $\{v,w\} \cap \{z',z''\} = \varnothing$, then we have $\{v,w\} \subseteq \mathrm{V}(\mathrm{Pr}_r)\backslash\{z\}$ and therefore each Hamilton path P in Pr_r or M_r linking v and w contains z as a vertex of degree two. This implies that P can be extended to a Hamilton path in $\mathrm{Pr}_r^{\boxminus}$ linking v and w.

If on the contrary $\{v,w\} \cap \{z',z''\} \neq \varnothing$, then there are subcases: if $\{v,w\} = \{z',z''\}$, then $z'x_0y_0y_1\dots y_{r-1}x_{r-1}x_{r-2}\dots x_1z''$ is—in Pr_r and in M_r as well—a Hamilton path linking v and w.

We are left with the case $|\{v,w\} \cap \{z',z''\}| = 1$. In the absence of information distinguishing v from w we may assume that $v \in \{z',z''\}$ and $w \notin \{z',z''\}$. One may treat this case, too, by re-using Hamilton paths in Pr_r or M_r, but now it can make a difference (for the extendability) *how* such Hamilton path looks like around the 'special' subgraph induced on the vertices $\{z,x_0,y_0,x_1,y_1\}$ and it therefore seems quicker to treat this case directly. Since the property '$v \in \{z',z''\}$ and $w \notin \{z',z''\}$', at face value, still comprises several cases, we should reduce their number via automorphisms. However—essentially due to x_0z'' and the unique degree-5-vertex x_0 caused by it—both $\mathrm{Aut}(\mathrm{Pr}_r^{\boxminus})$ and $\mathrm{Aut}(\mathrm{M}_r^{\boxminus})$ are trivial. But since Hamilton-connectedness is a monotone graph property, it suffices to prove that $\mathrm{Pr}_r^{\boxminus,-} := \mathrm{Pr}_r^{\boxminus} - x_0z''$ and $\mathrm{M}_r^{\boxminus,-} := \mathrm{M}_r^{\boxminus} - x_0z''$ are Hamilton-connected, and these graphs *do* have symmetries again, essentially the same as $\mathrm{Pr}_r^{\boxtimes}$ and $\mathrm{M}_r^{\boxtimes}$.

The automorphism group of both $\mathrm{Pr}_r^{\boxminus,-}$ and $\mathrm{M}_r^{\boxminus,-}$ is the group generated by the two unique homomorphic extensions of $\left(\substack{\{z',z'',x_0,y_0,x_1,y_1\}\to\{z',z'',x_0,y_0,x_1,y_1\} \\ z'\mapsto z',\ z''\mapsto z'',\ x_0\leftrightarrow y_0,\ x_1\leftrightarrow y_1}\right)$ and $\left(\substack{\{z',z'',x_0,y_0,x_1,y_1\}\to\{z',z'',x_0,y_0,x_1,y_1\} \\ z'\leftrightarrow z'',\ x_0\leftrightarrow x_1,\ y_0\leftrightarrow y_1}\right)$ to all of $\mathrm{V}(\mathrm{Pr}_r^{\boxminus,-}) = \mathrm{V}(\mathrm{M}_r^{\boxminus,-})$ (thus both $\mathrm{Aut}(\mathrm{Pr}_r^{\boxminus,-})$ and $\mathrm{Aut}(\mathrm{M}_r^{\boxminus,-})$ are isomorphic to the Klein four-group $\mathbb{F}_2 \oplus \mathbb{F}_2$). These extensions are involutions on $\mathrm{V}(\mathrm{Pr}_r^{\boxminus,-}) = \mathrm{V}(\mathrm{M}_r^{\boxminus,-})$ and will be denoted by Ξ_{xy} (the map with $z' \mapsto z'$, $z'' \mapsto z''$ and $x_i \leftrightarrow y_i$ for every $0 \leqslant i \leqslant r-1$) and Ξ_{xx} (the map with $z' \leftrightarrow z''$ and, for $u \in \{x,y\}$, $u_1 \leftrightarrow u_0$, $u_2 \leftrightarrow u_{r-1}$, $u_3 \leftrightarrow u_{r-2}$, $\dots$, $u_{\lfloor \frac{r+1}{2} \rfloor} \leftrightarrow u_{\lceil \frac{r+1}{2} \rceil}$. Both Ξ_{xy} and Ξ_{xx} are automorphisms of both $\mathrm{M}_r^{\boxminus,-}$ (for every $r \geqslant 5$) and $\mathrm{Pr}_r^{\boxminus,-}$ (for every $r \geqslant 4$).

Since Ξ_{xx} interchanges z' and z'', we may assume that $v = z'$. Then there are two cases left: $w \in \{x_0,y_0,x_1,y_1\}$ and its negation $w \in \{x_2,y_2,x_3,y_3\dots,x_{r-1},y_{r-1}\}$ (keep in mind that we already assumed $w \notin \{z',z''\}$ and therefore this indeed is the negation).

Case 1. $w \in \{x_0,y_0,x_1,y_1\}$. Then since Ξ_{xy} maps $x_0 \leftrightarrow y_0$ and $x_1 \leftrightarrow y_1$ while keeping z' fixed, we may assume that $w \in \{x_0,x_1\}$ and are left with two cases.

Case 1.1. If $w = x_0$, then $z'y_0y_1z''x_1x_2y_2y_3x_3\dots y_{r-2}y_{r-1}x_{r-1}x_0$ is a Hamilton path linking v and w in $\mathrm{Pr}_r^{\boxminus,-}$ for every even $r \geqslant 4$, and $z'y_0y_1z''x_1x_2y_2y_3x_3\dots x_{r-2}x_{r-1}y_{r-1}x_0$ is one in $\mathrm{M}_r^{\boxminus,-}$ for every odd $r \geqslant 5$.

Case 1.2. If $w = x_1$, then $z'x_0y_0y_{r-1}x_{r-1}x_{r-2}y_{r-2}y_{r-3}x_{r-3}\dots y_2y_1z''x_1$ is a Hamilton path linking v and w in $\mathrm{Pr}_r^{\boxminus,-}$ for every even $r \geqslant 4$, and $z'x_0y_0x_{r-1}y_{r-1}y_{r-2}x_{r-2}x_{r-3}y_{r-3}\dots y_2y_1z''x_1$ is one in $\mathrm{M}_r^{\boxminus,-}$ for every odd $r \geqslant 5$.

Case 2. $w \in \{x_2, y_2, x_3, y_3 \dots, x_{r-1}, y_{r-1}\}$. Then since Ξ_{xy} interchanges the sets $\{x_0, \dots, x_{r-1}\}$ and $\{y_0, \dots, y_{r-1}\}$ while fixing z', we may assume that $w = x_i$ with $2 \leqslant i \leqslant r-1$. If $i \geqslant 3$, then $z'x_0y_0y_1z''x_1x_2y_2y_3\dots y_{r-1}x_{r-1}x_{r-2}\dots x_i$ is—regardless of whether i is odd or even—a Hamilton path linking v and w in both $\mathrm{Pr}_r^{\boxminus,-}$ and $\mathrm{M}_r^{\boxminus,-}$. In the case that $i = 2$, the path $z'y_0x_0x_{r-1}y_{r-1}y_{r-2}x_{r-2}x_{r-3}\dots x_3y_3y_2y_1z''x_1x_2$ is a Hamilton path linking v and w in $\mathrm{Pr}_r^{\boxminus,-}$, and $z'y_0x_0y_{r-1}x_{r-1}x_{r-2}y_{r-2}y_{r-3}\dots x_3y_3y_2y_1z''x_1x_2$ is one in $\mathrm{M}_r^{\boxminus,-}$, completing Case 2, and also the proof of both (a18) and (a19).

As to (a20) in the case $G = \mathrm{Pr}_r^{\boxtimes}$, for every even $r \geqslant 4$, the (5×5)-minor indexed by x_0y_0, x_1y_1, zx_1, zy_1, y_0y_{r-1} of the $\bigl(\|\mathrm{Pr}_r^{\boxtimes}\| \times 5\bigr)$-matrix which represents the elements of $\{\mathrm{c}_C\colon C \in \mathcal{CB}^{(1)}_{\mathrm{Pr}_r^{\boxtimes}}\}$ as elements of $\mathrm{C}_1(\mathrm{Pr}_r^{\boxtimes};\mathbb{F}_2) \supseteq \mathrm{Z}_1(\mathrm{Pr}_r^{\boxtimes};\mathbb{F}_2)$ w.r.t. the standard basis of $\mathrm{C}_1(\mathrm{Pr}_r^{\boxtimes};\mathbb{F}_2)$, is the one shown in (2.2).

	$C_{\mathrm{ev},r,1}$	$C_{\mathrm{ev},r,2}$	$C_{\mathrm{ev},r,3}$	$C_{\mathrm{ev},r,4}$	$C_{\mathrm{ev},r,5}$
x_0y_0	1	1	1	0	1
x_1y_1	1	0	1	1	0
zx_1	0	1	1	0	1
zy_1	1	1	0	0	1
y_0y_{r-1}	0	0	1	1	1

(2.2)

The matrix in (2.2) is a nonsingular element of $\mathbb{F}_2^{[5]^2}$, its inverse being $\left(\begin{smallmatrix}1&0&1&0&0\\0&1&0&1&1\\1&0&0&1&0\\0&1&1&1&0\\1&1&1&0&1\end{smallmatrix}\right) \in \mathbb{F}_2^{[5]^2}$. The existence of one such minor by itself proves (a20) in the case $G = \mathrm{Pr}_r^{\boxtimes}$. As to (a20) in the case $G = \mathrm{M}_r^{\boxtimes}$, for every odd $r \geqslant 5$, the (5×5)-minor indexed by x_0y_0, x_1y_1, zx_1, zy_1, x_0y_{r-1} of the $\bigl(\|\mathrm{M}_r^{\boxtimes}\| \times 5\bigr)$-matrix which represents the elements of $\{\mathrm{c}_C\colon C \in \mathcal{CB}^{(1)}_{\mathrm{M}_r^{\boxtimes}}\}$ as elements of $\mathrm{C}_1(\mathrm{M}_r^{\boxtimes};\mathbb{F}_2) \supseteq \mathrm{Z}_1(\mathrm{M}_r^{\boxtimes};\mathbb{F}_2)$ w.r.t. the standard basis of $\mathrm{C}_1(\mathrm{M}_r^{\boxtimes};\mathbb{F}_2)$, is the one shown in (2.3).

	$C_{\mathrm{od},r,1}$	$C_{\mathrm{od},r,2}$	$C_{\mathrm{od},r,3}$	$C_{\mathrm{od},r,4}$	$C_{\mathrm{od},r,5}$
x_0y_0	1	1	1	0	1
x_1y_1	1	0	1	1	0
zx_1	0	1	1	0	1
zy_1	1	1	0	0	1
x_0y_{r-1}	1	1	0	0	0

(2.3)

The matrix in (2.3) is a nonsingular element of $\mathbb{F}_2^{[5]^2}$, its inverse being $\left(\begin{smallmatrix}1&0&1&0&0\\1&0&1&0&1\\1&0&0&1&0\\0&1&1&1&0\\0&0&0&1&1\end{smallmatrix}\right) \in \mathbb{F}_2^{[5]^2}$. The existence of one such minor by itself proves (a20) in the case $G = \mathrm{M}_r^{\boxtimes}$. As to (a20) in the case $G = \mathrm{Pr}_r^{\boxminus}$, for every even $r \geqslant 4$ the (5×5)-minor indexed by x_0y_0, x_1y_1, $z'x_0$, $z''y_1$ and x_0x_{r-1} of the $\bigl(\|\mathrm{Pr}_r^{\boxminus}\| \times 5\bigr)$-matrix which represents the elements of $\{\mathrm{c}_C\colon C \in \mathcal{CB}^{(1)}_{\mathrm{Pr}_r^{\boxminus}}\}$ as elements of $\mathrm{C}_1(\mathrm{Pr}_r^{\boxminus};\mathbb{F}_2) \subseteq \mathrm{Z}_1(\mathrm{Pr}_r^{\boxminus};\mathbb{F}_2)$ w.r.t. the standard basis of $\mathrm{C}_1(\mathrm{Pr}_r^{\boxminus};\mathbb{F}_2)$, is the one shown in (2.4).

	$C_{\boxminus,\mathrm{ev},r,1}$	$C_{\boxminus,\mathrm{ev},r,2}$	$C_{\boxminus,\mathrm{ev},r,3}$	$C_{\boxminus,\mathrm{ev},r,4}$	$C_{\boxminus,\mathrm{ev},r,5}$
x_0y_0	0	0	0	1	0
x_1y_1	0	1	1	0	0
$z'x_0$	1	0	1	1	1
$z''y_1$	0	0	0	0	1
x_0x_{r-1}	0	1	0	0	1

(2.4)

The matrix in (2.4) is a nonsingular element of $\mathbb{F}_2^{[5]^2}$ with inverse $\left(\begin{smallmatrix}1&1&1&0&1\\0&0&0&1&1\\0&1&0&1&1\\1&0&0&0&0\\0&0&0&1&0\end{smallmatrix}\right)$. The existence of one such minor by itself proves (a20) in the case $G = \mathrm{Pr}_r^{\boxminus}$. As to (a20) in the case $G = \mathrm{M}_r^{\boxminus}$, it suffices to note that if in the preceding paragraph '$\mathrm{Pr}_r^{\boxminus}$' is replaced by '$\mathrm{M}_r^{\boxminus}$', 'even $r \geqslant 4$' by 'odd $r \geqslant 5$' and 'x_0x_{r-1}' by 'x_0y_{r-1}', then the matrix obtained is exactly the one in (2.4). This completes the proof of (a20) in its entirety.

As to (a21) in the case $G = \mathrm{Pr}_r^{\boxtimes}$, for every even $r \geqslant 4$, the $\bigl((r-1)\times(r-1)\bigr)$-minor indexed by x_1y_1, x_2y_2, $\dots$, $x_{r-1}y_{r-1}$ of the $\bigl(\|\mathrm{Pr}_r^{\boxtimes}\| \times (r-1)\bigr)$-matrix which represents the elements of

$\{c_C \colon C \in \mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxtimes}}\}$ as elements of $\mathrm{C}_1(\mathrm{Pr}_r^{\boxtimes};\mathbb{F}_2) \supseteq \mathrm{Z}_1(\mathrm{Pr}_r^{\boxtimes};\mathbb{F}_2)$ w.r.t. the standard basis of $\mathrm{C}_1(\mathrm{Pr}_r^{\boxtimes};\mathbb{F}_2)$, is the element A of $\mathbb{F}_2^{[r-1]^2}$ which is defined by $A\big[x_1y_1, C^{x_1y_1}_{\mathrm{ev},r}\big] := 1$, $A\big[x_iy_i, C^{x_jy_j}_{\mathrm{ev},r}\big] := 1$ for every $(i,j) \in \bigsqcup_{2\leqslant\iota\leqslant r-1}\{(\iota,\iota-1),(\iota,\iota)\}$ and $A\big[x_iy_i, C^{x_jy_j}_{\mathrm{ev},r}\big] := 0$ for every other $(i,j) \in \{1,\ldots,r-1\}^2$. This is a band matrix which in particular is 'lower' triangular with its main diagonal filled entirely with ones, hence nonsingular. The existence of one such minor alone implies the claim in the case $G = \mathrm{Pr}_r^{\boxtimes}$. As to the case $G = \mathrm{Pr}_r^{\boxminus}$, due to the similar definition of $\mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxminus}}$ compared to $\mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxtimes}}$, a proof in this case is obtained if in the first paragraph '$\mathrm{Pr}_r^{\boxtimes}$' is replaced by '$\mathrm{Pr}_r^{\boxminus}$', '$C^{x_1y_1}_{\mathrm{ev},r}$' by '$C^{x_1y_1}_{\boxminus,\mathrm{ev},r}$' and '$C^{x_iy_i}_{\mathrm{ev},r}$' by '$C^{x_iy_i}_{\boxminus,\mathrm{ev},r}$'. As to (a21) in the cases $G = \mathrm{M}_r^{\boxtimes}$ (respectively, $G = \mathrm{M}_r^{\boxminus}$), due to the similar definition of $\mathcal{CB}^{(2)}_{\mathrm{M}_r^{\boxtimes}}$ compared to $\mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxtimes}}$, a proof of these two cases is obtained if in the first paragraph 'even $r \geqslant 4$' is replaced by 'odd $r \geqslant 5$', '$\mathrm{Pr}_r^{\boxtimes}$' by '$\mathrm{M}_r^{\boxtimes}$' (respectively, '$\mathrm{M}_r^{\boxminus}$'), '$C^{x_1y_1}_{\mathrm{ev},r}$' by '$C^{x_1y_1}_{\mathrm{od},r}$' (respectively, '$C^{x_1y_1}_{\boxminus,\mathrm{od},r}$'), and '$C^{x_iy_i}_{\mathrm{ev},r}$' by '$C^{x_iy_i}_{\mathrm{od},r}$' (respectively, '$C^{x_iy_i}_{\boxminus,\mathrm{od},r}$'). This completes the proof of (a21) in its entirety.

As to (a22) in the case $G = \mathrm{Pr}_r^{\boxtimes}$, for an arbitrary even $r \geqslant 4$ let $c \in \big\langle \mathcal{CB}^{(1)}_{\mathrm{Pr}_r^{\boxtimes}} \big\rangle_{\mathbb{F}_2} \cap \big\langle \mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxtimes}} \big\rangle_{\mathbb{F}_2}$ be arbitrary. Then there exist $(\lambda^{(1)}) \in \mathbb{F}_2^{[5]}$ and $(\lambda^{(2)}) \in \mathbb{F}_2^{[r-1]}$ such that

($\boxtimes$.Su 1) $c = \sum_{1\leqslant i\leqslant 5} \lambda_i^{(1)}\, c_{C_{\mathrm{ev},r,i}}$, ($\boxtimes$.Su 2) $c = \sum_{1\leqslant i\leqslant r-1} \lambda_i^{(2)}\, c_{C^{x_iy_i}_{\mathrm{ev},r}}$.

where c_M for some set of edges M denotes the element $c \in \mathrm{C}_1(\mathrm{Pr}_r^{\boxtimes};\mathbb{F}_2)$ with $\mathrm{Supp}(c) = M$. We now show by contradiction that $\lambda_1^{(2)} = \cdots = \lambda_{r-1}^{(2)} = 0$, hence $\big\langle \mathcal{CB}^{(1)}_{\mathrm{Pr}_r^{\boxtimes}} \big\rangle_{\mathbb{F}_2} \cap \big\langle \mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxtimes}} \big\rangle_{\mathbb{F}_2} = \{0\}$. To this end, we make the assumption that, on the contrary,

$$\lambda_i^{(2)} = 1 \qquad \text{(for at least one } 1 \leqslant i \leqslant r-1\text{)} . \tag{2.5}$$

Drawing on the facts (straightforward to check using the definitions (P.$\boxtimes$.ES.1) and (P.$\boxtimes$.ES.2)),

(F1) $\{x_2y_2, x_3y_3, \ldots, x_{r-1}y_{r-1}\} = C_{\mathrm{ev},r,1} \cap C_{\mathrm{ev},r,2} \cap C_{\mathrm{ev},r,3} \cap C_{\mathrm{ev},r,4} \cap C_{\mathrm{ev},r,5}$,
(F2) $x_0x_{r-1} \in C_{\mathrm{ev},r,1} \cap C_{\mathrm{ev},r,2}$, $x_0x_{r-1} \notin C_{\mathrm{ev},r,3} \cup C_{\mathrm{ev},r,4} \cup C_{\mathrm{ev},r,5}$,
(F3) $y_0y_{r-1} \notin C_{\mathrm{ev},r,1} \cup C_{\mathrm{ev},r,2}$, $y_0y_{r-1} \in C_{\mathrm{ev},r,3} \cap C_{\mathrm{ev},r,4} \cap C_{\mathrm{ev},r,5}$,
(F4) $\{x_2y_2, x_3y_3, \ldots, x_{r-1}y_{r-1}\} \cap C^{x_iy_i}_{\mathrm{ev},r} \neq \varnothing$ for every $1 \leqslant i \leqslant r-1$,
(F5) $\{i \in \{1,2,\ldots,r-1\} \colon x_1y_1 \in C^{x_iy_i}_{\mathrm{ev},r}\} = \{1\}$,
(F6) $\{i \in \{1,\ldots,r-1\} \colon zx_1 \in C^{x_iy_i}_{\mathrm{ev},r}\} = \{1,\ldots,r-2\}$,
(F7) $\{\iota \in \{1,2,\ldots,r-1\} \colon x_iy_i \in C^{x_\iota y_\iota}_{\mathrm{ev},r}\} = \{i-1,i\}$ for every $2 \leqslant i \leqslant r-1$,
(F8) $\{zy_1, x_0y_0\} \cap C^{x_iy_i}_{\mathrm{ev},r} = \varnothing$ for every $1 \leqslant i \leqslant r-1$,
(F9) $\{x_0x_{r-1}, y_0y_{r-1}\} \subseteq C^{x_iy_i}_{\mathrm{ev},r}$ for every $1 \leqslant i \leqslant r-2$,
(F10) $\{x_0x_{r-1}, y_0y_{r-1}\} \cap C^{x_{r-1}y_{r-1}}_{\mathrm{ev},r} = \varnothing$,

we can now reason as follows, distinguishing whether $x_2y_2 \in \mathrm{Supp}(c)$ or not:

Case 1. $x_2y_2 \in \mathrm{Supp}(c)$. Then ($\boxtimes$.Su 1) together with (F1) implies $|\{i \in \{1,\ldots,5\} \colon \lambda_i^{(1)} = 1\}|$ being odd, and that implies that exactly one of the two numbers $|\{i \in \{1,2\} \colon \lambda_i^{(1)} = 1\}|$ and $|\{i \in \{3,4,5\} \colon \lambda_i^{(1)} = 1\}|$ is odd, which combined with ($\boxtimes$.Su 1), (F2) and (F3) implies that $|\{x_0x_{r-1}, y_0y_{r-1}\} \cap \mathrm{Supp}(c)| = 1$. But this contradicts ($\boxtimes$.Su 2), (F9) and (F10), which when taken together imply that $|\{x_0x_{r-1}, y_0y_{r-1}\} \cap \mathrm{Supp}(c)| \in \{0,2\} \not\ni 1$. This contradiction proves that Case 1 cannot occur (and we have not used our assumption (2.5) to arrive at this conclusion).

Case 2. $x_2y_2 \notin \mathrm{Supp}(c)$. From this we deduce

(Co 1) $zy_1 \notin \mathrm{Supp}(c)$,
(Co 2) $|\{i \in \{1,\ldots,5\} \colon \lambda_i^{(1)} = 1\}|$ is even ,
(Co 3) $\{x_2y_2, x_3y_3, \ldots, x_{r-1}y_{r-1}\} \cap \mathrm{Supp}(c) = \varnothing$,
(Co 4) $\lambda_1^{(2)} = \cdots = \lambda_{r-1}^{(2)} = 1$,
(Co 5) $\{x_0x_{r-1}, y_0y_{r-1}\} \cap \mathrm{Supp}(c) = \varnothing$,
(Co 6) $zx_1 \notin \mathrm{Supp}(c)$,
(Co 7) $x_1y_1 \in \mathrm{Supp}(c)$,
(Co 8) $x_0y_0 \notin \mathrm{Supp}(c)$.

These claims can be justified thus: (Co 1) follows from ($\boxtimes$.Su 2) and (F8). (Co 2) follows from combining $x_2y_2 \notin \mathrm{Supp}(c)$ with ($\boxtimes$.Su 1) and (F1). (Co 3) follows from (Co 2), ($\boxtimes$.Su 1) and (F1). (Co 4) follows from (Co 3), ($\boxtimes$.Su 2), (F4) and (F7), together with our assumption (2.5). (At this instance we have learned that in (2.5)—if it is true—the existential quantifier must necessarily hold as a universal quantifier.) (Co 5) follows from (Co 4), ($\boxtimes$.Su 2), (F9), (F10) and the evenness of

$r-2$. (Co 6) follows from (Co 4), (F6), and the evenness of $r-2 = |\{1,\dots,r-2\}|$. (Co 7) follows from ($\boxtimes$.Su 2) and (F5). (Co 8) follows from ($\boxtimes$.Su 2) and (F8).

Now from (Co 5) combined with (F2) and (F3), it follows that (Co 2) cannot be true with both $n_{1,2} := |\{i \in \{1,2\} \colon \lambda_i^{(1)} = 1\}|$ and $n_{3,4,5} := |\{i \in \{3,4,5\} \colon \lambda_i^{(1)} = 1\}|$ being odd. Therefore both $n_{1,2}$ and $n_{3,4,5}$ must be even. To finish the proof, we use the abbreviations $S_{1,2} := \mathrm{Supp}(\lambda_1^{(1)} \cdot \mathrm{c}_{C_{\mathrm{ev},r,1}} + \lambda_2^{(1)} \cdot \mathrm{c}_{C_{\mathrm{ev},r,2}})$ and $S_{3,4,5} := \mathrm{Supp}(\lambda_3^{(1)} \cdot \mathrm{c}_{C_{\mathrm{ev},r,3}} + \lambda_4^{(1)} \cdot \mathrm{c}_{C_{\mathrm{ev},r,4}} + \lambda_5^{(1)} \cdot \mathrm{c}_{C_{\mathrm{ev},r,5}})$, with which we have

$$\mathrm{Supp}(c) = S_{1,2} \mathbin{\triangle} S_{3,4,5} \qquad \text{(symmetric difference)} , \tag{2.6}$$

and distinguish cases according to the value of $n_{1,2} \in \{0,2\}$.

Case 2.1. $n_{1,2} = 0$. Then in particular $x_1y_1 \notin S_{1,2}$, $zx_1 \notin S_{1,2}$ and $zy_1 \notin S_{1,2}$.

Case 2.1.1. $n_{3,4,5} = 0$. This implies that $S_{3,4,5} = \varnothing$, and this together with $x_1y_1 \notin S_{1,2}$ and (2.6) in particular implies $x_1y_1 \notin \mathrm{Supp}(c)$, contradicting (Co 7) and proving Case 2.1.1 to be impossible.

Case 2.1.2. $n_{3,4,5} = 2$. Let us distinguish whether $\lambda_5^{(1)} \in \mathbb{F}_2$ is 0 or 1 (the motivation for this being that $zy_1 \notin S_{1,2}$ and among $C_{\mathrm{ev},r,3}$, $C_{\mathrm{ev},r,4}$, $C_{\mathrm{ev},r,5}$ only $C_{\mathrm{ev},r,5}$ contains zy_1, making it possible to draw a conclusion from the value of $\lambda_5^{(1)}$). If $\lambda_5^{(1)} = 1$, then $zy_1 \in \mathrm{Supp}(\lambda_5^{(1)} \cdot \mathrm{c}_{C_{\mathrm{ev},r,5}})$ and moreover exactly one of $\lambda_3^{(1)}$ and $\lambda_4^{(1)}$ is $= 1$. Whichever it is, due to $zy_1 \notin \mathrm{Supp}(\lambda_3^{(1)} \cdot \mathrm{c}_{C_{\mathrm{ev},r,3}})$ and $zy_1 \notin \mathrm{Supp}(\lambda_4^{(1)} \cdot \mathrm{c}_{C_{\mathrm{ev},r,4}})$ it follows that $zy_1 \in S_{3,4,5}$, which combined with $zy_1 \notin S_{1,2}$ and (2.6) implies $zy_1 \in \mathrm{Supp}(c)$, contradicting (Co 1) and proving $\lambda_5^{(1)} = 1$ to be impossible. If on the contrary $\lambda_5^{(1)} = 0$, then $\lambda_3^{(1)} = \lambda_4^{(1)} = 1$ and it follows that $zx_1 \in S_{3,4,5}$. Being within Case 2.1 we know that $zx_1 \notin S_{1,2}$, hence in view of (2.6) we may conclude that $zx_1 \in \mathrm{Supp}(c)$, contradicting (Co 6), proving Case 2.1.2, and therefore Case 2.1 as a whole, to be impossible.

Case 2.2. $n_{1,2} = 2$. This implies $x_0y_0 \notin S_{1,2}$, $x_1y_1 \in S_{1,2}$ and $zx_1 \in S_{1,2}$. Again it remains to consider the possibilities for $n_{3,4,5} \in \{0,1,2,3\}$ to be even.

Case 2.2.1. $n_{3,4,5} = 0$. Then $S_{3,4,5} = \varnothing$, and this together with $zx_1 \in S_{1,2}$ and (2.6) in particular implies $zx_1 \in \mathrm{Supp}(c)$, contradicting (Co 6) and proving Case 2.2.1 to be impossible.

Case 2.2.2. $n_{3,4,5} = 2$. Again we analyse this case by distinguishing whether $\lambda_5^{(1)} \in \mathbb{F}_2$ is 0 or 1. If $\lambda_5^{(1)} = 1$, then exactly one of $\lambda_3^{(1)}$ and $\lambda_4^{(1)}$ is $= 1$ and, whichever it is, it follows that $x_1y_1 \in S_{3,4,5}$. Being within Case 2.2. we know $x_1y_1 \in S_{1,2}$, hence in view of (2.6) it follows that $x_1y_1 \notin \mathrm{Supp}(c)$, contradicting (Co 7) and proving $\lambda_5^{(1)} = 1$ to be impossible. If on the contrary $\lambda_5^{(1)} = 0$, then $\lambda_3^{(1)} = \lambda_4^{(1)} = 1$ and it follows that $x_0y_0 \in S_{3,4,5}$. Being within Case 2.2 we know that $x_0y_0 \in S_{1,2}$ which in view of (2.6) implies $x_0y_0 \in \mathrm{Supp}(c)$, contradicting (Co 8) and proving $\lambda_5^{(1)} = 0$ to be impossible. This proves Case 2.2.2, and therefore also Case 2.2 and the entire Case 2, to be impossible. Since the mutually exclusive Cases 1 and 2 both lead to contradictions, the assumption (2.5) is false, completing the proof of (a22) for $G = \mathrm{Pr}_r^{\boxtimes}$.

As to (a22) in the case $G = \mathrm{M}_r^{\boxtimes}$, the proof given for the case $G = \mathrm{Pr}_r^{\boxtimes}$ can be repeated with the appropriate minor changes to obtain a proof in the case $G = \mathrm{M}_r^{\boxtimes}$, these changes being the following: first of all, the statements (F1)–(F10) have been chosen in such a way that each of (F1)–(F10) becomes a true statement about the set $\mathcal{CB}^{(2)}_{\mathrm{M}_r^{\boxtimes}}$ if exactly the following changes are made in (F1)–(F10): 'ev' is to be replaced by 'od', 'x_0x_{r-1}' is to be replaced by 'x_0y_{r-1}' (all occurrences, i.e. in (F2), in (F9) and in (F10)), 'y_0y_{r-1}' is to be replaced by 'y_0x_{r-1}' (all occurrences, i.e. in (F3), in (F9) and in (F10)). With the references to (F1)–(F10) now referring to the statements thus modified, the only thing to be done in the entire remaining proof of the case $G = \mathrm{Pr}_r^{\boxtimes}$ (in order to arrive at a proof of the case $G = \mathrm{M}_r^{\boxtimes}$) is to replace '$x_0x_{r-1}$' by '$x_0y_{r-1}$' and '$y_0y_{r-1}$' by '$y_0x_{r-1}$' at all three occurences of these edges (twice in Case 1, once in (Co 5)), and moreover to replace 'ev' by 'od'. This completes the proof of (a22) for $G = \mathrm{M}_r^{\boxtimes}$.

As to (a22) in the case $G = \mathrm{Pr}_r^{\boxminus}$, for an arbitrary even $r \geqslant 4$ let $c \in \left\langle \mathcal{CB}^{(1)}_{\mathrm{Pr}_r^{\boxminus}} \right\rangle_{\mathbb{F}_2} \cap \left\langle \mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxminus}} \right\rangle_{\mathbb{F}_2}$ be arbitrary. Then there are $(\lambda^{(1)}) \in \mathbb{F}_2^{[5]}$ and $(\lambda^{(2)}) \in \mathbb{F}_2^{[r-1]}$ such that

($\boxminus$.Su 1) $c = \sum_{1 \leqslant i \leqslant 5} \lambda_i^{(1)} \cdot \mathrm{c}_{C_{\boxminus,\mathrm{ev},r,i}}$, ($\boxminus$.Su 2) $c = \sum_{1 \leqslant i \leqslant r-1} \lambda_i^{(2)} \cdot \mathrm{c}_{C^{x_iy_i}_{\boxminus,\mathrm{ev},r}}$.

where C_M for some set of edges M denotes the unique element $c \in \mathrm{C}_1(\mathrm{Pr}_r^{\boxminus}; \mathbb{F}_2)$ with $\mathrm{Supp}(c) = M$.

We will show directly (this time we will not have any use for making the assumption (2.5)) that $c = 0$, hence $\left\langle \mathcal{CB}^{(1)}_{\mathrm{Pr}^{\boxminus}_r} \right\rangle_{\mathbb{F}_2} \cap \left\langle \mathcal{CB}^{(2)}_{\mathrm{Pr}^{\boxminus}_r} \right\rangle_{\mathbb{F}_2} = \{0\}$. We can now use the evident facts

($\boxminus$.F1) $z'z'' \in \bigcap_{1\leqslant i\leqslant r-1} C^{x_i y_i}_{\boxminus,\mathrm{ev},r}$,
($\boxminus$.F2) $\{x_0 x_{r-1}, y_0 y_{r-1}\} \subseteq C^{x_i y_i}_{\boxminus,\mathrm{ev},r}$ for every $1 \leqslant i \leqslant r-2$,
($\boxminus$.F3) $z'z'' \notin C_{\boxminus,\mathrm{ev},r,1}$, $z'z'' \in C_{\boxminus,\mathrm{ev},r,2}$, $z'z'' \notin C_{\boxminus,\mathrm{ev},r,3}$, $z'z'' \in C_{\boxminus,\mathrm{ev},r,4}$, $z'z'' \notin C_{\boxminus,\mathrm{ev},r,5}$,
($\boxminus$.F4) for even $r \geqslant 4$,
the only circuit among the circuits in $\mathcal{CB}^{(2)}_{\mathrm{Pr}^{\boxminus}_r}$ to contain $x_0 z''$ is $C^{x_{r-1}y_{r-1}}_{\boxminus,\mathrm{ev},r}$,
($\boxminus$.F5) for even $r \geqslant 4$,
the only circuit among the circuits in $\mathcal{CB}^{(1)}_{\mathrm{Pr}^{\boxminus}_r} \sqcup \mathcal{CB}^{(2)}_{\mathrm{Pr}^{\boxminus}_r}$ to contain $y_1 z''$ is $C_{\boxminus,\mathrm{ev},r,5}$,
($\boxminus$.F6) for even $r \geqslant 4$,
the only circuit among the circuits in $\mathcal{CB}^{(1)}_{\mathrm{Pr}^{\boxminus}_r} \sqcup \mathcal{CB}^{(2)}_{\mathrm{Pr}^{\boxminus}_r}$ to contain $x_0 y_0$ is $C_{\boxminus,\mathrm{ev},r,4}$,
($\boxminus$.F7) for even $r \geqslant 4$,
the only circuits among the circuits in $\mathcal{CB}^{(1)}_{\mathrm{Pr}^{\boxminus}_r} \sqcup \mathcal{CB}^{(2)}_{\mathrm{Pr}^{\boxminus}_r}$ to contain an odd number of the two edges $x_0 x_{r-1}$ and $y_0 y_{r-1}$ are the two circuits $C_{\boxminus,\mathrm{ev},r,3}$ and $C_{\boxminus,\mathrm{ev},r,5}$,

to argue as follows. First of all, we immediately conclude that

($\boxminus$.Co 1) $\lambda^{(1)}_4 = 0$ because of ($\boxminus$.Su 1) and ($\boxminus$.Su 2) combined with ($\boxminus$.F6),
($\boxminus$.Co 2) $\lambda^{(1)}_5 = 0$ because of ($\boxminus$.Su 1) and ($\boxminus$.Su 2) combined with ($\boxminus$.F5).

Case 1. $|\{i \in \{1,\ldots,r-1\}\colon \lambda^{(2)}_i = 1\}|$ is odd. Then ($\boxminus$.Su 2) together with ($\boxminus$.F1) implies $z'z'' \in \mathrm{Supp}(c)$. Therefore, and because of ($\boxminus$.F3), it follows that exactly one of $\lambda^{(1)}_2$ and $\lambda^{(1)}_4$ is equal to 1, hence $\lambda^{(1)}_2 = 1$ because of ($\boxminus$.Co 1). Now let us consider $\lambda^{(1)}_3$. It cannot be true that $\lambda^{(1)}_3 = 1$, since then ($\boxminus$.F7) implies $\lambda^{(1)}_5 = 1$, contradicting ($\boxminus$.Co 2). Thus we may assume that $\lambda^{(1)}_3 = 0$. This implies $x_1 y_1 \in \mathrm{Supp}(c)$ due to ($\boxminus$.Su 1), $\lambda^{(1)}_2 = 1$, ($\boxminus$.Co 1) and the fact that for every even $r \geqslant 4$, the only circuits among the circuits in $\mathcal{CB}^{(1)}_{\mathrm{Pr}^{\boxminus}_r}$ to contain $x_1 y_1$ are $C_{\boxminus,\mathrm{ev},r,2}$ and $C_{\boxminus,\mathrm{ev},r,3}$. Among the coefficients $\lambda^{(1)}_i$, $1 \leqslant i \leqslant 5$, only the value of $\lambda^{(1)}_1$ is not yet known to us.

Case 1.1. $\lambda^{(1)}_1 = 0$. Then $z'y_0 \in C_{\boxminus,\mathrm{ev},r,2}$, $\lambda^{(1)}_2 = 1$ and $\lambda^{(1)}_1 = \lambda^{(1)}_3 = \lambda^{(1)}_4 = \lambda^{(1)}_5 = 0$ together with ($\boxminus$.Su 1) imply that $z'y_0 \in \mathrm{Supp}(c)$. Since for every even $r \geqslant 4$, the only circuit among the circuits in $\mathcal{CB}^{(2)}_{\mathrm{Pr}^{\boxminus}_r}$ to contain $y_0 z'$ is $C^{x_{r-1}y_{r-1}}_{\boxminus,\mathrm{ev},r}$, from $z'y_0 \in \mathrm{Supp}(c)$ it follows that $\lambda^{(2)}_{r-1} = 1$. Being within Case 1, this implies that $|\{i \in \{1,\ldots,r-2\}\colon \lambda^{(2)}_i = 1\}|$ is even, which by ($\boxminus$.F2) implies that $\{x_0 x_{r-1}, y_0 y_{r-1}\} \cap \mathrm{Supp}(c) = \varnothing$; but $\{x_0 x_{r-1}, y_0 y_{r-1}\} \subseteq C_{\boxminus,\mathrm{ev},r,2}$ together with ($\boxminus$.Su 1), $\lambda^{(1)}_1 = \lambda^{(1)}_3 = \lambda^{(1)}_4 = \lambda^{(1)}_5 = 0$ and $\lambda^{(1)}_2 = 1$ implies that, on the contrary, $\{x_0 x_{r-1}, y_0 y_{r-1}\} \subseteq \mathrm{Supp}(c)$. This contradiction proves Case 1.1 to be impossible.

Case 1.2. $\lambda^{(1)}_1 = 1$. Then $\lambda^{(1)}_3 = \lambda^{(1)}_4 = \lambda^{(1)}_5 = 0$, $\lambda^{(1)}_1 = \lambda^{(1)}_2 = 1$ and ($\boxminus$.Su 1) together imply $x_0 z'' \notin \mathrm{Supp}(c)$. Because of ($\boxminus$.F4), this implies $\lambda^{(2)}_{r-1} = 0$. Being within Case 1, it follows that $|\{i \in \{1,\ldots,r-2\}\colon \lambda^{(2)}_i = 1\}|$ is even, hence ($\boxminus$.F2) together with ($\boxminus$.Su 2) implies that $\{x_0 x_{r-1}, y_0 y_{r-1}\} \cap \mathrm{Supp}(c) = \varnothing$; but $\lambda^{(1)}_3 = \lambda^{(1)}_4 = \lambda^{(1)}_5 = 0$, $\lambda^{(1)}_1 = \lambda^{(1)}_2 = 1$, and ($\boxminus$.Su 2), together with the facts that $\{x_0 x_{r-1}, y_{r-1}\} \cap C_{\boxminus,r,1} = \varnothing$ and $\{x_0 x_{r-1}, y_{r-1}\} \subseteq C_{\boxminus,r,2}$ imply $\{x_0 x_{r-1}, y_0 y_{r-1}\} \subseteq \mathrm{Supp}(c)$, contradiction. Therefore Case 1.2 is impossible, too.

This proves the entire Case 1 to be impossible.

Case 2. $|\{i \in \{1,\ldots,r-1\}\colon \lambda^{(2)}_i = 1\}|$ is even. Then ($\boxminus$.Su 2) together with ($\boxminus$.F1) imply $z'z'' \notin \mathrm{Supp}(c)$, hence in view of ($\boxminus$.F3) it follows that either $\lambda^{(1)}_2 = \lambda^{(1)}_4 = 0$ or $\lambda^{(1)}_2 = \lambda^{(1)}_4 = 1$, the latter being impossible because of ($\boxminus$.Co 1). Therefore, $\lambda^{(1)}_2 = \lambda^{(1)}_4 = 0$.

Case 2.1. $\lambda^{(1)}_3 = 1$. This, together with ($\boxminus$.Su 1), ($\boxminus$.Su 2), ($\boxminus$.F7) and the fact that every $C \in \{C^{x_i y_i}_{\boxminus,\mathrm{ev},r}\colon 1 \leqslant i \leqslant r-1\}$ contains an even number of the edges $x_0 x_{r-1}$ and $y_0 y_{r-1}$, implies that we must have $\lambda^{(1)}_5 = 1$, contradicting ($\boxminus$.Co 2).

Case 2.2. $\lambda_3^{(1)} = 0$. Then ($\boxminus$.Su 1), $\lambda_2^{(1)} = 0$ and the fact that $C_{\boxminus,r,2}$ and $C_{\boxminus,r,3}$ are the only circuits among $C_{\boxminus,r,1},\dots,C_{\boxminus,r,5}$ to contain x_1y_1 imply that $x_1y_1 \notin \mathrm{Supp}(c)$. Hence from ($\boxminus$.Su 2), together with the fact that for every even $r \geqslant 4$, the only circuit among the circuits in $\mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxminus}}$ to contain x_1y_1 is $C^{x_1y_1}_{\boxminus,\mathrm{ev},r}$, it follows that $\lambda_1^{(2)} = 0$. Now let us consider $\lambda_{r-1}^{(2)}$. If we would have $\lambda_{r-1}^{(2)} = 1$, then—being within Case 2—the number $|\{i \in \{2,\dots,r-2\} \colon \lambda_i^{(2)} = 1\}|$ is odd, hence $\{x_0x_{r-1}, y_0y_{r-1}\} \subseteq \mathrm{Supp}(c)$ by ($\boxminus$.Su 2) and ($\boxminus$.F2); but this contradicts ($\boxminus$.Su 1), $\lambda_2^{(1)} = \lambda_3^{(1)} = 0$, $\{x_0x_{r-1}, y_0y_{r-1}\} \cap C_{\boxminus,r,1} = \varnothing$, $\{x_0x_{r-1}, y_0y_{r-1}\} \cap C_{\boxminus,r,4} = \varnothing$ and $\{x_0x_{r-1}, y_0y_{r-1}\} \cap C_{\boxminus,r,5} = \{x_0x_{r-1}\}$, which when taken together imply $\{x_0x_{r-1}, y_0y_{r-1}\} \cap \mathrm{Supp}(c) \in \big\{\varnothing, \{x_0, x_{r-1}\}\big\}$. Therefore we may assume $\lambda_{r-1}^{(2)} = 0$. Then—being within Case 2—the number $|\{i \in \{2,\dots,r-2\} \colon \lambda_i^{(2)} = 1\}|$ is even, hence ($\boxminus$.Su 2) and ($\boxminus$.F2) imply that $\{x_0x_{r-1}, y_0y_{r-1}\} \cap \mathrm{Supp}(c) = \varnothing$. Since among $C_{\boxminus,r,1},\dots,C_{\boxminus,r,5}$ only $C_{\boxminus,r,5}$ contains x_0x_{r-1}, this implies $\lambda_5^{(1)} = 0$. We now know that $\lambda_2^{(2)} = \lambda_3^{(2)} = \lambda_4^{(2)} = \lambda_5^{(2)} = 0$. Therefore, if we would have $\lambda_1^{(1)} = 1$, then $x_1z'' \in \mathrm{Supp}(c)$, contradicting the fact that ($\boxminus$.Su 2), $\lambda_{r-1}^{(2)} = 0$, the evenness of $|\{i \in \{2,\dots,r-2\} \colon \lambda_i^{(2)} = 1\}|$ and the property $x_1z'' \in C^{x_iy_i}_{\boxminus,\mathrm{ev},r}$ for every $1 \leqslant i \leqslant r-2$ together imply $x_1z'' \notin \mathrm{Supp}(c)$. Thus, $\lambda_1^{(1)} = \lambda_2^{(2)} = \lambda_3^{(2)} = \lambda_4^{(2)} = \lambda_5^{(2)} = 0$, hence $c = 0$ by ($\boxminus$.Su 1), completing the proof of $\left\langle \mathcal{CB}^{(1)}_{\mathrm{Pr}_r^{\boxminus}} \right\rangle_{\mathbb{F}_2} \cap \left\langle \mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxminus}} \right\rangle_{\mathbb{F}_2} = \{0\}$ in Case 2. This completes the proof of (a22) in the case $G = \mathrm{Pr}_r^{\boxminus}$. As to (a22) in the case $G = \mathrm{M}_r^{\boxminus}$, again the proof of the case $G = \mathrm{Pr}_r^{\boxminus}$ can be repeated with the necessary small changes, namely: throughout, 'Pr_r' is to be replaced by 'M_r', 'ev' by 'od', 'x_0x_{r-1}' by 'x_0y_{r-1}', 'y_0y_{r-1}' by 'y_0x_{r-1}'. Afterwards, ($\boxminus$.F1)—($\boxminus$.F2) are still true and the proof given for the case $G = \mathrm{Pr}_r^{\boxminus}$ has become a proof for the case $G = \mathrm{M}_r^{\boxminus}$. The proof of Lemma (a22) is now complete.

As to (a23).($\boxtimes$.(0)), note that $\dim_{\mathbb{F}_2} \mathrm{Z}_1(\mathrm{Pr}_r^{\boxtimes}; \mathbb{F}_2) = (3r+4) - (2r+1) + 1 = r+4$, and that (a20), (a21) and (a22) in the case $G = \mathrm{Pr}_r^{\boxtimes}$ together imply that for even $r \geqslant 4$ we have $\dim_{\mathbb{F}_2}\left(\left\langle \mathcal{CB}^{(1)}_{\mathrm{Pr}_r^{\boxtimes}} \right\rangle_{\mathbb{F}_2} + \left\langle \mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxtimes}} \right\rangle_{\mathbb{F}_2} \right) = r+4$. Therefore the set $\left\langle \mathcal{CB}^{(1)}_{\mathrm{Pr}_r^{\boxtimes}} \right\rangle_{\mathbb{F}_2} + \left\langle \mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxtimes}} \right\rangle_{\mathbb{F}_2}$ is an $\mathbb{F}_2$-linear subspace of $\mathrm{Z}_1(\mathrm{Pr}_r^{\boxtimes}; \mathbb{F}_2)$ having the same dimension as the ambient space. In a vector space this implies equality as a set. This proves ($\boxtimes$.(0)). An entirely analogous argument proves (a23).($\boxtimes$.(1)).

As to (a23).($\boxminus$.(0)), note that $\dim_{\mathbb{F}_2} \mathrm{Z}_1(\mathrm{Pr}_r^{\boxminus}; \mathbb{F}_2) = (3r+6) - (2r+2) + 1 = r+5$ and that (a20), (a21) and (a22) in the case $G = \mathrm{Pr}_r^{\boxtimes}$ together imply $\dim_{\mathbb{F}_2}\left(\left\langle \mathcal{CB}^{(1)}_{\mathrm{Pr}_r^{\boxtimes}} \right\rangle_{\mathbb{F}_2} + \left\langle \mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxtimes}} \right\rangle_{\mathbb{F}_2} \right) = r+4$ for even $r \geqslant 4$. Since $\dim_K(V/U) = \dim_K(V) - \dim_K(U)$ for finite-dimensional K-vectors spaces $U \subseteq V$, this implies ($\boxminus$.(0)). An analogous argument proves (a23).($\boxminus$.(1)).

As to (a23).($\boxminus$.$|\cdot|-1$.(0)), this claim follows quickly from ($\boxminus$.(0)): it suffices to note that in $\mathrm{Pr}_r^{\boxminus}$ there actually exists a circuit of length $|\cdot|-1$. Since $|\mathrm{Pr}_r^{\boxminus}| = |\mathrm{M}_r^{\boxminus}| = r+4$ is even for even r, and since the support of the sum of two circuits of even length is an edge-disjoint union of circuits of even length, any circuit of length $|\cdot|-1$ in $\mathrm{Pr}_r^{\boxminus}$ is not contained in $\left\langle \mathcal{CB}^{(1)}_{\mathrm{Pr}_r^{\boxtimes}} \right\rangle_{\mathbb{F}_2} + \left\langle \mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxtimes}} \right\rangle_{\mathbb{F}_2}$, hence after adding this circuit to the set $\mathcal{CB}^{(1)}_{\mathrm{Pr}_r^{\boxtimes}} \sqcup \mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxtimes}}$, the $\mathbb{F}_2$-linear span has dimension $(r+4)+1 = r+5 = \dim_{\mathbb{F}_2} \mathrm{Z}_1(\mathrm{Pr}_r^{\boxminus}; \mathbb{F}_2)$, proving ($\boxminus$.$|\cdot|-1$.(0)), since that finite-dimensional vector spaces do not contain proper subspaces of the same dimension. An entirely analogous argumentation proves (a23).($\boxminus$.$|\cdot|-1$.(1)), this time using ($\boxminus$.(1)).

We have now proved (a24)–(a29): property(a24) follows from ($\boxtimes$.(0)) (which is equivalent to $\mathrm{Pr}_r^{\boxtimes} \in \mathrm{Cd}_0\mathcal{C}_{\{|\cdot|\}}$), (a16) and Definition 204.(12); property (a25) follows from ($\boxtimes$.(1)) (which is equivalent to $\mathrm{M}_r^{\boxtimes} \in \mathrm{Cd}_0\mathcal{C}_{\{|\cdot|\}}$), (a17) and Definition 204.(12); property (a26) follows from ($\boxminus$.(0)) (which is equivalent to $\mathrm{Pr}_r^{\boxminus} \in \mathrm{Cd}_1\mathcal{C}_{\{|\cdot|\}}$), (a18) and Definition 204.(12); property (a27) follows from ($\boxminus$.(1)) (which is equivalent to $\mathrm{M}_r^{\boxminus} \in \mathrm{Cd}_1\mathcal{C}_{\{|\cdot|\}}$), (a19) and Definition 204.(12); property (a28) follows from ($\boxminus$.$|\cdot|-1$.(0)) (which is equivalent to $\mathrm{Pr}_r^{\boxminus} \in \mathrm{Cd}_0\mathcal{C}_{\{|\cdot|-1,|\cdot|\}}$), (a18) and Definition 204.(12); property (a29) follows follows from ($\boxminus$.$|\cdot|-1$.(1)) (which is equivalent to $\mathrm{M}_r^{\boxminus} \in \mathrm{Cd}_0\mathcal{C}_{\{|\cdot|-1,|\cdot|\}}$), (a19) and Definition 204.(12).

As to (a30), the bandwidth of any of C_n^2, CL_r, $\mathrm{Pr}_r^{\boxtimes}$, $\mathrm{Pr}_r^{\boxminus}$, $\mathrm{M}_r^{\boxtimes}$ and $\mathrm{M}_r^{\boxminus}$ is constant, i.e. does not grow with r or n. Therefore (a30) is true in stronger form than is stated here. Since knowing the exact bandwidths would profit us nothing given the proof technology that is available at present, knowing the statement (a30) is enough. To prove it, we employ a general characterisation [23, Theorem 8] of low-bandwidth graphs due to Böttcher, Pruessmann, Taraz and Würfl. This characterisation allows us to prove the smallness of the bandwidth for each of the rather different graphs C_n^2, CL_r, $\mathrm{Pr}_r^{\boxtimes}$, $\mathrm{Pr}_r^{\boxminus}$, $\mathrm{M}_r^{\boxtimes}$ and $\mathrm{M}_r^{\boxminus}$ without any close attention to the specifics of these graphs—simply by exhibiting small separators: in C_n^2 there does not exist any edge between the two sets $A := \{0, 1, \ldots, \lfloor \frac{n}{2} \rfloor - 2\}$ and $B := \{\lfloor \frac{n}{2} \rfloor + 1, \ldots, n-3\}$, and since both $|A|$ and $|B|$ are $\leqslant \frac{2}{3}|C_n^2|$, the existence of the separator $S := \{\lfloor \frac{n}{2} \rfloor - 1, \lfloor \frac{n}{2} \rfloor, n-2, n-1\}$ implies that the separation number (in the sense of [23, Definition 2]) of C_n^2 is at most 4. The claim (a30) in the case of $G = \mathrm{CL}_r$ now follows by [23, Theorem 8, equivalence (2) $\Leftrightarrow$ (4)]. To prove the case $G = \mathrm{CL}_r$ of (a30), in the first sentence of this paragraph use '$A := \bigsqcup_{1 \leqslant i \leqslant \lfloor \frac{r}{2} \rfloor - 1} \{a_i, b_i\}$', '$B := \bigsqcup_{\lfloor \frac{r}{2} \rfloor + 1 \leqslant i \leqslant r-1} \{a_i, b_i\}$' and '$S := \{a_0, b_0, a_{\lfloor \frac{r}{2} \rfloor}, b_{\lfloor \frac{r}{2} \rfloor}\}$'. To prove the cases $G \in \{\mathrm{Pr}_r^{\boxtimes}, \mathrm{M}_r^{\boxtimes}\}$ of (a30), in the first sentence of this paragraph use '$A := \{z\} \sqcup \bigsqcup_{1 \leqslant i \leqslant \lfloor \frac{r}{2} \rfloor - 1} \{x_i, y_i\}$', '$B := \bigsqcup_{\lfloor \frac{r}{2} \rfloor + 1 \leqslant i \leqslant r-1} \{x_i, y_i\}$' and '$S := \{a_0, b_0, a_{\lfloor \frac{r}{2} \rfloor}, b_{\lfloor \frac{r}{2} \rfloor}\}$'. To prove the cases $G \in \{\mathrm{Pr}_r^{\boxminus}, \mathrm{M}_r^{\boxminus}\}$ of (a30), use B and S as in the preceding sentence but '$A := \{z', z''\} \sqcup \bigsqcup_{1 \leqslant i \leqslant \lfloor \frac{r}{2} \rfloor - 1} \{x_i, y_i\}$'. This proves the statement about the bandwidth in (a30), for every $H \in \{C_n^2, \mathrm{CL}_r, \mathrm{Pr}_r^{\boxtimes}, \mathrm{Pr}_r^{\boxminus}, \mathrm{M}_r^{\boxtimes}, \mathrm{M}_r^{\boxminus}\}$.

As to the additional claims concerning $H \in \{\mathrm{Pr}_r^{\boxtimes}, \mathrm{Pr}_r^{\boxminus}, \mathrm{M}_r^{\boxtimes}, \mathrm{M}_r^{\boxminus}\}$, we explicitly give suitable maps b_H and h_H (thus for $\mathrm{Pr}_r^{\boxtimes}$, $\mathrm{Pr}_r^{\boxminus}$, $\mathrm{M}_r^{\boxtimes}$, $\mathrm{M}_r^{\boxminus}$ giving another proof of the small bandwidth).

As to $H = \mathrm{Pr}_r^{\boxtimes}$, for every even $r \geqslant 4$, the map b_H defined by $z \mapsto 1$, $x_0 \mapsto 2$ $x_i \mapsto 4i$ for $1 \leqslant i \leqslant \lfloor \frac{r}{2} \rfloor$, $x_i \mapsto 4(r-i) + 2$ for $\lfloor \frac{r}{2} \rfloor + 1 \leqslant i \leqslant r-1$, $y_0 \mapsto 3$, $y_i \mapsto 4i + 1$ for $1 \leqslant i \leqslant \lfloor \frac{r}{2} \rfloor$, and $y_i \mapsto 4(r-i) + 3$ for $\lfloor \frac{r}{2} \rfloor + 1 \leqslant i \leqslant r-1$ is a bandwidth-4-labelling of $\mathrm{Pr}_r^{\boxtimes}$. Moreover, the map h_H defined by $z \mapsto 0$, $x_i \mapsto 1$ and $y_i \mapsto 2$ for even $0 \leqslant i \leqslant r-1$, $x_i \mapsto 2$ and $y_i \mapsto 1$ for odd $0 \leqslant i \leqslant r-1$, is a 3-colouring of $\mathrm{Pr}_r^{\boxtimes}$ which for every r large enough to have simultaneously $\beta|H| = \beta(2r+1) \geqslant 1 = |\mathrm{h}_H^{-1}(0)|$ and $8 \cdot 2 \cdot \beta \cdot |H| = 16\beta(2r+1) \geqslant 2$ obviously satisfies the requirement in Theorem 38 of being $(8 \cdot 2 \cdot \beta \cdot |H|, 4 \cdot 2 \cdot \beta \cdot |H|)$-zero-free w.r.t. b_H and having $|\mathrm{h}_H^{-1}(0)| \leqslant \beta|H|$. This proves (a30) for $H = \mathrm{Pr}_r^{\boxtimes}$.

As to $H = \mathrm{M}_r^{\boxtimes}$, the same map b_H that was defined at the beginning of the preceding paragraph is (this being the reason for having used $\lfloor \cdot \rfloor$ despite even r) a bandwidth-5-labelling of $\mathrm{M}_r^{\boxtimes}$ (which has bandwidth 4, by the way), for every odd $r \geqslant 5$. Likewise, the same map h_H defined in the preceding paragraph is a 3-colouring of $\mathrm{M}_r^{\boxtimes}$ for which concerning $|\mathrm{h}_H^{-1}(0)|$ and zero-freeness w.r.t. b_H exactly the same can be said as in the previous paragraph. This proves (a30) for $H = \mathrm{M}_r^{\boxtimes}$.

As to $H = \mathrm{Pr}_r^{\boxminus}$, for every even $r \geqslant 4$, the map b_H defined by $z' \mapsto 1$, $z'' \mapsto 2$, $x_0 \mapsto 3$, $y_0 \mapsto 4$, $x_i \mapsto 4i + 1$ and $y_i \mapsto 4i + 2$ for $1 \leqslant i \leqslant \lfloor \frac{r}{2} \rfloor$, $x_i \mapsto 4(r-i) + 3$ and $y_i \mapsto 4(r-i) + 4$ for $\lfloor \frac{r}{2} \rfloor + 1 \leqslant i \leqslant r-1$ is a bandwidth-5-labelling of $\mathrm{Pr}_r^{\boxminus}$. Moreover, the map h_H defined by $z' \mapsto 2$, $z'' \mapsto 0$, $x_0 \mapsto 1$, $y_0 \mapsto 2$, $x_1 \mapsto 0$, $y_1 \mapsto 1$, $x_i \mapsto 1$ and $y_i \mapsto 2$ for even $2 \leqslant i \leqslant r-1$, and $x_i \mapsto 2$ and $y_i \mapsto 1$ for odd $2 \leqslant i \leqslant r-1$ is a 3-colouring of $\mathrm{Pr}_r^{\boxminus}$. In view of $|\mathrm{h}_H^{-1}(0)| = 2$ and in particular in view of the fact that $\mathrm{b}(\mathrm{h}_H^{-1}(0)) = \{2, 5\}$ for *every* even $r \geqslant 4$ (i.e. the distance along the bandwidth-5-labelling of the two 0-labelled vertices is constantly 3, i.e. independent of $|H|$), it is obvious that h_H is $(8 \cdot 2 \cdot \beta \cdot |H|, 4 \cdot 2 \cdot \beta \cdot |H|)$-zero-free w.r.t. b_H, provided that r is large enough to have $4 \cdot 2 \cdot \beta \cdot |H| = 8\beta(2r+2) \geqslant 5$ (when testing the zero-freeness-property for the vertex $z' = \mathrm{b}_H^{-1}(1)$, we have to make five steps forward in order to have a zero-free interval ahead of us—but this is also the highest number of necessary repositioning steps we can encounter). If r is large enough to have $\beta|H| = \beta(2r+2) \geqslant 2 = |\mathrm{h}_H^{-1}(0)|$, too, then both requirements about h_H are met. This completes the proof of (a30) in the case $H = \mathrm{Pr}_r^{\boxminus}$.

As to $H = \mathrm{M}_r^{\boxminus}$, replace '$\mathrm{M}_r^{\boxtimes}$' by '$\mathrm{M}_r^{\boxminus}$' throughout the paragraph before the last (and delete the comment about bandwidth equal to 4) in order to arrive at a proof of (a30) in the case $H = \mathrm{M}_r^{\boxminus}$.

Since n_0 can be chosen large enough to simultaneously satisfy the finitely many (and only β-dependent) requirements on r encountered in the above cases, we have now proved (a30) (where the n_0 is promised *before* the choice $H \in \{\mathrm{C}_n^2, \mathrm{CL}_r, \mathrm{Pr}_r^{\boxtimes}, \mathrm{Pr}_r^{\boxminus}, \mathrm{M}_r^{\boxtimes}, \mathrm{M}_r^{\boxminus}\}$ is made) in its entirety. $\square$

Let us close Section 2.1.2 with two comments. Firstly, our proof of ($\boxminus.|\cdot| - 1.(0)$) shows that out of the generating set $\mathcal{C}_{\{|\cdot|-1,|\cdot|\}}(\mathrm{Pr}_r^{\boxminus})$ it suffices to use only *one* circuit having the length $|\cdot| - 1$. The same is true for $\mathcal{C}_{\{|\cdot|-1,|\cdot|\}}(\mathrm{Pr}_r^{\boxminus})$. Since the monotonicity-argument used for proving Theorem 6 keeps adding Hamilton-circuits to the current generating set—but never adds a circuit of length $|\cdot| - 1$ to it—this also implies that in Theorem 6.(I2), a single circuit of length $|\cdot| - 1$ suffices in a generating set. For this reason our proof of Theorem 6.(I2) indeed proves the full mod-2-version obtained by reading Conjecture 3.(I.2) modulo 2 (which gives $\mathcal{M}_{|\cdot|,0}{}^{-}$, not only $\mathcal{M}_{\{|\cdot|-1,|\cdot|\},0}$, in the conclusion of Theorem 6.(I2)).

Secondly, with $\mathrm{Pr}_r^{\boxminus,-} := \mathrm{Pr}_r^{\boxminus} - x_0 z''$ and $\mathrm{M}_r^{\boxminus,-} := \mathrm{M}_r^{\boxminus} - x_0 z''$, the study of the special cases $r = 4$ and $r = 6$ strongly suggests that for every even $r \geqslant 4$,

($\boxminus, -.(0)$) $\dim_{\mathbb{F}_2}\big(\mathrm{Z}_1(\mathrm{Pr}_r^{\boxminus,-};\mathbb{F}_2)/\langle\mathcal{H}(\mathrm{Pr}_r^{\boxminus})\rangle_{\mathbb{F}_2}\big) = 2,$ ($\boxminus, -.(1)$) $\dim_{\mathbb{F}_2}\big(\mathrm{Z}_1(\mathrm{M}_r^{\boxminus,-};\mathbb{F}_2)/\langle\mathcal{H}(\mathrm{M}_r^{\boxminus})\rangle_{\mathbb{F}_2}\big) = 2,$

but this we will not prove. The statements ($\boxminus, -.(0)$) and ($\boxminus, -.(1)$), if true in general, provide a justification for adding the symmetry-destroying edge $x_0 z''$ into the rather symmetric graphs. Because of these two codimensions, the graphs $\mathrm{Pr}_r^{\boxminus,-}$ and $\mathrm{M}_r^{\boxminus,-}$ are unsuitable as auxiliary substructures for proving (I2) in Theorem 6: when adding an edge, the codimension of the span of Hamilton-circuits in the cycle space can at most stay the same, never decrease.

2.1.3 Details on steps ($\mathbb{F}_2$-St.2) and ($\mathbb{F}_2$-St.3)

Important tools for ($\mathbb{F}_2$-St.2) are the following theorems:

Theorem 38 (Böttcher–Schacht–Taraz [24, Theorem 2])**.** *For every $\gamma > 0$ and arbitrary $\rho \in \mathbb{Z}_{\geqslant 2}$ and $\Delta \in \mathbb{Z}_{\geqslant 2}$ there exist numbers $\beta = \beta(\gamma, \Delta) > 0$ and $n_0 = n_0(\gamma, \Delta)$ such that the following is true: for every graph G with $|G| \geqslant n_0$ and $\delta(G) \geqslant (\frac{\rho-1}{\rho} + \gamma)|G|$, and for every graph H having $|G| = |H|$, $\Delta(H) \leqslant \Delta$ and $\mathrm{bw}(H) \leqslant \beta|H|$, and admitting a bandwidth-$\beta|H|$-labelling $\mathrm{b}\colon \mathrm{V}(H) \to \{1, \ldots, |H|\}$ and a $(\rho+1)$-colouring $h\colon \mathrm{V}(H) \to \{0, 1, \ldots, \rho\}$ which is $\big(8\rho\beta|H|, 4\rho\beta|H|\big)$-zero-free w.r.t.* b *and has $|h^{-1}(0)| \leqslant \beta|H|$, there is an embedding $H \hookrightarrow G$.* □

Theorem 39 (Böttcher–Heinig–Taraz [22, Theorem 3])**.** *For every $\gamma > 0$ and every $\Delta \in \mathbb{Z}$ there exist numbers $\beta = \beta(\gamma, \Delta) > 0$ and $n_0 = n_0(\gamma, \Delta) \in \mathbb{Z}$ such that the following is true: for every balanced bipartite graph G with $|G| \geqslant n_0$ and $\delta(G) \geqslant (\frac{1}{4} + \gamma)|G|$, and for every balanced bipartite graph H with $|H| = |G|$, $\Delta(H) \leqslant \Delta$ and $\mathrm{bw}(H) \leqslant \beta|H|$, there is an embedding $H \hookrightarrow G$.* □

The lower bound of terrestrial magnitude provided by (I4) depends on a recent theorem of Châu, DeBiasio and Kierstead (who say [35, p. 17, Section 5, l. 5] that by optimising their proof one may not push the bound further down than to about $n_0 = 10^5$, but who do express optimism as to the possibility of getting rid of the lower bound on $|\cdot|$ altogether):

Theorem 40 (Komlós–Sárközy–Szemerédi [102, Theorem 1], Jamshed [90, Chapter 3]; explicit lower bound on $|G|$ proved by Châu–DeBiasio–Kierstead [35, Theorem 7])**.** *For every graph G with $|G| \geqslant 2 \cdot 10^8$ and $\delta(G) \geqslant \frac{2}{3}|G|$ there exists an embedding $\mathrm{C}^2_{|G|} \hookrightarrow G$.* □

2.1.3.1 Auxiliary statements for step ($\mathbb{F}_2$-St.3)

There is a simple algebraic lemma underlying Lemma 43, and for this lemma it appears that free modules over principal ideal domains provide the natural generality. With a view towards Section 2.2 which studies $\vec{\mathcal{H}}(G)$ vis-à-vis the $\mathbb{Z}$-module $\mathrm{Z}_1(G;\mathbb{Z})$, let us opt for this generality right-away, at negligible additional cost, but with more insight into the underlying mechanism. If R is a commutative ring, M a free R-module and $\mathcal{B} \subseteq M$ an R-basis of M, then for every $v \in M$ we write $(\lambda_{\mathcal{B},v,b})_{b\in\mathcal{B}} \in R^B$ for the unique element of R^B (cofinitely-many components zero) with $v = \sum_{b\in\mathcal{B}} \lambda_{\mathcal{B},v,b}\, b$. Then for every $b \in \mathcal{B}$ the map $\lambda_{\mathcal{B},\cdot,b}\colon v \mapsto \lambda_{\mathcal{B},v,b}$ is an element of $\mathrm{Hom}_R(M, R)$ (which elsewhere is often denoted by b^*). Moreover, we define $\mathrm{Supp}_{\mathcal{B}}(v) := \{b \in \mathcal{B}\colon \lambda_{\mathcal{B},v,b} \neq 0\} \subseteq \mathcal{B}$. We can now formulate a slight generalization of [113, Lemma 1] and [7, Corollary 3.2], which is the algebraic mechanism underlying Lemma 43:

Lemma 41. *If R is a principal ideal domain, $R^\times$ its group of units, M a finitely-generated free R-module, $\mathcal{B} \subseteq M$ an R-basis of M, $b_0 \in \mathcal{B}$ an arbitrary element, $U \subseteq M$ an arbitrary sub-R-module, and $u_0 \in U$ an arbitrary element with $\lambda_{\mathcal{B},u_0,b_0} \in R^\times$,*

$$U = \langle\{u \in U : b_0 \notin \mathrm{Supp}_{\mathcal{B}}(u)\}\rangle_R \oplus \langle u_0\rangle_R \ . \tag{2.7}$$

Proof of Lemma 41. The sum is obviously direct: $b_0 \in \mathrm{Supp}_{\mathcal{B}}(u_0)$ while $b_0 \notin \mathrm{Supp}_{\mathcal{B}}(v)$ for every $v \in \langle\{u \in U : b_0 \notin \mathrm{Supp}_{\mathcal{B}}(u)\}\rangle_R$, hence the intersection of the summands is $\{0\}$. What is to be justified is that $U \subseteq \langle\{u \in U : b_0 \notin \mathrm{Supp}_{\mathcal{B}}(u)\}\rangle_R + \langle u_0\rangle_R$. So let $v \in U$ be arbitrary. By standard theory (e.g. [69, Theorem 6.1]), M being free over a principal ideal domain implies that U is free, too, i.e., there exists a finite R-basis $\mathcal{E} \in \binom{U}{\mathrm{rk}_R(U)}$ of U. Let $\mathcal{E}_0 := \{e \in \mathcal{E} : b_0 \in \mathrm{Supp}_{\mathcal{B}}(e)\}$. Since $\lambda_{\mathcal{B},\cdot,b_0} \in \mathrm{Hom}_R(M,R)$, we know $\lambda_{\mathcal{B},\cdot,b_0}\ \big(\ (\ \sum_{e\in\mathcal{E}\setminus\mathcal{E}_0} \lambda_{\mathcal{E},v,e}\ e\ ) + (\ \sum_{e\in\mathcal{E}_0} \lambda_{\mathcal{E},v,e}\ (\ e - \lambda_{\mathcal{B},e,b_0}$ $(\lambda_{\mathcal{B},u_0,b_0})^{-1}\ u_0\ )\ )\ \big) = 0$, and therefore b_0 is not an element of $\mathrm{Supp}_{\mathcal{B}}(\cdot)$ of

$$v - \left((\lambda_{\mathcal{B},u_0,b_0})^{-1} \sum_{e\in\mathcal{E}_0} \lambda_{\mathcal{E},v,e}\ \lambda_{\mathcal{B},e,b_0}\right) u_0 = \left(\sum_{e\in\mathcal{E}\setminus\mathcal{E}_0} \lambda_{\mathcal{E},v,e}\ e\right) + \left(\sum_{e\in\mathcal{E}_0} \lambda_{\mathcal{E},v,e}\ (e - \lambda_{\mathcal{B},e,b_0}(\lambda_{\mathcal{B},u_0,b_0})^{-1}u_0)\right) . \tag{2.8}$$

Thus, writing $v = \big(v - \big((\lambda_{\mathcal{B},u_0,b_0})^{-1} \sum_{e\in\mathcal{E}_0} \lambda_{\mathcal{E},v,e}\ \lambda_{\mathcal{B},e,b_0}\ \big)\ u_0\big) + \big((\lambda_{\mathcal{B},u_0,b_0})^{-1} \sum_{e\in\mathcal{E}_0} \lambda_{\mathcal{E},v,e}\ \lambda_{\mathcal{B},e,b_0}\ \big)\ u_0$ shows that $v \in \langle\{u \in U : b_0 \notin \mathrm{Supp}_{\mathcal{B}}(u)\}\rangle_R + \langle u_0\rangle_R$, completing the proof of $U \subseteq \langle\{u \in U : b_0 \notin \mathrm{Supp}_{\mathcal{B}}(u)\}\rangle_R \oplus \langle u_0\rangle_R$. □

The proof of Lemma 41 does not work if the assumption of M being finitely generated is dropped: while U then still admits a basis, there is no reason why $\mathcal{E}_0$ should be a finite set, so the sums in (2.8) might not be defined. Within the unexplored realm of the linear algebra of Hamilton circles in infinite graphs, this obstacle to naïvely adapting the monotonicity argument might be a good point to start.

Definition 42 (the elementary chains $\mathrm{c}_{C,u,v}$)**.** *If G is a graph, $uv \in \mathrm{E}(G)$, and C a circuit in G, then we define $\mathrm{c}_{C,xy}$ as the element of $\mathrm{Z}_1(G)$ obtained by starting at x and then moving along C edge-by-edge, in the direction defined by moving from x to y, and each time adding a summand of the form $\varepsilon \cdot (u' \wedge v')$ where $\{u',v'\}$ is the edge traversed and $\varepsilon = +1$ if $\{u',v'\}$ is traversed from u' to v' and $\varepsilon = -1$ if it is traversed from v' to u'. (In particular, we start with adding $u \wedge v \in \mathrm{C}_1(G)$).*

Let us note that in the notation $\mathrm{c}_{C,u,v}$, the order of u and v matters: we have $\mathrm{c}_{C,u,v} = -\mathrm{c}_{C,v,u}$ for every G, uv and C as in Definition 42.

Some of the following statements are essential both for the proof via ($\mathbb{F}_2$-St.1)–($\mathbb{F}_2$-St.3), and the proof via ($\mathbb{Z}$-St.1)–($\mathbb{Z}$-St.3) in Section 2.2.2:

Lemma 43. *For any function $\mathfrak{L}$ mapping graphs to subsets of $\mathbb{Z}_{\geqslant 1}$, any $\xi \in \mathbb{Z}_{\geqslant 0}$, and any finitely-generated abelian group A, the sets from Definition 204.*(12)*–*(22) *are monotone; specifically:*

1. $\mathcal{M}_{\mathfrak{L},\xi}$ *is a monotone increasing graph property,*
2. $\mathrm{b}\mathcal{M}_{\mathfrak{L},\xi}$ *is a monotone increasing property of bipartite graphs,*
3. ${\mathcal{M}_{\mathfrak{L},\xi}}^-$ *is a monotone increasing graph property,*
4. $\mathrm{b}{\mathcal{M}_{\mathfrak{L},\xi}}^-$ *is a monotone increasing property of bipartite graphs,*
5. $\mathcal{M}^{\mathbb{Z}}_{\mathfrak{L},A}$ *is a monotone increasing graph property,*
6. $\mathrm{b}\mathcal{M}^{\mathbb{Z}}_{\mathfrak{L},A}$ *is a monotone increasing property of bipartite graphs,*
7. $\mathcal{M}^{\mathbb{Z}\mathrm{Bas}}_{\mathfrak{L}}$ *is a monotone increasing graph property,*
8. $\mathrm{b}\mathcal{M}^{\mathbb{Z}\mathrm{Bas}}_{\mathfrak{L}}$ *is a monotone increasing property of bipartite graphs,*
9. $\mathcal{M}^{\beta_0=1}_{\mathrm{B}\nsubseteq\mathrm{G}}$ *is a monotone increasing graph property,*
10. $\mathrm{b}\mathcal{M}^{\beta_0=1}_{\mathrm{B}\nsubseteq\mathrm{G}}$ *is a monotone increasing property of bipartite graphs.*

Proof of Lemma 43. Let us first note that for each of the sets of graphs named in (1)–(10) it is obvious that the sets are invariant (as sets) under any graph isomorphism, because the properties by which they are defined are obviously invariant.

Statement (1) (resp. (2)) is implied by statement (5) (resp. (6)) by setting $A := \bigoplus^{\beta_1(G)-\xi} \mathbb{F}_2$, so it suffices to prove those statements. As to (3) and (4), the proofs are entirely analogous to those of (5) and (6).

As to statement (5), if $\mathcal{M}^{\mathbb{Z}}_{\mathfrak{L},A} = \varnothing$, the claim is vacuously true. Otherwise, consider any $G \in \mathcal{M}^{\mathbb{Z}}_{\mathfrak{L},A}$ and any $e \in \binom{\mathrm{V}(G)}{2} \backslash \mathrm{E}(G)$. We will use the abbreviation $G + e := (\mathrm{V}(G), \mathrm{E}(G) \sqcup \{e\})$. We have to prove $G + e \in \mathcal{M}^{\mathbb{Z}}_{\mathfrak{L},A}$. Trivially, $G + e \in \mathcal{CO}_{\mathfrak{L}-1}$. What has to be justified is that $G + e \in \mathrm{Quo}_A\mathcal{C}_{\mathfrak{L}}$. Since $G \in \mathcal{CO}_{\mathfrak{L}-1}$, there exists in G a path P with length in $\{l - 1 \colon l \in \mathfrak{L}\}$ linking the endvertices of e and we have $e \notin \mathrm{E}(P)$ since $e \notin \mathrm{E}(G)$. Choose any such P. We now use Lemma 41 twice: let $R := \mathbb{Z}$, $M := \mathrm{C}_1(G + e)$, $\mathcal{B} := \{\mathrm{c}_{u<v} \colon \{u < v\} \in \mathrm{E}(G + e)\}$ (the standard basis of the 1-dimensional chain group $\mathrm{C}_1(G + e)$) and $b_0 := e$.

For $\{u < v\} := e$, the circuit $C := uPvu$ and the elementary 1-chain $\mathrm{c}_{C,u,v}$ (cf. Definition 42) we have both $\mathrm{c}_{C,u,v} \in \vec{\mathcal{C}}_{\mathfrak{L}}(G + e)$ and $\mathrm{c}_{C,u,v} \in \mathrm{Z}_1(G + e)$, hence, both with $U := \langle \vec{\mathcal{C}}_{\mathfrak{L}}(G + e)\rangle_{\mathbb{Z}}$ and with $U := \mathrm{Z}_1(G + e)$, in both cases we have $u_0 := \mathrm{c}_{C,u,v} \in U$, and therefore Lemma 41 gives us

(ds1) $\langle \vec{\mathcal{C}}_{\mathfrak{L}}(G + e)\rangle_{\mathbb{Z}} = \langle \vec{\mathcal{C}}_{\mathfrak{L}}(G)\rangle_{\mathbb{Z}} \oplus \langle \mathrm{c}_{C,u,v}\rangle_{\mathbb{Z}}$, (ds2) $\mathrm{Z}_1(G + e) = \mathrm{Z}_1(G) \oplus \langle \mathrm{c}_{C,u,v}\rangle_{\mathbb{Z}}$,

in particular since $\{u \in \mathrm{Z}_1(G + e) \colon b_0 \notin \mathrm{Supp}_{\mathcal{B}}(u)\} = \mathrm{Z}_1(G)$. The direct sum decompositions of abelian groups in (ds1) and (ds2) imply $\mathrm{Z}_1(G + e)/\langle \mathcal{C}_{\mathfrak{L}}(G + e)\rangle_{\mathbb{Z}} \cong \mathrm{Z}_1(G)/\langle \vec{\mathcal{C}}_{\mathfrak{L}}(G)\rangle_{\mathbb{Z}} \cong$ (since by hypothesis $G \in \mathcal{M}^{\mathbb{Z}}_{\mathfrak{L},A}$) $\cong A$, hence indeed $G + e \in \mathrm{Quo}_A\mathcal{C}_{\mathfrak{L}}$, completing the proof of (5).

As to (6), it suffices to note that the proof of (5) may be repeated to yield a proof of (6), the only change required being to restrict e to be an edge whose addition keeps the graph bipartite and to replace '$\mathcal{CO}_{\mathfrak{L}-1}$' by '$\mathcal{LA}_{\mathfrak{L}-1}$'. and '$\mathrm{Quo}_A\mathcal{C}_{\mathfrak{L}}$' by '$\mathrm{bQuo}_A\mathcal{C}_{\mathfrak{L}}$'.

As to (7), the proof parallels the proof of (5) and could, if you will, be considered even easier (one does not even need the quotient of two direct sum-decompositions here): if $\mathcal{M}^{\mathbb{Z}\mathrm{Bas}}_{\mathfrak{L}} = \varnothing$, the claim is vacuously true. Otherwise, consider any $G \in \mathcal{M}^{\mathbb{Z}\mathrm{Bas}}_{\mathfrak{L}}$ and any $e \in \binom{\mathrm{V}(G)}{2} \backslash \mathrm{E}(G)$. We will use the abbreviation $G + e := (\mathrm{V}(G), \mathrm{E}(G) \sqcup \{e\})$. We have to prove $G + e \in \mathcal{M}^{\mathbb{Z}\mathrm{Bas}}_{\mathfrak{L}}$. Trivially, $G + e \in \mathcal{CO}_{\mathfrak{L}-1}$. What has to be justified is that $G + e \in \mathrm{Bas}\mathcal{C}_{\mathfrak{L}}$. Now it suffices to use Lemma 41 once: setting $R := \mathbb{Z}$, $M := \mathrm{C}_1(G + e)$, $\mathcal{B} := \{\mathrm{c}_{u<v} \colon \{u < v\} \in \mathrm{E}(G + e)\}$ (the standard basis of the 1-dimensional chain group $\mathrm{C}_1(G + e)$), $b_0 := e$, $\{u < v\} := e$, $C := uPvu$, $U := \mathrm{Z}_1(G + e)$ and $u_0 := \mathrm{c}_{C,u,v} \in U$, we have $\lambda_{\mathcal{B},u_0,b_0} = +1 \in \{\pm\} = \mathbb{Z}^{\times}$, so Lemma 41 implies

$$\mathrm{Z}_1(G + e) = \mathrm{Z}_1(G) \oplus \langle \mathrm{c}_{C,u,v}\rangle_{\mathbb{Z}} \ . \tag{2.9}$$

By the assumption $G \in \mathcal{M}^{\mathbb{Z}\mathrm{Bas}}_{\mathfrak{L}}$, the group $\mathrm{Z}_1(G)$ admits a basis $\mathcal{B}$ with $z \in \vec{\mathcal{C}}_{\mathfrak{L}}(G)$ for each $z \in \mathcal{B}$. By construction, we know that $\mathrm{c}_{C,u,v} \in \vec{\mathcal{C}}_{\mathfrak{L}}(G)$, too. Thus, (2.9) implies that $\mathcal{B} \sqcup \{\mathrm{c}_{C,u,v}\}$ is a basis of $\mathrm{Z}_1(G + e)$ with the required property, completing the proof of (7).

As to (8), it suffices to note that the proof of (7) may be repeated to yield a proof of (8); the only necessary modification is to restrict e to be an edge whose addition keeps the graph bipartite, to replace '$\mathcal{CO}_{\mathfrak{L}-1}$' by '$\mathcal{LA}_{\mathfrak{L}-1}$', and '$\mathrm{Bas}\mathcal{C}_{\mathfrak{L}}$' by '$\mathrm{bBas}\mathcal{C}_{\mathfrak{L}}$'.

As to (9), this would be vacuously true if $\mathcal{M}^{\beta_0=1}_{\mathrm{B}\nsubseteq\mathrm{G}}$ was empty (the example of the 6-wheel in Section 1.2.1 shows it is not). Otherwise, consider any $G \in \mathcal{M}^{\beta_0=1}_{\mathrm{B}\nsubseteq\mathrm{G}}$ and any edge $\{u < v\} := e$ not in G. By hypothesis, there is at least one generating set $\mathcal{S} \subseteq \mathrm{Z}_1(G)$ of $\mathrm{Z}_1(G)$ not containing any basis. Aiming at a contradiction, suppose every generating set of $\mathrm{Z}_1(G + e)$ contained a basis. Since G is connected, there exists a path connecting the endvertices of e; choose any such path P and set $C := uPvu$. With this C, an application of Lemma 41 entirely analogous to the one in the proof of (7) above again yields the decomposition (2.9). Since $\mathcal{S}$ is a generating set of $\mathrm{Z}_1(G)$, it follows from (2.9) that the set $\{\mathrm{c}_{C,u,v}\} \sqcup \mathcal{S}$ is a generating set of $\mathrm{Z}_1(G + e)$. By our assumption, this implies the existence of a basis $\mathcal{B} \subseteq \{\mathrm{c}_{C,u,v}\} \sqcup \mathcal{S}$ of $\mathrm{Z}_1(G + e)$. By definition of 'basis',

$$|\mathcal{B}| = \mathrm{rank}(\mathrm{Z}_1(G + e)) = 1 + \mathrm{rank}(\mathrm{Z}_1(G)) \ . \tag{2.10}$$

Since $\mathrm{c}_{C,u,v}$ is the only element of $\{\mathrm{c}_{C,u,v}\} \sqcup \mathcal{S}$ whose support contains e, and since there are elements of $\mathrm{Z}_1(G + e)$ whose support contains e, necessarily $\mathrm{c}_{C,u,v} \in \mathcal{B}$. Therefore,

$$|\mathcal{B}\backslash\{\mathrm{c}_{C,u,v}\}| = |\mathcal{B}| - 1 \overset{(2.10)}{=} \mathrm{rank}(\mathrm{Z}_1(G)) \ . \tag{2.11}$$

We moreover show that $\mathcal{B}\backslash\{c_{C,u,v}\}$ is a generating set of $Z_1(G)$: let an arbitrary $z \in Z_1(G)$ be given. Since we know that $\mathcal{B}$ is a basis of $Z_1(G+e) \supseteq Z_1(G)$, there exists exactly one $\lambda \in \mathbb{Z}^{\mathcal{B}}$ with $z = \sum_{b\in\mathcal{B}} \lambda_b b$. From $z \in Z_1(G)$ it follows that $e \notin \mathrm{Supp}(z)$; since $c_{C,u,v}$ is the only element of $\{c_{C,u,v}\} \sqcup \mathcal{S}$ whose support contains e, it is also the only element of $\mathcal{B}$ whose support contains e, hence $e \notin \mathrm{Supp}(z)$ and $z = \sum_{b\in\mathcal{B}} \lambda_b b$ imply $\lambda_{c_{C,u,v}} = 0$, this is a $\mathbb{Z}$-linear combination of z in terms of $\mathcal{B}\backslash\{c_{C,u,v}\} \subseteq \mathcal{S}$. We have thus proved that $\mathcal{B}\backslash\{c_{C,u,v}\}$ is a generating set of $Z_1(G)$. This together with (2.11) proves $\mathcal{B}\backslash\{c_{C,u,v}\}$ to be a rank-sized generating set, i.e., a basis of $Z_1(G)$; but this and $\mathcal{B}\backslash\{c_{C,u,v}\} \subseteq \mathcal{S}$ is a contradiction to the definition of $\mathcal{S}$. We have proved it impossible that every generating set of $Z_1(G+e)$ contains a basis. Thus $G+e \in \mathcal{M}^{\beta_0=1}_{\mathrm{B}\nsubseteq\mathrm{G}}$, completing the proof of (9).

As to (10), it suffices to note that the proof of (9) may be repeated, the only necessary modification being not allow e to be any edge outside $\mathrm{E}(G)$, but just any edge outside $\mathrm{E}(G)$ whose addition keeps G bipartite. □

In words, Lemma 43.(7) says that if a graph is Hamilton-connected and has Hamilton-based flow lattice, then all its supergraphs on the same vertex-set have these properties, too. Let us note that, with $\langle 0\rangle$ denoting the one-element group, $\mathcal{M}^{\mathbb{Z}}_{\{|\cdot|\},\langle 0\rangle}$ is the set of all Hamilton-connected graphs whose flow lattice admits a generating set consisting of Hamilton-flows.

Lemma 43 can serve to elevate theorems guaranteeing the existence of spanning subgraphs with a certain property to theorems guaranteeing this property for the entire ambient graph:

Corollary 44 (lifting properties from spanning subgraphs to host graphs)**.** *For any function $\mathfrak{L}$ mapping graphs to subsets of $\mathbb{Z}_{\geqslant 1}$, any $\xi \in \mathbb{Z}_{\geqslant 0}$, any finitely-generated abelian group A, any set of graphs $\mathcal{G}$ and any set $\mathrm{b}\mathcal{G}$ of bipartite graphs:*

$$(1)\quad \begin{pmatrix} \text{if } G \in \mathcal{G}, \text{ then } \exists H \in \mathcal{M}_{\mathfrak{L},\xi} \text{ with} \\ |H| = |G| \text{ and } H \hookrightarrow G \end{pmatrix} \Longrightarrow \big(\text{ if } G \in \mathcal{G}, \text{ then } G \in \mathcal{M}_{\mathfrak{L},\xi}\ \big),$$

$$(2)\quad \begin{pmatrix} \text{if } G \in \mathrm{b}\mathcal{G}, \text{ then } \exists H \in \mathrm{b}\mathcal{M}_{\mathfrak{L},\xi} \text{ with} \\ |H| = |G| \text{ and } H \hookrightarrow G \end{pmatrix} \Longrightarrow \big(\text{ if } G \in \mathrm{b}\mathcal{G}, \text{ then } G \in \mathrm{b}\mathcal{M}_{\mathfrak{L},\xi}\ \big),$$

$$(3)\quad \begin{pmatrix} \text{if } G \in \mathcal{G}, \text{ then } \exists H \in \mathcal{M}^{\mathbb{Z}}_{\mathfrak{L},A} \text{ with} \\ |H| = |G| \text{ and } H \hookrightarrow G \end{pmatrix} \Longrightarrow \big(\text{ if } G \in \mathcal{G}, \text{ then } G \in \mathcal{M}^{\mathbb{Z}}_{\mathfrak{L},A}\ \big),$$

$$(4)\quad \begin{pmatrix} \text{if } G \in \mathrm{b}\mathcal{G}, \text{ then } \exists H \in \mathrm{b}\mathcal{M}^{\mathbb{Z}}_{\mathfrak{L},A} \text{ with} \\ |H| = |G| \text{ and } H \hookrightarrow G \end{pmatrix} \Longrightarrow \big(\text{ if } G \in \mathrm{b}\mathcal{G}, \text{ then } G \in \mathrm{b}\mathcal{M}^{\mathbb{Z}}_{\mathfrak{L},A}\ \big),$$

$$(5)\quad \begin{pmatrix} \text{if } G \in \mathcal{G}, \text{ then } \exists H \in \mathcal{M}^{\mathbb{Z}\mathrm{Bas}}_{\mathfrak{L}} \text{ with} \\ |H| = |G| \text{ and } H \hookrightarrow G \end{pmatrix} \Longrightarrow \big(\text{ if } G \in \mathcal{G}, \text{ then } G \in \mathcal{M}^{\mathbb{Z}\mathrm{Bas}}_{\mathfrak{L}}\ \big),$$

$$(6)\quad \begin{pmatrix} \text{if } G \in \mathrm{b}\mathcal{G}, \text{ then } \exists H \in \mathrm{b}\mathcal{M}^{\mathbb{Z}\mathrm{Bas}}_{\mathfrak{L}} \text{ with} \\ |H| = |G| \text{ and } H \hookrightarrow G \end{pmatrix} \Longrightarrow \big(\text{ if } G \in \mathrm{b}\mathcal{G}, \text{ then } G \in \mathrm{b}\mathcal{M}^{\mathbb{Z}\mathrm{Bas}}_{\mathfrak{L}}\ \big). \quad \square$$

Lemma 43 is what makes ($\mathbb{F}_2$-St.3) of the argument tick. It is very similar to a lemma of Locke [113, Lemma 1], but we will re-prove Lemma 43 in Section 2.1.2, for two reasons: First, Locke's assumption of 2-connectedness and the attendant appeal to Menger's theorem [113, p. 253, last line] was appropriate while being concerned with a (possibly small) subgraph of special nature within a larger 2-connected graph. But it seems out of place when dealing with *spanning* subgraphs. It feels more to the point to explicitly name a one-dimensional direct summand which is acquired as a result of the added edge. Second, we will need a version of Locke's lemma especially phrased for bipartite graphs, and this is not to be found in (but easily obtained from) [113].

Let us note that the 'monotonising' intersection

$$\begin{aligned}\mathcal{M}^{\mathbb{Z}\mathrm{Bas}}_{|\cdot|} &= \mathrm{Bas}\mathcal{C}_{|\cdot|} \cap \mathcal{CO}_{|\cdot|-1} \\ &= \{\text{graphs which admit a Hamilton-flow-basis of their flow lattice}\} \\ &\cap \{\text{Hamilton-connected graphs}\}\ , \end{aligned} \tag{2.12}$$

which to consider seems key to proving sufficient conditions for Hamilton-based flow lattices, is a non-trivial intersection in the sense that neither of the two intersectands is contained in the other. The non-inclusion

$$\mathcal{CO}_{|\cdot|-1} \nsubseteq \mathrm{Bas}\mathcal{C}_{|\cdot|} \tag{2.13}$$

is for example witnessed by the graph $\mathrm{CE}_{(\mathrm{I1})}$ from Definition 212: while $\mathrm{CE}_{(\mathrm{I1})} \in \mathcal{CO}_{|\cdot|-1}$ by Lemma 45, from Proposition 46 we know $\mathrm{CE}_{(\mathrm{I1})} \in \mathrm{Cd}_1\mathcal{C}_{|\cdot|}$, hence $\mathrm{CE}_{(\mathrm{I1})} \notin \mathrm{Cd}_0\mathcal{C}_{|\cdot|}$, hence $\mathrm{CE}_{(\mathrm{I1})} \notin \mathrm{Bas}\mathcal{C}_{|\cdot|}$, proving (2.13).

The non-inclusion

$$\mathrm{Bas}\mathcal{C}_{|\cdot|} \nsubseteq \mathcal{CO}_{|\cdot|-1} \tag{2.14}$$

is of course already witnessed by the trivial example of circuit graphs, but there are non-trivial examples, too, i.e. *graphs with a Hamilton-based flow lattice of arbitrarily large rank which are nevertheless not Hamilton-connected.* I.e., despite a Hamilton-based flow lattice, there can be non-adjacent vertices not connected by a Hamilton-path (adjacent vertices must of course be so connected). These (counter-)examples will not be treated in detail in this thesis (in particular this would involve another formal proof of being Hamilton-based for an infinite set of graphs), but we will describe one such non-trivial example witnessing (2.14): start with $\mathrm{C}_{13}^{2^-}$ from Definition 214, remove the two edges $0,1$ and $0,12$, then add the edges $0,3$ and $0,11$. The resulting graph, which we denote $\mathrm{C}_{13}^{2^-\wedge}$, has the same degree-sequence as $\mathrm{C}_{13}^{2^-}$, and has edit-distance 4 from it. According to calculations that will not be proved in this thesis, $\mathrm{C}_{13}^{2^-\wedge}$ has Hamilton-*based* flow lattice $\mathrm{Z}_1(\mathrm{C}_{13}^{2^-\wedge})$. Yet $\mathrm{C}_{13}^{2^-\wedge}$ is not Hamilton-connected: there does not exist any Hamilton-path linking the two non-adjacent vertices 2 and 3. Examples like $\mathrm{C}_{13}^{2^-\wedge}$, which look deceptively similar to the suitable seed graphs $\mathrm{C}_n^{2^-}$, provide some justification for the effort we will invest in proving Hamilton-connectedness of $\mathrm{C}_n^{2^-}$ for $n \equiv 3 \pmod 4$ in Lemma 66.(5) (in particular, counterexamples in such close proximity to $\mathrm{C}_n^{2^-}$ make it seem unlikely that there are neat sufficient criteria which would quickly prove $\mathrm{C}_n^{2^-}$ Hamilton-connected).

The two non-inclusions (2.13) and (2.14) shed some light on the role of the minimum-degree condition $\delta(G) \geqslant (\frac{1}{2}+\gamma)|G|$: this condition binds together (or rather, forces a graph into the intersection of) two graph properties which, all graphs considered, are logically independent; if a graph satisfies the minimum degree-condition, then because of *this* it must lie in the intersection

$$\mathrm{Bas}\mathcal{C}_{|\cdot|} \cap \mathcal{CO}_{|\cdot|-1} = \mathcal{M}_{|\cdot|}^{\mathbb{Z}\mathrm{Bas}}\ . \tag{2.15}$$

One could view this as an example of correlation without causation in the presence of another cause: if $\delta(G) \geqslant (\frac{1}{2}+\gamma)|G|$ holds, then because of this the properties $\mathrm{Bas}\mathcal{C}_{|\cdot|}$ and $\mathcal{CO}_{|\cdot|-1}$ co-appear, without a causal link between the two.

2.1.3.2 Proof of Theorem 6 from Chapter 1

As to (I1), let $\gamma > 0$ be given and invoke Theorem 38 with this γ, $\rho := 2$ and $\Delta := 4$ to get a $\beta > 0$ and an n_0, here denoted by n_0', with the property stated there. Give this β to Lemma 37.(a30) to get an $n_0 = n_0(\beta)$, here denoted by n_0'', with the properties stated there. We now argue that with $n_0 := \max(n_0', n_0'')$ the claim in (I1) is true. Let $\mathcal{G}$ be the set of all graphs G with odd $|G| \geqslant n_0$ and $\delta(G) \geqslant (\frac{1}{2}+\gamma)|G|$. Let $G \in \mathcal{G}$ be arbitrary, $r := \frac{1}{2}(|G|-1)$ and $H := \mathrm{Pr}_r^{\boxtimes}$ in case $|G| \equiv 1 \pmod 4$, resp. $H := \mathrm{M}_r^{\boxtimes}$ in case $|G| \equiv 3 \pmod 4$. Then $H \in \mathcal{M}_{\{|\cdot|\},0}$ in view of Lemmas 37.(a24) and 37.(a25), moreover $|H| = |G|$ and also $H \hookrightarrow G$ since $\Delta(H) = 4 \leqslant \Delta$ and Lemma 37.(a30) in the case '$H = \mathrm{Pr}_r^{\boxtimes}$' (resp. '$H = \mathrm{M}_r^{\boxtimes}$') allows us to apply Theorem 38—with the γ, ρ, Δ, β, n_0 we already fixed—to the graphs G and H. Therefore, by Corollary 44.(1) it follows that $G \in \mathcal{M}_{\{|\cdot|\},0}$, in particular $G \in \mathrm{Cd}_0\mathcal{C}_{\{|\cdot|\}}$, as claimed in (I1).

As to (I2), if throughout the preceding paragraph we replace '(I1)' by '(I2)', 'odd' by 'even', '$r := \frac{1}{2}(|G|-1)$' by '$r := \frac{1}{2}|G|$', '$\mathrm{Pr}_r^{\boxtimes}$' by '$\mathrm{Pr}_r^{\boxminus}$', '$\mathrm{M}_r^{\boxtimes}$' by '$\mathrm{M}_r^{\boxminus}$', '$\mathcal{M}_{\{|\cdot|\},0}$' by '$\mathcal{M}_{\{|\cdot|\},1}$', 'Lemma 37.(a24)' by 'Lemma 37.(a26)', 'Lemma 37.(a25)' by 'Lemma 37.(a27)', '$\Delta(H) = 4$' by '$\Delta(H) = 5$', and '$\mathrm{Cd}_0\mathcal{C}_{|\cdot|}$' by '$\mathrm{Cd}_1\mathcal{C}_{|\cdot|}$', then we obtain a proof of the codimension-one-statement in (I2). Moreover,

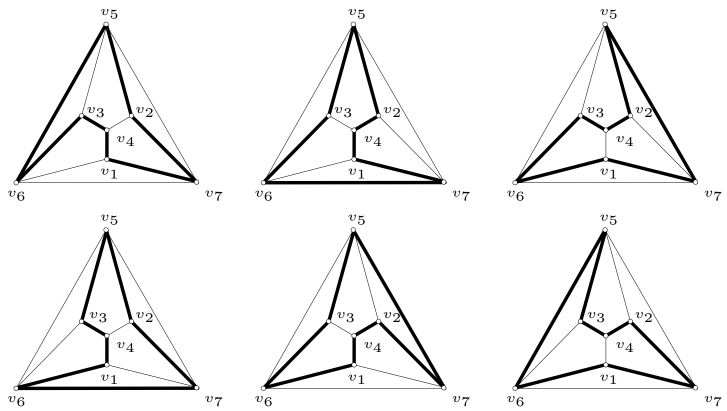


Figure 2.1: A counterexample which proves that a graph having several properties which intuitively may seem conducive to the property of being Hamilton-generated, can nevertheless fail to have it: the graph $\mathrm{CE}_{(\mathrm{I1})}$ underlying Figure 2.1 has an odd number of vertices, is 3-vertex-connected, pancyclic and Hamilton-connected (by, e.g., [164, Theorem 1.2]: the only independent set with three vertices is $\{v_1, v_2, v_3\}$, and for this vertex-set the criterion [164, Theorem 1.2] holds, as $\deg(v_1) + \deg(v_2) + \deg(v_3) - |\mathrm{N}(v_1) \cap \mathrm{N}(v_2) \cap \mathrm{N}(v_3)| = \deg(v_1) + \deg(v_2) + \deg(v_3) - |\{v_4\}| = 3 + 3 + 3 - 1 = 8 \geqslant 7 + 1 = |\mathrm{CE}_{(\mathrm{I1})}| + 1$), in particular has each of its edges contained in a Hamilton-circuit. And yet it has its cycle space not generated by its Hamilton-circuits (it is easy to see that the six shown in the figure are all that $\mathrm{CE}_{(\mathrm{I1})}$ has, and that there is a non-trivial $\mathbb{F}_2$-linear relation among them). Note that $\mathrm{CE}_{(\mathrm{I1})}$ (barely) fails the Dirac condition, so is not a counterexample to Conjecture 23.

if in these replacement instructions we change '$\mathcal{M}_{\{|\cdot|\},1}$' to '$\mathcal{M}_{\{|\cdot|-1,|\cdot|\},0}$', 'Lemma 37.(a26)' to 'Lemma 37.(a28)', and 'Lemma 37.(a27)' to 'Lemma 37.(a29)', and then apply the new instructions once more to the first paragraph, we obtain a proof of the second claim in (I2).

As to (I3), let $\gamma > 0$ be given and invoke Theorem 39 with this γ and $\Delta := 3$ to get a $\beta > 0$ and an n_0, here denoted by n_0', with the property stated there. Give this β to Lemma 37.(a30) to get an $n_0 = n_0(\beta)$, here denoted by n_0'', with the properties stated there. We now argue that with $n_0 := \max(n_0', n_0'')$ the claim in (I3) is true. Let $\mathrm{b}\mathcal{G}$ be the set of all balanced bipartite graphs G with $|G| \geqslant n_0$ and $\delta(G) \geqslant (\frac{1}{4} + \gamma)|G|$. Let $G \in \mathcal{G}$ be arbitrary and set $r := \frac{1}{2}|G|$ and $H := \mathrm{CL}_r$. Then $H \in \mathrm{b}\mathcal{M}_{\{|\cdot|\},0}$ in view of Lemma 37.(a15), moreover $|H| = |G|$ and also $H \hookrightarrow G$ since $\Delta(H) = 3 \leqslant \Delta$ and Lemma 37.(a30) in the case $H = \mathrm{CL}_r$ allows us to apply Theorem 39—with the γ, ρ, Δ, β, n_0 we already fixed—to the graphs G and H. Therefore, by Corollary 44.(2) it follows that $G \in \mathrm{b}\mathcal{M}_{\{|\cdot|\},0}$, in particular $G \in \mathrm{bcd}_0\mathcal{C}_{\{|\cdot|\}}$, which is what is claimed in (I3).

As to (I4), let $\mathcal{G}$ be the set of all graphs G with $|G| \geqslant 2 \cdot 10^8$ and $\delta(G) \geqslant \frac{2}{3}|G|$. Let $G \in \mathcal{G}$ be arbitrary. Then Theorem 40 guarantees that $\mathrm{C}^2_{|G|} \hookrightarrow G$. If $|G|$ is odd, then by combining Corollary 44.(1) and Lemma 37.(a4), it follows that $G \in \mathcal{M}_{\{|\cdot|\},0}$, in particular $G \in \mathrm{Cd}_0\mathcal{C}_{\{|\cdot|\}}$, which proves (I4) in the case of odd $|\cdot|$. If $|G|$ is even, then (I4) follows by combining Corollary 44.(1) with Lemma 37.(a3), resp. Lemma 37.(a5). All the implications in Theorem 6 have now been proved.

2.1.3.3 On weakening the hypotheses of Theorem 6 from Chapter 1

Lemma 45. *The graph* $\mathrm{CE}_{(I1)}$ *from Definition 212 is Hamilton-connected.*

Proof. We use a criterion of Bing Wei [164, Theorem 1.2]. The graph $\mathrm{CE}_{(I1)}$ is 3-connected, so what is left to check is the degree-condition, for every independent set of size 3 in $\mathrm{CE}_{(I1)}$. Since v_5, v_6, v_7 induce a complete graph, any independent set may intersect $\{v_5, v_6, v_7\}$ in at most one element; if so, then it is evident that the other two vertices cannot be chosen so as to be non-adjacent to the vertex in $\{v_5, v_6, v_7\}$ and be non-adjacent to each other. Therefore each independent set of size 3 in $\mathrm{CE}_{(I1)}$ must be contained in $\{v_1, v_2, v_3, v_4\}$, and then it must equal $\{v_1, v_2, v_3\}$. Therefore, the *only* independent set of size 3 in $\mathrm{CE}_{(I1)}$ is $\{v_1, v_2, v_3\}$. For this set, the degree condition in [164, Theorem 1.2] holds. □

The graph $\mathrm{CE}_{(I1)}$ has odd $|\cdot|$, is 3-vertex-connected, is pancyclic (i.e. contains at least one circuit of each of all possible lengths $3, \ldots, |G|$) and is Hamilton-connected. Moreover, $\frac{1}{2}|\mathrm{CE}_{(I1)}| = 3.5 \nleqslant 3 = \delta(\mathrm{CE}_{(I1)})$, i.e. $\mathrm{CE}_{(I1)}$ has minimum-degree only one less than what is required by the Dirac-threshold. Thus, the following facts (which prove the claim made in Theorem 6 about weakening (I1)) show that the open question (I1) can easily acquire a negative answer even if the hypotheses are only slightly weakened and a few other intuitively favorable assumptions, like Hamilton-connectedness (which at one less than the Dirac threshold is not yet implied by the minimum-degree alone) *added* to it:

Proposition 46. $\dim_{\mathbb{F}_2}\big(\mathrm{Z}_1(\mathrm{CE}_{(I1)};\mathbb{F}_2)/\big\langle\mathcal{H}(\mathrm{CE}_{(I1)})\big\rangle_{\mathbb{F}_2}\big) = 1$

Proof. The smallness of $\mathrm{CE}_{(I1)}$ makes it easy to check that $\mathrm{CE}_{(I1)}$ contains no other than for following six Hamilton-circuits (shown in Figure 2.1) $C_1 := v_1v_7v_2v_5v_6v_3v_4v_1$, $C_2 := v_1v_7v_6v_3v_5v_2v_4v_1$, $C_3 := v_1v_7v_5v_2v_4v_3v_6v_1$, $C_4 := v_1v_6v_7v_2v_5v_3v_4v_1$, $C_5 := v_1v_6v_3v_5v_7v_2v_4v_1$, $C_6 := v_1v_6v_5v_3v_4v_2v_7v_1$. If the standard basis of $\mathrm{C}_1(\mathrm{CE}_{(I1)};\mathbb{F}_2)$ is identified with the edges of $\mathrm{CE}_{(I1)}$ and these edges used as the row index set, then w.r.t. to this basis the Hamilton-circuits $C_1, \ldots, C_6$ give rise to the matrix shown in (2.16), which has $\mathbb{F}_2$-rank 5 (that it has $\mathbb{F}_2$-rank $\leqslant 5$ is obvious since every row contains an even number of ones):

$$\begin{array}{c|cccccc} & C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \\ \hline v_1v_4 & 1 & 1 & 0 & 1 & 1 & 0 \\ v_1v_6 & 0 & 0 & 1 & 1 & 1 & 1 \\ v_1v_7 & 1 & 1 & 1 & 0 & 0 & 1 \\ v_2v_4 & 0 & 1 & 1 & 0 & 1 & 1 \\ v_2v_5 & 1 & 1 & 1 & 1 & 0 & 0 \\ v_2v_7 & 1 & 0 & 0 & 1 & 1 & 1 \\ v_3v_4 & 1 & 0 & 1 & 1 & 0 & 1 \\ v_3v_5 & 0 & 1 & 0 & 1 & 1 & 1 \\ v_3v_6 & 1 & 1 & 1 & 0 & 1 & 0 \\ v_5v_6 & 1 & 0 & 0 & 0 & 0 & 1 \\ v_5v_7 & 0 & 0 & 1 & 0 & 1 & 0 \\ v_6v_7 & 0 & 1 & 0 & 1 & 0 & 0 \end{array} \tag{2.16}$$

Therefore $\big\langle\mathcal{H}(\mathrm{CE}_{(I1)})\big\rangle_{\mathbb{F}_2}$ is a 5-dimensional subspace of $\mathrm{Z}_1(\mathrm{CE}_{(I1)};\mathbb{F}_2)$. Since $\dim_{\mathbb{F}_2}\mathrm{Z}_1(\mathrm{CE}_{(I1)};\mathbb{F}_2) = \beta_1(\mathrm{CE}_{(I1)}) = \|\mathrm{CE}_{(I1)}\| - |\mathrm{CE}_{(I1)}| + 1 = 12 - 7 + 1 = 6$, this proves Proposition 46. □

2.2 Integral flows ($\mathbb{Z}$-coefficients)

2.2.1 A Hamilton-connected graph with a Hamilton-generated yet not Hamilton-based flow lattice

A priori, it cannot be dismissed out of hand that for any graph G, the condition of $\mathrm{Z}_1(G)$ being Hamilton-generated might be strong enough to always imply $\mathrm{Z}_1(G)$ being Hamilton-based (which would then answer the Cayley-graph-focused Question 8 in Chapter 1 from more general principles); Proposition 47 proves that this is not the case. All graphs considered, being Hamilton-generated is a weaker property than being Hamilton-based (thus, Question 8 probably cannot be answered without using properties specific to Cayley-graphs):

Proposition 47 (an example of $\mathcal{M}^{\mathbb{Z}\mathrm{Bas}}_{|\cdot|} \subsetneq \mathcal{M}^{\mathbb{Z}}_{|\cdot|,\langle 0\rangle}$)**.** *With* $\mathrm{X}^{\mathrm{hg}}_{\neg\mathrm{hb}}$ *as in Definition 221,*

(1) $\mathrm{X}^{\mathrm{hg}}_{\neg\mathrm{hb}}$ *is Hamilton-connected ,*

(2) $\mathrm{X}^{\mathrm{hg}}_{\neg\mathrm{hb}}$ *does not contain any other Hamilton-circuits than the following eleven:*

(H.1) $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 1$,
(H.2) $1, 2, 3, 4, 5, 6, 10, 11, 12, 13, 9, 8, 7, 1$,
(H.3) $1, 2, 3, 4, 5, 12, 11, 10, 6, 7, 8, 9, 13, 1$,
(H.4) $1, 2, 3, 4, 8, 7, 6, 5, 12, 11, 10, 9, 13, 1$,
(H.5) $1, 2, 3, 5, 4, 8, 9, 13, 12, 11, 10, 6, 7, 1$,
(H.6) $1, 2, 13, 9, 8, 4, 3, 5, 12, 11, 10, 6, 7, 1$,
(H.7) $1, 2, 13, 9, 10, 6, 11, 12, 5, 3, 4, 8, 7, 1$,
(H.8) $1, 2, 13, 12, 5, 3, 4, 8, 9, 10, 11, 6, 7, 1$,
(H.9) $1, 2, 13, 12, 11, 10, 9, 8, 4, 3, 5, 6, 7, 1$,
(H.10) $1, 7, 6, 5, 12, 11, 10, 9, 8, 4, 3, 2, 13, 1$,
(H.11) $1, 7, 8, 9, 10, 6, 11, 12, 5, 4, 3, 2, 13, 1$.

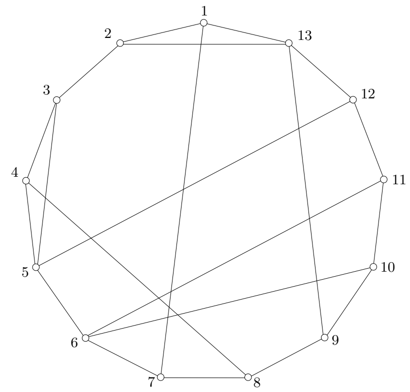


Figure 2.2: The graph $\mathrm{X}^{\mathrm{hg}}_{\neg\mathrm{hb}}$ from Definition 221, which has its flow lattice $\mathrm{Z}_1(\mathrm{X}^{\mathrm{hg}}_{\neg\mathrm{hb}})$ Hamilton-generated, but not Hamilton-based. According to Lemma 47.(2), there are exactly 11 Hamilton-circuits in $\mathrm{X}^{\mathrm{hg}}_{\neg\mathrm{hb}}$. The rank of the abelian group $\mathrm{Z}_1(\mathrm{X}^{\mathrm{hg}}_{\neg\mathrm{hb}})$ is 9. None of the $55 = \binom{11}{9}$ rank-sized subsets of Hamilton-flows is a generating set for $\mathrm{Z}_1(\mathrm{X}^{\mathrm{hg}}_{\neg\mathrm{hb}})$, hence $\mathrm{Z}_1(\mathrm{X}^{\mathrm{hg}}_{\neg\mathrm{hb}})$ is not Hamilton-based. (Incidentally, there do exist 10-element sets of Hamilton-flows that generate $\mathrm{Z}_1(\mathrm{X}^{\mathrm{hg}}_{\neg\mathrm{hb}})$, i.e. (rank+1)-sized generating sets *do* exist for this graph.) By Lemma 47.(1), the graph $\mathrm{X}^{\mathrm{hg}}_{\neg\mathrm{hb}}$ is moreover Hamilton-connected; this proves that the possibility of having the flow lattice Hamilton-generated yet not Hamilton-based persists also under the assumption of Hamilton-connectedness.

(3) *The following is the incidence matrix of all simple Hamilton-flows in* $\mathrm{Z}_1(\mathrm{X}^{\mathrm{hg}}_{\neg\mathrm{hb}})$ *which can be obtained from a Hamilton-circuit* H *of* $\mathrm{X}^{\mathrm{hg}}_{\neg\mathrm{hb}}$ *and then giving each of them the orientation from* 1 *to the smaller of* 1*'s neighbours in* H*:*

$\mathrm{E}(\mathrm{X}^{\mathrm{hg}}_{\neg\mathrm{hb}})$:	1,2	1,7	1,13	2,3	2,13	3,4	3,5	4,5	4,8	5,6	5,12	6,7	6,10	6,11	7,8	8,9	9,10	9,13	10,11	11,12	12,13
$\vec{H}_1$:	+	0	−	+	0	+	0	+	0	+	0	+	0	0	+	+	+	0	+	+	+
$\vec{H}_2$:	+	−	0	+	0	+	0	+	0	+	0	0	+	0	−	−	0	−	+	+	+
$\vec{H}_3$:	+	0	−	+	0	+	0	+	0	0	+	+	−	0	+	+	0	+	−	−	0
$\vec{H}_4$:	+	0	−	+	0	+	0	0	+	−	+	−	0	0	−	0	−	+	−	−	0
$\vec{H}_5$:	+	−	0	+	0	0	+	−	+	0	0	+	−	0	0	+	0	+	−	−	−
$\vec{H}_6$:	+	−	0	0	+	−	+	0	−	0	+	+	−	0	0	−	0	−	−	−	0
$\vec{H}_7$:	+	−	0	0	+	+	−	0	+	0	−	0	−	+	−	0	+	−	0	+	0
$\vec{H}_8$:	+	−	0	0	+	+	−	0	+	0	−	+	0	−	0	+	+	0	+	0	−
$\vec{H}_9$:	+	−	0	0	+	−	+	0	−	+	0	+	0	0	0	−	−	0	−	−	−
$\vec{H}_{10}$:	0	+	−	−	+	−	0	0	−	−	+	−	0	0	0	−	−	0	−	−	0
$\vec{H}_{11}$:	0	+	−	−	+	−	0	−	0	0	−	0	−	+	+	+	+	0	0	+	0

(2.17)

(4) *the abelian group* $\mathrm{Z}_1(\mathrm{X}^{\mathrm{hg}}_{\neg\mathrm{hb}})$ *is Hamilton-generated,*

(5) *the abelian group* $\mathrm{Z}_1(\mathrm{X}^{\mathrm{hg}}_{\neg\mathrm{hb}})$ *is not Hamilton-based.*

Proof of Proposition 47. As for statement (1), this is true since the following paths are examples of Hamilton-paths of $\mathrm{X}^{\mathrm{hg}}_{\neg\mathrm{hb}}$ in all $78 = \binom{13}{2}$ instances of Hamilton-connectedness of $\mathrm{X}^{\mathrm{hg}}_{\neg\mathrm{hb}}$:

1, 2 :	1, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2 ,
1, 3 :	1, 2, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3 ,
1, 4 :	1, 2, 3, 5, 12, 13, 9, 10, 11, 6, 7, 8, 4 ,
1, 5 :	1, 2, 3, 4, 8, 7, 6, 11, 10, 9, 13, 12, 5 ,
1, 6 :	1, 2, 13, 9, 10, 11, 12, 5, 3, 4, 8, 7, 6 ,
1, 7 :	1, 13, 2, 3, 4, 8, 9, 10, 11, 12, 5, 6, 7 ,
1, 8 :	1, 7, 6, 5, 4, 3, 2, 13, 12, 11, 10, 9, 8 ,
1, 9 :	1, 7, 8, 4, 5, 3, 2, 13, 12, 11, 6, 10, 9 ,
1, 10 :	1, 13, 2, 3, 4, 5, 12, 11, 6, 7, 8, 9, 10 ,
1, 11 :	1, 2, 13, 9, 10, 6, 7, 8, 4, 3, 5, 12, 11 ,
1, 12 :	1, 13, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 ,
1, 13 :	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13 ,
2, 3 :	2, 1, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3 ,
2, 4 :	2, 1, 13, 12, 11, 10, 9, 8, 7, 6, 5, 3, 4 ,
2, 5 :	2, 1, 7, 6, 10, 11, 12, 13, 9, 8, 4, 3, 5 ,
2, 6 :	2, 3, 4, 5, 12, 13, 1, 7, 8, 9, 10, 11, 6 ,
2, 7 :	2, 3, 4, 8, 9, 10, 11, 6, 5, 12, 13, 1, 7 ,
2, 8 :	2, 3, 4, 5, 6, 7, 1, 13, 12, 11, 10, 9, 8 ,
2, 9 :	2, 3, 4, 8, 7, 1, 13, 12, 5, 6, 11, 10, 9 ,
2, 10 :	2, 1, 7, 6, 5, 3, 4, 8, 9, 13, 12, 11, 10 ,
2, 11 :	2, 3, 4, 5, 6, 10, 9, 8, 7, 1, 13, 12, 11 ,
2, 12 :	2, 13, 1, 7, 6, 5, 3, 4, 8, 9, 10, 11, 12 ,
2, 13 :	2, 1, 7, 6, 5, 3, 4, 8, 9, 10, 11, 12, 13 ,
3, 4 :	3, 2, 1, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4 ,
3, 5 :	3, 4, 8, 9, 10, 11, 12, 13, 2, 1, 7, 6, 5 ,
3, 6 :	3, 4, 8, 7, 1, 2, 13, 9, 10, 11, 12, 5, 6 ,
3, 7 :	3, 5, 4, 8, 9, 10, 6, 11, 12, 13, 2, 1, 7 ,
3, 8 :	3, 4, 5, 6, 7, 1, 2, 13, 12, 11, 10, 9, 8 ,
3, 9 :	3, 5, 4, 8, 7, 1, 2, 13, 12, 11, 6, 10, 9 ,
3, 10 :	3, 2, 1, 7, 6, 5, 4, 8, 9, 13, 12, 11, 10 ,
3, 11 :	3, 2, 1, 7, 8, 4, 5, 12, 13, 9, 10, 6, 11 ,
3, 12 :	3, 4, 5, 6, 11, 10, 9, 8, 7, 1, 2, 13, 12 ,
3, 13 :	3, 2, 1, 7, 6, 5, 4, 8, 9, 10, 11, 12, 13 ,
4, 5 :	4, 3, 2, 1, 13, 12, 11, 10, 9, 8, 7, 6, 5 ,
4, 6 :	4, 5, 3, 2, 1, 13, 12, 11, 10, 9, 8, 7, 6 ,
4, 7 :	4, 8, 9, 10, 6, 11, 12, 5, 3, 2, 13, 1, 7 ,
4, 8 :	4, 3, 5, 6, 7, 1, 2, 13, 12, 11, 10, 9, 8 ,
4, 9 :	4, 3, 5, 6, 10, 11, 12, 13, 2, 1, 7, 8, 9 ,
4, 10 :	4, 3, 2, 1, 13, 9, 8, 7, 6, 5, 12, 11, 10 ,
4, 11 :	4, 3, 5, 12, 13, 2, 1, 7, 8, 9, 10, 6, 11 ,
4, 12 :	4, 5, 3, 2, 1, 13, 9, 8, 7, 6, 10, 11, 12 ,
4, 13 :	4, 5, 3, 2, 1, 7, 8, 9, 10, 6, 11, 12, 13 ,
5, 6 :	5, 4, 3, 2, 1, 13, 12, 11, 10, 9, 8, 7, 6 ,
5, 7 :	5, 12, 11, 6, 10, 9, 13, 1, 2, 3, 4, 8, 7 ,
5, 8 :	5, 4, 3, 2, 1, 7, 6, 10, 11, 12, 13, 9, 8 ,
5, 9 :	5, 4, 3, 2, 1, 13, 12, 11, 10, 6, 7, 8, 9 ,
5, 10 :	5, 4, 3, 2, 1, 13, 12, 11, 6, 7, 8, 9, 10 ,
5, 11 :	5, 3, 4, 8, 9, 10, 6, 7, 1, 2, 13, 12, 11 ,
5, 12 :	5, 4, 3, 2, 1, 13, 9, 8, 7, 6, 10, 11, 12 ,
5, 13 :	5, 12, 11, 10, 6, 7, 1, 2, 3, 4, 8, 9, 13 ,
6, 7 :	6, 5, 4, 3, 2, 1, 13, 12, 11, 10, 9, 8, 7 ,
6, 8 :	6, 7, 1, 13, 2, 3, 4, 5, 12, 11, 10, 9, 8 ,
6, 9 :	6, 7, 8, 4, 5, 3, 2, 1, 13, 12, 11, 10, 9 ,
6, 10 :	6, 7, 8, 9, 13, 1, 2, 3, 4, 5, 12, 11, 10 ,
6, 11 :	6, 10, 9, 8, 7, 1, 13, 2, 3, 4, 5, 12, 11 ,
6, 12 :	6, 5, 4, 3, 2, 13, 1, 7, 8, 9, 10, 11, 12 ,
6, 13 :	6, 5, 4, 3, 2, 1, 7, 8, 9, 10, 11, 12, 13 ,
7, 8 :	7, 6, 5, 4, 3, 2, 1, 13, 12, 11, 10, 9, 8 ,
7, 9 :	7, 1, 2, 13, 12, 11, 10, 6, 5, 3, 4, 8, 9 ,
7, 10 :	7, 8, 9, 13, 1, 2, 3, 4, 5, 12, 11, 6, 10 ,
7, 11 :	7, 8, 9, 10, 6, 5, 4, 3, 2, 1, 13, 12, 11 ,
7, 12 :	7, 8, 9, 10, 11, 6, 5, 4, 3, 2, 1, 13, 12 ,
7, 13 :	7, 8, 9, 10, 6, 11, 12, 5, 4, 3, 2, 1, 13 ,
8, 9 :	8, 7, 6, 5, 4, 3, 2, 1, 13, 12, 11, 10, 9 ,
8, 10 :	8, 4, 5, 3, 2, 1, 7, 6, 11, 12, 13, 9, 10 ,
8, 11 :	8, 9, 10, 6, 7, 1, 13, 2, 3, 4, 5, 12, 11 ,
8, 12 :	8, 7, 6, 11, 10, 9, 13, 1, 2, 3, 4, 5, 12 ,
8, 13 :	8, 4, 3, 2, 1, 7, 6, 5, 12, 11, 10, 9, 13 ,
9, 10 :	9, 8, 7, 6, 5, 4, 3, 2, 1, 13, 12, 11, 10 ,
9, 11 :	9, 10, 6, 7, 8, 4, 5, 3, 2, 1, 13, 12, 11 ,
9, 12 :	9, 13, 1, 2, 3, 5, 4, 8, 7, 6, 10, 11, 12 ,
9, 13 :	9, 8, 7, 6, 10, 11, 12, 5, 4, 3, 2, 1, 13 ,
10, 11 :	10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 13, 12, 11 ,
10, 12 :	10, 11, 6, 7, 8, 9, 13, 1, 2, 3, 4, 5, 12 ,
10, 13 :	10, 9, 8, 7, 6, 11, 12, 5, 4, 3, 2, 1, 13 ,
11, 12 :	11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 13, 12 ,
11, 13 :	11, 6, 10, 9, 8, 7, 1, 2, 3, 4, 5, 12, 13 ,
12, 13 :	12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 13 .

As for (2), suppose C is a Hamilton-circuit of $\mathrm{X}^{\mathrm{hg}}_{\neg\mathrm{hb}} =: G$. We will use the following conventions:

(1) We write paths as sequences of vertices; for example, $u, v, w \subseteq C$ is an abbreviation for the path $(\{u, v, w\}, \{\{u, v\}, \{v, w\}\})$ being a subgraph of C.

(2) By 'P' we always mean the subpath of C already uncovered at the point in the proof when 'P' occurs. Writing $G - P$ means the graph obtained from G by removing all vertices of P (and all their incident edges).

(3) Any occurrence of the phrase 'there is an i.r.p.', shorthand for 'there is an incompatible remaining path', means the following: at that time of the proof, the graph $G - P$ is a path graph, where P has its meaning from (2). To complete P to a Hamilton-circuit of G, we have to find a Hamilton-path of the graph $G-P$ whose endvertices are, respectively, neighbours of the two endvertices of P. If $G-P$ is itself a path, it has only one Hamilton-path, $G-P$ itself, so the condition for extendability of P to a Hamilton-circuit of G is then simply whether the endvertices of P are, respectively, neighbours of the endvertices of $G - P$. In the situation that we summarise by 'there is an i.r.p.', this condition is violated, often[3] because at least one of the endvertices of P has only inner vertices of the path $G - P$ as neighbours. In such a situation, P cannot be a subpath of a Hamilton-circuit of G, and we then give up on the relevant subcase, now known to be impossible.

(4) The phrase 'because of an e.-f.r.p. via x, y', shorthand for 'because of an extension-forcing remaining path via x, y', means the following: at that point of the proof, $G - P$ is a path graph, so the condition stated in (3) decides whether P can be extended to a Hamilton-circuit of G, but now, unlike in (3), one of the endvertices x of P is adjacent to *exactly one* endvertex y of $G - P$; the other endvertex y of P is adjacent to *at least* the *other* endvertex of $G - P$; in that situation, there obviously is exactly one extension of P to a Hamilton-circuit of G: move from v to its unique endvertex-of-$(G-P)$-neighbour, then traverse the path $G-P$, and

[3]But not always; e.g. in the case (2).(2).(2).(2) below, the reason is that both endvertices of P are adjacent to one and the same endvertex of the path $G - P$.

finally move from the other endvertex of $G-P$ to the other endvertex of P.

We know that $1 \in \mathrm{V}(C)$. Because of $\mathrm{N}_G(1) = \{2,7,13\}$ and C being a circuit, there are exactly the following cases (1), (2) and (3):

(1) $1,2 \subseteq C$, $1,13 \subseteq C$. Then $\mathrm{N}_G(2)-P = \{3\}$, hence $2,3 \subseteq C$. We then know that $13,1,2,3 \subseteq C$. Because of $\mathrm{N}_G(3)-P = \{4,5\}$, so (1) splits into exactly two subcases:

(1).(1) $13,1,2,3,5 \subseteq C$. Because of $\mathrm{N}_G(5)-P = \{4,6,12\}$, there are three further cases. However, both in the case $13,1,2,3,5,6 \subseteq C$ and in the case $13,1,2,3,5,12 \subseteq C$ we have $\deg_{G-P}(4) = 1$. Since none of the two endvertices of P is adjacent to 4, vertex 4 cannot lie in the hypothetical Hamilton-circuit C, a contradiction. Therefore, the only case we have to analyse further is $13,1,2,3,5,4 \subseteq C$. Then, because of $\mathrm{N}_G(4)-P = \{8\}$, necessarily $13,1,2,3,5,4,8 \subseteq C$. Because of $\mathrm{N}_G(8)-P = \{7,9\}$, two cases remain:

(1).(1).(1) $13,1,2,3,5,4,8,7 \subseteq C$. Then $\mathrm{N}_G(7)-P = \{6\}$, hence $13,1,2,3,5,4,8,7,6 \subseteq C$. Now there is an i.r.p.

(1).(1).(2) $13,1,2,3,5,4,8,9 \subseteq C$. Then $\mathrm{N}_G(9)-P = \{10\}$, hence $13,1,2,3,5,4,8,9,10 \subseteq C$. Now there is an i.r.p.

This completes the case (1).(1), which was found to be impossible.

(1).(2) $13,1,2,3,4 \subseteq C$. Because of $\mathrm{N}_G(4)-P = \{5,8\}$, there are only two further cases.

(1).(2).(1) $13,1,2,3,4,5 \subseteq C$. Then $\mathrm{N}_G(5)-P = \{6,12\}$, so there are two further cases.

- $13,1,2,3,4,5,6 \subseteq C$. Now because of an e.-f.r.p. via $6,7$, while 13 is adjacent to 12 while not to 7, there is exactly one extension of P to a Hamilton-circuit of G, namely 13,1,2,3,4,5,6,7,8,9,10,11,12,13 $= C$, which is (H.1) in (2).
- $13,1,2,3,4,5,12 \subseteq C$. Then $\mathrm{N}_G(12)-P = \{11\}$ and $\mathrm{N}_G(13)-P = \{9\}$, hence we then know 9,13,1,2,3,4,5,12,11 $\subseteq C$.
  Now it follows that $C = 9,13,1,2,3,4,5,12,11,10,6,7,8,9$, since there is an e.-f.r.p. via $11,10$. We have reached (H.3) in (2).

This completes the case (1).(2).(1).

(1).(2).(2) $13,1,2,3,4,8 \subseteq C$. Because of $\mathrm{N}_G(8)-P = \{7,9\}$, only two cases remain:

- $13,1,2,3,4,8,7 \subseteq C$. Then $\mathrm{N}_G(7)-P = \{6\}$, hence $13,1,2,3,4,8,7,6 \subseteq C$. Now because of an e.-f.r.p. via $6,9$ we conclude that $C =$ 13,1,2,3,4,8,7,6,5,12,11,10,9,13, which is (H.4) in (2).
- $13,1,2,3,4,8,9 \subseteq C$. Then 7 has degree 1 in $G-P$ while not being adjacent to an endvertex of P; this precludes extendability of P to a Hamilton-circuit of G and shows the present case to be impossible.

This completes the case (1).(2).(2).

This completes the case (1).(2).

This completes the proof of (1).

(2) $1,2 \subseteq C$, $1,7 \subseteq C$. Then $\mathrm{N}_G(2)-P = \{3,13\}$, so there are the following two subcases.

(2).(1) $7,1,2,3 \subseteq C$. Because of $\mathrm{N}_G(3)-P = \{4,5\}$, there are two subcases.

(2).(1).(1) $7,1,2,3,4 \subseteq C$. Because of $\mathrm{N}_G(4)-P = \{5,8\}$, there are two subcases.

(2).(1).(1).(1) $7,1,2,3,4,5 \subseteq C$. Because of $\mathrm{N}_G(5) = \{6,12\}$, there are two subcases.

(2).(1).(1).(1).(1) 7,1,2,3,4,5,6 $\subseteq C$. Then 8,7,1,2,3,4,5,6 $\subseteq C$ because of $\mathrm{N}_G(7)-P = \{8\}$. Then $\mathrm{N}_G(8)-P = \{9\}$, hence 9,8,7,1,2,3,4,5,6 $\subseteq C$. Now because of an e.-f.r.p. via $9,13$ it follows that $C =$ 6,10,11,12,13,9,8,7,1,2,3,4,5,6, which is (H.2) in (2).

(2).(1).(1).(1).(2) $7,1,2,3,4,5,12 \subseteq C$. Because otherwise 12 would become a vertex of degree 1 in $G-P$ and not adjacent to an endvertex of P, at this point we know that, necessarily, $7,1,2,3,4,5,12,13 \subseteq C$. Then $\mathrm{N}_G(13)-P = \{9\}$, hence

$7, 1, 2, 3, 4, 5, 12, 13, 9 \subseteq C$. At this point, again to avoid having a degree-1-vertex in $G - P$ not connected to an endvertex of P, we would have to have both 7 and 8 adjacent to 7 in P, which is impossible since P is path. This proves (2).(1).(1).(1).(2) to be impossible.

This completes the case (2).(1).(1).(1).

(2).(1).(1).(2) $7, 1, 2, 3, 4, 8 \subseteq C$. Then $\mathrm{N}_G(8) = \{9\}$ and $\mathrm{N}_G(7) = \{6\}$, hence $6,7,1,2,3,4,8,9 \subseteq C$. Although 6 still has three neighbours in $G - P$, we know that necessarily $5,6,7,1,2,3,4,8,9 \subseteq C$, otherwise 5 would have been left having degree 1 in $G-P$, while not being adjacent to an endvertex of P. Now $\mathrm{N}_G(5) - P = \{12\}$, hence $12, 5, 6, 7, 1, 2, 3, 4, 8, 9 \subseteq C$. Then $\mathrm{N}_G(12) - P = \{11, 13\}$; if $12, 13 \subseteq C$, then $\mathrm{N}_G(13) - P = \{9\}$ and it follows that the circuit $9, 13, 12, 5, 6, 7, 1, 2, 3, 4, 8, 9$ is a subgraph of C, a contradiction; therefore we may assume that $12, 11 \subseteq C$. Then $\mathrm{N}_G(11) - P = \{10\}$, hence $10, 11, 12, 5, 6, 7, 1, 2, 3, 4, 8, 9 \subseteq C$. Then $\mathrm{N}_G - P = \{13\}$, hence $10, 11, 12, 5, 6, 7, 1, 2, 3, 4, 8, 9, 13 \subseteq C$, a Hamilton-path of G; since 10 and 13 are not adjacent in G, this is a contradiction (every Hamilton-path of a circuit-graph has its ends adjacent). We have thus shown (2).(1).(1).(2) to be impossible.

This completes the case (2).(1).(1).

(2).(1).(2) $7, 1, 2, 3, 5 \subseteq C$. Since otherwise 4 would become a degree-1-vertex in $G - P$ not adjacent to an endvertex of P, we then know that $7, 1, 2, 3, 5, 4 \subseteq C$. Then $\mathrm{N}_G(4) - P = \{8\}$, hence $7, 1, 2, 3, 5, 4, 8 \subseteq C$. Then $\mathrm{N}_G(7) - P = \{6\}$ and $\mathrm{N}_G(8) - P = \{9\}$, hence $6, 7, 1, 2, 3, 5, 4, 8, 9 \subseteq C$. Then because of an e.-f.r.p. via 6, 10 (let us note that 9 is adjacent to *both* endvertices of the path $G - P$, so it would be false to claim an e.-f.r.p. via 9, 13 here, even though $9, 13 \subseteq C$ is implied by the e.-f.r.p. via 6, 10) it follows that $C = 13, 12, 11, 10, 6, 7, 1, 2, 3, 5, 4, 8, 9, 13$, which is (H.5) in (2).

This completes the case (2).(1).

(2).(2) $7, 1, 2, 13 \subseteq C$. Because of $\mathrm{N}_G(13) - P = \{9, 12\}$, there are two subcases.

(2).(2).(1) $7, 1, 2, 13, 9 \subseteq C$. Because of $\mathrm{N}_G(9) - P = \{8, 10\}$, there are two subcases.

(2).(2).(1).(1) $7, 1, 2, 13, 9, 8 \subseteq C$. Then $\mathrm{N}_G(8) - P = \{4\}$ and $\mathrm{N}_G(7) - P = \{6\}$, hence $6, 7, 1, 2, 13, 9, 8, 4 \subseteq C$. Then because of an e.-f.r.p. via 6, 10 it follows that $C = 4, 3, 5, 12, 11, 10, 6, 7, 1, 2, 13, 9, 8, 4$, which is (H.6) in (2).

(2).(2).(1).(2) $7, 1, 2, 13, 9, 10 \subseteq C$. Then $\mathrm{N}_G(10) - P = \{6, 11\}$, so there are two subcases.

(2).(2).(1).(2).(1) $7, 1, 2, 13, 9, 10, 6 \subseteq C$. Then $\mathrm{N}_G(7) - P = \{8\}$, so $8,7,1,2,13,9,10,6 \subseteq C$, and then $\mathrm{N}_G(8) = \{4\}$, hence $4,8,7,1,2,13,9,10,6 \subseteq C$. Now, because of an e.-f.r.p. via 6, 11, it follows that $C = 6,11,12,5,3,4,8,7,1,2,13,9,10,6$, which is (H.7) in (2).

(2).(2).(1).(2).(2) $7, 1, 2, 13, 9, 10, 11 \subseteq C$. We then can assume $11, 12 \subseteq C$, for otherwise the vertex 12 would acquire degree 1 in $G - P$ while not being adjacent to an endvertex of P, which is a contradiction. For the same reason, we can also assume $7, 8 \subseteq C$. At this point of the proof we know $8,7,1,2,1,3,9,10,11,12 \subseteq C$. Then $\mathrm{N}_G(12) - P = \{5\}$ and $\mathrm{N}_G(8) = \{4\}$, $4,8,7,1,2,13,9,10,11,12,5 \subseteq C$. Then $\mathrm{N}_G(4) - P = \{3\}$ and $\mathrm{N}_G(5) - P = \{6\}$, hence $3,4,8,7,1,2,13,9,10,11,12,5,6 \subseteq C$, which is a Hamilton-path of G whose endvertices are non-adjacent, a contradiction to the hypothesis of C being Hamilton-circuit. This shows (2).(2).(1).(2).(2) to be impossible.

This completes the case (2).(2).(1).(2).

This completes the case (2).(2).(1).

(2).(2).(2) $7, 1, 2, 13, 12 \subseteq C$. Then because of $\mathrm{N}_G(7) - P = \{6, 8\}$, there are two subcases.

(2).(2).(2).(1) $6, 7, 1, 2, 13, 12 \subseteq C$. Then because of $\mathrm{N}_G(6) - P = \{5, 10, 11\}$, there are three subcases, two of which are quickly dealt with: if $6, 10 \subseteq C$, i.e. $10, 6, 7, 1, 2, 13, 12 \subseteq C$, then 11 is an isolated vertex in $G - P$, which precludes extendability of P to a Hamilton-circuit of G, making this case impossible; if $6, 5 \subseteq C$, i.e. $5, 6, 7, 1, 2, 13, 12 \subseteq C$, then because of an e.-f.r.p. via $5, 3$ we know that $C = 12, 11, 10, 9, 8, 4, 3, 5, 6, 7, 1, 2, 13, 12$, which is (H.9) in (2). As to the third case, if $6, 11 \subseteq C$, i.e. $11, 6, 7, 1, 2, 13, 12 \subseteq C$, then $\mathrm{N}_G(12) - P = \{5\}$, so $11, 6, 7, 1, 2, 13, 12, 5 \subseteq C$, and now because of an e.-f.r.p. via $5, 3$ it follows that $C = 11, 6, 7, 1, 2, 13, 12, 5, 3, 4, 8, 9, 10, 11 \subseteq C$, which is (H.8) in (2).

(2).(2).(2).(2) $8, 7, 1, 2, 13, 12 \subseteq C$. Then because of $\mathrm{N}_G(8) - P = \{4, 9\}$, there are two subcases, both of which are quickly dealt with: if $8, 9 \subseteq C$, i.e. $9, 8, 7, 1, 2, 13, 12 \subseteq C$, then $\mathrm{N}_G(9) - P = \{10\}$, hence $10, 9, 8, 7, 1, 2, 13, 12 \subseteq C$; then $X - P$ has only two Hamilton-paths anymore, one with endvertices $\{11, 3\}$, the other with endvertices $\{11, 4\}$, and since none of the two endvertices 10 and 12 of P are adjacent to either 3 or 4, an extension of P to a Hamilton-path of G is not possible, proving this case to be impossible. If on the contrary $8, 4 \subseteq C$, i.e. $4, 8, 7, 1, 2, 13, 12 \subseteq C$, then necessarily $3, 4, 8, 7, 1, 2, 13, 12 \subseteq C$ (for choosing the only other non-P-neighbour of 4 would leave 3 isolated in $X - P$). Then $\mathrm{N}_G(3) - P = \{5\}$, hence $5, 3, 4, 8, 7, 1, 2, 13, 12 \subseteq C$, and then $\mathrm{N}_G(5) - P = \{6\}$, hence $6, 5, 3, 4, 8, 7, 1, 2, 13, 12 \subseteq C$. Then there is an i.r.p.

This completes the case (2).(2).(2).

This completes the case (2).(2).

This completes the case (2).

(3) $1, 7 \subseteq C$, $1, 13 \subseteq C$. Then $\mathrm{N}_G(13) - P = \{2, 9, 12\}$, but, of these vertices, at most 2 can keep P extendible to a Hamilton-circuit of G. I.e., we may immediately assume $13, 2 \subseteq C$, the reason being that if $13, 9 \subseteq C$ or $13, 12 \subseteq C$, either way the vertex 2 would have degree 1 in $G - P$ while *not* being adjacent to an endvertex of P, which makes it impossible to ever include 2 into a Hamilton-circuit of G. So we now know that $7, 1, 13, 2 \subseteq C$. Then $\mathrm{N}_G(2) - P = \{3\}$, hence $7, 1, 13, 2, 3 \subseteq C$. Now $\mathrm{N}_G(7) - P = \{6, 8\}$, so there are two subcases.

(3).(1) $3, 2, 13, 1, 7, 6 \subseteq C$. Then $\mathrm{N}_G(6) - P = \{5, 10, 11\}$, so there are three subcases.

(3).(1).(1) $3, 2, 13, 1, 7, 6, 5 \subseteq C$. Then we have already reached the situation that there is an e.-f.r.p., this time via $5, 12$, so it follows that $C = 13, 1, 7, 6, 5, 12, 11, 10, 9, 8, 4, 3, 2, 1, 13$, which is (H.10) in (2).

(3).(1).(2) $3, 2, 13, 1, 7, 6, 10 \subseteq C$. Then there is an i.r.p.

(3).(1).(3) $3, 2, 13, 1, 7, 6, 11 \subseteq C$. Then there is an i.r.p.

This completes the case (3).(1).

(3).(2) $3, 2, 13, 1, 7, 8 \subseteq C$. Because of $\mathrm{N}_G(8) - P = \{4, 9\}$, there are two subcases.

(3).(2).(1) $3, 2, 13, 1, 7, 8, 4 \subseteq C$. Then $\mathrm{N}_G(4) - P = \{5\}$, hence $3, 2, 13, 1, 7, 8, 4, 5 \subseteq C$, implying that $\mathrm{N}_G(3) - P = \varnothing$, which, since 3 is an endvertex of P, makes it impossible to extend P to a Hamilton-circuit of G.

(3).(2).(2) $3, 2, 13, 1, 7, 8, 9 \subseteq C$. Then $\mathrm{N}_G(9) - P = \{10\}$, hence $3, 2, 13, 1, 7, 8, 9, 10 \subseteq C$. Then $\mathrm{N}_G(10) - P = \{6, 11\}$, hence there are two subcases.

(3).(2).(2).(1) $3, 2, 13, 1, 7, 8, 9, 10, 6 \subseteq C$. Then there is an e.-f.r.p. via $6, 11$, hence $C = 3, 2, 13, 1, 7, 8, 9, 10, 6, 11, 12, 5, 4, 3$, which is (H.11) in (2).

(3).(2).(2).(2) $3, 2, 13, 1, 7, 8, 9, 10, 11 \subseteq C$. Then $\mathrm{N}_G(11) - P = \{6, 12\}$, so there are two subcases; however, both in the case $3, 2, 13, 1, 7, 8, 9, 10, 11, 6 \subseteq C$ and in the case $3, 2, 13, 1, 7, 8, 9, 10, 11, 12 \subseteq C$ there already is an i.r.p.

This completes the case (3).(2).(2).

This completes the case (3).(2).

This completes the case (3).

Since the cases were exhaustive and only the Hamilton-circuits (H.1)–(H.11) have been found, this completes the proof of (2). Statement (3) is immediate from (2).

As for (4), it suffices to note that the Smith Normal Form of e.g. the (10×21)-submatrix obtained from (2.17) by leaving out the first row has elementary divisors $(1^{\times 9})$, i.e., the 10-element set $\mathcal{G} := \{\vec{H}_2, \ldots, \vec{H}_{11}\}$, with the $\vec{H}_i$ as defined in (2.17), is a generating set of the rank-9 abelian group $\mathrm{Z}_1(\mathrm{X}^{\mathrm{hg}}_{\neg\mathrm{hb}})$.

As for (5), we first note that according to (2), there are (up to the choice of the orientations, which does not matter) no other Hamilton-flows in $\mathrm{X}^{\mathrm{hg}}_{\neg\mathrm{hb}}$ than the flows $\vec{H}_1, \ldots, \vec{H}_{11}$ defined by the matrix in (2.17). Computing the Smith Normal Form of each of the $55 = \binom{11}{9}$ distinct (9×21)-incidence matrices of Hamilton-flows obtained by choosing a 9-element subset of the rows of the matrix in (2.17) shows that of those 55 submatrices, 52 have full rank 9, but of these 52 matrices none has all elementary divisors equal to 1, the lexicographically smallest vector of elementary divisors that occurs being $(1^{\times 8}, 2^{\times 1})$. Therefore, the lexicographically smallest vector of invariant factors of a full sublattice of $\mathrm{Z}_1(\mathrm{X}^{\mathrm{hg}}_{\neg\mathrm{hb}})$ spanned by a rank-sized set of Hamilton-flows is $(1^{\times 8}, 2^{\times 1})$, i.e., the smallest index of a subgroup of $\mathrm{Z}_1(\mathrm{X}^{\mathrm{hg}}_{\neg\mathrm{hb}})$ generated by a rank-sized set of Hamilton-flows is 2, which proves (5).

The claims in this proof can easily be checked by inputting the matrix from (2.17) into any computer algebra system offering Smith Normal Forms. □

Let us note that the generating set $\mathcal{G}$ used in the proof of (4) in Proposition 47 does not contain any basis of $\mathrm{Z}_1(\mathrm{X}^{\mathrm{hg}}_{\neg\mathrm{hb}})$, and it *cannot*, for the strong reason that by (5) in Proposition 47 there does not exist *any* such basis; leaving out exactly one of the ten elements of $\mathcal{G}$ leaves sets which generate rank-9-subgroups with the following indices in the group $\mathrm{Z}_1(\mathrm{X}^{\mathrm{hg}}_{\neg\mathrm{hb}})$: leaving out $\vec{H}_2$ yields the index 32, leaving out $\vec{H}_3$ yields the index 9, leaving out $\vec{H}_4$ yields the index 55, leaving out $\vec{H}_5$ yields the index 60, leaving out $\vec{H}_6$ yields the index 31, leaving out $\vec{H}_7$ yields the index 13, leaving out $\vec{H}_8$ yields the index 6, leaving out $\vec{H}_9$ yields the index 22, leaving out $\vec{H}_{10}$ yields the index 65, leaving out $\vec{H}_{11}$ yields the index 19. The set $\mathcal{G}$ used in the proof of Proposition 47.(4) is not the only 10-element generating set of $\mathrm{Z}_1(\mathrm{X}^{\mathrm{hg}}_{\neg\mathrm{hb}})$ consisting of Hamilton-flows: of the (by Proposition 47.(2) and arbitrarily choosing orientations) $11 = \binom{11}{10}$ distinct 10-element sets of Hamilton-flows in $\mathrm{X}^{\mathrm{hg}}_{\neg\mathrm{hb}}$, precisely 6 generate $\mathrm{Z}_1(\mathrm{X}^{\mathrm{hg}}_{\neg\mathrm{hb}})$. This can easily be computed from the matrix in (2.17).

2.2.2 Plan of the proof of Theorem 4 from Chapter 1

Our proof of Theorem 4 from Chapter 1 is conceptually analogous to the one for Theorem 6, outlined in ($\mathbb{F}_2$-St.1)–($\mathbb{F}_2$-St.3), naturally breaking into three steps. To carry out ($\mathbb{Z}$-St.1) is a much harder task than ($\mathbb{F}_2$-St.1) (one measure for that being that Section 2.1.2 extends from p. 28 to p. 39, while Section 2.2.3 extends from p. 52 to p. 105), in particular since arbitrary choices have to be made which can easily result in unmanageably complex linear algebra (with regard to writing an argument valid for infinitely-many orders n):

- ($\mathbb{Z}$-St.1) Analogous to ($\mathbb{F}_2$-St.1), the difference being that now the auxiliary spanning substructures have to have the properties claimed in Conjecture 3,
- ($\mathbb{Z}$-St.2) Same as ($\mathbb{F}_2$-St.2) in Section 2.1.1.
- ($\mathbb{Z}$-St.3) By using Lemma 41 with $R := \mathbb{Z}$, argue that the properties proved in ($\mathbb{Z}$-St.1) transfer from the subgraph H to the host graph G, completing the proof of Theorem 4.

One can summarise the differences between the proof of Theorem 6 (published in [82]) and the proof of Theorem 4 (not published before this thesis) by saying that the former proves surjectivity via injectivity, while the latter proves injectivity via surjectivity. The proof of Theorem 4 that we give in the present chapter will (and must) prove surjectivity directly, a much more exacting task, yet a necessary one: while in Section 2.1.1 it was sufficient to prove injectivity of the relevant coefficient map and have surjectivity follow by linear-algebraic reasons alone, in the non-vector-space setting of Section 2.2, it would no longer be sufficient to merely give a proof of injectivity of the coefficient map $\mathbb{Z}^{\frac{1}{2}(n+5)} \ni (\lambda_1, \ldots, \lambda_{\frac{1}{2}(n+5)}) \mapsto \sum_{\vec{C} \in \mathcal{B}_n} \lambda_i \cdot \vec{C}$; we have to go for its surjectivity outright (we then at least have injectivity follow, i.e. the rank-sized set $\mathcal{B}_n$ from Proposition 69 will be known to be a basis as soon as it is known to be generating).

There being no reason to expect the change-of-basis-matrices to be highly structured, the author would be surprised if the complexity of the proof could be much reduced. The overall plan of the proof is conceptually rather simple, yet it is challenging not to bungle when carrying it out, e.g. by selecting auxiliary structures that lead to unmanageably complex linear algebra which then make writing a general argument impossible.

2.2.3 Details on step ($\mathbb{Z}$-St.1)

2.2.3.1 Justification for using other seed graphs than for $\mathbb{F}_2$-coefficients

In Section 2.1 it was sufficient to use the seed graphs $\mathrm{M}_r^{\boxtimes}$, $\mathrm{M}_r^{\boxtimes}$, $\mathrm{Pr}_r^{\boxtimes}$ and $\mathrm{Pr}_r^{\boxminus}$. For integer coefficients, these graphs are not enough: they are *both* denser than necessary (hence undesirable if we resolve to construct auxiliary graphs which are applicable also under hypotheses different from Dirac-type minimum-degree-conditions) *and* for some r they simply do not have their flow lattices generated by Hamilton-flows. One example is explained in Figure 2.3 and Conjecture 73; the incidence matrix of, to all appearances, *all* (we do not prove the exhaustiveness in this thesis) Hamilton-flows that exist in $\vec{\mathrm{Pr}}_6^{\boxtimes}$ from Figure 2.3 is in (2.18):

E($\vec{\mathrm{Pr}}_6^{\boxtimes}$):	zx_0	zy_0	zx_1	zy_1	x_0y_0	x_1y_1	x_2y_2	x_3y_3	x_4y_4	x_5y_5	x_0x_1	x_1x_2	x_2x_3	x_3x_4	x_4x_5	x_5x_0	y_0y_1	y_1y_2	y_2y_3	y_3y_4	y_4y_5	y_5y_0
$\vec{C}_1$:	0	−	0	+	+	−	+	−	+	−	0	+	0	+	0	+	0	0	+	0	+	0
$\vec{C}_2$:	−	0	+	0	−	+	−	+	−	+	0	0	+	0	+	0	0	+	0	+	0	+
$\vec{C}_3$:	0	0	−	+	−	0	−	+	−	+	0	−	0	−	0	−	−	0	−	0	−	0
$\vec{C}_4$:	0	0	−	+	−	0	−	+	−	+	+	0	+	0	+	0	0	+	0	+	0	+
$\vec{C}_5$:	−	+	0	0	0	−	+	−	+	−	0	+	0	+	0	+	+	0	+	0	+	0
$\vec{C}_6$:	−	+	0	0	0	−	+	−	+	−	−	0	−	0	−	0	0	−	0	−	0	−
$\vec{C}_7$:	−	0	+	0	0	+	−	0	0	0	0	0	+	+	+	+	−	0	−	−	−	−
$\vec{C}_8$:	−	0	+	0	0	0	+	−	0	0	0	+	0	+	+	+	−	−	0	−	−	−
$\vec{C}_9$:	−	0	+	0	0	0	0	+	−	0	0	+	+	0	+	+	−	−	−	0	−	−
$\vec{C}_{10}$:	−	0	+	0	0	0	0	0	+	−	0	+	+	+	0	+	−	−	−	−	0	−
$\vec{C}_{11}$:	0	−	0	+	0	−	+	0	0	0	−	0	−	−	−	−	0	0	+	+	+	+
$\vec{C}_{12}$:	0	−	0	+	0	0	−	+	0	0	−	−	0	−	−	−	0	+	0	+	+	+
$\vec{C}_{13}$:	0	−	0	+	0	0	0	−	+	0	−	−	−	0	−	−	0	+	+	0	+	+
$\vec{C}_{14}$:	0	−	0	+	0	0	0	0	−	+	−	−	−	−	0	−	0	+	+	+	0	+
$\vec{C}_{15}$:	0	0	−	+	0	0	−	0	0	0	+	0	+	+	+	+	−	0	−	−	−	−
$\vec{C}_{16}$:	−	+	0	0	0	−	0	0	0	0	0	+	+	+	+	+	0	−	−	−	−	−
$\vec{C}_{17}$:	0	0	−	+	−	0	0	0	0	0	0	−	−	−	−	−	0	+	+	+	+	+
$\vec{C}_{18}$:	−	0	+	0	−	0	0	0	0	+	0	+	+	+	+	0	−	−	−	−	−	0
$\vec{C}_{19}$:	−	+	0	0	0	0	0	0	0	−	−	−	−	−	−	0	+	+	+	+	+	0
$\vec{C}_{20}$:	0	−	0	+	+	0	0	0	0	−	−	−	−	−	−	0	0	+	+	+	+	0

(2.18)

According to e.g. the software package Sage (using the command `.elementary_divisors()`), the elementary divisors of the matrix in (2.18) are $(1^{\times 9}, 11^{\times 1})$, hence

$$\mathrm{Z}_1\ (\ \mathrm{Pr}_6^{\boxtimes}\ )\ /\ \langle \vec{\mathcal{H}}(\mathrm{Pr}_6^{\boxtimes}) \rangle_{\mathbb{Z}} \cong \mathbb{Z}/11\ . \tag{2.19}$$

2.2.3.2 A sparse seed graph for $n \equiv 3 \pmod 4$, proved for $n \equiv 3 \pmod 8$

In Section 2.2.3.2 we will prove the graph $\mathrm{C}_n^{2^-}$ from Definition 214 on p. 199 to be a suitable seed graph, for every $n \geqslant 11$ with $n \equiv 3 \pmod 8$. It is very likely also a suitable seed graph for every $n \equiv 3 \pmod 4$, but will give a complete proof for $n \equiv 3 \pmod 8$ only. Most of the difficulty comes with defining and certifying an explicit Hamilton-flow-basis for every such n; this is done in Section 2.2.3.3. We sometimes use the symbol $=_n$ to denote equality of integers modulo n.

Let us recall that, formally, we treat the chain-group $\mathrm{C}_1(G)$ as is traditionally[4] done in homology theory of abstract simplicial complexes, i.e., as a subgroup of the second exterior power of the free abelian group generated by the vertices of G. In particular, we use the notation $v_1 \wedge v_2$ to indicate

[4]At least since [78]. See [151, p. 875] for an appraisal of the modernity of Hausdorff's treatment of the topic.

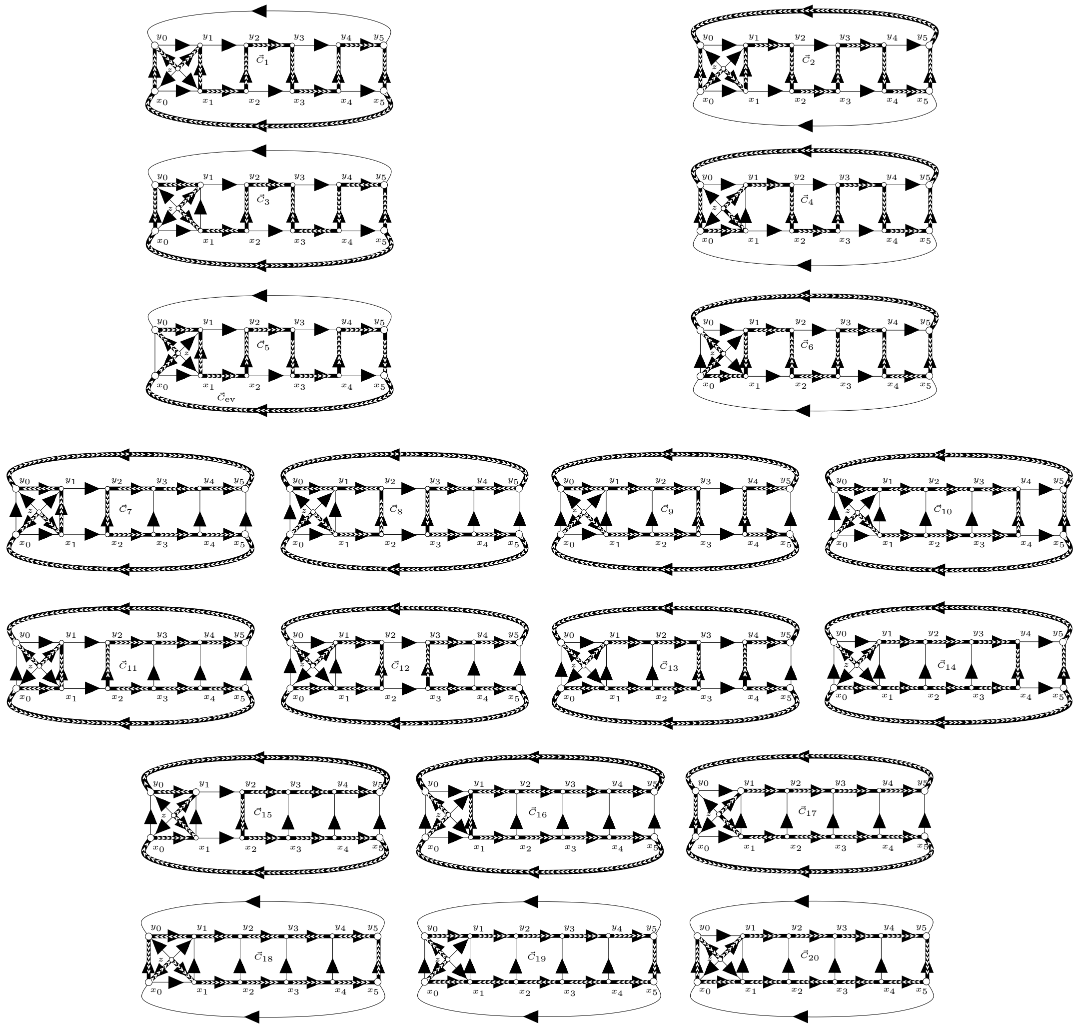


Figure 2.3: One of the reasons why in Section 2.2 we resort to other seed graphs than prisms and Möbius-ladders is the following: the prism-derived graph $\mathrm{Pr}_r^{\boxtimes}$ from Definition 207, while a suitable seed graph for proving Theorem 6.(I1) from Chapter 6, is not only a few edges denser than necessary, but, worse, simply does not work as a seed graph for the $\mathbb{Z}$-coefficient-version of Theorem 6. Its Hamilton-flows generate a sublattice with index larger than 1. The present figure gives an example in the case $r = 6$: the Hamilton-supported simple flows $\vec{C}_1, \ldots, \vec{C}_{20}$ shown in Figure 2.3 are all such flows that Pr_6 has (up to the irrelevant choice of orientation); yet these twenty flows do not generate the flow lattice: $\langle \vec{C}_1, \ldots, \vec{C}_{20} \rangle_{\mathbb{Z}} \subseteq \mathrm{Z}_1(\mathrm{Pr}_6)$ is a full sublattice of index 11. This is fine after reducing modulo 2, but for $\mathbb{Z}$-coefficients it is not enough. The incidence-matrix of these Hamilton-flows is in (2.18). The white arrows inside the flows indicate the arbitrary orientation selected; the arrow-heads in Figure 2.3 are not giving information about the Hamilton-flows, but give the arbitrary edge-orientation that is used for determining the signs in the matrix in (2.18). The rationale behind choosing arbitrary orientations of the circuits was to let it be determined by the lexicographically smallest z-containing edge, w.r.t. the ordering induced by the ordering $x_0 < x_1 < y_0 < y_1 < z$; e.g. $\vec{C}_1$ is oriented as it is because of $y_0 < z$, $y_1 < z$, and $y_0 z \ < y_1 z$. There is no connection of this convention to the arbitrary orientations of the *edges* selected; for them, we take $x_i < x_{i+1}$, $y_i < y_{i+1}$, except $x_5 < x_0$ and y_5, and $z < x_i$ and $z < y_i$; there is no necessity behind these conventions, but, like choosing a coordinate-system, agreeing on *some* such convention *is* necessary to compute an incidence-matrix and the Smith Normal Form. Let us emphasise that in the situation of Figure 2.3 we are using 20 Hamilton-flow-generators, i.e. twice as many as $\mathrm{rank}_{\mathbb{Z}}(\mathrm{Z}_1(\mathrm{Pr}_6)) = 10$, and yet we cannot generate every flow in $\mathrm{Z}_1(\mathrm{Pr}_6)$. In particular, Pr_6 provides a counterexample to the perhaps plausible conjecture 'every odd-order, Hamilton-connected graph G with minimum degree *three* and at least *twice as many* Hamilton-circuits as the rank of its flow lattice has its flow lattice generated by Hamilton-flows'. The author does not know how much 'twice as many' can be increased here to still leave a false statement. It seems plausible that no constant multiple is sufficient in general, cf. Conjecture 22.

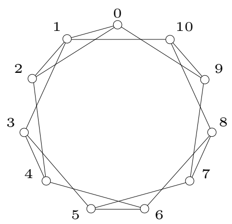


Figure 2.4: The graph C_n^{2--} from Definition 215 in the case $n = 11$. The $\mathbb{F}_2$-cycle space of C_n^{2--} *is* generated by its Hamilton-circuits. However, C_n^{2--} has not all of its non-adjacent pairs of vertices connected by a Hamilton-path (1 and 4 are not; cf. Proposition 71). This makes C_n^{2--} unsuitable as a seed graph for the monotonicity arguments from Corollary 44, for whatever coefficients. By the way, while C_n^{2--} has $\mathrm{Z}_1(\mathrm{C}_n^{2--};\mathbb{F}_2)$ generated by Hamilton-circuits, the flow lattice $\mathrm{Z}_1(\mathrm{C}_n^{2--})$ is not Hamilton-generated, let alone Hamilton-based: the $\mathbb{Z}$-linear span of all Hamilton-flows of C_{11}^{2--} is a full sublattice of index 5 in $\mathrm{Z}_1(\mathrm{C}_{11}^{2--})$. This is justification for why we employ the graphs C_n^{2-} as our seed graphs, which have one edge more than C_n^{2--}, but are Hamilton-connected and Hamilton-based in return.

(the 1-chain defined by) an oriented edge. Most of the time, this formalism remains out of sight, but sometimes it looms large, as in Definition 48 and in the explicit calculations to come:

Definition 48 (the Hamilton-flows $\vec{C}_{0,1,2;n}$, $\vec{C}_{0,1,n-1;n}$, $\vec{C}_{i;n}$)**.** *If* $n \geqslant 11$ *and* $n \equiv 3 \pmod 4$*, and*

$$\Sigma_n := \sum_{j\in\{2+4k\colon\, k\in\{0,1,\dots,\frac{1}{4}(n-7)\}\}} j \wedge j+2 \quad + \quad j+2 \wedge j+1 \quad + \quad j+1 \wedge j+3 \quad + \quad j+3 \wedge j+4$$
$$\in \mathrm{C}_1(\mathrm{C}_n^{2-}) \ , \tag{2.20}$$

we define $\vec{C}_{0,1,2;n}$, $\vec{C}_{0,1,n-1;n}$, $\vec{C}_{i;n} \in \mathrm{C}_1(\mathrm{C}_n^{2-})$ *(avoiding a claim, we do not write* $\mathrm{Z}_1(\mathrm{C}_n^{2-})$ *here), by*

(1) $\vec{C}_{0,1,2;n} \quad := \quad 0 \wedge 1 \quad + \quad 1 \wedge 2 \quad + \quad \Sigma_n \quad + \quad n-1 \wedge 0 \ ,$
(2) $\vec{C}_{0,1,n-1;n} \quad := \quad 0 \wedge 1 \quad + \quad 1 \wedge n-1 \quad - \quad \Sigma_n \quad + \quad 2 \wedge 0 \ ,$
(3) *for every* $i \in \{0,1,\dots,\frac{1}{2}(n-3)\}$,

$$\begin{aligned}\vec{C}_{i;n} := {} & 2i+2 \quad \wedge \quad 2i+1 \quad + \quad 2i \quad \wedge \quad 2i-1 \\ & + \sum_{j\in\{0,1,\dots,\frac{1}{2}(n-3)\}} 2(i+j)+1 \quad \wedge \quad 2(i+j)+3 \\ & + \sum_{j\in\{1,2,\dots,\frac{1}{2}(n-3)\}} 2(i+j)+2 \quad \wedge \quad 2(i+j) \ ,\end{aligned}$$

(4) $\vec{C}_{\frac{1}{2}(n-1);n} := 0 \wedge 1 \ + \ 17 \wedge 18$
$\qquad + \sum_{j\in\{0,1,\dots,\frac{1}{2}(n-5)\}}(2j+1 \quad \wedge \quad 2j+3) + \sum_{j\in\{0,1,\dots,\frac{1}{2}(n-3)\}}(2j+2 \quad \wedge \quad 2j) \ .$

In Definition 48, all numbers are to be taken modulo n, e.g. for $n = 15$ and $(i,j) = (0,6)$ we have $2(i+j)+1 \quad \wedge \quad 2(i+j)+3 = 13 \wedge 0$ and for $(i,j) = (1,6)$ we have $2(i+j)+1 \quad \wedge \quad 2(i+j)+3 = 0 \wedge 2$. (These are examples that, after reducing modulo an odd number n, an odd-seeming expression like $2(i+j)+1$ might represents an non-odd element of $\mathbb{Z}/n$.)

For $n = 11$ (resp. for $n = 19$), the incidence-matrices of the flows in Definition 48 is shown in (2.87) (resp. for (2.92)).

For $i = \frac{1}{2}(n-3)$, the flow in Definition 48.(3) is $\vec{C}_{\frac{1}{2}(n-3);n} = n-1 \wedge n-2 + n-3 \wedge n-4 + \sum_{j\in\{0,1,\dots,\frac{1}{2}(n-3)\}} 2j-2 \wedge 2j + \sum_{j\in\{0,1,\dots,\frac{1}{2}(n-3)\}} 2j-1 \wedge 2j-3$. For $i = \frac{1}{2}(n-1)$, the flow in Definition 48.(3) is $\vec{C}_{\frac{1}{2}(n-1);n} = 1 \wedge 0 + n-1 \wedge n-2 + \sum_{j\in\{0,1,\dots,\frac{1}{2}(n-3)\}} 2j \wedge 2j+2 + \sum_{j\in\{0,1,\dots,\frac{1}{2}(n-3)\}} 2j+1 \wedge 2j-1$.

Definition 49 (f_T)**.** *If T is a spanning tree of a graph $G=(V,E)$, then for every $(u,v)\in V^2$ with $uv \in \mathrm{E}(G)\backslash\mathrm{E}(T)$ we define $\mathrm{f}_T(u,v)\in \mathrm{Z}_1(G)$ as the simple flow having its support equal to the unique circuit in $T+uv$ and its orientation defined by orienting uv from u to v.*

In particular, we have $\mathrm{f}_T(u,v)+\mathrm{f}_T(v,u)=0$ for any (u,v).

Lemma 50. *The $\vec{C}_{0,1,2;n}$, $\vec{C}_{0,1,n-1;n}$, $\vec{C}_{i;n}$ from Definition 48 are Hamilton-flows, with supports the Hamilton-circuits $C_{0,1,2;n}$, $C_{0,1,n-1;n}$, $C_{i;n}$ from Definition 219 in Chapter 5, respectively.*

Proof. As to the claim about the supports, by comparing (5.1) (resp. (1)) in Definition 219 with (2.20) (resp. (1)) in Definition 48 it is easily checked that $\mathrm{Supp}(\vec{C}_{0,1,2;n}) = \mathrm{E}(C_{0,1,2;n})$, and comparing (5.1) (resp. (2)) in Definition 219 with (2.20) (resp. (2)) in Definition 48 yields $\mathrm{Supp}(\vec{C}_{0,1,n-1;n}) = \mathrm{E}(C_{0,1,n-1;n})$. On account of the different descriptions of the 2-sets underlying Definition 219.(3) and Definition 48.(3), work is needed to prove $\mathrm{Supp}(\vec{C}_{i;n}) = \mathrm{E}(C_{i;n})$. By Definition 48.(3),

$$\begin{aligned}\mathrm{Supp}(\vec{C}_{i;n}) := \{\{2i+1,2i+2\},\{2i-1,2i\}\} \\ \cup \bigcup_{j\in\{0,1,\dots,\frac{1}{2}(n-3)\}} \big\{\{2(i+j)+1,2(i+j)+3\}\big\} \\ \cup \bigcup_{j\in\{1,2,\dots,\frac{1}{2}(n-3)\}} \big\{\{2(i+j),2(i+j)+2\}\big\}\ .\end{aligned} \tag{2.21}$$

From Definition 219.(3) we know that $\{2i+1,2i+2\}\in\mathrm{E}(C_{i;n})$ and $\{2i-1,2i\}\in\mathrm{E}(C_{i;n})$, and it remains to prove that for any $i\in\{0,1,\dots,\frac{1}{2}(n-1)\}$,

$$\begin{aligned}\bigsqcup_{k\in\mathbb{Z}/n\backslash\{2i,2i+1\}} \{\{k-1,k+1\}\} =_n \bigcup_{j\in\{0,1,\dots,\frac{1}{2}(n-3)\}} \big\{\{2(i+j)+1,2(i+j)+3\}\big\} \\ \cup \bigcup_{j\in\{1,2,\dots,\frac{1}{2}(n-3)\}} \big\{\{2(i+j),2(i+j)+2\}\big\}\ .\end{aligned} \tag{2.22}$$

Let $i\in\{0,1,\dots,\frac{1}{2}(n-1)\}$ and $k\in\mathbb{Z}/n\backslash\{2i,2i+1\}$ be given. We would like to show that $\{k-1,k+1\}$ is in the right-hand side of (2.22). For this we have to distinguish cases according to parity and size of k.

Case 1. Even k.

Case 1.1. $\frac{1}{2}k-1-i\geqslant 0$. Then with $j_k := \frac{1}{2}k-1-i$ we have $j_k\geqslant 0$. Moreover, $j_k\leqslant\frac{1}{2}(n-1)-1-i \leqslant \frac{1}{2}(n-1)-1 = \frac{1}{2}(n-3)$, so j_k occurs as an index of the first union on the right of (2.22), and indeed $\{2(i+j_k)+1, 2(i+j_k)+3\} = \{k-1,k+1\}$.

Case 1.2. $\frac{1}{2}k-1-i<0$. We now show that with $j_k := \frac{1}{2}k+\frac{1}{2}(n-1)-i$, we can realise $\{k-1,k+1\}$ via the *second* union on the right-hand side of (2.22). Let us first show that j_k occurs as an index of that union. We first show $j_k\geqslant 1$. We have

$$j_k \geqslant 1 \Leftrightarrow i \leqslant \tfrac{1}{2}(k+n-3)\ . \tag{2.23}$$

If $k=0$, then the right-hand side of (2.23) holds, as $i=\frac{1}{2}(n-1)\Leftrightarrow 2i+1=n=_n 0$ would contradict $0=k\in\mathbb{Z}/n\backslash\{2i,2i+1\}$, so the choice of $i\in[\frac{1}{2}(n-1)]_0$ actually implies $i\leqslant\frac{1}{2}(n-3)$. If on the contrary $k\geqslant 1$, then $k\geqslant 2$ since k is even, and then (2.23) merely demands $i\leqslant\frac{1}{2}(n-1)$, which is implied by the hypothesis $i\in\{0,1,\dots,\frac{1}{2}(n-1)\}$, so then the right-hand side of (2.23) holds, too. Either way, (2.23) yields $j_k\geqslant 1$. We now show $j_k\leqslant\frac{1}{2}(n-3)$. The hypothesis of Case 1.2 is $k<2i+2$, which, being in Case 1, can be strengthened to $k<2i+1$. Therefore $j_k = \frac{1}{2}k+\frac{1}{2}(n-1)-i < \frac{1}{2}(n-2)$, which, since n is odd and j_k integral, can be strengthened to

$j_k \leqslant \frac{1}{2}(n-2)-\frac{1}{2} = \frac{1}{2}(n-3)$, as needed. This completes the proof that the j_k of Case 1.2 occurs as an index of the union. Finally, with this j_k, indeed $\{2(i+j_k), 2(i+j_k)+2\} =_n \{k+n-1, k+n+1\} =_n \{k-1, k+1\}$. This completes Case 1.

Case 2. Odd k.

Case 2.1. $\frac{1}{2}(k-1)-i \geqslant 1$. Then with $j_k := \frac{1}{2}(k-1)-i$ we have $j_k \geqslant 1$. Moreover, $j_k = \frac{1}{2}(k-1)-i \leqslant \frac{1}{2}(n-1-1)-i \leqslant \frac{1}{2}(n-2)$, which because of n being odd and j_k an integer can be sharpened to $j_k \leqslant \frac{1}{2}(n-3)$. Thus, j_k occurs as an index of the second union in (2.22), and, indeed, $\{2(i+j_k), 2(i+j_k)+2\} = \{k-1, k+1\}$.

Case 2.2. $\frac{1}{2}(k-1)-i < 1$. We now show that with $j_k := \frac{1}{2}(k-1)+\frac{1}{2}(n-1)-i$ we realise $\{k-1, k+1\}$ via the first union on the right-hand side of (2.22): j_k occurs as an index of that union, since $j_k \geqslant 0 \Leftrightarrow i \leqslant \frac{1}{2}(n+k-2) \Leftarrow$ (since k is odd, hence $k \geqslant 1$) $\Leftarrow i \leqslant \frac{1}{2}(n-1)$, which is true by hypothesis about i. Morever, $j_k \leqslant \frac{1}{2}(n-3) \Leftrightarrow i \geqslant \frac{1}{2}(k+1)$; being in case Case 2.2, we already know $i > \frac{1}{2}(k-3)$, i.e. $i \geqslant \frac{1}{2}(k-1)$. Since i is integer and k odd, the needed $i \geqslant \frac{1}{2}(k+1)$ follows once we can eliminate the possibility $i = \frac{1}{2}(k-1)$; but this is equivalent to $k = 2i+1$ which is excluded by the initial hypothesis about k. Therefore, indeed $j_k \leqslant \frac{1}{2}(n-3)$, completing the proof that j_k occurs as an index. Finally, indeed $\{2(i+j_k)+1, 2(i+j_k)+3\} =_n \{k-1+n, k+1+n\} =_n \{k-1, k+1\}$. This completes Case 2. The proof of (2.22), and therefore of $\mathrm{Supp}(\vec{C}_{i;n}) = \mathrm{E}(C_{i;n})$ is now complete.

Now knowing that the supports of $\vec{C}_{0,1,2;n}$, $\vec{C}_{0,1,n-1;n}$, $\vec{C}_{i;n} \in \mathrm{C}_1(\mathrm{C}_n^{2^-})$ are indeed Hamilton-circuits, it remains to prove that they are flows, i.e. elements of $\mathrm{Z}_1(\mathrm{C}_n^{2^-}) = \ker(\partial_1)$.

We first prove $\vec{C}_{0,1,2;n}$, $\vec{C}_{0,1,n-1;n} \in \mathrm{Z}_1(\mathrm{C}_n^{2^-})$. The chain Σ_n in (2.20) involves only the $4 \cdot |\{0, \ldots, \frac{1}{4}(n-7)\}| + 1 = (n-3)+1 = n-2$ vertices $\{2, 3, \ldots, n-1\}$ (to see this, we imagine the vertex-4-set $\{j, j+2, j+1, j+3\}$ underlying the summand, with index j moving in steps of 4, and note that finally, for $j = \frac{1}{4}(n-7)$, there is one instance, where the vertex $j+4$ is not mentioned by the next iteration via the term j). It is obvious from the definition of the boundary operator $\partial_1 \colon \mathrm{C}_1(\mathrm{C}_n^{2^-}) \to \mathrm{C}_0(\mathrm{C}_n^{2^-})$ and the definition of Σ_n in (2.20) that $\langle \partial_1(\pm\Sigma_n), k\rangle = 0$ holds for each such $k \in \{2, 3, \ldots, n-1\}$, except for the two boundary-vertices 2 and $n-1$, which are mentioned only once within Σ_n. For them,

(Σ_n.1) $\langle \partial_1(\pm\Sigma_n), 2\rangle = \mp$,
(Σ_n.2) $\langle \partial_1(\pm\Sigma_n), n-1\rangle = \pm$.

In the chain $\vec{C}_{0,1,2;n}$ (resp. $\vec{C}_{0,1,n-1;n}$), that we would like to prove to be a flow, the chain Σ_n (resp. $-\Sigma_n$) occurs as a summand, and there are only three other summands, namely $0 \wedge 1$, $1 \wedge 2$, $n-1 \wedge 0$ (resp. $0 \wedge 1$, $1 \wedge n-1$, $2 \wedge 0$). From this we see $\langle \vec{C}_{0,1,2;n}, 0\rangle = 0$ and $\langle \vec{C}_{0,1,2;n}, 1\rangle = 0$ (resp. $\langle \vec{C}_{0,1,n-1;n}, 0\rangle = 0$ and $\langle \vec{C}_{0,1,n-1;n}, 1\rangle = 0$). Moreover, $\langle 0 \wedge 1 + 1 \wedge 2 + n-1 \wedge 0, 2\rangle = +$ and $\langle 0 \wedge 1 + 1 \wedge 2 + n-1 \wedge 0, n-1\rangle = -$, so, taken together with the $+\Sigma_n$ version of (Σ_n.1) and (Σ_n.2), it follows that $\langle \vec{C}_{0,1,2;n}, 2\rangle = 0$ and $\langle \vec{C}_{0,1,2;n}, n-1\rangle = 0$. Furthermore, $\langle 0 \wedge 1 + 1 \wedge n-1 + 2 \wedge 0, 2\rangle = -$ and $\langle 0 \wedge 1 + 1 \wedge n-1 + 2 \wedge 0, n-1\rangle = +$, hence together with the $-\Sigma_n$ version of (Σ_n.1) and (Σ_n.2), it follows that $\langle \vec{C}_{0,1,n-1;n}, 2\rangle = 0$ and $\langle \vec{C}_{0,1,n-1;n}, n-1\rangle = 0$. Now we have proved the flow-condition to hold, both for $\vec{C}_{0,1,2;n}$ and $\vec{C}_{0,1,n-1;n}$, for each of the n vertices of $\mathrm{C}_n^{2^-}$. This completes our proof that $\vec{C}_{0,1,2;n}$, $\vec{C}_{0,1,n-1;n} \in \mathrm{Z}_1(\mathrm{C}_n^{2^-})$.

We finally prove $\vec{C}_{i;n} \in \mathrm{Z}_1(\mathrm{C}_n^{2^-})$. Let an arbitrary $i \in \{0, 1, \ldots, \frac{1}{2}(n-1)\}$ be given. For every $j \in \{0, 1, \ldots, \frac{1}{2}(n-3)-1\}$, we have $j+1 \in \{1, 2, \ldots, \frac{1}{2}(n-3)\}$, i.e. $j+1$ is permitted as an index of the first sum in Definition 48.(3). The vertex $2(i+j)+3$ therefore appears twice in $\vec{C}_{i;n}$, once in the first (for index j), and once in the second (for index $j+1$) argument of the $\wedge$ in the first sum of Definition 48.(3); therefore, $\langle \partial_1(\vec{C}_{i;n}), 2(i+j)+3\rangle = 0$ for every $j \in \{0, 1, \ldots, \frac{1}{2}(n-3)-1\}$. For every $j \in \{2, \ldots, \frac{1}{2}(n-3)\}$, we have $j-1 \in \{1, \ldots, \frac{1}{2}(n-3)-1\}$, i.e. $j-1$ is permitted as an index of the second sum in Definition 48.(3). The vertex $2(i+j)$ appears twice in $\vec{C}_{i;n}$, once in the first (for index $j-1$), and once in the second (for index j) argument of the $\wedge$ in the first sum of Definition 48.(3). Therefore, $\langle \partial_1(\vec{C}_{i;n}), 2(i+j)\rangle = 0$ for $j \in \{1, \ldots, \frac{1}{2}(n-3)\}$. We have thus proved the flow-condition at each vertex $2(i+j)+3$ with $j \in \{0, 1, \ldots, \frac{1}{2}(n-3)-1\}$ and each vertex

$2(i+j)$ with $j \in \{1, \ldots, \frac{1}{2}(n-3)\}$. These are the $|\{0, 1, \ldots, \frac{1}{2}(n-3)-1\}| + |\{2, \ldots, \frac{1}{2}(n-3)\}| = n-4$ distinct vertices $2i+3, 2i+5, \ldots, 2i-2$ and $2i+4, 2i+6, \ldots, 2i-3$. It remains to prove the condition at the three missing vertices $2i-1$, $2i$, $2i+1$, $2i+2$. Vertex $2i-1$ appears as the second argument of $\wedge$ in $2i \wedge 2i-1$ and as the first argument in $2(i+j)+2 \wedge 2(i+j)$ when $j = \frac{1}{2}(n-3)$ in the second sum in Definition 48.(3). Vertex $2i$ appears as the first argument of $\wedge$ in $2i \wedge 2i-1$ and as the second argument in $2(i+j)+1 \wedge 2(i+j+3)$ when $j = \frac{1}{2}(n-3)$ in the first sum in (3). Vertex $2i+1$ appears as the second argument of $\wedge$ in $2i+2 \wedge 2i+1$ and as the first argument in $2(i+j)+1 \wedge 2(i+j)+3$ when $j = 0$ in the first sum in (3). Vertex $2i+2$ appears as the first argument of $\wedge$ in $2i+2 \wedge 2i+1$ and as the second argument in $2(i+j)$ when $j = 1$ in the second sum in (3). This proves that $\langle \partial_1(\vec{C}_{i;n}), v\rangle = 0$ for each $v \in \{2i-1, 2i, 2i+1, 2i+2\}$, and completes the proof that $\vec{C}_{i;n} \in \mathrm{Z}_1(\mathrm{C}_n^{2^-})$. The proof of Lemma 50 is now complete. $\square$

We will now prove, in the form of Lemmas 51–65, some auxiliary statements that we will use many times for looking-up values during the calculations made to establish (2.50) in the proof of Proposition 69.

Lemma 51 (how the edges of $\mathrm{C}_n^{2^-}$ relate to the Hamilton-circuit $C_{0,1,2;n}$ and $C_{0,1,n-1;n}$). *For every $n \geqslant 11$ with $n \equiv 3 \pmod 4$,*

(1) $\{2k+1, 2k+2\} \in \mathrm{E}(C_{0,1,2;n}) \cap \mathrm{E}(C_{0,1,n-1;n})$ *for every* $k \in [\frac{1}{2}(n-3)]$,
(2) $\{k-1, k+1\} \in \mathrm{E}(C_{0,1,2;n}) \cap \mathrm{E}(C_{0,1,n-1;n})$ *if* $k \in [n-1]$ *with* $k \equiv 0 \pmod 4$,
$\{k-1, k+1\} \notin \mathrm{E}(C_{0,1,2;n}) \cup \mathrm{E}(C_{0,1,n-1;n})$ *if* $k \in [n-1]$ *with* $k \equiv 1 \pmod 4$ *and* $k \geqslant 5$,
$\{k-1, k+1\} \notin \mathrm{E}(C_{0,1,2;n}) \cup \mathrm{E}(C_{0,1,n-1;n})$ *if* $k \in [n-1]$ *with* $k \equiv 2 \pmod 4$,
$\{k-1, k+1\} \in \mathrm{E}(C_{0,1,2;n}) \cap \mathrm{E}(C_{0,1,n-1;n})$ *if* $k \in [n-1]$ *with* $k \equiv 3 \pmod 4$.

Proof. Immediate from Definition 219. $\square$

Let us note that the following Lemma 52 makes statements about exactly $1+1+1+|[\frac{1}{2}(n-3)]_0| + |[n-1]| = \frac{3}{2}(n+1) =$ (by Lemma 66.(2)) $= \|\mathrm{C}_n^{2^-}\|$ distinct edges, as is necessary to make a statement about every edge of $\mathrm{C}_n^{2^-}$:

Lemma 52 (how the edges of $\mathrm{C}_n^{2^-}$ relate to the Hamilton-circuits $C_{i;n}$). *For every $n \geqslant 11$ with $n \equiv 3 \pmod 4$, abbreviating $[m]_0 := \{0, 1, \ldots, m\}$, and with $C_{i;n}$ as in Definition 219.*(3),

(1) $\{i \in [\frac{1}{2}(n-1)]_0 \colon \{0,1\} \in \mathrm{E}(C_{i;n})\} = \{\frac{1}{2}(n-1)\}$,
(2) $\{i \in [\frac{1}{2}(n-1)]_0 \colon \{0,n-1\} \in \mathrm{E}(C_{i;n})\} = \{0\}$,
(3) $\{i \in [\frac{1}{2}(n-1)]_0 \colon \{1,n-1\} \in \mathrm{E}(C_{i;n})\} = [\frac{1}{2}(n-1)]_0 \backslash \{0, \frac{1}{2}(n-1)\}$,
(4) $\{i \in [\frac{1}{2}(n-1)]_0 \colon \{2k+1,2k+2\} \in \mathrm{E}(C_{i;n})\} = \{k, k+1\}$ *for every* $k \in [\frac{1}{2}(n-3)]_0$,
(5) $\{i \in [\frac{1}{2}(n-1)]_0 \colon \{k-1,k+1\} \in \mathrm{E}(C_{i;n})\} = [\frac{1}{2}(n-1)]_0 \backslash \{\lfloor \frac{1}{2}k \rfloor\}$ *for every* $k \in [n-1]$.

Proof. From their definition we see that the only two edges in $C_{i;n}$ whose vertices differ by 1 (modulo n) are $\{2i+1, 2i+2\}$ and $\{2i-1, 2i\}$. Therefore, for every $\{a,b\} \in \mathrm{E}(\mathrm{C}_n^{2^-})$:

$$\begin{aligned}&(i \in [\tfrac{1}{2}(n-1)]_0 \text{ and } \{a,b\} \in \mathrm{E}(C_{i;n})) \Leftrightarrow \\ &(i \in [\tfrac{1}{2}(n-1)]_0 \text{ and } \{2i+1,2i+2\} =_n \{a,b\}) \text{ or } (i \in [\tfrac{1}{2}(n-1)]_0 \text{ and } \{2i-1,2i\} =_n \{a,b\})\end{aligned} \tag{2.24}$$

We have $\{2i+1, 2i+2\} \cup \{2i-1, 2i\} \subseteq \{1, 2, \ldots, n-1\}$ for $i \in \{1, 2, \ldots, \frac{1}{2}(n-1)-1\}$, so calculating modulo n is relevant only for $i = 0$ and $i = \frac{1}{2}(n-1)$. We will therefore habitually treat these candidate-values of the index i separately. Moreover, if we reach a condition using $=_n$ and involving k as the only variable, then because of $k \leqslant \frac{1}{2}(n-3) \Leftrightarrow 2k+2 \leqslant n-1$ we may equivalently replace $=_n$ by $=$. This will be done without further justification on several occasions below.

As for (1), we use (2.24) with $a := 0$ and $b := 1$. If $i = 0$, then $\{2i+1, 2i+2\} =_n \{0,1\} \Leftrightarrow \{1,2\} =_n \{0,1\} \Leftrightarrow$ false, and $\{2i-1, 2i\} =_n \{0,1\} \Leftrightarrow \{0, n-1\} = \{0,1\} \Leftrightarrow$ false; if $i = \frac{1}{2}(n-1)$, then $\{2i+1, 2i+2\} =_n \{0,1\} \Leftrightarrow \{0,1\} =_n \{0,1\} \Leftrightarrow$ true, and $\{2i-1, 2i\} =_n \{0,1\} \Leftrightarrow \{n-2, n-1\} =_n \{0,1\} \Leftrightarrow$ false; if finally $i \in \{1, 2, \ldots, \frac{1}{2}(n-1)-1\}$, we can replace '$=_n$' with '$=$', so $\{2i+1, 2i+2\} =_n \{0,1\}$

$\Leftrightarrow \{2i+1, 2i+2\} = \{0,1\} \Rightarrow 2i+1 = 1 \Rightarrow i = 0$, contradicting $i \geqslant 1$, hence then $\{2i+1, 2i+2\}$ cannot equal $\{0,1\}$, and $\{2i-1, 2i\} =_n \{0,1\} \Leftrightarrow \{2i-1, 2i\} = \{0,1\} \Rightarrow 2i = 0 \Rightarrow i = 0$, again contradicting that at present $i \geqslant 1$. Since the only possibility to realise $\{0,1\}$ by $\{2i+1, 2i+2\}$ or $\{2i-1, 2i\}$ was found to be $i = \frac{1}{2}(n-1)$, this proves (1).

As for (2), we use (2.24) with $a := 0$ and $b := n-1$. If $i = 0$, then $\{2i+1, 2i+2\} =_n \{0, n-1\} \Leftrightarrow \{1,2\} =_n \{0, n-1\} \Leftrightarrow$ false, and $\{2i-1, 2i\} =_n \{0, n-1\} \Leftrightarrow \{0, n-1\} = \{0, n-1\} \Leftrightarrow$ true; if $i = \frac{1}{2}(n-1)$, then $\{2i+1, 2i+2\} =_n \{0, n-1\} \Leftrightarrow \{0,1\} =_n \{0, n-1\} \Leftrightarrow$ false, and $\{2i-1, 2i\} =_n \{0, n-1\} \Leftrightarrow \{n-2, n-1\} =_n \{0, n-1\} \Leftrightarrow$ false; if finally $i \in \{1, 2, \ldots, \frac{1}{2}(n-1)-1\}$, we can replace '$=_n$' with '$=$', so $\{2i+1, 2i+2\} =_n \{0, n-1\} \Leftrightarrow \{2i+1, 2i+2\} = \{0, n-1\} \Rightarrow$ false, since both 0 and $n-1$ are even, so then $\{2i+1, 2i+2\}$ cannot equal $\{0, n-1\}$, and $\{2i-1, 2i\} =_n \{0, n-1\} \Leftrightarrow \{2i-1, 2i\} = \{0, n-1\} \Rightarrow$ false, since both 0 and $n-1$ are even. Since the only possibility to realise $\{0, n-1\}$ by $\{2i+1, 2i+2\}$ or $\{2i-1, 2i\}$ was found to be $i = 0$, this proves (2).

As for (3), we first note that $\{1, n-1\}$ is a difference-2-edge, so has to be realised via the parametrised union in Definition 219.(3); we have $\{j-1, j+1\} =_n \{1, n-1\} \Rightarrow$ (($\exists \ell \in \mathbb{Z}$: $j-1 = 1 + \ell n$) or ($\exists \ell \in \mathbb{Z}$: $j-1 = n-1+\ell n$)) $\Rightarrow$ ($j = 2$ or $j = 0$). For $j = 2$, the other equality needed for $\{j-1, j+1\} =_n \{1, n-1\}$ does not hold, since $2+1 \not=_n n-1$. For $j = 0$, though, we indeed have $\{j-1, j+1\} =_n \{1, n-1\}$. Thus, $\{j-1, j+1\} =_n \{1, n-1\} \Leftrightarrow j = 0$. It now remains to note that the only values of $i \in [\frac{1}{2}(n-1)]_0$ for which the value $j = 0$ does *not* occur as an index of the union in Definition 219.(3) are $i = 0$ and $i = \frac{1}{2}(n-1)$. This completes the proof of (3).

As for (4), we use (2.24) with $a := 2k+1$ and $b := 2k+2$. If $i = 0$, then $\{2i+1, 2i+2\} =_n \{2k+1, 2k+2\} \Leftrightarrow \{1,2\} =_n \{2k+1, 2k+2\} \Leftrightarrow \{1,2\} = \{2k+1, 2k+2\} \Leftrightarrow k = 0$, and $\{2i-1, 2i\} =_n \{2k+1, 2k+2\} \Leftrightarrow \{0, n-1\} =_n \{2k+1, 2k+2\} \Leftrightarrow \{0, n-1\} = \{2k+1, 2k+2\} \Rightarrow k = -1$ contradicting $k \in \{0, 1, \ldots, \frac{1}{2}(n-3)\}$; if $i = \frac{1}{2}(n-1)$, then $\{2i+1, 2i+2\} =_n \{2k+1, 2k+2\} \Leftrightarrow \{0,1\} =_n \{2k+1, 2k+2\} \Leftrightarrow \{0,1\} = \{2k+1, 2k+2\} \Leftrightarrow 0 = 2k+2 \Rightarrow k = -1$ contradicting $k \in \{0, 1, \ldots, \frac{1}{2}(n-3)\}$; if finally $i \in \{1, 2, \ldots, \frac{1}{2}(n-1)-1\}$, we can replace '$=_n$' with '$=$', so $\{2i+1, 2i+2\} =_n \{2k+1, 2k+2\} \Leftrightarrow \{2i+1, 2i+2\} = \{2k+1, 2k+2\} \Leftrightarrow i = k$ and $\{2i-1, 2i\} =_n \{2k+1, 2k+2\} \Leftrightarrow \{2i-1, 2i\} = \{2k+1, 2k+2\} \Leftrightarrow i = k+1$. This completes the proof of (4).

As for (5), we first note that $\{k-1, k+1\}$ is a difference-2-edge, so has to be realised via the parametrised union in Definition 219.(3); we have $\{j-1, j+1\} =_n \{k-1, k+1\} \Leftrightarrow j = k$. For every $k \in \{1, 2, \ldots, n-1\}$ there is exactly one $i \in [\frac{1}{2}(n-1)]_0$ with $k \in \{2i, 2i+1\}$, namely $i = \lfloor \frac{1}{2}k \rfloor$. Hence for every $i \in [\frac{1}{2}(n-1)]_0 \backslash \{\lfloor \frac{1}{2}k \rfloor\}$ the value $j = k$ occurs among the indices of the union in (3), realising the edge $\{j-1, j+1\} = \{k-1, k+1\}$, while only for $i = \lfloor \frac{1}{2}k \rfloor$ the edge $\{k-1, k+1\}$ does not occur in $C_{i;n}$. This proves (5). $\square$

Lemma 53 (how the oriented edges of $\mathrm{C}_n^{2^-}$ relate to the Hamilton-flows $\vec{C}_{0,1,2;n}$ and $\vec{C}_{0,1,n-1;n}$)**.** *For every $n \geqslant 11$ with $n \equiv 3 \pmod 4$, abbreviating $[m]_0 := \{0, 1, \ldots, m\}$, and with $\vec{C}_{i;n}$ as in Definition 48.(2),*

(1) $\langle 0 \wedge 1, \vec{C}_{0,1,2;n} \rangle = +$ *and* $\langle 0 \wedge 1, \vec{C}_{0,1,n-1;n} \rangle = +$,
(2) $\langle 0 \wedge 2, \vec{C}_{0,1,2;n} \rangle = 0$ *and* $\langle 0 \wedge 2, \vec{C}_{0,1,n-1;n} \rangle = -$,
(3) $\langle 0 \wedge n-1, \vec{C}_{0,1,2;n} \rangle = -$ *and* $\langle 0 \wedge n-1, \vec{C}_{0,1,n-1;n} \rangle = 0$,
(4) $\langle 1 \wedge 2, \vec{C}_{0,1,2;n} \rangle = +$ *and* $\langle 1 \wedge 2, \vec{C}_{0,1,n-1;n} \rangle = 0$,
(5) $\langle 1 \wedge n-1, \vec{C}_{0,1,2;n} \rangle = 0$ *and* $\langle 1 \wedge n-1, \vec{C}_{0,1,n-1;n} \rangle = +$,
(6) $\langle 2k+1 \wedge 2k+2, \vec{C}_{0,1,2;n} \rangle = +$ *and* $\langle 2k+1 \wedge 2k+2, \vec{C}_{0,1,n-1;n} \rangle = -$
for every even $k \in [\frac{1}{2}(n-3)]_0$ *with* $k \geqslant 2$,
$\langle 2k+1 \wedge 2k+2, \vec{C}_{0,1,2;n} \rangle = -$ *and* $\langle 2k+1 \wedge 2k+2, \vec{C}_{0,1,n-1;n} \rangle = +$
for every odd $k \in [\frac{1}{2}(n-3)]_0$,
(7) $\langle k-1 \wedge k+1, \vec{C}_{0,1,2;n} \rangle = +$ *and* $\langle k-1 \wedge k+1, \vec{C}_{0,1,n-1;n} \rangle = -$
if $k \in [n-1]$ *with* $k \equiv 0 \pmod 4$, *(where this implies* $k \geqslant 2$*; for* $k = 0$ *see* (5)*)* ,
$\langle k-1 \wedge k+1, \vec{C}_{0,1,2;n} \rangle = 0$ *and* $\langle k-1 \wedge k+1, \vec{C}_{0,1,n-1;n} \rangle = 0$
if $k \in [n-1]$ *with* $k \equiv 1 \pmod 4$ *and* $k \geqslant 5$ *(for* $k = 1$ *see* (2)*)* ,

$\langle k-1 \wedge k+1, \vec{C}_{0,1,2;n}\rangle = 0$ *and* $\langle k-1 \wedge k+1, \vec{C}_{0,1,n-1;n}\rangle = 0$
if $k \in [n-1]$ *with* $k \equiv 2 \pmod 4$,
$\langle k-1 \wedge k+1, \vec{C}_{0,1,2;n}\rangle = +$ *and* $\langle k-1 \wedge k+1, \vec{C}_{0,1,n-1;n}\rangle = -$
if $k \in [n-1]$ *with* $k \equiv 3 \pmod 4$,

(8) $\langle k-1 \wedge k+1, \vec{C}_{0,1,2;n}\rangle + \langle k-1 \wedge k+1, \vec{C}_{0,1,n-1;n}\rangle = 0$ *for every* $k \in \{2,3,\ldots,n-1\}$.

Proof. Each of (1)—(5), holds in view of (1) and (2) of Definition 48, and since none of $0 \wedge 1$, $0 \wedge 2$, $0 \wedge n-1$, $1 \wedge 2$ or $1 \wedge n-1$ appears in Σ_n.

As for (6), if $k \in [\frac{1}{2}(n-3)]$ is even with $k \geqslant 2$, the number $j_k := 2k-2 \geqslant 2$ is an element of the index-set $\{2+4\kappa \colon \kappa \in \{0,1,\ldots,\frac{1}{4}(n-7)\}\}$ of the sum Σ_n from (2.20), hence $j_k+3 \wedge j_k+4 = 2k+1 \wedge 2k+2$ is a summand of Σ_n, hence $\langle 2k+1 \wedge 2k+2, \Sigma_n\rangle = +$, proving the first, and $\langle 2k+1 \wedge 2k+2, -\Sigma_n\rangle = -$, proving the second statement in (6). Moreover, if $k \in [\frac{1}{2}(n-3)]_0$ is odd, then $k \geqslant 1$, hence $j_k := 2k \geqslant 2$, hence j_k is an element of $\{2+4\kappa \colon \kappa \in \{0,1,\ldots,\frac{1}{4}(n-7)\}\}$, hence $j_k+2 \wedge j_k+1 = 2k+2 \wedge 2k+1$ is a summand of Σ_n, hence $\langle 2k+1 \wedge 2k+2, \Sigma_n\rangle = -$, proving the third, and $\langle 2k+1 \wedge 2k+2, -\Sigma_n\rangle = +$, proving the fourth statement in (6).

As for (6), if $k \in [n-1]$ with $k \equiv 0 \pmod 4$, then $k \geqslant 4$, $j_k := k-2 \geqslant 2$ lies in $\{2+4\kappa \colon \kappa \in \{0,1,\ldots,\frac{1}{4}(n-7)\}\}$ in (2.20), and therefore $j_k+1 \wedge j_k+3 = k-1 \wedge k+1$ a summand of Σ_n, hence $\langle k-1 \wedge k+1, \Sigma_n\rangle = +$ and $\langle k-1 \wedge k+1, -\Sigma_n\rangle = -$, which (since $\vec{C}_{0,1,2;n}$ contains Σ_n while $\vec{C}_{0,1,n-1;n}$ contains $-\Sigma_n$) proves the first statement in (6). If $k \in [n-1]$ with $k \equiv 1 \pmod 4$ and $k \geqslant 5$, or with $k \equiv 2 \pmod 4$, then, since from Lemma 50 we know $\mathrm{Supp}(\vec{C}_{0,1,2;n}) = \mathrm{E}(C_{0,1,2;n})$ and $\mathrm{Supp}(\vec{C}_{0,1,n-1;n}) = \mathrm{E}(C_{0,1,n-1;n})$, and since by the second and third lines of Lemma 51.(2) the edge $\{k-1,k+1\}$ is neither in $\mathrm{Supp}(\vec{C}_{0,1,2;n})$ nor $\mathrm{Supp}(\vec{C}_{0,1,n-1;n})$, the second and third statements of (6) follow. Finally, if $k \in [n-1]$ with $k \equiv 3 \pmod 4$, then $k \geqslant 3$, hence $j_k := k-1 \geqslant 2$ is an element of $\{2+4\kappa \colon \kappa \in \{0,1,\ldots,\frac{1}{4}(n-7)\}\}$ from (2.20), and therefore $j_k \wedge j_k+2 = k-1 \wedge k+1$ is a summand of Σ_n, so $\langle k-1 \wedge k+1, \Sigma_n\rangle = +$ and $\langle k-1 \wedge k+1, -\Sigma_n\rangle = -$, implying the fourth statement in (7). Statement (8) is immediate from (6). $\square$

Lemma 54 (how the oriented edges of $\mathrm{C}_n^{2^-}$ relate to the Hamilton-flows $\vec{C}_{i;n}$)**.** *For every* $n \geqslant 11$ *with* $n \equiv 3 \pmod 4$*, abbreviating* $[m]_0 := \{0,1,\ldots,m\}$*, and with* $\vec{C}_{i;n}$ *as in Definition 48.*(3)*,*

(1) (i) $\langle 0 \wedge 1, \vec{C}_{\frac{1}{2}(n-1);n}\rangle = +$,
(ii) $\langle 0 \wedge 1, \vec{C}_{i;n}\rangle = 0$ *for all* $i \in [\frac{1}{2}(n-1)]_0 \backslash \{\frac{1}{2}(n-1)\}$,

(2) (i) $\langle 0 \wedge n-1, \vec{C}_{0;n}\rangle = +$,
(ii) $\langle 0 \wedge n-1, \vec{C}_{i;n}\rangle = 0$ *for all* $i \in [\frac{1}{2}(n-1)]_0 \backslash \{0\}$,

(3) (i) $\langle 1 \wedge n-1, \vec{C}_{0;n}\rangle = \langle 1 \wedge n-1, \vec{C}_{\frac{1}{2}(n-1);n}\rangle = 0$,
(ii) $\langle 1 \wedge n-1, \vec{C}_{i;n}\rangle = +$ *for all* $i \in [\frac{1}{2}(n-1)]_0 \backslash \{0, \frac{1}{2}(n-1)\}$,

(4) *for every* $k \in [\frac{1}{2}(n-3)]_0$,
(i) $\langle 2k+1 \wedge 2k+2, \vec{C}_{k;n}\rangle = \langle 2k+1 \wedge 2k+2, \vec{C}_{k+1;n}\rangle = -$,
(ii) $\langle 2k+1 \wedge 2k+2, \vec{C}_{i;n}\rangle = 0$ *for all* $i \in [\frac{1}{2}(n-1)]_0 \backslash \{k, k+1\}$,

(5) *for every* $k \in [n-1]$,
(i) $\langle k-1 \wedge k+1, \vec{C}_{\lfloor \frac{1}{2}k \rfloor;n}\rangle = 0$,
(ii) *if* k *is even,*
$\langle k-1 \wedge k+1, \vec{C}_{i;n}\rangle = +$ *for all* $i \in \left[\lfloor \frac{1}{2}k \rfloor - 1\right]_0$,
$\langle k-1 \wedge k+1, \vec{C}_{i;n}\rangle = -$ *for all* $i \in \{\lfloor \frac{1}{2}k \rfloor + 1, \lfloor \frac{1}{2}k \rfloor + 2, \ldots, \frac{1}{2}(n-3)\}$,
and
$\langle k-1 \wedge k+1, \vec{C}_{\frac{1}{2}(n-1);n}\rangle = +$,
(iii) *if* k *is odd,*
$\langle k-1 \wedge k+1, \vec{C}_{i;n}\rangle = -$ *for all* $i \in \left[\lfloor \frac{1}{2}k \rfloor - 1\right]_0$,

$\langle k-1 \wedge k+1, \vec{C}_{i;n}\rangle = +$ *for all* $i \in \{\lfloor \frac{1}{2}k\rfloor + 1, \lfloor \frac{1}{2}k\rfloor + 2, \ldots, \frac{1}{2}(n-3)\}$,
and
$\langle k-1 \wedge k+1, \vec{C}_{\frac{1}{2}(n-1);n}\rangle = -$.

Proof. Lemma 52.(1) implies (1).(ii), while Definition 48.(4) obviously implies (1).(i). Lemma 52.(2) implies (2).(ii); moreover, according to Definition 48.(3), the chain $2\cdot 0 \wedge (2\cdot 0 - 1) =_n 0 \wedge n-1$ is a summand of $\vec{C}_{0;n}$, proving (2).(i).

Lemma 52.(3) implies (3).(i); moreover, for every $i \in [\frac{1}{2}(n-1)]_0 \backslash \{0, \frac{1}{2}(n-1)\}$ we have $j_i := \frac{1}{2}(n-1) - i \in \{1, 2, \ldots, \frac{1}{2}(n-3)\}$, $2(i+j_i)+2 =_n 1$ and $2(i+j_i) =_n n-1$, so the chain $1 \wedge n-1$ appears as the j_i-indexed summand of the second sum in Definition 48.(3), proving (3).(ii).

Lemma 52.(4) implies (4).(ii); if $i = k$, the chain $2i+2\wedge 2i+1 = 2k+2\wedge 2k+1$ is the only summand of $\vec{C}_{i;n}$ with support $\{2k+1, 2k+2\}$, and if $i = k+1$, then the chain $2i \wedge 2i-1 = 2k+2 \wedge 2k+1$ is the only summand of $\vec{C}_{i;n}$ with support $\{2k+1, 2k+2\}$, proving (4).(i).

Lemma 52.(5) implies (5).(i). If $k \in [n-1]$ is even, then for every $i \in \left[\lfloor \frac{1}{2}k\rfloor - 1\right]_0$ we have $\frac{1}{2}(k-2)-i \leqslant \frac{1}{2}(k-2)-0 = \frac{1}{2}k-1 \leqslant \frac{1}{2}(n-1)-1 = \frac{1}{2}(n-3)$ and $\frac{1}{2}(k-2)-i \geqslant \frac{1}{2}(k-2)-(\lfloor\frac{1}{2}k\rfloor-1) = 0$, hence indeed $j_i := \frac{1}{2}(k-2) - i \in \{0, 1, \ldots, \frac{1}{2}(n-3)\}$, so $2(i+j_i)+1 \wedge 2(i+j_i)+3 = k-1 \wedge k+1$ is the summand of $\vec{C}_{i;n}$ with support $\{k-1, k+1\}$, proving the first statement in (5).(ii). Since $k \in [n-1]$ is even, $k \geqslant 2$, so for every $i \in \{\lfloor\frac{1}{2}k\rfloor+1, \lfloor\frac{1}{2}k\rfloor+2, \ldots, \frac{1}{2}(n-1)\}$ we have $\frac{1}{2}(n+k-1)-i \geqslant \frac{1}{2}(n+k-1) - \frac{1}{2}(n-1) = \frac{1}{2}k \geqslant 1$. Since also $\frac{1}{2}(n+k-1)-i \leqslant \frac{1}{2}(n+k-1) - (\lfloor\frac{1}{2}k\rfloor+1) = \frac{1}{2}(n-1)-1 = \frac{1}{2}(n-3)$, we indeed have $j_i := \frac{1}{2}(n+k-1) - i \in \{1, 2, \ldots, \frac{1}{2}(n-3)\}$. So the summand $2(i+j_i)+2 \wedge 2(i+j_i) =_n n+k+1 \wedge n+k-1 =_n k+1 \wedge k-1$ of the second sum in Definition 48.(3) is the summand of $\vec{C}_{i;n}$ with support $\{k-1, k+1\}$, proving the second statement in (5).(ii). The third statement in (5).(ii) holds in view of Definition 48.(4).

If $k \in [n-1]$ is odd, then since n is odd, too, we know $k \leqslant n-2$, so for every $i \in \left[\lfloor\frac{1}{2}k\rfloor - 1\right]_0$ we have $\frac{1}{2}(k-1)-i \leqslant \frac{1}{2}(k-1) \leqslant \frac{1}{2}((n-2)-1) = \frac{1}{2}(n-3)$, and $\frac{1}{2}(k-1)-i \geqslant \frac{1}{2}(k-1) - (\lfloor\frac{1}{2}k\rfloor - 1) = 1$, hence $j_i := \frac{1}{2}(k-1) - i \in \{1, 2, \ldots, \frac{1}{2}(n-3)\}$ and the summand $2(i+j_1)+2 \wedge 2(i+j_i) = k+1 \wedge k-1$ of the second sum in Definition 48.(3) is the summand with support $\{k-1, k+1\}$ of $\vec{C}_{i;n}$, proving the first statement in (5).(iii). Moreover, if $i \in \{\lfloor\frac{1}{2}k\rfloor+1, \lfloor\frac{1}{2}k\rfloor+2, \ldots, \frac{1}{2}(n-1)\}$, then $\frac{1}{2}(k-2+n)-i \leqslant \frac{1}{2}(k-2+n) - (\lfloor\frac{1}{2}k\rfloor+1) = \frac{1}{2}(n-3)$ and $\frac{1}{2}(k-2+n)-i \geqslant \frac{1}{2}(k-2+n) - \frac{1}{2}(n-1) = \frac{1}{2}(k-1) \geqslant \frac{1}{2}(1-1) = 0$, so $j_i := \frac{1}{2}(k-2+n) - i \in \{0, 1, \ldots, \frac{1}{2}(n-3)\}$, hence the chain $2(i+j_i)+1 \wedge 2(i+j_i)+3 =_n k-1+n \wedge k+1+n =_n k-1 \wedge k+1$ is a summand of the first sum in Definition 48.(3), and we have again found the summand of $\vec{C}_{i;n}$ with support $\{k-1, k+1\}$, proving the second statement in (5).(iii). The third statement in (5).(iii) holds in view of Definition 48.(4). $\square$

Definition 55 ($\Sigma_a^b(v \mid c_1, c_2, c_3, c_4)$)**.** *For* $n \in \mathbb{N}$, $0 \leqslant a \leqslant b \leqslant n-1$, $v \in \mathbb{Z}/n$, K^n *denoting the complete graph on* $\mathbb{Z}/n$, *and with* $v(j) := v + 4j$, *we define the following element of* $\mathrm{C}_1(\mathrm{K}^n)$*:*

$$\Sigma_a^b(v \mid c_1, c_2, c_3, c_4) := \sum_{j=a}^{b} \begin{aligned}[t] & v(j) \wedge (v(j)+c_1) + \\ & (v(j)+c_1) \wedge (v(j)+c_1+c_2) + \\ & (v(j)+c_1+c_2) \wedge (v(j)+c_1+c_2+c_3) + \\ & (v(j)+c_1+c_2+c_3) \wedge (v(j)+c_1+c_2+c_3+c_4) \end{aligned} \tag{2.25}$$

Let us note that the sum $\Sigma_a^b(v \mid c_1, c_2, c_3, c_4)$ involves $4(b-a+1)$ elementary 1-chains $x \wedge y$, and the largest vertex in them has label $v + (b+1)\cdot(c_1+c_2+c_3+c_4)$.

For later calculations, we need general explicit expressions of the fundamental flows defined by the spanning trees in Definition 218:

Lemma 56 (the fundamental flows pertaining to the spanning trees from Definition 218 and Figure 5.3)**.** *For every* $n \geqslant 11$ *such that* $n \equiv 3 \pmod 8$, *and with* T_n *denoting the spanning-tree defined in Definition 218 and* $\mathrm{f}_{\mathrm{T}_n}$ *as in Definition 49, and with the notation* $\Sigma_a^b(v \mid c_1, c_2, c_3, c_4)$ *from Definition 55, the following are all the fundamental flows defined by* T_n*:*

(ff.1) $\mathrm{f}_{\mathrm{T}_n}(n-2,0) = n-2 \wedge 0 + 0 \wedge n-1 + n-1 \wedge n-2 \quad \in \quad \mathrm{C}_1(\mathrm{C}_n^{2^-})$,
(ff.2) $\mathrm{f}_{\mathrm{T}_n}(n-1,1) = n-1 \wedge 1 + 1 \wedge 0 + 0 \wedge n-1 \quad \in \quad \mathrm{C}_1(\mathrm{C}_n^{2^-})$,
(ff.3) $\mathrm{f}_{\mathrm{T}_n}(0,2) = 0 \wedge 2 + 2 \wedge 1 + 1 \wedge 0 \quad \in \quad \mathrm{C}_1(\mathrm{C}_n^{2^-})$,
(ff.4) *for every* $\ell \in \{0,1,\ldots,\frac{1}{4}(n-3)-1\}\backslash\{\frac{1}{8}(n-3)\}$,

$$\begin{aligned}\mathrm{f}_{\mathrm{T}_n}(1+4\ell,3+4\ell) = (1+4\ell \wedge 3+4\ell) \quad &+ \quad (3+4\ell \wedge 4+4\ell) \\ + (4+4\ell \wedge 2+4\ell) \quad &+ \quad (2+4\ell \wedge 1+4\ell) \quad \in \mathrm{C}_1(\mathrm{C}_n^{2^-}) \ ,\end{aligned}$$

(ff.5) *with* $i_n := \frac{1}{2}(n+1)$,

$$\begin{aligned}\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n+1) = i_n-1 \wedge i_n+1 \quad &+ \quad \Sigma_0^{\frac{1}{8}(n-19)}(i_n+1 \mid {\scriptstyle +2,+1,+2,-1}) \\ + n-4 \wedge n-2 \quad + \quad n-2 \wedge n-1 \quad &+ \quad n-1 \wedge 0 \quad + \quad 0 \wedge 1 \\ + \Sigma_0^{\frac{1}{8}(n-11)}(1 \mid {\scriptstyle +1,+2,-1,+2}) \quad &\in \mathrm{C}_1(\mathrm{C}_n^{2^-}) \ ,\end{aligned}$$

(ff.6) *for every* $\ell \in \{0,1,\ldots,\frac{1}{4}(n-3)-1\}\backslash\{\frac{1}{8}(n-3)-1\}$,

$$\begin{aligned}\mathrm{f}_{\mathrm{T}_n}(4+4\ell,6+4\ell) = (4+4\ell \wedge 6+4\ell) \quad &+ \quad (6+4\ell \wedge 5+4\ell) \\ + (5+4\ell \wedge 3+4\ell) \quad &+ \quad (3+4\ell \wedge 4+4\ell) \quad \in \mathrm{C}_1(\mathrm{C}_n^{2^-}) \ ,\end{aligned}$$

(ff.7) *with* $i_n := \frac{1}{2}(n+1)$,

$$\begin{aligned}\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n) = i_n-2 \wedge i_n \quad &+ \quad \Sigma_0^{\frac{1}{8}(n-11)}(i_n \mid {\scriptstyle +2,-1,+2,+1}) \\ + n-1 \wedge 0 \quad + \quad 0 \wedge 1 \quad &+ \quad \Sigma_0^{\frac{1}{8}(n-19)}(1 \mid {\scriptstyle +1,+2,-1,+2}) \\ + i_n-5 \wedge i_n-4 \quad &+ \quad i_n-4 \wedge i_n-2 \quad \in \mathrm{C}_1(\mathrm{C}_n^{2^-}) \ ,\end{aligned}$$

(ff.8) *with* $i_n := \frac{1}{2}(n+1)$,

$$\begin{aligned}\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n) = i_n-1 \wedge i_n \quad &+ \quad \Sigma_0^{\frac{1}{8}(n-11)}(i_n \mid {\scriptstyle +2,-1,+2,+1}) \\ + n-1 \wedge 0 \quad + \quad 0 \wedge 1 \quad &+ \quad \Sigma_0^{\frac{1}{8}(n-11)}(1 \mid {\scriptstyle +1,+2,-1,+2}) \quad \in \mathrm{C}_1(\mathrm{C}_n^{2^-}) \ .\end{aligned}$$

Proof. This can be read off from Definitions 49 and 218. We note the following consistency checks:

(1) The value $\ell = \frac{1}{8}(n-3)$ excluded in (ff.4) corresponds to the separately given fundamental flow $\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n+1)$ in (ff.5), since $1 + 4\cdot\frac{1}{8}(n-3) = \frac{1}{2}(n-3)+1 = i_n-1$.
(2) The value $\ell = \frac{1}{8}(n-3)-1$ excluded in (ff.6) corresponds to the separately given fundamental flow $\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n)$ in (ff.7), since $4 + 4\cdot(\frac{1}{8}(n-3)-1) = \frac{1}{2}(n-3) = i_n-2$.
(3) In (ff.5), $|\mathrm{Supp}(\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n+1))| = 1 + (\frac{1}{8}(n-19)+1)\cdot 4 + 1 + 1 + 1 + 1 + (\frac{1}{8}(n-11)+1)\cdot 4 = n-2$, consistent with the fact that $\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n+1)$ is a simple flow of length $n-2$.
(4) In (ff.8), $|\mathrm{Supp}(\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n))| = 1 + 4\cdot|\{0,1,\ldots,\frac{1}{8}(n-11)\}| + 2 + 4\cdot|\{0,1,\ldots,\frac{1}{8}(n-11)\}| + 1 + \frac{1}{2}(n-3) + 2 + \frac{1}{2}(n-3) = n$, consistent with $\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n)$ being the only among the fundamental flows in Lemma 56 which is itself a Hamilton-flow.
(5) In Lemma 56 we list $(1)+(1)+(1)+(\frac{1}{4}(n-3)-1)+(1)+(\frac{1}{4}(n-3)-1)+(1)+(1) = \frac{1}{2}(n-3)+4 = \frac{1}{2}(n+5) = \frac{3}{2}(n+1)-n+1 =$ (by Lemma 66.(2)) $= \|\mathrm{C}_n^{2^-}\| - |\mathrm{C}_n^{2^-}| + 1$ fundamental flows, which is consistent with the claim that these are all the fundamental flows defined by the spanning-tree T_n.
(6) The number of elementary 1-chains in the 1-chain $\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n)$ from Lemma 56.(ff.7) is $1 + 4\cdot(\frac{1}{8}(n-11)+1) + 2 + 4\cdot(\frac{1}{8}(n-19)+1) + 2 = 1 + \frac{1}{2}(n-3) + 2 + \frac{1}{2}(n-11) + 2 = n-2$, which is consistent with the fact that the fundamental circuit of T_n obtained by adding the edge $\{i_n-2,i_n\}$ has $n-2$ edges (it misses the vertices i_n-3 and i_n-1).

(7) In (ff.5), the largest label of the elementary chain in the sum $\Sigma_0^{\frac{1}{8}(n-11)}(1 \mid {\scriptstyle +1,+2,-1,+2})$ is $1+(\frac{1}{8}(n-11)+1)\cdot(+1+2-1+2) = \frac{1}{2}(n-1) = i_n-1$, consistent with the fact that the first label in the first elementary 1-chain in $\mathrm{f}_{\mathrm{T}_n}(i_n-1, i_n+1)$ is i_n-1. The largest label of the elementary chain in the sum $\Sigma_0^{\frac{1}{8}(n-19)}(i_n+1 \mid {\scriptstyle +2,+1,+2,-1})$ is $i_n+1+(\frac{1}{8}(n-19)+1)\cdot(+2+1+2-1) = \frac{1}{2}(n+3)+\frac{1}{2}(n-19)+4 = n-4$, consistent with the fact that in $\mathrm{f}_{\mathrm{T}_n}(i_n-1, i_n+1)$, the first label in the first elementary 1-chain after this sum is $n-4$.

(8) In Lemma 56, $n = 11$ is expressly allowed, and for $n = 11$ we have $\{0, 1, \ldots, \frac{1}{8}(n-19)\} = \varnothing$, hence $\Sigma_0^{\frac{1}{8}(n-19)}(i_n+1 \mid {\scriptstyle +2,+1,+2,-1})$ in (ff.5) is an empty sum. This is intended. With this sum vanishing, the fundamental flow defined in (ff.5) is correct. This can be seen in Figure 5.3: for $n = 11$, the portion from i_n-1 to 0 of the fundamental flow created by adding $(5,7)$ to T_{11} is already described by the 1-chains $i_n-1 \wedge i_n+1 = 5 \wedge 7$, $n-4 \wedge n-2 = 7 \wedge 9$, $n-2 \wedge n-1 = 9 \wedge 10$ and $n-1 \wedge 0 = 10 \wedge 0$; the sum $\Sigma_0^{\frac{1}{8}(n-19)}(i_n+1 \mid {\scriptstyle +2,+1,+2,-1})$ has to stay zero here. A similar remark applies to the sum $\Sigma_0^{\frac{1}{8}(n-19)}(1 \mid {\scriptstyle +1,+2,-1,+2})$ in (ff.7).

□

In the following, we will use an abbreviation for the inner product on $\mathrm{C}_1(\mathrm{C}_n^{2^-})$: for every k and every 1-chain $c \in \mathrm{C}_1(\mathrm{C}_n^{2^-})$ we define

$$[c]_k := \langle k-1 \wedge k+1, c\rangle \; . \tag{2.26}$$

We have $i_n - 2 \wedge i_n = \frac{1}{2}(n-3) \wedge \frac{1}{2}(n+1)$, hence $\langle k-1 \wedge k+1, i_n-2 \wedge i_n\rangle$ if and only if $k = \frac{1}{2}(n-1) = i_n-1$. In that case, $\langle k-1 \wedge k+1, n-1 \wedge 0\rangle = \langle k-1 \wedge k+1, 0 \wedge 1\rangle = 0$. Of the four elementary 1-chains in each summand of the first sum in Lemma 56.(ff.7), only $i_n+4j \wedge i_n+2+4j$ and $i_n+4j+1 \wedge i_n+4j+3$, having vertex-difference two, can possibly result in a non-zero inner product with $k-1 \wedge k+1$. Because of $n \equiv 3 \pmod 4$, we know $k-1 = \frac{1}{2}(n-3)$ to be even, hence $[i_n+4j+1 \wedge i_n+4j+3]_k = 0$, while $[i_n+4j \wedge i_n+2+4j]_k = 1$ if and only if $k-1 =_n i_n-2 =_n i_n+4j$, i.e. $n-2 =_n 4j$, contradicting oddness of n. Thus, if $k = i_n - 1$, then $k-1 \wedge k+1$ has zero inner product with the first sum in Lemma 56.(ff.7).

Lemma 57 (some inner products with 1-chains of the form $\Sigma_a^b(v \mid {\scriptstyle c_1, c_2, c_3, c_4})$ appearing in Lemma 56)**.** *If $n \equiv 3 \pmod 8$ and $k \in [n-1]_0$, and with $[\cdot]_k$ as in* (2.26)*, and with the notation $\Sigma_a^b(v \mid {\scriptstyle c_1, c_2, c_3, c_4})$ from Definition 55,*

(1) $[\Sigma_a^b(1 \mid {\scriptstyle +1,+2,-1,+2})]_k = 1$
if ($(k \equiv 0 \pmod 4)$ and $a \leqslant \frac{1}{4}(k-4) \leqslant b$)
or ($k \equiv 3 \pmod 4$ and $a \leqslant \frac{1}{4}(k-3) \leqslant b$)),
while it is $= 0$ otherwise,

(2) $[\Sigma_a^b(i_n \mid {\scriptstyle +2,-1,+2,+1})]_k = 1$
if (($k - \frac{1}{2}(n+3) \equiv 1 \pmod 4$ and $a \leqslant \frac{1}{4}(k - \frac{1}{2}n - \frac{5}{2}) \leqslant b$)
or ($k - \frac{1}{2}(n+3) \equiv 0 \pmod 4$ and $a \leqslant \frac{1}{4}(k - \frac{1}{2}n - \frac{3}{2}) \leqslant b$)),
while it is $= 0$ otherwise,

(3) $[\Sigma_a^b(i_n+1 \mid {\scriptstyle +2,+1,+2,-1})]_k = 1$
if (($k - \frac{1}{2}(n+3) \equiv 1 \pmod 4$ and $a \leqslant \frac{1}{4}(k - \frac{1}{2}n - \frac{5}{2}) \leqslant b$)
or ($k - \frac{1}{2}(n+3) \equiv 0 \pmod 4$ and $a \leqslant \frac{1}{4}(k - \frac{1}{2}n - \frac{11}{2}) \leqslant b$)),
while it is $= 0$ otherwise.

Proof. For the inner products we use the abbreviation from (2.26). Since the 1-chain $k-1 \wedge k+1$ has vertex-difference 2, we know for label-difference-reasons alone,

(s.1) $[\Sigma_a^b(1 \mid {\scriptstyle +1,+2,-1,+2})]_k = \sum_a^b \, [(2+4j) \wedge (4+4j)]_k + [(3+4j) \wedge (5+4j)]_k$,

(s.2) $[\Sigma_a^b(i_n \mid {\scriptstyle +2,-1,+2,+1})]_k = \sum_a^b \, [(i_n+4j) \wedge (i_n+4j+2)]_k + [(i_n+4j+1) \wedge (i_n+4j+3)]_k$,

(s.3) $[\Sigma_a^b(i_n+1 \mid {\scriptstyle +2,+1,+2,-1})]_k$
$= \sum_a^b \, [(i_n+4j+1) \wedge (i_n+4j+3)]_k + [(i_n+4j+4) \wedge (i_n+4j+6)]_k$.

We can now combine (s.1)–(s.3) with Lemma 61 to deduce the conditions given in (1)–(3).

The conditions $k \equiv 0 \pmod 4$ and $k \equiv 3 \pmod 4$, which according to (1) and (2) in Lemma 61, decide if and when a summand in (s.1) vanishes, are mutually exclusive; this immediately translates into the condition given in (1).

The conditions $k - \frac{1}{2}(n+3) \equiv 0 \pmod 4$ and $k - \frac{1}{2}(n+3) \equiv 1 \pmod 4$, which according to (3) and (4) in Lemma 61, decide if and when a summand in (s.2) vanishes, are mutually exclusive; this immediately translates into the condition given in (2).

The conditions $k - \frac{1}{2}(n+3) \equiv 1 \pmod 4$ and $k - \frac{1}{2}(n+3) \equiv 0 \pmod 4$, which according to (4) and (5) in Lemma 61, decide if and when a summand in (s.3) vanishes, are mutually exclusive; this immediately translates into the condition given in (3). $\square$

Lemma 58 (values of the inner product of $k-1 \wedge k+1$ with $\mathrm{f}_{\mathrm{T}_n}(i_n - 2, i_n)$)**.** *With* $\mathrm{f}_{\mathrm{T}_n}(i_n - 2, i_n)$ *as in* (ff.7) *of Lemma 56, if* $0 \leqslant k \leqslant n-1$ *and* $n \equiv 3 \pmod 8$,

(1) *if* $k - \frac{1}{2}(n+3) \equiv 0 \pmod 4$, *then* $\langle k-1 \wedge k+1, \mathrm{f}_{\mathrm{T}_n}(i_n - 2, i_n)\rangle = 1$, *without exceptions,*

(2) *if* $k - \frac{1}{2}(n+3) \equiv 1 \pmod 4$, *then* $\langle k-1 \wedge k+1, \mathrm{f}_{\mathrm{T}_n}(i_n - 2, i_n)\rangle = 1$, *except when* $k = \frac{1}{2}(n-3)$, *which implies* $\langle k-1 \wedge k+1, \mathrm{f}_{\mathrm{T}_n}(i_n - 2, i_n)\rangle = 0$,

(3) *if* $k - \frac{1}{2}(n+3) \equiv 2 \pmod 4$, *then* $\langle k-1 \wedge k+1, \mathrm{f}_{\mathrm{T}_n}(i_n - 2, i_n)\rangle = 0$, *except when* $k = \frac{1}{2}(n-1)$, *which implies* $\langle k-1 \wedge k+1, \mathrm{f}_{\mathrm{T}_n}(i_n - 2, i_n)\rangle = 1$,

(4) *if* $k - \frac{1}{2}(n+3) \equiv 3 \pmod 4$, *then* $\langle k-1 \wedge k+1, \mathrm{f}_{\mathrm{T}_n}(i_n - 2, i_n)\rangle = 0$, *without exceptions.*

Proof. In general, by (ff.7), and using the abbreviation from (2.26),

$$\begin{aligned} \langle k-1 \wedge k+1, \mathrm{f}_{\mathrm{T}_n}(i_n - 2, i_n)\rangle &= [i_n - 2 \wedge i_n]_k + [\Sigma_0^{\frac{1}{8}(n-11)}(i_n \mid {\scriptstyle +2,-1,+2,+1})]_k \\ &\quad + [n-1 \wedge 0]_k + [0 \wedge 1]_k + [\Sigma_0^{\frac{1}{8}(n-19)}(1 \mid {\scriptstyle +1,+2,-1,+2})]_k \\ &\quad + [i_n - 5 \wedge i_n - 4]_k + [i_n - 4 \wedge i_n - 2]_k \\ &= [i_n - 2 \wedge i_n]_k + [i_n - 4 \wedge i_n - 2]_k \\ &\quad + [\Sigma_0^{\frac{1}{8}(n-11)}(i_n \mid {\scriptstyle +2,-1,+2,+1})]_k + [\Sigma_0^{\frac{1}{8}(n-19)}(1 \mid {\scriptstyle +1,+2,-1,+2})]_k\ , \end{aligned} \tag{2.27}$$

where for the latter equality we used that for reasons of vertex-label-differences alone, $[n-1 \wedge 0]_k = [0 \wedge 1]_k = [i_n - 5 \wedge i_n - 4]_k = 0$ for any k. In each of the cases (1)–(4) we will determine the four relevant values of the summands in (2.27) and thus compute the value of $\langle k-1 \wedge k+1, \mathrm{f}_{\mathrm{T}_n}(i_n - 2, i_n)\rangle$.

As to (1), let us first mention that because of $\frac{1}{2}(n-5) - \frac{1}{2}(n+3) = -4 \equiv 0 \pmod 4$, the case $k = \frac{1}{2}(n-5)$, which prompts the following case-analysis, can indeed occur. That the value 1 for $\langle k-1 \wedge k+1, \mathrm{f}_{\mathrm{T}_n}(i_n - 2, i_n)\rangle$ is arrived at via three different calculations in (2.28), (2.29) and (2.30) shows that the following case analysis, despite the three identical results, is in some sense necessary:

((1).i) if $k - \frac{1}{2}(n+3) \equiv 0 \pmod 4$ and $k < \frac{1}{2}(n-5)$, then $[\mathrm{f}_{\mathrm{T}_n}(i_n - 2, i_n)]_k = 1$,

((1).ii) if $k - \frac{1}{2}(n+3) \equiv 0 \pmod 4$ and $k = \frac{1}{2}(n-5)$, then $[\mathrm{f}_{\mathrm{T}_n}(i_n - 2, i_n)]_k = 1$,

((1).iii) if $k - \frac{1}{2}(n+3) \equiv 0 \pmod 4$ and $k > \frac{1}{2}(n-5)$, then $[\mathrm{f}_{\mathrm{T}_n}(i_n - 2, i_n)]_k = 1$.

For ((1).i), assume $k - \frac{1}{2}(n+3) \equiv 0 \pmod 4$ and $k < \frac{1}{2}(n-5)$. Then $[i_n - 2 \wedge i_n]_k = 0$ because of $k < \frac{1}{2}(n-5) < \frac{1}{2}(n-1)$ and Lemma 60.(2). Further, $[i_n - 4 \wedge i_n - 2]_k = 0$ because of $k < \frac{1}{2}(n-5)$ and Lemma 60.(3). The assumption $k < \frac{1}{2}(n-5)$ implies $\frac{1}{4}(k - \frac{1}{2}n - \frac{3}{2}) < -1$, which is enough to know that both disjunctions in Lemma 57.(2) (with $a := 0$ and $b := \frac{1}{8}(n-11)$) are false, hence $[\Sigma_0^{\frac{1}{8}(n-11)}(i_n \mid {\scriptstyle +2,-1,+2,+1})]_k = 0$. For the summand $[\Sigma_0^{\frac{1}{8}(n-19)}(1 \mid {\scriptstyle +1,+2,-1,+2})]_k$ we use Lemma 57.(1) with $a = 0$ and $b := \frac{1}{8}(n-19)$, which we now show to be possible. First we note that, given our current assumptions, what we already know about the size of k, i.e., $k < \frac{1}{2}(n-5)$, can be strengthened to $k \leqslant \frac{1}{2}(n-13)$: for every $\iota \in \{1,2,3\}$, if $k = \frac{1}{2}(n-5) - \iota$, then $k - \frac{1}{2}(n+3) = \iota \not\equiv 0 \pmod 4$, contradicting the assumption about $k - \frac{1}{2}(n+3)$ in (1). Therefore, $k < \frac{1}{2}(n-5)$ implies $k < \frac{1}{2}(n-5) - 3 = \frac{1}{2}(n-11)$, hence, by integrality, $k \leqslant \frac{1}{2}(n-13)$. We now know that $\frac{1}{4}(k-4) \leqslant \frac{1}{4}(\frac{1}{2}(n-13)-4) = \frac{1}{8}(n-21) < \frac{1}{8}(n-19)$, hence, with $a = 0$ and $b := \frac{1}{8}(n-19)$, the first clause of the disjunction in Lemma 57.(2) is true, and it follows that $[\Sigma_0^{\frac{1}{8}(n-19)}(1 \mid {\scriptstyle +1,+2,-1,+2})]_k = 1$. Having worked out the four inner products, we can now prove ((1).i) by the calculation (along the last line of (2.27)),

$$\langle k-1 \wedge k+1, \mathrm{f}_{\mathrm{T}_n}(i_n - 2, i_n)\rangle = 0+0+0+1 = 1\ . \tag{2.28}$$

For ((1).ii), assume $k - \frac{1}{2}(n+3) \equiv 0 \pmod 4$ and $k = \frac{1}{2}(n-5)$. Then $[i_n - 2 \wedge i_n]_k = 0$ because of $k = \frac{1}{2}(n-5) \neq \frac{1}{2}(n-1)$ and Lemma 60.(2). Further, $[i_n - 4 \wedge i_n - 2]_k = 1$ by Lemma 60.(3). Moreover, $k = \frac{1}{2}(n-5)$ implies $\frac{1}{4}(k - \frac{1}{2}n - \frac{3}{2}) = -1 < 0$, which by itself suffices to make both clauses of the disjunction in Lemma 57.(2) (with $a := 0$ and $b := \frac{1}{8}(n-11)$) false, hence $[\Sigma_0^{\frac{1}{8}(n-11)}(i_n \mid {\scriptstyle +2,-1,+2,+1})]_k = 0$. Finally, $k = \frac{1}{2}(n-5)$ implies $\frac{1}{4}(k-3) = \frac{1}{4}(\frac{1}{2}(n-5)-3) = \frac{1}{8}(n-11) > \frac{1}{8}(n-19)$, hence when in Lemma 57.(1) we set $a := 0$ and $b := \frac{1}{8}(n-19)$, as we have to in order to determine $[\Sigma_0^{\frac{1}{8}(n-19)}(1 \mid {\scriptstyle +1,+2,-1,+2})]_k$, then the second clause in the disjunction is false. Since the first clause is false by $k \not\equiv 0 \pmod 4$, it follows that $[\Sigma_0^{\frac{1}{8}(n-19)}(1 \mid {\scriptstyle +1,+2,-1,+2})]_k = 0$. We can now prove ((1).ii) by calculating, along the last line of (2.27),

$$\langle k-1 \wedge k+1, \mathrm{f}_{\mathrm{T}_n}(i_n - 2, i_n)\rangle = 0+1+0+0 = 1\ . \tag{2.29}$$

For ((1).iii), assume $k - \frac{1}{2}(n+3) \equiv 0 \pmod 4$ and $k > \frac{1}{2}(n-5)$. Since $\frac{1}{2}(n-1) - \frac{1}{2}(n+3) = -2 \equiv 2 \not\equiv 0 \pmod 4$, the unique possibility for $[i_n - 2 \wedge i_n] = 1$ mentioned in Lemma 60.(2), namely $k = \frac{1}{2}(n-1)$, cannot occur under our present assumption of $k - \frac{1}{2}(n+3) \equiv 0 \pmod 4$, although it does not conflict with the assumption $k > \frac{1}{2}(n-5)$; therefore, $[i_n - 2 \wedge i_n]_k = 0$. Moreover, $[i_n - 4 \wedge i_n - 2]_k = 0$ because of Lemma 60.(3) and $k > \frac{1}{2}(n-5)$. From $k > \frac{1}{2}(n-5)$ it follows that $\frac{1}{4}(k - \frac{1}{2}n - \frac{3}{2}) > \frac{1}{4}(\frac{1}{2}(n-5) - \frac{1}{2}n - \frac{3}{2}) = -1$, i.e. $\frac{1}{4}(k - \frac{1}{2}n - \frac{3}{2}) \geqslant 0$. We now need to apply Lemma 57.(2) with $a := 0$ and $b := \frac{1}{8}(n-11)$, and for the clause containing $k - \frac{1}{2}(n+3) \equiv 0 \pmod 4$ we need $\frac{1}{4}(k - \frac{1}{2}n - \frac{3}{2}) \leqslant \frac{1}{8}(n-11)$. For this our assumption $k \leqslant n-1$ is not enough (on the face of it, it only implies $\frac{1}{4}(k - \frac{1}{2}n - \frac{3}{2}) \leqslant \frac{1}{4}(\frac{1}{2}(n-11)))$, but taking our divisibility assumption into account we can strengthen the upper bound: for every $\iota \in \{0,1,2\}$, if $k = n-1-\iota$, then, using that $n = 8\mu+3$ with $\mu \in \mathbb{N}$, $k - \frac{1}{2}(n+3) = 4\mu - 1 - \iota \equiv 3 - \iota \not\equiv 0 \pmod 4$, contradicting $k - \frac{1}{2}(n+3) \equiv 0 \pmod 4$. Thus we know that $k \leqslant n-1-3 = n-4$, and this implies $\frac{1}{4}(k - \frac{1}{2}n - \frac{3}{2}) \leqslant \frac{1}{4}(n - 4 - \frac{1}{2}n - \frac{3}{2}) = \frac{1}{8}(n-11)$. Now we can use Lemma 57.(2) with $a := 0$ and $b := \frac{1}{8}(n-11)$ to conclude $[\Sigma_0^{\frac{1}{8}(n-11)}(i_n \mid {\scriptstyle +2,-1,+2,+1})]_k = 1$. For $[\Sigma_0^{\frac{1}{8}(n-19)}(1 \mid {\scriptstyle +1,+2,-1,+2})]_k$ we have to use Lemma 57.(1) with $a := 0$ and $b := \frac{1}{8}(n-19)$: however, the assumption $k > \frac{1}{2}(n-5)$ implies $\frac{1}{4}(k-4) > \frac{1}{4}(\frac{1}{2}(n-5)-4) = \frac{1}{8}(n-13) > \frac{1}{8}(n-19)$, which by itself suffices to make both clauses of the disjunction false. Therefore, in the present case, $[\Sigma_0^{\frac{1}{8}(n-19)}(1 \mid {\scriptstyle +1,+2,-1,+2})]_k = 0$. Now we can prove ((1).iii) by calculating, along the last line of (2.27),

$$\langle k-1 \wedge k+1, \mathrm{f}_{\mathrm{T}_n}(i_n - 2, i_n)\rangle = 0+0+1+0 = 1\ , \tag{2.30}$$

completing the proof of ((1).iii).

As to (2), let us first mention that because of $\frac{1}{2}(n-3) - \frac{1}{2}(n+3) = -3 \equiv 1 \pmod 4$, the case $k = \frac{1}{2}(n-3)$, around which revolves the following case analysis, can indeed occur.

((2).i) if $k - \frac{1}{2}(n+3) \equiv 1 \pmod 4$ and $k < \frac{1}{2}(n-3)$, then $[\mathrm{f}_{\mathrm{T}_n}(i_n - 2, i_n)]_k = 1$,
((2).ii) if $k - \frac{1}{2}(n+3) \equiv 1 \pmod 4$ and $k = \frac{1}{2}(n-3)$, then $[\mathrm{f}_{\mathrm{T}_n}(i_n - 2, i_n)]_k = 0$,
((2).iii) if $k - \frac{1}{2}(n+3) \equiv 1 \pmod 4$ and $k > \frac{1}{2}(n-3)$, then $[\mathrm{f}_{\mathrm{T}_n}(i_n - 2, i_n)]_k = 1$.

In each of the cases ((2).i)–((2).iii), we now evaluate the summands in the last line of (2.27).

For ((2).i), we assume $k - \frac{1}{2}(n+3) \equiv 1 \pmod 4$ and $k < \frac{1}{2}(n-3)$. Then $[i_n - 2 \wedge i_n]_k = 0$ because of $k < \frac{1}{2}(n-3) < \frac{1}{2}(n-1)$ and Lemma 60.(2). The possibility for $[i_n - 4 \wedge i_n - 2]_k = 1$ given in Lemma 60.(3), i.e. $k = \frac{1}{2}(n-5)$, is not ruled out by $k < \frac{1}{2}(n-3)$, but is ruled out by our other current assumption $k - \frac{1}{2}(n+3) \equiv 1 \pmod 4$. Thus, $[i_n - 4 \wedge i_n - 2]_k = 0$ by Lemma 60.(3). The assumption $k < \frac{1}{2}(n-3)$ implies $\frac{1}{4}(k - \frac{1}{2}n - \frac{3}{2}) < \frac{1}{4}(\frac{1}{2}(n-3) - \frac{1}{2}n - \frac{3}{2}) = -\frac{3}{4}$, which is enough to know that both disjunctions in Lemma 57.(2) are false (with $a := 0$ and $b := \frac{1}{8}(n-11)$, we would in particular have needed $\frac{1}{4}(k - \frac{1}{2}n - \frac{3}{2})$ to be nonnegative), hence $[\Sigma_0^{\frac{1}{8}(n-11)}(i_n \mid {\scriptstyle +2,\,-1,\,+2,\,+1})]_k = 0$. For the summand $[\Sigma_0^{\frac{1}{8}(n-19)}(1 \mid {\scriptstyle +1,\,+2,\,-1,\,+2})]_k$ we use Lemma 57.(2) with $a = 0$ and $b := \frac{1}{8}(n-19)$, which we now show to be possible. First we note that, given our current assumptions, what we already know about the size of k, i.e., $k < \frac{1}{2}(n-3)$, can be strengthened to $k \leqslant \frac{1}{2}(n-11)$, for the following reasons: for every $\iota \in \{1,2,3\}$, if $k = \frac{1}{2}(n-3) - \iota$, then $k - \frac{1}{2}(n+3) = -3 - \iota \equiv 1 - \iota \not\equiv 1 \pmod 4$, contradicting the assumption $k - \frac{1}{2}(n+3) \equiv 1 \pmod 4$ in (2). Therefore, $k < \frac{1}{2}(n-3)$ in fact implies $k < \frac{1}{2}(n-9)$, hence, by integrality, $k \leqslant \frac{1}{2}(n-11)$. We now know that $\frac{1}{4}(k-4) \leqslant \frac{1}{4}(\frac{1}{2}(n-11) - 4) = \frac{1}{8}(n-19)$ hence, with $a = 0$ and $b := \frac{1}{8}(n-19)$, the first clause of the disjunction in Lemma 57.(1) is true, and it follows that $[\Sigma_0^{\frac{1}{8}(n-19)}(1 \mid {\scriptstyle +1,\,+2,\,-1,\,+2})]_k = 1$. We can now prove ((2).i) by the calculation (along the last line of (2.27)),

$$\langle k-1 \wedge k+1, \mathrm{f}_{\mathrm{T}_n}(i_n - 2, i_n)\rangle = 0 + 0 + 0 + 1 = 1 \ . \tag{2.31}$$

For ((2).ii), we assume $k = \frac{1}{2}(n-3)$. Then $[i_n - 2 \wedge i_n]_k = 0$ (resp. $[i_n - 4 \wedge i_n - 2]_k = 0$) by Lemma 60.(2) (resp. Lemma 60.(3)). Moreover, $k = \frac{1}{2}(n-3)$ implies $\frac{1}{4}(k - \frac{1}{2}n - \frac{3}{2}) = \frac{1}{4}(\frac{1}{2}(n-3) - \frac{1}{2}n - \frac{3}{2}) = -\frac{3}{4} < 0$, thus $\frac{1}{4}(k - \frac{1}{2}n - \frac{5}{2}) < 0$, too, hence when setting $a := 0$ and $b := \frac{1}{8}(n-11)$ in Lemma 57.(2), as we *have* to in order to determine $[\Sigma_0^{\frac{1}{8}(n-11)}(i_n \mid {\scriptstyle +2,\,-1,\,+2,\,+1})]_k$, both clauses of the disjunction there are false, hence $[\Sigma_0^{\frac{1}{8}(n-11)}(i_n \mid {\scriptstyle +2,\,-1,\,+2,\,+1})]_k = 0$. Finally, $k = \frac{1}{2}(n-3)$ and $n \equiv 3 \pmod 8$ implies $k \equiv 0 \pmod 4$, hence we can determine $[\Sigma_0^{\frac{1}{8}(n-19)}(1 \mid {\scriptstyle +1,\,+2,\,-1,\,+2})]_k = 1$ via Lemma 57.(1); Because of $k = \frac{1}{2}(n-3)$ we have $\frac{1}{4}(k-4) = \frac{1}{4}(\frac{1}{2}(n-3) - 4) = \frac{1}{8}(n-11) \not\leqslant \frac{1}{8}(n-19)$, hence with $a := 0$ and $b := \frac{1}{8}(n-19)$ both clauses of Lemma 57.(1) are false. In this case we found all four summands in the last line of (2.27) to be zero, proving ((2).ii).

For ((2).iii), we assume $k - \frac{1}{2}(n+3) \equiv 1 \pmod 4$ and $k > \frac{1}{2}(n-3)$. Because of $k - \frac{1}{2}(n+3) \equiv 1 \pmod 4$, the assumption $k > \frac{1}{2}(n-3)$ can be strengthened: for every $\iota \in \{1,2,3\}$, if $k = \frac{1}{2}(n-3)+\iota$, then $k - \frac{1}{2}(n+3) = -3+\iota \equiv 1+\iota \not\equiv 1 \pmod 4$, contradicting $k - \frac{1}{2}(n+3) \equiv 1 \pmod 4$. Thus we know that $k \geqslant \frac{1}{2}(n-3)+4 = \frac{1}{2}(n+5)$. Then $[i_n - 2 \wedge i_n]_k = 0$ because of $\frac{1}{2}(n+5) > \frac{1}{2}(n-1)$ and Lemma 60.(2), $[i_n - 4 \wedge i_n - 2]_k = 0$ because of $\frac{1}{2}(n+5) > \frac{1}{2}(n-5)$ and Lemma 60.(3).

We now again have to strengthen the bound $k \leqslant n-1$: for every $\iota \in \{0,1\}$, if $k = n - 1 - \iota$, then, using $n = 8\mu + 3$ with $\mu \in \mathbb{N}$, $k - \frac{1}{2}(n+3) = 4\mu - 1 - \iota \equiv 3 - \iota \not\equiv 1 \pmod 4$, contradicting the current assumption $k - \frac{1}{2}(n+3) \equiv 1 \pmod 4$. Thus we know that $k \leqslant n - 1 - 2 = n - 3$, and this implies $\frac{1}{4}(k - \frac{1}{2}n - \frac{5}{2}) \leqslant \frac{1}{8}(n-11)$. Moreover, $k \geqslant \frac{1}{2}(n+5)$ implies $0 = \leqslant \frac{1}{4}(k - \frac{1}{2}n - \frac{5}{2})$. Now we know exactly what is sufficient to make the first clause of Lemma 57.(2) with $a := 0$ and $b := \frac{1}{8}(n-11)$ come true and it follows that $[\Sigma_0^{\frac{1}{8}(n-11)}(i_n \mid {\scriptstyle +2,\,-1,\,+2,\,+1})]_k = 1$. Since we know that $k \geqslant \frac{1}{2}(n+5)$, and therefore $\frac{1}{4}(k-4) \geqslant \frac{1}{4}(\frac{1}{2}(n+5) - 4) = \frac{1}{8}(n-3) > \frac{1}{8}(n-19)$, both clauses in Lemma 57.(1) with $a := 0$ and $b := \frac{1}{8}(n-19)$ are false, hence $[\Sigma_0^{\frac{1}{8}(n-19)}(1 \mid {\scriptstyle +1,\,+2,\,-1,\,+2})]_k = 0$. We can now prove ((2).iii) by the calculation (along the last line of (2.27)),

$$\langle k-1 \wedge k+1, \mathrm{f}_{\mathrm{T}_n}(i_n - 2, i_n)\rangle = 0 + 0 + 1 + 0 = 1 \ . \tag{2.32}$$

This completes the proof of (2).

As to (3) and (4) we first note that, independently of the case analysis that we will make for (3),

> both in case (3) and in case (4) each of the permitted k satisfies $[\Sigma_0^{\frac{1}{8}(n-11)}(i_n \mid {\scriptstyle +2,\,-1,\,+2,\,+1})]_k = [\Sigma_0^{\frac{1}{8}(n-19)}(1 \mid {\scriptstyle +1,\,+2,\,-1,\,+2})]_k = 0.$ (2.33)

The reasons for (2.33) are that, firstly, either of the assumptions $k - \frac{1}{2}(n+3) \equiv 2 \pmod 4$ in (3) and $k - \frac{1}{2}(n+3) \equiv 3 \pmod 4$ in (3) immediately implies that both clauses in the disjunction in

Lemma 57.(2) (with $a := 0$ and $b := \frac{1}{8}(n-11)$) are false, and, secondly, together with the hypothesis $n \equiv 3 \pmod 8$ these assumptions imply $k \equiv 1 \pmod 4$ or $k \equiv 2 \pmod 4$, respectively, so both clauses in Lemma 57.(1) are false, too.

As to (3), let us note that because of $\frac{1}{2}(n-1) - \frac{1}{2}(n+3) = -2 \equiv 2$, the exception $k = \frac{1}{2}(n-1)$ mentioned in (3), around which we structure the following case-disctinction, can occur:

((3).i) if $k - \frac{1}{2}(n+3) \equiv 2 \pmod 4$ and $k < \frac{1}{2}(n-1)$, then $[\mathrm{f}_{\mathrm{T}_n}(i_n-2, i_n)]_k = 0$,
((3).ii) if $k - \frac{1}{2}(n+3) \equiv 2 \pmod 4$ and $k = \frac{1}{2}(n-1)$, then $[\mathrm{f}_{\mathrm{T}_n}(i_n-2, i_n)]_k = 1$,
((3).iii) if $k - \frac{1}{2}(n+3) \equiv 2 \pmod 4$ and $k > \frac{1}{2}(n-1)$, then $[\mathrm{f}_{\mathrm{T}_n}(i_n-2, i_n)]_k = 0$.

In each of the cases ((3).i)–((3).iii), we now evaluate the summands in the last line of (2.27).

For ((3).i), we assume $k - \frac{1}{2}(n+3) \equiv 2 \pmod 4$ and $k < \frac{1}{2}(n-1)$. Then $[i_n - 2 \wedge i_n]_k = 0$ because of $k < \frac{1}{2}(n-1)$ and Lemma 60.(2). The possibility for $[i_n - 4 \wedge i_n - 2]_k = 1$ given in Lemma 60.(3), i.e. $k = \frac{1}{2}(n-5)$, is not ruled out by $k < \frac{1}{2}(n-1)$, but is ruled out by our other current assumption $k - \frac{1}{2}(n+3) \equiv 2 \pmod 4$, since $\frac{1}{2}(n-5) - \frac{1}{2}(n+3) = -8 \equiv 0 \not\equiv 2 \pmod 4$. Thus, $[i_n - 4 \wedge i_n - 2]_k = 0$ by Lemma 60.(3).

So, in case ((3).i) we have, keeping in mind (2.33) for the fourth summand's value, found all four summands at the end of (2.27) to be zero, proving ((3).i).

For ((3).ii), we assume $k = \frac{1}{2}(n-1)$. Then $[i_n - 2 \wedge i_n]_k = 1$ by Lemma 60.(2), and $[i_n - 4 \wedge i_n - 2]_k = 0$ by Lemma 60.(3). Moreover, $k = \frac{1}{2}(n-1)$ implies $k - \frac{1}{2}(n+3) = -2 \equiv 2 \pmod 4$, making both clauses in Lemma 57.(2) impossible; therefore, by Lemma 57.(2) with $a := 0$ and $b := \frac{1}{8}(n-11)$, it follows that $[\Sigma_0^{\frac{1}{8}(n-11)}(i_n \mid {\scriptstyle +2,-1,+2,+1})]_k = 0$. We can now prove ((3).ii) by the calculation (along the last line of (2.27), and using (2.33)),

$$\langle k-1 \wedge k+1, \mathrm{f}_{\mathrm{T}_n}(i_n-2, i_n)\rangle = 1 + 0 + 0 + 0 = 1 \ . \tag{2.34}$$

For ((3).iii), we assume $k - \frac{1}{2}(n+3) \equiv 2 \pmod 4$ and $k > \frac{1}{2}(n-1)$. Then $[i_n - 2 \wedge i_n]_k = 0$ and $[i_n - 4 \wedge i_n - 2]_k = 0$, by Lemma 60.(2) and Lemma 60.(3). In view of (2.33), all four summands at the end of (2.27) have been found to be zero, proving ((3).iii). This completes the proof of (3).

As to (4), we assume $k - \frac{1}{2}(n+3) \equiv 3 \pmod 4$. Then $\frac{1}{2}(n-1) - \frac{1}{2}(n+3) = -2 \equiv 2 \not\equiv 3 \pmod 4$ implies $[i_n - 2 \wedge i_n]_k = 0$, while $\frac{1}{2}(n-5) - \frac{1}{2}(n+3) = -4 \equiv 0 \not\equiv 3 \pmod 4$ implies $[i_n - 4 \wedge i_n - 2]_k = 0$, each time by Lemma 60. In view of (2.33), we then know all summands in the last line of (2.27) to be zero, proving (4). $\square$

Lemma 59 (values of the inner product of $k-1 \wedge k+1$ with $\mathrm{f}_{\mathrm{T}_n}(i_n-1, i_n+1)$)**.** *With* $\mathrm{f}_{\mathrm{T}_n}(i_n-1, i_n+1)$ *as in* (ff.5) *of Lemma 56, if* $0 \leqslant k \leqslant n-1$ *and* $n \equiv 3 \pmod 8$*, then*

(1) *if* $k - \frac{1}{2}(n+3) \equiv 0 \pmod 4$*, then* $\langle k-1 \wedge k+1, \mathrm{f}_{\mathrm{T}_n}(i_n-1, i_n+1)\rangle = 1$*,*
except when $k = i_n + 1$*, which implies* $\langle k-1 \wedge k+1, \mathrm{f}_{\mathrm{T}_n}(i_n-1, i_n+1)\rangle = 0$*,*
(2) *if* $k - \frac{1}{2}(n+3) \equiv 1 \pmod 4$*, then* $\langle k-1 \wedge k+1, \mathrm{f}_{\mathrm{T}_n}(i_n-1, i_n+1)\rangle = 1$*,*
except when $k = 0$*, which implies* $\langle k-1 \wedge k+1, \mathrm{f}_{\mathrm{T}_n}(i_n-1, i_n+1)\rangle = 0$*,*
(3) *if* $k - \frac{1}{2}(n+3) \equiv 2 \pmod 4$*, then* $\langle k-1 \wedge k+1, \mathrm{f}_{\mathrm{T}_n}(i_n-1, i_n+1)\rangle = 0$*,*
without exceptions,
(4) *if* $k - \frac{1}{2}(n+3) \equiv 3 \pmod 4$*, then* $\langle k-1 \wedge k+1, \mathrm{f}_{\mathrm{T}_n}(i_n-1, i_n+1)\rangle = 0$*,*
except when $k = i_n$*, which implies* $\langle k-1 \wedge k+1, \mathrm{f}_{\mathrm{T}_n}(i_n-1, i_n+1)\rangle = 1$*.*

Proof. In general, using the abbreviation from (2.26),

$$\begin{aligned}\langle k-1 \wedge k+1, \mathrm{f}_{\mathrm{T}_n}(i_n-1, i_n+1)\rangle \overset{\text{(ff.5)}}{=} {} & [i_n-1 \wedge i_n+1]_k \quad + \quad [\Sigma_0^{\frac{1}{8}(n-19)}(i_n+1 \mid {\scriptstyle +2,+1,+2,-1})]_k \\ & + [n-4 \wedge n-2]_k \quad + \quad [n-2 \wedge n-1]_k \quad + \quad [n-1 \wedge 0]_k \\ & + [0 \wedge 1]_k + [\Sigma_0^{\frac{1}{8}(n-11)}(1 \mid {\scriptstyle +1,+2,-1,+2})]_k \\ = {} & [i_n-1 \wedge i_n+1]_k \quad + \quad [\Sigma_0^{\frac{1}{8}(n-19)}(i_n+1 \mid {\scriptstyle +2,+1,+2,-1})]_k \\ & + [n-4 \wedge n-2]_k \quad + \quad [\Sigma_0^{\frac{1}{8}(n-11)}(1 \mid {\scriptstyle +1,+2,-1,+2})]_k \ ,\end{aligned} \tag{2.35}$$

where the latter equality holds since independently of k, for reasons of vertex-label-differences alone, $[n-2 \wedge n-1]_k = [n-1 \wedge 0]_k = [0 \wedge 1]_k = 0$.

In each of the cases (1)–(4), we will determine the four relevant values of the summands in (2.35) and thus compute the value of $\langle k-1 \wedge k+1, \mathrm{f}_{\mathrm{T}_n}(i_n-1, i_n+1)\rangle$.

As to (1), we first note that the assumption $k - \frac{1}{2}(n+3) \equiv 0 \pmod 4$ that we make there implies both $[i_n - 1 \wedge i_n + 1]_k = 0$ and $[n-4 \wedge n-2]_k = 0$: by Lemma 60.(1) we know that $[i_n - 1 \wedge i_n + 1]_k = 1$ if and only if $k = \frac{1}{2}(n+1)$, but because of $n \equiv 3 \pmod 8$ this contradicts the assumption $k - \frac{1}{2}(n+3) \equiv 0 \pmod 4$; moreover, since $n = 8\mu + 3$, $\mu \in \mathbb{N}$, we have $[n-4 \wedge n-2]_k = 1$ if and only if $k = n-3$ if and only if $k - \frac{1}{2}(n+3) = \frac{1}{2}(n-9) = 4\mu - 3 \equiv 1 \not\equiv 0 \pmod 4$. Thus, (2.35) simplifies to

$$\begin{aligned} \langle k-1 \wedge k+1, \mathrm{f}_{\mathrm{T}_n}(i_n-1, i_n+1)\rangle = {} & [\Sigma_0^{\frac{1}{8}(n-19)}(i_n+1 \mid {\scriptstyle +2,+1,+2,-1})]_k \\ & + [\Sigma_0^{\frac{1}{8}(n-11)}(1 \mid {\scriptstyle +1,+2,-1,+2})]_k \ . \end{aligned} \tag{2.36}$$

Let us note that in the following we are not missing cases, neither between ((1).i) and ((1).ii), nor between ((1).ii) and ((1).iii): none of the six values of $k \in [n-1]_0$ which are not covered by $k \leqslant \frac{1}{2}(n-5)$, $k = \frac{1}{2}(n+3)$ and $k \geqslant (n+11)$ is consistent with the accompanying divisibility-assumption $k - \frac{1}{2}(n+3) \equiv 0 \pmod 4$.

((1).i) if $k - \frac{1}{2}(n+3) \equiv 0 \pmod 4$ and $k \leqslant \frac{1}{2}(n-5)$, then $[\mathrm{f}_{\mathrm{T}_n}(i_n-1, i_n+1)]_k = 1$,
((1).ii) if $k - \frac{1}{2}(n+3) \equiv 0 \pmod 4$ and $k = \frac{1}{2}(n+3)$, then $[\mathrm{f}_{\mathrm{T}_n}(i_n-1, i_n+1)]_k = 0$,
((1).iii) if $k - \frac{1}{2}(n+3) \equiv 0 \pmod 4$ and $k \geqslant \frac{1}{2}(n+11)$, then $[\mathrm{f}_{\mathrm{T}_n}(i_n-1, i_n+1)]_k = 1$.

For ((1).i), assume $k - \frac{1}{2}(n+3) \equiv 0 \pmod 4$ and $k \leqslant \frac{1}{2}(n-5)$. From $k \leqslant \frac{1}{2}(n-5)$ it follows that $\frac{1}{4}(k - \frac{1}{2}n - \frac{11}{2}) < 0$, hence the hypothesis of Lemma 57.(3) with $a := 0$ (a value which we *have* to set to evaluate $[\Sigma_0^{\frac{1}{8}(n-19)}(i_n+1 \mid {\scriptstyle +2,+1,+2,-1})]_k$) cannot be true, hence $[\Sigma_0^{\frac{1}{8}(n-19)}(i_n+1 \mid {\scriptstyle +2,+1,+2,-1})]_k = 0$. For the summand $[\Sigma_0^{\frac{1}{8}(n-11)}(1 \mid {\scriptstyle +1,+2,-1,+2})]_k$ we can use Lemma 57.(1) with $a := 0$ and $b := \frac{1}{8}(n-11)$, in particular since $k \leqslant \frac{1}{2}(n-5)$ implies $\frac{1}{4}(k-4) \leqslant \frac{1}{8}(n-13) < \frac{1}{8}(n-11) = b$, to conclude that $[\Sigma_0^{\frac{1}{8}(n-11)}(1 \mid {\scriptstyle +1,+2,-1,+2})]_k = 1$. Thus, by (2.36),

$$\langle k-1 \wedge k+1, \mathrm{f}_{\mathrm{T}_n}(i_n-1, i_n+1)\rangle = 0 + 1 = 1 \ , \tag{2.37}$$

proving ((1).i).

For ((1).ii), assume $k - \frac{1}{2}(n+3) \equiv 0 \pmod 4$ and $k = \frac{1}{2}(n+3)$. From $k = \frac{1}{2}(n+3)$ it follows that $\frac{1}{4}(k - \frac{1}{2}n - \frac{11}{2}) < 0$, hence the hypothesis of Lemma 57.(3) with $a := 0$ (a parameter that we *have* to set to evaluate $[\Sigma_0^{\frac{1}{8}(n-19)}(i_n+1 \mid {\scriptstyle +2,+1,+2,-1})]_k$) cannot be true, hence $[\Sigma_0^{\frac{1}{8}(n-19)}(i_n+1 \mid {\scriptstyle +2,+1,+2,-1})]_k = 0$. From $k = \frac{1}{2}(n+3)$ it follows that $\frac{1}{4}(k-4) = \frac{1}{8}(n-5) > \frac{1}{8}(n-11)$, hence the hypothesis of Lemma 57.(1) with $b := \frac{1}{8}(n-11)$ cannot be true, hence for $[\Sigma_0^{\frac{1}{8}(n-11)}(1 \mid {\scriptstyle +1,+2,-1,+2})]_k = 0$. Hence, by (2.36),

$$\langle k-1 \wedge k+1, \mathrm{f}_{\mathrm{T}_n}(i_n-1, i_n+1)\rangle = 0 + 0 = 0 \ , \tag{2.38}$$

proving ((1).ii).

For ((1).iii), assume $k - \frac{1}{2}(n+3) \equiv 0 \pmod 4$ and $k \geqslant \frac{1}{2}(n+11)$. For the summand $[\Sigma_0^{\frac{1}{8}(n-19)}(i_n+1 \mid {\scriptstyle +2,+1,+2,-1})]_k$ we use Lemma 57.(3) with $a := 0$ and $b := \frac{1}{8}(n-19)$, with its clause containing $k - \frac{1}{2}(n+3) \equiv 0 \pmod 4$; for this we need

(1) $\frac{1}{4}(k - \frac{1}{2}n - \frac{11}{2}) \leqslant \frac{1}{8}(n-19)$,
(2) $\frac{1}{4}(k - \frac{1}{2}n - \frac{11}{2}) \geqslant 0$.

Our default hypothesis $k \leqslant n-1$ alone is not enough for (1), but we can strengthen it, by taking the divisibility assumption into account: for every $\iota \in \{0,1,2\}$, if $k = n-1-\iota$, then, using $n = 8\mu + 3$

with $\mu \in \mathbb{N}$, $k - \frac{1}{2}(n+3) = \frac{1}{2}(n-5-2\iota) = 4\mu - 1 - \iota \equiv 3 - \iota \not\equiv 0$, contradicting the assumption. Therefore, we know that in fact $k \leqslant n-1-3 = n-4$, and this indeed implies $\frac{1}{4}(k - \frac{1}{2}n - \frac{11}{2}) \leqslant \frac{1}{8}(n-19)$. Moreover, $k \geqslant \frac{1}{2}(n+11)$ implies (2). Thus, we may indeed use Lemma 57.(3) to conclude $[\Sigma_0^{\frac{1}{8}(n-19)}(i_n + 1 \mid {\scriptstyle +2,+1,+2,-1})]_k = 1$. For the summand $[\Sigma_0^{\frac{1}{8}(n-11)}(1 \mid {\scriptstyle +1,+2,-1,+2})]_k$ we apply Lemma 57.(1); because of $k \geqslant \frac{1}{2}(n+11)$ we know $\frac{1}{4}(k-4) \geqslant \frac{1}{4}(\frac{1}{2}(n+11)-4) = \frac{1}{8}(n+3) > \frac{1}{8}(n-11)$, hence none of the clauses in the hypothesis of Lemma 57.(1) is true, hence $[\Sigma_0^{\frac{1}{8}(n-11)}(1 \mid {\scriptstyle +1,+2,-1,+2})]_k = 0$. So, by (2.36),

$$\langle k-1 \wedge k+1, \mathrm{f}_{\mathrm{T}_n}(i_n - 1, i_n + 1)\rangle = 1 + 0 = 1 \ , \tag{2.39}$$

proving ((1).iii). This completes the proof of (1).

As to (2), we first note that the assumptions $n \equiv 3 \pmod 8$ and $k - \frac{1}{2}(n+3) \equiv 1 \pmod 4$ imply $k \neq \frac{1}{2}(n+1)$, hence $[i_n - 1 \wedge i_n + 1]_k = 0$ by Lemma 60.(1). The summand $[n-4 \wedge n-2]_k$ in (2.35) can be non-zero—but only for one value of k. Thus, we get this one case out of the way first, so as to be able to calculate with (2.36) afterwards: we have $[n-4 \wedge n-2]_k = 1$ if and only if $k = n-3$, and for $k = n-3$ the size-conditions in the clauses of Lemma 57.(3) and Lemma 57.(1) then cannot hold, hence both $[\Sigma_0^{\frac{1}{8}(n-19)}(i_n+1 \mid {\scriptstyle +2,+1,+2,-1})]_k = 0$ and $[\Sigma_0^{\frac{1}{8}(n-11)}(1 \mid {\scriptstyle +1,+2,-1,+2})]_k = 0$. Thus, if $k = n-3$, then $\langle k-1 \wedge k+1, \mathrm{f}_{\mathrm{T}_n}(i_n - 1, i_n + 1)\rangle = [n-4 \wedge n-2]_k = 1$, in agreement with the claim in (2). So we may now assume that $k \in [n-1]_0$ and $k - \frac{1}{2}(n+3) \equiv 1 \pmod 4$, and $k \neq n-3$, so (2.36) holds, and all we have to work out are the values of $[\Sigma_0^{\frac{1}{8}(n-19)}(i_n + 1 \mid {\scriptstyle +2,+1,+2,-1})]_k$, and $[\Sigma_0^{\frac{1}{8}(n-11)}(1 \mid {\scriptstyle +1,+2,-1,+2})]_k$.

If $k = 0$, which is a case satisfying $k - \frac{1}{2}(n+3) \equiv 1 \pmod 4$, in Lemma 57.(1) with $a := 0$, the condition $a \leqslant \frac{1}{4}(k-4)$ does not hold, hence $[\Sigma_0^{\frac{1}{8}(n-11)}(1 \mid {\scriptstyle +1,+2,-1,+2})]_k = 0$. Since for $k = 0$, also $[\Sigma_0^{\frac{1}{8}(n-19)}(i_n + 1 \mid {\scriptstyle +2,+1,+2,-1})]_k = 0$, this proves the exception mentioned in (2). So we may assume $k \in [n-1]_0 \backslash \{0, n-3\}$ and $k - \frac{1}{2}(n+3) \equiv 1 \pmod 4$. Using Lemma 57.(3), by entirely analogous arguments as were sufficient for (1) we then find

(1) $[\Sigma_0^{\frac{1}{8}(n-19)}(i_n + 1 \mid {\scriptstyle +2,+1,+2,-1})]_k = 1$ and $[\Sigma_0^{\frac{1}{8}(n-11)}(1 \mid {\scriptstyle +1,+2,-1,+2})]_k = 0$
for every $k \in [n-1]_0 \backslash \{0, n-3\}$ with $k - \frac{1}{2}(n+3) \equiv 1 \pmod 4$ and $k \geqslant \frac{1}{2}(n+5)$,
(2) $[\Sigma_0^{\frac{1}{8}(n-19)}(i_n + 1 \mid {\scriptstyle +2,+1,+2,-1})]_k = 0$ and $[\Sigma_0^{\frac{1}{8}(n-11)}(1 \mid {\scriptstyle +1,+2,-1,+2})]_k = 1$
for every $k \in [n-1]_0 \backslash \{0, n-3\}$ with $k - \frac{1}{2}(n+3) \equiv 1 \pmod 4$ and $k \leqslant \frac{1}{2}(n-3)$.

No cases for k are missing in (1) and (2) since each of $k = \frac{1}{2}(n+5) - 1$, $k = \frac{1}{2}(n+5) - 2$ and $k = \frac{1}{2}(n+5) - 3 = \frac{1}{2}(n-1)$ are impossible in view of $k - \frac{1}{2}(n+3) \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 8$. This proves (2).

As to (3), we first analyse the assumptions $k - \frac{1}{2}(n+3) \equiv 2 \pmod 4$ and $n \equiv 3 \pmod 8$ in the light of Lemma 60.(1): with $\mu, \nu \in \mathbb{N}$ we have $n = 8\mu + 3$ and $k = \frac{1}{2}(n+3) + 2 + 4\nu = 4(\mu+\nu) + 5 \equiv 1 \pmod 4$, while $\frac{1}{2}(n+1) = 4\mu + 2 \equiv 2 \pmod 4$, so Lemma 60.(1) implies $[i_n - 1, i_n + 1]_k = 0$. Moreover, we have $n - 4 = 8\mu - 1 \equiv 3 \pmod 4$ while $k - 1 = 4(\mu+\nu) + 4 \equiv 0 \pmod 4$, hence $[n-4 \wedge n-2]_k \, \langle k-1 \wedge k+1, n-4 \wedge n-2\rangle = 0$ in (2.35). Thus, (2.35) simplifies to

$$\langle k-1 \wedge k+1, \mathrm{f}_{\mathrm{T}_n}(i_n - 1, i_n + 1)\rangle = [\Sigma_0^{\frac{1}{8}(n-19)}(i_n+1 \mid {\scriptstyle +2,+1,+2,-1})]_k + [\Sigma_0^{\frac{1}{8}(n-11)}(1 \mid {\scriptstyle +1,+2,-1,+2})]_k \ . \tag{2.40}$$

Since currently $k - \frac{1}{2}(n+3) \equiv 2 \pmod 4$, Lemma 57.(3) tells us $[\Sigma_0^{\frac{1}{8}(n-19)}(i_n+1 \mid {\scriptstyle +2,+1,+2,-1})]_k = 0$. Since currently $k \equiv 1 \pmod 4$, Lemma 57.(1) tells us $[\Sigma_0^{\frac{1}{8}(n-11)}(1 \mid {\scriptstyle +1,+2,-1,+2})]_k = 0$, too. Thus, $\langle k-1 \wedge k+1, \mathrm{f}_{\mathrm{T}_n}(i_n - 1, i_n + 1)\rangle = 0$, completing the proof of (3).

As to (4), we first note that if $k = i_n$, then indeed $\langle k-1 \wedge k+1, \mathrm{f}_{\mathrm{T}_n}(i_n - 1, i_n + 1)\rangle =$ (directly by (ff.5)) $= 1$. Now we assume, in addition to $k - \frac{1}{2}(n+3) \equiv 3 \pmod 4$, that $k \neq i_n$. Then in (2.35) we have $[i_n - 1 \wedge i_n + 1]_k = 0$. Moreover, $n - 4 = 8\mu - 1 \equiv 3 \pmod 4$, while $k = \frac{1}{2}(n+3) + 3 + 4\nu = 4(\mu+\nu) + 6 \equiv 2 \pmod 4$ implies $k - 1 \equiv 1 \pmod 4$, hence in (2.35) we have $[n-4 \wedge n-2]_k = 0$

and (2.35) simplifies again to (2.40). Now, $k - \frac{1}{2}(n+3) \equiv 3 \pmod 4$, so in (2.40) we have $[\Sigma_0^{\frac{1}{8}(n-19)}(i_n + 1 \mid {\scriptstyle +2,+1,+2,-1})]_k = 0$ by Lemma 57.(3). Moreover, we currently have $k \equiv 2$ (mod 4), hence in (2.40) we have $[\Sigma_0^{\frac{1}{8}(n-11)}(1 \mid {\scriptstyle +1,+2,-1,+2})]_k = 0$ by Lemma 57.(1), completing the proof of (4). □

Lemma 60 (some further values). *With* $[\cdot]_k := \langle k-1 \wedge k+1, \cdot\rangle$ *and* $i_n := \frac{1}{2}(n+1)$, *for every* $k \in [n-1]_0$,

(1) $[i_n - 1 \wedge i_n + 1]_k = 1$ *if* $k = \frac{1}{2}(n+1)$, *else* $= 0$,
(2) $[i_n - 2 \wedge i_n]_k = 1$ *if* $k = \frac{1}{2}(n-1)$, *else* $= 0$,
(3) $[i_n - 4 \wedge i_n - 2]_k = 1$ *if* $k = \frac{1}{2}(n-5)$, *else* $= 0$.

Proof. Statement (1) holds since $\{k-1, k+1\} = \{i_n - 1, i_n + 1\}$ if and only if $k = i_n = \frac{1}{2}(n+1)$. Statement (2) holds since $\{k-1, k+1\} = \{i_n - 2, i_n\}$ if and only if $k = \frac{1}{2}(n-1)$. Statement (3) holds since $\{k-1, k+1\} = \{i_n - 4, i_n - 2\}$ if and only if $k = \frac{1}{2}(n-5)$. □

Lemma 61 (values of inner products of $(k-1 \wedge k+1)$ with some elementary 1-chains appearing in the parametrised sums used in Lemma 56). *If* $i_n := \frac{1}{2}(n+1)$, $k \in [n-1]_0$, $j \in \mathbb{Z}$ *and* $n \equiv 3 \pmod 8$, *and with* $[\cdot]_k := \langle k-1 \wedge k+1, \cdot\rangle$,

(1) $[(2+4j) \wedge (4+4j)]_k = 1$ *if and only if* $k - 3 \equiv 0 \pmod 4$ *and* $j = \frac{1}{4}(k-3)$,
(2) $[(3+4j) \wedge (5+4j)]_k = 1$ *if and only if* $k \equiv 0 \pmod 4$ *and* $j = \frac{1}{4}(k-4)$,
(3) $[(i_n + 4j) \wedge (i_n + 4j + 2)]_k = 1$
if and only if $k - \frac{1}{2}(n+3) \equiv 0 \pmod 4$ *and* $j = \frac{1}{4}(k - \frac{1}{2}n - \frac{3}{2})$, *otherwise* $= 0$,
(4) $[(i_n + 4j + 1) \wedge (i_n + 4j + 3)]_k = 1$
if and only if $k - \frac{1}{2}(n+3) \equiv 1 \pmod 4$ *and* $j = \frac{1}{4}(k - \frac{1}{2}n - \frac{5}{2})$, *otherwise* $= 0$,
(5) $[(i_n + 4j + 4) \wedge (i_n + 4j + 6)]_k = 1$
if and only if $k - \frac{1}{2}(n+3) \equiv 0 \pmod 4$ *and* $j = \frac{1}{4}(k - \frac{1}{2}n - \frac{11}{2})$, *otherwise* $= 0$.

Proof. We use that the standard inner product of two elementary 1-chains is non-zero if and only if the two 2-sets of vertices in the elementary chains are equal. Statement (1) holds since $(k-1, k+1) = (2+4j, 4+4j)$ if and only if $k-3 = 4j$, (2) holds since $(k-1, k+1) = (3+4j, 5+4j)$ if and only if $k-4 = 4j$, (3) holds since $(k-1, k+1) = (i_n + 4j, i_n + 4j + 2)$ if and only if $k - 1 - i_n = k - \frac{1}{2}(n+3) = 4j$, (4) holds since $(k-1, k+1) = (i_n + 4j + 1, i_n + 4j + 3)$ if and only if $k - 1 - i_n = k - \frac{1}{2}(n+3) = 4j + 1$, (5) holds since $(k-1, k+1) = (i_n + 4j + 4, i_n + 4j + 6)$ if and only if $k - 5 - i_n = k - \frac{1}{2}(n+11) = 4j$, explaining the given value of j, and the divisibility condition holds since $k - 5 - i_n = k - \frac{1}{2}(n+11) = 4j$ is equivalent to $k - 1 - i_n = k - \frac{1}{2}(n+3) = 4(j+1)$. □

Lemma 62. *For every* $n \equiv 3 \pmod 4$, *every* $\ell \in \{0, 1, \ldots, \frac{1}{8}(n-3) - 2\}$ *and every* even $k \in [n-1]$, *and with* $\vec{C}_{i;n}$ *as in Definition 219,*

(1) *if* $3 + 2\ell \leqslant \lfloor \frac{1}{2}k \rfloor - 1$, *then* $\sum_{3+2\ell \leqslant i \leqslant \frac{1}{2}(n-3)} (-1)^{i-1} \cdot 2 \cdot \langle k-1 \wedge k+1, \vec{C}_{i;n}\rangle = 2$,
(2) *if* $3 + 2\ell \geqslant \lfloor \frac{1}{2}k \rfloor$, *then* $\sum_{3+2\ell \leqslant i \leqslant \frac{1}{2}(n-3)} (-1)^{i-1} \cdot 2 \cdot \langle k-1 \wedge k+1, \vec{C}_{i;n}\rangle = 0$,
(3) *if* $2 + 2\ell \leqslant \lfloor \frac{1}{2}k \rfloor - 1$, *then* $\sum_{2+2\ell \leqslant i \leqslant \frac{1}{2}(n-3)} (-1)^{i} \cdot 2 \cdot \langle k-1 \wedge k+1, \vec{C}_{i;n}\rangle = 0$.

Proof. We use the abbreviation $[c]_k := \langle k-1 \wedge k+1, c\rangle$, for every k and every 1-chain $c \in \mathrm{C}_1(\mathrm{C}_n^{2^-})$.

As to (1), we split the sum in (1) at the zero-summand $\langle k-1 \wedge k+1, \vec{C}_{\lfloor \frac{1}{2}k \rfloor;n}\rangle = 0$ and calculate as follows (the sum $\sum_{3+2\ell\leqslant i\leqslant\lfloor\frac{1}{2}k\rfloor-1}(-1)^{i-1}\cdot 2\cdot[\vec{C}_{i;n}]_k$ is non-empty because of $3+2\ell\leqslant\lfloor\frac{1}{2}k\rfloor-1$):

$$\begin{aligned}
\sum_{3+2\ell\leqslant i\leqslant\frac{1}{2}(n-3)}(-1)^{i-1}\cdot 2\cdot\left[\vec{C}_{i;n}\right]_k &= 2\cdot\left(\sum_{i=3+2\ell}^{\lfloor\frac{1}{2}k\rfloor-1}(-1)^{i-1}\cdot\left[\vec{C}_{i;n}\right]_k+\sum_{i=\lfloor\frac{1}{2}k\rfloor+1}^{\frac{1}{2}(n-3)}(-1)^{i-1}\cdot\left[\vec{C}_{i;n}\right]_k\right)\\
\text{(by Lemma 54.(5).(ii))} &= 2\cdot\left(\sum_{i=3+2\ell}^{\lfloor\frac{1}{2}k\rfloor-1}(-1)^{i-1}\cdot(+)+\sum_{i=\lfloor\frac{1}{2}k\rfloor+1}^{\frac{1}{2}(n-3)}(-1)^{i-1}\cdot(-)\right)\\
&= 2\cdot\sum_{3+2\ell\leqslant i\leqslant\lfloor\frac{1}{2}k\rfloor-1}(-1)^{i-1}+2\cdot\sum_{\lfloor\frac{1}{2}k\rfloor+1\leqslant i\leqslant\frac{1}{2}(n-3)}(-1)^{i}\\
&= 2\cdot\left(\sum_{\lfloor\frac{1}{2}k\rfloor+1\leqslant i\leqslant\frac{1}{2}(n-3)}(-1)^{i}-\sum_{3+2\ell\leqslant i\leqslant\lfloor\frac{1}{2}k\rfloor-1}(-1)^{i}\right)\\
&= 2\cdot\begin{cases}0-(-1) & \text{by (3) and (4) in Lemma 65}\\ & \text{if } \lfloor\frac{1}{2}k\rfloor \text{ is even}\\ (+)-0 & \text{by (1) and (2) in Lemma 65}\\ & \text{if } \lfloor\frac{1}{2}k\rfloor \text{ is odd}\end{cases} = 2\ .
\end{aligned}\tag{2.41}$$

As to (2), because of $3+2\ell\geqslant\lfloor\frac{1}{2}k\rfloor$ and $[\vec{C}_{\lfloor\frac{1}{2}k\rfloor;n}]_k=0$, we may assume that $3+2\ell\geqslant\lfloor\frac{1}{2}k\rfloor+1$, hence by Lemma 54.(5).(ii), every $[\vec{C}_{i;n}]_k$ in (2) equals -1, so

$$\begin{aligned}
\sum_{3+2\ell\leqslant i\leqslant\frac{1}{2}(n-3)}(-1)^{i-1}\cdot 2\cdot[\vec{C}_{i;n}]_k = 2\cdot\sum_{3+2\ell\leqslant i\leqslant\frac{1}{2}(n-3)}(-1)^{i-1}\cdot(-) &= 2\cdot\sum_{3+2\ell\leqslant i\leqslant\frac{1}{2}(n-3)}(-1)^{i}\\
\text{(by Lemma 65.(3))} &= 0\ .
\end{aligned}\tag{2.42}$$

As to (3), we split the sum at the zero-summand $[\vec{C}_{\lfloor\frac{1}{2}k\rfloor;n}]_k=0$ and calculate as follows (let us note that the sum $\sum_{2+2\ell\leqslant i\leqslant\lfloor\frac{1}{2}k\rfloor-1}(-1)^{i}\cdot 2\cdot[\vec{C}_{i;n}]_k$ is non-empty because of $2+2\ell\leqslant\lfloor\frac{1}{2}k\rfloor-1$):

$$\begin{aligned}
\sum_{2+2\ell\leqslant i\leqslant\frac{1}{2}(n-3)}(-1)^{i}\cdot 2\cdot[\vec{C}_{i;n}]_k &= \sum_{2+2\ell\leqslant i\leqslant\lfloor\frac{1}{2}k\rfloor-1}(-1)^{i}\cdot 2\cdot[\vec{C}_{i;n}]_k+\sum_{\lfloor\frac{1}{2}k\rfloor+1\leqslant i\leqslant\frac{1}{2}(n-3)}(-1)^{i}\cdot 2\cdot[\vec{C}_{i;n}]_k\\
\text{(by (5).(ii) in Lemma 54)} &= 2\cdot\sum_{2+2\ell\leqslant i\leqslant\lfloor\frac{1}{2}k\rfloor-1}(-1)^{i}\cdot(+)+2\cdot\sum_{\lfloor\frac{1}{2}k\rfloor+1\leqslant i\leqslant\frac{1}{2}(n-3)}(-1)^{i}\cdot(-)\\
&= 2\cdot\sum_{i=2+2\ell}^{\lfloor\frac{1}{2}k\rfloor-1}(-1)^{i+1}+2\cdot\sum_{i=\lfloor\frac{1}{2}k\rfloor+1}^{\frac{1}{2}(n-3)}(-1)^{i+1}\\
&= 2\cdot\left(\sum_{i=\lfloor\frac{1}{2}k\rfloor+1}^{\frac{1}{2}(n-3)}(-1)^{i+1}-\sum_{i=2+2\ell}^{\lfloor\frac{1}{2}k\rfloor-1}(-1)^{i+1}\right)\\
&= 2\cdot\begin{cases}0-(0) & \text{by (3) and (2) in Lemma 65 if } \lfloor\frac{1}{2}k\rfloor \text{ is even}\\ (-)-(-) & \text{by (1) in Lemma 65 if } \lfloor\frac{1}{2}k\rfloor \text{ is odd}\end{cases}\\
&= 0\ .
\end{aligned}\tag{2.43}$$

□

The following simple observation will save us some work with the calculations to come:

Lemma 63. *If $G=(V,E)$ is a graph, $z_1,z_2\in\mathrm{Z}_1(G)$, and if $E'\subseteq E$ is any subset such that $(V,E\backslash E')$ is a forest, then, with $\langle\cdot,\cdot\rangle$ the standard bilinear form on $\mathrm{C}_1(G)\subseteq\mathrm{Z}_1(G)$,*

$$\langle e',z_1\rangle=\langle e',z_2\rangle \text{ for every } e'\in E' \quad\Longrightarrow\quad z_1=z_2\ . \tag{2.44}$$

Proof of Lemma 63. Let z_1, z_2 and E' be given as stated, and suppose they satisfy the hypothesis of (2.44). From $z_1, z_2 \in \mathrm{Z}_1(G)$ it follows that $z_1 - z_2 \in \mathrm{Z}_1(G)$. By (2.44) we know that $z_1 - z_2$ is non-zero at most on edges in $E \backslash E'$. Since $E \backslash E'$ is a forest, and every integral flow on a forest is the zero flow, it follows that $z_1 - z_2$ is zero on E', too, hence $z_1 = z_2$. $\square$

Corollary 64 (a sufficient criterion for equality of two flows on $\mathrm{C}_n^{2^-}$)**.** *If* $n \equiv 3 \pmod 8$*, and with* $i_n := \frac{1}{2}(n+1)$ *and* $\mathrm{C}_n^{2^-}$ *the graph from Definition 214: if* $z_1, z_2 \in \mathrm{Z}_1(\mathrm{C}_n^{2^-})$ *and* $\langle k-1 \wedge k+1, z_1 \rangle = \langle k-1 \wedge k+1, z_2 \rangle$ *for every* $k \in [n-1]_0$*, then* $z_1 = z_2$*.*

Proof. This follows from Lemma 63 since, with $E'_n := \{\{k-1, k+1\} \colon k \in [n-1]_0\}$, the graph $(\mathrm{V}(\mathrm{C}_n^{2^-}), \mathrm{E}(\mathrm{C}_n^{2^-}) \backslash E'_n)$ is a forest (consisting of the $\frac{1}{2}(n-5)$ length-1-paths in $\{\{2k+1, 2k+2\} \colon k \in \{1, 2, \ldots, \frac{1}{2}(n-5)\}\}$ and the one length-4-path $n-2, n-1, 0, 1, 2$). $\square$

Let us note that the set E'_n underlying Corollary 64, and which we will use to prove equality of the various flows $\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(a,b))$ (in the sense of Definition 68 below) and $\mathrm{f}_{\mathrm{T}_n}(a,b)$ (in the sense of Definition 49 above), is by far not a minimum-cardinality set E' to use Lemma 63 with; the set E'_n was rather selected for reasons of uniformity and convenience w.r.t. our later arguments: thus, we thus only ever have to evaluate inner products of the form $\langle k-1 \wedge k+1, \cdot \rangle$, which we will abbreviate as $[\cdot]_k$. There exist sets E' with $\mathrm{E}(\mathrm{C}_n^{2^-}) \backslash E'$ a forest and E' having almost half as many elements as E'_n, but these E' contain both difference-2-edges $\{k-1, k+1\}$ *and* difference-1-edges $\{2k-1, 2k\}$, hence these E' were judged less convenient for our purposes (in particular, we would then have to establish auxiliary statements like Lemma 59 with $\langle 2k-1 \wedge 2k, \cdot \rangle$ instead of $\langle k-1 \wedge k+1, \cdot \rangle$).

We state the following as a reference when justifying simplifications in later calculations:

Lemma 65. *If* $0 \leqslant b_1 \leqslant b_2$ *are integers,*

(1) $\sum_{b_1 \leqslant i \leqslant b_2} (-1)^i = +1$ *for even* b_1 *and* b_2*,*

(2) $\sum_{b_1 \leqslant i \leqslant b_2} (-1)^i = 0$ *for even* b_1 *and odd* b_2*,*

(3) $\sum_{b_1 \leqslant i \leqslant b_2} (-1)^i = 0$ *for odd* b_1 *and even* b_2*,*

(4) $\sum_{b_1 \leqslant i \leqslant b_2} (-1)^i = -1$ *for odd* b_1 *and* b_2*.* $\square$

In the following Lemma 66, we state properties of the auxiliary substructures $\mathrm{C}_n^{2^-}$ from Definition 66 that we need to know for our later arguments.

Statement (3) in Lemma 66 is given as justification for why, in the proof of case $(1+4k.(+3+4i))$ below, we do not simply apply a 'rotation by $4k$' (i.e. adding $4k$ to every vertex) to the i-parametrised path (as was done in $(1+4k.(+2+4i))$, for instance): for a Hamilton-path of $\mathrm{C}_n^{2^-}$ between 1 and 4 to be rotatable in that way, it would have to have the properties in (2.45). In that local sense, our use of another k-dependence in $(1+4k.(+3+4i))$ is necessary. Statement (4) in Lemma 66 is given as an example that in the proof of Hamilton-connectedness of $\mathrm{C}_n^{2^-}$ (cf. (5) in Lemma 66), we do not have much choice (and it is probably not possible to significantly shorten the proof): in all cases where the author tried to determine how many Hamilton-paths there were in total for a given Hamilton-connectedness-instance, it turned out that there were only *two* Hamilton-paths on offer. The graph $\mathrm{C}_n^{2^-}$ is only *just* Hamilton-connected, and moreover there exist graphs with the same degree-sequence, and at edit-distance only four from it, which are *not* Hamilton-connected (cf. the discussion after (2.14) on p. 43).

Lemma 66 (properties of $\mathrm{C}_n^{2^-}$)**.** *For every odd* $n \geqslant 11$*, and with* $\mathrm{C}_n^{2^-}$ *as in Definition 214,*

(1) $\mathrm{C}_n^{2^-}$ *has* $n-3$ *vertices of degree three and* 3 *vertices of degree four,*

(2) $|\mathrm{C}_n^{2^-}| = n$ *and* $\|\mathrm{C}_n^{2^-}\| = \frac{3}{2}(n+1)$*,*

(3) *there does not exist a Hamilton-path* P *of* $\mathrm{C}_n^{2^-}$ *linking* 1 *and* 4 *and moreover satisfying*

$$\{n-1, 0\} \in \mathrm{E}(P), \quad \{n-2, n-1\} \notin \mathrm{E}(P), \quad \{0,1\} \in \mathrm{E}(P)\ , \tag{2.45}$$

(4) *for every* $n \equiv 3 \pmod 4$*, in* $\mathrm{C}_n^{2^-}$ *there are only two Hamilton-paths connecting* 4 *and* 8*, namely*
$4, 3, 1, 2, 0, n-1, n-2, {}_{(-1,-2,+1,-2)^{\frac{1}{4}(n-3)-2}}, 9, 7, 5, 6, 8$

and
$4, 6, 5, 3, 1, 2, 0, n-1, n-2, {}_{(-2,+1,-2,-1)^{\frac{1}{4}(n-3)-2}}, 9, 7, 8$,

(5) *for every* $n \equiv 3 \pmod 4$*, the graph* $\mathrm{C}_n^{2^-}$ *is Hamilton-connected,*
(6) $\mathrm{C}_n^{2^-}$ *is not bipartite, while for every* $n \geqslant 7$ *with* $n \equiv 3 \pmod 4$ *it admits a* 3*-colouring which uses the third colour only once,*
(7) $\mathrm{C}_n^{2^-} \subseteq \mathrm{C}_n^2$ *for every* n*, with* C_n^2 *the square of a circuit from Lemma 37.*(a1)*,*
(8) $\mathrm{C}_n^{2^-}$ *has sublinear bandwidth in the sense that, for every* $\beta > 0$ *there exists* $n_0 = n_0(\beta)$ *such that* $\mathrm{bw}(\mathrm{C}_n^{2^-}) \leqslant \beta \cdot n$ *whenever* $n_0 \leqslant n \equiv 3 \pmod 4$*,*
(9) $\mathrm{rank}_{\mathbb{Z}}(\mathrm{Z}_1(\mathrm{C}_n^{2^-})) = \frac{1}{2}(n+5)$*,*
(10) *for every* $n \equiv 3 \pmod 8$*, the graph* T_n *from Definition 218 is a spanning tree of* $\mathrm{C}_n^{2^-}$*.*

Proof of Lemma 66. The properties (1) and (2) are obvious from Definition 214.

As to (3), suppose that P were such a path. Then necessarily $\{2,0\} \in \mathrm{E}(P)$ since $\deg_P(2) = 2$ and $\mathrm{N}_{\mathrm{C}_n^{2^-}}(2) = \{0,1,4\}$, so $\{2,0\} \notin \mathrm{E}(P)$ would imply the contradiction $1,2,4 \subseteq P$. But then $\{n-2,0\} \notin \mathrm{E}(P)$ since otherwise (2.45) and $\{2,0\} \in \mathrm{E}(P)$ imply the contradiction $\deg_P(0) \geqslant 3$. Because of $\{n-2,0\} \notin \mathrm{E}(P)$ and (2.45), it then follows that $\deg_P(n-2) \leqslant 1$, contradicting the assumption that P is a Hamilton path of $\mathrm{C}_n^{2^-}$ with $n-2$ one of its inner vertices. This proves such P to be impossible.

As to (4), suppose P is a Hamilton-path of C_n^{2-} connecting 4 and 8. There are three possibilities for the neighbour of 4 in P: either 2, 3 or 6.
Case 1. $4,3 \subseteq P$. Then, since otherwise 6 could only have degree $\leqslant 1$ in P, despite being an inner vertex of the hypothetical Hamilton-path P, we know that $8,6 \subseteq P$, and hence $6,5 \subseteq P$. This implies the existence of the subpath $8,6,5,7,9, {}_{(+1,+2,-1,+2)^{\frac{1}{4}(n-3)-2}}, n-2, n-1, 0 \subseteq P$. Since it is then no longer possible that $3,5 \subseteq P$, it follows that we have the subpath $4,3,1,2,0$. Putting together the two subpaths, it follows that $P = 4,3,1,2,0,n-1,n-2, {}_{(-1,-2,+1,-2)^{\frac{1}{4}(n-3)-2}}, 9,7,5,6,8$, which is the first path given in (4).
Case 2. $4,3 \nsubseteq P$. Then either $4,2 \subseteq P$ or $4,6 \subseteq P$.
Case 2.1. $4,2 \subseteq P$. Then necessarily $5,3,1 \subseteq P$, since 3 is an inner vertex of the Hamilton-path P. Then necessarily $8,6,5 \subseteq P$, since 6, too, is such a vertex. But then 7 can have at most degree 1 in P, contradicting that 7, too, is such a vertex. Therefore, Case 2.1 is impossible.
Case 2.2. $4,6 \subseteq P$. Since $6,8 \subseteq P$ is obviously impossible, it then follows that $6,5 \subseteq P$, this being the only other possibility for 6 having degree 2 in P. Then necessarily $4,6,5,3,1,2,0 \subseteq P$ and $8,7,9 \subseteq P$. Considering vertex 9, it is clear that, necessarily, $9, {}_{(+1,+2,-1,+2)^{\frac{1}{4}(n-3)-2}}, n-2, n-1, 0, 2, 1, 3, 5, 6 \subseteq P$. Putting together the two subpaths, $P = 8,7,9, {}_{(+1,+2,-1,+2)^{\frac{1}{4}(n-3)-2}}, n-2, n-1, 0, 2, 1, 3, 5, 6, 4$, completing the proof of (4).

As to (5), let us first justify recourse to an elementary proof by checking cases: none of the sufficient criteria for Hamilton-connectedness known to the author applies to $\mathrm{C}_n^{2^-}$. The various criteria in the literature can be classified by their hypotheses as either (1) having a hypothesis of high-minimum degree, or (2) having a hypothesis of being a Cayley-graph. The graph $\mathrm{C}_n^{2^-}$, however, (1) has bounded-degree, and (2) is (for obvious reasons) not a Cayley-graph, escaping twice. Moreover, having Hamilton-based flow lattice by itself does not imply Hamilton-connectedness (while it of course *does* imply that any edge is contained in a Hamilton-circuit, in general it does not imply that every non-edge is connected by a Hamilton-path), cf. the brief discussion of the example $\mathrm{C}_{13}^{2^- \wedge}$ after (2.14) on p. 43. So it would not suffice to just prove Proposition 69 alone. By the way, according to [47, Theorem 1], the decision problem whether an arbitrary given graph is Hamilton-connected, is NP-hard. Moreover, the fact that $\mathrm{C}_n^{2^-}$ is still one edge sparser than the auxiliary graph $\mathrm{M}_r^{\boxtimes}$ means that the Hamilton-paths per instance are in shorter supply and more affected by parity issues.

Thus, proving the Hamilton-connectedness of $\mathrm{C}_n^{2^-}$, which is indispensable part of our proof of Theorem 3.(I.1), seems to necessitate explicitly describing Hamilton paths between any two vertices of $C_{11}^{2^-}$. In a preliminary step, we will separately do away with the $\Omega(n)$-many Hamilton-connectedness-instances involving vertex 0 (for contingent reasons not worth detailing it appears not to be possible to absorb these instances into $(4+4k.(+1+4i))$–$(4+4k.(+4+4i))$ below, neither by permitting $k=-1$ in those very parametrisations, nor by trying to write new ones which would work for all endvertices $a = 0, 4, \ldots, n-7, n-3$):

$(0.(+1+4i))$ For every $i \in [\frac{1}{4}(n-3)]_0$, the path
$0,_{(+2,-1,+2,+1)^i}, 4i,_{(+2)^{\frac{1}{2}(n-3)-2i+1}}, n-1, n-2,_{(-2)^{\frac{1}{2}(n-3)-2i}}, 1+4i$
is a Hamilton-path of $C_n^{2^-}$ connecting $a := 0$ and $b_i := 1+4i$.

$(0.(+2+4i))$ For every $i \in [\frac{1}{4}(n-3)]_0$, the path
$0, 1,_{(+1,+2,-1,+2)^i}, 1+4i,_{(+2)^{\frac{1}{2}(n-3)-2i}}, n-2, n-1,_{(-2)^{\frac{1}{2}(n-3)-2i}}, 2+4i$
is a Hamilton-path of $C_n^{2^-}$ connecting $a := 0$ and $b_i := 2+4i$.

$(0.(+3+4i))$ For every $i \in [\frac{1}{4}(n-3)-1]_0$, the path
$0, 1, 2,_{(+2,-1,+2,+1)^i}, 2+4i,_{(+2)^{\frac{1}{2}(n-3)-2i}}, n-1, n-2,_{(-2)^{\frac{1}{2}(n-3)-2i-1}}, 3+4i$
is a Hamilton-path of $C_n^{2^-}$ connecting $a := 0$ and $b_i := 3+4i$.

$(0.(+4+4i))$ For every $i \in [\frac{1}{4}(n-3)-1]_0$, the path
$0, 2, 1, 3,_{(+1,+2,-1,+2)^i}, 3+4i,_{(+2)^{\frac{1}{2}(n-3)-2i-1}}, n-2, n-1,_{(-2)^{\frac{1}{2}(n-3)-2i-1}}, 4+4i$
is a Hamilton-path of $C_n^{2^-}$ connecting $a := 0$ and $b_i := 4+4i$.

Now we proceed to doubly-parametrised Hamilton-paths, making it possible to do away with the remaining $\Omega_n(n^2)$ instances in space $O_n(1)$. The notation used for the periodic portions of the paths is self-explanatory (in particular, zero-exponents mean that the respective periodic movement does not occur). As an example, when $i=2$, $k=1$ and $n=19$, the path in $(1+4k.(+4+4i))$ is 5, 6, 4, 3, 1, 2, 0, 18, 16, 14, 12, 10, 8, 7, 9, 11, 13, 15, 17, a path that is shown at position $(k,i) = (1,2)$ in Figure 2.5.

For the instances containing $1+4k$, we claim the following:

$(1+4k.(+1+4i))$ For every $k \in [\frac{1}{4}(n-3)]_0$ and $i \in [\frac{1}{4}(n-3)-k]_0$, the path
$1+4k,\ _{(+1,+2,-1,+2)^i},\ 1+4(k+i),_{(+2)^{\frac{1}{2}(n-3)-2i+1}},$
$4k,\ 4k+n-1,\ _{(-2)^{\frac{1}{2}(n-3)-2i}},\ 2+4(k+i)$
is a Hamilton-path of $C_n^{2^-}$ connecting $a_k := 1+4k$ and $b_{k,i} := 2+4k+4i$.

$(1+4k.(+2+4i))$ For every $k \in [\frac{1}{4}(n-3)-1]_0$ and $i \in [\frac{1}{4}(n-3)-1-k]_0$, the path
$1+4k,\ 2+4k,\ _{(+2,-1,+2,+1)^i},\ 2+4(k+i),\ _{(+2)^{\frac{1}{2}(n-3)-2i}},\ 4k+n-1,$
$4k,\ _{(-2)^{\frac{1}{2}(n-3)-2i}},\ 3+4(k+i)$
is a Hamilton-path of $C_n^{2^-}$ connecting $a_k := 1+4k$ and $b_{k,i} := 3+4k+4i$.

$(1+4k.(+3+4i))$ For every $k \in [\frac{1}{4}(n-3)-1]_0$ and $i \in [\frac{1}{4}(n-3)-1-k]_0$, the path
$1+4k,\ _{(+2)^{1+2i}},\ 3+4(k+i),\ _{(+2,+1,+2,-1)^{\frac{1}{4}(n-3)-i-k-1}},\ n-4,$
$n-2,\ n-1,\ 0,\ _{(+2,-1,+2,+1)^k},\ 4k,\ _{(+2)^{2i+2}},\ 4+4(k+i)$
is a Hamilton-path of $C_n^{2^-}$ connecting $a_k := 1+4k$ and $b_{k,i} := 4+4i+4k$.

$(1+4k.(+4+4i))$ For every $k \in [\frac{1}{4}(n-3)-1]_0$ and $i \in [\frac{1}{4}(n-3)-1-k]_0$, the path
$1+4k,\ 2+4k,\ 4k,\ _{(-1,-2,+1,-2)^k},\ 0,\ n-1,\ _{(-1,-2,+1,-2)^{\frac{1}{4}(n-3)-k-i-1}},\ 6+4(k+i),$
$_{(-2)^{2i+1}},\ 4+4k,\ 3+4k,\ _{(+2)^{2i+1}},\ 5+4(k+i)$
is a Hamilton-path of $C_n^{2^-}$ connecting $a_k := 1+4k$ and $b_{k,i} := 5+4i+4k$.

For the instances containing $2+4k$, we claim the following:

$(2+4k.(+1+4i))$ For every $k \in [\frac{1}{4}(n-3)-1]_0$ and $i \in [\frac{1}{4}(n-3)-1-k]_0$, the path
$2+4k,\ _{(-1,-2,+1,-2)^k},\ 2,\ 1,\ _{(-1,-2,+1,-2)^{\frac{1}{4}(n-3)-k-i}},\ 4+4(k+i),\ _{(-2)^{2i}},\ 4+4k,$

$3+4k$, ${}_{(+2)^{2i}}$, $3+4(k+i)$
is a Hamilton-path of $\mathrm{C}_n^{2^-}$ connecting $a_k := 2+4k$ and $b_{k,i} := 3+4k+4i$.

$(2+4k.(+2{+}4i))$ For every $k \in [\frac{1}{4}(n-3)-1]_0$ and $i \in [\frac{1}{4}(n-3)-1-k]_0$, the path
$2+4k$, ${}_{(-1,-2,+1,-2)^k}$, 2, 1, 0, $n-1$, $n-2$, ${}_{(-2,+1,-2,-1)^{\frac{1}{4}(n-3)-k-i-1}}$, $5+4(k+i)$, ${}_{(-2)^{2i+1}}$, $3+4k$, $4+4k$, ${}_{(+2)^{2i}}$, $4+4(k+i)$
is a Hamilton-path of $\mathrm{C}_n^{2^-}$ connecting $a_k := 2+4k$ and $b_{k,i} := 4+4k+4i$.

$(2+4k.(+3{+}4i))$ For every $k \in [\frac{1}{4}(n-3)-1]_0$ and $i \in [\frac{1}{4}(n-3)-1-k]_0$, the path
$2+4k$, ${}_{(-1,-2,+1,-2)^k}$, 2, 1, 0, $n-1$, ${}_{(-1,-2,+1,-2)^{\frac{1}{4}(n-3)-k-i-1}}$, $6+4(k+i)$, ${}_{(-2)^{2i+1}}$, $4+4k$, $3+4k$, ${}_{(+2)^{2i+1}}$, $5+4(k+i)$
is a Hamilton-path of $\mathrm{C}_n^{2^-}$ connecting $a_k := 2+4k$ and $b_{k,i} := 5+4k+4i$.

$(2+4k.(+4{+}4i))$ For every $k \in [\frac{1}{4}(n-3)-1]_0$ and $i \in [\frac{1}{4}(n-3)-1-k]_0$, the path
$2+4k$, ${}_{(-1,-2,+1,-2)^k}$, 2, 1, ${}_{(-1,-2,+1,-2)^{\frac{1}{4}(n-3)-k-i-1}}$, $8+4(k+i)$, $7+4(k+i)$, ${}_{(-2)^{2i+2}}$, $3+4k$, $4+4k$, ${}_{(+2)^{2i+1}}$, $6+4(k+i)$
is a Hamilton-path of $\mathrm{C}_n^{2^-}$ connecting $a_k := 2+4k$ and $b_{k,i} := 6+4k+4i$.

For the instances containing $3+4k$, we claim the following:

$(3+4k.(+1{+}4i))$ For every $k \in [\frac{1}{4}(n-3)-1]_0$ and $i \in [\frac{1}{4}(n-3)-1-k]_0$, the path
$3+4k$, ${}_{(+2)^{2i+1}}$, $5+4k+4i$, ${}_{(+1,+2,-1,+2)^{\frac{1}{4}(n-3)-k-i-1}}$, $n-2$, $n-1$, 0, 1, ${}_{(+1,+2,-1,+2)^k}$, $1+4k$, $2+4k$, ${}_{(+2)^{2i+1}}$, $4+4(k+i)$
is a Hamilton-path of $\mathrm{C}_n^{2^-}$ connecting $a_k := 3+4k$ and $b_{k,i} := 4+4k+4i$.

$(3+4k.(+2{+}4i))$ For every $k \in [\frac{1}{4}(n-3)-1]_0$ and $i \in [\frac{1}{4}(n-3)-1-k]_0$, the path
$3+4k$, $4+4k$, ${}_{(+2,-1,+2,+1)^i}$, $4+4(k+i)$, ${}_{(+2)^{\frac{1}{2}(n-3)-2i}}$, $1+4k$, $2+4k$, ${}_{(-2)^{\frac{1}{2}(n-3)-2i}}$, $5+4(k+i)$
is a Hamilton-path of $\mathrm{C}_n^{2^-}$ connecting $a_k := 3+4k$ and $b_{k,i} := 5+4k+4i$.

$(3+4k.(+3{+}4i))$ For every $k \in [\frac{1}{4}(n-3)-1]_0$ and $i \in [\frac{1}{4}(n-3)-1-k]_0$, the path
$3+4k$, $4+4k$, $2+4k$, ${}_{(+1,+2,-1,+2)^k}$, 2, 1, 0, ${}_{(-2,+1,-2,-1)^{\frac{1}{4}(n-3)-k-i-1}}$, $7+4k+4i$, ${}_{(-2)^{2i+1}}$, $5+4k$, $6+4k$, ${}_{(+2)^{2i}}$, $6+4(k+i)$
is a Hamilton-path of $\mathrm{C}_n^{2^-}$ connecting $a_k := 3+4k$ and $b_{k,i} := 6+4k+4i$.

$(3+4k.(+4{+}4i))$ For every $k \in [\frac{1}{4}(n-3)-2]_0$ and $i \in [\frac{1}{4}(n-3)-2-k]_0$, the path
$3+4k$, $4+4k$, $2+4k$, ${}_{(-1,-2,+1,-2)^k}$, 2, 1, ${}_{(-1,-2,+1,-2)^{\frac{1}{4}(n-3)-k-i-1}}$, $8+4(k+i)$, ${}_{(-2)^{2i+1}}$, $6+4k$, $5+4k$, ${}_{(+2)^{2i+1}}$, $7+4(k+i)$
is a Hamilton-path of $\mathrm{C}_n^{2^-}$ connecting $a_k := 3+4k$ and $b_{k,i} := 7+4k+4i$.

For the instances containing $4+4k$ with $0 \leqslant k \leqslant \frac{1}{4}(n-3)-1$, we claim the following (for those instances containing the vertex 0, see $(0.(+1{+}4i))$–$(0.(+4{+}4i))$):

$(4+4k.(+1{+}4i))$ For every $k \in [\frac{1}{4}(n-3)-1]_0$ and $i \in [\frac{1}{4}(n-3)-1-k]_0$, the path
$4+4k$, ${}_{(+2)^{2i+1}}$, $6+4(k+i)$, ${}_{(+2,-1,+2,+1)^{\frac{1}{4}(n-3)-k-i-1}}$, $n-1$, ${}_{(+1,+2,-1,+2)^{k+1}}$, $4k+3$, ${}_{(+2)^{2i+1}}$, $5+4(k+i)$
is a Hamilton-path of $\mathrm{C}_n^{2^-}$ connecting $a_k := 4+4k$ and $b_{k,i} := 5+4k+4i$.

$(4+4k.(+2{+}4i))$ For every $k \in [\frac{1}{4}(n-3)-1]_0$ and $i \in [\frac{1}{4}(n-3)-1-k]_0$, the path
$4+4k$, ${}_{(-2)^{2k}}$, 4, 3, ${}_{(+2)^{2k+1}}$ $5+4k$, ${}_{(+1,+2,-1,+2)^i}$, $5+4(k+i)$, ${}_{(+2)^{\frac{1}{2}(n-3)-2k-2i}}$, 2, 1, ${}_{(-2)^{\frac{1}{2}(n-3)-2k-2i-1}}$, $6+4(k+i)$
is a Hamilton-path of $\mathrm{C}_n^{2^-}$ connecting $a_k := 4+4k$ and $b_{k,i} := 6+4k+4i$.

$(4+4k.(+3{+}4i))$ For every $k \in [\frac{1}{4}(n-3)-2]_0$ and $i \in [\frac{1}{4}(n-3)-2-k]_0$, the path
$4+4k$, ${}_{(+2)^{2i+1}}$, $6+4(k+i)$, $5+4(k+i)$, ${}_{(-2)^{2i+1}}$, $3+4k$, ${}_{(-2,+1,-2,-1)^k}$, 3, 1, 2, 0, ${}_{(-2,+1,-2,-1)^{\frac{1}{4}(n-3)-k-i-1}}$, $7+4(k+i)$
is a Hamilton-path of $\mathrm{C}_n^{2^-}$ connecting $a_k := 4+4k$ and $b_{k,i} := 7+4k+4i$.

$(4+4k.(+4{+}4i))$ For every $k \in [\frac{1}{4}(n-3)-2]_0$ and $i \in [\frac{1}{4}(n-3)-2-k]_0$, the path
$4+4k$, ${}_{(-1,-2,+1,-2)^{k+1}}$, 0, $n-1$, $n-2$, ${}_{(-2,+1,-2,-1)^{\frac{1}{4}(n-3)-k-i-2}}$, $9+4k+4i$, ${}_{(-2)^{2i+2}}$, $5+4k$, $6+4k$, ${}_{(+2)^{2i+1}}$, $8+4(k+i)$
is a Hamilton-path of $\mathrm{C}_n^{2^-}$ connecting $a_k := 4+4k$ and $b_{k,i} := 8+4k+4i$.

We first check that each of the vertex-sequences in $(1+4k.(+1{+}4i))$–$(4+4k.(+4{+}4i))$ actually is a *subpath* of $\mathrm{C}_n^{2^-}$. In doing so, all difference-2-edges can be ignored, both insofar as existence of those edges is concerned (since $\mathrm{C}_n^{2^-}$ contains each such egdge), and insofar as parity is concerned, since adding or subtracting 2 does not change parity (on which the existence of the difference-1-edges *does* depend, after all). It remains to argue that in each difference-1-edge, the smaller vertex is the odd one, except when the odd vertex is in $\{n-2, n-1, 0, 1, 2\}$. Proof for $(1+4k.(+1{+}4i))$: the vertex in front of the ${}_{(+1,+2,-1,+2)}$, namely $a_k = 1+4k$, is odd, which implies that the $+1$ and -1 within the upcoming ${}_{(+1,+2,-1,+2)}$-steps are permissible. The remaining difference-1-edge, namely $\{4k, 4k+n-1\}$ is permissible, too: if $k=0$, then this edge is a subset of the exception-set $\{n-2, n-1, 0, 1, 2\}$ where according to Definition 214, the requirement about difference-1-edges is suspended; if $k \geqslant 1$, then $\{4k, 4k+n-1\} =_n \{4k, 4k-1\}$ satisfies the requirement that the smaller vertex is the odd one. Proof for $(1+4k.(+2{+}4i))$: the vertex preceding the ${}_{(+2,-1,+2,+1)}$ there, namely $2+4k$, is even, hence the upcoming -1 is permissible, as is the following $+1$; the remaining difference-1-edge, namely $\{4k+n-1, 4k\}$ is permissible, too, since if $k=0$, then this edge is a subset of $\{n-2, n-1, 0, 1, 2\}$, while if $k \geqslant 1$, then in $\{4k+n-1, 4k\} =_n \{4k-1, 4k\}$, both representatives $4k-1$ and $4k$ are positive, making this an edge with the smaller vertex odd and the larger even. Proof for $(1+4k.(+3{+}4i))$: the vertex immediately before the ${}_{(+2,-1,+2,+1)}$ is $3+4(k+i)$, hence odd, so the $(+1)$–(-1)-alternations do indeed yield edges of $\mathrm{C}_n^{2^-}$. Moreover, the vertex preceding the ${}_{(+2,-1,+2,+1)}$ is 0, hence even, making the (-1)–$(+1)$-alternation permissible, too. Proof for $(1+4k.(+4{+}4i))$: the vertex preceding the ${}_{(-1,-2,+1,-2)^k}$ is $4k$; if $k=0$, then this means that the edge $\{0,-1\} =_n \{0, n-1\}$ is a subset of the exception-set $\{n-2, n-1, 0, 1, 2\}$, where the smaller-vertex-be-odd-requirement is suspended, whereas for every $k \in [\frac{1}{4}(n-3)-i-1]_0 \backslash \{0\}$ the edge $\{4k, 4k-1\}$ satisfies that requirement. Therefore, the (-1)–$(+1)$-alternations in ${}_{(-1,-2,+1,-2)^k}$ are permitted by the edges of $\mathrm{C}_n^{2^-}$. Moreover, the vertex in front of the ${}_{(-1,-2,+1,-2)^{\frac{1}{4}(n-3)-k-i-1}}$ is $n-1$, which is even because of $n \equiv 3 \pmod 4$, hence the (-1)–$(+1)$-alternations there are permissible, too. Proof for $(2+4k.(+1{+}4i))$: the vertex in front of the first ${}_{(-1,-2,+1,-2)}$-portion is the even number $2+4k$, hence all edges demanded by the (-1)–$(+1)$-alternations are indeed edges of $\mathrm{C}_n^{2^-}$. The vertex in front of the second ${}_{(-1,-2,+1,-2)}$-portion is the non-even number 1, but the edges $\{1,0\}$ demanded by the first -1 of that portion is a subset of the exception-set $\{n-2, n-1, 0, 1, 2\}$ where the smaller-vertex-odd-requirement is suspended; then comes the edge $\{0,-2\} =_n \{0, n-2\}$, hence the vertex in the coming $(+1)$-edge $\{n-2, n-1\}$ is the odd number $n-2$, as it has to be, and since $n-1$ is even, from then on, the (-1)-edge is always begun with an even number, as it has to be. Proof for $(2+4k.(+2{+}4i))$: the vertex in front of the ${}_{(-1,-2,+1,-2)}$ is $2+4k$, hence even, so the alternating (-1)- and $(+1)$-edges that follow are edges of $\mathrm{C}_n^{2^-}$; moreover, the vertex preceding the $+1$ in ${}_{(-2,+1,-2,-1)}$ is $n-4$, which is odd, hence the following $+1$, and more generally all the $(+1)$–(-1)-alternations in that portion of the path are in fact edges of $\mathrm{C}_n^{2^-}$. Proof for $(2+4k.(+3{+}4i))$: both the vertex $2+4k$ preceding the first ${}_{(-1,-2,+1,-2)}$-portion, and the vertex $n-1$ preceding the second ${}_{(-1,-2,+1,-2)}$-portion of that path are even, hence (-1)-edges with which these portions start, and, more generally, the entire (-1)–$(+1)$-alternation only use edges of $\mathrm{C}_n^{2^-}$. Proof for $(2+4k.(+4{+}4i))$: the vertex preceding the first ${}_{(-1,-2,+1,-2)}$-portion is $2+4k$, hence even, so the (-1)- and $(+1)$-edges in that portion all are edges of $\mathrm{C}_n^{2^-}$. For the second ${}_{(-1,-2,+1,-2)}$-portion, the same can be said as in the case of $(2+4k.(+3{+}4i))$. Proof for $(3+4k.(+1{+}4i))$: the vertices immediately before the two ${}_{(+1,+2,-1,+2)}$-portions, namely $5+4k+4i$ and 1, are both odd, hence so are the $(+1)$–(-1)-alternations. Proof for $(3+4k.(+2{+}4i))$: the vertex preceding the -1 in the ${}_{(+2,-1,+2,-1)}$-portion is $6+4k$, hence even, hence the (-1)–$(+1)$-alternation is permissible. Proof for $(3+4k.(+3{+}4i))$: the vertex preceding the $+1$ in the ${}_{(-2,+1,-2,-1)}$-portion of the path is

$n-2$, hence odd, hence $(+1)$–(-1)-alternation is permissible. Proof for $(3+4k.(+4+4i))$: while the vertex preceding the first $_{(-1,-2,+1,-2)}$-portion is $2+4k$, hence even, making the first (-1)–$(+1)$-alternation use only permissible edges, for the second $_{(-1,-2,+1,-2)}$-portion the vertex preceding the (-1)-step is 1, which is odd; but 1 is an element of the exception-set $\{n-2, n-1, 0, 1, 2\}$ where the smaller-vertex-odd-requirement is suspended, and the next $(+1)$-step is $n-2$, hence odd, making the following $(+1)$-step, and the remaining (-1)–$(+1)$-alternations permissible. Proof for $(4+4k.(+1+4i))$: the vertex preceding the (-1)-step in the $_{(+2,-1,+2,+1)}$-portion is $8+4(k+i)$, hence even, so this step uses an existing edge, as do all the (-1)–$(+1)$-alternations in that portion; the vertex preceding the first $(+1)$-step in the $_{(+1,+2,-1,+2)}$-portion is $n-1$, hence even, but an element of the exception set, while already the following (-1)-step, from 2 to 1, again abides by the smaller-vertex-odd-requirement. Proof for $(4+4k.(+2+4i))$: the vertex immediately in front of the $(+1)$-step in the $_{(+1,+2,-1,+2)}$-portion is $5+4k$, hence odd, hence the $(+1)$–(-1)-alternation is permissible. Proof for $(4+4k.(+3+4i))$: the vertex preceding the $(+1)$-step in the first $_{(-2,+1,-2,-1)}$-portion is the odd number $1+4k$, hence the $(+1)$–(-1)-alternations only use existing edges of $\mathrm{C}_n^{2^-}$; in the second such portion the first vertex before an $(+1)$-step is the odd number $n-2$, so all the $(+1)$–(-1)-alternations in that portion are permissible, too. Proof for $(4+4k.(+4+4i))$: the vertex preceding the (-1)-step in the $_{(-1,-2,+1,-2)}$-portion is $4+4k$, hence even, making this step, and all the other difference-1-steps there, permissible; the vertex preceding the first $(+1)$-step in the $_{(-2,+1,-2,-1)}$-portion is the odd number $n-4$, hence all the $(+1)$–(-1)-alternations occurring there use only edges of $\mathrm{C}_n^{2^-}$.

We now check a necessary condition for the above parametrisations to define *Hamilton*-paths: to define $n-1$ edges. (We will not explicitly show that in each of the parametrisations all n vertices are indeed pairwise distinct, considering this evident, and attempting to write a proof of it unhelpful; granting this additional fact, our counting of edges is both necessary and sufficient for the sequences to be Hamilton-paths.)

In $(0.(+1+4i))$, each path lists $4i+\frac{1}{2}(n-3)-2i+1+1+\frac{1}{2}(n-3)-2i = n-1$ edges. In $(0.(+2+4i))$, each lists $1+4i+\frac{1}{2}(n-3)-2i+1+\frac{1}{2}(n-3)-2i = n-1$ edges. In $(0.(+3+4i))$, each lists $2+4i+\frac{1}{2}(n-3)-2i+1+\frac{1}{2}(n-3)-2i-1 = n-1$ edges. In $(0.(+4+4i))$, each lists $3+4i+\frac{1}{2}(n-3)-2i-1+1\frac{1}{2}(n-3)-2i-1 = n-1$ edges.

In $(1+4k.(+1+4i))$, each path lists $4i+\frac{1}{2}(n-3)-2i+1+1+\frac{1}{2}(n-3)-2i = n-1$ edges. In $(1+4k.(+2+4i))$, each lists $1+4i+\frac{1}{2}(n-3)-2i+1+\frac{1}{2}(n-3)-2i = n-1$ edges. In $(1+4k.(+3+4i))$, each lists $1+2i+4\cdot(\frac{1}{4}(n-3)-i-k-1))+3+4k+2i+2 = n-1$ edges. In $(1+4k.(+4+4i))$, each lists $2+4k+1+4\cdot(\frac{1}{4}(n-3)-k-i-1)+2i+1+1+2i+1 = n-1$ edges.

In $(2+4k.(+1+4i))$, each path lists $4k+1+4\cdot(\frac{1}{4}(n-3)-k-i)+2i+1+2i = n-1$ edges. In $(2+4k.(+2+4i))$, each lists $4k+1+1+1+1+n-3-4k-4i-4+2i+1+1+2i = n-1$ edges. In $(2+4k.(+3+4i))$, each lists $4k+1+1+1+n-3-4k-4i-4+2i+1+1+2i+1 = n-1$ edges. In $(2+4k.(+4+4i))$, each lists $4k+1+(n-3)-4k-4i-4+1+2i+2+1+2i+1 = n-1$ edges.

In $(3+4k.(+1+4i))$, each path lists $2i+1+n-3-4k-4i-4+1+1+1+4k+1+2i+1 = n-1$ edges. In $(3+4k.(+2+4i))$, each lists $1+4i+\frac{1}{2}(n-3)-2i+1+\frac{1}{2}(n-3)-2i = n-1$ edges. In $(3+4k.(+3+4i))$, each lists $1+1+4k+1+1+n-3-4k-4i-4+2i+1+1+2i = n-1$ edges. In $(3+4k.(+4+4i))$, each lists $1+1+4k+1+n-3-4k-4i-4+2i+1+1+2i+1 = n-1$ edges.

In $(4+4k.(+1+4i))$, each path lists $2i+1+n-3-4k-4i-4+4k+4+2i+1 = n-1$ edges. In $(4+4k.(+2+4i))$, each lists $2k+1+2k+1+4i+\frac{1}{2}(n-3)-2k-2i+1+\frac{1}{2}(n-3)-2k-2i-1 = n-1$ edges. In $(4+4k.(+3+4i))$, each lists $2i+1+1+2i+1+4k+1+1+1+(n-3)-4k-4i-4 = n-1$ edges. In $(4+4k.(+4+4i))$, each lists $4k+4+1+1+n-3-4k-4i-8+2i+2+1+2i+1 = n-1$ edges.

We now complete the proof of Hamilton-connectedness of $\mathrm{C}_n^{2^-}$. As mentioned before, we take the view that we have already given enough evidence that each of the vertex sequences above is indeed a Hamilton-path of $\mathrm{C}_n^{2^-}$; what we still have to justify is to have given enough Hamilton-paths to prove that for each of the Hamilton-connectedness-instances for $\mathrm{C}_n^{2^-}$, i.e. for each $(a,b) \in \{0,1,\ldots,n-1\}^2$,

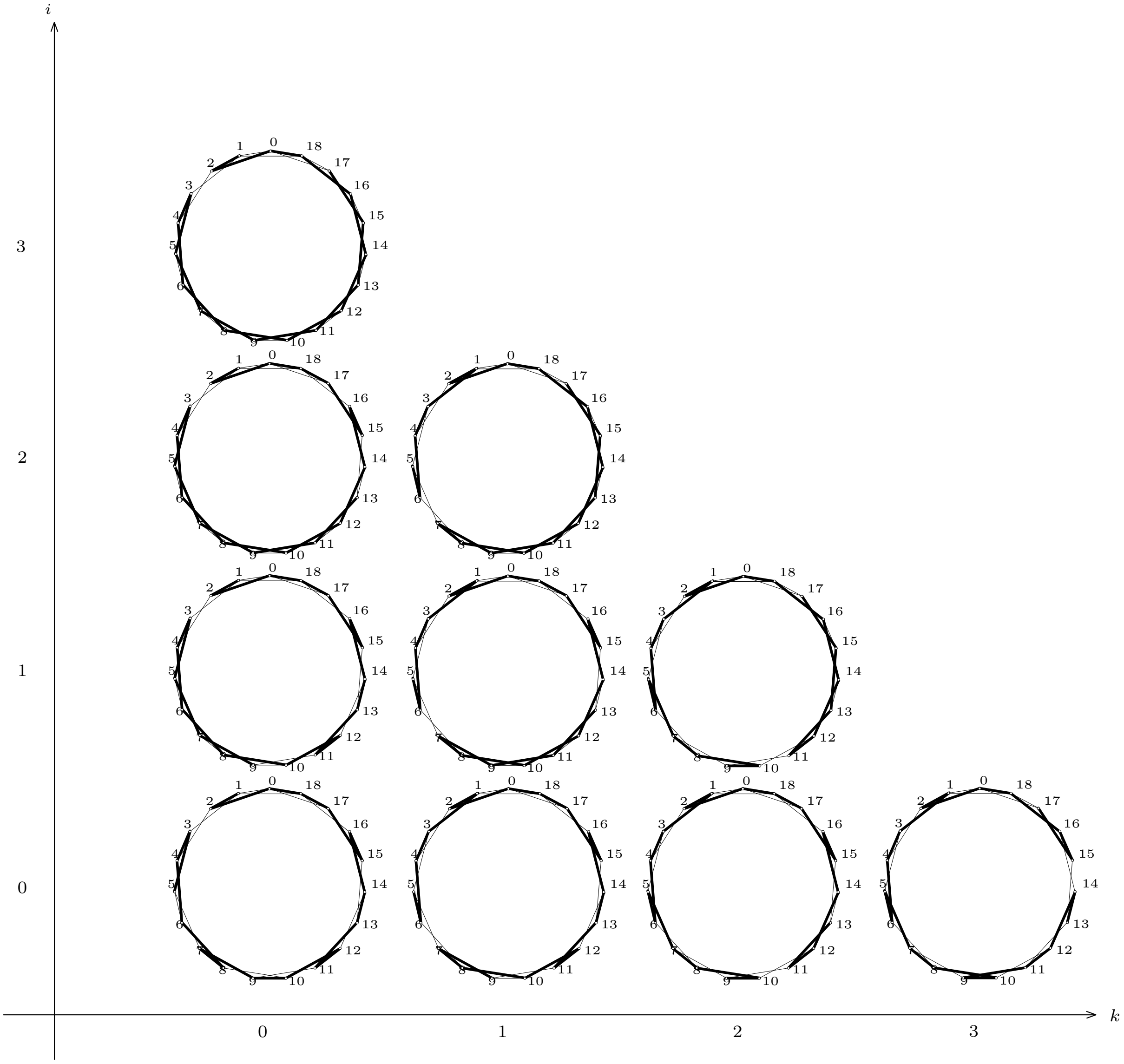


Figure 2.5: A part of the proof that the auxiliary graph $\mathrm{C}_n^{2^-}$ from Definition 214 is Hamilton-connected, illustrated for $n = 19$: these are the Hamilton-paths used for the case $(1 + 4k.(+4{+}4i))$. Each of the graphs in Figure 2.5 is $\mathrm{C}_{19}^{2^-}$ with one Hamilton-path indicated. There exist graphs with the same degree-sequence as $\mathrm{C}_{19}^{2^-}$, and having an edge-edit-distance of merely 4 from it, which are *not* Hamilton-connected.

there is a Hamilton-path of $\mathrm{C}_n^{2^-}$ with ends a and b. For every $(x,y) \in [3]_0^2 = \{0,1,2,3\}^2$ we define

$$\mathcal{I}_{x,y} := \{(a,b) \in \{0,1,\ldots,n-1\}^2 \colon a \equiv x \pmod 4, \quad b-a \equiv y \pmod 4\} \,. \tag{2.46}$$

For each $j \in \{1,2\}$ we let $\mathrm{h}_{(0.(+j+4i))}$ denote the map $[\frac{1}{4}(n-3)]_0 \to \{0,1,\ldots,n-1\}^2$, $i \mapsto (0,b_i)$, while for $j \in \{3,4\}$ we let $\mathrm{h}_{(0.(+j+4i))}$ denote the map $[\frac{1}{4}(n-3)-1]_0 \to \{0,1,\ldots,n-1\}^2$, $i \mapsto (0,b_i)$, each time with b_i as in $(0.(+j+4i))$ from $(0.(+1+4i))$–$(0.(+4+4i))$.

For each $(\ell,j) \in [4]^2$ we let $\mathrm{h}_{(\ell+4k.(j+4i))}$ denote the map $(i,k) \mapsto (a_k, b_{k,i})$, with a_k and $b_{k,i}$ as in $(\ell+4k.(j+4i))$ on p. 73–74 above.
We structure the remaining part of the proof into two steps:

(hcs.1) For each $j \in \{1,2\}$ we show that the map $\mathrm{h}_{(0.(+j+4i))}$ is an injection $[\frac{1}{4}(n-3)]_0 \to \mathcal{I}_{0,j}$, while for each $j \in \{3,4\}$ the map $\mathrm{h}_{(0.(+j+4i))}$ is an injection $[\frac{1}{4}(n-3)-1]_0 \to \mathcal{I}_{0,j}$ For each $(\ell,j) \in [4]^2$ we show that the map $\mathrm{h}_{(\ell+4k.(j+4i))} \colon (i,k) \mapsto (a_k, b_{k,i})$, from case $(\ell+4k.(j+4i))$ above, its domain being the cartesian product of the two sets for i and k in the respective $(\ell+4k.(j+4i))$, is an injection to $\mathcal{I}_{\ell,j}$.

(hcs.2) We show $\sum_{j\in[4]} |\mathrm{Dom}(\mathrm{h}_{(0.(j+4i))})| + \sum_{(\ell,j)\in[4]^2} |\mathrm{Dom}(\mathrm{h}_{(\ell+4k.(j+4i))})| = \binom{n}{2}$.

It then follows from (hcs.1) and (hcs.2) that $\mathrm{C}_n^{2^-}$ is Hamilton-connected: any injection between two finite sets whose domain has the same cardinality as its co-domain is surjective, i.e., a bijection. It thus follows from (hcs.1) and (hcs.2) that the union of all the maps $\mathrm{h}_{(0.(+j+4i))}$ and $\mathrm{h}_{(\ell+4k.(j+4i))}$ is a bijection from the union of all their respective domains to the set of instances $\{0,1,\ldots,n-1\}^2 = \bigsqcup_{(x,y)\in[3]_0\times[3]_0} \mathcal{I}_{x,y}$, i.e. each Hamilton-connectedness-instance has been considered by us.

We now carry out step (hcs.1). Let us recall that the purpose of the proof of injectivity is to prove surjectivity, which will follow after step (hcs.2). In view of the definition of the $\mathrm{h}_{(0.(+j+4i))}$ and $\mathrm{h}_{(\ell+4k.(j+4i))}$, and in view of the mutual disjointness of the $\mathcal{I}_{x,y}$, it is evident that to prove injectivity it suffices to argue that the various maps indeed have their image in the respective $\mathcal{I}_{x,y}$. For each separate $\mathcal{I}_{x,y}$, the injectivity of the respective map is evident, the only thing we have to rule out is that non-injectivity occurs by some of the maps having its image intersecting more than one $\mathcal{I}_{x,y}$. To complete step (hcs.1) we will therefore not do more than prove bounds on the various b_i, a_k and $b_{k,i}$.

We consider the functions $\mathrm{h}_{(0.(+j+4i))}$ from $(0.(+1+4i))$–$(0.(+4+4i))$: in each of $(0.(+1+4i))$–$(0.(+4+4i))$ we have $a=0$, which is one necessary condition for the images of the functions $\mathrm{h}_{(0.(+j+4i))}$ to lie in $\mathcal{I}_{0,j}$. We now consider the b_i. For the b_i from $(0.(+j+4i))$ we have $b_i - a = b_i = j+4i \equiv j \pmod 4$, for each $j \in [4]$. It remains to prove that in each of $(0.(+1+4i))$–$(0.(+4+4i))$ we have $b_i \in \{0,1,\ldots,n-1\}$. Indeed, in $(0.(+1+4i))$ and $(0.(+2+4i))$ we have $1 \leqslant j \leqslant 2$ and $0 \leqslant i \leqslant \frac{1}{4}(n-3)$, so $0 \leqslant b_i = j+4i \leqslant j+n-3 \leqslant n-1$, while in $(0.(+3+4i))$ and $(0.(+4+4i))$ we have $3 \leqslant j \leqslant 4$ and $0 \leqslant i \leqslant \frac{1}{4}(n-3)-1$, so $0 \leqslant b_i = j+4i \leqslant j+n-7 \leqslant n-3 \leqslant n-1$. This completes the proof that $\mathrm{h}_{(0.(+j+4i))}$ is a map $[\frac{1}{4}(n-3)]_0 \to \mathcal{I}_{0,j}$ for both $j \in \{1,2\}$, while $\mathrm{h}_{(0.(+j+4i))}$ is indeed a map $[\frac{1}{4}(n-3)-1]_0 \to \mathcal{I}_{0,j}$ for both $j \in \{3,4\}$.

As to $\ell = 1$, in each of $(1+4k.(+1+4i))$–$(1+4k.(+4+4i))$ we indeed have $a_k = 1+4k \equiv 1 \pmod 4$ and $0 \leqslant a_k = 1+4k \leqslant 1+4(\frac{1}{4}(n-3)-1) = n-6 \leqslant n-1$. Moreover, for every $j \in [4]$, the $b_{k,i}$ from $(1+4k.(+j+4i))$ satisfies $b_{k,i} - a_k = j+4i \equiv j \pmod 4$ and $0 \leqslant b_k = j+4(k+i) \leqslant j+4(k+\frac{1}{4}(n-3)-1-k) = j+n-5 \leqslant n-1$, so, indeed, $\mathrm{h}_{(1+4k.(j+4i))}$ for every $j \in [4]$ is a map $[\frac{1}{4}(n-3)-1]_0 \times [\frac{1}{4}(n-3)-1-k]_0 \to \mathcal{I}_{1,j}$.

As to $\ell = 2$, in each of $(2+4k.(+1+4i))$–$(2+4k.(+4+4i))$ we indeed know $a_k = 2+4k \equiv 2 \pmod 4$ and $0 \leqslant a_k = 2+4k \leqslant 2+4(\frac{1}{4}(n-3)-1) = 2+n-3-4 = n-5 \leqslant n-1$. Moreover, for every $j \in [4]$, the $b_{k,i}$ from $(2+4k.(+j+4i))$ satisfies $b_{k,i} - a_k = (2+j+4(k+i)) - (2+4k) = j+4i \equiv j \pmod 4$ and $0 \leqslant b_{k,i} = 2+j+4(k+i) \leqslant 2+j+4(k+\frac{1}{4}(n-3)-1-k) = j+n-5 \leqslant n-1$, so $\mathrm{h}_{(2+4k.(j+4i))}$ is indeed a map $[\frac{1}{4}(n-3)-1]_0 \times [\frac{1}{4}(n-3)-1-k]_0 \to \mathcal{I}_{2,j}$.

As to $\ell = 3$, in each of $(3+4k.(+1+4i))$–$(3+4k.(+4+4i))$ we indeed know $a_k = 3+4k \equiv 3 \pmod 4$ and $0 \leqslant a_k = 3+4k \leqslant 3+4(\frac{1}{4}(n-3)-1) = n-4 \leqslant n-1$. Moreover, for every $j \in [4]$, the $b_{k,i} = (3+j)+4(k+i)$ from $(3+4k.(+j+4i))$ indeed satisfies $b_{k,i} - a_k = j+4i \equiv j \pmod 4$.

Since for every $j \in [3]$ we have $0 \leqslant b_{k,i} = 3 + j + 4(k + i) \leqslant 3 + j + n - 7 = n - 4 + j \leqslant n - 1$, it follows that $\mathrm{h}_{(3+4k.(j+4i))}$ for every $j \in [3]$ is a map $[\frac{1}{4}(n-3)-1]_0 \times [\frac{1}{4}(n-3)-1-k]_0 \to \mathcal{I}_{3,j}$. Finally, for $j = 4$, i.e. in $(3 + 4k.(+4+4i))$, there is the smaller bound $i \leqslant \frac{1}{4}(n-3) - 2 - k$, so then we have $0 \leqslant b_{k,i} = 3 + 4 + 4(k + i) \leqslant 3 + 4 + n - 11 = n - 4 \leqslant n - 1$, hence $\mathrm{h}_{(3+4k.(j+4i))}$, in the case of $j = 4$, too, is a map $[\frac{1}{4}(n-3)-2]_0 \times [\frac{1}{4}(n-3)-2-k]_0 \to \mathcal{I}_{3,j}$.

As to $\ell = 4$, in each of $(4 + 4k.(+1+4i))$–$(4 + 4k.(+4+4i))$ we indeed know $a_k = 4 + 4k \equiv 4 \pmod 4$ and $0 \leqslant a_k = 4 + 4k \leqslant$ (here we do not need the smaller lower bound available for k in $(4+4k.(+3+4i))$ and $(4+4k.(+4+4i))$) $\leqslant 4+4(\frac{1}{4}(n-3)-1) = n-3 \leqslant n-1$. Moreover, for every $j \in [4]$, with $b_{k,i} = (4+j)+4(k+i)$ from $(4+4k.(+j+4i))$ we have $b_{k,i} - a_k = j + 4i \equiv j \pmod 4$. For every $j \in \{1,2\}$ we know that $b_{k,i} = (4+j)+4(k+i) \leqslant (4+2)+4(k+\frac{1}{4}(n-3)-1-k) = n-1$, while for $j \in \{3,4\}$, using the smaller upper bound on i in $(4+4k.(+3+4i))$ and $(4+4k.(+4+4i))$, we know that $b_{k,i} = (4+j)+4(k+i) \leqslant (4+4)+4(k+\frac{1}{4}(n-3)-2-k) = n-3 \leqslant n-1$. Therefore, $\mathrm{h}_{(4+4k.(j+4i))}$ for $j \in \{1,2\}$ is a map $[\frac{1}{4}(n-3)-1]_0 \times [\frac{1}{4}(n-3)-1-k]_0 \to \{0,1,\dots,n-1\}^2$, while for $j \in \{3,4\}$ it is a map $[\frac{1}{4}(n-3)-2]_0 \times [\frac{1}{4}(n-3)-2-k]_0 \to \{0,1,\dots,n-1\}^2$.

We now carry out step (hcs.2). Let $\mathrm{Dom}(f)$ denote the domain of a function f. Then from $(0.(+1+4i))$–$(0.(+4+4i))$ we see $\sum_{1 \leqslant j \leqslant 4} |\mathrm{Dom}(\mathrm{h}_{(0.(+j+4i))})| = 2 \cdot (\frac{1}{4}(n-3)+1) + 2 \cdot \frac{1}{4}(n-3) = n-3+2 = n-1$, i.e. we have treated $n-1$ instances there. This is consistent with $(0.(+1+4i))$–$(0.(+4+4i))$ claiming to take care of all instances $(a,b) \in \{0,1,\dots,n-1\}^2$ with $a = 0$.

Abbreviating $N_n := \frac{1}{4}(n-3)$, $g(x) := \frac{1}{2}x(x+1)$, and keeping in mind that $|[x]_0| = x+1$, we infer from $(1 + 4k.(+1+4i))$–$(4 + 4k.(+4+4i))$ that

$$\begin{aligned}
&(1.1)\ \ |\mathrm{Dom}(\mathrm{h}_{1+4k.(1+4i)})| = \textstyle\sum_{0 \leqslant k \leqslant N_n} (N_n - k + 1) = g(N_n + 1)\ , \\
&(1.2)\ \ |\mathrm{Dom}(\mathrm{h}_{1+4k.(j+4i)})| = \textstyle\sum_{0 \leqslant k \leqslant N_n - 1} (N_n - k) = g(N_n) \text{ for each } j \in \{2,3,4\}\ , \\[1ex]
&(2.1)\ \ |\mathrm{Dom}(\mathrm{h}_{2+4k.(j+4i)})| = \textstyle\sum_{0 \leqslant k \leqslant N_n - 1} (N_n - k) = g(N_n) \text{ for each } j \in \{1,2,3,4\}\ , \\[1ex]
&(3.1)\ \ |\mathrm{Dom}(\mathrm{h}_{3+4k.(j+4i)})| = \textstyle\sum_{0 \leqslant k \leqslant N_n - 1} (N_n - k) = g(N_n) \text{ for each } j \in \{1,2,3\}\ , \\
&(3.2)\ \ |\mathrm{Dom}(\mathrm{h}_{3+4k.(4+4i)})| = \textstyle\sum_{0 \leqslant k \leqslant N_n - 2} (N_n - 1 - k) = g(N_n - 1)\ , \\[1ex]
&(4.1)\ \ |\mathrm{Dom}(\mathrm{h}_{4+4k.(j+4i)})| = \textstyle\sum_{0 \leqslant k \leqslant N_n - 1} (N_n - k) = g(N_n) \text{ for each } j \in \{1,2\}\ , \\
&(4.2)\ \ |\mathrm{Dom}(\mathrm{h}_{4+4k.(j+4i)})| = \textstyle\sum_{0 \leqslant k \leqslant N_n - 2} (N_n - 1 - k) = g(N_n - 1) \text{ for each } j \in \{3,4\}\ .
\end{aligned}$$

By (1.1)–(4.2), $\sum_{(\ell,j) \in [4]^2} |\mathrm{Dom}(\mathrm{h}_{\ell+4k.(j+4i)})| = g(N_n+1) + 12 \cdot g(N_n) + 3 \cdot \frac{1}{2}(N_n - 1)N_n = 8 \cdot N_n^2 - 10 \cdot N_n + 3 = \binom{n}{2} - (n-1)$. Summing up, indeed $\sum_{1 \leqslant j \leqslant 4} |\mathrm{Dom}(\mathrm{h}_{(0.(+j+4i))})| + \sum_{(\ell,j) \in [4]^2} |\mathrm{Dom}(\mathrm{h}_{\ell+4k.(j+4i)})| = n - 1 + \binom{n}{2} - (n-1) = \binom{n}{2}$. This completes step (hcs.1), and, as explained above, the proof of Lemma 66.(5).

As for (6), its triangles witness the non-bipartiteness of the graph. For every $n \geqslant 7$ with $n \equiv 3 \pmod 4$, the parentheses in

$$\Bigl(\{0\}\Bigr) \sqcup \Biggl(\{1\} \sqcup \bigsqcup_{i \in [\frac{1}{4}(n-3)]} \{4i, 4i+1\}\Biggr) \sqcup \Biggl(\{n-1\} \sqcup \bigsqcup_{i \in [\frac{1}{4}(n-3)]} \{4i-2, 4i-1\}\Biggr) \tag{2.47}$$

define the colour-classes of a proper 3-colouring of $\mathrm{C}_n^{2^-}$.

As to (7), this is immediate from Definition 214: all edges given there are edges of the Cayley graph $\mathrm{C}_n^2 \cong \mathrm{Cay}(\mathbb{Z}/n; \{1, 2, n-2, n-1\})$.

Statement (8) follows from (7), together with Lemma 37.(a30) and the obvious fact that bandwidth is non-increasing w.r.t. taking subgraphs.

As to (9), $\mathrm{rank}_{\mathbb{Z}}(\mathrm{Z}_1(\mathrm{C}_n^{2^-})) = \|\mathrm{C}_n^{2^-}\| - |\mathrm{C}_n^{2^-}| + 1 =$ (by (2)) $= \frac{3}{2}(n+1) - n + 1 = \frac{1}{2}(n+5)$.

As for (10), let us first note that $(+2)+(-1)+(+2)+(+1) = 4$ and $\frac{1}{2}(n+1)+\frac{1}{8}(n-3)\cdot 4 = n-1$, $(+1)+(+2)+(-1)+(+2) = 4$ and $1+\frac{1}{8}(n-3)\cdot 4 = \frac{1}{2}(n-1)$, $(+2)+(+1)+(+2)+(-1) = 4$ and $\frac{1}{2}(n-1)+\frac{1}{8}(n-7)\cdot 4 = n-4$, and $(-1)+(+2)+(+1)+(+2) = 4$ and $4+\frac{1}{8}(n-7)\cdot 4 = \frac{1}{2}(n+1)$, hence the instructions in Definition 218 do indeed move in between the vertices stated there. We have to show that each of the edges described in Definition 218 is indeed an edge of $\mathrm{C}_n^{2^-}$: by Definition 214 we have $\{i, i+2\} \in \mathrm{E}(\mathrm{C}_n^{2^-})$ for every i, so each of the (+2)-steps uses an edge. What remains to be shown is that each of the (−1)-steps occurs at even, and each of the (+1)-steps at *odd* vertices: since we know $n = 8\mu + 3$, $\mu \in \mathbb{N}$, we know $\frac{1}{2}(n+1)$ to be even, hence the vertex is even after the (+2)-step, and then comes the (−1)-step; the vertex now being odd, the following (+2)-step keeps it odd, and then comes the (+1)-step, making the current vertex even again; this shows that the magnitude-1-steps occur at permitted times during a full iteration of the phase from $\frac{1}{2}(n+1)$ to $n-1$. The traversal of the path $n-1, 0, 1$ is possible since in Definition 214, the edge $\{2j, 2j+1\}$ is not removed when $j = 0$, nor when $j = \frac{1}{2}(n-1)$. The following (+1)-step uses an edge since it happens at the odd vertex 1, making it even, and after an evenness-preserving (+2)-step the (−1)-step of the iteration again goes via an edge, proving also the second loop described in Definition 218 to only use edges of $\mathrm{C}_n^{2^-}$, completing the proof of (10). The proof of Lemma 66 is now complete. □

The author is aware that the graph $\mathrm{C}_n^{2^-}$ admits one (and, to all appearances, *only* one) nontrivial automorphism, yet found this not helpful in proving Hamilton-connectedness of $\mathrm{C}_n^{2^-}$.

We will not consider the question of Hamilton-connectedness of $\mathrm{C}_n^{2^-}$ in the case $n \equiv 1 \pmod 4$, since according to Conjecture 73, the graph $\mathrm{C}_n^{2^-}$ is not suitable for our purposes (in this thesis, $\mathrm{C}_n^{2^-}$ is but a means to an end). Moreover, in this thesis, we will use the Hamilton-connectedness of $\mathrm{C}_n^{2^-}$, which by Lemma 66 we know for every $n \equiv 3 \pmod 4$, only for $n \equiv 3 \pmod 8$.

2.2.3.3 An explicit Hamilton-flow-basis for the auxiliary substructures $\mathrm{C}_n^{2^-}$

The following lemma gives the edges defining the fundamental flows on $\mathrm{C}_n^{2^-}$ w.r.t. the spanning tree T_n from Definition 218 in Chapter 5; along these edges we will structure the proof in the present Section 2.2.3.3, where we will construct the change-of-bases between a Hamilton-flow-basis of $\mathrm{C}_n^{2^-}$ and the spanning-tree-basis w.r.t. T_n:

Lemma 67 (the edges in $\mathrm{E}(\mathrm{C}_n^{2^-})\backslash\mathrm{E}(\mathrm{T}_n)$)**.** *If $n \equiv 3 \pmod 8$, and with $\mathrm{C}_n^{2^-}$ as in Definition 214 and T_n as in Definition 218, we have*

$$\begin{aligned}
\mathrm{E}(\mathrm{C}_n^{2^-})\backslash\mathrm{E}(\mathrm{T}_n) = {} & \{\{n-2, 0\}, \{n-1, 1\}, \{0, 2\}\} \\
& \sqcup \bigsqcup_{\ell \in \{0,1,\ldots,\frac{1}{4}(n-3)-1\}\backslash\{\frac{1}{8}(n-3)\}} \{\{1+4\ell, 3+4\ell\}\} \\
& \sqcup \{\{i_n - 1, i_n + 1\}\} \\
& \sqcup \bigsqcup_{\ell \in \{0,1,\ldots,\frac{1}{4}(n-3)-1\}\backslash\{\frac{1}{8}(n-3)-1\}} \{\{4+4\ell, 6+4\ell\}\} \\
& \sqcup \{\{i_n - 2, i_n\}\} \sqcup \{\{i_n - 1, i_n\}\}
\end{aligned} \tag{2.48}$$

Proof. By definition of $\mathrm{C}_n^{2^-}$ and T_n. Let us note that (2.48) mentions $3 + (\frac{1}{4}(n-3)) - 1 + 1 + \frac{1}{4}(n-3) - 1 + 2 = \frac{1}{2}(n+5)$ edges, which is consistent with T_n being a spanning tree of $\mathrm{C}_n^{2^-}$ and $\|\mathrm{C}_n^{2^-}\| - (|\mathrm{C}_n^{2^-}| - 1) =$ (by Lemma 66.(2)) $= \frac{3}{2}(n+1) - (n-1) = \frac{1}{2}(n+5)$. □

We now proceed to the heart of the proof: we define the coordinates (which only at the end we will know to be unique) of the fundamental flows w.r.t. T_n in terms of $\mathcal{B}_n$, and then we will proceed to show that they are indeed what we have just claimed them to be:

Definition 68 (the coordinates of the fundamental flows of T_n w.r.t. the basis $\mathcal{B}_n$)**.** *For every $n \geqslant 11$ with $n \equiv 3 \pmod 8$, and with $i_n := \frac{1}{2}(n+1)$ and $\vec{C}_{i;n}$, $\vec{C}_{0,1,2;n}$ and $\vec{C}_{0,1,n-1;n}$ as in Definition 48, we define the following elements of $\mathrm{Z}_1(\mathrm{C}_n^{2^-})$:*

(hs.1) $\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(n-2,0)) := \sum_{i=0}^{\frac{1}{2}(n-3)}(-1)^i \cdot \vec{C}_{i;n}$

(hs.2) $\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(n-1,1)) := (\sum_{i=1}^{\frac{1}{2}(n-3)}(-1)^i \cdot \vec{C}_{i;n}) + \vec{C}_{\frac{1}{2}(n-1);n} - \vec{C}_{0,1,2;n} - \vec{C}_{0,1,n-1;n}$

(hs.3) $\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(0,2)) := (\sum_{i=1}^{\frac{1}{2}(n-3)}(-1)^{i-1} \cdot \vec{C}_{i;n}) - \vec{C}_{\frac{1}{2}(n-1);n}$

(hs.4) *for every* $\ell \in \{0,1,\ldots,\frac{1}{4}(n-3)-1\} \backslash \{\frac{1}{8}(n-3)\}$,
$\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(1+4\ell,3+4\ell)) := \vec{C}_{0;n} + \vec{C}_{1+2\ell;n} + (\sum_{i=2+2\ell}^{\frac{1}{2}(n-3)}(-1)^{i-1} \cdot 2 \cdot \vec{C}_{i;n})$
$- 2 \cdot \vec{C}_{\frac{1}{2}(n-1);n} + \vec{C}_{0,1,2;n} + \vec{C}_{0,1,n-1;n}$

(hs.5) $\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n+1)) := \vec{C}_{0;n} + \vec{C}_{\frac{1}{4}(n+1);n} + (\sum_{i=\frac{1}{4}(n+5)}^{\frac{1}{2}(n-3)}(-1)^{i-1} \cdot 2 \cdot \vec{C}_{i;n})$
$- 2 \cdot \vec{C}_{\frac{1}{2}(n-1);n} + 2 \cdot \vec{C}_{0,1,2;n} + \vec{C}_{0,1,n-1;n}$

(hs.6) *for every* $\ell \in \{0,1,\ldots,\frac{1}{4}(n-3)-1\} \backslash \{\frac{1}{8}(n-3)-1\}$,
$\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(4+4\ell,6+4\ell)) := \vec{C}_{0;n} - \vec{C}_{2+2\ell;n} + (\sum_{i=3+2\ell}^{\frac{1}{2}(n-3)}(-1)^{i-1} \cdot 2 \cdot \vec{C}_{i;n})$
$- 2 \cdot \vec{C}_{\frac{1}{2}(n-1);n} + \vec{C}_{0,1,2;n} + \vec{C}_{0,1,n-1;n}$

(hs.7) $\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n)) := \vec{C}_{0;n} - \vec{C}_{\frac{1}{4}(n-3);n} + (\sum_{i=\frac{1}{4}(n+1)}^{\frac{1}{2}(n-3)}(-1)^{i-1} \cdot 2 \cdot \vec{C}_{i;n})$
$- 2 \cdot \vec{C}_{\frac{1}{2}(n-1);n} + 2 \cdot \vec{C}_{0,1,2;n} + \vec{C}_{0,1,n-1;n}$

(hs.8) $\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n)) := \vec{C}_{0,1,2;n}$

The following result seems to provide the first non-trivial examples of a *basis* of a flow lattice consisting only of Hamilton-flows:

Proposition 69 (an explicit basis proving $\mathrm{Z}_1(\mathrm{C}_n^{2^-})$ to be Hamilton-based for every $n \equiv 3 \pmod 8$)**.** *For every* $n \geqslant 11$ *with* $n \equiv 3 \pmod 8$,

$$\mathcal{B}_n := \left\{\vec{C}_{i;n} \colon i \in \{0,1,\ldots,\tfrac{1}{2}(n-1)\}\right\} \sqcup \{\vec{C}_{0,1,2;n}\} \sqcup \{\vec{C}_{0,1,n-1;n}\} \ , \tag{2.49}$$

with $\vec{C}_{i;n}$*,* $\vec{C}_{0,1,2;n}$ *and* $\vec{C}_{0,1,n-1;n}$ *the Hamilton-circuit-supported flows from Definition 48, is a basis of the abelian group* $\mathrm{Z}_1(\mathrm{C}_n^{2^-})$*, the flow lattice of the graph* $\mathrm{C}_n^{2^-}$ *from Definition 214.*

Proof. Let us first note that $|\mathcal{B}_n| = 2 + |\{0,1,\ldots,\frac{1}{2}(n-1)\}| = \frac{1}{2}(n+5)$. By standard theory (e.g., [5, Proposition (6.22)] with $R := \mathbb{Z}$), every rank-sized generating set of a finitely-generated abelian group is a basis.[5] Therefore, if we show that $\mathcal{B}$ is a generating set for $\mathrm{Z}_1(\mathrm{C}_n^{2^-})$, then in view of Lemma 66.(9) on p. 72 we will know it to be a rank-sized generating set, and it then follows for algebraic reasons alone that $\mathcal{B}_n$ is a basis. We need not prove that $\mathcal{B}_n$ is $\mathbb{Z}$-linearly independent, equivalently, we need not argue why the linear combinations constructed in this proof are unique, since this follows automatically at the end.

That $\mathcal{B}_n$ is a generating set of $\mathrm{Z}_1(\mathrm{C}_n^{2^-})$ will be shown as follows: it is well-documented (e.g. [9, Lemma 2] [68, Proposition 3.1]) that for any spanning tree of a connected graph G, the fundamental

[5] A much more general principle is pulling the strings here (not that we needed to know this for our purposes): by a fundamental algebraic fact, known since at least the 1970's, if R is *any* commutative ring, M any (not necessarily free) finitely-generated R-module, and $f \colon M \to M$ any R-module endomorphism, then surjectivity of f implies injectivity of f. One reference for this is [125, Theorem 2.4], where the statement is quickly deduced from Nakayama's lemma. This theorem can be applied to our (almost maximally specialised) situation of the concretely given, free $\mathbb{Z}$-module $\mathrm{Z}_1(\mathrm{C}_n^{2^-})$, as follows: first choose an arbitrary spanning tree T_n of $\mathrm{C}_n^{2^-}$ and let $\mathcal{B}_{T_n}$ denote a fundamental-flow-basis of $\mathrm{Z}_1(\mathrm{C}_n^{2^-})$ obtained from T_n after arbitrarily choosing one of the two orientations for each of its fundamental circuits. Let $r := \mathrm{rank}(\mathrm{Z}_1(\mathrm{C}_n^{2^-}))$ and let $\phi_{\mathcal{B}_{T_n}} \colon \mathrm{Z}_1(\mathrm{C}_n^{2^-}) \to \mathbb{Z}^r$ denote the coordinate-isomorphism w.r.t. the basis $\mathcal{B}_{T_n}$. Let $\vec{C}_1,\ldots,\vec{C}_r$ be any enumeration of the Hamilton-flows in $\mathcal{B}_n$ from Proposition 69, and let $\psi_{\mathcal{B}_n} \colon \mathbb{Z}^r \to \mathrm{Z}_1(\mathrm{C}_n^{2^-})$, $(\lambda_1,\ldots,\lambda_r) \mapsto \lambda_1\vec{C}_1 + \cdots + \lambda_r\vec{C}_r$. Then we define the endomorphism $f_{\mathcal{B}_n} := \psi_{\mathcal{B}_n} \circ \phi_{\mathcal{B}_{T_n}} \colon \mathrm{Z}_1(\mathrm{C}_n^{2^-}) \to \mathrm{Z}_1(\mathrm{C}_n^{2^-})$. Our proof of Proposition 69 will show $f_{\mathcal{B}_n}$ to be surjective. Since any two distinct non-trivial $\mathbb{Z}$-linear combinations of the same flow $z \in \mathrm{Z}_1(\mathrm{C}_n^{2^-})$ in terms of $\vec{C}_1,\ldots,\vec{C}_r$ would imply non-injectivity of the endomorphism $f_{\mathcal{B}_n}$, the theorem just mentioned implies that any such linear combination is unique, i.e., $\mathcal{B}_n$ is a basis. In a word, our proof that $\mathcal{B}_n$ is a basis makes use of the fact that the abelian group $\mathrm{Z}_1(\mathrm{C}_n^{2^-})$ is *Hopfian*.

flows defined by it constitute a basis for $\mathrm{Z}_1(G)$. We will show that if $n \equiv 3 \pmod 4$, each of the fundamental flows pertaining to the spanning tree T_n in Definition 218 is a $\mathbb{Z}$-linear sum of the Hamilton-flows obtained from $\mathcal{B}$. This can be done with bearable effort for general $n \equiv 3 \pmod 4$ only with a careful choice of a (spanning-tree)-(Hamilton-basis)-pair. This choice ensures a matrix for the change-of-basis which is relatively (compared with most other choices of spanning trees and Hamilton-bases) pattern-rich (cf. (2.92) on p. 102 for the case $n = 19$). Of course, the patterns could be simpler (note, for instance, that the $(+1)$-(-1)-alternations in both the $(\mathrm{f}_{\mathrm{T}_{19}}(0,2))$- and the $(\mathrm{f}_{\mathrm{T}_{19}}(18,1))$-indexed row of (2.92) cease already at the $\vec{C}_{8;19}$-indexed column), yet this base-change-matrix is the simplest the author could find. Its form for general $n \equiv 3 \pmod 4$ is given by the linear combinations in (hs.1)–(hs.8) in Chapter 5.

With $\mathrm{f}_{\mathrm{T}_n}(a,b)$ as in Lemma 56, we will prove

$$\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(a,b)) = \mathrm{f}_{\mathrm{T}_n}(a,b) \text{ for every } \{a,b\} \in \mathrm{E}(\mathrm{C}_n^{2^-})\backslash\mathrm{E}(\mathrm{T}_n)\ , \tag{2.50}$$

and we will use the description of the set $\mathrm{E}(\mathrm{C}_n^{2^-})\backslash\mathrm{E}(\mathrm{T}_n)$ from Lemma 67. For each such (a,b), the question arises how many checks have to be performed to show $\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(a,b)) = \mathrm{f}_{\mathrm{T}_n}(a,b)$. Let us recall that, by the very definition of the free $\mathbb{Z}$-module $\mathrm{C}_1(\mathrm{C}_n^{2^-})$, two elements $z', z'' \in \mathrm{C}_1(\mathrm{C}_n^{2^-})$ are equal if and only if $\langle a \wedge b, z'\rangle = \langle a \wedge b, z''\rangle$ for every $\{a < b\} \in \mathrm{E}(\mathrm{C}_n^{2^-})$. Obviously, for two general elements z' and z'', it is necessary to check equality on every edge $\{a < b\}$ to be sure that $z' = z''$. Given the information that z' and z'' are elements of $\mathrm{Z}_1(\mathrm{C}_n^{2^-}) \subseteq \mathrm{C}_1(\mathrm{C}_n^{2^-})$, though, one can save some work by using Lemma 63, i.e., by using Corollary 64.

We will prove each instance of (2.50), by going along the elements of $\mathrm{E}(\mathrm{C}_n^{2^-})\backslash\mathrm{E}(\mathrm{T}_n)$ in the order given in Lemma 67 (below, these edges are given in bold face at the beginning of each instance). Each time we will prove the claimed equality of flows with the help of Corollary 64. When looking up values of inner products calculated in previous lemmas, we will sometimes tacitly use that in the abelian group $\mathrm{C}_1(\mathrm{C}_n^{2^-}) \subseteq \bigwedge^2 \bigoplus_{v \in \mathrm{V}(\mathrm{C}_n^{2^-})} \mathbb{Z}v$ we have $u \wedge v = - v \wedge u$.

As for (2.50) with $(\mathbf{a},\mathbf{b}) = (\mathbf{n-2},\mathbf{0})$, among all the cases $k \in [n-1]_0$ required by Corollary 64, we first—for reasons explained after the calculation—bar the possibilities $k = 0$, $k = n-2$ and $k = n-1$. Moreover, we distinguish the parities of k. I.e., for every even $k \in \{1,2,\dots,n-3\}$ we first calculate as follows, using the abbreviation $[\cdot]_k = \langle k-1 \wedge k+1, \cdot\rangle$, partitioning the index set $[\frac{1}{2}(n-3)]$ into the three sets $\{1,2,\dots,\lfloor\frac{1}{2}k\rfloor - 1\}$, $\{\lfloor\frac{1}{2}k\rfloor\}$, $\{\lfloor\frac{1}{2}k\rfloor+1, \lfloor\frac{1}{2}k\rfloor+2,\dots,\frac{1}{2}(n-3)\}$, and using $[\vec{C}_{\lfloor\frac{1}{2}k\rfloor;n}]_k = 0$ from Lemma 54.(5).(i), and also Lemma 54.(5).(ii),

$$\begin{aligned}
[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(n-2,0))]_k &\overset{\text{(hs.1)}}{=} \sum_{0 \leqslant i \leqslant \frac{1}{2}(n-3)} (-1)^i \cdot [\vec{C}_{i;n}]_k \\
&= \sum_{0 \leqslant i \leqslant \lfloor\frac{1}{2}\rfloor - 1} (-1)^{i-1} \cdot (+) + \sum_{\lfloor\frac{1}{2}k\rfloor+1 \leqslant i \leqslant \frac{1}{2}(n-3)} (-1)^{i-1} \cdot (-) \\
&= -\left(\sum_{0 \leqslant i \leqslant \lfloor\frac{1}{2}k\rfloor - 1} (-1)^i - \sum_{\lfloor\frac{1}{2}k\rfloor+1 \leqslant i \leqslant \frac{1}{2}(n-3)} (-1)^i \right) \\
&= -\begin{cases} (0)-(0) & \begin{array}{l}\text{by (2) and (3)}\\ \text{in Lemma 65}\\ \text{if } \lfloor\frac{1}{2}k\rfloor \text{ is even}\end{array} \\ (1)-(1) & \begin{array}{l}\text{by (1) in Lemma 65}\\ \text{if } \lfloor\frac{1}{2}k\rfloor \text{ is odd}\end{array} \end{cases} \left(\begin{array}{l}\text{because of}\\ n \equiv 3 \pmod 4,\\ \text{we know that}\\ \frac{1}{2}(n-3) \text{ is even}\end{array}\right) \\
&= 0 \\
(\text{because of } k \neq n-1) &= \langle k-1 \wedge k+1\ ,\quad n-2 \wedge 0 + 0 \wedge n-1 + n-1 \wedge n-2 \rangle \\
&\overset{\text{(ff.1)}}{=} [\mathrm{f}_{\mathrm{T}_n}(n-2,0)]_k\ .
\end{aligned} \tag{2.51}$$

For every odd $k \in \{1,2,\dots,n-3\}$, we may repeat the calculation in (2.51), this time using Lemma 54.(5).(iii) for the evaluations, whereupon all signs of the inner products flip, still leading

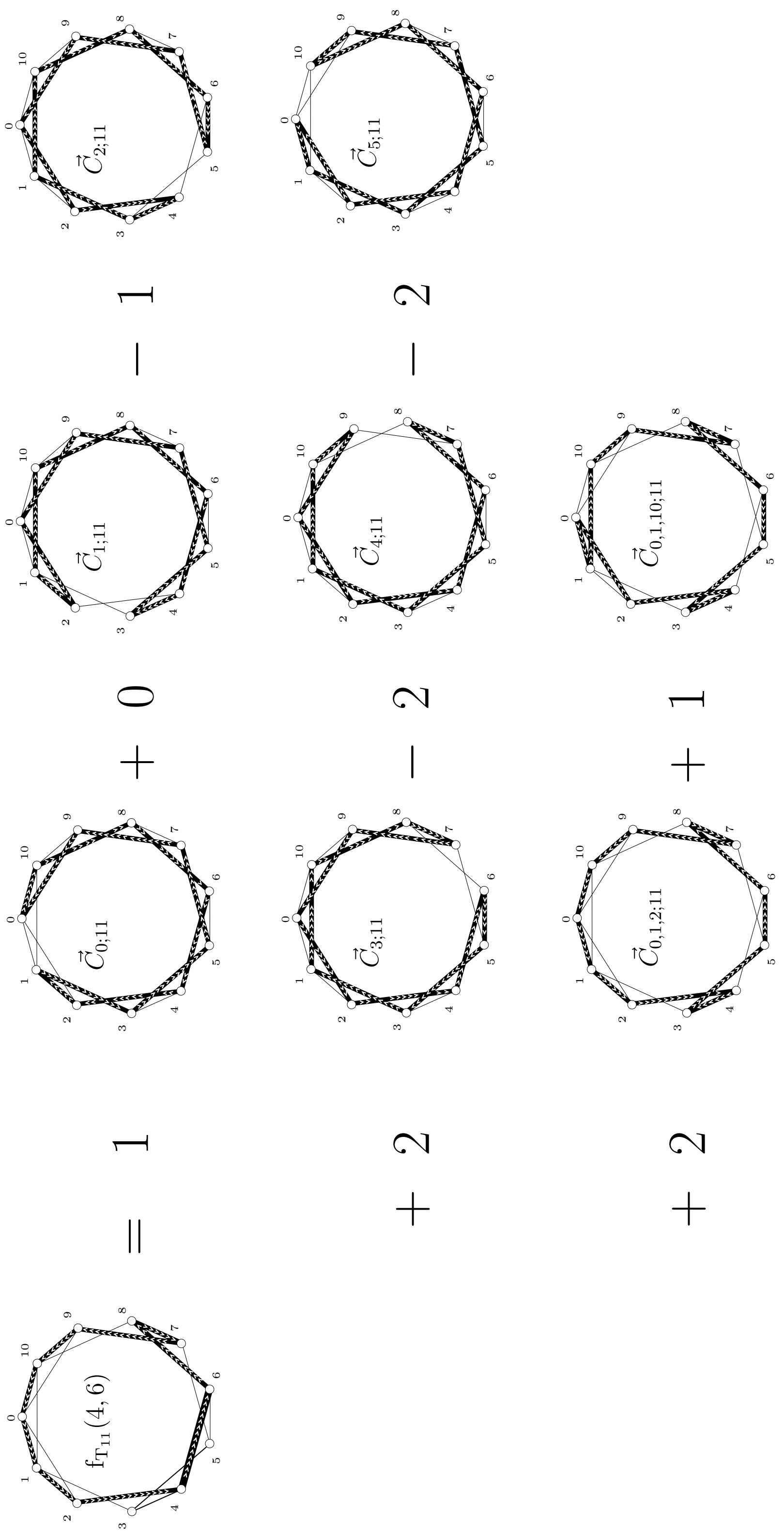


Figure 2.6: This is the unique $\mathbb{Z}$-linear combination of the fundamental flow $\mathrm{f}_{\mathrm{T}_{11}}(4,6) = \mathrm{f}_{\mathrm{T}_{11}}(i_{11}-2, i_{11})$ from (hs.7) in Lemma 56 in terms of the basis of the Hamilton-circuit-supported-simple-flow-basis $\mathcal{B}_{11}$ from Proposition 69. The arrows indicate the orientations of the Hamilton-circuit-supported simple flows in $\mathcal{B}_{11}$. Taking, as we always do, the lexicographic ordering of an edge as its positive orientation, the calculation on the right-hand side of the equation, restricted to, e.g., the edge $\{7,8\}$ is: $1\cdot(0) + 0\cdot(0) - 1\cdot(0) + 2\cdot(-) - 2\cdot(-) - 2\cdot(0) + 2\cdot(-) + 1\cdot(+)$, which is -1, in agreement with the orientation of $\{7,8\}$ being from 8 to 7 in $\mathrm{f}_{\mathrm{T}_{11}}(4,6)$. The small magnitude of the coefficients is not an accident: the spanning trees T_n from Definition 218 and the basis $\mathcal{B}_n$ from Proposition 69 have been carefully selected so as to make the matrix describing the change-of-bases from $\mathcal{B}_n$ to $\{\mathrm{f}_{\mathrm{T}_n}(u,v)\colon \{u<v\} \in \mathrm{E}(\mathrm{C}_n^{2n})\backslash\mathrm{E}(\mathrm{T}_n)\}$ be manageably pattern-rich, in particular, to make it have all entries of magnitude at most two. In the cases $n = 11$ (resp. $n = 19$) that matrix is shown in (2.89) (resp. (2.92)).

to the result 0. In the case $k = 0$ (resp. $k = n-2$, resp. $k = n-1$) the sum $\sum_{0\leqslant i\leqslant\lfloor\frac{1}{2}k\rfloor-1}(-1)^i$ (resp. $\sum_{\lfloor\frac{1}{2}k\rfloor+1\leqslant i\leqslant\frac{1}{2}(n-3)}(-1)^i$, which is empty for both $k = n-2$ and $k = n-1$) is empty, hence then (2.51) does not hold as is, yet can still be read as a proof that $[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(n-2,0))]_0 = 0$ (resp. $[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(n-2,0))]_{n-2} = 0$, resp. $[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(n-2,0))]_{n-1} \overset{\text{(ff.1)}}{=} 1$). For the value $[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(n-2,0))]_{n-1} = 1$ keep in mind that $k = n-1$ is even, hence the sums after the second equality in (2.51) now read $\sum_{0\leqslant i\leqslant\lfloor\frac{1}{2}\rfloor-1}(-1)^{i-1}\cdot(-) + \sum_{\lfloor\frac{1}{2}k\rfloor+1\leqslant i\leqslant\frac{1}{2}(n-3)}(-1)^{i-1}\cdot(+)$, so there then is no minus-sign in front of the difference of the two sums; moreover, $n \equiv 3 \pmod 8$ implies that, if $k = n-1$, then $\lfloor\frac{1}{2}k\rfloor = \lfloor\frac{1}{2}(8\mu+3-1)\rfloor = 4\mu+1$ so $\lfloor\frac{1}{2}k\rfloor - 1$ is even, so $[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(n-2,0))]_{n-1} = 1$ computes as $((+)-(0))$, the zero coming from the empty sum. A similar explanation applies to the value $[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(n-2,0))]_{n-2} = 0$. Now (2.50) with $(a,b) = (n-2,0)$ is proved, according to Corollary 64.

As for (2.50) with $(\mathbf{a},\mathbf{b}) = (\mathbf{n-1},\mathbf{1})$, we again check the criterion in Corollary 64, distinguishing parities of k. For reasons explained after the calculation, we first exclude $k = 0$ and $k = 1$. For every even $k \geqslant 2$, by partitioning the index set $[\frac{1}{2}(n-3)]$ into the three sets $\{1,2,\ldots,\lfloor\frac{1}{2}k\rfloor-1\}$, $\{\lfloor\frac{1}{2}k\rfloor\}$ and $\{\lfloor\frac{1}{2}k\rfloor+1,\lfloor\frac{1}{2}k\rfloor+2,\ldots,\frac{1}{2}(n-1)\}$, using Lemma 53.(8) for the last two summands $[\vec{C}_{0,1,2;n}]_k$ and $[\vec{C}_{0,1,n-1;n}]_k$ (which sum to zero), and using Lemma 54.(5).(ii) for the $[\vec{C}_{i;n}]_k$, it follows that

$$
\begin{aligned}
[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(n-1,1))]_k &\overset{\text{(hs.2)}}{=} \Big(\sum_{1\leqslant i\leqslant\frac{1}{2}(n-3)}(-1)^i\cdot[\vec{C}_{i;n}]_k\Big) + [\vec{C}_{\frac{1}{2}(n-1);n}]_k - [\vec{C}_{0,1,2;n}]_k - [\vec{C}_{0,1,n-1;n}]_k \\
\Big(\text{since } [\vec{C}_{\lfloor\frac{1}{2}k\rfloor;n}]_k = 0\Big) &= \left(\sum_{1\leqslant i\leqslant\lfloor\frac{1}{2}k\rfloor-1}(-1)^i\cdot(+) + \sum_{\lfloor\frac{1}{2}k\rfloor+1\leqslant i\leqslant\frac{1}{2}(n-3)}(-1)^i\cdot(-)\right) + (+) - 0 \\
&= 1 + \left(\sum_{1\leqslant i\leqslant\lfloor\frac{1}{2}k\rfloor-1}(-1)^i - \sum_{\lfloor\frac{1}{2}k\rfloor+1\leqslant i\leqslant\frac{1}{2}(n-3)}(-1)^i\right) \\
&= 1 + \begin{cases} (-1)-(0) & \text{by (4) and (3) in Lemma 65} \\ & \text{if } \lfloor\frac{1}{2}k\rfloor \text{ is even} \\ (0)-(+1) & \text{by (3) and (1) in Lemma 65} \\ & \text{if } \lfloor\frac{1}{2}k\rfloor \text{ is odd} \end{cases} \quad \begin{pmatrix} \text{because of} \\ n\equiv 3 \pmod 4, \\ \text{we know that} \\ \frac{1}{2}(n-3) \text{ is even} \end{pmatrix} \\
&= 0 \\
(\text{since } k \neq 0\ ) &= \langle k-1 \wedge k+1\ ,\quad n-1\wedge 1 + 1\wedge 0 + 0\wedge n-1\rangle \\
&\overset{\text{(ff.2)}}{=} [\mathrm{f}_{\mathrm{T}_n}(n-1,1)]_k\ .
\end{aligned}
\tag{2.52}
$$

For every odd $k \in [n-1]\backslash\{1\}$ we can repeat the calculation (2.52), using Lemma 54.(5).(iii) for the values of the inner products, and still find the value 0, in agreement with $[\mathrm{f}_{\mathrm{T}_n}(n-1,1)]_k$. As to the excluded cases, if $k = 0$, the sum $\sum_{1\leqslant i\leqslant\lfloor\frac{1}{2}k\rfloor-1}$ is empty, so the fourth equality in (2.52) does not hold anymore, but the calculation in (2.52) then still can be read as a proof of $[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(n-1,1))]_0 = 1 = \langle\ n-1\wedge 1\ ,\ n-1\wedge 1+1\wedge 0+0\wedge n-1\ \rangle = [\mathrm{f}_{\mathrm{T}_n}(n-1,1)]_0$. Finally, if $k = 1$, we again have to use Lemma 54.(5).(iii) for the values of the inner products of the form $[\vec{C}_{i;n}]_1$, but now one thing is diffferent compared with the other calculations done for (2.50): we cannot use Lemma 53.(8) for the two summands $-[\vec{C}_{0,1,2;n}]_1 - [\vec{C}_{0,1,n-1;n}]_1$ but rather have to separately look up the values in Lemma 53.(2) to find $-[\vec{C}_{0,1,2;n}]_1 - [\vec{C}_{0,1,n-1;n}]_1 = -(0)-(-) = 1$. Now, calculating like in (2.52), we arrive at (immediately leaving out the empty sum) $[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(n-1,1))]_1 = \sum_{\lfloor\frac{1}{2}k\rfloor+1\leqslant i\leqslant\frac{1}{2}(n-3)}(-1)^i\cdot(+) + (-1) + (+1) = 0 + (-1) + (+1) = 0 = [\mathrm{f}_{\mathrm{T}_n}(n-1,1)]_1$, again in agreement. According to Corollary 64, we have now proved (2.50) with $(a,b) = (n-1,1)$.

As for (2.50) with $(\mathbf{a},\mathbf{b}) = (\mathbf{0},\mathbf{2})$, we once again check the criterion in Corollary 64, distinguishing the parities of k. For reasons explained after the calculation we will treat the cases $k = 0$ and $k = 1$ separately at the end. For every even $k \in [n-1]_0$ with $k \geqslant 2$, partitioning the index set $[\frac{1}{2}(n-3)]$ into the three sets $\{1,2,\ldots,\lfloor\frac{1}{2}k\rfloor-1\}$, $\{\lfloor\frac{1}{2}k\rfloor\}$ and $\{\lfloor\frac{1}{2}k\rfloor+1,\lfloor\frac{1}{2}k\rfloor+2,\ldots,\frac{1}{2}(n-1)\}$, and using

Lemma 54.(5).(ii), we have

$$
\begin{aligned}
[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(0,2))]_k &\overset{(\mathrm{hs}.3)}{=} \left(\sum_{i\in[\frac{1}{2}(n-3)]} (-1)^{i-1}\cdot[\vec{C}_{i;n}]_k \right) - [\vec{C}_{\frac{1}{2}(n-1);n}]_k \\
&= \left(\sum_{1\leqslant i\leqslant\lfloor\frac{1}{2}k\rfloor-1} (-1)^{i-1}\cdot(+) + \sum_{\lfloor\frac{1}{2}k\rfloor+1\leqslant i\leqslant\frac{1}{2}(n-3)} (-1)^{i-1}\cdot(-) \right) - (+) \\
&= -1 - \left(\sum_{1\leqslant i\leqslant\lfloor\frac{1}{2}k\rfloor-1} (-1)^{i} - \sum_{\lfloor\frac{1}{2}k\rfloor+1\leqslant i\leqslant\frac{1}{2}(n-3)} (-1)^{i} \right) \\
&= -1 - \begin{cases} (-1)-(0) & \text{by (4) and (3) in Lemma 65 if } \lfloor\frac{1}{2}k\rfloor \text{ is even} \\ (0)-(+1) & \text{by (3) and (4) in Lemma 65 if } \lfloor\frac{1}{2}k\rfloor \text{ is odd} \end{cases} \\
&= 0 \\
&= \langle k-1 \wedge k+1\ ,\quad 0\wedge 2 + 2\wedge 1 + 1\wedge 0\rangle \overset{(\mathrm{ff}.3)}{=} [\mathrm{f}_{\mathrm{T}_n}(0,2)]_k\ .
\end{aligned}
\tag{2.53}
$$

For every odd $k \in [n-1]\backslash\{1\}$, we may repeat the calculation in (2.53), but this time using Lemma 54.(5).(iii) for the evaluations, which makes all the signs of the respective inner products flip, yet still leads to the result 0. There is only one thing to look out for, namely the penultimate equality in (2.53): for odd k, and solely in the case $k = 1$, the inner product $\langle k-1\wedge k+1\ ,\ 0\wedge 2+2\wedge 1+1\wedge 0\rangle$ does not equal 0, but 1. This is a cue for explaining the role of the cases $k = 0$ and $k = 1$: because of $k \geqslant 2$, in the calculation (2.53) we knew the sum $\sum_{1\leqslant i\leqslant\lfloor\frac{1}{2}k\rfloor-1}(-1)^i$ to be non-empty; both for $k = 0$ and $k = 1$, it is empty, and in these two cases the fourth equality in (2.53) does not hold. The end-result then becomes 0 in case of $k = 0$ and $+1$ (keeping in mind the sign-flips in the odd-k-version of (2.53)) in case of $k = 1$. This agrees with the value $\langle k-1\wedge k+1, 0\wedge 2+2\wedge 1+1\wedge 0\rangle$ just mentioned. At this point, (2.50) with $(a, b) = (0, 2)$ is proved, in view of Corollary 64.

As for (2.50) with $(\mathbf{a},\mathbf{b}) = (\mathbf{1}+\mathbf{4}\ell, \mathbf{3}+\mathbf{4}\ell)$ and $\ell = \frac{1}{8}(n-3)$, this means $(a, b) = (\frac{1}{2}(n-1), \frac{1}{2}(n+1)) = (i_n-1, i_n+1)$, and this case is dealt with on p. 88. As for (2.50) with $(\mathbf{a},\mathbf{b}) = (\mathbf{1}+\mathbf{4}\ell, \mathbf{3}+\mathbf{4}\ell)$ and $\ell \in \{0, 1, \ldots, \frac{1}{4}(n-3)-1\}\backslash\{\frac{1}{8}(n-3)\}$, we will carry out the following case analysis, necessitated by the structure of Lemmas 53 and 54:

(0) even k,
- (0).(i) even k and $1 + 2\ell \leqslant \lfloor\frac{1}{2}k\rfloor - 1$,
- (0).(ii) even k and $1 + 2\ell > \lfloor\frac{1}{2}k\rfloor - 1$,

(1) odd k,
- (1).(i) odd k and $1 + 2\ell \leqslant \lfloor\frac{1}{2}k\rfloor - 1$,
- (1).(ii) odd k and $1 + 2\ell > \lfloor\frac{1}{2}k\rfloor - 1$.

As to (0).(i), we will in particularly have occasion to use Lemma 62, which needs the hypothesis $2+2\ell \leqslant \lfloor\frac{1}{2}k\rfloor-1$, so we treat the case $1+2\ell = \lfloor\frac{1}{2}k\rfloor-1$ separately: since k is even, this is equivalent to $k = 4+4\ell$. In the following calculation, we immediately write $\sum_{i=\lfloor\frac{1}{2}k\rfloor+1}^{\frac{1}{2}(n-3)} (-1)^{i-1}\cdot 2\cdot\left[\vec{C}_{i;n}\right]_k$ for the (in view of Lemma 54.(i)) equal sum $\sum_{i=\lfloor\frac{1}{2}k\rfloor}^{\frac{1}{2}(n-3)}(-1)^{i-1}\cdot 2\cdot\big[\vec{C}_{i;n}\big]_k$:

$$
\begin{aligned}
\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(1+4\ell,3+4\ell))\big]_k &\overset{\text{(hs.4)}}{=} \big[\vec{C}_{0;n}\big] + \big[\vec{C}_{\lfloor\frac{1}{2}k\rfloor-1;n}\big]_k + \Big(\sum_{\lfloor\frac{1}{2}k\rfloor+1\leqslant i\leqslant\frac{1}{2}(n-3)}(-1)^{i-1}\cdot 2\cdot\big[\vec{C}_{i;n}\big]_k\Big)\\
&\quad - 2\cdot\big[\vec{C}_{\frac{1}{2}(n-1);n}\big]_k + \big[\vec{C}_{0,1,2;n}\big]_k + \big[\vec{C}_{0,1,n-1;n}\big]_k\\
\left(\begin{array}{l}\text{by Lemma 54.(ii),}\\ \text{and, as far as the}\\ \text{last two summands}\\ \text{are concerned, by}\\ \text{Lemma 53.(8),}\\ \text{applicable since}\\ 1+2\ell\leqslant\lfloor\frac{1}{2}k\rfloor-1\\ \text{implies } k\geqslant 2\end{array}\right) &\overset{(2.74)}{=} (+)+(+)+\Big(\sum_{\lfloor\frac{1}{2}k\rfloor+1\leqslant i\leqslant\frac{1}{2}(n-3)}(-1)^{i-1}\cdot 2\cdot(-)\Big) - 2\cdot(+) + 0\\
&= 2\cdot\sum_{\lfloor\frac{1}{2}k\rfloor+1\leqslant i\leqslant\frac{1}{2}(n-3)}(-1)^i = 2\cdot\sum_{3+2\ell\leqslant i\leqslant\frac{1}{2}(n-3)}(-1)^i = 2\cdot 0 = 0\\
&= \big\langle 3+4\ell\wedge 5+4\ell\ ,\\
&\qquad (1+4\ell)\wedge(3+4\ell) \quad + \quad (3+4\ell)\wedge(4+4\ell)\\
&\qquad + (4+4\ell)\wedge(2+4\ell) \quad + \quad (2+4\ell)\wedge(1+4\ell)\big\rangle\\
&\overset{\text{(ff.4)}}{=} \big[\mathrm{f}_{\mathrm{T}_n}(1+4\ell,3+4\ell)\big]_{4+4\ell} = \big[\mathrm{f}_{\mathrm{T}_n}(1+4\ell,3+4\ell)\big]_k\ . \qquad (2.54)
\end{aligned}
$$

So let us now assume that k is even, and $2+2\ell\leqslant\lfloor\frac{1}{2}k\rfloor-1$. Then in particular $1+2\ell\leqslant\lfloor\frac{1}{2}k\rfloor-1$, hence $\big[\vec{C}_{1+2\ell;n}\big]_k = +$ by Lemma 54.(5).(ii), and we can calculate

$$
\begin{aligned}
\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(1+4\ell,3+4\ell))\big]_k &\overset{\text{(hs.4)}}{=} \big[\vec{C}_{0;n}\big] + \big[\vec{C}_{1+2\ell;n}\big]_k + \Big(\sum_{2+2\ell\leqslant i\leqslant\frac{1}{2}(n-3)}(-1)^{i-1}\cdot 2\cdot\big[\vec{C}_{i;n}\big]_k\Big)\\
&\quad - 2\cdot\big[\vec{C}_{\frac{1}{2}(n-1);n}\big]_k + \big[\vec{C}_{0,1,2;n}\big]_k + \big[\vec{C}_{0,1,n-1;n}\big]_k\\
\left(\begin{array}{l}\text{because of } 2+2\ell\\ \leqslant\lfloor\frac{1}{2}k\rfloor-1,\text{ we can}\\ \text{appeal to Lemma 62.(3)}\\ \text{for evaluating the al-}\\ \text{ternating sum; the last}\\ \text{two summands sum}\\ \text{to zero according to}\\ \text{Lemma 53.(8)}\end{array}\right) &= (+)+(+)+(0)-2\cdot(+)+0 = 2-2\\
&= 0\\
\left(\begin{array}{l}\text{since } 2+2\ell\\ \leqslant\lfloor\frac{1}{2}k\rfloor-1\text{ implies}\\ k-1\geqslant 5+4\ell\end{array}\right) &= \big\langle k-1\wedge k+1,\\
&\qquad (1+4\ell\wedge 3+4\ell) \quad + \quad (3+4\ell\wedge 4+4\ell)\\
&\qquad + (4+4\ell\wedge 2+4\ell) \quad + \quad (2+4\ell\wedge 1+4\ell)\big\rangle\\
&\overset{\text{(ff.4)}}{=} \big[\mathrm{f}_{\mathrm{T}_n}(1+4\ell,3+4\ell)\big]_k\ . \qquad (2.55)
\end{aligned}
$$

This completes the proof of (0).(i).

As to (0).(ii), in order to make Lemma 53.(8) available, we treat the case $k=0$ separately: $\big[\vec{C}_{0;n}\big]_0 = 0$ by Lemma 54.(3).(i). Because of $\ell\leqslant\frac{1}{8}(n-3)-1$, we have $1+2\ell\neq\frac{1}{2}(n-1)$, hence $\langle n-1\wedge 1,\vec{C}_{1+2\ell;n}\rangle = -$ and $\langle n-1\wedge 1,\vec{C}_{i;n}\rangle = -$ for $2+2\ell\leqslant i\leqslant\frac{1}{2}(n-3)<\frac{1}{2}(n-1)$, by

Lemma 54.(3).(ii).Therefore,

$$
\begin{aligned}
\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(1+4\ell,3+4\ell))\big]_k &= \big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(1+4\ell,3+4\ell))\big]_0 \\
&\overset{(\mathrm{hs}.4)}{=} \big[\vec{C}_{0;n}\big]_0 + \big[\vec{C}_{1+2\ell;n}\big]_0 + 2\cdot \sum_{2+2\ell\leqslant i\leqslant \frac{1}{2}(n-3)} (-1)^{i-1}\cdot\big[\vec{C}_{i;n}\big]_0 \\
&\quad - 2\cdot\big[\vec{C}_{\frac{1}{2}(n-1);n}\big]_0 + \big[\vec{C}_{0,1,2;n}\big]_0 + \big[\vec{C}_{0,1,n-1;n}\big]_0 \\
\begin{pmatrix}\text{using Lemma 53.(5)}\\ \text{for the last two}\\ \text{summands}\end{pmatrix} &= (0) + (-) + 2\cdot(\sum_{2+2\ell\leqslant i\leqslant \frac{1}{2}(n-3)} (-1)^{i-1}\cdot(-)) - 2\cdot(0) + (0) + (-) \\
&= -1 + 2\cdot(\sum_{2+2\ell\leqslant i\leqslant \frac{1}{2}(n-3)} (-1)^{i}) - 1 \\
(\text{by Lemma 65.(1) }) &= -1 + 2 + -1 = 0 \\
&= \langle n-1\wedge 1, \\
&\qquad (1+4\ell\wedge 3+4\ell) \quad + \quad (3+4\ell\wedge 4+4\ell) \\
&\quad + (4+4\ell\wedge 2+4\ell) \quad + \quad (2+4\ell\wedge 1+4\ell)\big\rangle \\
&\overset{(\mathrm{ff}.4)}{=} \big[\mathrm{f}_{\mathrm{T}_n}(1+4\ell,3+4\ell)\big]_0 = \big[\mathrm{f}_{\mathrm{T}_n}(1+4\ell,3+4\ell)\big]_k \ .
\end{aligned} \tag{2.56}
$$

So we may now assume that k is even and $1+2\ell > \lfloor\frac{1}{2}k\rfloor - 1 = \frac{1}{2}k-1$, i.e. $k < 4+4\ell$, and in addition to that may assume $k\geqslant 2$. Then

$$
\begin{aligned}
\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(1+4\ell,3+4\ell))\big]_k &\overset{(\mathrm{hs}.4)}{=} \big[\vec{C}_{0;n}\big]_k + \big[\vec{C}_{1+2\ell;n}\big]_k + 2\cdot \sum_{2+2\ell\leqslant i\leqslant \frac{1}{2}(n-3)} (-1)^{i-1}\cdot\big[\vec{C}_{i;n}\big]_k \\
&\quad - 2\big[\vec{C}_{\frac{1}{2}(n-1);n}\big]_k + \big[\vec{C}_{0,1,2;n}\big]_k + \big[\vec{C}_{0,1,n-1;n}\big]_k \\
\begin{pmatrix}\text{by Lemma 54.(ii), and}\\ \text{since } 1+2\ell > \lfloor\frac{1}{2}k\rfloor - 1\\ \text{in particular implies } 2+\\ 2\ell \geqslant \lfloor\frac{1}{2}k\rfloor + 1\text{; the last}\\ \text{two summands sum to}\\ \text{zero by Lemma 53.(8),}\\ \text{applicable because of our}\\ \text{assumption } k\geqslant 2.\end{pmatrix} &= (+) + (-) + 2\cdot(\sum_{2+2\ell\leqslant i\leqslant \frac{1}{2}(n-3)} (-1)^{i-1}\cdot(-)) - 2\cdot(+) + 0 \\
&= 2\cdot(\sum_{2+2\ell\leqslant i\leqslant \frac{1}{2}(n-3)} (-1)^{i}) - 2 \\
(\text{by Lemma 65.(1) }) &= 2\cdot(+1) - 2 \\
&= 0 \\
&= \big\langle k-1\wedge k+1, \\
&\qquad (1+4\ell\wedge 3+4\ell) \quad + \quad (3+4\ell\wedge 4+4\ell) \\
&\quad + (4+4\ell\wedge 2+4\ell) \quad + \quad (2+4\ell\wedge 1+4\ell)\big\rangle \\
&\overset{(\mathrm{ff}.4)}{=} [\mathrm{f}_{\mathrm{T}_n}(1+4\ell,3+4\ell)]_k \ ,
\end{aligned} \tag{2.57}
$$

where the penultimate equality holds since k being even implies $1+2\ell > \lfloor\frac{1}{2}k\rfloor - 1 = \frac{1}{2}k-1$, hence $k-1 < 3+4\ell$, so the smaller vertex in $k-1\wedge k+1$ is smaller than $3+4\ell$, which is the smallest vertex in the two elementary 1-chains with label-difference two inside the 1-chain $\mathrm{f}_{\mathrm{T}_n}(3+4\ell,1+4\ell)$, which implies that for each of the ℓ under consideration, the two 1-chains in the arguments of the inner product have disjoint supports. This completes the proof of (0).(ii), and the proof of (0).

As for (1).(i), we would like to re-use the above analysis of the even-k-case. In the separately-treated case $1+2\ell = \lfloor\frac{1}{2}k\rfloor - 1$ we used evenness of k. If k is odd, then $1+2\ell = \lfloor\frac{1}{2}k\rfloor - 1$ is not equivalent to $k = 4+4\ell$ but equivalent to $k = 5+4\ell$, i.e. we are then taking the inner product with the elementary 1-chain $4+4\ell \wedge 6+4\ell$, not $3+4\ell \wedge 5+4\ell$. Since the calculation in (2.54) in its structure only depends on the size-hypothesis $1+2\ell \leqslant \lfloor \frac{1}{2}k \rfloor - 1$, not the parity of k, and since the sign-flips demanded by Lemma 54 on account of the oddness of k again lead to the result 0, the only thing that remains to be changed in that calculation is to replace '$3+4\ell \wedge 5+4\ell$' after the penultimate equality in (2.54) with '$4+4\ell \wedge 6+4\ell$'. The rest of our analysis of the even-k-case only depended on size-assumptions and can be read (flipping all signs of relevant values of inner products) as a proof for (1).(i).

As for (1).(ii), we again can use the above analysis of the case of even $k \geqslant 2$; there, we did not use evenness of k except when Lemma 54 was invoked for values of inner products of the form $\langle k-1 \wedge k+1, \quad \cdot \quad \rangle$ with $k \geqslant 1$. Since the values for odd k are obtained from those for even k by flipping all signs, the calculation (2.57) can be read as a proof of (1).(ii), with the necessary modifications. This completes the proof of (2.50) in the case $(a,b) = (1+4\ell, 3+4\ell)$.

As for (2.50) with $(\mathbf{a},\mathbf{b}) = (\mathbf{i_n}-\mathbf{1}, \mathbf{i_n}+\mathbf{1})$, we will distinguish the following cases:

(0) $k=0$,
(1) $k=1$,
(2) $k \geqslant 2$,
  (1) even $k \geqslant 2$
    (1) even $k \geqslant 2$ and $\frac{1}{4}(n+5) \leqslant \lfloor \frac{1}{2}k \rfloor - 1$,
    (2) even $k \geqslant 2$ and $\frac{1}{4}(n+5) > \lfloor \frac{1}{2}k \rfloor - 1$, and $\frac{1}{4}(n+5) > \lfloor \frac{1}{2}k \rfloor$,
    (3) even $k \geqslant 2$ and $\frac{1}{4}(n+5) > \lfloor \frac{1}{2}k \rfloor - 1$, and $\frac{1}{4}(n+5) \leqslant \lfloor \frac{1}{2}k \rfloor$,
  (2) odd $k \geqslant 2$
    (1) odd $k \geqslant 2$ and $\frac{1}{4}(n+5) \leqslant \lfloor \frac{1}{2}k \rfloor - 1$,
    (2) odd $k \geqslant 2$ and $\frac{1}{4}(n+5) > \lfloor \frac{1}{2}k \rfloor - 1$, and $\frac{1}{4}(n+5) > \lfloor \frac{1}{2}k \rfloor$,
    (3) odd $k \geqslant 2$ and $\frac{1}{4}(n+5) > \lfloor \frac{1}{2}k \rfloor - 1$, and $\frac{1}{4}(n+5) \leqslant \lfloor \frac{1}{2}k \rfloor$.

As for (0), if $k=0$, then $\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n+1))\big]_k = \big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n+1))\big]_0 \overset{(\mathrm{hs}.5)}{=} [\vec{C}_{0;n}]_0 + [\vec{C}_{\frac{1}{4}(n+1);n}]_0 + (\sum_{i=\frac{1}{4}(n+5)}^{\frac{1}{2}(n-3)} (-1)^{i-1} \cdot 2 \cdot [\vec{C}_{i;n}]_0) - 2 \cdot [\vec{C}_{\frac{1}{2}(n-1);n}]_0 + 2 \cdot [\vec{C}_{0,1,2;n}]_0 + [\vec{C}_{0,1,n-1;n}]_0 = (0) + (-) + (\sum_{i=\frac{1}{4}(n+5)}^{\frac{1}{2}(n-3)} (-1)^{i-1} \cdot 2 \cdot (-)) - 2 \cdot (0) + 2 \cdot (0) + (-) = -2 + 2 \cdot \sum_{i=\frac{1}{4}(n+5)}^{\frac{1}{2}(n-3)} (-1)^i =$ (since both $\frac{1}{4}(n+5)$ and $\frac{1}{2}(n-3)$ are even) $= -2+2 = 0$. Here we used Lemma 54.(3).(ii) for the $[\vec{C}_{i;n}]_0$. Let us note that $0 < \frac{1}{4}(n+5) \leqslant i \leqslant \frac{1}{2}(n-3) < \frac{1}{2}(n-1)$, so the exceptional values in Lemma 54.(3).(i) play no role inside the alternating sum. Moreover, we used Lemma 53.(5) for $[\vec{C}_{0,1,n-1;n}]_0$ (e.g. $[\vec{C}_{0,1,n-1;n}]_0 = \langle n-1 \wedge 1, \vec{C}_{0,1,n-1;n} \rangle = -\langle 1 \wedge n-1, \vec{C}_{0,1,n-1;n} \rangle = (-)$). The value found for $\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n+1))\big]_0$ agrees with the value $\big[\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n+1)\big]_0 = 0$ found in Lemma 59.(2) (note that for $k=0$ we have $k - \frac{1}{2}(n+3) \equiv 1 \pmod 4$).

As for (1), if $k=1$, then $\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n+1))\big]_k = \big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n+1))\big]_1 \overset{(\mathrm{hs}.5)}{=} [\vec{C}_{0;n}]_1 + [\vec{C}_{\frac{1}{4}(n+1);n}]_1 + (\sum_{i=\frac{1}{4}(n+5)}^{\frac{1}{2}(n-3)} (-1)^{i-1} \cdot 2 \cdot [\vec{C}_{i;n}]_1) - 2 \cdot [\vec{C}_{\frac{1}{2}(n-1);n}]_1 + 2 \cdot [\vec{C}_{0,1,2;n}]_1 + [\vec{C}_{0,1,n-1;n}]_1 = (0) + (+) + (\sum_{i=\frac{1}{4}(n+5)}^{\frac{1}{2}(n-3)} (-1)^{i-1} \cdot 2 \cdot (+)) - 2 \cdot (-) + 2 \cdot (0) + (-) = 0$, in particular, by Lemma 54.(5): if $k=1$, then $[\lfloor \frac{1}{2}k \rfloor - 1]_0 = \varnothing$, so $[\vec{C}_{0;n}]_1$ has to be looked up in Lemma 54.(5) and is $[\vec{C}_{0;n}]_1 = 0$. The value found for $\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n+1))\big]_0$ agrees with the value $\big[\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n+1)\big]_0 = 0$ found in Lemma 59.(3) (note that for $k=1$ we have $k - \frac{1}{2}(n+3) \equiv 2 \pmod 4$).

As for case (2), if $k \geqslant 2$, then, independently of the cases to come, we know from Lemma 53.(8) that

$$2 \cdot \big[\vec{C}_{0,1,2;n}\big]_k + [\vec{C}_{0,1,n-1;n}]_k = \big[\vec{C}_{0,1,2;n}\big]_k \ . \tag{2.58}$$

The relation (2.58), which only needs $k \geqslant 2$ as its hypothesis and does not depend on us currently trying to prove (2.50) with $(a,b) = (i_n-1, i_n+1)$, will also be referenced in the proof of (2.50) with $(a,b) = (i_n-2, i_n)$ further below.

As for (2).(1).(1), if $k \geqslant 2$ is even and $\frac{1}{4}(n+5) \leqslant \lfloor \frac{1}{2}k \rfloor - 1$, then the sum $\sum_{i=\frac{1}{4}(n+5)}^{\frac{1}{2}(n-3)} (-1)^{i-1} \cdot 2 \cdot [\vec{C}_{i;n}]_k$ in the following calculation is non-empty: splitting the alternating sum around the $([\vec{C}_{i;n}]_k = 0)$-implying index-value $i = \lfloor \frac{1}{2}k \rfloor$,

$$
\begin{aligned}
\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n+1))\big]_k &\stackrel{(\mathrm{hs}.5)}{=} [\vec{C}_{0;n}]_k + [\vec{C}_{\frac{1}{4}(n+1);n}]_k + \Big(\sum_{i=\frac{1}{4}(n+5)}^{\frac{1}{2}(n-3)} (-1)^{i-1}\cdot 2\cdot [\vec{C}_{i;n}]_k\Big)\\
&\quad - 2\cdot[\vec{C}_{\frac{1}{2}(n-1);n}]_k + 2\cdot[\vec{C}_{0,1,2;n}]_k + [\vec{C}_{0,1,n-1;n}]_k\\
&= [\vec{C}_{0;n}]_k + [\vec{C}_{\frac{1}{4}(n+1);n}]_k + \Big(\sum_{i=\frac{1}{4}(n+5)}^{\lfloor\frac{1}{2}k\rfloor-1} (-1)^{i-1}\cdot 2\cdot [\vec{C}_{i;n}]_k\Big)\\
&\quad + \Big(\sum_{i=\lfloor\frac{1}{2}k\rfloor+1}^{\frac{1}{2}(n-3)} (-1)^{i-1}\cdot 2\cdot [\vec{C}_{i;n}]_k\Big)\\
&\quad - 2\cdot[\vec{C}_{\frac{1}{2}(n-1);n}]_k + 2\cdot[\vec{C}_{0,1,2;n}]_k + [\vec{C}_{0,1,n-1;n}]_k\\
\left(\begin{array}{l}\tfrac{1}{4}(n+5)\leqslant\lfloor\tfrac{1}{2}k\rfloor-1 \text{ implies}\\ \tfrac{1}{4}(n+1)\leqslant\lfloor\tfrac{1}{2}k\rfloor-1,\\ \text{hence } [\vec{C}_{\frac{1}{4}(n+1);n}]_k = +\\ \text{by Lemma 54.(5).(ii); more-}\\ \text{over, that lemma also gives}\\ [\vec{C}_{\frac{1}{2}(n-1);n}]_k\end{array}\right) &\stackrel{(2.58)}{=} (+)+(+)+\Big(\sum_{i=\frac{1}{4}(n+5)}^{\lfloor\frac{1}{2}k\rfloor-1}(-1)^{i-1}\cdot 2\cdot(+)\Big)\\
&\quad + \Big(\sum_{i=\lfloor\frac{1}{2}k\rfloor+1}^{\frac{1}{2}(n-3)}(-1)^{i-1}\cdot 2\cdot(-)\Big) - 2\cdot(+) + \big[\vec{C}_{0,1,2;n}\big]_k\\
&= \big[\vec{C}_{0,1,2;n}\big]_k - 2\left(\sum_{i=\frac{1}{4}(n+5)}^{\lfloor\frac{1}{2}k\rfloor-1}(-1)^i - \sum_{i=\lfloor\frac{1}{2}k\rfloor+1}^{\frac{1}{2}(n-3)}(-1)^i\right)\\
&= \begin{cases} 1-2((0)-(0)) = 1 & \text{if } k\equiv 0 \pmod 4\text{, by Lemma 53 and Lemma 65.(2)}\\ 0-2((+)-(+)) = 0 & \text{if } k\equiv 2 \pmod 4\text{, by Lemma 53 and Lemma 65.(1)}\end{cases}.
\end{aligned}
\tag{2.59}
$$

We now compare the values found in (2.59) with the values of $[\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n+1)]_k$ in the cases $k\equiv 0 \pmod 4$ and $k\equiv 2 \pmod 4$, using Lemma 59. If $k=4\nu$, $\nu\in\mathbb{Z}$, then, using $n=8\mu+3$ with $\mu\in\mathbb{Z}$, $k-\frac{1}{2}(n+3)=4(\nu-\mu)-3\equiv 1 \pmod 4$, hence $[\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n+1)]_k = 1$ by Lemma 59.(2), except when $k=0$; the latter exception being irrelevant since here we are assuming $k\geqslant 2$, this result is in agreement with the first case at the end of (2.59). If $k=4\nu+2$, $\nu\in\mathbb{Z}$, then $k-\frac{1}{2}(n+3)=4(\nu-\mu)-1\equiv 3 \pmod 4$, hence $[\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n+1)]_k=0$, except when $k=i_n$; the latter exception is again irrelevant since we are currently assuming $\frac{1}{4}(n+1)\leqslant\lfloor\frac{1}{2}k\rfloor-1=\frac{1}{2}k-1$, hence $k\geqslant\frac{1}{2}(n+5)>\frac{1}{2}(n+1)=i_n$. Thus, in the case $k\equiv 2 \pmod 4$, too, the result agrees with (the second case at the end of) the calculation (2.59).

As for (2).(1).(2) and (2).(1).(3), i.e. if $k\geqslant 2$ is even and $\frac{1}{4}(n+5)>\lfloor\frac{1}{2}k\rfloor-1$, then splitting the sum is not necessary: by Lemma 54.(5).(ii), every summand $[\vec{C}_{i;n}]_k$ in the sum $\sum_{i=\frac{1}{4}(n+5)}^{\frac{1}{2}(n-3)}(-1)^{i-1}\cdot 2\cdot[\vec{C}_{i;n}]_k$ is $(-)$ then. However, for the evaluation of the alternating sum, we have to distinguish whether $\frac{1}{4}(n+5)=\lfloor\frac{1}{2}k\rfloor$ (in which case the non-zero summands start from $\frac{1}{4}(n+5)+1=\frac{1}{4}(n+9)$), this being the reasons why there are the cases (2) and (3) within case (2).(1).

As for (2).(1).(2), if $k\geqslant 2$ is even, $\frac{1}{4}(n+5)>\lfloor\frac{1}{2}k\rfloor-1$ and in addition to that, $\frac{1}{4}(n+5)>\lfloor\frac{1}{2}k\rfloor$, then

$$\begin{aligned}
\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n+1))\big]_k &\overset{(\mathrm{hs}.5)}{=} [\vec{C}_{0;n}]_k + [\vec{C}_{\frac{1}{4}(n+1);n}]_k + (\sum_{i=\frac{1}{4}(n+5)}^{\frac{1}{2}(n-3)} (-1)^{i-1}\cdot 2\cdot [\vec{C}_{i;n}]_k) \\
&\quad - 2\cdot [\vec{C}_{\frac{1}{2}(n-1);n}]_k + 2\cdot [\vec{C}_{0,1,2;n}]_k + [\vec{C}_{0,1,n-1;n}]_k \\
&= (+) + [\vec{C}_{\frac{1}{4}(n+1);n}]_k + (\sum_{i=\frac{1}{4}(n+5)}^{\frac{1}{2}(n-3)} (-1)^{i-1}\cdot 2\cdot (-)) \\
&\quad - 2\cdot (+) + 2\cdot [\vec{C}_{0,1,2;n}]_k + [\vec{C}_{0,1,n-1;n}]_k \\
&\overset{(2.58)}{=} -1 + [\vec{C}_{\frac{1}{4}(n+1);n}]_k + [\vec{C}_{0,1,2;n}]_k + 2\cdot (\sum_{i=\frac{1}{4}(n+5)}^{\frac{1}{2}(n-3)} (-1)^{i}) \ . \qquad (2.60)
\end{aligned}$$

Because of $n \equiv 3 \pmod 8$, both $\frac{1}{4}(n+5)$ and $\frac{1}{2}(n-3)$ are even, so $2\cdot(\sum_{i=\frac{1}{4}(n+5)}^{\frac{1}{2}(n-3)}(-1)^i) = 2$ and (2.60) simplifies to

$$\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n+1))\big]_k = 1 + [\vec{C}_{\frac{1}{4}(n+1);n}]_k + [\vec{C}_{0,1,2;n}]_k \ , \qquad (2.61)$$

hence,

(1) if $\frac{1}{4}(n+1) = \lfloor\frac{1}{2}k\rfloor$, then $[\vec{C}_{\frac{1}{4}(n+1);n}]_k = 0$ by Lemma 54.(5).(i), and $[\vec{C}_{0,1,2;n}]_k = 0$ because of Lemma 53.(7) and since in the current case we have $k = \frac{1}{2}(n+1) \equiv 2 \pmod 4$. Thus $\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n+1))\big]_k = 1$, by (2.61). This then agrees with $\big[\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n+1)\big]_k$ since for $k = \frac{1}{2}(n+1) = i_n$, and because of $-1 \equiv 3 \pmod 4$, case (4) in Lemma 59 applies and tells us that indeed $\big[\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n+1)\big]_k = 1$.

(2) if $\frac{1}{4}(n+1) \geqslant \lfloor\frac{1}{2}k\rfloor + 1$, then $[\vec{C}_{\frac{1}{4}(n+1);n}]_k = -1$ by Lemma 54.(5).(ii), so (2.61) implies

$$\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n+1))\big]_k = [\vec{C}_{0,1,2;n}]_k = \begin{cases} 1 & \text{if } k \equiv 0 \pmod 4, \\ 0 & \text{if } k \equiv 2 \pmod 4, \end{cases} \qquad (2.62)$$

the cases according to Lemma 53.(7). The values in (2.62) agree with the values for $\big[\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n+1)\big]_k$: if $k \equiv 0 \pmod 4$, then $k - \frac{1}{2}(n+3) \equiv 1 \pmod 4$, hence $\big[\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n+1)\big]_k = 1$ by Lemma 59.(2); if $k \equiv 2 \pmod 4$, then $k - \frac{1}{2}(n+3) \equiv 3 \pmod 4$, hence $\big[\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n+1)\big]_k = 0$ by Lemma 59.(4).

As for (2).(1).(3), i.e. if $k \geqslant 2$ is even and $\frac{1}{4}(n+5) > \lfloor\frac{1}{2}k\rfloor - 1$ but $\frac{1}{4}(n+1) \leqslant \lfloor\frac{1}{2}k\rfloor - 1$, then, by integrality, $\frac{1}{4}(n+1) = \lfloor\frac{1}{2}k\rfloor - 1$. Since k is assumed to be even, $\lfloor\frac{1}{2}k\rfloor - 1 = \frac{1}{2}k - 1$, so we are speaking about the sole case $k = \frac{1}{2}(n+1) + 2 = \frac{1}{2}(n+5) = i_n + 1$ here. Then we cannot re-use (2.61), which was predicated on $\sum_{i=\frac{1}{4}(n+5)}^{\frac{1}{2}(n-3)}(-1)^{i-1}\cdot[\vec{C}_{i;n}]_k = +$, while this sum is now 0, because of $\frac{1}{4}(n+5) = \lfloor\frac{1}{2}k\rfloor$. We therefore calculate again:

$$\begin{aligned}
\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n+1))\big]_{i_n+1} &\overset{(\mathrm{hs}.5)}{=} [\vec{C}_{0;n}]_{i_n+1} + [\vec{C}_{\frac{1}{4}(n+1);n}]_{i_n+1} + (\sum_{i=\frac{1}{4}(n+5)}^{\frac{1}{2}(n-3)} (-1)^{i-1}\cdot 2\cdot [\vec{C}_{i;n}]_{i_n+1}) \\
&\quad - 2\cdot [\vec{C}_{\frac{1}{2}(n-1);n}]_{i_n+1} + 2\cdot [\vec{C}_{0,1,2;n}]_{i_n+1} + [\vec{C}_{0,1,n-1;n}]_{i_n+1} \\
&= (+) + (+) + 2\cdot(0) - 2\cdot(+) + 2\cdot(+) + (-) = 1 \ . \qquad (2.63)
\end{aligned}$$

We compare (2.63) with $\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n+1))\big]_{i_n+1}$: because of $\frac{1}{2}(n+5) - \frac{1}{2}(n+3) = 1 \equiv 1 \pmod 4$, so Lemma 59.(2) tells us that $\big[\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n+1)\big]_{i_n+1} = 1$, in agreement with (2.63).

As for (1).(2).(1)—(3), the calculations in case (1) constitute a proof for this case, too, with the necessary small modfications on where to look up the individual values of the scalar product. E.g. for (1).(2).(1), using Lemma 54.(5).(iii) for the individual values we then arrive at

$$\begin{aligned}\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n+1))\big]_k &= \big[\vec{C}_{0,1,2;n}\big]_k + 2\cdot\Big(\sum_{i=\frac{1}{4}(n+5)}^{\lfloor\frac{1}{2}k\rfloor-1}(-1)^i - \sum_{i=\lfloor\frac{1}{2}k\rfloor+1}^{\frac{1}{2}(n-3)}(-1)^i\Big)\\ &= \begin{cases}(0)-2\cdot((0)-(0))=0 & \text{if } k\equiv 1 \pmod 4\text{, by Lemma 53,}\\ & \text{and (2) and (3) in Lemma 65}\\ (+)-2\cdot((+)-(+))=1 & \text{if } k\equiv 3 \pmod 4\text{, by Lemma 53}\\ & \text{and (1) in Lemma 65}\end{cases}.\end{aligned} \tag{2.64}$$

We now compare the values found in (2.64) with the values of $[\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n+1)]_k$ in the cases $k\equiv 1 \pmod 4$ and $k\equiv 3 \pmod 4$, using Lemma 59. If $k=4\nu+1$, $\nu\in\mathbb{Z}$, then, using $n=8\mu+3$ with $\mu\in\mathbb{Z}$, $k-\frac{1}{2}(n+3)=4(\nu-\mu)-2\equiv 2 \pmod 4$, hence $[\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n+1)]_k=0$ by Lemma 59.(3), except when $k=0$; the latter exception being irrelevant since we are assuming $k\geqslant 2$, this result is in agreement with the first case at the end of (2.64). If $k=4\nu+3$, $\nu\in\mathbb{Z}$, then $k-\frac{1}{2}(n+3)=4(\nu-\mu)\equiv 0 \pmod 4$, hence $[\mathrm{f}_{\mathrm{T}_n}(i_n-1,i_n+1)]_k=1$, except when $k=i_n$; the latter exception is again irrelevant since we are currently assuming $\frac{1}{4}(n+1)\leqslant\lfloor\frac{1}{2}k\rfloor-1=\frac{1}{2}(k-1)-1$, i.e. $k\geqslant\frac{1}{2}(n+7)>\frac{1}{2}(n+1)=i_n$. Thus, in the case $k\equiv 3 \pmod 4$, too, the result agrees with (the second case at the end of) (2.64). This completes the proof of (2.50) in the case $(a,b)=(i_n-1,i_n+1)$.

As for (2.50) with $(\mathbf{a},\mathbf{b})=(\mathbf{4}+\mathbf{4}\ell,\mathbf{6}+\mathbf{4}\ell)$ and $\ell=\frac{1}{8}(n-3)-1$, this is the case $(a,b)=(i_n-2,i_n)$, that we deal with on p. 94. As for (2.50) with $(\mathbf{a},\mathbf{b})=(\mathbf{4}+\mathbf{4}\ell,\mathbf{6}+\mathbf{4}\ell)$ and $\ell\in\{0,1,\dots,\frac{1}{8}(n-3)-1\}\backslash\{\frac{1}{8}(n-3)-1\}$, we carry out a case analysis analogous to the one for $(a,b)=(1+4\ell,3+4\ell)$; since the linear combination $\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(4+4\ell,6+4\ell))$ in (hs.6) has its alternating sum start at $3+2\ell$, instead of the $2+2\ell$ for $\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(1+4\ell,3+4\ell))$, the cases now pivot around $2+2\ell$:

- (0) even k,
  - (0).(i) even k and $2+2\ell\leqslant\lfloor\frac{1}{2}k\rfloor-1$,
  - (0).(ii) even k and $2+2\ell>\lfloor\frac{1}{2}k\rfloor-1$,
- (1) odd k,
  - (1).(i) odd k and $2+2\ell\leqslant\lfloor\frac{1}{2}k\rfloor-1$,
  - (1).(ii) odd k and $2+2\ell>\lfloor\frac{1}{2}k\rfloor-1$.

As for (0).(i), since we will later apply Lemma 62.(1), for which we need to know $3+2\ell\leqslant\lfloor\frac{1}{2}k\rfloor-1$, we treat the case $2+2\ell=\lfloor\frac{1}{2}k\rfloor-1$ separately: since k is even, this is equivalent to $k=6+4\ell$. Therefore, in this case (dropping the summand with $[\vec{C}_{\lfloor\frac{1}{2}k\rfloor;n}]_k=0$, we immediately write $\sum_{\lfloor\frac{1}{2}k\rfloor+1\leqslant i\leqslant\frac{1}{2}(n-3)}(-1)^{i-1}\cdot 2\cdot[\vec{C}_{i;n}]_k$ instead of the (equal) sum $\sum_{\lfloor\frac{1}{2}k\rfloor\leqslant i\leqslant\frac{1}{2}(n-3)}(-1)^{i-1}\cdot 2\cdot[\vec{C}_{i;n}]_k$ that one gets when directly substituting the definition of $\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(4+4\ell,6+4\ell))$ from (hs.6)):

$$\begin{aligned}
[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(4+4\ell,6+4\ell))]_k &= \big[\vec{C}_{0;n}\big] - [\vec{C}_{2+2\ell;n}]_k + (\sum_{3+2\ell\leqslant i\leqslant \frac{1}{2}(n-3)} (-1)^{i-1}\cdot 2\cdot [\vec{C}_{i;n}]_k) \\
&\quad - 2\cdot\big[\vec{C}_{\frac{1}{2}(n-1);n}\big]_k + \big[\vec{C}_{0,1,2;n}\big]_k + \big[\vec{C}_{0,1,n-1;n}\big]_k \\
&= \big[\vec{C}_{0;n}\big] - [\vec{C}_{\lfloor\frac{1}{2}k\rfloor-1;n}]_k + (\sum_{\lfloor\frac{1}{2}k\rfloor+1\leqslant i\leqslant \frac{1}{2}(n-3)} (-1)^{i-1}\cdot 2\cdot [\vec{C}_{i;n}]_k) \\
&\quad - 2\cdot\big[\vec{C}_{\frac{1}{2}(n-1);n}\big]_k + \big[\vec{C}_{0,1,2;n}\big]_k + \big[\vec{C}_{0,1,n-1;n}\big]_k \\
&= (+) - (+) + 2\cdot \sum_{\lfloor\frac{1}{2}k\rfloor+1\leqslant i\leqslant \frac{1}{2}(n-3)} (-1)^{i-1}\cdot(-) - 2\cdot(+) + 0 \\
&= 2\cdot(\sum_{i=\lfloor\frac{1}{2}k\rfloor+1}^{\frac{1}{2}(n-3)} (-1)^i) - 2 = 2\cdot(\sum_{i=4+2\ell}^{\frac{1}{2}(n-3)}(-1)^i) - 2 = 2-2 = 0 \\
&= \big\langle 5+4\ell \wedge 7+4\ell, \\
&\qquad (4+4\ell\wedge 6+4\ell) \quad + \quad (6+4\ell\wedge 5+4\ell) \\
&\quad + (5+4\ell\wedge 3+4\ell) \quad + \quad (3+4\ell\wedge 4+4\ell)\big\rangle \\
&= [\mathrm{f}_{\mathrm{T}_n}(4+4\ell,6+4\ell)]_k\ .
\end{aligned} \tag{2.65}$$

So now suppose that $3+2\ell \leqslant \lfloor\frac{1}{2}k\rfloor - 1$. Then $[\vec{C}_{2+2\ell;n}]_k = +$ by Lemma 54.(5).(ii), and we can calculate as follows (for the second equality in particular using Lemma 62.(1), which is applicable because of the assumption $3+2\ell \leqslant \lfloor\frac{1}{2}k\rfloor - 1$, and using Lemma 53.(8), which is applicable since $2+2\ell \leqslant 3+2\ell \leqslant \lfloor\frac{1}{2}k\rfloor - 1$ implies $k \geqslant 2$):

$$\begin{aligned}
[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(4+4\ell,6+4\ell))]_k &= \big[\vec{C}_{0;n}\big] - [\vec{C}_{2+2\ell;n}]_k \\
&\quad + (\sum_{3+2\ell\leqslant i\leqslant \frac{1}{2}(n-3)} (-1)^{i-1}\cdot 2\cdot [\vec{C}_{i;n}]_k) \\
&\quad - 2\cdot\big[\vec{C}_{\frac{1}{2}(n-1);n}\big]_k + \big[\vec{C}_{0,1,2;n}\big]_k + \big[\vec{C}_{0,1,n-1;n}\big]_k \\
&= (+) - (+) + 2 - 2\cdot(+) + 0 \\
&= 0 \\
\begin{pmatrix}\text{since } 2+2\ell \leqslant \lfloor\frac{1}{2}k\rfloor - 1 \\ \leqslant \frac{1}{2}(k-1) - 1 \\ \text{implies } k+1 \geqslant 8+4\ell\end{pmatrix} &= \big\langle k-1 \wedge k+1, \\
&\qquad (4+4\ell\wedge 6+4\ell) \quad + \quad (6+4\ell\wedge 5+4\ell) \\
&\quad + (5+4\ell\wedge 3+4\ell) \quad + \quad (3+4\ell\wedge 4+4\ell)\big\rangle \\
(\text{by Lemma 56.(ff.6)}) &= [\mathrm{f}_{\mathrm{T}_n}(4+4\ell,6+4\ell)]_k\ ,
\end{aligned} \tag{2.66}$$

completing the prooof in the case (0).(i).

As for (0).(ii), then, $\frac{1}{2}(k-1) \leqslant \lfloor\frac{1}{2}k\rfloor \leqslant 2+2\ell$, hence $k \leqslant 5+4\ell < 8+4\ell$, so, by Lemma 62.(2),

$$\sum_{3+2\ell\leqslant i\leqslant \frac{1}{2}(n-3)} (-1)^{i-1}\cdot 2\cdot [\vec{C}_{i;n}]_k = 0\ . \tag{2.67}$$

Since Lemma 54.(5) requires $k \geqslant 1$, whereas $k = 0$ is treated as a special case in Lemma 54, we have to treat this case separately: if $k = 0$, then (using Lemma 54.(3) and noticing that the hypothesis $n \equiv 3 \pmod 4$ implies both $2+2\ell \neq 0$ and $2+2\ell \neq \frac{1}{2}(n-1)$) it follows that $[\vec{C}_{2+2\ell;n}]_0$

$= \langle n-1 \wedge 1, \vec{C}_{2+2\ell;n} \rangle = -$, and we can calculate

$$\begin{aligned}
[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(4+4\ell,6+4\ell))]_k &= [\vec{C}_{0;n}]_0 - [\vec{C}_{2+2\ell;n}]_0 + (\sum_{3+2\ell \leqslant i \leqslant \frac{1}{2}(n-3)} (-1)^{i-1} \cdot 2 \cdot [\vec{C}_{i;n}]_0) \\
&\quad - 2 \cdot [\vec{C}_{\frac{1}{2}(n-1);n}]_0 + [\vec{C}_{0,1,2;n}]_0 + [\vec{C}_{0,1,n-1;n}]_0 \\
&= 0 - (-) + (0) - 2 \cdot (0) + 0 + (-) = 1 - 1 \\
&= 0 \\
&= \langle n-1 \wedge 1, \\
&\qquad (4+4\ell \wedge 6+4\ell) \quad + \quad (6+4\ell \wedge 5+4\ell) \\
&\quad + (5+4\ell \wedge 3+4\ell) \quad + \quad (3+4\ell \wedge 4+4\ell)\rangle \\
\text{(by Lemma 56.(ff.6))} &= [\mathrm{f}_{\mathrm{T}_n}(4+4\ell,6+4\ell)]_0 \ .
\end{aligned} \tag{2.68}$$

For $2+2\ell \geqslant \lfloor \frac{1}{2}k \rfloor$ and $k \geqslant 2$, though, we can use Lemma 54.(5).(ii). Because of the structure of Lemma 54.(5) we have to further distinguish whether $2+2\ell = \lfloor \frac{1}{2}k \rfloor$ or $2+2\ell \geqslant \lfloor \frac{1}{2}k \rfloor + 1$.

If $2+2\ell = \lfloor \frac{1}{2}k \rfloor$, then $[\vec{C}_{2+2\ell;n}]_k = 0$, and, using (2.67),

$$\begin{aligned}
[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(4+4\ell,6+4\ell))]_k &= \big[\vec{C}_{0;n}\big]_k - [\vec{C}_{2+2\ell;n}]_k + (\sum_{3+2\ell \leqslant i \leqslant \frac{1}{2}(n-3)} (-1)^{i-1} \cdot 2 \cdot [\vec{C}_{i;n}]_k) \\
&\quad - 2 \cdot \big[\vec{C}_{\frac{1}{2}(n-1);n}\big]_k + \big[\vec{C}_{0,1,2;n}\big]_k + \big[\vec{C}_{0,1,n-1;n}\big]_k \\
\begin{pmatrix}\text{the last two summands} \\ \text{sum to 0 by (8) in Lemma 53}\end{pmatrix} &= (+) - (0) + (0) - 2 \cdot (+) + 0 \\
&= -1 \\
&= \langle 3+4\ell \wedge 5+4\ell, \\
&\qquad (4+4\ell \wedge 6+4\ell) \quad + \quad (6+4\ell \wedge 5+4\ell) \\
&\quad + (5+4\ell \wedge 3+4\ell) \quad + \quad (3+4\ell \wedge 4+4\ell)\rangle \\
\text{(by Lemma 56.(ff.6))} &= \langle (4+4\ell)-1 \wedge (4+4\ell)+1, \mathrm{f}_{\mathrm{T}_n}(4+4\ell,6+4\ell)\rangle \\
\begin{pmatrix}\text{by Lemma 56.(ff.6),} \\ \text{and since in the} \\ \text{present case } 2+2\ell = \\ \lfloor \frac{1}{2}k \rfloor = \frac{1}{2}k, \text{ i.e., } k = \\ 4+4\ell\end{pmatrix} &= \langle k-1 \wedge k+1, \mathrm{f}_{\mathrm{T}_n}(4+4\ell,6+4\ell)\rangle \ .
\end{aligned} \tag{2.69}$$

In case of $2+2\ell \geqslant \lfloor \frac{1}{2}k \rfloor + 1$, however, Lemma 54.(5) tells us that $[\vec{C}_{2+2\ell;n}]_k = -$, hence (for the second equality we use that $2+2\ell \geqslant \lfloor \frac{1}{2}k \rfloor + 1$ implies $3+2\ell \geqslant \lfloor \frac{1}{2}k \rfloor$, so we can use Lemma 62.(2) for the alternating sum and Lemma 53.(8) for the sum of the last two summands),

$$\begin{aligned}
[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(4+4\ell,6+4\ell))]_k &= \big[\vec{C}_{0;n}\big] - [\vec{C}_{2+2\ell;n}]_k + (\sum_{3+2\ell \leqslant i \leqslant \frac{1}{2}(n-3)} (-1)^{i-1} \cdot 2 \cdot [\vec{C}_{i;n}]_k) \\
&\quad - 2 \cdot \big[\vec{C}_{\frac{1}{2}(n-1);n}\big]_k + \big[\vec{C}_{0,1,2;n}\big]_k + \big[\vec{C}_{0,1,n-1;n}\big]_k \\
&= (+) - (-) + (0) - 2 \cdot (+) + 0 = 2 - 2 \\
&= 0 \\
&= [\mathrm{f}_{\mathrm{T}_n}(4+4\ell,6+4\ell)]_k \ ,
\end{aligned} \tag{2.70}$$

the penultimate equality since, for even k, by Lemma 56.(ff.6), the only chance for a non-zero inner product between $k-1 \wedge k+1$ and $\mathrm{f}_{\mathrm{T}_n}(4+4\ell,6+4\ell)$ is $k-1 = 3+4\ell$, i.e. $k = 4+4\ell$, which is impossible since we are currently assuming $2+2\ell \geqslant \lfloor \frac{1}{2}k \rfloor + 1$. This completes the proof of (0).(ii).

As for (1), most of that case can be deduced from the above analysis of the even-k-case. This was the reason why, whenever sufficient, we did not dispense with the $\lfloor \cdot \rfloor$ in $\lfloor \frac{1}{2}k \rfloor$ despite evenness of k, and used estimates like $\frac{1}{2}(k-1) \leqslant \lfloor \frac{1}{2}k \rfloor$ instead: this way, most of the calculations already done can be re-used for the odd-k-case. Let us first note that

(1) the values of the inner products in the odd-k-case (iii) of Lemma 54.(5) are obtained from the even-k-case (ii) just by flipping all signs,
(2) each of
 (i) the case-analyses within the above treatment of the even-k-case,
 (ii) the reasons given for $\langle k-1 \wedge k+1, \mathrm{f}_{\mathrm{T}_n}(4+4\ell, 6+4\ell)\rangle = 0$ in (2.70),

 depend on the *relative sizes* of ℓ and k only, not on further parity considerations,
(3) if in (2.66), (2.68) and (2.70) all signs are flipped, the calculation is still correct.

The only cases in the analysis of the even-k-case which are not immediately re-usable for the odd-k-case is $2+2\ell = \lfloor \frac{1}{2}k \rfloor$, with the attendant calculation (2.69). For odd $k \in [n-1]$, the hypothesis $2+2\ell = \lfloor \frac{1}{2}k \rfloor$ now means $2+2\ell = \frac{1}{2}(k-1) = \lfloor \frac{1}{2}k \rfloor$, i.e. $5+4\ell = k$, and when using this value for k, the values of the inner products work out correctly: it is then still true that $\langle k-1 \wedge k+1, \vec{C}_{2+2\ell;n}\rangle = 0$, and (by flipping each sign of an inner-product-value), the calculation (2.69) then leads to $\langle k-1 \wedge k+1, \mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(4+4\ell, 6+4\ell))\rangle = +1$. Moreover, because we now have $k = 5+4\ell$, the remaining part of the calculation works out, too, since indeed $\langle 4+4\ell \wedge 6+4\ell, \mathrm{f}_{\mathrm{T}_n}(4+4\ell, 6+4\ell)\rangle = +1$. This completes the proof of (2.50) for $(a,b) = (4+4\ell, 6+4\ell)$.

As for (2.50) with $(\mathbf{a},\mathbf{b}) = (\mathbf{i_n-2}, \mathbf{i_n})$, $i_n = \frac{1}{2}(n+1)$, in general we have

$$\left[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-2, i_n))\right]_k \overset{(\mathrm{hs}.7)}{=} \left[\vec{C}_{0;n}\right]_k - \left[\vec{C}_{\frac{1}{4}(n-3);n}\right]_k + 2 \cdot \sum_{\frac{1}{4}(n+1) \leqslant i \leqslant \frac{1}{2}(n-3)} (-1)^{i-1} \cdot \left[\vec{C}_{i;n}\right]_k - 2 \cdot \left[\vec{C}_{\frac{1}{2}(n-1);n}\right]_k + 2 \cdot \left[\vec{C}_{0,1,2;n}\right]_k + \left[\vec{C}_{0,1,n-1;n}\right]_k . \tag{2.71}$$

We will now prove (2.50) by splitting (2.71) into the following (exhaustive and mutually exclusive) cases, which seem to be dictated by the structure of the graphs $\mathrm{C}_n^{2^-}$ and the auxiliary lemmas about values of the inner products $[\cdot]_k$. In each case we evaluate $\left[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-2, i_n))\right]_k$ via (2.71) and find the result to agree with $\left[\mathrm{f}_{\mathrm{T}_n}(i_n-2, i_n)\right]_k$. This then constitutes a proof of (2.50), in view of Corollary 64. For various reasons we do away with the cases $k=0$ and $k=1$ separately:

(0) $k = 0$,
(1) $k = 1$,
(2) $k \geqslant 2$,
 (1) even $k \geqslant 2$
  (1) even $k \geqslant 2$ and $\frac{1}{4}(n-3) \geqslant \lfloor \frac{1}{2}k \rfloor + 1$,
  (2) even $k \geqslant 2$ and $\frac{1}{4}(n-3) = \lfloor \frac{1}{2}k \rfloor$,
  (3) even $k \geqslant 2$ and $\frac{1}{4}(n-3) = \lfloor \frac{1}{2}k \rfloor - 1$,
  (4) even $k \geqslant 2$ and $\frac{1}{4}(n-3) \leqslant \lfloor \frac{1}{2}k \rfloor - 2$,
 (2) odd $k \geqslant 2$
  (1) odd $k \geqslant 2$ and $\frac{1}{4}(n-3) \geqslant \lfloor \frac{1}{2}k \rfloor + 1$,
  (2) odd $k \geqslant 2$ and $\frac{1}{4}(n-3) = \lfloor \frac{1}{2}k \rfloor$,
  (3) odd $k \geqslant 2$ and $\frac{1}{4}(n-3) = \lfloor \frac{1}{2}k \rfloor - 1$,
  (4) odd $k \geqslant 2$ and $\frac{1}{4}(n-3) \leqslant \lfloor \frac{1}{2}k \rfloor - 2$.

Case (0). If $k = 0$, then $[\vec{C}_{0;n}]_k = \langle n-1 \wedge 1, \vec{C}_{0;n}\rangle = 0$ by Lemma 54.(3).(i), $[\vec{C}_{\frac{1}{4}(n-3);n}]_k = \langle n-1 \wedge 1, \vec{C}_{\frac{1}{4}(n-3);n}\rangle = -$ by Lemma 54.(3).(ii) (since $\frac{1}{4}(n-3) \notin \{0, \frac{1}{2}(n-1)\}$), $[\vec{C}_{i;n}]_k = \langle n-1 \wedge 1, \vec{C}_{i;n}\rangle = -$ for all $\frac{1}{4}(n+1) \leqslant i \leqslant \frac{1}{2}(n-3)$ by Lemma 54.(3).(ii) (since $0 < \frac{1}{4}(n+1)$ and $\frac{1}{2}(n-3) < \frac{1}{2}(n-1)$), $[\vec{C}_{\frac{1}{2}(n-1);n}]_k = \langle n-1 \wedge 1, \vec{C}_{\frac{1}{2}(n-1);n}\rangle = 0$ by Lemma 54.(3).(i), and $[\vec{C}_{0,1,2;n}]_k = \langle n-1 \wedge 1, \vec{C}_{0,1,2;n}\rangle = 0$ and $[\vec{C}_{0,1,n-1;n}]_k = \langle n-1 \wedge 1, \vec{C}_{0,1,n-1;n}\rangle = -$ by Lemma 53.(5), so we can evaluate (2.71) as follows:

$$\begin{aligned}
\bigl[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n))\bigr]_k &= \bigl[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n))\bigr]_0 \\
&\overset{(\mathrm{hs}.7)}{=} \langle n-1\wedge 1,\vec{C}_{0;n}\rangle - \langle n-1\wedge 1,\vec{C}_{\frac14(n-3);n}\rangle \\
&\quad + 2\cdot\sum_{i=\frac14(n+1)}^{\frac12(n-3)}(-1)^{i-1}\cdot\langle n-1\wedge 1,\vec{C}_{i;n}\rangle - 2\cdot\langle n-1\wedge 1,\vec{C}_{\frac12(n-1);n}\rangle \\
&\quad + 2\cdot\langle n-1\wedge 1,\vec{C}_{0,1,2;n}\rangle + \langle n-1\wedge 1,\vec{C}_{0,1,n-1;n}\rangle \\
&= (0)-(-)+2\cdot\Bigl(\sum_{\frac14(n+1)\leqslant i\leqslant\frac12(n-3)}(-1)^{i-1}\cdot(-)\Bigr)-2\cdot 0+2\cdot 0+(-) \\
&= 1+2\cdot 0-0+0-1 \\
&= 0 \\
&= \langle n-1\wedge 1,\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n)\rangle = [\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n)]_k\ , \qquad (2.72)
\end{aligned}$$

the penultimate equality in view of Lemma 56.(ff.7), where obviously the elementary 1-chain $n-1\wedge 1$ does not appear in the 1-chain given there (in fact, the only elementary 1-chain in that chain which contains the vertex $n-1$ is $n-1\wedge 0 \neq n-1\wedge 1$).

Case (1). If $k=1$, then $\lfloor\frac12 k\rfloor = 0$, so Lemma 54.(5).(i) tells us $\langle k-1\wedge k+1,\vec{C}_{0;n}\rangle = \langle 0\wedge 2,\vec{C}_{0;n}\rangle = 0$. Moreover, $\lfloor\frac12 k\rfloor+1 = 1$, hence the second line of Lemma 54.(5).(iii) applies to every $\vec{C}_{i;n}$ except $\vec{C}_{\frac12(n-1);n}$, hence, in particular, $[\vec{C}_{\frac14(n-3);n}]_k = \langle 0\wedge 2,\vec{C}_{\frac14(n-3);n}\rangle = +$, and $\langle 0\wedge 2,\vec{C}_{i;n}\rangle = +$ for every $i\in\{\frac14(n+1),\ldots,\frac12(n-3)\}$. Moreover, $\langle 0\wedge 2,\vec{C}_{0,1,2;n}\rangle = 0$ and $\langle 0\wedge 2,\vec{C}_{0,1,n-1;n}\rangle = -$ by Lemma 53.(2). Therefore, we can evaluate (2.71) as follows:

$$\begin{aligned}
\bigl[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n))\bigr]_k &= \bigl[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n))\bigr]_1 \\
&= \langle 0\wedge 2,\vec{C}_{0;n}\rangle - \langle 0\wedge 2,\vec{C}_{\frac14(n-3);n}\rangle \\
&\quad + 2\cdot\sum_{i=\frac14(n+1)}^{\frac12(n-3)}(-1)^{i-1}\cdot\langle 0\wedge 2,\vec{C}_{i;n}\rangle - 2\cdot\langle 0\wedge 2,\vec{C}_{\frac12(n-1);n}\rangle \\
&\quad + 2\cdot\langle 0\wedge 2,\vec{C}_{0,1,2;n}\rangle + \langle 0\wedge 2,\vec{C}_{0,1,n-1;n}\rangle \\
&= (0)-(+)+2\cdot\left(\sum_{i=\frac14(n+1)}^{\frac12(n-3)}(-1)^{i-1}\cdot(+)\right)-2\cdot(-)+2\cdot(0)+(-) \\
&= -1+0+2+0-1 \\
&= 0 \\
&= \langle 0\wedge 2,\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n)\rangle = [\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n)]_k\ , \qquad (2.73)
\end{aligned}$$

where the alternating sum equals (0) because $n\equiv 3 \pmod 8$ implies that $\frac14(n+1)$ is odd while $\frac12(n-3)$ is even, and $0 = \langle 0\wedge 2,\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n)\rangle$ holds since in Lemma 56.(ff.7), the explicit 1-chain given for $\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n)$ evidently does not contain the elementary 1-chain $0\wedge 2$ (note that $\Sigma_0^{\frac18(n-11)}(i_n \mid {\scriptstyle +2,-1,+2,+1})$, which is the only summand in $\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n)$ for which it is not obvious whether the vertex 0 occurs in it, has $i_n+4\cdot\frac18(n-11) = \frac12(n+1)+\frac12(n-11) = n-5$ as its largest vertex. This completes the analysis of the case $k=1$.

Case (2). This case needs further analysis.

Case (2).(1). Since $k\geqslant 2$ is even, by (5).(ii) in Lemma 54 it follows that

$$[\vec{C}_{0;n}]_k = \bigl[\vec{C}_{\frac12(n-1);n}\bigr]_k = +\ . \qquad (2.74)$$

Case (2).(1).(1). If $k \geqslant 2$ is even and $\frac{1}{4}(n-3) \geqslant \lfloor \frac{1}{2}k \rfloor + 1$, then $\frac{1}{4}(n+1) \geqslant \lfloor \frac{1}{2}k \rfloor + 1$, too, hence Lemma 54.(ii) implies that both $\big[\vec{C}_{\frac{1}{4}(n-3);n}\big]_k = -$ and $\big[\vec{C}_{i;n}\big]_k = -$ for all $\big[\vec{C}_{i;n}\big]_k$ in the sum $\sum_{\frac{1}{4}(n+1) \leqslant i \leqslant \frac{1}{2}(n-3)} (-1)^{i-1} \cdot \big[\vec{C}_{i;n}\big]_k$. Since $n \equiv 3 \pmod 8$, we know that $\frac{1}{4}(n+1)$ is odd and $\frac{1}{2}(n-3)$ is even, hence $\sum_{\frac{1}{4}(n+1) \leqslant i \leqslant \frac{1}{2}(n-3)} (-1)^{i-1} \cdot \big[\vec{C}_{i;n}\big]_k = \sum_{\frac{1}{4}(n+1) \leqslant i \leqslant \frac{1}{2}(n-3)} (-1)^{i-1} \cdot (-) = \sum_{\frac{1}{4}(n+1) \leqslant i \leqslant \frac{1}{2}(n-3)} (-1)^{i} = 0$ by Lemma 65.(3); therefore,

$$\begin{aligned} \big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n - 2, i_n))\big]_k &\overset{(2.71)}{=} \big[\vec{C}_{0;n}\big]_k - \big[\vec{C}_{\frac{1}{4}(n-3);n}\big]_k + 2 \cdot \sum_{\frac{1}{4}(n+1) \leqslant i \leqslant \frac{1}{2}(n-3)} (-1)^{i-1} \cdot \big[\vec{C}_{i;n}\big]_k \\ &\quad - 2 \cdot \big[\vec{C}_{\frac{1}{2}(n-1);n}\big]_k + 2 \cdot \big[\vec{C}_{0,1,2;n}\big]_k + \big[\vec{C}_{0,1,n-1;n}\big]_k \\ &\overset{(2.58),(2.74)}{=} (+) - (-) + 2 \cdot (0) - 2 \cdot (+) + \big[\vec{C}_{0,1,2;n}\big]_k = \big[\vec{C}_{0,1,2;n}\big]_k \,, \end{aligned} \tag{2.75}$$

hence, by Lemma 53.(7) and (2.75),

(0) if $k \geqslant 2$ even, $\frac{1}{4}(n-3) \geqslant \lfloor \frac{1}{2}k \rfloor + 1$ and $k \equiv 0 \pmod 4$, then $\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n - 2, i_n))\big]_k = \big[\vec{C}_{0,1,2;n}\big]_k = 1$,
(1) if $k \geqslant 2$ even, $\frac{1}{4}(n-3) \geqslant \lfloor \frac{1}{2}k \rfloor + 1$ and $k \equiv 2 \pmod 4$, then $\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n - 2, i_n))\big]_k = \big[\vec{C}_{0,1,2;n}\big]_k = 0$.

Now we compare, again assuming $k \geqslant 2$ even and $\frac{1}{4}(n-3) \geqslant \lfloor \frac{1}{2}k \rfloor + 1$, the two evaluations of $\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n - 2, i_n))\big]_k$ in (0) and (1) with the values for $\big[\mathrm{f}_{\mathrm{T}_n}(i_n - 2, i_n)\big]_k$. Let us recall that we are assuming $n = 8\mu + 3$ with $\mu \in \mathbb{Z}$. Under these assumptions,

(f.0) if $k \equiv 0 \pmod 4$, i.e. $k = 4\nu$ with $\nu \in \mathbb{Z}$, then $k - \frac{1}{2}(n+3) = 4\nu - \frac{1}{2}(8\mu + 6) = 4(\nu - \mu) - 3 \equiv 1 \pmod 4$, so $[\mathrm{f}_{\mathrm{T}_n}(i_n - 2, i_n)]_k = 1$ by Lemma 58.(2), in agreement with (0),
(f.1) if $k \equiv 2 \pmod 4$, i.e. $k = 4\nu + 2$ with $\nu \in \mathbb{Z}$, then $k - \frac{1}{2}(n+3) = 4\nu + 2 - (4\mu + 3) = -1 \equiv 3 \pmod 4$, so $[\mathrm{f}_{\mathrm{T}_n}(i_n - 2, i_n)]_k = 0$ by Lemma 58.(4), in agreement with (1),

which completes the proof of Case (2).(1).(1) in our proof of (2.50).

As to (2).(1).(2)–(2).(1).(4), in all these three cases we have $k \geqslant 2$ even and $\frac{1}{4}(n-3) < \lfloor \frac{1}{2}k \rfloor + 1$, and we have to split the sum in (2.71) around $i = \lfloor \frac{1}{2}k \rfloor$. We now simplify (2.71) from p. 94 as much as we presently can. We immediately leave out $i = \lfloor \frac{1}{2}k \rfloor$ from the sum (as the summand is $[\vec{C}_{i;n}]_k = 0$ then); moreover, we use (2.58) and (2.74), and find:

$$\begin{aligned} \big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n - 2, i_n))\big]_k &\overset{(2.71)}{=} \big[\vec{C}_{0;n}\big]_k - \big[\vec{C}_{\frac{1}{4}(n-3);n}\big]_k + 2 \cdot \sum_{\frac{1}{4}(n+1) \leqslant i \leqslant \frac{1}{2}(n-3)} (-1)^{i-1} \cdot \big[\vec{C}_{i;n}\big]_k \\ &\quad - 2 \cdot \big[\vec{C}_{\frac{1}{2}(n-1);n}\big]_k + 2 \cdot \big[\vec{C}_{0,1,2;n}\big]_k + [\vec{C}_{0,1,n-1;n}]_k \\ \begin{pmatrix} \text{using Lemma 54.(5).(ii)} \\ \text{for the } [\vec{C}_{i;n}]_k \end{pmatrix} &\overset{(2.58),(2.74)}{=} (+) - \big[\vec{C}_{\frac{1}{4}(n-3);n}\big]_k \\ &\quad + 2 \cdot \sum_{i=\frac{1}{4}(n+1)}^{\lfloor \frac{1}{2}k \rfloor - 1} (-1)^{i-1} \cdot (+) + 2 \cdot \sum_{i=\lfloor \frac{1}{2}k \rfloor + 1}^{\frac{1}{2}(n-3)} (-1)^{i-1} \cdot (-) \\ &\quad - 2 \cdot (+) + [\vec{C}_{0,1,2;n}]_k \\ &= -1 - \big[\vec{C}_{\frac{1}{4}(n-3);n}\big]_k + [\vec{C}_{0,1,2;n}]_k \\ &\quad + 2 \cdot \left(\sum_{i=\frac{1}{4}(n+1)}^{\lfloor \frac{1}{2}k \rfloor - 1} (-1)^{i-1} - \sum_{i=\lfloor \frac{1}{2}k \rfloor + 1}^{\frac{1}{2}(n-3)} (-1)^{i-1} \right) , \end{aligned} \tag{2.76}$$

To evaluate (2.76), we have to treat the cases (2).(1).(2)–(2).(1).(4) separately.

Case (2).(1).(2). If $\frac{1}{4}(n-3) = \lfloor \frac{1}{2}k \rfloor$, then $\big[\vec{C}_{\frac{1}{4}(n-3);n}\big]_k = 0$ by Lemma 54.(5).(i), the first sum in (2.76) is empty, and the second sum equal to $\sum_{\frac{1}{4}(n+1)\leqslant i\leqslant \frac{1}{2}(n-3)}(-1)^{i-1}$. Because of $n \equiv 3 \pmod 8$, $\frac{1}{4}(n+1)$ is odd and $\frac{1}{2}(n-3)$ is even, so $\sum_{\frac{1}{4}(n+1)\leqslant i\leqslant \frac{1}{2}(n-3)}(-1)^{i-1} = 0$. Hence, by (2.76),

$$\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n))\big]_k = -1 - 0 + [\vec{C}_{0,1,2;n}]_k + 2\cdot(0-0) = -1 + [\vec{C}_{0,1,2;n}]_k = 0, \tag{2.77}$$

the last equality since under the present assumptions (in particular $n = 8\mu+3$, $\mu \in \mathbb{N}$) we know that $\frac{1}{2}k = \lfloor\frac{1}{2}k\rfloor = \frac{1}{4}(n-3) = 2\mu$, i.e. $k = 4\mu$, hence $-1+[\vec{C}_{0,1,2;n}]_k = -1+(+) = 0$ by Lemma 53.(7). We now compare the value in (2.77) with $\big[\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n)\big]_k$. We have $k-\frac{1}{2}(n+3) = 4\mu-(4\mu+3) = -3 \equiv 1 \pmod 4$, and $\frac{1}{4}(n-3) = \lfloor\frac{1}{2}k\rfloor = \frac{1}{2}k$, i.e. $k = \frac{1}{2}(n-3)$, so this is the exceptional case in Lemma 58.(2); thus, $\big[\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n)\big]_k = 0$, in agreement with (2.77), completing the proof of (2).(1).(2).

Case (2).(1).(3). If $\frac{1}{4}(n-3) = \lfloor\frac{1}{2}k\rfloor - 1$, then $\big[\vec{C}_{\frac{1}{4}(n-3);n}\big]_k = +$ by Lemma 54.(5), the first sum in (2.76) is still empty, and the second sum equal to $\sum_{\frac{1}{4}(n+5)\leqslant i\leqslant \frac{1}{2}(n-3)}(-1)^{i-1}$. Our assumption $n \equiv 3 \pmod 8$ implies both $\frac{1}{4}(n+5)$ and $\frac{1}{2}(n-3)$ are even, hence $\sum_{\frac{1}{4}(n+5)\leqslant i\leqslant \frac{1}{2}(n-3)}(-1)^{i-1} = -$. Now (2.76) gives

$$\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n))\big]_k = -1 - (+) + [\vec{C}_{0,1,2;n}]_k + 2\cdot(0-(-)) = [\vec{C}_{0,1,2;n}]_k = 0, \tag{2.78}$$

the last equality since in this paragraph we know $\frac{1}{2}k - 1 = \lfloor\frac{1}{2}k\rfloor - 1 = \frac{1}{4}(n-3)$, i.e. $k = \frac{1}{2}(n+1)$, which because of $n \equiv 3 \pmod 8$ implies $k \equiv 2 \pmod 4$, hence $[\vec{C}_{0,1,2;n}]_k = 0$ by Lemma 53.(7). Now we again compare the value in (2.78) with $[\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n)]_k$. We now have $k - \frac{1}{2}(n+3) = \frac{1}{2}(n+1) - \frac{1}{2}(n+3) = -1 \equiv 3 \pmod 4$, hence by Lemma 58.(4) we know $\big[\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n)\big]_k = 0$, which agrees with (2.78), completing the proof of (2).(1).(3).

Case (2).(1).(4). If $\frac{1}{4}(n-3) \leqslant \lfloor\frac{1}{2}k\rfloor - 2$, then $[\vec{C}_{\frac{1}{4}(n-3);n}]_k = +$ by Lemma 54.(5); moreover, $\frac{1}{4}(n+1) \leqslant \lfloor\frac{1}{2}k\rfloor - 1$, hence the first sum in (2.76) is not empty. Our assumption $n \equiv 3 \pmod 8$ implies that $\frac{1}{4}(n+1)$ is odd while $\frac{1}{2}(n-3)$ is even. If $\lfloor\frac{1}{2}k\rfloor-1$ is even (resp. odd), then $\lfloor\frac{1}{2}k\rfloor+1$ is even (resp. odd), too, and then $\sum_{i=\frac{1}{4}(n+1)}^{\lfloor\frac{1}{2}k\rfloor-1}(-1)^{i-1} = 0$ (resp. $= +$), while $\sum_{i=\lfloor\frac{1}{2}k\rfloor+1}^{\frac{1}{2}(n-3)}(-1)^{i-1} = -$ (resp. $= 0$), and, using (2.76), we can then calculate $\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n))\big]_k = -1-(+)+[\vec{C}_{0,1,2;n}]_k+2\cdot(0-(-)) = [\vec{C}_{0,1,2;n}]_k$ (resp. $\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n))\big]_k = -1-(+)+[\vec{C}_{0,1,2;n}]_k+2\cdot(+-(0)) = [\vec{C}_{0,1,2;n}]_k$. Hence, independent of the parity of $\lfloor\frac{1}{2}k\rfloor-1$,

$$\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n))\big]_k = [\vec{C}_{0,1,2;n}]_k\ . \tag{2.79}$$

In view of the structure of Lemma 53.(7), we have to distinguish the two possible remainders of the even k modulo 4, but both these cases can be dealt with analogously: if $k \equiv 0 \pmod 4$ (resp. if $k \equiv 2 \pmod 4$), then, by (2.79) and Lemma 53.(7), $\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n))\big]_k = 1$ (resp. $\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n))\big]_k = 0$). We now compare this with $\big[\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n)\big]_k$: at present we know $\frac{1}{2}k-2 = \lfloor\frac{1}{2}k\rfloor-2 \geqslant \frac{1}{4}(n-3)$, i.e., $k \geqslant \frac{1}{2}(n+5)$, while $k = 4\mu$ (resp. $k = 4\mu+2$) and $n = 8\nu+3$ imply $k-\frac{1}{2}(n+3) = 4(\mu-\nu)-3 \equiv 1 \pmod 4$ (resp. $k-\frac{1}{2}(n+3) = 4(\mu-\nu)-1 \equiv 3 \pmod 4$); therefore $\big[\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n)\big]_k = 1$ (resp. $\big[\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n)\big]_k = 0$), by Lemma 58.(2) (resp. by Lemma 58.(4)), in particular, because $k \geqslant \frac{1}{2}(n+5)$ implies that the exceptional case in Lemma 58.(2) does not occur here. For both even remainders of k modulo 4 we have found the value of $\big[\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n)\big]_k$ to be in agreement with the value just obtained for $\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n))\big]_k$, completing the proof of (2).(1).(4). This completes the proof of Case (2).(1).

Case (2).(2). Now we know that $k \geqslant 2$ is odd. Therefore, by (5).(iii) in Lemma 54,

$$[\vec{C}_{0;n}]_k = \big[\vec{C}_{\frac{1}{2}(n-1);n}\big]_k = -\ . \tag{2.80}$$

Case (2).(2).(1). If $k \geqslant 2$ is odd and $\frac{1}{4}(n-3) \geqslant \lfloor\frac{1}{2}k\rfloor + 1$, then $\frac{1}{4}(n+1) \geqslant \lfloor\frac{1}{2}k\rfloor + 1$, too, hence Lemma 54.(iii) implies that both $\big[\vec{C}_{\frac{1}{4}(n-3);n}\big]_k = +$ and $\big[\vec{C}_{i;n}\big]_k = +$ for all $\big[\vec{C}_{i;n}\big]_k$ in the sum $\sum_{\frac{1}{4}(n+1)\leqslant i\leqslant\frac{1}{2}(n-3)}(-1)^{i-1}\cdot\big[\vec{C}_{i;n}\big]_k$. Since $n\equiv 3 \pmod 8$, we know that $\frac{1}{4}(n+1)$ is odd and $\frac{1}{2}(n-3)$ is even, hence $\sum_{\frac{1}{4}(n+1)\leqslant i\leqslant\frac{1}{2}(n-3)}(-1)^{i-1}\cdot\big[\vec{C}_{i;n}\big]_k=0$, by Lemma 65.(3); therefore,

$$\begin{aligned}\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n))\big]_k &\overset{(2.71)}{=} \big[\vec{C}_{0;n}\big]_k-\big[\vec{C}_{\frac{1}{4}(n-3);n}\big]_k+2\cdot\sum_{\frac{1}{4}(n+1)\leqslant i\leqslant\frac{1}{2}(n-3)}(-1)^{i-1}\cdot\big[\vec{C}_{i;n}\big]_k\\ &\quad-2\cdot\big[\vec{C}_{\frac{1}{2}(n-1);n}\big]_k+2\cdot\big[\vec{C}_{0,1,2;n}\big]_k+[\vec{C}_{0,1,n-1;n}]_k\\ &\overset{(2.58),(2.80)}{=}(-)-(+)+2\cdot 0-2\cdot(-)+[\vec{C}_{0,1,2;n}]_k\\ &=[\vec{C}_{0,1,2;n}]_k\ ,\end{aligned}\tag{2.81}$$

hence, by Lemma 53.(7) and (2.81),

(0) if $k\geqslant 2$ odd, $\frac{1}{4}(n-3)\geqslant\lfloor\frac{1}{2}k\rfloor+1$ and $k\equiv 1 \pmod 4$, then $\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n))\big]_k=[\vec{C}_{0,1,2;n}]_k=0$,
(1) if $k\geqslant 2$ odd, $\frac{1}{4}(n-3)\geqslant\lfloor\frac{1}{2}k\rfloor+1$ and $k\equiv 3 \pmod 4$, then $\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n))\big]_k=[\vec{C}_{0,1,2;n}]_k=1$.

Now we compare, assuming that $k\geqslant 2$ is odd and that $\frac{1}{4}(n-3)\geqslant\lfloor\frac{1}{2}k\rfloor+1$, the two evaluations of $\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n))\big]_k$ in (0) and (1) with the values for $\big[\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n)\big]_k$. We will use that we are assuming $n=8\mu+3$ with some $\mu\in\mathbb{N}$. Under these assumptions,

(f.0) if $k\equiv 1 \pmod 4$, i.e. $k=4\nu+1$, $\nu\in\mathbb{Z}$, then $k-\frac{1}{2}(n+3)=4(\nu-\mu)-2\equiv 2 \pmod 4$, so $[\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n)]_k=0$ by Lemma 58.(3) (in particular, since $\frac{1}{4}(n-3)\ngeqslant\lfloor\frac{1}{4}(n-1)\rfloor+1$, the sole exception mentioned there cannot occur under our present assumption $\frac{1}{4}(n-3)\geqslant\lfloor\frac{1}{2}k\rfloor+1$); this agrees with (0),
(f.1) if $k\equiv 3 \pmod 4$, i.e. $k=4\nu+3$ with $\nu\in\mathbb{Z}$, then $k-\frac{1}{2}(n+3)=0 \pmod 4$, so $[\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n)]_k=1$ by Lemma 58.(1), which agrees with (1).

This completes the proof of Case (2).(2).(1).

As to (2).(2).(2)–(2).(2).(4) on p. 94, in all these three cases we have $k\geqslant 2$ odd and $\frac{1}{4}(n-3)<\lfloor\frac{1}{2}k\rfloor+1$, and we have to split the sum in (2.71) around $i=\lfloor\frac{1}{2}k\rfloor$. Analogously to (2.76), we again simplify (2.71) as much as possible with the information currently at hand. We immediately leave out $i=\lfloor\frac{1}{2}k\rfloor$ from the sum (since then $[\vec{C}_{i;n}]_k=0$); moreover, we use (2.80) and (2.58), and find:

$$\begin{aligned}\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n))\big]_k &\overset{(2.71)}{=} \big[\vec{C}_{0;n}\big]_k-\big[\vec{C}_{\frac{1}{4}(n-3);n}\big]_k+2\cdot\sum_{\frac{1}{4}(n+1)\leqslant i\leqslant\frac{1}{2}(n-3)}(-1)^{i-1}\cdot\big[\vec{C}_{i;n}\big]_k\\ &\quad-2\cdot\big[\vec{C}_{\frac{1}{2}(n-1);n}\big]_k+2\cdot\big[\vec{C}_{0,1,2;n}\big]_k+[\vec{C}_{0,1,n-1;n}]_k\\ \begin{pmatrix}\text{using}\\ \text{Lemma 54.(5).(iii)}\\ \text{for the }[\vec{C}_{i;n}]_k\end{pmatrix}&\overset{(2.58),(2.80)}{=}(-)-\big[\vec{C}_{\frac{1}{4}(n-3);n}\big]_k\\ &\quad+2\cdot\sum_{i=\frac{1}{4}(n+1)}^{\lfloor\frac{1}{2}k\rfloor-1}(-1)^{i-1}\cdot(-)+2\cdot\sum_{i=\lfloor\frac{1}{2}k\rfloor+1}^{\frac{1}{2}(n-3)}(-1)^{i-1}\cdot(+)\\ &\quad-2\cdot(-)+[\vec{C}_{0,1,2;n}]_k\\ &=1-\big[\vec{C}_{\frac{1}{4}(n-3);n}\big]_k+[\vec{C}_{0,1,2;n}]_k\\ &\quad-2\cdot\left(\sum_{i=\frac{1}{4}(n+1)}^{\lfloor\frac{1}{2}k\rfloor-1}(-1)^{i-1}-\sum_{i=\lfloor\frac{1}{2}k\rfloor+1}^{\frac{1}{2}(n-3)}(-1)^{i-1}\right)\ .\end{aligned}\tag{2.82}$$

To evaluate (2.82), we have to treat the cases (2).(2).(2)–(2).(2).(4) separately.

Case (2).(2).(2). If $k \geqslant 2$ is odd and $\frac{1}{4}(n-3) = \lfloor \frac{1}{2}k \rfloor$, then $[\vec{C}_{\frac{1}{4}(n-3);n}]_k = 0$ by Lemma 54.(5).(i), the first sum in (2.82) is empty, and the second sum is equal to $\sum_{i=\frac{1}{4}(n+1)}^{\frac{1}{2}(n-3)}(-1)^{i-1}$. Because of $n \equiv 3 \pmod 8$, we know $\frac{1}{4}(n+1)$ to be odd and $\frac{1}{2}(n-3)$ to be even, hence $\sum_{i=\frac{1}{4}(n+1)}^{\frac{1}{2}(n-3)}(-1)^{i-1} = 0$. Therefore, by (2.82),

$$\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n))\big]_k = 1 + [\vec{C}_{0,1,2;n}]_k = 1, \tag{2.83}$$

the last equality since under the present assumptions (in particular $n = 8\mu+3$, $\mu \in \mathbb{N}$) we know that $\frac{1}{2}(k-1) = \lfloor\frac{1}{2}k\rfloor = \frac{1}{4}(n-3) = 2\mu$, i.e. $k = 4\mu+1$, hence $1 + [\vec{C}_{0,1,2;n}]_k = 1 + 0 = 1$ by Lemma 53.(7). We now compare the value in (2.83) with $\big[\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n)\big]_k$. We have $k - \frac{1}{2}(n+3) = (4\mu+1)-(4\mu+3) = -2 \equiv 2 \pmod 4$, and $\frac{1}{4}(n-3) = \lfloor\frac{1}{2}k\rfloor = \frac{1}{2}(k-1)$, i.e. $k = \frac{1}{2}(n-1)$, hence, by Lemma 58.(3) in its exceptional case, $\big[\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n)\big]_k = 1$. This agrees with (2.83), completing the proof of (2).(2).(2).

Case (2).(2).(3). If $\frac{1}{4}(n-3) = \lfloor\frac{1}{2}k\rfloor - 1$, then $\big[\vec{C}_{\frac{1}{4}(n-3);n}\big]_k = -$ by Lemma 54.(5), the first sum in (2.82) is still empty, the second equal to $\sum_{\frac{1}{4}(n+5)\leqslant i\leqslant\frac{1}{2}(n-3)}(-1)^{i-1}$. Our assumption $n \equiv 3 \pmod 8$ implies both $\frac{1}{4}(n+5)$ and $\frac{1}{2}(n-3)$ are even, hence $\sum_{\frac{1}{4}(n+5)\leqslant i\leqslant\frac{1}{2}(n-3)}(-1)^{i-1} = -$. Now (2.82) gives

$$\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n))\big]_k = 1-(-)+[\vec{C}_{0,1,2;n}]_k - 2\cdot(0-(-)) = [\vec{C}_{0,1,2;n}]_k = 1\ , \tag{2.84}$$

the last equality since in this paragraph we know $\frac{1}{2}(k-1)-1 = \lfloor\frac{1}{2}k\rfloor-1 = \frac{1}{4}(n-3)$, i.e. $k = \frac{1}{2}(n+3)$, which because of $n \equiv 3 \pmod 8$ implies $k \equiv 3 \pmod 4$, hence $[\vec{C}_{0,1,2;n}]_k = 1$ by Lemma 53.(7).

Now we again compare the value in (2.84) with $[\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n)]_k$. Here we have $k - \frac{1}{2}(n+3) = \frac{1}{2}(n+3) - \frac{1}{2}(n+3) = 0 \equiv 3 \pmod 4$, hence $\big[\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n)\big]_k = 1$ by Lemma 58.(1). This agrees with (2.84), completing the proof of (2).(2).(3).

Case (2).(2).(4). If $\frac{1}{4}(n-3) \leqslant \lfloor\frac{1}{2}k\rfloor - 2$, then $[\vec{C}_{\frac{1}{4}(n-3);n}]_k = -$ by Lemma 54.(5).(iii); moreover, $\frac{1}{4}(n+1) \leqslant \lfloor\frac{1}{2}k\rfloor - 1$, hence the first sum in (2.82) is not empty. Our assumption $n \equiv 3 \pmod 8$ implies that $\frac{1}{4}(n+1)$ is odd while $\frac{1}{2}(n-3)$ is even. If $\lfloor\frac{1}{2}k\rfloor - 1$ is even (resp. odd), then $\lfloor\frac{1}{2}k\rfloor + 1$ is even (resp. odd), too, and then $\sum_{i=\frac{1}{4}(n+1)}^{\lfloor\frac{1}{2}k\rfloor-1}(-1)^{i-1} = 0$ (resp. $= +$), while $\sum_{i=\lfloor\frac{1}{2}k\rfloor+1}^{\frac{1}{2}(n-3)}(-1)^{i-1} = -$ (resp. $= 0$), and, using (2.82), we can then calculate $\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n))\big]_k = 1-(-)+[\vec{C}_{0,1,2;n}]_k - 2\cdot(0-(-)) = [\vec{C}_{0,1,2;n}]_k$ (resp. $\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n))\big]_k = 1-(-)+[\vec{C}_{0,1,2;n}]_k - 2\cdot(+-(0)) = [\vec{C}_{0,1,2;n}]_k$. Hence, independent of the parity of $\lfloor\frac{1}{2}k\rfloor - 1$,

$$\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n))\big]_k = [\vec{C}_{0,1,2;n}]_k\ . \tag{2.85}$$

In view of the structure of Lemma 53.(7), we have to distinguish the possible remainders of the odd k modulo 4. Both these cases can be dealt with analogously: if $k \equiv 1 \pmod 4$ (resp. if $k \equiv 3 \pmod 4$), then, by (2.85) and Lemma 53.(7), $\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n))\big]_k = 0$ (resp. $\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n))\big]_k = 1$). We now compare this with $\big[\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n)\big]_k$: at present we know $\frac{1}{2}(k-1)-2 = \lfloor\frac{1}{2}k\rfloor - 2 \geqslant \frac{1}{4}(n-3)$, i.e. $k \geqslant \frac{1}{2}(n+9)$, while $k = 4\mu+1$ with some $\mu \in \mathbb{N}$ (resp. $k = 4\mu+3$ with some $\mu \in \mathbb{N}$) and $n = 8\nu+3$ with some $\nu \in \mathbb{N}$ imply $k - \frac{1}{2}(n+3) = 4(\mu-\nu)-2 \equiv 2 \pmod 4$ (resp. $k-\frac{1}{2}(n+3) = 4(\mu-\nu) \equiv 0 \pmod 4$); therefore $\big[\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n)\big]_k = 0$ (resp. $\big[\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n)\big]_k = 1$), by Lemma 58.(3) (resp. by Lemma 58.(1)), where we used that the present lower bound $k \geqslant \frac{1}{2}(n+9)$ rules out the exceptional case in Lemma 58.(3). For both odd remainders of k modulo 4 we have now found the value of $\big[\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n)\big]_k$ to be in agreement with the value of $\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n))\big]_k$, completing the proof of (2).(2).(4). This proves Case (2).(2). The proof of Case (2) is now complete. Since we found $\big[\mathrm{hs}(\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n))\big]_k$ to agree with $\big[\mathrm{f}_{\mathrm{T}_n}(i_n-2,i_n)\big]_k$, we have now completed the proof of (2.50) with $(a,b) = (i_n-2,i_n)$.

As for (2.50) with $(\mathbf{a},\mathbf{b}) = (\mathbf{i_n-1},\mathbf{i_n})$, it suffices to note that, with the abbreviations from (2.20) and Definition 55,

$$1\wedge 2+\Sigma_n = \Sigma_0^{\frac{1}{8}(n-11)}(1 \mid {\scriptstyle +1,+2,-1,+2}) \quad + \quad i_n-1\wedge i_n \quad + \quad \Sigma_0^{\frac{1}{8}(n-11)}(i_n \mid {\scriptstyle +2,-1,+2,+1})\ , \tag{2.86}$$

hence comparing Lemma 56.(ff.8) with (1) in Definition 48 shows that indeed $\mathrm{f}_{\mathrm{T}_n}(i_n - 1, i_n) = \vec{C}_{0,1,2;n}$. This completes the proof of (2.50). As announced above, we have now considered all fundamental flows w.r.t. T_n, in the order given by Definition 68 on p. 80. This proves that $\mathcal{B}_n$ is a rank-sized generating set of $\mathrm{Z}_1(\mathrm{C}_n^{2^-})$, i.e. a basis, completing the proof of Proposition 69. □

Proposition 69 immediately implies a basic fact that apparently has not been pointed out before, and which is a special case of the conjectural strengthenings (cf. Question 8) of the results of [117] (there, Corollary 70 is proved only with 'generating set of 'Hamilton-based' and the proof seems not to allow to control the total number of Hamilton-flow generators used; in return, the result of [117] is of course much more general than Corollary 70):

Corollary 70 (squares of n-circuits have Hamilton-based flow lattice if $n \equiv 3 \pmod 8$)**.** *For every* $n \geqslant 11$ *with* $n \equiv 3 \pmod 8$ *there exists a* basis *of the abelian group* $\mathrm{Z}_1(\mathrm{C}_n^2) = \mathrm{Z}_1(\mathrm{Cay}(\mathbb{Z}/n; \{1, 2, n-2, n-1\}))$ *consisting only of Hamilton-flows.*

Proof. By Proposition 69 we know $\mathrm{C}_n^{2^-} \in \mathcal{M}_{|\cdot|}^{\mathbb{Z}\mathrm{Bas}}$, by Lemma 43.(7) with $\mathfrak{L} := \{|\cdot|\}$ we know $\mathcal{M}_{|\cdot|}^{\mathbb{Z}\mathrm{Bas}}$ to be monotone increasing, and from Definition 214 on p. 199 we see $\mathrm{C}_n^{2^-} \subseteq \mathrm{C}_n^2$. □

Corollary 70 is very likely to be true for every odd n, the divisibility condition is a side-effect of using Proposition 69. The proof of Corollary 70 might already give a sufficient strategy to answer all of Question 8, but one would need statements about spanning embeddings into circulants.

The incidence matrix of the Hamilton-flows in $\mathcal{B}_n$ from Proposition 69, for $n = 11$, is in (2.87).

$\mathrm{E}(\mathrm{C}_{11}^{2^-})$	0,1	0,2	0,9	0,10	1,2	1,3	1,10	2,4	3,4	3,5	4,6	5,6	5,7	6,8	7,8	7,9	8,10	9,10
$\vec{C}_{0;11}$	0	0	−	+	−	+	0	−	0	+	−	0	+	−	0	+	−	0
$\vec{C}_{1;11}$	0	+	−	0	−	0	+	0	−	+	−	0	+	−	0	+	−	0
$\vec{C}_{2;11}$	0	+	−	0	0	−	+	+	−	0	0	−	+	−	0	+	−	0
$\vec{C}_{3;11}$	0	+	−	0	0	−	+	+	0	−	+	−	0	0	−	+	−	0
$\vec{C}_{4;11}$	0	+	−	0	0	−	+	+	0	−	+	0	−	+	−	0	0	−
$\vec{C}_{5;11}$	+	−	0	0	0	+	0	−	0	+	−	0	+	−	0	+	−	+
$\vec{C}_{0,1,2;11}$	+	0	0	−	+	0	0	+	−	+	0	+	0	+	−	+	0	+
$\vec{C}_{0,1,10;11}$	+	−	0	0	0	0	+	−	+	−	0	−	0	−	+	−	0	−

(2.87)

The incidence matrix of the fundamental flows in $\mathrm{C}_{11}^{2^-}$ w.r.t. T_n from Definition 218 is in (2.88).

$\mathrm{E}(\mathrm{C}_{11}^{2^-})$	0,1	0,2	0,9	0,10	1,2	1,3	1,10	2,4	3,4	3,5	4,6	5,6	5,7	6,8	7,8	7,9	8,10	9,10
$\mathrm{f}_{\mathrm{T}_{11}}$(9,0)=**9**,**0**,10,9	0	0	−	+	0	0	0	0	0	0	0	0	0	0	0	0	0	−
$\mathrm{f}_{\mathrm{T}_{11}}$(10,1)=**10**,**1**,0,10	−	0	0	+	0	0	−	0	0	0	0	0	0	0	0	0	0	0
$\mathrm{f}_{\mathrm{T}_{11}}$(0,2)=**0**,**2**,1,0	−	+	0	0	−	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathrm{f}_{\mathrm{T}_{11}}$(1,3)=**1**,**3**,4,2,1	0	0	0	0	−	+	0	−	+	0	0	0	0	0	0	0	0	0
$\mathrm{f}_{\mathrm{T}_{11}}$(5,7)=**5**,**7**,9,10,0,1,2,4,3,5	+	0	0	−	+	0	0	+	−	+	0	0	+	0	0	+	0	+
$\mathrm{f}_{\mathrm{T}_{11}}$(8,10)=**8**,**10**,9,7,8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	+	−	+	−
$\mathrm{f}_{\mathrm{T}_{11}}$(4,6)=**4**,**6**,8,7,9,10,0,1,2,4	+	0	0	−	+	0	0	+	0	0	+	0	0	+	−	+	0	+
$\mathrm{f}_{\mathrm{T}_{11}}$(5,6)=**5**,**6**,8,7,9,10,0,1,2,4,3,5	+	0	0	−	+	0	0	+	−	+	0	+	0	+	−	+	0	+

(2.88)

The unique coordinates of the eight fundamental flows from (2.88) in terms of the Hamilton-basis from (2.87) are given in (2.89).

$\mathrm{E}(\mathrm{C}_{11}^{2^-})$	$\vec{C}_{0;11}$	$\vec{C}_{1;11}$	$\vec{C}_{2;11}$	$\vec{C}_{3;11}$	$\vec{C}_{4;11}$	$\vec{C}_{5;11}$	$\vec{C}_{0,1,2;11}$	$\vec{C}_{0,1,10;11}$
$\mathrm{f}_{\mathrm{T}_{11}}$(9,0)	+1	−1	+1	−1	+1	0	0	0
$\mathrm{f}_{\mathrm{T}_{11}}$(10,1)	0	−1	+1	−1	+1	+1	−1	−1
$\mathrm{f}_{\mathrm{T}_{11}}$(0,2)	0	+1	−1	+1	−1	−1	0	0
$\mathrm{f}_{\mathrm{T}_{11}}$(1,3)	+1	+1	−2	+2	−2	−2	+1	+1
$\mathrm{f}_{\mathrm{T}_{11}}$(5,7)	+1	0	0	+1	−2	−2	+2	+1
$\mathrm{f}_{\mathrm{T}_{11}}$(8,10)	+1	0	0	0	−1	−2	+1	+1
$\mathrm{f}_{\mathrm{T}_{11}}$(4,6)	+1	0	−1	+2	−2	−2	+2	+1
$\mathrm{f}_{\mathrm{T}_{11}}$(5,6)	0	0	0	0	0	0	+1	0

(2.89)

The incidence matrix of the $\mathrm{rank}(\mathrm{Z}_1(\mathrm{C}_{19}^{2^-})) = 12$ fundamental flows defined by the spanning tree 9,7,8,6,5,3,4,2,1,0,18,17,15,16,14,13,11,12,10 is shown in (2.90) (a fundamental flow like e.g. $18 \wedge 1$

$+\ 1 \wedge 0 + 0 \wedge 18$ is abbreviated to $\mathbf{18}, \mathbf{1}, 0, 18$, the bold numbers indicating the non-tree edge). The incidence matrix of the Hamilton-flow-basis $\mathcal{B}_{19}$ from Proposition 69 is shown in (2.91).

$\mathrm{E}(\mathrm{C}_{19}^{2-})$	0,1	0,2	0,17	0,18	1,2	1,3	1,18	2,4	3,4	3,5	4,6	5,6	5,7	6,8	7,8	7,9	8,10	9,10	9,11	10,12	11,12	11,13	12,14	13,14	13,15	14,16	15,16	15,17	16,18	17,18
$\mathrm{f}_{\mathrm{T}_{19}}(17,0)=\mathbf{17},\mathbf{0},18,0,17$	0	0	−	+	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−
$\mathrm{f}_{\mathrm{T}_{19}}(18,1)=\mathbf{18},\mathbf{1},0,18$	−	0	0	+	0	0	−	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathrm{f}_{\mathrm{T}_{19}}(0,2)=\mathbf{0},\mathbf{2},1,0$	−	+	0	0	−	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathrm{f}_{\mathrm{T}_{19}}(1,3)=\mathbf{1},\mathbf{3},4,2,1$	0	0	0	0	−	+	0	−	+	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathrm{f}_{\mathrm{T}_{19}}(5,7)=\mathbf{5},\mathbf{7},8,6,5$	0	0	0	0	0	0	0	0	0	0	0	−	+	−	+	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathrm{f}_{\mathrm{T}_{19}}(9,11)=\mathbf{9},\mathbf{11},13,14,16,15,17,18,0,1,2,4,3,5,6,8,7,9$	+	0	0	−	+	0	0	+	−	+	0	+	0	+	−	+	0	0	+	0	0	+	0	+	0	+	−	+	0	+
$\mathrm{f}_{\mathrm{T}_{19}}(13,15)=\mathbf{13},\mathbf{15},16,14,13$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−	+	−	+	0	0	0
$\mathrm{f}_{\mathrm{T}_{19}}(4,6)=\mathbf{4},\mathbf{6},5,3,4$	0	0	0	0	0	0	0	0	+	−	+	−	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathrm{f}_{\mathrm{T}_{19}}(12,14)=\mathbf{12},\mathbf{14},13,11,12$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	+	−	+	−	0	0	0	0	0	0
$\mathrm{f}_{\mathrm{T}_{19}}(16,18)=\mathbf{16},\mathbf{18},17,15,16$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	+	−	+	−
$\mathrm{f}_{\mathrm{T}_{19}}(8,10)=\mathbf{8},\mathbf{10},12,11,13,14,16,15,17,18,0,1,2,4,3,5,6,8$	+	0	0	−	+	0	0	+	−	+	0	+	0	+	0	0	+	0	0	+	−	+	0	+	0	+	−	+	0	+
$\mathrm{f}_{\mathrm{T}_{19}}(9,10)=\mathbf{9},\mathbf{10},12,11,13,14,16,15,17,18,0,1,2,4,3,5,6,8,7,9$	+	0	0	−	+	0	0	+	−	+	0	+	0	+	−	+	0	+	0	+	−	+	0	+	0	+	−	+	0	+

(2.90)

$\mathrm{E}(\mathrm{C}_{19}^{2-})$	0,1	0,2	0,17	0,18	1,2	1,3	1,18	2,4	3,4	3,5	4,6	5,6	5,7	6,8	7,8	7,9	8,10	9,10	9,11	10,12	11,12	11,13	12,14	13,14	13,15	14,16	15,16	15,17	16,18	17,18
$\vec{C}_{0;19}$	0	0	−	+	−	+	0	−	0	+	−	0	+	−	0	+	−	0	+	−	0	+	−	0	+	−	0	+	−	0
$\vec{C}_{1;19}$	0	+	−	0	−	0	+	0	−	+	−	0	+	−	0	+	−	0	+	−	0	+	−	0	+	−	0	+	−	0
$\vec{C}_{2;19}$	0	+	−	0	0	−	+	+	−	0	0	−	+	−	0	+	−	0	+	−	0	+	−	0	+	−	0	+	−	0
$\vec{C}_{3;19}$	0	+	−	0	0	−	+	+	0	−	+	−	0	0	−	+	−	0	+	−	0	+	−	0	+	−	0	+	−	0
$\vec{C}_{4;19}$	0	+	−	0	0	−	+	+	0	−	+	0	−	+	−	0	0	−	+	−	0	+	−	0	+	−	0	+	−	0
$\vec{C}_{5;19}$	0	+	−	0	0	−	+	+	0	−	+	0	−	+	0	−	+	−	0	0	−	+	−	0	+	−	0	+	−	0
$\vec{C}_{6;19}$	0	+	−	0	0	−	+	+	0	−	+	0	−	+	0	−	+	0	−	+	−	0	0	−	+	−	0	+	−	0
$\vec{C}_{7;19}$	0	+	−	0	0	−	+	+	0	−	+	0	−	+	0	−	+	0	−	+	0	−	+	−	0	0	−	+	−	0
$\vec{C}_{8;19}$	0	+	−	0	0	−	+	+	0	−	+	0	−	+	0	−	+	0	−	+	0	−	+	0	−	+	−	0	0	−
$\vec{C}_{9;19}$	+	−	0	0	0	+	0	−	0	+	−	0	+	−	0	+	−	0	+	−	0	+	−	0	+	−	0	+	−	+
$\vec{C}_{0,1,2;19}$	+	0	0	−	+	0	0	+	−.	+	0	+	0	+	−	+	0	+	0	+	−	+	0	+	0	+	−	+	0	+
$\vec{C}_{0,1,18;19}$	+	−	0	0	0	0	+	−	+	−	0	−	0	−	+	−	0	−	0	−	+	−	0	−	0	−	+	−	0	−

(2.91)

The matrices in (2.90) and (2.91) should be compared with the matrices for the case $n = 11$ shown in (2.88) and (2.89). The unique coordinates of the twelve fundamental flows from (2.90) in terms of the Hamilton-basis from (2.91) are given in (2.92).

$$\begin{array}{c|cccccccccccc}
\mathrm{E}(\mathrm{C}_{19}^{2^-}) & \vec{C}_{0;19} & \vec{C}_{1;19} & \vec{C}_{2;19} & \vec{C}_{3;19} & \vec{C}_{4;19} & \vec{C}_{5;19} & \vec{C}_{6;19} & \vec{C}_{7;19} & \vec{C}_{8;19} & \vec{C}_{9;19} & \vec{C}_{0,1,2;19} & \vec{C}_{0,1,18;19} \\
\mathrm{f}_{\mathrm{T}_{19}}(17,0) & +1 & -1 & +1 & -1 & +1 & -1 & +1 & -1 & +1 & 0 & 0 & 0 \\
\mathrm{f}_{\mathrm{T}_{19}}(18,1) & 0 & -1 & +1 & -1 & +1 & -1 & +1 & -1 & +1 & +1 & -1 & -1 \\
\mathrm{f}_{\mathrm{T}_{19}}(0,2) & 0 & +1 & -1 & +1 & -1 & +1 & -1 & +1 & -1 & -1 & 0 & 0 \\
\mathrm{f}_{\mathrm{T}_{19}}(1,3) & +1 & +1 & -2 & +2 & -2 & +2 & -2 & +2 & -2 & -2 & +1 & +1 \\
\mathrm{f}_{\mathrm{T}_{19}}(5,7) & +1 & 0 & 0 & +1 & -2 & +2 & -2 & +2 & -2 & -2 & +1 & +1 \\
\mathrm{f}_{\mathrm{T}_{19}}(9,11) & +1 & 0 & 0 & 0 & 0 & +1 & -2 & +2 & -2 & -2 & +2 & +1 \\
\mathrm{f}_{\mathrm{T}_{19}}(13,15) & +1 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & -2 & -2 & +1 & +1 \\
\mathrm{f}_{\mathrm{T}_{19}}(4,6) & +1 & 0 & -1 & +2 & -2 & +2 & -2 & +2 & -2 & -2 & +1 & +1 \\
\mathrm{f}_{\mathrm{T}_{19}}(12,14) & +1 & 0 & 0 & 0 & 0 & 0 & -1 & +2 & -2 & -2 & +1 & +1 \\
\mathrm{f}_{\mathrm{T}_{19}}(16,18) & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & +1 & +1 \\
\mathrm{f}_{\mathrm{T}_{19}}(8,10) & +1 & 0 & 0 & 0 & -1 & +2 & -2 & +2 & -2 & -2 & +2 & +1 \\
\mathrm{f}_{\mathrm{T}_{19}}(9,10) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & 0
\end{array} \tag{2.92}$$

We do not publish the Mathematica-code that was used to semi-automatically mine rank-sized sets of Hamilton-flows, looking for bases with manageably-structured change-of-basis-matrices. This is to emphasise that this thesis contains a complete proof of Theorem 4 from p. 6, not depending on machine-computations. The incidence matrices in (2.89) and (2.92) are certificates for the two special cases $n = 11$ and $n = 19$; to check these certificates, it suffices to convince oneself of the semantics of those matrices, and then compute their Smith Normal Forms.

To all appearances, it is far from true that bases like the one provided by Proposition 69 are easily found by choosing some more or less random rank-sized set of Hamilton-circuits. There was human intuition involved in steering the search in a direction leading to a general argument of manageable complexity. Without having conducted systematic large scale experiments, the experience of the author was that most rank-sized sets are $\mathbb{Z}$-linearly dependent, and among the $\mathbb{Z}$-linearly *in*dependent sets $\mathcal{S}$ of Hamilton-flows, most result in rather large torsion, i.e. rather large order of the finite group $|\mathrm{Z}_1(\mathrm{C}_n^{2^-})/\langle\mathcal{S}\rangle_{\mathbb{Z}}|$, and among those which do not, again the majority leads to matrices for the change of bases not pattern-rich enough to write a general argument. Trying to find bases with humanly manageable structure (allowing to write a readable proof) seems a not easily automated task, and the basis $\mathcal{B}_n$ from Proposition 69 was the best such basis the author could find. It was found by a semi-automatic mixture of human (symmetry-seeking) intuition and a random, computer-assisted mining of possibilities.

The author has exercised considerable care in trying to keep the proof that $\mathcal{B}_n$ from Proposition 69 constitutes a generating set as short as possible, in particular searching for a Hamilton-basis of $\mathrm{Z}_1(\mathrm{C}_n^{2^-})$ keeping the coefficients manageably low and pattern-rich (while Hamilton-bases appear to be rather rare among subsets with full-rank span, Hamilton-bases with manageable base-change-matrices are again rare among those Hamilton-bases); finding such a basis (and an accompanying spanning-tree-basis) was the main technical challenge in the proof of Proposition 69; the vast majority of pairs of tree-bases and Hamilton-flow-bases appear to result in coefficients with so little structure that writing a proof of general $n \equiv 3 \pmod 8$ would be unbearable.

It is because of the decision to use (close-to) sparsest auxiliary graphs that the divisibility-conditions on n arise. Near the 'boundaries' of a set (in this case, the *set of all suitable seed graphs*) there is restricted freedom of movement. The bandwidth-theorem (cf. Theorem 38) *would* allow to 'stay away' from the rugged boundary of this set, by using slightly denser seed graphs, which work for every odd n, and can be uniformly certified to do so without additional divisibility conditions. The author knows such sets of seed graphs H with edge-density $\|H\|/|H| = 2$. The existence of such denser seed graphs, working for every odd n, combined with the permissiveness of the bandwidth-theorem, make it exceedingly likely that Conjecture 3 is true in full, despite being formally proved in this thesis only for $n \equiv 3 \pmod 8$. However, seed graphs with edge-density two are neither suitable for proving the Conjecture 79 about $G(n, n^{-2/3+\varepsilon})$ (see end of the present chapter), nor for future attempts at proving a best-possible minimum-degree-threshold of the form $\delta(G) \geqslant \frac{1}{2}|G| + c$. In order to do technical work that can be re-used for other purposes in the future, it was decided to go for the (very-close-to-) sparsest-possible seed graphs $\mathrm{C}_n^{2^-}$, thus incurring dependencies on the remainder of n modulo 8.

The use of the not very symmetrical graphs $\mathrm{C}_n^{2^-}$ as auxiliary substructures is a more or less necessary consequence of the conflicting[6] demands of the bandwidth theorem from [24]: (1) low bandwidth, (2) constant maximum degree, (3) three-colourability with a constant-sized (and sufficiently zero-free) third colour class, (4) Hamilton-connectedness, and (5) a Hamilton-based flow lattice have all to be combined in one and the same graph; the simplest solution that the author found is $\mathrm{C}_n^{2^-}$ from Definition 214; this is a graph which for every $n \geqslant 11$ with $n \equiv 3 \pmod 4$ simultaneously meets all five desiderata (in the present thesis, for non-mathematical reasons, this is proved only for $n \geqslant 11$ with $n \equiv 3 \pmod 8$). The graphs $\mathrm{C}_n^{2^-}$ are (at most *one* edge away from) sparsest-possible seed graphs for the monotonicity argument. This is another reason for the care invested in $\mathrm{C}_n^{2^-}$: the author is working on a strengthening of Dirac's theorem on Hamilton-circuits, best-possible of its kind (i.e., one minimum-degree hypothesis only, and the conclusion being a statement about Hamilton-supported flows generating the flow lattice),with a hypothesis of $\delta \geqslant \frac{1}{2}|G| + c$ instead of $\delta \geqslant (\frac{1}{2} + \gamma)|G|$; in doing so, one has to prove a dedicated embedding-lemma, and before committing oneself to one spanning subgraph to embed, one had better be sure that it is a sparsest-possible one. It should be noted that

(nm.1) $\mathrm{C}_n^{2^-}$ is not sparsest-possible among all n-vertex graphs both Hamilton-connected and have Hamilton-based flow lattice (cf. Proposition 72),
(nm.2) yet $\mathrm{C}_n^{2^-}$ has only *one edge more* than graphs as in (nm.1) must have,
(nm.3) if not only being Hamilton-generated and being Hamilton-based are required, but also sublinear bandwidth and three-colourability with a sublinearly-sized third colour-class, then the graph $\mathrm{C}_n^{2^-}$ *might* be sparsest-possible in a strict sense, but this is not known; but it is known that $\mathrm{C}_n^{2^-}$ is *at most* one edge 'too' dense.

To explain (nm.3) in more detail, let $\mathbb{N}_{\mathrm{odd}}$ denote the odd positive integers. A proof that the $\mathrm{C}_n^{2^-}$ are in fact sparsest-possible seed graphs for a monotonicity argument founded on [24, Theorem 2], would have to show the following: for *every* infinite set of graphs $\{H_n \subseteq \mathrm{K}^n \colon\ n \in \mathbb{N}_{\mathrm{odd}}\}$ with

(ap.1) H_n Hamilton-connected for all sufficiently large $n \in \mathbb{N}_{\mathrm{odd}}$,
(ap.2) $\mathrm{Z}_1(H_n)$ Hamilton-based for all sufficiently large $n \in \mathbb{N}_{\mathrm{odd}}$,
(ap.3) $\mathrm{bw}(H_n) \in o(n)$,
(ap.4) for every $\beta > 0$ and for all sufficiently large $n \in \mathbb{N}_{\mathrm{odd}}$ there exists a proper three-colouring $\mathrm{c}_n \colon \mathrm{V}(H_n) \to \{0,1,2\}$ such that $|\mathrm{c}_n^{-1}(0)| \in o(n)$ and c_n is $(8r\beta n, 4r\beta n)$-zero-free,

it follows that the degree-sequence of H_n eventually majorises $(4^{\times 3}, 3^{\times(n-3)})$ for all sufficiently large $n \in \mathbb{N}_{\mathrm{odd}}$. In particular, the claim that the $\mathrm{C}_n^{2^-}$ are sparsest-possible seed graphs is a statement about a complicated optimisation over an infinite set of graphs. The author did not attempt to prove this. Even if the aforementioned implication should be false, there remains the human factor that all infinite sets $\{H_n \colon\ n \in \mathbb{N}_{\mathrm{odd}}\}$ achieving (ap.1)–(ap.4) with degree-sequences $(4^{\times 1}, 3^{\times(n-1)})$, thus beating $(4^{\times 3}, 3^{\times(n-3)})$, might turn out to be much more complicated to work with than the set $\{\mathrm{C}_n^{2^-} : n \equiv 3 \pmod 8\}$ used in this thesis.

So the $\mathrm{C}_n^{2^-}$ are very close to optimal, and in particular, they are at most one edge away from sparsest-possible seed graphs for the monotonicity argument. Still sparser seed graphs, i.e. seed graphs with degree-sequence below $(4^{\times 1}, 3^{\times(n-1)})$ cannot exist: any suitable seed graph for the monotonicity argument must be Hamilton-connected (cf. the footnote on p. 27), hence have minimum-degree 3; this leaves only the question whether there might be *cubic* seed graphs for the monotonicity argument; but there is a trivial reason why cubic graphs are never suitable: any cubic graph has even order, and this alone makes it impossible to use it as a seed graph.

To all appearances, the graph $\mathrm{C}_{11}^{2^-}$ is a suitable seed graph for the monotonicity argument which is minimal w.r.t. edge-deletion. (As we just discussed, $\mathrm{C}_{11}^{2^-}$ might be at most one edge 'too' dense, so studying the deletion of *one* edge from it makes sense.) This is not formally proved in this thesis,

[6] This is not to say that (1)–(5) mutually conflict; rather, {(1),(2),(3)} tend conflict with {(4),(5)}.

but let us at least consider one of the possible edge-deletions: the graph $\mathrm{C}_{11}^{2^{--}}$ of Definition 215, obtained by deleting the edge $\{n-1, 0\}$ from $\mathrm{C}_n^{2^-}$, leaves a non-Hamilton-connected graph:

Proposition 71. *The graph $\mathrm{C}_{11}^{2^{--}}$ has non-adjacent vertices not connected by a Hamilton-path.*

Proof. We show that there does not exist a Hamilton-path in $\mathrm{C}_{11}^{2^{--}}$ with end-vertices 1 and 4. Suppose P is such a path. We denote by $\tilde{P}$ the subpath of P already uncovered at some point in the proof. We examine the four possibilities for the neighbour of 1 in P.

If $1, 2 \subseteq P$, then, since otherwise 0 (which is not among the two endvertices of the hypothetical path P) would acquire degree 1 in $\mathrm{C}_{11}^{2^{--}} - \tilde{P}$, it is then necessary that $2, 0 \subseteq P$. Then, obviously, $1, 2, 0, 9 \subseteq P$. We now turn to vertex 4: since otherwise 3 would acquire degree 1 in $\mathrm{C}_{11}^{2^{--}} - \tilde{P}$, necessarily $4, 3, 5 \subseteq P$. Now there are two cases: if $5, 7 \subseteq P$, then necessarily $7, 8 \subseteq P$ and it then has become evident that $\tilde{P}$ cannot become a Hamilton-path anymore; if on the other hand $5, 6 \subseteq P$, then $\mathrm{N}(6)\backslash\mathrm{V}(\tilde{P}) = \{8\}$ implies $6, 8 \subseteq P$, and no matter whether 7 or 10 is the other neighbour of 8 in P, it has then again become evident that $\tilde{P}$ cannot be extended to a Hamilton-path. These contradictions show that $1, 2 \subseteq P$ is impossible.

If $1, 3 \subseteq P$, then, since 4 is an endvertex of the hypothetical Hamilton-path P, necessarily $3, 5 \subseteq P$. Now since $5, 7 \subseteq P$ would leave 6 with degree one in $\mathrm{C}_{11}^{2^{--}} - \tilde{P}$, while not being an end-vertex of P, it is necessary that $5, 7 \nsubseteq P$ and $5, 6 \subseteq P$. Then necessarily $6, 8 \subseteq P$, and, lest 7 acquire degree one in $\mathrm{C}_{11}^{2^{--}} - \tilde{P}$, necessarily $8, 7 \subseteq P$. Then $7, 9 \subseteq P$ since no other neighbour is left. Turning to vertex 4, since 2 is the only neighbour of 4 not yet in $\tilde{P}$, necessarily $4, 2 \subseteq P$, which in turn implies $2, 0 \subseteq P$. At this point, only 10 is missing from P, hence necessarily $9, 10 \subseteq P$. Since the edge $0, 10$ is not in $\mathrm{C}_{11}^{2^{--}}$, it is impossible to close $\tilde{P}$ to a Hamilton-path. This contradiction proves $1, 3 \subseteq P$ to be impossible.

If $1, 10 \subseteq P$, then there are the following cases: if $10, 9 \subseteq P$, then necessarily $9, 0 \subseteq P$, for otherwise 0 would acquire degree one in $\mathrm{C}_{11}^{2^{--}} - \tilde{P}$. Then, necessarily, $0, 2 \subseteq P$, and then 4 is the only available neighbour of 2 in $\mathrm{C}_{11}^{2^{--}} - \tilde{P}$; but 4 is an endvertex of the hypothetical Hamilton-path P, and we did not yet visit every vertex of $\mathrm{C}_{11}^{2^{--}}$. This contradiction proves that $1, 10 \subseteq P$ is an impossible case, too.

If $1, 0 \subseteq P$, then $0, 2 \subseteq P$ is impossible since this would necessitate visiting 4 before all other vertices are visited, hence $0, 9 \subseteq P$. Since otherwise 10 would acquire degree one in $\mathrm{C}_{11}^{2^{--}} - \tilde{P}$, necessarily $9, 10 \subseteq P$, implying $10, 8 \subseteq P$. For similar reasons, it then follows that necessarily $8, 7, 5, 6, 4 \subseteq P$. Since 4 is an endvertex of the hypothetical P, but 2 and 3 have not yet been visited, we have reached a contradiction, proving $1, 0 \subseteq P$ to be impossible, too.

Now all possibilities for the neighbour of 1 in P have been shown to be impossible and the proof of Proposition 71 is complete. □

Let us finally give an explicit example proving that $\mathrm{C}_n^{2^-}$ are not sparsest-possible seed graphs in the simplistic sense of having least degree-sequence among graphs which are simultaneously Hamilton-connected and have Hamilton-based flow lattice (but see the discussion given on p. 103 for a more complex sense in which they are):

Proposition 72 (Hamilton-connected graphs with Hamilton-based flow lattice and degree-sequence $(4^{\times 1}, 3^{\times(n-1)})$ exist)**.** *The graph $\mathrm{X}_9^{\triangle}$ from Definition 213*

(1) *has degree-sequence $(4^{\times 1}, 3^{\times(n-1)})$, where $n = 9$,*
(2) *is Hamilton-connected,*
(3) *and has $\mathrm{Z}_1(\mathrm{X}_9^{\triangle})$ Hamilton-based.*

Proof. Statement (1) is obvious from Definition 213. As to (2), we exhibit a Hamilton-path of $\mathrm{X}_9^{\triangle}$ with end-vertices x and y for each of the $36 = \binom{9}{2}$ distinct 2-sets $\{x, y\} \in \binom{\mathrm{V}(\mathrm{X}_9^{\triangle})}{2}$:

(1) $(x,y) = (1,2)$: $1,9,8,7,6,5,4,3,2$,
(2) $(x,y) = (1,3)$: $1,9,8,4,5,6,7,2,3$,
(3) $(x,y) = (1,4)$: $1,2,3,9,8,7,6,5,4$,
(4) $(x,y) = (1,5)$: $1,6,7,2,3,9,8,4,5$,
(5) $(x,y) = (1,6)$: $1,9,8,7,2,3,4,5,6$,
(6) $(x,y) = (1,7)$: $1,6,5,4,8,9,3,2,7$,
(7) $(x,y) = (1,8)$: $1,9,3,2,7,6,5,4,8$,
(8) $(x,y) = (1,9)$: $1,2,3,4,5,6,7,8,9$,
(9) $(x,y) = (2,3)$: $2,1,9,8,7,6,5,4,3$,
(10) $(x,y) = (2,4)$: $2,1,5,6,7,8,9,3,4$,
(11) $(x,y) = (2,5)$: $2,1,9,3,4,8,7,6,5$,
(12) $(x,y) = (2,6)$: $2,3,4,5,1,9,8,7,6$,
(13) $(x,y) = (2,7)$: $2,3,4,5,6,1,9,8,7$,
(14) $(x,y) = (2,8)$: $2,1,9,3,4,5,6,7,8$,
(15) $(x,y) = (2,9)$: $2,1,5,6,7,8,4,3,9$,
(16) $(x,y) = (3,4)$: $3,2,1,9,8,7,6,5,4$,
(17) $(x,y) = (3,5)$: $3,2,7,6,1,9,8,4,5$,
(18) $(x,y) = (3,6)$: $3,9,1,2,7,8,4,5,6$,
(19) $(x,y) = (3,7)$: $3,2,1,9,8,4,5,6,7$,
(20) $(x,y) = (3,8)$: $3,9,1,2,7,6,5,4,8$,
(21) $(x,y) = (3,9)$: $3,2,7,8,4,5,6,1,9$,
(22) $(x,y) = (4,5)$: $4,3,2,1,9,8,7,6,5$,
(23) $(x,y) = (4,6)$: $4,5,1,2,3,9,8,7,6$,
(24) $(x,y) = (4,7)$: $4,5,6,1,2,3,9,8,7$,
(25) $(x,y) = (4,8)$: $4,5,1,6,7,2,3,9,8$,
(26) $(x,y) = (4,9)$: $4,3,2,1,5,6,7,8,9$,
(27) $(x,y) = (5,6)$: $5,4,3,2,1,9,8,7,6$,
(28) $(x,y) = (5,7)$: $5,4,8,9,3,2,1,6,7$,
(29) $(x,y) = (5,8)$: $5,4,3,2,7,6,1,9,8$,
(30) $(x,y) = (5,9)$: $5,6,7,8,4,3,2,1,9$,
(31) $(x,y) = (6,7)$: $6,5,4,3,2,1,9,8,7$,
(32) $(x,y) = (6,8)$: $6,7,2,3,4,5,1,9,8$,
(33) $(x,y) = (6,9)$: $6,1,5,4,8,7,2,3,9$,
(34) $(x,y) = (7,8)$: $7,6,5,4,3,2,1,9,8$,
(35) $(x,y) = (7,9)$: $7,8,4,5,6,1,2,3,9$,
(36) $(x,y) = (8,9)$: $8,7,6,5,4,3,2,1,9$.

As to (3), it suffices to check that

$$\begin{array}{rcccccccccccccccc}
\mathrm{E}(\mathrm{X}_9^{\triangle}) & : & 1,2 & 1,5 & 1,6 & 1,9 & 2,3 & 2,7 & 3,4 & 3,9 & 4,5 & 4,8 & 5,6 & 6,7 & 7,8 & 8,9 \\
\vec{C}_1 & : & + & 0 & 0 & - & + & 0 & + & 0 & + & 0 & + & + & + & + \\
\vec{C}_2 & : & + & 0 & - & 0 & 0 & + & + & - & + & 0 & + & 0 & + & + \\
\vec{C}_3 & : & 0 & + & - & 0 & - & + & 0 & - & - & + & 0 & - & 0 & + \\
\vec{C}_4 & : & 0 & + & 0 & - & + & - & + & 0 & 0 & + & + & + & 0 & + \\
\vec{C}_5 & : & 0 & 0 & + & - & - & + & - & 0 & - & 0 & - & 0 & + & + \\
\vec{C}_6 & : & 0 & 0 & + & - & + & - & 0 & + & - & + & - & 0 & - & 0
\end{array} \tag{2.93}$$

is the incidence-matrix of six Hamilton-flows of $\mathrm{X}_9^{\triangle}$, and its elementary divisors are $(1^{\times 6})$. □

Let us note that the graph $\mathrm{X}_9^{\triangle}$ contains *exactly* six Hamilton-circuits in total. For a proof of Proposition 72.(3) it is not necessary to know this; there, it was sufficient to exhibit *at least* six Hamilton-circuits, orient them to get Hamilton-flows, and check that the submodule spanned by these flows has six invariant factors (i.e., the elementary divisors of the flows' incidence matrix) equal to 1. The fact (not proved here) that there *do not exist more* than these six Hamilton-circuits shows that $\mathrm{X}_9^{\triangle}$ is not only an example of a Hamilton-connected graph with Hamilton-based flow lattice and smallest-possible degree-sequence, but also of an unexplored property that one might call *globally-minimal Hamilton-based*: graphs G for which there exist exactly $\mathrm{rank}(\mathrm{Z}_1(G))$-many Hamilton-circuits in total, i.e., not more than the bare minimum of what one needs to hope to generate $\mathrm{Z}_1(G)$ by Hamilton-flows, and for which these Hamilton-flows actually *do* generate $\mathrm{Z}_1(G)$.

One could accuse our approach to (I.1) in Conjecture 3 on p. 5 of overfitting noise (in the set of all suitable seed graphs): if the bandwidth theorem is permissive enough to embed seed graph denser than $\mathrm{C}_n^{2^-}$ (with minimum degree 4 instead of 3, for example), which then work uniformly for every odd n, why, of all seed graphs, insist on sparsest-possible seed graphs and then incur divisibility conditions for one's pains? One answer to this is: these sparse and demanding seed graphs can be *simultaneously useful for several other hypotheses* in future work, in particular when working with hypotheses less permissive than $\delta(G) \geqslant (\frac{1}{2} + \gamma)|G|$. Ongoing work of the author aims at using one and the same set of seed graphs, namely the $\mathrm{C}_n^{2^-}$ for $n \equiv 3 \pmod 4$, and for $n \equiv 1 \pmod 4$ a similar set not disclosed in the thesis, to prove versions of Theorem 4 having several different kinds of hypotheses, each less permissive than $\delta(G) \geqslant (\frac{1}{2} + \gamma)|G|$, namely

(hy.1) $\delta(G) > \frac{1}{2}|G|$,
(hy.2) $G \sim G(n, n^{-2/3+\varepsilon})$,
(hy.3) random geometric graphs,
(hy.4) G a pseudo-random graph with sufficient parameters,
(hy.5) G a connected Cayley-graph on a finite abelian groups,

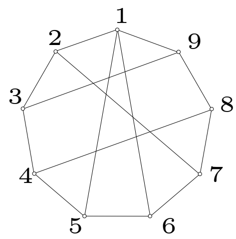


Figure 2.7: The graph $X_9^{\triangle}$ from Definition 213. This is a sparsest-possible graph among the graphs which are both Hamilton-connected and have Hamilton-based flow lattice; cf. Proposition 72. I.e., it has smallest possible number of edges among all graphs on 9 vertices which are both Hamilton-connected and have the flow lattice Hamilton-based. (Reason for $\|\cdot\|$ being minimum: $X_9^{\triangle}$ has degree-sequence $(4^{\times 1}, 3^{\times 8})$, Hamilton-connectedness alone implies minimum-degree three, and cubic graphs on nine vertices do not exist.) By Lemma 72, it is Hamilton-connected, too, so it could be the first graph in an infinite family of suitable seed graphs with degree-sequence $(4^{\times 1}, 3^{\times 8})$; it is not known to the author whether such a family exists (which would show the degree-$(4^{\times 1}, 3^{\times 8})$-graphs employed in this thesis to be *one* edge denser than necessary). The question here is whether one can extend $X_9^{\triangle}$ to an infinite family which besides Hamilton-connected and Hamilton-based all are *three-colourable with sublinearly-sized third colour class* and have *sublinear bandwith* (preferably both 'sublinear' properties holding as 'constant-sized'). All graphs ever seen by the author which are Hamilton-connected with Hamilton-based flow lattice, and have degree-sequence $(4^{\times 1}, 3^{\times (n-1)})$ seem to be much less structured than the $C_n^{2^-}$ from Definition 214. For these reasons, it was decided to allow the slightly more generous degree-sequences $(4^{\times 3}, 3^{\times (n-3)})$, i.e., allow one edge more, and settle for the seed graphs $C_n^{2^-}$. Thus, the use of seed graphs with degree-sequence $(4^{\times 3}, 3^{\times (n-3)})$ instead of $(4^{\times 1}, 3^{\times (n-1)})$ in the proof of Theorem 4 is not necessitated by the properties of Hamilton-connectedness and being Hamilton-based, but rather by the additional bandwidth- and colourability-requirements, and considerations of having enough structure to write a general proof: the one additional edge is spent to have graphs with more structure and smaller bandwidth available. The graphs $C_n^{2^-}$ from Definition 214 which are used for the proof of Theorem 4 are *at most one edge away* from being sparsest-possible seed graphs for the monotonicity argument, and might actually *be* sparsest-possible in a strict sense.
The graph $X_9^{\triangle}$ is moreover *globally-minimal Hamilton-based*, i.e., it has Hamilton-based flow lattice and only $\mathrm{rank}_{\mathbb{Z}}(X_9^{\triangle}) = 6$ distinct Hamilton-circuits in total.

and none of these (for interesting parameters) seems to brook seed graphs H with edge-density $\|H\|/|H| = 2$ (which minimum-degree-4-graphs would be). Thus, the point in proving the $C_n^{2^-}$ from Definition 214 from Chapter 5 to be suitable seed graphs is that they are (at most one edge away from) sparsest-possible seed graphs, hence are *simultaneously* tools for proving each of (hy.1)–(hy.5). Thus, in *any* future argument proving a Hamilton-based flow lattice via spanning subgraphs and monotonicity, using the $C_n^{2^-}$ is likely to be optimal. The $C_n^{2^-}$ make it possible to push the spanning-subgraph-argument to its limit. The author sought, but regrettably did not find, a set of other seed graphs with edge-density $(1 + 1/n)\frac{3}{2}$ (the edge-density of $C_n^{2^-}$) which would work uniformly for every odd n.

2.2.4 Details on steps ($\mathbb{Z}$-St.2) and ($\mathbb{Z}$-St.3)

2.2.4.1 Proof of Theorem 4, i.e., proof of Conjecture 3.(I.1) restricted to the congruence class $n \equiv 3 \pmod 8$

By combining the auxiliary statements proved so far, we now write down a short proof of Theorem 4. As mentioned before, the proof is conceptually analogous to the proofs in Section 2.1.3.2.

Proof of Theorem 4. Let $\gamma > 0$ be given and invoke Theorem 38 with this γ, $\rho := 2$ and $\Delta := 4$ to get some $\beta > 0$ and n_0, which for the present purposes will be denoted by n_0', with the property stated in Theorem 38. Give this β to Lemma 66.(8) to get an $n_0 = n_0(\beta)$, for the present purposes denoted by n_0'', with the properties stated there. We now argue that with $n_0 := \max(n_0', n_0'')$ the claim of Theorem 4 is true. Let $\mathcal{G}$ denote the set of all graphs G with $n_0 \leqslant |G| \equiv 3 \pmod 8$ and $\delta(G) \geqslant (\frac{1}{2} + \gamma)|G|$. Let an arbitary $G \in \mathcal{G}$ be given. We then choose, with $\mathrm{C}_n^{2^-}$ from Definition 214,

$$H := \mathrm{C}_{|G|}^{2^-} \; . \tag{2.94}$$

Then $|H| = |G|$, and, in the terms of Definition 204, by combining Lemma 66.(5) and Proposition 69,

$$H \in \mathcal{M}_{|\cdot|}^{\mathbb{Z}\mathrm{Bas}} \; . \tag{2.95}$$

Moreover, $\Delta(H) = \Delta(\mathrm{C}_{|G|}^{2^-}) = 4 \leqslant \Delta$, and by choice of n_0 we know $\mathrm{bw}(\mathrm{C}_n^{2^-}) \leqslant \beta n$ for every $n_0 \leqslant n \equiv 3 \pmod 4$. Moreover, $\mathrm{C}_n^{2^-}$ by Lemma 66.(6) admits a proper 3-colouring with constant-sized third colour class, which in particular is $\big(8{\cdot}2{\cdot}\beta|\mathrm{C}_n^{2^-}|, 4{\cdot}\beta|\mathrm{C}_n^{2^-}|\big)$-zero-free. Thus, the bandwidth theorem (i.e., Theorem 38 on p. 39) with our numbers γ, ρ, Δ, β and n_0 as parameters, guarantees the existence of an embedding $H \hookrightarrow G$. Because of $G \in \mathcal{G}$, $|H| = |G|$ and $H \hookrightarrow G$, Corollary 44.(5) with $\mathfrak{L} = \{|\cdot|\}$ now implies $G \in \mathcal{M}_{|\cdot|}^{\mathbb{Z}\mathrm{Bas}}$, in particular $G \in \mathrm{Bas}\mathcal{C}_{|\cdot|}$, which is the claim of Theorem 4. □

2.2.4.2 Proof of Conjecture 3.(I.4)

Proof of Theorem 5. We start as in the proof of Theorem 4, with the same γ, ρ, Δ, β and n_0, where this time the β comes from Lemma 37.(a30). We then let $\mathcal{G}$ denote the set of all graphs G with $2 \cdot 10^8 \leqslant n := |G| \equiv 3 \pmod 8$ and $\delta(G) \geqslant \frac{2}{3}n$. Let an arbitrary $G \in \mathcal{G}$ be given. We choose

$$H := \mathrm{C}_n^2 \; . \tag{2.96}$$

Then $|H| = |G|$. By Lemma 37.(a1) and Theorem 35 we know $H \in \mathcal{CO}_{\{|\cdot|-1\}}$ for every n. By Corollary 70, we know $H \in \mathrm{Bas}\mathcal{C}_{|\cdot|}$. Taken together, in the terms of Definition 204, we therefore know $H \in \mathcal{M}_{|\cdot|}^{\mathbb{Z}\mathrm{Bas}}$. As in the proof of Theorem 4, the bandwidth theorem from p. 39 guarantees an embedding $H \hookrightarrow G$, and because of $G \in \mathcal{G}$, $|H| = |G|$ and $H \hookrightarrow G$, Corollary 44.(5) with $\mathfrak{L} = \{|\cdot|\}$ implies $G \in \mathcal{M}_{|\cdot|}^{\mathbb{Z}\mathrm{Bas}}$, in particular $G \in \mathrm{Bas}\mathcal{C}_{|\cdot|}$, as claimed in Theorem 5. □

2.2.5 Comments on the case $n \equiv 1 \pmod 4$

The regrettable fact in Claim 73 is the reason why for those odd n with $n \equiv 1 \pmod 4$ one has to choose not merely[7] a slightly different pair

$$(\text{ Hamilton-flow-basis}, \text{spanning-tree-basis })$$

within the seed graph, but a different seed graph altogether:

Claim 73 (why for $n \equiv 1 \pmod 4$ one has to use an auxiliary graph other than $\mathrm{C}_n^{2^-}$)**.** *With $\vec{\mathcal{H}}(G)$ denoting the set of all Hamilton-flows on a graph G, for every $k \geqslant 3$ we have the cyclic quotient $\mathrm{Z}_1(\mathrm{C}_{4k+1}^{2^-})/\langle\vec{\mathcal{H}}(\mathrm{C}_{4k+1}^{2^-})\rangle_{\mathbb{Z}} \cong \mathbb{Z}/(2k+1)$. Hence for every $k \geqslant 3$, the graph $\mathrm{C}_{4k+1}^{2^-}$ is not a suitable seed graph for proving a $(4k+1)$-vertex graph G to have Hamilton-generated flow lattice $\mathrm{Z}_1(G)$, let alone to prove G to have Hamilton-based flow lattice.*

Claim 73 is the reason why the proof of Conjecture 3.(I.1) from Chapter 1 is not carried out for $n \equiv 1 \pmod 4$ in this thesis (but, to repeat, the thesis does contain a complete proof for the case $n \equiv 3 \pmod 8$ of Conjecture 3.(I.1)).

[7] Of that more trifling kind is the difference between the cases $n \equiv 3 \pmod 8$ and $n \equiv 7 \pmod 8$.

Claim 73 is made on the grounds of an exhaustive machine-search of all rank-sized sets of Hamilton-flows in $\mathrm{C}_{13}^{2^-}$ and $\mathrm{C}_{17}^{2^-}$, done with Mathematica. We do not attempt a humanly-checkable proof of Claim 73; to do so seems an unimportant and pointlessly negativistic pursuit which moreover would take much effort: it would require formally proving that $\mathrm{C}_{13}^{2^-}$ and $\mathrm{C}_{17}^{2^-}$ do *not* work as seed graphs. In particular, it would necessitate formally enumerating all their Hamilton-circuits and dismissing all rank-sized candidate sets of Hamilton-supported flows—unless it is possible to give short certificates for a flow lattice being *not* Hamilton-generated, which seems unlikely (cf. Conjecture 19 in Section 1.2.3 of Chapter 1).

While in the case $n \equiv 1 \pmod 4$, too, there are many suitable seed graphs, the author could not yet find a triple

$$(\text{suitable seed graph}, \text{Hamilton-flow basis}, \text{spanning tree}) \tag{2.97}$$

such that—and this is something we *did* find for $n \equiv 3 \pmod 8$, thus should aspire to for other residue classes as well—the matrix describing the change of basis between $\mathcal{B}_n$ from Proposition 69 and the fundamental-flow-basis defined by the trees T_n from Definition 218 has all its entries of magnitude at most 2. Even the most convenient such triples that the author could find so far need some coefficients of magnitude 3 within the matrices describing the change of basis. While these matrices seem manageable, too, they are unsatisfactory compared with the simplicity achieved in the case $n \equiv 3 \pmod 8$ (cf. the coefficients in Definition 68). Moreover, the author just cannot believe that the use of integer coefficients of magnitude *larger* than 2 is a mathematical necessity in the case $n \equiv 1 \pmod 4$, and still suspects that there is a triple (2.97) avoiding it. This is a reason why it was decided to only give a complete proof for the case $n \equiv 3 \pmod 8$, and to rather go on searching a triple on par with the relative simplicity of the proof in Section 2.2.3.3.

2.3 Random graphs with Hamilton-generated cycle space

In the face of the apparent intractability of efficiently describing the set of all finite hamiltonian graphs, one realistic strategy to acquire more knowledge is the study of *slices* of that complicated set: under the assumption that a known sufficient condition for hamiltonicity holds, one tries to find proofs that such a graph has *extra properties*, such as having *many* Hamilton-circuits, or having many *well-distributed* Hamilton-circuits. There are known examples of this paradigm in a random setting (e.g., [19] [39] [40] [100]), and in the present Section 2.3 we will give another variation on that theme. We will combine some recent results in order to prove a new result, Theorem 74, teaching us that sufficiently dense binomial random graphs a.a.s. have the above universality-property (and for a vastly smaller edge-probability than what would a.a.s. force the minimum-degree-condition from [82, Theorem 1] to hold). With more work, the threshold in Theorem 74 can apparently be lowered to $n^{-2/3+\varepsilon}$. The techniques in Section 2.3 appear to cut no ice with Conjecture 12 in Chapter 1 though, where $p(n)$ is so small that it does not allow embedding a *preselected* substructure. In Section 2.3.2 below we prove that Conjecture 12 would become false if $p(n)$ were lowered further.

2.3.1 An upper bound for the smallest sufficient p

In this section we prove an upper bound of $n^{-1/2+\varepsilon}$ on the smallest p sufficient for $\mathrm{Z}_1(G;\mathbb{F}_2) = \langle\mathcal{H}(G)\rangle_{\mathbb{F}_2}$ to a.a.s. hold in $G_{n,p}$:

Theorem 74. *If $\varepsilon > 0$ and $p \in [0,1]^{\mathbb{N}}$ with $p(n) \geqslant n^{-\frac{1}{2}+\varepsilon}$, then for $n \to \infty$ a random graph $G \sim G_{n,p}$ has the following properties a.a.s. (cf. Definition 204 in Chapter 5 for notation):*
(1) $G \in \mathrm{Cd}_0\mathcal{C}_{\{|\cdot|\}}{}^-$, (2) *if n is even, then* $G \in \mathrm{Cd}_1\mathcal{C}_{|\cdot|}$, (3) *if n is odd, then* $G \in \mathrm{Cd}_0\mathcal{C}_{|\cdot|}$.

Proof of Theorem 74. By a recent theorem of Kühn and Osthus [107, Theorem 1.2 specialised to $k = 2$], we know that, asymptotically almost surely, $G_{n,p}$ contains C_n^2 as a spanning subgraph. By Lemma 37.(a5), we know that $\mathrm{C}_n^2 \in \mathcal{M}_{\{|\cdot|\},0}{}^-$. By Lemma 37.(a3) we know $\mathrm{C}_n^2 \in \mathcal{M}_{\{|\cdot|\},1}$ if n

is even. By Lemma 37.(a4) we know $\mathrm{C}_n^2 \in \mathcal{M}_{\{|\cdot|\},0}$ if n is odd. Moreover, setting $\mathfrak{L} = \{|\cdot|\}$ and $\xi = 0$ (resp. $\mathfrak{L} = \{|\cdot|\}$ and $\xi = 1$) in (3) (resp. (1)) of Lemma 43, we know that each of the graph properties $\mathcal{M}_{\{|\cdot|\},0}{}^{-}$, $\mathcal{M}_{\{|\cdot|\},0}$, and $\mathcal{M}_{\{|\cdot|\},1}$ is monotone. Therefore, $G_{n,p}$ itself is a.a.s. in these graph properties. This completes the proof of Theorem 74. □

We have phrased Theorem 74 in linear-algebraic language, using the notation from Definition 204. The results can also be stated in terms of symmetric differences. The statements (2) and (3) in Theorem 74 amount to saying that, *a.a.s.*, the set of cycles which can be constructed as a symmetric difference of Hamilton-circuits is as large as parity permits. I.e., *all* cycles if n is odd, and all *even* cycles (i.e., elements of the cycle space having a support of even size, of which circuits are examples) if n is even. One of these equivalences is a deterministic one: statement (3) is deterministically equivalent to saying that for odd n, every cycle of G is a symmetric difference of (edge-sets of) Hamilton-circuits of G. The other of these two implications holds only after restricting to non-bipartite models, and hence also holds in an a.a.s. sense: statement (2) is not deterministically equivalent to saying that for even n, every even cycle is a symmetric difference of Hamilton-circuits: if n is even and G happens to be *bipartite*, then every even cycle being a symmetric difference of Hamilton-circuits is equivalent to $\dim_{\mathbb{F}_2}(\mathrm{Z}_1(G;\mathbb{F}_2)/\langle\mathcal{H}(G)\rangle_{\mathbb{F}_2}) = 0$, not $= 1$. But when restricted to *non-bipartite* graphs only—and $G_{n,p}$ with the present $p(n)$ *is* non-bipartite a.a.s.—*then* statement (2) is equivalent to saying that for even n, every even cycle is a symmetric difference of Hamilton-circuits: $G \sim G_{n,p}$ with $p(n) \geqslant n^{-1/2+\varepsilon}$ a.a.s. contains triangles. Since n is even, if there were an even cycle $z \in \mathrm{Z}_1(G;\mathbb{F}_2)$ not in the $\mathbb{F}_2$-span of the Hamilton-circuits of G, then this cycle together with some triangle would represent two linearly independent elements modulo $\langle\mathcal{H}(G)\rangle_{\mathbb{F}_2}$, in contradiction to $\dim_{\mathbb{F}_2}(\mathrm{Z}_1(G;\mathbb{F}_2)/\langle\mathcal{H}(G)\rangle_{\mathbb{F}_2}) = 1$, the latter being true a.a.s. by (2).

2.3.2 A lower bound for the smallest sufficient p

In this section we will derive a lower bound (larger than the hamiltonicity threshold) that any p which a.a.s. ensures $\mathrm{Z}_1(G_{n,p};\mathbb{F}_2) = \langle\mathcal{H}(G_{n,p})\rangle_{\mathbb{F}_2}$ must satisfy.

All graphs considered, the property of graphs G with $\mathrm{Z}_1(G;\mathbb{F}_2) = \langle\mathcal{H}(G)\rangle_{\mathbb{F}_2}$ (no Hamilton-connectedness required) is not monotone, so a priori one should be cautious of speaking of a threshold for that property in $G_{n,p}$. However, restricted to binomial random graphs $G_{n,p}$, the property can only arise 'within a monotone property', hence there *is* a threshold: by Theorem 11 in Section 2.3.2 we know that if $G_{n,p}$ a.a.s. has the property, then p is (on infinitely-many odd integers) at least as large as the known (cf. [119, Theorem 1 with $k = 2$]) threshold $(\log n + 2\log\log n + \omega(1))/n$ for Hamilton-*connectedness* of $G_{n,p}$. I.e., for any p which a.a.s. ensures $\mathrm{Z}_1(G;\mathbb{F}_2) = \langle\mathcal{H}(G)\rangle_{\mathbb{F}_2}$, this property necessarily comes together with Hamilton-connectedness, i.e., a.a.s. $G_{n,p}$ is in the 'monotonised' intersection $\{G \subseteq \mathrm{K}^n\colon\ \mathrm{Z}_1(G;\mathbb{F}_2) = \langle\mathcal{H}(G)\rangle_{\mathbb{F}_2}\ \} \cap \{G \subseteq \mathrm{K}^n\colon\ G \text{ is Hamilton-connected }\}$, i.e., in a monotone graph property. Thus, by [58, Theorem 1.1], the property $\mathrm{Z}_1(G;\mathbb{F}_2) = \langle\mathcal{H}(G)\rangle_{\mathbb{F}_2}$ has a sharp threshold. One can see this as another example (cf. the discussion on p. 43) for correlation without causation in the presence of another cause (this time, being a random graph) between the two graph-properties $\mathrm{Bas}\mathcal{C}_{|\cdot|}$ and $\mathcal{CO}_{|\cdot|-1} = \{G \subseteq \mathrm{K}^n\colon\ G \text{ Hamilton-connected }\}$: if a.a.s. $G_{n,p} \in \mathrm{Bas}\mathcal{C}_{|\cdot|}$, then in particular a.a.s. $G_{n,p} \in \{G \subseteq \mathrm{K}^n\colon\ \mathrm{Z}_1(G;\mathbb{F}_2) = \langle\mathcal{H}(G)\rangle_{\mathbb{F}_2}\ \} = \mathrm{Cd}_0\mathcal{C}_{|\cdot|} \supseteq \mathrm{Bas}\mathcal{C}_{|\cdot|}$, hence by Theorem 11 it follows that $p(n) > (\log n + 2\log\log n + \omega(1))/n$, hence by [119, Theorem 1 with $k = 2$] also a.a.s. $G_{n,p} \in \mathcal{CO}_{|\cdot|-1}$.

We will use the following known fact:

Lemma 75 ([18, Exercise 3.2])**.** *For any $k \geqslant 0$ and any $\omega \in \mathbb{R}_{>0}^{\mathbb{N}}$ with $\omega_n \xrightarrow{n\to\infty} \infty$,*

(1) *if $p_n \geqslant \frac{\log n + k\log\log n + \omega_n}{n}$, then $\mathsf{P}_{G_{n,p}}[\delta \geqslant k+1] \xrightarrow{n\to\infty} 1$,*
(2) *if $p_n = \frac{\log n + k\log\log n + c + h(n)}{n}$ with some constant $c \in \mathbb{R}$ and some function $h \in o(1)$, then $\mathsf{P}_{G_{n,p}}[\delta = k] \xrightarrow{n\to\infty} 1-\exp(-\exp(-c)/k!)$ and $\mathsf{P}_{G_{n,p}}[\delta = k+1] \xrightarrow{n\to\infty} \exp(-\exp(-c)/k!)$,*
(3) *if $p_n \leqslant \frac{\log n + k\log\log n - \omega_n}{n}$, then $\mathsf{P}_{G_{n,p}}[\delta \leqslant k] \xrightarrow{n\to\infty} 1$.*

We now prepare with a structural lemma:

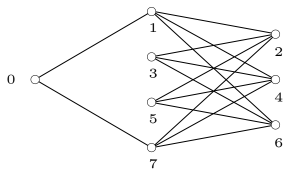


Figure 2.8: The graph $K^{\hat{4},3}$, a small example that being neither a forest nor a circuit and having every cycle a symmetric difference of Hamilton-circuits does not *always* imply minimum degree $\geqslant 3$. This implication holds *a.a.s.* in $G_{n,p}$, though, see Lemma 78.

Lemma 76. *If $G = (V, E)$ is a graph such that*

(1) *G is neither a forest nor a circuit,*
(2) *every cycle in G is a symmetric difference of Hamilton-circuits,*
(3) *G contains a vertex of degree 2,*

then for every vertex v as in (3), the graph $G - v$ obtained by deleting v is bipartite.

Proof. Let G be any such graph. Being a non-forest, the cycle space of G is non-trivial. Since otherwise (2) fails (for trivial reasons), we may assume that G contains a Hamilton-circuit, in particular, G is 2-connected. By (3), we can choose some $v \in V$ with $\deg(v) = 2$. If all vertices of G had degree 2, then G would be one single circuit, contradicting hypothesis (1). We can therefore choose some $w' \in V$ with $\deg(w') \geqslant 3$. By connectedness, there exists a v-w'-path P, and by finiteness there exists a vertex w on this path such that $\deg(w) \geqslant 3$ and all vertices between v and w have degree 2 in G. (Possibly, $w' = w$ is a neighbour of v and there are no vertices between v and w at all.) Let v^- and v^+ denote the two neighbours of v, with v^+ the one in the direction of w along P (possibly, $v^+ = w$). Then all vertices from v up to and including the predecessor w^- of w on P have degree 2, hence

$$\text{every circuit in } G \text{ either contains all or none of the edges } v^-v,\ vv^+,\ \ldots,\ w^-w\ . \tag{2.98}$$

Now consider some circuit C in G which does not contain any of the edges v^-v, $vv^+, \ldots, w^-w$. Such circuits exist, since for example any two neighbours w', w'' of the $\geqslant 2$ neighbours of w other than w^- are connected by a path $\tilde{P}$ which neither contains w (by 2-connectedness of G and Menger's theorem) nor any of the vertices v, v^+, $\ldots$, w^- (since these all have degree 2), so the circuit $ww'\tilde{P}w''w$ is an example. By hypothesis (2), there exist Hamilton-circuits $H_1, \ldots, H_t$ of G such that C equals their symmetric difference. Each H_i contains v, hence vv^+, hence by (2.98) contains all edges v^-v, vv^+, $\ldots$, w^-w. Since C itself does not contain any of these edges, t is even. We have shown that every circuit C in G which does not contain any of the edges v^-v, $vv^+, \ldots, w^-w$ is the symmetric difference of an even number of Hamilton-circuits; since every such circuit has an even number of edges (no matter whether the Hamilton-circuits have odd or even length). Since every circuit in $G - v$ *is* a circuit not containing any of the edges v^-v, vv^+, $\ldots$, w^-w, we have shown that $G - v$ is bipartite. $\square$

Let us note that the proof of Lemma 76 would not work if in (2) we would merely require 'for even $|G|$, every *even* cycle, and for odd $|G|$, *every* cycle is a symmetric difference of Hamilton-circuits of G': with this weakened hypothesis, if $|G|$ is even, then at the point of the proof where we show that the circuit C must be even, we could only apply hypothesis (2) to a C already known to be even, but this is what we would then be trying to prove. Thus, with that weakened hypothesis, the non-existence of odd circuits in $G - v$ could not be shown anymore. And not only did the proof break down, Lemma 76 with the above-mentioned weakened hypothesis (2) would be false: take G to be K^4 with one edge removed. This non-bipartite graph with even $|G|$ indeed has its every even cycle (of which there is only one) a symmetric difference of Hamilton-circuits; yet deleting a degree-2-vertex leaves the non-bipartite graph K^3.

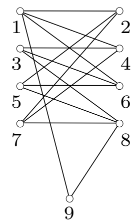


Figure 2.9: A non-bipartite example illustrating Lemma 76: the graph in Figure 2.9 is neither a forest nor a circuit, has every cycle a symmetric difference of Hamilton-circuits, *and* has 9 as a vertex of degree 2. Hence, by Lemma 76, the graph left after deleting 9 is bipartite. Unlike the example $\mathrm{K}^{\hat{4},3}$ from Definition 222, the present graph is non-bipartite. However, the graph G in Figure 2.9 does not have a Hamilton-generated flow lattice $\mathrm{Z}_1(G)$: the elementary divisors of the submodule $\langle\mathcal{H}(G)\rangle_{\mathbb{Z}} \subseteq \mathrm{Z}_1(G)$ are $(1,1,1,1,1,5)$, hence $\mathrm{Z}_1(G)/\langle\mathcal{H}(G)\rangle_{\mathbb{Z}} \cong \mathbb{Z}/5$. The oddness of 5 ensures the above-mentioned property $\mathrm{Z}_1(G;\mathbb{F}_2) = \langle\mathcal{H}(G)\rangle_{\mathbb{F}_2}$, though.

Proposition 77. *If $G = (V, E)$ is a graph such that*

(1) *G is neither a forest nor a circuit,*
(2) *every cycle in G is a symmetric difference of Hamilton-circuits,*

then it does not *follow that G has minimum degree $\geqslant 3$.*

Proof. We prove that the graph $\mathrm{K}^{\hat{4},3}$ from Definition 222 is an example of this non-implication. Evidently, it is neither a forest nor a circuit and it does not have minimum degree $\geqslant 3$. So all we have to show is that (2) holds for $G = \mathrm{K}^{\hat{4},3}$.

The cycle space of G has dimension $\|\mathrm{K}^{\hat{4},3}\| - |\mathrm{K}^{\hat{4},3}| + 1 = 14 - 8 + 1 = 7$. Since a vector space does not contain proper subspaces of its own dimension, to prove (2) it suffices to exhibit seven Hamilton-circuits linearly independent over $\mathbb{F}_2$: the circuits

(1) $C_1 := 0,1,4,5,2,3,6,7,0$,
(2) $C_2 := 0,1,6,3,4,5,2,7,0$,
(3) $C_3 := 0,1,4,3,2,5,6,7,0$,
(4) $C_4 := 0,1,2,5,4,3,6,7,0$,
(5) $C_5 := 0,1,6,5,2,3,4,7,0$,
(6) $C_6 := 0,1,2,3,6,5,4,7,0$,
(7) $C_7 := 0,1,2,5,6,3,4,7,0$,

are Hamilton-circuits of $\mathrm{K}^{\hat{4},3}$, the matrix of $C_1, \ldots, C_7$ w.r.t. the standard basis of the edge-space of $\mathrm{K}^{\hat{4},3}$ is

$$\begin{array}{c|ccccccc} & C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 \\ 0,1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0,7 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1,2 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1,4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1,6 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 2,3 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 2,5 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 2,7 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3,4 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 3,6 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 4,5 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 4,7 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 5,6 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 6,7 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{array} \tag{2.99}$$

and this matrix indeed has $\mathbb{F}_2$-rank 7. This completes the proof of Proposition 77. $\square$

If G is a forest, the property $\mathrm{Z}_1(G;\mathbb{F}_2) = \langle\mathcal{H}(G)\rangle_{\mathbb{F}_2}$ vacuously holds, and then no conclusions can be drawn from it. This is the reason for (1) in the following lemma. As to the second hypothesis in (1), of course, $G_{n,p}$ is a.a.s. not a circuit, for any p, so we could just leave out 'nor a circuit'; but we leave it in, for better analogy with the deterministic Proposition 77:

Lemma 78 (the non-implication from Proposition 77 holds for $G_{n,p}$)**.** *If $p \in [0,1]^{\mathbb{N}}$ is such that for $G \sim G_{n,p}$ a.a.s. for odd n,*

(1) *G is neither a forest nor a circuit,*

(2) *every cycle in G is a symmetric difference of Hamilton-circuits,*

then a.a.s. G has minimum degree $\geqslant 3$.

Proof. Suppose that we do *not* have

$$\mathsf{P}_{G_{n,p}}[\delta(G) \geqslant 3] \xrightarrow{n\to\infty} 1 \ . \tag{2.100}$$

Then by the Bolzano–Weierstraß-theorem there is $0 \leqslant \xi < 1$ and a subsequence $(n_i)_{i\in\mathbb{N}}$ with $\mathsf{P}_{G_{n_i,p}}[\{G \subseteq \mathrm{K}^{n_i} \colon \delta(G) \geqslant 3\}] \xrightarrow{i\to\infty} \xi$, equivalently,

$$\mathsf{P}_{G_{n_i,p}}[\{G \subseteq \mathrm{K}^{n_i} \colon \delta(G) \leqslant 2\}] \xrightarrow{i\to\infty} 1-\xi \in (0,1] \quad . \tag{2.101}$$

Since with our p, a.a.s. G contains some circuit and has property (2), in particular we a.a.s. have at least one Hamilton-circuit in G, hence in particular we a.a.s. have $\delta(G) \geqslant 2$, i.e.,

$$\mathsf{P}_{G_{n_i,p}}[\{G \subseteq \mathrm{K}^{n_i} \colon \delta(G) \geqslant 2\}] \xrightarrow{i\to\infty} 1 \quad . \tag{2.102}$$

Using the fact that intersecting with an a.a.s. property does not change an asymptotic probability, from (2.101) and (2.102) it follows that

$$\mathsf{P}_{G_{n_i,p}}[\{G \subseteq \mathrm{K}^{n_i} \colon \delta(G) = 2\}] \xrightarrow{i\to\infty} 1-\xi \in (0,1] \quad , \tag{2.103}$$

and now (2.103) and the a.a.s. property $\mathrm{Z}_1(G;\mathbb{F}_2) = \langle \mathcal{H}(G) \rangle_{\mathbb{F}_2}$ from hypothesis (2) imply

$$\mathsf{P}_{G_{n_i,p}}[\{G \subseteq \mathrm{K}^{n_i} \colon \delta(G) = 2 \quad \text{and} \quad \mathrm{Z}_1(G;\mathbb{F}_2) = \langle \mathcal{H}(G) \rangle_{\mathbb{F}_2}\}] \xrightarrow{i\to\infty} 1-\xi \in (0,1] \ . \tag{2.104}$$

All in all, we now know that with our p,

$$\mathsf{P}_{G_{n_i,p}}\left[\left\{ G \subseteq \mathrm{K}^{n_i} \colon \begin{array}{l} G \text{ is neither a forest nor a circuit} \\ \text{and } \mathrm{Z}_1(G;\mathbb{F}_2) = \langle \mathcal{H}(G) \rangle_{\mathbb{F}_2} \\ \text{and } G \text{ contains a vertex of degree } 2 \end{array} \right\}\right] \xrightarrow{i\to\infty} 1-\xi \in (0,1] \ . \tag{2.105}$$

We abbreviate

$$\mathcal{B}_{G-v,n} := \{\ G \subseteq \mathrm{K}^n \colon \text{ for every } v \in \mathrm{V}(G) \text{ with } \deg(v) = 2 \text{ the graph } G - v \text{ is bipartite } \} \ . \tag{2.106}$$

By Lemma 76, there is the deterministic implication that, for any $n \in \mathbb{N}$,

$$\left\{ G \subseteq \mathrm{K}^{n} \colon \begin{array}{l} G \text{ is neither a forest nor a circuit} \\ \text{and } \mathrm{Z}_1(G;\mathbb{F}_2) = \langle \mathcal{H}(G) \rangle_{\mathbb{F}_2} \\ \text{and } G \text{ contains a vertex of degree } 2 \end{array} \right\} \subseteq \mathcal{B}_{G-v,n} \ . \tag{2.107}$$

Applying $\mathsf{P}_{G_{n_i,p}}$ to (2.107), taking lim sup on both sides and using (2.105), it follows that

$$0 < 1-\xi \leqslant \limsup_{i\to\infty} \mathsf{P}_{G_{n_i,p}}\left[\mathcal{B}_{G-v,n_i}\right] \ . \tag{2.108}$$

We now claim that from what we know about p, it also follows that

$$\limsup_{i\to\infty} \mathsf{P}_{G_{n_i,p}}\left[\mathcal{B}_{G-v,n_i}\right] = 0 \ , \tag{2.109}$$

contradicting (2.108) and completing the proof of Lemma 78.

To prove (2.109), we first note that, with the abbreviation $\mathcal{E}_{\delta=2,n} := \{\ G \subseteq \mathrm{K}^n \colon$ there exists in G a vertex v with $\deg(v) = 2\ \}$, the limit (2.103) can be written

$$\mathsf{P}_{G_{n_i,p}}[\mathcal{E}_{\delta=2,n_i}] \xrightarrow{i\to\infty} 1-\xi > 0 \ . \tag{2.110}$$

Conditioning on the event $\mathcal{E}_{\delta=2,n_i}$, we now write

$$\mathsf{P}_{G_{n_i,p}}\left[\mathcal{B}_{G-v,n_i}\right] = \mathsf{P}_{G_{n_i,p}}\left[\mathcal{B}_{G-v,n_i} \mid \mathcal{E}_{\delta=2,n_i}\right] \cdot \mathsf{P}_{G_{n_i,p}}\left[\mathcal{E}_{\delta=2,n_i}\right] \ , \tag{2.111}$$

and claim that from what we know about p it follows that,

$$\limsup_{i\to\infty} \mathsf{P}_{G_{n_i,p}}\left[\mathcal{B}_{G-v,n_i} \mid \mathcal{E}_{\delta=2,n_i}\right] = 0 \ . \tag{2.112}$$

To prove (2.112), we show $\limsup_{i\to\infty} \mathsf{P}_{G_{n_i,p}}\left[\mathcal{B}_{G-v,n_i} \cap \mathcal{E}_{\delta=2,n_i}\right] \xrightarrow{i\to\infty} 0$. We first define $\mathcal{E}_{\forall v\colon \triangle\in G-v,n}$:= $\{\ G \subseteq \mathrm{K}^{n_i} :$ for every vertex v of G the graph $G - v$ contains a triangle $\}$ and note that $\mathcal{B}_{G-v,n}$ $\cap\ \mathcal{E}_{\forall v\colon \triangle\in G-v,n} = \varnothing$, hence, for any n,

$$\mathcal{B}_{G-v,n} \cap \mathcal{E}_{\delta=2,n} \cap \mathcal{E}_{\forall v\colon \triangle\in G-v,n} = \varnothing \ . \tag{2.113}$$

We now use that (cf. e.g. [91, Theorem 3.4]) if $p(n-1) \gg \frac{1}{n-1}$ then $G_{n-1,p}$ a.a.s. contains a triangle. Because of (2.102) and Lemma 75.(3) with $k = 1$, it follows that $p_n \gg \frac{1}{n-1}$. Since for our $G \sim G_{n,p}$ we have $G - v \sim G_{n-1,p}$ for every $v \in \mathrm{V}(G)$, it follows that with our $p_n \gg \frac{1}{n-1}$,

$$\mathsf{P}_{G_{n_i,p}}\left[\mathcal{E}_{\forall v\colon \triangle\in G-v,n_i}\right] \xrightarrow{i\to\infty} 1 \quad . \tag{2.114}$$

Again using that intersecting with an a.a.s. property does not change an asymptotic probability,

$$\begin{aligned}\limsup_{i\to\infty} \mathsf{P}_{G_{n_i,p}}\left[\mathcal{B}_{G-v,n_i} \cap \mathcal{E}_{\delta=2,n_i}\right] &\overset{(2.114)}{=} \limsup_{i\to\infty} \mathsf{P}_{G_{n_i,p}}\left[\mathcal{B}_{G-v,n_i} \cap \mathcal{E}_{\delta=2,n_i} \cap \mathcal{E}_{\forall v\colon \triangle\in G-v,n_i}\right] \\ &\overset{(2.113)}{=} 0 \ . \end{aligned} \tag{2.115}$$

Now (2.115) and (2.110) imply (2.112), which via (2.111) implies (2.109). As already mentioned, this completes the proof of Lemma 78. □

We can now give a proof of Theorem 11 in Chapter 1:

Proof of Theorem 11. Aiming at a contradiction, we assume the contrary of the conclusion, i.e., we assume that

for every infinite sequence $(n_k)_{k\in\mathbb{N}}$ of odd numbers, and for every infinite sequence $(\omega_{n_k})_{k\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ with $\omega_{n_k} \xrightarrow{k\to\infty} \infty$, there exist infinitely-many $k \in \mathbb{N}$ with $p_{n_k} \leqslant \frac{\log n_k + 2\log\log n_k + \omega_{n_k}}{n_k}$. (2.116)

Case 0. For every constant $c \in \mathbb{R}$ there exist infinitely-many odd numbers $n = n(c)$ with

$$p_n > \frac{\log n + 2\log\log n + c}{n} \ . \tag{2.117}$$

Then for every $k \in \mathbb{N}$ we can choose any such odd number $n = n(k)$, and by $n_k := n(k)$ define a sequence of odd numbers $\mathfrak{n} := (n_k)_{k\in\mathbb{N}}$ having the property that, for every $k \in \mathbb{N}$,

$$p_{n_k} > \frac{\log n_k + 2\log\log n_k + k}{n_k} \ . \tag{2.118}$$

The sequence $(\omega_{n_k})_{k\in\mathbb{N}}$ defined by $\omega_{n_k} := k$ then satisfies $\omega_{n_k} \xrightarrow{k\to\infty} \infty$. Therefore, the present sequences $(n_k)_{k\in\mathbb{N}}$ and $(\omega_{n_k})_{k\in\mathbb{N}}$, and (2.118) holding for every $k \in \mathbb{N}$, contradict (2.116). This proves Case 0 to be impossible.

Case 1. The negation of Case 0: there exists a constant $c \in \mathbb{R}$ such that for all but at most finitely-many odd numbers n we have $p_n \leqslant \frac{\log n + 2\log\log n + c}{n}$. Then, choosing $c \in \mathbb{R}$ large enough, we may just as well assume that for *all* odd numbers n we have

$$p_n \leqslant \frac{\log n + 2\log\log n + c}{n} \ . \tag{2.119}$$

We will prove that *not* a.a.s. $\delta(G) \geqslant 3$. This then contradicts Lemma 78, completing the proof of Theorem 11. It we set $p^+(n) := (\log n + 2\log\log n + c)/n \geqslant p_n$ for every n, then because of (2.119) and since the graph property $\{ G\colon\ \delta(G) \geqslant 3 \}$ is monotone increasing, it follows (cf. [91, Lemma 1.10]) that $\mathsf{P}_{G(n,p_n)}[\delta \geqslant 3] \leqslant \mathsf{P}_{G(n,p^+(n))}[\delta \geqslant 3]$ for every odd n, so

$$\limsup_{n\to\infty} \mathsf{P}_{G(n,p_n)}[\delta \geqslant 3] \leqslant \limsup_{n\to\infty} \mathsf{P}_{G(n,p^+(n))}[\delta \geqslant 3]\ . \tag{2.120}$$

Because of $\log n + 2\log\log n + c = \log n + 3\log\log n - (\log\log n - c)$, Lemma 75.(3) with $k := 3$ and $\omega_n := \log\log n - c$ implies

$$\mathsf{P}_{G(n,p^+(n))}[\delta \leqslant 3] \xrightarrow{n\to\infty} 1\ . \tag{2.121}$$

It now follows from (2.120) and (2.121) that

$$\begin{aligned}\limsup_{n\to\infty} \mathsf{P}_{G(n,p^+(n))}[\delta \geqslant 3] &= \limsup_{n\to\infty} \mathsf{P}_{G(n,p^+(n))}[\delta \geqslant 3 \text{ and } \delta \leqslant 3] \\ &= \limsup_{n\to\infty} \mathsf{P}_{G(n,p^+(n))}[\delta = 3]\ . \end{aligned}\tag{2.122}$$

By definition of p^+, and by Lemma 75.(2) with $k := 2$ and $h := 0$, we have $\mathsf{P}_{G(n,p^+(n))}[\delta = 3] \xrightarrow{n\to\infty} \exp(-\exp(-c)/2)$ which when substituted into (2.122) yields

$$\limsup_{n\to\infty} \mathsf{P}_{G(n,p^+(n))}[\delta \geqslant 3] = \exp(-\exp(-c)/2)\ . \tag{2.123}$$

From (2.123) and (2.120) we finally get a conclusion involving p itself:

$$\limsup_{n\to\infty} \mathsf{P}_{G_{n,p_n}}[\delta \geqslant 3] \leqslant \exp(-\exp(-c)/2) < 1\ . \tag{2.124}$$

The bound (2.124) shows that for p it is not the case that $\mathsf{P}_{G_{n,p_n}}[\delta \geqslant 3] \xrightarrow{n\to\infty} 1$, contradicting Lemma 78. This completes the proof. $\square$

It it very likely that the condition $p(n) \geqslant n^{-1/2+\varepsilon}$ in Theorem 74 can be much improved. By using (instead of the C_n^2) the sparser graphs $\mathrm{C}_n^{2^-}$ from Definition 214 as seed graphs (this is one of the motivations for investing so much effort in that special set of graphs), and by adapting the embedding technology of [92] and [107], it seems possible to not only strengthen Theorem 74 from $-1/2$ to $-2/3$ (in view of Łuczak's argument, which apparently reaches all the way down to the lower bound that we give in Theorem 11, *this* particular strengthening is pointless), but rather prove a stronger conclusion:

Conjecture 79. *For every $\varepsilon > 0$, if $G \sim G(n, n^{-2/3+\varepsilon})$, then* $\mathrm{Z}_1(G)$ *is a.a.s. Hamilton*-based.

Working on Conjecture 79, despite Conjecture 12.(gnp.1) from Chapter 1 apparently settled by an argument of T. Łuczak is not superfluous: Conjecture 79 is about Hamilton-*bases* for $\mathrm{Z}_1(G)$, while Łuczak's argument, to all appearances, seems only to work for $\mathrm{Z}_1(G;\mathbb{F}_2)$, hence not to work for Conjecture 79, nor Conjecture 12.(gnp.2), let alone Conjecture 12.(gnp.3). In return, Conjecture 79 has a stronger hypothesis than Conjecture 12, which is somewhat in between 'sparse' and 'dense' random graphs. The adaption of the embedding methods of [92] and [107] in particular requires the construction of $\mathrm{C}_n^{2^-}$-dedicated 'absorbers', sparser than the absorbers used in [107]. This is work in progress and left out of this thesis.

It seems very likely that $p(n) \geqslant n^{-2/3+\varepsilon}$ is the utmost of what can be achieved with the technique of proving the existence of some spanning subgraph, pre-selected ahead of time: every seed-graph suitable for the monotonicity argument must be Hamilton-connected (cf. the footnote on p. 27), hence must have minimum degree three, which makes the embedding-techniques from [92] and [107] demand $p(n) \geqslant n^{-2/3+\varepsilon}$.

3 Logical limit laws for minor-closed classes of graphs

> **construe** ▸ **verb (construes, construing, construed)** [with obj.] **1** interpret (a word or action) in a particular way [...] ORIGIN late Middle English: from Latin *construere* (see **construct**), in late Latin 'analyse the construction of a sentence'
>
> Oxford Dictionary of English. Second Edition 2003. ISBN 0-19-8613474. p. 371

This chapter contains proofs for the results introduced in Section 1.3 of Chapter 1.

3.1 How to determine explicit sets of probability limits for an arbitrary addable class of graphs

The key to our explicit formulas for probability limits is the realisation that for any given addable class $\mathcal{A}$ and any given statement φ in MSO-logic about graphs, the limit of its probabilities for φ w.r.t. $\mathcal{A}$ can be described by one infinite series corresponding to the *fragment* $\mathrm{Frag}(G)$ of a random graph G from $\mathcal{A}$. The series is indexed by a set of isomorphism types in $\mathcal{A}$ which when disjointly united with a large random connected graph of $\mathcal{A}$ result in a model of φ, see Theorem 87 on p. 118 and its proof. We will then be fortunate enough to find the relevant mathematics sufficiently well-developed to go on from Theorem 87 to explicit formulas: firstly, the asymptotic distribution of non-largest components of an arbitrary uniformly random element of an addable class is precisely known, thanks to work of McDiarmid (see Theorem 83), turning the formidable problem of describing the tail-sums of a convergent series indexed by isomorphism types of graphs into the comparatively straightforward task of describing the tail-sums of an ordinary $\mathbb{N}$-indexed convergent series of real numbers; secondly, explicit conditions which decide about the structure of a set of tail-sums of a convergent series have long since been worked out (see Corollary 95), and what will then remain to do is the (technically demanding) task of checking this criterion.

All told, the proofs of our results about probability limit sets are rather complex, and harness several results from the literature. E.g., for the proof of the general Theorem 87 on p. 118 we will use a construction inspired by work of McColm [126], providing what one might call 'universal models' for equivalence classes w.r.t. the Ehrenfeucht-Fraïssé-relation for MSO-logic. As another example, for a reduction from convergent series indexed by isomorphism types to ordinary convergent series indexed by $\mathbb{N}$ we use a theorem of McDiarmid (Theorem 83) about the asymptotic distribution of the non-largest components of a random graph from an addable class of graphs. For the analysis of the $\mathbb{N}$-indexed convergent series we use lemmas about the structure of tail-sums of convergent series, and to convert the information provided by those lemmas into an explicit description of the the closure of the probability limits, we use (verified approximations based on) the precise analytic determination in [63] of two real numbers related to the set of planar graphs.

In Section 3.1.3 we present a general method to determine a set of probability limits for any addable minor-closed class, and then carry out its specific steps for the addable class of *forests*, and for the addable class of *planar graphs*.

3.1.1 Asymptotic MSO-probabilities can be described by the tail-sums of *one* convergent series

The statement in the title of the present section is explained by Theorem 87 below, together with the definition of $\mathsf{P}_{\mathrm{BP}\mathcal{A}}$ in Theorem 81 below.

As mentioned before, the probability that a uniformly random n-vertex forest is a tree converges to $1/A(\rho) = \mathrm{e}^{-1/2}$, with A the egf of forests and ρ its radius of convergence. (A general explanation for this limit is [128, Corollary 1.6(c)].) McDiarmid, Steger and Welsh conjectured [130, Conjecture 2.4] that $\mathrm{e}^{-1/2}$ is the minimum, over *all bridge-addable* classes $\mathcal{A}$ of graphs of the probability limit for an n-vertex uniformly random element of $\mathcal{A}$ to be connected. That conjecture is still open, but it was recently proved under the additional hypothesis of $\mathcal{A}$ being not only bridge-addable, but addable:

Theorem 80 ([4] [96, Theorem 1.1])**.** *If $\mathcal{A}$ is an addable, minor-closed class of graphs, and A its egf with radius of convergence ρ, then $\frac{1}{A(\rho)} \geqslant \mathrm{e}^{-\frac{1}{2}}$.*

An explanation for the role of expressions like $\frac{1}{A(\rho)}$ in Theorem 80 is provided by what will be the main quantitative tool in our determination of limit-sets: McDiarmid's *Boltzmann–Poisson-measure* for isomorphism types:

Theorem 81 (McDiarmid; the Boltzmann–Poisson measure on the isomorphism types in a decomposable class of graphs; cf. [128, Theorem 1.3])**.** *If $\mathcal{D}$ denotes a decomposable class of graphs, $\mathcal{UD}$ the set of all isomorphism types of elements of $\mathcal{D}$, D the exponential generating function of $\mathcal{D}$, and ρ its radius of convergence, then*

$$\mathsf{P}_{\mathrm{BP}\mathcal{D}}[H] := \frac{1}{D(\rho)} \, \frac{\rho^{|H|}}{|\mathrm{Aut}(H)|} \qquad \textit{for every } H \in \mathcal{UD} \tag{3.1}$$

defines a probability measure on $\mathcal{UD}$.

Moreover, w.r.t. $\mathsf{P}_{\mathrm{BP}\mathcal{D}}$ *the set* $\{\ \#_H :=$ *number of connected components isomorphic to H: H a* connected *element of $\mathcal{UD}$ $\}$ is a set of independent random variables with* $\#_H \sim \mathrm{Poi}(\rho^{|H|}/|\mathrm{Aut}(H)|)$.

We will need to know the following:

Theorem 82 (cf. [128, Theorem 1.2 and Lemma 2.4])**.** *Let $\mathcal{A}$ denote an addable, minor-closed class, and $\mathcal{A} \subseteq \mathcal{C}$ its connected elements. Then both $\mathcal{C}$ and $\mathcal{A}$ are smooth.* □

An essential tool in our proof of Theorem 87 below, allowing a reduction from summations indexed by infinite sets of substructures to summations indexed by $\mathbb{N}$, is the following recent result about the distribution of the fragment of a graph from an addable class (see p. 193 for $\mathrm{Frag}(G_n)$):

Theorem 83 (McDiarmid; cf. [128, Theorem 1.5])**.** *If $\mathcal{A}$ denotes an addable minor-closed class other than the class of all graphs, A the egf of $\mathcal{A}$ and ρ its radius of convergence, then the series $A(\rho)$ converges and, if $G_n \in \mathcal{A}_n$ is a graph drawn uniformly at random from the n-vertex elements of $\mathcal{A}$, the distribution of the isomorphism type of the fragment* $\mathrm{Frag}(G_n)$ *for $n \to \infty$ converges in total variation to* $\mathsf{P}_{\mathrm{BP}\mathcal{A}}$ *of Theorem 81.* □

As an aside, let us mention that, in general, non-addable minor-closed graph classes can display very different behaviour: see, for example, the recent preprint [27], where several non-addable graph classes are analysed in detail. In particular, the size of the largest component can happen to be a.a.s. *sublinear* (as opposed to a.a.s. $n - O(1)$ for the special non-addable class of graphs on a fixed surface). One therefore cannot expect a result like Theorem 83 to hold for arbitrary decomposable minor-closed classes, even if assumed to be smooth. We will see below, however, that for the special non-addable class of graphs embeddable on a fixed surface, one finds behaviour similar to the addable class of planar graphs. In particular, despite being a non-addable class, the *size* (questions about its structure are more difficult) of the largest component essentially behaves like the largest component of a random planar graph (the number of vertices not in the largest component is described by the Boltzmann–Poisson random graph for planar graphs, cf. Theorem 134 on p. 150).

Theorem 83 has interesting corollaries, in particular a formula expressing the probability for connectedness of a random element of an arbitrary addable minor-closed class in terms of its egf:

Lemma 84 ([128, Lemma 4.3]; by Theorem 82 the class $\mathcal{A}$ in the present statement is smooth, hence [128, Lemma 4.3] applicable)**.** *If $\mathcal{A}$ is an addable and minor-closed class of graphs, and $G_n \in \mathcal{A}_n$ a uniformly random element, then a.a.s. $|\mathrm{Big}(G_n)| = n - O(1)$ and*

$$\lim_{n\to\infty} \mathsf{P}[G_n \quad \text{connected}] = \lim_{n\to\infty} \mathsf{P}[\mathrm{Frag}(G_n) = \varnothing] = \mathsf{P}[R = \varnothing] = \frac{1}{G(\rho)}\ . \tag{3.2}$$

The following theorem is an important tool for proving our results on addable classes:

Theorem 85 (McDiarmid; [128, Theorem 1.7])**.** *For every addable minor-closed class of graphs $\mathcal{A}$ there exists $\alpha > 0$ such that for any fixed connected graph $H \in \mathcal{A}$, a uniformly random element of the n-vertex graphs in $\mathcal{A}$ a.a.s. contains more than αn pendant copies of H.* □

We will use a consequence of Theorem 85, explicitly stating that the substructure appears within the giant component:

Lemma 86. *For every addable minor-closed class of graphs $\mathcal{A}$ there exists $\alpha > 0$ such that if $H \in \mathcal{A}$ is any fixed connected graph and $G_n \in \mathcal{A}_n$ a uniformly random element, then* $\mathrm{Big}(G_n)$ *a.a.s. contains more than αn pendant copies of H.*

Proof. Let P denote the uniform measure on $\mathcal{A}_n$, $\tilde{\alpha} > 0$ the constant guaranteed by Theorem 85 on input $\mathcal{A}$, and $\mathcal{E}_n$ the event that G_n contains more than $\tilde{\alpha}n$ pendant copies of H.

Let $\mathcal{E}_n'$ denote the event that $\mathrm{Big}(G_n)$ contains more than $\frac{1}{2}\tilde{\alpha}n$ pendant copies of H, and $\mathcal{E}_n''$ the event that $\mathrm{Frag}(G_n)$ contains more than $\frac{1}{2}\tilde{\alpha}n$ copies of H. Then $\mathsf{P}[\mathcal{E}_n'] + \mathsf{P}[\mathcal{E}_n''] \geqslant \mathsf{P}[\mathcal{E}_n' \cup \mathcal{E}_n''] =$ (since $\mathcal{E}_n' \cup \mathcal{E}_n'' \supseteq \mathcal{E}_n$) $\geqslant \mathsf{P}[\mathcal{E}_n] \xrightarrow{n\to\infty} 1$, the latter by Theorem 85, hence

$$\liminf_{n\to\infty} \quad \mathsf{P}[\mathcal{E}_n'] + \mathsf{P}[\mathcal{E}_n''] \geqslant 1\ . \tag{3.3}$$

Moreover, for every fixed $K > 0$, for sufficiently large n we have $\mathcal{E}_n'' \subseteq \{G \in \mathcal{A}_n \colon |\mathrm{Frag}(G)| > K\}$, hence

$$\mathsf{P}[\mathcal{E}_n''] \leqslant \mathsf{P}[|\mathrm{Frag}(G_n)| > K]\ . \tag{3.4}$$

Furthermore, by Theorem 83 with A and ρ as defined there and $\mathcal{UA}$ denoting the set of all isomorphism types in $\mathcal{A}$, and since convergence in total variation implies convergence in distribution,

$$\lim_{K\to\infty}\lim_{n\to\infty} \mathsf{P}[|\mathrm{Frag}(G_n)| > K] = \lim_{K\to\infty} \mathsf{P}_{\mathrm{BP}\mathcal{A}}[|\cdot| > K] = \frac{1}{A(\rho)} \lim_{K\to\infty} \sum_{H\in\mathcal{UA}\colon |H|>K} \frac{\rho^{|H|}}{|\mathrm{Aut}(H)|} = 0\ . \tag{3.5}$$

For every $K > 0$ we have $\limsup_{n\to\infty} \mathsf{P}[\mathcal{E}_n''] \leqslant$ (by (3.4)) $\leqslant \limsup_{n\to\infty} \mathsf{P}[|\mathrm{Frag}(G_n)| > K] =$ (since this is known to converge by Theorem 83) $= \lim_{n\to\infty} \mathsf{P}[|\mathrm{Frag}(G_n)| > K]$. Since K is arbitrary, in view of (3.5) it now follows that $\limsup_{n\to\infty} \mathsf{P}[\mathcal{E}_n''] = 0$, hence, being a probability, $\mathsf{P}[\mathcal{E}_n''] \xrightarrow{n\to\infty} 0$. This, together with (3.3), implies $\liminf_{n\to\infty} \mathsf{P}[\mathcal{E}_n'] \geqslant 1$, which is equivalent to $\mathsf{P}[\mathcal{E}_n'] \xrightarrow{n\to\infty} 1$ since $\mathsf{P}[\mathcal{E}_n']$ is a probability. With $\alpha := \frac{1}{2}\tilde{\alpha}$, the claim in Lemma 86 is proved. □

Here we prove a more explicit version of our convergence law in Theorem 24 announced in Chapter 1; it is another global consequence of containing a local structure:

Theorem 87 (convergence law for MSO-sentences about an addable minor-closed class; joint work with T. Müller, M. Noy and A. Taraz)**.** *Suppose $\mathcal{A}$ denotes an addable, minor-closed class of graphs, $\mathcal{UA}$ the set of its isomorphism types and $\mathsf{P}_{\mathrm{BP}\mathcal{A}}$ the Boltzmann–Poisson measure from Theorem 81. Then for every MSO-sentence φ about graphs there exists a set $\mathcal{F}(\varphi) \subseteq \mathcal{UA}$ with*

$$\mathsf{P}_{\mathcal{A}_n}[G_n \models \varphi] := \frac{|\{G \in \mathcal{A}_n : G \models \varphi\}|}{|\mathcal{A}_n|} \xrightarrow{n\to\infty} \mathsf{P}_{\mathrm{BP}\mathcal{A}}[\mathcal{F}(\varphi)] \ . \tag{3.6}$$

Proof. Let $\varphi \in \mathrm{MSO}$ be given and set $r := \mathrm{qr}(\varphi)$. By Lemma 86, for a uniformly random element $G_n \in \mathcal{A}_n$ the giant component $\mathrm{Big}(G_n)$ a.a.s. contains a pendant copy of a graph $\mathrm{X}_{r,\mathcal{A}}$ as guaranteed by Lemma 226 in Chapter 5. Thus, a.a.s.

$$\mathrm{Big}(G_n) \equiv_r^{\mathrm{MSO}} \mathrm{X}_{r,\mathcal{A}} \ . \tag{3.7}$$

By a standard fact (cf. e.g. [31, Lemma 6.20]) about disjoint unions of $\equiv_r^{\mathrm{MSO}}$-equivalent structures, (3.7) implies that a.a.s.,

$$G_n = \mathrm{Big}(G_n) \sqcup \mathrm{Frag}(G_n) \equiv_r^{\mathrm{MSO}} \mathrm{X}_{r,\mathcal{A}} \sqcup \mathrm{Frag}(G_n) \ . \tag{3.8}$$

Hence, $\mathsf{P}_{\mathcal{A}_n}[G_n \models \varphi] = \mathsf{P}_{\mathcal{A}_n}[\mathrm{X}_{r,\mathcal{A}} \sqcup \mathrm{Frag}(G_n) \models \varphi] + o(1)$, the $o(1)$ term since w.r.t. $\mathsf{P}_{\mathcal{A}_n}$, the equivalence (3.8) holds only a.a.s. Now let $\mathcal{H}(\varphi) := \{H \in \mathcal{A} \colon \mathrm{X}_{r,\mathcal{A}} \sqcup H \models \varphi\}$, where $\sqcup$ denotes vertex-disjoint union, and $\mathcal{UH}(\varphi)$ the set of isomorphism types in $\mathcal{H}(\varphi)$. Then

$$\begin{aligned}\lim_{n\to\infty} \mathsf{P}_{\mathcal{A}_n}[G_n \models \varphi] &\overset{(3.8)}{=} \lim_{n\to\infty} \mathsf{P}_{\mathcal{A}_n}[\mathrm{X}_{r,\mathcal{A}} \sqcup \mathrm{Frag}(G_n) \models \varphi] \\ &= \lim_{n\to\infty} \mathsf{P}_{\mathcal{A}_n}[\mathrm{Frag}(G_n) \in \mathcal{H}(\varphi)] = \mathsf{P}_{\mathrm{BP}\mathcal{A}}[\mathcal{UH}(\varphi)] \ ,\end{aligned} \tag{3.9}$$

the last equality by Theorem 83. With $\mathcal{F}(\varphi) := \mathcal{UH}(\varphi)$, the proof of Theorem 87 is complete. $\square$

While not explicit in [128], Lemma 86 carries over to random *connected* graphs from $\mathcal{G}$:

Corollary 88. *For every addable minor-closed class of graphs $\mathcal{A}$ there exists $\alpha > 0$ such that for any fixed connected graph $H \in \mathcal{A}$, a uniformly random element C_n of the set of n-vertex* connected *elements of $\mathcal{A}$ contains more than αn pendant copies of H, a.a.s. if $n \to \infty$.*

Proof. Let $\mathcal{E}_{n,\alpha} := \{\ G \in \mathcal{A}_n \colon\ G$ contains more than αn pendant copies of $H\ \}$. Let $\mathsf{P}_{\mathcal{A}_n}$ denote the uniform measure on $\mathcal{A}_n$. By Theorem 85, there exists $\alpha > 0$ with

$$\mathsf{P}_{\mathcal{A}_n}[\mathcal{E}_{n,\alpha}] \xrightarrow{n\to\infty} 1 \ . \tag{3.10}$$

Let $\mathcal{C}_n := \{\ G \in \mathcal{A}_n \colon\ G$ connected $\}$ and $\mathcal{F}_{n,\alpha} := \{\ G \in \mathcal{C}_n \colon\ G$ contains more than αn pendant copies of $H\ \} \subseteq \mathcal{E}_{n,\alpha}$.

By way of contradiction, suppose $(\mathsf{P}_{\mathcal{C}_n}[\mathcal{F}_{n,\alpha}])_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$ for $n \to \infty$ does *not* converge to 1. Then, by the Bolzano–Weierstraß-theorem, there exists a subsequence $(n_i)_{i\in\mathbb{N}}$ and $\xi \in [0,1)$ with

$$\mathsf{P}_{\mathcal{C}_{n_i}}[\mathcal{F}_{n_i,\alpha}] \xrightarrow{i\to\infty} \xi < 1 \ . \tag{3.11}$$

After conditioning on $\mathcal{C}_n$, a uniformly random element G_n of $(\mathcal{A}_S)_n$ is distributed like C_n, i.e, $\mathsf{P}_{\mathcal{A}_n}[\mathcal{E}_{n,\alpha} \mid \mathcal{C}_n] = \mathsf{P}_{\mathcal{C}_n}[\mathcal{F}_{n,\alpha}]$, hence the first equality in

$$\begin{aligned}\lim_{i\to\infty} \mathsf{P}_{\mathcal{A}_{n_i}}[\mathcal{E}_{n_i,\alpha}] &= \lim_{i\to\infty} \mathsf{P}_{\mathcal{A}_{n_i}}[\mathcal{E}_{n_i,\alpha} \mid \mathcal{C}_{n_i}] \cdot \mathsf{P}_{\mathcal{A}_{n_i}}[\mathcal{C}_{n_i}] + \mathsf{P}_{\mathcal{A}_{n_i}}[\mathcal{E}_{n_i,\alpha} \mid \mathcal{C}_{n_i}^c] \cdot (1 - \mathsf{P}_{\mathcal{A}_{n_i}}[\mathcal{C}_{n_i}]) \\ &= \lim_{i\to\infty} \mathsf{P}_{\mathcal{C}_{n_i}}[\mathcal{F}_{n_i,\alpha}] \cdot \mathsf{P}_{\mathcal{A}_{n_i}}[\mathcal{C}_{n_i}] + \mathsf{P}_{\mathcal{A}_{n_i}}[\mathcal{E}_{n_i,\alpha} \mid \mathcal{C}_{n_i}^c] \cdot (1 - \mathsf{P}_{\mathcal{A}_{n_i}}[\mathcal{C}_{n_i}]) \\ &\leqslant \lim_{i\to\infty} \mathsf{P}_{\mathcal{C}_{n_i}}[\mathcal{F}_{n_i,\alpha}] \cdot \mathsf{P}_{\mathcal{A}_{n_i}}[\mathcal{C}_{n_i}] + 1 \cdot (1 - \mathsf{P}_{\mathcal{A}_{n_i}}[\mathcal{C}_{n_i}]) \\ \text{(by (3.2) in Lemma 84)} &= \xi \cdot 1/G(\rho) + (1 - 1/G(\rho)) \\ &= 1 - (1-\xi) \cdot \tfrac{1}{G(\rho)} < 1 \ ,\end{aligned} \tag{3.12}$$

$\cdot^{\mathrm{c}}$ denoting complementation in $\mathcal{A}_{n_i}$. Inequality (3.12) contradicts (3.10), completing the proof. $\square$

We can now prove a zero-one law for MSO-probability-limits w.r.t. any addable minor-closed class of graphs:

Theorem 89. *If φ denotes any statement in MSO-logic about graphs, $\mathcal{A}$ any addable, minor-closed class of graphs, and $\mathcal{C}_n \subseteq \mathcal{A}_n$ the set of all n-vertex* connected *graphs in $\mathcal{A}$,*

$$\lim_{n\to\infty} \frac{|\{G \in \mathcal{C}_n \colon G \models \varphi\}|}{|\mathcal{C}_n|} \in \{0,1\} \ . \tag{3.13}$$

Proof. Let an arbitrary $\varphi \in \mathrm{MSO}$ be given. With $r := \mathrm{qr}(\varphi)$, Corollary 88 guarantees that a uniformly random element of $\mathcal{C}_n$ a.a.s. contains a pendant copy of a graph $\mathrm{X}_{r,\mathcal{A}}$ as provided by Lemma 226 on p. 205. Hence, by Lemma 226, a uniformly random element of $\mathcal{C}_n$ a.a.s. is $\equiv_r^{\mathrm{MSO}}$-equivalent to $\mathrm{X}_{r,\mathcal{A}}$. Thus, if $\mathrm{X}_{r,\mathcal{A}} \models \varphi$ then the limit in (3.13) is 1, while otherwise it is 0. $\square$

Here we prove the more detailed version of Theorem 26 announced in Section 1.3.1 of Chapter 1. Let $\mathcal{A}$ denote an arbitrary addable minor-closed class of graphs. For brevity,

(1) $\mathbb{L}_{\mathcal{A},\mathrm{MSO}} := \left\{\lim_{n\to\infty} \frac{|\{G\in\mathcal{A}_n \colon G\models\varphi\}|}{|\mathcal{A}_n|} \colon \varphi \in \mathrm{MSO}\right\}$, (2) $\mathbb{L}_{\mathcal{A},\mathrm{FO}} := \left\{\lim_{n\to\infty} \frac{|\{G\in\mathcal{A}_n \colon G\models\varphi\}|}{|\mathcal{A}_n|} \colon \varphi \in \mathrm{FO}\right\}$.

Both $\mathbb{L}_{\mathcal{A},\mathrm{MSO}}$ and $\mathbb{L}_{\mathcal{A},\mathrm{FO}}$ are countable sets since the set of FO- (resp. MSO-) sentences is countable. Moreover, $\mathbb{L}_{\mathcal{A},\mathrm{FO}} \subseteq \mathbb{L}_{\mathcal{A},\mathrm{MSO}} \subseteq \{\mathsf{P}_{\mathrm{BP}\mathcal{A}}[\mathcal{F}] \colon \mathcal{F} \subseteq \mathcal{U}\mathcal{A}\}$, the latter by Theorem 87 on p. 118. The following shows that these inclusions are dense:

Lemma 90 (the Boltzmann–Poisson-measure of any set of isomorphism types in an addable class can be approximated by the asymptotic uniform probability of a single FO-sentence; joint work with T. Müller, M. Noy and A. Taraz)**.** *For every addable minor-closed class $\mathcal{A}$ of graphs, every $\mathcal{F} \subseteq \mathcal{U}\mathcal{A}$ and every $\varepsilon > 0$, there exists $\varphi \in \mathrm{FO}$ such that*

$$\left| \mathsf{P}_{\mathrm{BP}\mathcal{A}}[\mathcal{F}] - \lim_{n\to\infty} \frac{|\{G \in \mathcal{A}_n \colon G \models \varphi\}|}{|\mathcal{A}_n|} \right| \leqslant \varepsilon \ . \tag{3.14}$$

Proof. It suffices to consider *finite* $\mathcal{F} \subseteq \mathcal{U}\mathcal{A}$, since by (3.1) in Theorem 81 on p. 116, and the vanishing of tail-sums of convergent series, for every $0 < \varepsilon' < \varepsilon$ there always is a finite subset $\mathcal{F}' \subseteq \mathcal{F}$ with $\mathsf{P}_{\mathrm{BP}\mathcal{A}}[\mathcal{F}'] \geqslant \mathsf{P}_{\mathrm{BP}\mathcal{A}}[\mathcal{F}] - \varepsilon'$. We therefore assume $\mathcal{F}$ to be finite. Denoting by $\{|\cdot| > K\}$ the event that an isomorphism type in $\mathcal{A}$ has more than K vertices, in view of the formula for the measure in Theorem 81 it is possible to choose $K > 0$ with $\mathsf{P}_{\mathrm{BP}\mathcal{A}}[|\cdot| > K] < \varepsilon$. For $G \in \mathcal{A}_n$, let $\mathrm{Frag}_K(G)$ denote the union of all those components of G with order at most K. For every $F \in \mathcal{F}$, the event $\{G \in \mathcal{A}_n \colon \mathrm{Frag}_K(G) \cong F\}$ can be defined in FO-logic. Since $\mathcal{F}$ is finite, the event $\{G \in \mathcal{A}_n \colon \mathrm{Frag}_K(G) \in \mathcal{F}\}$ can thus be defined by a single FO-sentence φ, i.e., we define φ as any FO-formula with

$$\{G \in \mathcal{A}_n \colon \mathrm{Frag}_K(G) \in \mathcal{F}\} = \{G \in \mathcal{A}_n \colon G \models \varphi\} \ . \tag{3.15}$$

Suppose $G \in \mathcal{A}$ and $\mathrm{Frag}_K(G) \in \mathcal{F}$. Then $|\mathrm{Frag}(G)| \leqslant K$ implies $\mathrm{Frag}_K(G) = \mathrm{Frag}(G) \in \mathcal{F}$. Equivalently, we then know that [$|\mathrm{Frag}(G)| > K$ or $\mathrm{Frag}_K(G) \in \mathcal{F}$]. To summarise,

$$\{G \in \mathcal{A}_n \colon \mathrm{Frag}_K(G) \in \mathcal{F}\} \subseteq \{G \in \mathcal{A}_n \colon \mathrm{Frag}(G) \in \mathcal{F} \quad \text{or} \quad |\mathrm{Frag}(G)| > K\} \ . \tag{3.16}$$

Abbreviating $\mathsf{P}_{\mathcal{A}_n}[\cdot] := |\{G \in \mathcal{A}_n \colon \cdot\}|/|\mathcal{A}_n|$, we have the estimates

$$\begin{aligned}
\lim_{n\to\infty} \mathsf{P}_{\mathcal{A}_n}[\mathrm{Frag}_K \in \mathcal{F}] &\overset{(3.16)}{\leqslant} \lim_{n\to\infty} \mathsf{P}_{\mathcal{A}_n}\left[\mathrm{Frag} \in \mathcal{F} \quad \text{or} \quad |\mathrm{Frag}| > K\right] \\
&\leqslant \lim_{n\to\infty} \mathsf{P}_{\mathcal{A}_n}\left[\mathrm{Frag} \in \mathcal{F}\right] \quad + \quad \lim_{n\to\infty} \mathsf{P}_{\mathcal{A}_n}\left[|\mathrm{Frag}| > K\right] \\
\text{(by Theorem 81)} &= \mathsf{P}_{\mathrm{BP}\mathcal{A}}[\mathcal{F}] + \mathsf{P}_{\mathrm{BP}\mathcal{A}}[|\cdot| > K] \\
&< \mathsf{P}_{\mathrm{BP}\mathcal{A}}[\mathcal{F}] + \varepsilon \ ,
\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}
\lim_{n\to\infty} \mathsf{P}_{\mathcal{A}_n}[\mathrm{Frag}_K \in \mathcal{F}] &\geqslant \lim_{n\to\infty} \mathsf{P}_{\mathcal{A}_n}[\mathrm{Frag} \in \mathcal{F} \quad \text{and} \quad |\mathrm{Frag}| \leqslant K] \\
&\geqslant \lim_{n\to\infty} \mathsf{P}_{\mathcal{A}_n}[\mathrm{Frag} \in \mathcal{F}] \quad - \quad \lim_{n\to\infty} \mathsf{P}_{\mathcal{A}_n}[|\mathrm{Frag}| > K] \\
\text{(by Theorem 81)} &= \mathsf{P}_{\mathrm{BP}\mathcal{A}}[\mathcal{F}] - \mathsf{P}_{\mathrm{BP}\mathcal{A}}[|\cdot| > K] \\
&> \mathsf{P}_{\mathrm{BP}\mathcal{A}}[\mathcal{F}] - \varepsilon \ ,
\end{aligned} \tag{3.18}$$

which because of (3.15) prove (3.14) in Lemma 90. □

As a consequence of our Theorem 89, we now prove the following result, which sheds some light on the structure of the set of probability limits w.r.t. a general addable minor-closed class; it is the more detailed version of Proposition 25 announced in Chapter 1:

Theorem 91 (the middle gap of the truth spectrum for a general addable class of graphs; joint work with T. Müller, M. Noy and A. Taraz)**.** *If $\mathcal{A}$ denotes an addable, minor-closed class of graphs, A its egf, and ρ the radius of convergence of A, then*

$$\left\{\lim_{n\to\infty} \frac{|\{G \in \mathcal{A}_n : G \models \varphi\}|}{|\mathcal{A}_n|} : \varphi \in \mathrm{MSO}\right\} \cap \left(1 - \frac{1}{A(\rho)}, \frac{1}{A(\rho)}\right) = \varnothing \ . \tag{3.19}$$

Proof. Let $\varphi \in \mathrm{MSO}$ be given. Let $\mathcal{C}_n := \{\, G \in \mathcal{A}_n : \ G \text{ connected} \,\}$ and $\mathsf{P}_{\mathcal{A}_n}$ (resp. $\mathsf{P}_{\mathcal{C}_n}$) the uniform distribution on $\mathcal{A}_n$ (resp. $\mathcal{C}_n$). Then $\mathsf{P}_{\mathcal{A}_n}[\cdot \mid \mathcal{C}_n] = \mathsf{P}_{\mathcal{C}_n}[\cdot]$. By Theorem 89, either $\lim_{n\to\infty} \mathsf{P}_{\mathcal{C}_n}[\varphi] = 1$ or $\lim_{n\to\infty} \mathsf{P}_{\mathcal{C}_n}[\varphi] = 0$.

If $\lim_{n\to\infty} \mathsf{P}_{\mathcal{C}_n}[\varphi] = 1$, then, with $\cdot^{\mathrm{c}}$ denoting complementation in $\mathcal{A}_n$,

$$\begin{aligned}
\lim_{n\to\infty} \mathsf{P}_{\mathcal{A}_n}[\varphi] &= \lim_{n\to\infty} \mathsf{P}_{\mathcal{A}_n}[\varphi \mid \mathcal{C}_n] \cdot \mathsf{P}_{\mathcal{A}_n}[\mathcal{C}_n] + \mathsf{P}_{\mathcal{A}_n}[\varphi \mid \mathcal{C}_n^{\mathrm{c}}] \cdot \mathsf{P}_{\mathcal{A}_n}[\mathcal{C}_n^{\mathrm{c}}] \\
&\geqslant \lim_{n\to\infty} \mathsf{P}_{\mathcal{C}_n}[\varphi] \cdot \mathsf{P}_{\mathcal{A}_n}[\mathcal{C}_n] = (\lim_{n\to\infty} \mathsf{P}_{\mathcal{C}_n}[\varphi]) \cdot (\lim_{n\to\infty} \mathsf{P}_{\mathcal{A}_n}[\mathcal{C}_n]) = 1/A(\rho) \ ,
\end{aligned} \tag{3.20}$$

where for the last equality we used Lemma 84. If on the contrary $\lim_{n\to\infty} \mathsf{P}_{\mathcal{C}_n}[\varphi] = 0$, then $\lim_{n\to\infty} \mathsf{P}_{\mathcal{C}_n}[\neg\varphi] = 1$, hence the above argument with φ replaced by $\neg\varphi$ yields $\lim_{n\to\infty} \mathsf{P}_{\mathcal{C}_n}[\neg\varphi] \geqslant 1/A(\rho)$, which is equivalent to $\lim_{n\to\infty} \mathsf{P}_{\mathcal{C}_n}[\varphi] \leqslant 1 - 1/A(\rho)$, completing the proof. □

We now prove the more detailed version of Theorem 26 from Section 1.3.1 of Chapter 1:

Theorem 92 (joint work with T. Müller, M. Noy and A. Taraz)**.** *If $\mathcal{A}$ denotes any addable, minor-closed class of graphs,* FO *(resp.* MSO*) the set of all FO-sentences (resp. MSO-sentences) about graphs,* cl *closure of subsets of* $\mathbb{R}$ *w.r.t. the usual metric topology,* $\mathbb{L}_{\mathcal{A},\cdot}$ *the sets from* (1) *and* (2) *on p. 119,* $\mathcal{UA}$ *the set of all isomorphism types in* $\mathcal{A}$ *and* $\mathsf{P}_{\mathrm{BP}\mathcal{A}}$ *the measure from Theorem 81, then we have the following equality, and the set is equal to a union of finitely-many intervals:*

$$\mathrm{cl}\,(\mathbb{L}_{\mathcal{A},\mathrm{FO}}) = \mathrm{cl}\,(\mathbb{L}_{\mathcal{A},\mathrm{MSO}}) = \{\mathsf{P}_{\mathrm{BP}\mathcal{A}}[\mathcal{F}] : \mathcal{F} \subseteq \mathcal{UA}\} \ . \tag{3.21}$$

Proof. From Lemma 90 we know that already the set $\mathbb{L}_{\mathcal{A},\mathrm{FO}}$ of probability limits w.r.t. $\mathcal{A}$ for FO-statements about graphs is dense in the set $\mathbb{T} := \{\mathsf{P}_{\mathrm{BP}\mathcal{A}}[\mathcal{F}] : \mathcal{F} \subseteq \mathcal{UA}\}$. Thus, to prove Theorem 92, it remains to show that the latter set is a finite union of intervals. This is possible since by definition of $\mathsf{P}_{\mathrm{BP}\mathcal{A}}$ we know $\mathbb{T}$ to be a set of tail-sums of a convergent series, so Lemma 94 provides a sufficient criterion for $\mathbb{T}$ to be a finite union of intervals: while Corollary 95 does not require this, for handling the tail-bounds-term-condition it is convenient to have the summands of the series non-increasing. We therefore assume that $G_1, G_2, \ldots \in \mathcal{UA}$ is any total ordering of the isomorphism types in $\mathcal{A}$ which makes $(p_i)_{i\in\mathbb{N}}$ with

$$p_i := \mathsf{P}_{\mathrm{BP}\mathcal{A}}[\mathrm{G}_i] \tag{3.22}$$

a non-increasing sequence. By Corollary 95, for Theorem 92 it suffices to show that there exists $i_0 \in \mathbb{N}$ with $p_i \leqslant \sum_{j>i} p_j$ for all $i > i_0$. For every $k \in \mathbb{N}$ we define the event $E_k := \{$ every component has $\geqslant k$ vertices and exactly one has exactly k } and the random variable $Z_k :=$ total number of k-vertex components. Moreover, we set $q_k := \mathsf{P}_{\mathrm{BP}\mathcal{A}}[E_k]$ and $\mu_k := \sum_{H \in \mathcal{UC}_k} \frac{\rho^k}{|\mathrm{Aut}(H)|}$, where

$$\mathcal{UC}_k \subseteq \mathcal{UA}$$

denotes the set of isomorphism types of *connected* graphs in $\mathcal{A}$ with exactly k vertices. Since $\mathcal{UC}_k$ is a finite set for every k, and since a sum of independent Poisson-distributed random variables is again Poisson-distributed, it follows from McDiarmid's Theorem 81 with $\mathcal{D} := \mathcal{A}$ and $Z_k = \sum_{H \in \mathcal{UC}_k} \#_H$ that $Z_1, \dots, Z_k$ are again independent and Poisson-distributed, with a mean that can be computed by linearity of expectation as $\mathsf{E}_{\mathrm{BP}\mathcal{A}}[Z_i] = \sum_{H \in \mathcal{UC}_i} \mathsf{E}_{\mathrm{BP}\mathcal{A}}[\#_H] =$ (by Theorem 81 and since the mean of a Poisson is its parameter) $= \sum_{H \in \mathcal{UC}_i} \rho^i / |\mathrm{Aut}(H)| = \mu_i$. Therefore, for every $k \in \mathbb{N}$,

$$q_k = \mathsf{P}[\mathrm{Poi}(\mu_1) = 0] \cdots \mathsf{P}[\mathrm{Poi}(\mu_{k-1}) = 0] \cdot \mathsf{P}[\mathrm{Poi}(\mu_k) = 1] = \mu_k \cdot \mathrm{e}^{-(\mu_1 + \cdots + \mu_k)} \ . \tag{3.23}$$

We now prove

(1) $\lim_{k\to\infty} q_k = 0$, (2) $\lim_{k\to\infty} q_{k+1}/q_k = 1$.

As for (1), it suffices to note that with $\mathcal{C}_n \subseteq \mathcal{A}_n$ denoting the set of all connected n-vertex elements of $\mathcal{A}$ we have $\mu_k = \sum_{H \in \mathcal{UC}_k} \frac{\rho^k}{|\mathrm{Aut}(H)|} = \sum_{G \in \mathcal{C}_k} \frac{\rho^k}{k!} = \frac{|\mathcal{C}_k|}{k!} \cdot \rho^k$, hence

$$\sum_{k=1}^{\infty} \mu_k = C(\rho) \leqslant A(\rho) \ , \tag{3.24}$$

with $C(z)$ (resp. $A(z)$) the exponential generating function of $\mathcal{C}$ (resp. $\mathcal{A}$). By Theorem 83, $A(\rho)$ is convergent, hence (3.24) proves that $\sum_{k=1}^{\infty} \mu_k$ converges, too. This, (3.23) and continuity of exp imply $\lim_{k\to\infty} q_k = (\lim_{k\to\infty} \mu_k) \cdot \exp(-\sum_{k=1}^{\infty} \mu_k) = 0 \cdot \exp(-C(\rho)) = 0$, proving (1).

As for (2), Theorem 82 and the formula for the radius of convergence of a series corresponding to the 'ratio test' tell us $\frac{(k+1)|\mathcal{C}_k|}{|\mathcal{C}_{k+1}|} \xrightarrow{k\to\infty} \rho$. Therefore, $\lim_{k\to\infty} \frac{q_{k+1}}{q_k} =$ (by (3.23)) $= \lim_{k\to\infty} \frac{\mu_{k+1}}{\mu_k} \cdot \mathrm{e}^{-\mu_{k+1}} = \lim_{k\to\infty} \frac{\rho|\mathcal{C}_{k+1}|}{(k+1)|\mathcal{C}_k|} \cdot \mathrm{e}^{-\mu_{k+1}} = 1$, proving (2). Because of (2), there is k_0 such that

$$q_{k+1} \geqslant \tfrac{2}{3} \cdot q_k \qquad \text{for all } k \geqslant k_0 \ . \tag{3.25}$$

We now choose any index $i_0 \in \mathbb{N}$ with $p_{i_0} < q_{k_0}$, which is possible as $p_i = \mathsf{P}_{\mathrm{BP}\mathcal{A}} \xrightarrow{i\to\infty} 0$ and k_0 has been fixed. Let $i > i_0$ be arbitrary and $k_i := \max\ \{\ k \geqslant k_0 \colon\ p_i \leqslant q_k\ \}$, which exists since $0 < p_i \leqslant p_{i_0} < q_{k_0}$, and $q_k \to 0$ by (1). By definition of q_k, for every $k \in \mathbb{N}$ there exists a subset $A_k \subseteq \mathbb{N}$ with $q_k = \sum_{i \in A_k} p_i$, and these A_k are necessarily pairwise disjoint. Now we can argue that $p_i < 2p_i = (\sum_{j\geqslant 1} (\frac{2}{3})^j) \cdot p_i \leqslant$ (since the assumptions imply that $(\frac{2}{3})^\ell\ p_i \leqslant q_{k_i+\ell}$ for every $\ell \in \mathbb{N}$) $\leqslant \sum_{j\geqslant 1} q_{k_i+j} = \sum_{j\in A} p_j \leqslant$ (since $p_i \geqslant p_{i+1}$, and since $p_i > q_{k_i+\ell}$ by choice of k_i, it follows that $j > i$ for every $j \in A := \bigsqcup_{\ell > k_i} A_\ell$, so $\sum_{j\in A} p_j$ is a sub-series of $\sum_{j\in\mathbb{N}\colon\, j>i} p_j) \leqslant \sum_{j\in\mathbb{N}\colon\, j>i} p_j$. Since $i > i_0$ was arbitrary, we have proved the condition in Lemma 94, hence Theorem 92. $\square$

Finally, let us mention some further questions arising from Chapter 3. There is a conjecture [33, Conjecture 5.1] positing that for every surface S, the random graph embeddable on S will have a.a.s. chromatic number equal to four. (Compare the discussion on pp. 21–22). Since being k-colourable is expressible in MSO-logic (and even in MSO's subset *EMSO*) for every fixed k, establishing Conjecture 31 in Section 1.3.3 of Chapter 1 would be a step in the direction of [33, Conjecture 5.1]. In [33] it was already shown that the chromatic number is a.a.s. an element of $\{4, 5\}$. Proving the MSO-zero-one law will imply that the chromatic number of the random graph is either four a.a.s., or five a.a.s. (as opposed to probability mass both on 4 and on 5, or oscillating between the two).

For every addable, minor-closed class $\mathcal{A}$ with egf A of radius of convergence ρ it is known that $A(\rho) \leqslant \mathrm{e}^{1/2}$ by a result of Addario-Berry, McDiarmid and Reed [4], and independently Kang and Panagiotou [96]. With $p_1 := \mathsf{P}_{\mathrm{BP}\mathcal{A}}[\varnothing]$ the probability that a large uniformly random isomorphism type in $\mathcal{A}$ has empty non-largest components, i.e. is connected, we know from Theorem 81 on p. 116 that $p_1 = 1/A(\rho)$, and $A(\rho) \leqslant \mathrm{e}^{1/2}$ implies that $p_1 > 1 - p_1$. By Corollary 95, this alone implies that there is at least one gap in the closure of the limiting probabilities. In the case of forests there are in fact three gaps in total. It seems probable that w.r.t. *any* proper addable, minor-closed class the closure of the set of probability limits has at least three gaps, and moreover that the reason is the following:

Conjecture 93. *If A is the exponential generating function of any addable minor-closed class of graphs other than the class of all graphs, and ρ its radius of covergence, then $A(\rho) < 1 + 2\rho$.*

Any minor-closed class contains the one-vertex graph $\bullet$, and with $p_2 := \mathsf{P}_{\mathrm{BP}\mathcal{A}}[\bullet]$, Theorem 81 tells us that $p_2 = \rho/A(\rho)$, so Conjecture 93 amounts to $\mathsf{P}_{\mathrm{BP}\mathcal{A}}[\bullet] > 1 - \mathsf{P}_{\mathrm{BP}\mathcal{A}}[\varnothing] - \mathsf{P}_{\mathrm{BP}\mathcal{A}}[\bullet]$, i.e., the statement that the probability of a random element of a minor-closed class having

$$\text{exactly one isolated vertex and the rest of the graph connected} \tag{3.26}$$

is larger than its being non-connected for any other reason. If true, by Corollary 95 this would imply that the closure of the set of probability limits always consists of at least four intervals, i.e., that it has at least three gaps.

3.1.2 The structure of the set of tail-sums of a convergent series in the tail-bounds-term case

The following basic observation, which could be more present in introductory calculus classes than it currently seems to be, has been on record for more than a century (cf. [95] [73] [144] [143] for more difficult questions raised by it). In both Lemma 94 and Corollary 95, $\mathbb{N} := \{0, 1, 2, \dots\}$:

Lemma 94 (set of all tail-sums of a convergent series in the case that tail sums bound their term)**.** *If $p \in \mathbb{R}_{\geqslant 0}^{\mathbb{N}}$ with $\sum_{i\geqslant 1} p_i < \infty$ and $p_i \leqslant \sum_{j>i} p_j$ for all $i \in \mathbb{N}$, then $\{\sum_{i\in A} p_i \colon A \subseteq \mathbb{N}\} = \big[0, \sum_{i=1}^{\infty} p_i\big]$.*

From Lemma 94 it follows that (cf. [143, Equation (3) and Proposition 6]):

Corollary 95 (set of all tail-sums of a convergent series if tails eventually all bound)**.** *Suppose $p \in \mathbb{R}_{\geqslant 0}^{\mathbb{N}}$ with $\sum_{i\geqslant 1} p_i < \infty$. If there is $i_0 \in \mathbb{N}$ with $p_i \leqslant \sum_{j>i} p_j$ for every $i > i_0$, then*

$$\left\{\sum_{i\in A} p_i \colon A \subseteq \mathbb{N}\right\} = \bigcup_{I\subseteq[i_0]} \left[\sum_{i\in I} p_i \ , \quad \sum_{i\in I} p_i + \sum_{i>i_0} p_i\right] \ . \tag{3.27}$$

□

At face value, the expression for the tail-sums given in (3.27) seems to depend on i_0, while the left-hand side does not. The following simple fact shows that the right-hand side does not either:

Lemma 96 (well-definedness of the union in Corollary 95)**.** *With the abbreviation*

$$U_i := \bigcup_{I\subseteq[i]} \left[\sum_{\imath\in I} p_\imath \ , \quad \sum_{\imath\in I} p_\imath + \sum_{j>i} p_j\right] \ , \tag{3.28}$$

there is the following implication (which by considering the smallest such $i_0 \in \mathbb{N}$ proves well-definedness of the union in (3.27) *of Corollary 95): if $i_0 \in \mathbb{N}$ with $p_i \leqslant \sum_{j>i} p_j$ for every $i > i_0$, then $U_{i_0} = U_i$ for every $i > i_0$.*

Proof. If $i_0 \in \mathbb{N}$ with $p_i \leqslant \sum_{j>i} p_j$ for every $i > i_0$, then also $p_i \leqslant \sum_{j>i} p_j$ for every $i > i_0 + 1$. Thus, iterating the statement $U_{i_0} = U_{i_0+1}$, taking $i_0 + 1$ as the new i_0, shows that it suffices to prove Lemma 96 in the special case $i = i_0 + 1$.

Let any i_0 as in Lemma 96 be given. Set $\Sigma_{>i} := \sum_{j>i} p_j$ and $\Sigma_I := \sum_{i\in I} p_i$ for any $I \subseteq \mathbb{N}$. We prove $U_{i_0} = U_{i_0+1}$.

As to $U_{i_0+1} \subseteq U_{i_0}$, let any subset $I \subseteq [i_0+1]$ be given. We have to show that $[\Sigma_I, \Sigma_I + \Sigma_{>i_0+1}]$ is a subset of one of the 2^{i_0} intervals united in U_{i_0}. Because of $\Sigma_{>i_0+1} \leqslant \Sigma_{>i_0}$, we know $[\Sigma_I, \Sigma_I + \Sigma_{>i_0+1}] \subseteq [\Sigma_I, \Sigma_I + \Sigma_{>i_0}]$. If $I \subseteq [i_0]$, then the latter interval is among the invervals united in U_{i_0} and we are done. Therefore we may assume $i_0+1 \in I$. With $I^- := I \backslash \{i_0+1\}$ then have

$$[\Sigma_I, \Sigma_I + \Sigma_{>i_0+1}] = [\Sigma_{I^-} + p_{i_0+1}, \Sigma_{I^-} + p_{i_0+1} + \Sigma_{>i_0+1}] \ . \tag{3.29}$$

Because of $\Sigma_{I^-} \leqslant \Sigma_{I^-} + p_{i_0+1}$ and $\Sigma_{I^-} + p_{i_0+1} + \Sigma_{>i_0+1} = \Sigma_{I^-} + \Sigma_{j>i_0}$, it follows from (3.29) that $[\Sigma_I, \Sigma_I + \Sigma_{>i_0+1}] \subseteq [\Sigma_{I^-}, \Sigma_{I^-} + \Sigma_{j>i_0}]$. Because of $I^- \subseteq [i_0]$, the latter interval is among the intervals being united to form U_{i_0}, proving $U_{i_0+1} \subseteq U_{i_0}$.

As to $U_{i_0} \subseteq U_{i_0+1}$, let $\xi \in U_{i_0}$ be arbitrary. Then there is $I \subseteq [i_0]$ with $\xi \in [\Sigma_I, \Sigma_I + \Sigma_{>i_0}]$. Every interval united in U_{i_0+1} has the form $[\Sigma_{I'}, \Sigma_{I'} + \Sigma_{>i_0+1}]$, with some $I' \subseteq [i_0+1]$. Hence we may assume $\xi > \Sigma_I + \Sigma_{>i_0+1}$, for otherwise setting $I' := I \subseteq [i_0]$ already proves our $\xi \in [\Sigma_I, \Sigma_I + \Sigma_{>i_0}]$ to lie in one of them. We then know

$$\Sigma_I + \Sigma_{>i_0+1} < \xi \leqslant \Sigma_I + \Sigma_{>i_0} = \Sigma_I + p_{i_0+1} + \Sigma_{>i_0+1} = \Sigma_{I\cup\{i_0+1\}} + \Sigma_{>i_0+1} \ . \tag{3.30}$$

Because of $I \cup \{i_0+1\} \subseteq [i_0+1]$, from (3.30) we can conclude (setting $I' := I \cup \{i_0+1\}$) that $\xi \in U_{i_0+1}$, provided that we know $\Sigma_{I\cup\{i_0+1\}} \leqslant \Sigma_I + \Sigma_{>i_0+1}$. But this is equivalent to $p_{i_0+1} \leqslant \Sigma_{>i_0+1}$, which holds by hypothesis. This proves $U_{i_0} \subseteq U_{i_0+1}$. □

3.1.3 Three steps for finding the set of probability limits of an arbitrary addable minor-closed class of graphs

Let $\mathcal{A}$ denote an arbitrary addable minor-closed class of graphs, and let $\rho_{\mathcal{A}}$ denote the radius of convergence of its exponential generating function $A = \sum_{n\geqslant 0} |\mathcal{A}_n| \frac{z^n}{n!}$.

(limset-St.1) Find a total ordering $G_1, G_2, \ldots$ of the set of isomorphism types $\mathcal{UA}$ and some index $i_0 \in \mathbb{N}$ such that with $\mathsf{P}_{\mathrm{BP}\mathcal{A}}$ from Theorem 81 and $p_i := \mathsf{P}_{\mathrm{BP}\mathcal{A}}[G_i]$,

$$p_i \leqslant \sum_{j\in\mathbb{N}\colon j>i} p_j \qquad \text{for every } i > i_0 \ . \tag{3.31}$$

For this it is sufficient to guess an initial finite part of such an ordering, hence in particular guess some i_0, and then prove by general estimates that (3.31) holds.

(limset-St.2) By explicit computations with $\rho_{\mathcal{A}}$ and $A(\rho_{\mathcal{A}})$ determine

$$I_>(i_0) := \{i \in [i_0] \colon p_i > \sum_{j\in\mathbb{N}\colon j>i} p_j\} \ . \tag{3.32}$$

(limset-St.3) Compute $i_{\mathrm{m}} := \max I_>(i_0)$. Then $U_{i_{\mathrm{m}}}$ (in the sense of (3.28)) is the closure of the set of probability limits w.r.t. $\mathcal{A}$. For writing explicit expressions of $U_{i_{\mathrm{m}}}$ as a parametrised union of pairwise disjoint intervals one needs to know a non-increasing ordering of the set $\{p_\imath \colon \imath \in [i_{\mathrm{m}}]\}$, in particular, one needs to know for which pairs $\{\imath', \imath''\} \subseteq [i_{\mathrm{m}}]$ one has $p_{\imath'} = p_{\imath''}$.

3.2 Carrying out the three steps in Section 3.1.3 for forests

Here we prove the more explicit version of Theorem 27 in Chapter 1:

Theorem 97 (joint work with T. Müller, M. Noy and A. Taraz)**.** *If $\mathcal{F}$ denotes the set of all labelled finite forests, then*

$$\operatorname{cl}\left(\left\{\lim_{n\to\infty}\frac{|\{G\in\mathcal{F}_n : G\models\varphi\}|}{|\mathcal{F}_n|} : \varphi\in\mathrm{MSO}\right\}\right)$$
$$= \left[0, 1-\mathrm{e}^{-\frac{1}{2}}-\mathrm{e}^{-\frac{3}{2}}\right]\sqcup\left[\mathrm{e}^{-\frac{3}{2}}, 1-\mathrm{e}^{\frac{1}{2}}\right]\sqcup\left[\mathrm{e}^{-\frac{1}{2}}, 1-\mathrm{e}^{-\frac{3}{2}}\right]\sqcup\left[\mathrm{e}^{-\frac{1}{2}}+\mathrm{e}^{-\frac{3}{2}}, 1\right]\ . \tag{3.33}$$

Proof of Theorem 97. We prove Theorem 97 following (limset-St.1)–(limset-St.3) in Section 3.1.3.

As to (limset-St.1), for the initial part of the total ordering we choose the isomorphism types $\mathrm{F}_1,\ldots,\mathrm{F}_9$ from Definition 229 in Chapter 5 and set $p_i := \mathsf{P}_{\mathrm{BP}\mathcal{F}}[\mathrm{F}_i]$ for every $i\in[9]$. The radius of convergence of the egf D of forests is $\rho = \mathrm{e}^{-1}$ (see, e.g., [57, Theorem IV.8] for a framework for constructing such explicit constants, and [14, p. 470, Section 2] for an explanation of that particular value), and $1/D(\rho)$ equals the limit (as $n\to\infty$) of the probability that a uniformly random n-vertex forest is a tree, i.e. $1/D(\rho) = 1/\sqrt{\mathrm{e}}$ (cf. e.g. [129, p. 189] and [128, p. 585] for that limit, and [128, Corollary 1.6(c)] for a more general setting). Substituting this into (3.1) of Theorem 81 yields (after working out the size of the automorphism group of each isomorphism type),

$$\begin{aligned} &p_1 = \mathrm{e}^{-1/2}, && p_2 = \mathrm{e}^{-3/2}, && p_3 = p_4 = \tfrac{1}{2}\mathrm{e}^{-5/2},\\ &p_5 = p_6 = \tfrac{1}{2}\mathrm{e}^{-7/2}, && p_7 = \tfrac{1}{6}\mathrm{e}^{-7/2}, && p_8 = p_9 = \tfrac{1}{2}\mathrm{e}^{-9/2}. \end{aligned} \tag{3.34}$$

Then $p_1 \geqslant \ldots \geqslant p_9$. Similarly to Lemma 100 in the proof for planar graphs further below, we will use the following quantitative statement to prove that all tails eventually bound their term:

Lemma 98. $\sum_{F\in\mathcal{UF}_n}\frac{1}{|\mathrm{Aut}(F)|} > \mathrm{e} = \exp(1)$ *for every* $n\geqslant 6$*, where* $\mathcal{UF}_n$ *denotes the set of isomorphism types of* n*-vertex forests.*

Proof. We will describe enough unlabelled forests on $n\geqslant 6$ vertices with small automorphism groups to make the sum exceed e. For every $n\geqslant 6$, the following five forests each have at most two automorphisms: a path on n vertices, the union of a path on $n-1$ vertices and an isolated vertex, a path on $n-1$ vertices with a leaf attached to its second vertex, and a path on $n-1$ vertices with a leaf attached to its third vertex, the union of an isolated vertex and a path on $n-2$ vertices with a leaf attached to its second vertex. The following forests both have exactly four automorphisms for every $n\geqslant 6$: the union of a path on $n-2$ vertices with two isolated vertices, the union of a path on $n-2$ vertices with a path on two vertices. For every $n\geqslant 6$, the seven forests just described are pairwise non-isomorphic. Thus, $\sum_{F\in\mathcal{UF}_n}\frac{1}{|\mathrm{Aut}(F)|}\geqslant 5\cdot(1/2)+2\cdot(1/4) = 3 > \mathrm{e}$. □

We now consider any extension $\mathrm{F}_1,\ldots,\mathrm{F}_9,\mathrm{F}_{10},\ldots$ of the ordering $\mathrm{F}_1,\ldots,\mathrm{F}_9$ to a total ordering of *all* isomorphism types of forests and set $p_i := \mathsf{P}_{\mathrm{BP}\mathcal{F}}[\mathrm{F}_i]$ for every $i\in\mathbb{N}$. It does not matter for our purposes how the forests other than $\mathrm{F}_1,\ldots,\mathrm{F}_9$ are ordered, so we do not have to specify the $\mathrm{F}_{10},\mathrm{F}_{11},\ldots$. We now show that $i_0 := 9$ is an index as in (3.31) of (limset-St.1). Let any $i\geqslant 10$ be given. By inspection, the only 4-vertex isomorphism types of forests not among $\mathrm{F}_1,\ldots,\mathrm{F}_9$ are the union of four isolated vertices, the union of an isolated edge with two isolated vertices, the union of two isolated edges, and $\mathrm{K}^{1,3}$. Each of these has at least four automorphisms (with the second-mentioned type having only four). Hence, every isomorphism type F of a forest with $F\notin\{\mathrm{F}_1,\ldots,\mathrm{F}_9\}$ either has at least five vertices and an unknown number of automorphisms, or four vertices and at least four automorphisms, so, by (3.1) in Theorem 81 we know that $\mathsf{P}_{\mathrm{BP}\mathcal{F}}[F] \leqslant \frac{1}{G(\rho)}\frac{\rho^5}{1} = \mathrm{e}^{-1/2}\cdot\mathrm{e}^{-5} = \mathrm{e}^{-11/2}$ or $\mathsf{P}_{\mathrm{BP}\mathcal{F}}[F]\leqslant\frac{1}{G(\rho)}\frac{\rho^4}{4} = \mathrm{e}^{-1/2}\cdot\frac{1}{4}\cdot\mathrm{e}^{-4} = \frac{1}{4}\cdot\mathrm{e}^{-9/2} < \mathrm{e}^{-11/2}$, for every forest-isomorphism-type $F\notin\{\mathrm{F}_1,\ldots,\mathrm{F}_9\}$, i.e.

$$\mathsf{P}_{\mathrm{BP}\mathcal{F}}[F]\leqslant\mathrm{e}^{-\frac{11}{2}}\ \text{for every forest-isomorphism-type } F\notin\{\mathrm{F}_1,\ldots,\mathrm{F}_9\}\ . \tag{3.35}$$

It follows from (3.35) that for every $i\geqslant 10 = i_0+1$ we have $\mathsf{P}_{\mathrm{BP}\mathcal{F}}[\mathrm{F}_i] = p_i\leqslant\mathrm{e}^{-\frac{11}{2}} = \mathrm{e}^{-6+\frac{1}{2}}$. Thus, and since $n\mapsto\mathrm{e}^{-n+\frac{1}{2}}$ is strictly decreasing, we know there is $n = n(i)\geqslant 6$ with

$$\mathrm{e}^{-n(i)-\frac{1}{2}} < p_i \leqslant \mathrm{e}^{-n(i)+\frac{1}{2}}\ . \tag{3.36}$$

For every $F \in \mathcal{UF}_{n(i)}$ we have $\mathsf{P}_{\mathrm{BP}\mathcal{F}}[F] = \frac{1}{G(\rho)} \frac{\rho^{n(i)}}{|\mathrm{Aut}(F)|} = \frac{\mathrm{e}^{-n(i)-\frac{1}{2}}}{|\mathrm{Aut}(F)|} \leqslant \mathrm{e}^{-n(i)-\frac{1}{2}} <$ (by (3.36)) $< p_i$, hence for every $F \in \mathcal{UF}_{n(i)}$ there is exactly one $j > i$ with $\mathrm{F}_j = F$, so $p_j = \mathsf{P}_{\mathrm{BP}\mathcal{F}}[\mathrm{F}_j] = \mathsf{P}_{\mathrm{BP}\mathcal{F}}[F]$. Hence the first '$\geqslant$' in $\sum_{j>i} p_j \geqslant \sum_{F \in \mathcal{UF}_{n(i)}} \mathsf{P}_{\mathrm{BP}\mathcal{F}}[F] >$ (by Lemma 98 and $n(i) \geqslant 6$) $> \mathrm{e}^{-n(i)+1/2} \geqslant$ (by (3.36)) $\geqslant p_i$, which proves that $p_i \leqslant \sum_{j>i} p_j$ holds for $i \geqslant i_0 + 1$. This completes (limset-St.1).

As for (limset-St.2), explicit computations (not described here), using numerical approximations to $1/\sqrt{\mathrm{e}}$, prove that the only $i \leqslant 9$ with $p_i > \sum_{j>i} p_j$ are $i = 1, 2$, i.e., $I_>(i_0) = \{1, 2\}$, completing (limset-St.2).

As for (limset-St.3), we have $i_{\mathrm{m}} = \max I_>(i_0) = \max\{1, 2\} = 2$. We know that $p_1 > p_2$, and it follows that U_{i_0} is the union of the following $2^{|[i_0]|} = 4$ pairwise-disjoint intervals (we immediately use that $\sum_{j>i_0} p_j = 1 - \sum_{1 \leqslant \imath \leqslant i_0} p_\imath = 1 - p_1 - p_2 = 1 - \mathrm{e}^{-1/2} - \mathrm{e}^{-3/2}$): $[\sum_{i \in \varnothing} p_i \ , \ \sum_{i \in \varnothing} p_i + 1 - \mathrm{e}^{-1/2} - \mathrm{e}^{-3/2}] = [0, 1 - \mathrm{e}^{-1/2} - \mathrm{e}^{-3/2}]$, $[\sum_{i \in \{1\}} p_i \ , \ \sum_{i \in \{1\}} p_i + 1 - \mathrm{e}^{-1/2} - \mathrm{e}^{-3/2}] = [\mathrm{e}^{-1/2}, 1 - \mathrm{e}^{-3/2}]$, $[\sum_{i \in \{2\}} p_i \ , \ \sum_{i \in \{2\}} p_i + 1 - \mathrm{e}^{-1/2} - \mathrm{e}^{-3/2}] = [\mathrm{e}^{-3/2}, 1 - \mathrm{e}^{-1/2}]$, and $[\sum_{i \in [2]} p_i \ , \ \sum_{i \in [2]} p_i + 1 - \mathrm{e}^{-1/2} - \mathrm{e}^{-3/2}] = [\mathrm{e}^{-1/2} + \mathrm{e}^{-3/2}, 1]$. Let us mention that all the interval-ends other than 0 and 1 are irrational numbers (cf. e.g. [142]). The proof of Theorem 97 is now complete. $\square$

3.3 Carrying out the steps in Section 3.1 for planar graphs

Here we prove the more explicit version of Theorem 28 mentioned in Section 1.3.2 of Chapter 1:

Theorem 99 (joint work with T. Müller, M. Noy and A. Taraz)**.** *If $\mathcal{P}$ denotes the set of labelled finite planar graphs, ρ the radius of convergence of $G := \mathrm{egf}(\mathcal{P})$, and*

$$\lambda_{a,b,c,d,e} := \frac{a + b\rho + \frac{1}{2}c\rho^2 + (\frac{1}{2}d + \frac{1}{6}e)\rho^3}{G(\rho)} \ , \qquad \ell := 1 - \frac{1 + \rho + \rho^2 + \frac{4}{3}\rho^3}{G(\rho)} \ , \tag{3.37}$$

then

$$\mathrm{cl}\left(\left\{\lim_{n\to\infty} \frac{|\{G \in \mathcal{P}_n \colon G \models \varphi\}|}{|\mathcal{P}_n|} : \varphi \in \mathrm{MSO}\right\}\right) = \bigsqcup_{\substack{a,b \in \{0,1\}, \\ c,d,e \in \{0,1,2\}}} [\lambda_{a,b,c,d,e}, \lambda_{a,b,c,d,e} + \ell] \ , \tag{3.38}$$

which is a union of $2^2 \cdot 3^3$ disjoint intervals each having length exactly $1 - \frac{1}{G(\rho)}(1 + \rho + \rho^2 + \frac{4}{3}\rho^3)$, a number with decimal order of magnitude 10^{-6}.

Proof of Theorem 99. As to (limset-St.1), for the initial part of the total ordering we choose the isomorphism-types $\mathrm{G}_1, \ldots, \mathrm{G}_{19}$ from Definition 229 in Chapter 5. With $p_i := \mathsf{P}_{\mathrm{BP}\mathcal{P}}[\mathrm{G}_i]$, and by determining the sizes of the automorphism groups of each G_i, and by applying the formula in Theorem 81,

$$\begin{aligned} &p_1 = \tfrac{1}{G(\rho)} \ , && p_2 = \tfrac{\rho}{G(\rho)} \ , && p_3 = p_4 = \tfrac{\rho^2}{2G(\rho)} \ , && p_5 = p_6 = \tfrac{\rho^3}{2G(\rho)} \ , \\ &p_7 = p_8 = \tfrac{\rho^3}{6G(\rho)} \ , && p_9 = p_{10} = p_{11} = \tfrac{\rho^4}{2G(\rho)} \ , && p_{12} = p_{13} = \tfrac{\rho^4}{4G(\rho)} \ , && p_{14} = p_{15} = \tfrac{\rho^4}{6G(\rho)} \ , \\ &p_{16} = p_{17} = \tfrac{\rho^4}{8G(\rho)} \ , && p_{18} = p_{19} = \tfrac{\rho^4}{24G(\rho)} \ . \end{aligned} \tag{3.39}$$

Furthermore, we consider an arbitrary extension $\mathrm{G}_{20}, \mathrm{G}_{21}, \ldots$, i.e. *any* total ordering of the countably infinite number of isomorphism-types of planar graphs not among $\mathrm{G}_1, \ldots, \mathrm{G}_{19}$; it does not matter for our purposes how the G_i with $i \geqslant 20$ are selected. We now show that $i_0 := 19$ is an index as in (3.31) of (limset-St.1).

Similarly to Lemma 98 above, we will use the following quantitative statement to prove that all tails eventually bound their term:

Lemma 100. $\sum\limits_{H \in \mathcal{UP}_n} \frac{1}{|\mathrm{Aut}(H)|} > 30$ *for every $n \geqslant 6$.*

Proof. With $\mathcal{UP}_n$ the set of isomorphism-types of n-vertex planar graphs we have

$$|\mathcal{P}_n| = \sum_{H \in \mathcal{UP}_n} \frac{n!}{|\mathrm{Aut}(H)|} \ , \tag{3.40}$$

hence $\varphi(n) = |\mathcal{P}_n|/n!$, with $\varphi(n) := \sum_{H \in \mathcal{UP}_n} \frac{1}{|\mathrm{Aut}(H)|}$. Moreover, $n|\mathcal{P}_{n-1}| \leqslant |\mathcal{P}_n|$ for every $n \geqslant 2$, since given an arbitrary element of $\mathcal{P}_{n-1}$, already the possibility to add the vertex n and then either join it to exactly one existing vertex, or leave it isolated, creates $n-1+1 = n$ planar graphs on $[n]$, and all $n|\mathcal{P}_{n-1}|$ elements of $\mathcal{P}_n$ thus created are distinct. Hence φ is monotone non-decreasing in n, and it suffices to prove $\varphi(6) > 30$, equivalently, $|\mathcal{P}_6| > 21600$. Now we use work of Bodirsky, Kang and Gröpl: the number of all labelled planar graphs with six vertices and m edges is given, for all possible values $0 \leqslant m \leqslant 12$, in the fifth row of the table in [16, Fig. 1] (the notation $G^{(0)}(n,m)$ is defined on p. 379), and their sum is $|\mathcal{P}_6| = 1 + 15 + 105 + 455 + 1365 + 3003 + 5005 + 6435 + 6435 + 4995 + 2937 + 1125 + 195 = 32071 > 21600$, completing the proof. □

As for (limset-St.2) on p. 123, we need precise numerical approximations to the real numbers $\rho_{\mathcal{A}}$ and $G(\rho_{\mathcal{A}})$ with $\mathcal{A} = \mathcal{P}$ the class of planar graphs. Such approximations are considerably harder to come by than in the case of $\mathcal{A} = \mathcal{F}$, the class of forests. Fortunately, such approximations have been made possible by the work of O. Giménez and M. Noy [63], who determined $\rho_{\mathcal{A}}$ as a solution of a (non-polynomial) system of equations. This makes it possible to compute the numbers ρ and $G(\rho)$ to any desired precision. The approximations in Lemma 101 will suffice for our present purpose.

Lemma 101. *With G the exponential generating function of labelled planar graphs, and ρ its radius of convergence,*

$$0.03672841251 \leqslant \rho \leqslant 0.03672841266 \quad \textit{and} \quad 0.96325282112 \leqslant 1/G(\rho) \leqslant 0.96325282254 \ . \tag{3.41}$$

Our proof of Theorem 99 depends in a delicate way on numerical approximation to the real numbers ρ and $G(\rho)$. We did not rigorously determine the least number of decimal digits sufficient to deduce Theorem 99, only that the accuracy provided in the following Section 3.3.1 is sufficient, and that five decimal digits are not sufficient. If we would use approximations with only five significant decimal digits, and if the approximation to $1/G(\rho)$ would differ from the true value of $1/G(\rho)$ in the fifth significant digit while all previous digits agree, and if the approximation to ρ agrees with ρ to five significant digits, then this would already result in a different set of intervals than given in Theorem 99: if we would use $\tilde{\rho} := 0.036728$ as an approximation of ρ and $\tilde{g} := 0.96326$ for $1/G(\rho)$ (compare this with both bounds for $1/G(\rho)$ in (3.41) starting as 0.96325), then, defining $\tilde{p_i}$ to be the number obtained by replacing, in (3.39), the number ρ with $\tilde{\rho}$ and $1/G(\rho)$ with $\tilde{g}$, it can be checked that with these approximations the claim in Lemma 133 on p. 149 would become false, and Corollary 95 then would imply a set of intervals different from those in Theorem 99; in particular, in (3.38) of Theorem 99 one would then be led to a union having a number of 'independent indices' other than five. Of course, in our proof of Lemma 133 below, we will not just replace ρ and $1/G(\rho)$ with some particular approximation, but rather make appropriate use of the correct lower and upper bounds from Lemma 101 from the upcoming Section 3.3.1.

While it is of course possible to input the system of equations from [63] into a computer algebra package and use some command for the solution of implicit equations to obtain approximations to ρ and $G(\rho)$, it is debatable whether this amounts to a proof of Lemma 101, even if ordering a large number of decimal digits. With commercial software we do not even have access to the source code, not to speak of formally verifying it. So we certainly could not prove that the output of the computation is correct to the desired degree of accuracy. This is the motivation for Section 3.3.1; there, a human being is left with either checking some elementary computations with integers, or finally entrusting *such* simple computations to a machine.

3.3.1 Proven bounds

for the numerical values of the numbers ρ and $G(\rho)$ from [63]

Definition 102 (B_0, B_2; cf. [63, p. 327])**.** *We have to work with the following functions:*

(1) $B_0 = \frac{(3t-1)^2(t+1)^6\log(t+1)}{512t^6} - \frac{(3t^4-16t^3+6t^2-1)\log(3t+1)}{32t^3} - \frac{(3t+1)^2(-t+1)^6\log(2t+1)}{1024t^6}$
$+ \frac{1}{4}\log(t+3) - \frac{1}{2}\log(t) - \frac{3}{8}\log(16) - \frac{(217t^6+920t^5+972t^4+1436t^3+205t^2-172t+6)(-t+1)^2}{2048t^4(3t+1)(t+3)}$,

(2) $B_2 = \frac{(-t+1)^3(3t-1)(3t+1)(t+1)^3\log(t+1)}{256t^6} - \frac{(-t+1)^3(3t+1)\log(3t+1)}{32t^3} + \frac{(3t+1)^2(-t+1)^6\log(2t+1)}{512t^6}$
$+ \frac{(t-1)^4(185t^4+698t^3-217t^2-160t+6)}{1024t^4(3t+1)(t+3)}$.

Definition 103 ($h_1(t)$, $h_2(t)$)**.** *For every* $t \in (0,1)$ *we define*

(1) $h_1(t) := \frac{2t+1}{(3t+1)(-t+1)}$,
(2) $h_2(t) := -\frac{t^2(-t+1)\left(5t^2+36t+18\right)}{2(t+3)(2t+1)(3t+1)^2}$.

Definition 104 ($Y(t)$; cf. [63, p. 310])**.** *For every* $t \in (0,1)$*, and with* h_1 *and* h_2 *as in Definition 103, we define* $Y(t) := -1 + h_1(t)\ \exp(h_2(t))$*.*

Lemma 105. *The function* $t \mapsto Y(t)$ *is strictly monotone increasing in the open interval* $(0,1)$*.*

Proof. The derivative of Y is

$$\frac{\mathrm{d}}{\mathrm{d}t}Y(t) = \frac{3t^2(144+736t+1256t^2+799t^3+141t^4+t^5-5t^6)}{(2t+1)(3t+1)^4(t^2+2t-3)^2}\ \exp\left(-\frac{t^2(-t+1)\left(5t^2+36t+18\right)}{2(t+3)(2t+1)(3t+1)^2}\right)\ . \tag{3.42}$$

The exponential function being a strictly positive real number for any real argument, (3.42) implies

$$t > 0 \quad \text{and} \quad \tfrac{\mathrm{d}}{\mathrm{d}t}Y(t) > 0 \qquad \Leftrightarrow \qquad 5t^6 < t^5 + 141t^4 + 799t^3 + 1256t^2 + 736t + 144\ , \tag{3.43}$$

the latter of which is obviously true since already $5t^6 < 144$ for every $0 < t < 1$. □

Lemma 106. *With* h_1 *and* h_2 *as in Definition 103 we have*

(1) $2.0941746325 - 10^{-10} < h_1(0.6263716633 - 10^{-10}) < 2.0941746325 + 10^{-10}$,
(2) $2.0941746335 - 10^{-10} < h_1(0.6263716633 + 10^{-10}) < 2.0941746335 + 10^{-10}$,
(3) $-0.0460123254 - 10^{-10} < h_2(0.6263716633 - 10^{-10}) < -0.0460123254 + 10^{-10}$,
(4) $-0.0460123253 - 10^{-10} < h_2(0.6263716633 + 10^{-10}) < -0.0460123253 + 10^{-10}$.

Proof. Checking these statements is left to the reader, who is advised to entrust this entirely routine task to an electronic computer. The functions h_1 and h_2 being rational, the statements can be checked via exact computations with arbitrary long integers, a standard functionality of several computer algebra systems (note that to check (3) and (4) one of course does not have to compute fractions, but one can rewrite (3) and (4) as a statement about adding, subtracting and multiplying integers).

Let us add that for reaching certainty about the equalities (3) and (4), non-commercial automated alternatives to hand-evaluation are `C` libraries for arbitrary precision arithmetic like `GMP` or `iRRAM`. According to [138], the code in the `iRRAM` package itself is currently in the process of being formally verified. □

We now derive Taylor polynomials specifically for our purposes (the approximation in (II) is designed to be used twice: both for the evaluations of exp within Y, and later on for evaluations $\exp(-\tilde{\nu})$ with $\tilde{\nu}$ an approximation of ν):

Lemma 107 (some Taylor approximations to exp)**.**

(I) *for every* $x \in (0.48, 0.49)$*,*

(1) $\left|\exp(x) - \sum_{0\leqslant i\leqslant 11} \frac{x^i}{i!}\right| < 0.11998784433 \cdot 10^{-11}$

(2) $0.39995948109 \cdot 10^{-12} + \sum_{0\leqslant i\leqslant 11} \frac{x^i}{i!} < \exp(x) < 0.11998784433 \cdot 10^{-11} + \sum_{0\leqslant i\leqslant 11} \frac{x^i}{i!}$

(II) *for every* $x \in (-0.05, 0)$,

(1) $\left|\exp(x) - \sum_{0\leqslant i\leqslant 5} \frac{x^i}{i!}\right| < 2.1701388889 \cdot 10^{-11}$

(2) $1.0850694444 \cdot 10^{-11} + \sum_{0\leqslant i\leqslant 5} \frac{x^i}{i!} < \exp(x) < 2.1701388889 \cdot 10^{-11} + \sum_{0\leqslant i\leqslant 5} \frac{x^i}{i!}$

Proof. As to (I), we develop exp around[1] 0 and use Lagrange's error term for Taylor's theorem: for every k and every $x \in (0, 0.49)$ there exists $\xi_x \in (0, 0.49)$ such that $\exp(x) = \sum_{0\leqslant i\leqslant k-1} \frac{x^i}{i!} + \frac{\exp(\xi_x)}{k!} x^k$. Because of $1 = \exp(0) < \exp(\xi_x) < \exp(0.49) < \exp(1) < 3$, we therefore know

$$\tfrac{1}{k!} x^k < \exp(x) - \sum_{0\leqslant i\leqslant k-1} \tfrac{x^i}{i!} < \tfrac{3}{k!} x^k \ , \tag{3.44}$$

for every $x \in (0, 0.49)$. In particular,

$$\left|\exp(x) - \sum_{0\leqslant i\leqslant k-1} \tfrac{x^i}{i!}\right| < \tfrac{3}{k!} x^k \quad \text{for every } x \in (0, 0.49)\ . \tag{3.45}$$

As for (I), we require k to be large enough to have $\frac{3}{k!} x^k < 10^{-11}$ for every $x \in (0.48, 0.49) \subseteq (0, 0.49)$, i.e., we require k to satisfy $\frac{3}{k!} 0.49^k < 10^{-11}$. The smallest such k is $k = 12$. Since $\frac{3}{12!} 0.49^{12} < 0.11998784433 \cdot 10^{-11}$ and $0.39995948109 \cdot 10^{-12} < \frac{1}{12!} 0.49^{12}$, (3.44) implies (I).(2), and hence (I).(1).

As for (II), for every $x \in (-0.05, 0)$, there exists $\xi_x \in (-0.05, 0)$ such that $\exp(x) = \sum_{0\leqslant i\leqslant k-1} \frac{x^i}{i!} + \frac{\exp(\xi_x)}{k!} x^k$. Since $\frac{1}{2} < \exp(-0.05) < \exp(\xi_x) < \exp(0) = 1$, we know that for every even k, and any $x \in (-0.05, 0)$ we have $x^k > 0$ and

$$\tfrac{1}{2k!} x^k < \exp(x) - \sum_{0\leqslant i\leqslant k-1} \tfrac{x^i}{i!} < \tfrac{1}{k!} x^k \ , \tag{3.46}$$

while for every odd k and any $x \in (-0.05, 0)$ we have $x^k < 0$ and

$$\tfrac{1}{k!} x^k < \exp(x) - \sum_{0\leqslant i\leqslant k-1} \tfrac{x^i}{i!} < \tfrac{1}{2k!} x^k \ . \tag{3.47}$$

In particular we now know that for every k (of whatever parity) and any $x \in (-0.05, 0)$,

$$\left|\exp(x) - \sum_{0\leqslant i\leqslant k-1} \tfrac{x^i}{i!}\right| < \tfrac{1}{k!} |x|^k \ . \tag{3.48}$$

We require k to be large enough to have $\frac{1}{k!} |x|^k < 10^{-10}$ for every $x \in (-0.05, 0)$, i.e., we require k to satisfy $\frac{1}{k!} 0.05^k < 10^{-10}$. The smallest such k is $k = 6$. Since $k = 6$ is even, (3.46) together with $1.0850694444 \cdot 10^{-11} < \frac{1}{2} \frac{1}{6!} 0.05^6$ and $\frac{1}{6!} 0.05^6 < 2.1701388889 \cdot 10^{-11}$ imply (II).(2), and hence (II).(1). In particular we know that $\sum_{0\leqslant i\leqslant 5} \frac{x^i}{i!}$ underestimates $\exp(x)$ for every $x \in (-0.05, 0)$. □

[1] If we would develop exp around a rational number x_0 inside the interval we are interested in, we'd need fewer than eleven terms to achieve the desired accuracy (w.r.t. arithmetic with arbitrary elements of $\mathbb{R}$). But we would then stray from our path towards a set of finite 'certificates' for the p_i-inequalities, consisting of rational computations only: Taylor's theorem would then require us to know $\exp(x_0)$ in order to compute the coefficients of the approximating polynomial. Since $\exp(x_0)$ is known to be irrational for every rational x_0 (e.g., [142]), another approximation would be necessary, resulting in additional complexity outweighing the gain in simplicity due to a lower-degree polynomial. Same for developing around an irrational number of the form $\log(x_0)$ with rational x_0 inside the respective intervals (which would keep the constant term rational yet necessitate approximations for what value to substitute into the variable). So developing around 0 seems the only sensible choice for the purpose of deriving rational certificates.

Lemma 108 (verified bounds for t_0)**.** *There exists exactly one* $t_0 \in (0,1)$ *with* $Y(t_0) = 1$*, and*

$$0.6263716633 - 10^{-10} < t_0 < 0.6263716633 + 10^{-10} \ . \tag{3.49}$$

Proof. Since all factors in denominators within $Y(t)$ are non-zero for $t \in (0,1)$, the function $t \mapsto Y(t)$ is continuous as a composition of continuous functions. By Lemma 105, it is moreover strictly monotone increasing in $(0,1)$. Therefore the claim follows (existence from continuity, uniqueness from monotonicity) via the Intermediate Value Theorem, provided we can show that

(1) $Y(0.6263716633 - 10^{-10}) < 1$, (2) $Y(0.6263716633 + 10^{-10}) > 1$.

A finite certificate for (1) is given by the calculation

$$\begin{aligned}
& Y(0.6263716633 - 10^{-10}) = -1 + h_1(0.6263716633 - 10^{-10}) \cdot \exp(h_2(0.6263716633 - 10^{-10})) \\
& \text{(upper bounds in (1) and (3) in Lemma 106, and since exp is monotone increasing)} \quad < -1 + 2.0941746326 \cdot \exp(-0.0460123253) \\
& \text{(upper bound in (1))} \quad < -1 + 2.0941746326 \cdot \\
& \qquad \left(2.1701388889 \cdot 10^{-11} + \sum_{0 \leqslant i \leqslant 5} \frac{(-0.0460123253)^i}{i!} \right) \\
& \qquad = 0.99999999995544408263310738324518\backslash \\
& \qquad 2705870208185832244853853496068 + \tfrac{1}{3} \cdot 10^{-62} < 1 \quad ,
\end{aligned} \tag{3.50}$$

while a finite certificate for (2) is given by the calculation

$$\begin{aligned}
& Y(0.6263716633 + 10^{-10}) = -1 + h_1(0.6263716633 + 10^{-10}) \cdot \exp(h_2(0.6263716633 + 10^{-10})) \\
& \text{(lower bounds in (2) and (4) in Lemma 106, and since exp is monotone increasing)} \quad > -1 + 2.0941746334 \cdot \exp(-0.0460123254) \\
& \text{(lower bound in (2))} \quad > -1 + 2.0941746334 \cdot \\
& \qquad \left(1.0850694444 \cdot 10^{-11} + \sum_{0 \leqslant i \leqslant 5} \frac{(-0.0460123254)^i}{i!} \right) \\
& \qquad = 1.00000000000957417297668951405800\backslash \\
& \qquad 480697033915364640304336242832 > 1 \ ,
\end{aligned} \tag{3.51}$$

where in each case $\backslash$ denotes that a number contiguously continues in the next line. $\square$

Definition 109 (the function t from [63, p. 317, paragraph 2], with explicit values for the 'suitable small neighbourhood of 1')**.** *For every* $y \in (0.9999999996, 1.00000000009)$ *we define* $t(y)$ *to be the unique* $t \in (0.6263716633 - 10^{-10}, 0.6263716633 + 10^{-10})$ *with* $Y(t) = y$*.*

Let us note that $t_0 = t(1)$.

Remark 110 (correctness of Definition 109)**.** *Definition 109 does indeed define a function*

$$t \colon (0.9999999996, 1.00000000009) \to (0.6263716633 - 10^{-10}, 0.6263716633 + 10^{-10}) \ . \tag{3.52}$$

Proof. Uniqueness of the $t(y)$ of Definition 109 follows from Lemma 105, while for existence we have to show that the argument in the proof of Lemma 108 can be carried out with any $y \in (0.9999999996, 1.00000000009)$ replacing the 1 in the conditions (1) and (2) there: this follows from (3.50) and (3.51): $0.999999999955444082633107383245182705870208185832244853853496068 + \frac{1}{3} \cdot 10^{-62} < 0.9999999996$ and $1.00000000000957417297668951405800480697033915364640304336242832 > 1.00000000009$, so each of these calculations can be used for proving the existence of any $t(y)$ with $y \in (0.9999999996, 1.00000000009)$. $\square$

Definition 111 (R; cf. [63, (2.6)])**.** *With t as in Definition 109, we define the function*

$$\begin{aligned} R\colon (0.9999999996, 1.00000000009) &\longrightarrow \mathbb{R} \\ y \longmapsto R(y) &:= \frac{(3 \cdot t(y) + 1)\,(-t(y) + 1)^3}{16 \cdot t(y)^3} \ . \end{aligned} \tag{3.53}$$

Lemma 112. *With $\xi(t) := \frac{(3 \cdot t+1)(-t+1)^3}{16 \cdot t^3}$,*

(1) $0.03819109771 < \xi(0.6263716633 - 10^{-10}) < 0.03819109772$,
(2) $0.03819109762 < \xi(0.6263716633 + 10^{-10}) < 0.03819109763$.

Proof. Finite statements about integers. The same comments as in the proof of Lemma 106 apply. □

Lemma 113 (some pointwise bounds for $B_0(t)$)**.** *With B_0 as in Definition 102.*(1)*,*

(1) $0.00073969957 < B_0(0.6263716633 - 10^{-10}) < 0.00073969958$,
(2) $0.00073969956 < B_0(0.6263716633 + 10^{-10}) < 0.00073969957$.

Proof. Finite statements about integers. The same comments as in the proof of Lemma 106 apply. □

Lemma 114 (uniform bounds for $B_0(t)$)**.** *With B_0 as in Definition 102.*(1)*,*

$$0.00073969896 < B_0(t) < 0.00073970019 \tag{3.54}$$

for every $t \in I := (0.6263716633 - 10^{-10}, 0.6263716633 + 10^{-10})$.

Proof. If we had a proof that B_0 is monotone decreasing in I, then (3.54) would follow from the slightly stronger pointwise bounds in Lemma 113—but the (known) continuity of B_0 alone is of course not enough to use Lemma 113. Unfortunately, a complete proof of this monotonicity seems to require at least as much work as the proof of (3.54) that follows.

The plan of the proof is the following: for each of the seven summands in B_0 we will derive both upper and lower bounds which uniformly hold in I. In the end, we add these bounds to derive the bounds in (3.54).

In the following paragraph, we prove the uniform bounds

$$0.22495616614 < \tfrac{(3t-1)^2(t+1)^6 \log(t+1)}{512t^6} < 0.22495616711 \quad \text{for every } t \in I \ . \tag{3.55}$$

Since $3 \cdot t > 1$ for every $t \in I$, the function $t \mapsto (3t-1)^2$ is evidently monotone increasing in I. So are the two functions $t \mapsto (t+1)^6$ and $t \mapsto \log(t+1)$. Therefore, $t \mapsto (3t-1)^2(t+1)^6 \log(t+1)$ is monotone increasing in I as a product of three such functions. Hence, for every $t \in I$,

$$(3t-1)^2(t+1)^6 \log(t+1) < (3t-1)^2(t+1)^6 \log(t+1)\Big|_{t=0.6263716633+10^{-10}} < 6.95601448698 \tag{3.56}$$

and

$$(3t-1)^2(t+1)^6 \log(t+1) > (3t-1)^2(t+1)^6 \log(t+1)\Big|_{t=0.6263716633-10^{-10}} > 6.95601447059 \ . \tag{3.57}$$

The function $t \mapsto 512t^6$ is evidently monotone increasing in I. Hence, for every $t \in I$,

$$512t^6 > 512t^6\Big|_{t=0.6263716633-10^{-10}} > 30.92164387643 \tag{3.58}$$

and

$$512t^6 < 512t^6\Big|_{t=0.6263716633+10^{-10}} < 30.92164393568\ . \tag{3.59}$$

Since (3.57) and (3.59) hold in all of I, it follows that, for every $t \in I$,

$$\tfrac{(3t-1)^2(t+1)^6\log(t+1)}{512t^6} > \tfrac{6.95601447059}{30.92164393568} > 0.22495616614\ , \tag{3.60}$$

proving the lower bound in (3.55).

Since (3.56) and (3.58) hold in all of I, it follows that, for every $t \in I$,

$$\tfrac{(3t-1)^2(t+1)^6\log(t+1)}{512t^6} < \tfrac{6.95601448698}{30.92164387643} < 0.22495616711\ , \tag{3.61}$$

proving the upper bound in (3.55).

In the following paragraph, we prove the uniform bounds

$$-0.28456395530 < \tfrac{(3t^4-16t^3+6t^2-1)\log(3t+1)}{32t^3} < -0.28456395528 \quad \text{for every } t \in I\ . \tag{3.62}$$

Since $2+\sqrt{3} > 1$, $2-\sqrt{3} < 0.5$ and $\frac{\mathrm{d}}{\mathrm{d}t}(12t^3-48t^2+12t) = 36t^2-96t+12 = 12t(t-(2+\sqrt{3}))(t-(2-\sqrt{3}))$, it is evident that $\frac{\mathrm{d}}{\mathrm{d}t}(12t^3-48t^2+12t) < 0$ for every $t \in I$, i.e., $t \mapsto 3t^4-16t^3+6t^2-1$ is strictly monotone decreasing in I, so

$$\begin{aligned} 3t^4-16t^3+6t^2-1 &> 3t^4-16t^3+6t^2-1\Big|_{t=0.6263716633+10^{-10}} \\ &= -2.1161809442159711262496568523624448554192 \quad \text{for every } t \in I\ , \end{aligned} \tag{3.63}$$

and

$$\begin{aligned} 3t^4-16t^3+6t^2-1 &< 3t^4-16t^3+6t^2-1\Big|_{t=0.6263716633-10^{-10}} \\ &= -2.1161809425425888723475949656101944348672 \quad \text{for every } t \in I\ . \end{aligned} \tag{3.64}$$

The function $t \mapsto \log(3t+1)$ is evidently strictly monotone increasing in I, hence

$$\log(3t+1) > \log(3t+1)\Big|_{t=0.6263716633-10^{-10}} > 1.05748295164 \quad \text{for every } t \in I\ , \tag{3.65}$$

and

$$\log(3t+1) < \log(3t+1)\Big|_{t=0.6263716633+10^{-10}} < 1.05748295186 \quad \text{for every } t \in I\ . \tag{3.66}$$

The function $t \mapsto 32t^3$ is evidently monotone increasing in I. Hence, for every $t \in I$,

$$32t^3 > 32t^3\Big|_{t=0.6263716633-10^{-10}} = 7.864050340179393384432870014976\ , \tag{3.67}$$

and

$$32t^3 < 32t^3\Big|_{t=0.6263716633+10^{-10}} = 7.864050347712349427668874499328\ . \tag{3.68}$$

It follows that, for every $t \in I$,

$$\begin{aligned} & \tfrac{(3t^4-16t^3+6t^2-1)\log(3t+1)}{32t^3} \\ \text{(by (3.63))} \quad &> \tfrac{(-2.1161809442159711262496568523624448554192)\cdot\log(3t+1)}{32t^3} \\ \text{(by (3.66); we recall that multiplying with a negative number flips an inequality)} \quad &> \tfrac{(-2.1161809442159711262496568523624448554192)\cdot 1.05748295186}{32t^3} \\ \text{(by (3.68))} \quad &> \tfrac{(-2.1161809442159711262496568523624448554192)\cdot 1.05748295186}{7.864050347712349427668874499328} \\ &> -0.28456395530\ , \end{aligned} \tag{3.69}$$

proving the lower bound in (3.62), and also that, for every $t \in I$,

$$\begin{aligned}
& \tfrac{(3t^4-16t^3+6t^2-1)\log(3t+1)}{32t^3} \\
\text{(by (3.64))} \quad & < \tfrac{(-2.116180942542588872347594965610194434867 2)\cdot\log(3t+1)}{32t^3} \\
\text{(by (3.65); we recall that multiplying with a negative number flips an inequality)} \quad & < \tfrac{(-2.1161809425425888723475949656101944348672)\cdot 1.05748295164}{32t^3} \\
\text{(by (3.67))} \quad & < \tfrac{(-2.1161809425425888723475949656101944348672)\cdot 1.05748295164}{7.8640503401793933844328700149 76} \\
& < -0.28456395528 \ , \qquad (3.70)
\end{aligned}$$

which proves the upper bound in (3.62).

In the following paragraph, we prove the uniform bounds

$$0.00029614190 < \tfrac{(3t+1)^2(-t+1)^6\log(2t+1)}{1024t^6} < 0.00029614191 \quad \text{for every } t \in I \ . \qquad (3.71)$$

While it is evident that $t \mapsto (3t+1)^2$ is strictly monotone increasing, and $t \mapsto (-t+1)^6$ strictly monotone decreasing in I, it is not evident whether the product $t \mapsto (3t+1)^2(-t+1)^6$ decreases or increases in I. To decide this, we note that $\frac{\mathrm{d}}{\mathrm{d}t}(3t+1)^2(-t+1)^6 = 24(-1+t)^5 t(1+3t)$, and from this factorization it *is* evident that $\frac{\mathrm{d}}{\mathrm{d}t}(3t+1)^2(-t+1)^6 < 0$ for every $t \in I$, hence that $t \mapsto (3t+1)^2(-t+1)^6$ is indeed strictly monotone decreasing in I. Therefore,

$$\begin{aligned}
(3t+1)^2(-t+1)^6 &< (3t+1)^2(-t+1)^6\Big|_{t=0.6263716633-10^{-10}} \\
&< 0.02255053559 \quad \text{for every } t \in I \ , \qquad (3.72)
\end{aligned}$$

and

$$\begin{aligned}
(3t+1)^2(-t+1)^6 &> (3t+1)^2(-t+1)^6\Big|_{t=0.6263716633+10^{-10}} \\
&> 0.02255053553 \quad \text{for every } t \in I \ . \qquad (3.73)
\end{aligned}$$

Moreover, since function $t \mapsto \log(2t+1)$ evidently is strictly monotone increasing in I, we know that

$$\begin{aligned}
\log(2t+1) &> \log(2t+1)\Big|_{t=0.6263716633-10^{-10}} \\
&> 0.81214872970 \quad \text{for every } t \in I \ , \qquad (3.74)
\end{aligned}$$

and

$$\begin{aligned}
\log(2t+1) &< \log(2t+1)\Big|_{t=0.6263716633+10^{-10}} \\
&< 0.81214872989 \quad \text{for every } t \in I \ . \qquad (3.75)
\end{aligned}$$

Furthermore, since the function $t \mapsto 1024t^6$ evidently is strictly monotone increasing in I, we know that

$$\begin{aligned}
1024t^6 &> 1024t^6\Big|_{t=0.6263716633-10^{-10}} \\
&> 61.84328775287 \quad \text{for every } t \in I \ , \qquad (3.76)
\end{aligned}$$

and

$$\begin{aligned}
1024t^6 &< 1024t^6\Big|_{t=0.6263716633+10^{-10}} \\
&< 61.84328787136 \quad \text{for every } t \in I \ . \qquad (3.77)
\end{aligned}$$

It follows that, for every $t \in I$,

$$\begin{aligned}
& \tfrac{(3t+1)^2(-t+1)^6\log(2t+1)}{1024t^6} \\
\text{(by (3.73))} \qquad & > \tfrac{0.02255053553\cdot\log(2t+1)}{1024t^6} \\
\text{(by (3.74))} \qquad & > \tfrac{0.02255053553\cdot 0.81214872970}{1024t^6} \\
\text{(by (3.77))} \qquad & > \tfrac{0.02255053553\cdot 0.81214872970}{61.84328787136} > 0.00029614190 \ , \qquad (3.78)
\end{aligned}$$

proving the lower bound in (3.71), and also that, for every $t \in I$,

$$\begin{aligned}
& \tfrac{(3t+1)^2(-t+1)^6\log(2t+1)}{1024t^6} \\
\text{(by (3.72))} \qquad & < \tfrac{0.02255053559\cdot\log(2t+1)}{1024t^6} \\
\text{(by (3.75))} \qquad & < \tfrac{0.02255053559\cdot 0.81214872989}{1024t^6} \\
\text{(by (3.76))} \qquad & < \tfrac{0.02255053559\cdot 0.81214872989}{61.84328775287} < 0.00029614191 \ , \qquad (3.79)
\end{aligned}$$

which proves the upper bound in (3.71).

Since $t \mapsto \frac{1}{4}\log(t+3)$ is evidently strictly monotone increasing in I, we know that, for every $t \in I$,

$$\begin{aligned}
\tfrac{1}{4}\log(t+3) &> \tfrac{1}{4}\log(t+3)\Big|_{t=0.6263716633-10^{-10}} \\
&> 0.32205815164 \quad \text{for every } t \in I \ , \qquad (3.80)
\end{aligned}$$

and

$$\begin{aligned}
\tfrac{1}{4}\log(t+3) &< \tfrac{1}{4}\log(t+3)\Big|_{t=0.6263716633+10^{-10}} \\
&< 0.32205815165 \quad \text{for every } t \in I \ . \qquad (3.81)
\end{aligned}$$

Since $t \mapsto \frac{1}{2}\log(t)$ is evidently strictly monotone increasing in I, we know that, for every $t \in I$,

$$\begin{aligned}
\tfrac{1}{2}\log(t) &> \tfrac{1}{2}\log(t)\Big|_{t=0.6263716633-10^{-10}} \\
&> -0.23390568644 \quad \text{for every } t \in I \ , \qquad (3.82)
\end{aligned}$$

and

$$\begin{aligned}
\tfrac{1}{2}\log(t) &< \tfrac{1}{2}\log(t)\Big|_{t=0.6263716633+10^{-10}} \\
&< -0.23390568627 \quad \text{for every } t \in I \ . \qquad (3.83)
\end{aligned}$$

As to the summand $\frac{3}{8}\log(16)$ in B_0, there are the bounds

$$1.03972077083 < \tfrac{3}{8}\log(16) < 1.03972077084 \ . \qquad (3.84)$$

In the following paragraph, we prove the uniform bounds

$$0.02472734758 < \tfrac{(217t^6+920t^5+972t^4+1436t^3+205t^2-172t+6)(-t+1)^2}{2048t^4(3t+1)(t+3)} < 0.02472734762 \quad \text{for every } t \in I \ . \qquad (3.85)$$

We have $\frac{\mathrm{d}}{\mathrm{d}t}\ (217t^6 + 920t^5 + 972t^4 + 1436t^3 + 205t^2 - 172t + 6)(-t+1)^2 = 2(t-1)(868t^6 + 2569t^5 + 616t^4 + 1646t^3 - 1744t^2 - 463t + 92)$, and since $2(t-1) < 0$ for every $t \in I$, to prove that $t \mapsto (217t^6 + 920t^5 + 972t^4 + 1436t^3 + 205t^2 - 172t + 6)(-t+1)^2$ is strictly monotone increasing in

I it suffices to show that $868t^6 + 2569t^5 + 616t^4 + 1646t^3 - 1744t^2 - 463t + 92 < 0$ for every $t \in I$. This is equivalent to

$$868t^6 + 2569t^5 + 616t^4 + 1646t^3 + 92 \quad < \quad 1744t^2 + 463t \quad \text{for every } t \in I \ . \tag{3.86}$$

Since both $t \mapsto 868t^6 + 2569t^5 + 616t^4 + 1646t^3 + 92$ and $t \mapsto 1744t^2 + 463t$, are strictly monotone increasing in I, we have, for every $t \in I$,

$$\begin{aligned} 868t^6 + 2569t^5 + 616t^4 + 1646t^3 + 92 &< 868t^6 + 2569t^5 + 616t^4 + 1646t^3 + 92\Big|_{t=0.6263716633+10^{-10}} \\ &< 891.450148292474 \\ &< 974.25358710372530451456 \\ &= 1744t^2 + 463t\Big|_{t=0.6263716633-10^{-10}} \quad < \quad 1744t^2 + 463t \ , \end{aligned} \tag{3.87}$$

proving (3.86). Since we now know that $t \mapsto (217t^6+920t^5+972t^4+1436t^3+205t^2-172t+6)(-t+1)^2$ is strictly monotone increasing in I, it follows that, for every $t \in I$,

$$\begin{aligned} &(217t^6 + 920t^5 + 972t^4 + 1436t^3 + 205t^2 - 172t + 6)(-t + 1)^2 \\ >&(217t^6 + 920t^5 + 972t^4 + 1436t^3 + 205t^2 - 172t + 6)(-t + 1)^2\Big|_{t=0.6263716633-10^{-10}} \\ >&81.3892822256 \end{aligned} \tag{3.88}$$

and

$$\begin{aligned} &(217t^6 + 920t^5 + 972t^4 + 1436t^3 + 205t^2 - 172t + 6)(-t + 1)^2 \\ <&(217t^6 + 920t^5 + 972t^4 + 1436t^3 + 205t^2 - 172t + 6)(-t + 1)^2\Big|_{t=0.6263716633+10^{-10}} \\ <&81.3892822381 \ . \end{aligned} \tag{3.89}$$

Since $t \mapsto 2048t^4(3t + 1)(t + 3)$ is evidently strictly monotone increasing in I, it follows that, for every $t \in I$,

$$2048t^4(3t + 1)(t + 3) > 2048t^4(3t + 1)(t + 3)\Big|_{t=0.6263716633-10^{-10}} > 3291.4683555 \tag{3.90}$$

and

$$2048t^4(3t + 1)(t + 3) < 2048t^4(3t + 1)(t + 3)\Big|_{t=0.6263716633+10^{-10}} < 3291.4683606 \quad . \tag{3.91}$$

It follows that, for every $t \in I$,

$$\begin{aligned} & \tfrac{(217t^6+920t^5+972t^4+1436t^3+205t^2-172t+6)(-t+1)^2}{2048t^4(3t+1)(t+3)} \\ \text{(by (3.88))} \qquad & > \tfrac{81.3892822256}{2048t^4(3t+1)(t+3)} \\ \text{(by (3.91))} \qquad & > \tfrac{81.3892822256}{3291.4683606} > 0.02472734758 \ , \end{aligned} \tag{3.92}$$

proving the lower bound in (3.85), and

$$\begin{aligned} & \tfrac{(217t^6+920t^5+972t^4+1436t^3+205t^2-172t+6)(-t+1)^2}{2048t^4(3t+1)(t+3)} \\ \text{(by (3.89))} \qquad & < \tfrac{81.3892822381}{2048t^4(3t+1)(t+3)} \\ \text{(by (3.90))} \qquad & < \tfrac{81.3892822381}{3291.4683555} < 0.02472734762 \ , \end{aligned} \tag{3.93}$$

proving the upper bound in (3.85).

We now add the uniform bounds for the summands in B_0 to prove the uniform bounds in (3.54), paying attention to which summand appears with a minus-sign in the definition of B_0.

From the lower bound in (3.55), the upper bounds in (3.62) and (3.71), the lower bound in (3.80), and the upper bounds in (3.82), (3.84) and (3.85), it follows that, for every $t \in I$,

$$\begin{aligned}B_0(t) &> 0.22495616614 - (-0.28456395528) - 0.00029614191\\ &\quad + 0.32205815164 - (-0.23390568627) - 1.03972077084 - 0.02472734762\\ &= 0.00073969896\ ,\end{aligned} \tag{3.94}$$

proving the lower bound in (3.54).

From the upper bound in (3.55), the lower bounds in (3.62) and (3.71), the upper bound in (3.80) and the lower bounds in (3.82), (3.84) and (3.85) it follows that, for every $t \in I$,

$$\begin{aligned}B_0(t) &< 0.22495616711 - (-0.28456395530) - 0.00029614190\\ &\quad + 0.32205815165 - (-0.23390568644) - 1.03972077083 - 0.02472734758\\ &= 0.00073970019\ ,\end{aligned} \tag{3.95}$$

proving the upper bound in (3.54). This completes the proof of Lemma 114. □

Lemma 115 (bounds for $B_0(t_0)$)**.** *With B_0 as in Definition 102.*(1)*,*

$$0.00073969896 < B_0(t_0) < 0.00073970019 \tag{3.96}$$

Proof. In view of Lemma 108, the bounds in (3.96) follow from the uniform bounds in Lemma 114. □

Lemma 116 (pointwise bounds for $B_2(t_0)$)**.** *With B_2 as in Definition 102.*(2)*,*

(1) $-0.0014914312 - 10^{-10} < B_2(0.6263716633 - 10^{-10}) < -0.0014914312 + 10^{-10}$,
(2) $-0.0014914312 - 10^{-10} < B_2(0.6263716633 + 10^{-10}) < -0.0014914312 + 10^{-10}$.

Proof. Left to the reader. The same comments as in the proof of Lemma 106 apply. □

Lemma 117 (uniform bounds for $B_2(t)$)**.** *With B_2 as in Definition 102.*(2)*,*

$$-0.001491431277 < B_2(t) < -0.001491431155\ . \tag{3.97}$$

for every $t \in I := (0.6263716633 - 10^{-10}, 0.6263716633 + 10^{-10})$.

Proof. The plan of the proof is the same as for Lemma 114: for each of the four summands in B_2, derive both upper and lower bounds which uniformly hold in I. In the end, we add these bounds to derive the bounds in (3.97).

In the following paragraph, we prove the uniform bounds

$$0.01786492701 < \tfrac{(-t+1)^3(3t-1)(3t+1)(t+1)^3\log(t+1)}{256t^6} < 0.01786492706 \quad \text{for every } t \in I\ . \tag{3.98}$$

Since $t \mapsto -1+3t^2$ is strictly monotone increasing in I, it follows that $-1+3t^2 > -1+3\cdot(0.6263716633 - 10^{-10})^2 = 0.17702438137980270272 > 0$ for every $t \in I$, and now it is evident from $\frac{\mathrm{d}}{\mathrm{d}t}(-t+1)^3(3t-1)(3t+1)(t+1)^3 = -24(-1+t)^2t(1+t)^2(-1+3t^2)$ that $\frac{\mathrm{d}}{\mathrm{d}t}(-t+1)^3(3t-1)(3t+1)(t+1)^3 < 0$ for every $t \in I$, i.e., that $t \mapsto (-t+1)^3(3t-1)(3t+1)(t+1)^3$ is strictly monotone decreasing in I, so

$$\begin{aligned}(-t+1)^3(3t-1)(3t+1)(t+1)^3 &> (-t+1)^3(3t-1)(3t+1)(t+1)^3\Big|_{t=0.6263716633+10^{-10}}\\ &> 0.56791522564 \quad \text{for every } t \in I\ ,\end{aligned} \tag{3.99}$$

and

$$(-t+1)^3(3t-1)(3t+1)(t+1)^3 < (-t+1)^3(3t-1)(3t+1)(t+1)^3\Big|_{t=0.6263716633-10^{-10}}$$
$$< 0.56791522584 \quad \text{for every } t \in I \ . \tag{3.100}$$

Since $t \mapsto \log(t+1)$ is strictly monotone increasing in I, it follows that

$$\log(t+1) > \log(t+1)\Big|_{t=0.6263716633-10^{-10}} > 0.48635156016 \quad \text{for every } t \in I \ , \tag{3.101}$$

and

$$\log(t+1) < \log(t+1)\Big|_{t=0.6263716633+10^{-10}} < 0.48635156029 \quad \text{for every } t \in I \ . \tag{3.102}$$

Since $t \mapsto 256t^6$ is strictly monotone increasing in I, it follows that

$$256t^6 > 256t^6\Big|_{t=0.6263716633-10^{-10}} > 15.46082193821 \quad \text{for every } t \in I \ , \tag{3.103}$$

and

$$256t^6 < 256t^6\Big|_{t=0.6263716633+10^{-10}} < 15.46082196784 \quad \text{for every } t \in I \ . \tag{3.104}$$

It follows that, for every $t \in I$,

$$\begin{aligned} & \tfrac{(-t+1)^3(3t-1)(3t+1)(t+1)^3 \log(t+1)}{256t^6} \\ \text{(by (3.99))} \quad & > \tfrac{0.56791522564 \cdot \log(t+1)}{256t^6} \\ \text{(by (3.101))} \quad & > \tfrac{0.56791522564 \cdot 0.48635156016}{256t^6} \\ \text{(by (3.104))} \quad & > \tfrac{0.56791522564 \cdot 0.48635156016}{15.46082196784} \\ & > 0.01786492701 \ , \end{aligned} \tag{3.105}$$

proving the lower bound in (3.98), and also that, for every $t \in I$,

$$\begin{aligned} & \tfrac{(-t+1)^3(3t-1)(3t+1)(t+1)^3 \log(t+1)}{256t^6} \\ \text{(by (3.100))} \quad & < \tfrac{0.56791522584 \cdot \log(t+1)}{256t^6} \\ \text{(by (3.102))} \quad & < \tfrac{0.56791522584 \cdot 0.48635156029}{256t^6} \\ \text{(by (3.103))} \quad & < \tfrac{0.56791522584 \cdot 0.48635156029}{15.46082193821} < 0.01786492706 \ , \end{aligned} \tag{3.106}$$

proving the upper bound in (3.98).

In the following paragraph, we prove the uniform bounds

$$0.02019321732 < \tfrac{(-t+1)^3(3t+1)\log(3t+1)}{32t^3} < 0.02019321738 \quad \text{for every } t \in I \ . \tag{3.107}$$

Since $\frac{\mathrm{d}}{\mathrm{d}t}(-t+1)^3(3t+1) = -12(-1+t)^2 t < 0$ for every $t \in I$, we know that $t \mapsto (-t+1)^3(3t+1)$ is strictly monotone decreasing in I, hence

$$(-t+1)^3(3t+1) > (-t+1)^3(3t+1)\Big|_{t=0.6263716633+10^{-10}}$$
$$> 0.15016835728 \quad \text{for every } t \in I \ , \tag{3.108}$$

and

$$(-t+1)^3(3t+1) < (-t+1)^3(3t+1)\Big|_{t=0.6263716633-10^{-10}}$$
$$< 0.15016835750 \quad \text{for every } t \in I \ . \tag{3.109}$$

Since $t \mapsto \log(3t+1)$ is strictly monotone increasing, it follows that for every $t \in I$,

$$\log(3t+1) > \log(3t+1)\Big|_{t=0.6263716633-10^{-10}} > 1.05748295164 \quad \text{for every } t \in I \ , \tag{3.110}$$

and

$$\log(3t+1) < \log(3t+1)\Big|_{t=0.6263716633+10^{-10}} < 1.05748295186 \quad \text{for every } t \in I \quad . \tag{3.111}$$

Since $t \mapsto 32t^3$ is strictly monotone increasing in I, it follows that

$$32t^3 > 32t^3\Big|_{t=0.6263716632} > 7.86405034017 \tag{3.112}$$

and

$$32t^3 < 32t^3\Big|_{t=0.6263716634} < 7.86405034771 \ . \tag{3.113}$$

It follows that, for every $t \in I$,

$$\begin{aligned} & \tfrac{(-t+1)^3(3t+1)\log(3t+1)}{32t^3} \\ \text{(by (3.108))} \quad & > \tfrac{0.15016835728\cdot\log(3t+1)}{32t^3} \\ \text{(by (3.110))} \quad & > \tfrac{0.15016835728\cdot 1.05748295164}{32t^3} \\ \text{(by (3.113))} \quad & > \tfrac{0.15016835728\cdot 1.05748295164}{7.86405034771} \\ & > 0.02019321732 \ , \end{aligned} \tag{3.114}$$

proving the lower bound in (3.107), and also that, for every $t \in I$,

$$\begin{aligned} & \tfrac{(-t+1)^3(3t+1)\log(3t+1)}{32t^3} \\ \text{(by (3.109))} \quad & < \tfrac{0.15016835750\cdot\log(3t+1)}{32t^3} \\ \text{(by (3.111))} \quad & < \tfrac{0.15016835750\cdot 1.05748295186}{32t^3} \\ \text{(by (3.112))} \quad & < \tfrac{0.15016835750\cdot 1.05748295186}{7.86405034017} \\ & < 0.02019321738 \ , \end{aligned} \tag{3.115}$$

proving the upper bound in (3.107).

In the following paragraph, we prove the uniform bounds

$$0.00059228380 < \tfrac{(3t+1)^2(-t+1)^6\log(2t+1)}{512t^6} < 0.00059228381 \quad \text{for every } t \in I \quad . \tag{3.116}$$

Since $\frac{\mathrm{d}}{\mathrm{d}t}\,(3t+1)^2(-t+1)^6 = 24(t-1)^5t(3t+1) < 0$ for every $t \in I$, we know that $t \mapsto (3t+1)^2(-t+1)^6$ is strictly monotone decreasing in I, hence

$$(3t+1)^2(-t+1)^6 > (3t+1)^2(-t+1)^6\Big|_{t=0.6263716633+10^{-10}} > 0.02255053553 \quad \text{for every } t \in I \ , \tag{3.117}$$

and

$$(3t+1)^2(-t+1)^6 < (3t+1)^2(-t+1)^6\Big|_{t=0.6263716633-10^{-10}} < 0.02255053560 \quad \text{for every } t \in I \ . \tag{3.118}$$

Since $t \mapsto \log(2t+1)$ is strictly monotone increasing in I, it follows that

$$\log(2t+1) > \log(2t+1)\Big|_{t=0.6263716633-10^{-10}} > 0.81214872970 \quad \text{for every } t \in I \tag{3.119}$$

and

$$\log(2t+1) < \log(2t+1)\Big|_{t=0.6263716633+10^{-10}} < 0.81214872989 \quad \text{for every } t \in I \ . \tag{3.120}$$

Since $t \mapsto 512t^6$ is strictly monotone increasing in I, it follows that

$$512t^6 > 512t^6\Big|_{t=0.6263716633-10^{-10}} > 30.92164387643 \quad \text{for every } t \in I \tag{3.121}$$

and

$$512t^6 < 512t^6\Big|_{t=0.6263716633+10^{-10}} < 30.92164393568 \quad \text{for every } t \in I \ . \tag{3.122}$$

It follows that, for every $t \in I$,

$$\begin{aligned} & && \tfrac{(3t+1)^2(-t+1)^6\log(2t+1)}{512t^6} \\ &\text{(by (3.117))} && > \tfrac{0.02255053553\cdot\log(2t+1)}{512t^6} \\ &\text{(by (3.119))} && > \tfrac{0.02255053553\cdot 0.81214872970}{512t^6} \\ &\text{(by (3.122))} && > \tfrac{0.02255053553\cdot 0.81214872970}{30.92164393568} \\ & && > 0.00059228380 \ , \end{aligned} \tag{3.123}$$

proving the lower bound in (3.116), and, for every $t \in I$,

$$\begin{aligned} & && \tfrac{(3t+1)^2(-t+1)^6\log(2t+1)}{512t^6} \\ &\text{(by (3.118))} && < \tfrac{0.02255053560\cdot\log(2t+1)}{512t^6} \\ &\text{(by (3.120))} && < \tfrac{0.02255053560\cdot 0.81214872989}{512t^6} \\ &\text{(by (3.121))} && < \tfrac{0.02255053560\cdot 0.81214872989}{30.92164387643} \\ & && < 0.00059228381 \ , \end{aligned} \tag{3.124}$$

proving the upper bound in (3.116).

In the following paragraph, we prove the uniform bounds

$$0.000244575293 < \tfrac{(t-1)^4(185t^4+698t^3-217t^2-160t+6)}{1024t^4(3t+1)(t+3)} < 0.000244575295 \quad \text{for every } t \in I \ . \tag{3.125}$$

For every $t \in I$, evidently $(t-1)^3 < 0$. Moreover, since both $t \mapsto 740t^4 + 2073t^3 + 92$ and $t \mapsto 1698t^2 + 183t$ are strictly monotone increasing in I, we have, for every $t \in I$,

$$\begin{aligned} 740t^4 + 2073t^3 + 92 &< 740t^4 + 2073t^3 + 92\Big|_{t=0.6263716633+10^{-10}} \\ &= 715.35255971414282994995344082730893566326 40 \\ &< 780.82181422656832973952 \\ &= 1698t^2 + 183t\Big|_{t=0.6263716633-10^{-10}} \\ &< 1698t^2 + 183t \ , \end{aligned} \tag{3.126}$$

i.e., $(740t^4 + 2073t^3 - 1698t^2 - 183t + 92) < 0$ for every $t \in I$. Taken together, it follows that $\frac{\mathrm{d}}{\mathrm{d}t}(t-1)^4(185t^4 + 698t^3 - 217t^2 - 160t + 6) = 2(t-1)^3(740t^4 + 2073t^3 - 1698t^2 - 183t + 92) > 0$ for every $t \in I$, hence $t \mapsto (t-1)^4(185t^4 + 698t^3 - 217t^2 - 160t + 6)$ is strictly monotone increasing in I, so

$$\begin{aligned} &(t-1)^4(185t^4 + 698t^3 - 217t^2 - 160t + 6) \\ >&(t-1)^4(185t^4 + 698t^3 - 217t^2 - 160t + 6)\Big|_{t=0.6263716633-10^{-10}} \\ >&0.40250592053 \quad \text{for every } t \in I \ , \end{aligned} \tag{3.127}$$

and

$$\begin{aligned} &(t-1)^4(185t^4 + 698t^3 - 217t^2 - 160t + 6) \\ <&(t-1)^4(185t^4 + 698t^3 - 217t^2 - 160t + 6)\Big|_{t=0.6263716633+10^{-10}} \\ <&0.40250592191 \quad \text{for every } t \in I \ . \end{aligned} \tag{3.128}$$

Since $t \mapsto 1024t^4(3t+1)(t+3)$ is strictly monotone increasing, we furthermore know

$$\begin{aligned} &1024t^4(3t+1)(t+3) \\ >&1024t^4(3t+1)(t+3)\Big|_{t=0.6263716633-10^{-10}} \\ >&1645.7341777 \quad \text{for every } t \in I \ , \end{aligned} \tag{3.129}$$

and

$$\begin{aligned} &1024t^4(3t+1)(t+3) \\ <&1024t^4(3t+1)(t+3)\Big|_{t=0.6263716633+10^{-10}} \\ <&1645.7341803 \quad \text{for every } t \in I \ . \end{aligned} \tag{3.130}$$

It follows that, for every $t \in I$,

$$\begin{aligned} & && \tfrac{(t-1)^4(185t^4+698t^3-217t^2-160t+6)}{1024t^4(3t+1)(t+3)} \\ &\text{(by (3.127))} && > \tfrac{0.40250592053}{1024t^4(3t+1)(t+3)} \\ &\text{(by (3.130))} && > \tfrac{0.40250592053}{1645.7341803} \\ & && > 0.000244575293 \ , \end{aligned} \tag{3.131}$$

proving the lower bound in (3.125), and, for every $t \in I$,

$$\begin{aligned} & && \tfrac{(t-1)^4(185t^4+698t^3-217t^2-160t+6)}{1024t^4(3t+1)(t+3)} \\ &\text{(by (3.128))} && < \tfrac{0.40250592191}{1024t^4(3t+1)(t+3)} \\ &\text{(by (3.129))} && < \tfrac{0.40250592191}{1645.7341777} < 0.000244575295 \ , \end{aligned} \tag{3.132}$$

proving the upper bound in (3.125).

From the lower bound in (3.98), the upper bound in (3.107), and the lower bounds in (3.116) and (3.125), it follows that, for every $t \in I$,

$$\begin{aligned} B_2(t) &> 0.01786492701 - 0.02019321738 + 0.00059228380 + 0.000244575293 \\ &= -0.001491431277 \ , \end{aligned} \tag{3.133}$$

proving the lower bound in (3.97).

From the upper bound in (3.98), the lower bound in (3.107), and the upper bounds in (3.116) and (3.125), it follows that, for every $t \in I$,

$$\begin{aligned} B_2(t) &< 0.01786492706 - 0.02019321732 + 0.00059228381 + 0.000244575295 \\ &= -0.001491431155 \ , \end{aligned} \tag{3.134}$$

proving the upper bound in (3.97). This completes the proof of Lemma 117. □

Lemma 118 (bounds for $B_2(t_0)$)**.** *With B_2 as in Definition 102.*(1),

$$-0.001491431277 < B_2(t_0) < -0.001491431155 \tag{3.135}$$

Proof. In view of Lemma 108, the bounds in (3.135) follow from the uniform bounds in Lemma 117. □

Lemma 119.

$$0.0381910976 = 0.0381910977 - 10^{-10} < R(1) < 0.0381910976 + 10^{-10} < 0.0381910977 \ . \tag{3.136}$$

Proof. By Definition 111, we know that with t_0 as in Lemma 109 we have $R(1) = \frac{(3 \cdot t_0+1)(-t_0+1)^3}{16 \cdot t_0^3}$. It is routine to check that the function $t \mapsto \xi(t) := \frac{(3 \cdot t+1)(-t+1)^3}{16 \cdot t^3}$ is strictly monotone decreasing for $t \in (0,1)$, hence $R(1) = \xi(t_0)$ together with the bounds on t_0 from (3.49) in Lemma 108 implies

$$\xi(0.6263716633 - 10^{-10}) < R(1) < \xi(0.6263716633 + 10^{-10}) \ , \tag{3.137}$$

so in (3.136) the lower bound follows from the lower bound in Lemma 112.(1), while the upper bound follows from the upper bound in Lemma 112.(2). □

Lemma 120 (exact formula for ν in terms of t_0)**.** *With $\rho = \gamma^{-1}$ as in [63, p. 310], C the exponential generating function of connected labelled planar graphs, and with B_0 and B_2 as in Definition 102, and with R as in [63, (2.6)] and B_0 and B_2 as in Definition 102,*

$$\nu := C(\rho) = R(1) + B_0(t_0) + B_2(t_0) \ . \tag{3.138}$$

Proof. See [63, p. 321, (4.7)], together with the equation immediately above that. □

Lemma 121 (verified bounds for ν)**.** *The real number ν defined in [63] satisfies*

$$0.037439365283 < \nu < 0.037439366735 \ . \tag{3.139}$$

Proof. The lower bound follows from

$$\begin{aligned} \nu &\overset{(3.138)}{=} R(1) + B_0(t_0) + B_2(t_0) \\ \text{(by Lemmas 119, 115 and 118)} \quad &> 0.0381910976 + 0.00073969896 + (-0.001491431277) \\ &= 0.037439365283 \end{aligned} \tag{3.140}$$

and the upper bound from

$$\begin{aligned} \nu &\overset{(3.138)}{=} R(1) + B_0(t_0) + B_2(t_0) \\ \text{(by Lemmas 119, 115 and 118)} \quad &< 0.0381910977 + 0.00073970019 + (-0.001491431155) \\ &= 0.037439366735 \ . \end{aligned} \tag{3.141}$$

□

Definition 122 ($A(t)$, $\rho(t)$)**.** *With*

$$A(t) := \tfrac{(3t-1)(t+1)^3 \log(t+1)}{16t^3} + \tfrac{(3t+1)(-t+1)^3 \log(2t+1)}{32t^3} + \tfrac{(-t+1)(185t^4+698t^3-217t^2-160t+6)}{64t(3t+1)^2(t+3)} \tag{3.142}$$

and

$$r(t) := \tfrac{1}{16}(3t+1)^{\frac{1}{2}}(t^{-1}-1)^3 \exp(A(t)) \tag{3.143}$$

we define

$$\rho := r(t_0) \tag{3.144}$$

Proof. See [63, p. 310]. □

Lemma 123 (uniform bounds for $A(t)$)**.** *With A as in Definition 122,*

$$0.48968967248 < A(t) < 0.48968967363 \tag{3.145}$$

for every $t \in I := (0.6263716633 - 10^{-10}, 0.6263716633 + 10^{-10})$.

Proof. The structure of the proof is analogous to the proofs of Lemmas 114 and 117.

In the following paragraph, we prove the uniform bounds

$$0.46777725975 < \tfrac{(3t-1)(t+1)^3 \log(t+1)}{16t^3} < 0.46777726082 \quad \text{for every } t \in I\ . \tag{3.146}$$

Because of $\frac{\mathrm{d}}{\mathrm{d}t}\ (3t-1)(t+1)^3 = 12t(t+1)^2 > 0$ for every $t \in I$, we know that $t \mapsto (3t-1)(t+1)^3$ is strictly monotone increasing in I and hence

$$(3t-1)(t+1)^3 > (3t-1)(t+1)^3\Big|_{t=0.6263716632} > 3.78185681259 \quad \text{for every } t \in I\ , \tag{3.147}$$

and

$$(3t-1)(t+1)^3 < (3t-1)(t+1)^3\Big|_{t=0.6263716634} < 3.78185681657 \quad \text{for every } t \in I\ . \tag{3.148}$$

Since $t \mapsto 16t^3$ is strictly monotone increasing,

$$16t^3 > 16t^3\Big|_{t=0.6263716632} > 3.93202517008 \quad \text{for every } t \in I\ , \tag{3.149}$$

and

$$16t^3 < 16t^3\Big|_{t=0.6263716634} < 3.93202517386 \quad \text{for every } t \in I\ . \tag{3.150}$$

From (3.147), (3.101) and (3.150) follows $((3t-1)(t+1)^3 \cdot \log(t+1))/(16t^3) > 3.78185681259 \cdot 0.48635156016\ /\ 3.93202517386 > 0.46777725975$ for every $t \in I$, hence the lower bound in (3.146). From (3.148), (3.102) and (3.149) follows $((3t-1)(t+1)^3 \cdot \log(t+1))/(16t^3) < 3.78185681657 \cdot 0.48635156029\ /\ 3.93202517008 < 0.46777726082$ for every $t \in I$, hence the upper bound in (3.146).

In the following paragraph, we prove the uniform bounds

$$0.01550842571 < \tfrac{(3t+1)(-t+1)^3 \log(2t+1)}{32t^3} < 0.01550842575 \quad \text{for every } t \in I\ . \tag{3.151}$$

Since $\frac{\mathrm{d}}{\mathrm{d}t}\ (3t+1)(-t+1)^3 = -12t(t-1)^2 < 0$ for every $t \in I$, we know that $t \mapsto (3t+1)(-t+1)^3$ is strictly monotone decreasing in I, so

$$(3t+1)(-t+1)^3 > (3t+1)(-t+1)^3\Big|_{t=0.6263716634} > 0.15016835728 \quad \text{for every } t \in I\ , \tag{3.152}$$

and

$$(3t+1)(-t+1)^3 < (3t+1)(-t+1)^3\Big|_{t=0.6263716632} < 0.15016835750 \quad \text{for every } t \in I\ . \tag{3.153}$$

From (3.152), (3.74) and (3.113) it follows that $((3t+1)(-t+1)^3 \cdot \log(2t+1))/(32t^3) > 0.15016835728 \cdot 0.81214872970\ /\ 7.86405034771 > 0.01550842571$, proving the lower bound in (3.151). From (3.153), (3.75) and (3.112) it follows that $((3t+1)(-t+1)^3 \cdot \log(2t+1))/(32t^3) < 0.15016835750 \cdot 0.81214872989\ /\ 7.86405034017 < 0.01550842575$.

In the following paragraph, we prove the uniform bounds

$$0.00640398702 < \tfrac{(-t+1)(185t^4+698t^3-217t^2-160t+6)}{64t(3t+1)^2(t+3)} < 0.00640398706 \quad \text{for every } t \in I\ . \tag{3.154}$$

Since both $t \mapsto 2745t^2$ and $t \mapsto 925t^4 + 2052t^3 + 114t + 166$ are strictly monotone increasing in I, we have $2745t^2 > 2745t^2\Big|_{t=0.6263716632} > 1076.97730896 > 884.07553334 > 925t^4 + 2052t^3 + 114t + 166\Big|_{t=0.6263716634} > 925t^4 + 2052t^3 + 114t + 166$ for every $t \in I$, hence $\frac{\mathrm{d}}{\mathrm{d}t}(-t+1)(185t^4 + 698t^3 - 217t^2 - 160t + 6) = -925t^4 - 2052t^3 + 2745t^2 - 114t - 166 > 0$ for every $t \in I$. Therefore, $t \mapsto (-t+1)(185t^4 + 698t^3 - 217t^2 - 160t + 6)$ is strictly monotone increasing in I, so

$$\begin{aligned}
&(-t+1)(185t^4 + 698t^3 - 217t^2 - 160t + 6)\\
>&(-t+1)(185t^4 + 698t^3 - 217t^2 - 160t + 6)\Big|_{t=0.6263716632}\\
>&7.71707734263 \quad \text{for every } t \in I\ ,
\end{aligned} \tag{3.155}$$

and

$$\begin{aligned}
&(-t+1)(185t^4 + 698t^3 - 217t^2 - 160t + 6)\\
<&(-t+1)(185t^4 + 698t^3 - 217t^2 - 160t + 6)\Big|_{t=0.6263716634}\\
<&7.71707738122 \quad \text{for every } t \in I\ .
\end{aligned} \tag{3.156}$$

Since $t \mapsto 64t(3t+1)^2(t+3)$ is strictly monotone increasing in I, we have

$$64t(3t+1)^2(t+3) > 64t(3t+1)^2(t+3)\Big|_{t=0.6263716633-10^{-10}} > 1205.0426269 \quad \text{for every } t \in I\ , \tag{3.157}$$

and

$$64t(3t+1)^2(t+3) < 64t(3t+1)^2(t+3)\Big|_{t=0.6263716633+10^{-10}} < 1205.0426279 \quad \text{for every } t \in I\ . \tag{3.158}$$

From (3.155) and (3.158) it follows that $(-t+1)(185t^4+698t^3-217t^2-160t+6)/(64t(3t+1)^2(t+3)) > 7.71707734263\ /\ 1205.0426279 > 0.00640398702$, proving the lower bound in (3.154). From (3.156) and (3.157) it follows that $(-t+1)(185t^4 + 698t^3 - 217t^2 - 160t + 6)/(64t(3t+1)^2(t+3)) < 7.71707738122\ /\ 1205.0426269 < 0.00640398706$, proving the upper bound in (3.154).

In view of Definition 122, the lower bounds in (3.146), (3.151) and (3.154) imply that for every $t \in I$,

$$A(t) > 0.46777725975 + 0.01550842571 + 0.00640398702 = 0.48968967248\ , \tag{3.159}$$

proving the lower bound in (3.145), while the upper bounds in (3.146), (3.151) and (3.154) imply that for every $t \in I$,

$$A(t) < 0.46777726082 + 0.01550842575 + 0.00640398706 = 0.48968967363 \; , \tag{3.160}$$

proving the upper bound in (3.145). $\square$

Lemma 124 (bounds for $A(t_0)$)**.** *With $A(t)$ as in Definition 122 and t_0 as in Definition 108,*

$$0.48968967248 < A(t_0) < 0.48968967363 \; . \tag{3.161}$$

Proof. Immediate from Lemmas 108 and 123. $\square$

Lemma 125 (uniform bounds for $r(t)$)**.** *With $r(t)$ as in Definition 122 and I as in Lemma 114,*

$$0.03672841251 < r(t) < 0.03672841266 \quad \textit{for every } t \in I \; . \tag{3.162}$$

Proof. Since $t \mapsto \frac{1}{16}(3t+1)^{\frac{1}{2}}$ is evidently strictly monotone increasing in I,

$$\tfrac{1}{16}(3t+1)^{\frac{1}{2}} > \tfrac{1}{16}(3t+1)^{\frac{1}{2}}\Big|_{t=0.6263716633-10^{-10}} > 0.10604971913 \quad \text{for every } t \in I \tag{3.163}$$

and

$$\tfrac{1}{16}(3t+1)^{\frac{1}{2}} < \tfrac{1}{16}(3t+1)^{\frac{1}{2}}\Big|_{t=0.6263716633+10^{-10}} < 0.10604971915 \quad \text{for every } t \in I \; . \tag{3.164}$$

Since $t \mapsto t^{-1} - 1$ is strictly monotone decreasing in I, so is $t \mapsto (t^{-1}-1)^3$, hence

$$(t^{-1}-1)^3 > (t^{-1}-1)^3\Big|_{t=0.6263716633+10^{-10}} > 0.21223798428 \quad \text{for every } t \in I \tag{3.165}$$

and

$$(t^{-1}-1)^3 < (t^{-1}-1)^3\Big|_{t=0.6263716633-10^{-10}} < 0.21223798483 \quad \text{for every } t \in I \; . \tag{3.166}$$

Combining Lemma 124 with (I).(2) in Lemma 107, and since exp is strictly monotone increasing, it follows that, for every $t \in I$,

$$\begin{aligned} \exp(A(t)) &> \exp(0.48968967248) \\ &> 0.39995948109 \cdot 10^{-12} + \sum_{0 \leqslant i \leqslant 11} (0.48968967248)^i / i! \\ &> 1.63180974590 \; , \end{aligned} \tag{3.167}$$

and, again for every $t \in I$,

$$\begin{aligned} \exp(A(t)) &< \exp(0.48968967363) \\ &< 0.11998784433 \cdot 10^{-11} + \sum_{0 \leqslant i \leqslant 11} (0.48968967363)^i / i! \\ &< 1.63180974778 \; . \end{aligned} \tag{3.168}$$

It follows that, for every $t \in I$,

$$\begin{aligned} r(t) &= \tfrac{1}{16}(3t+1)^{\frac{1}{2}}(t^{-1}-1)^3 \exp(A(t)) \\ \text{(3.163)} \quad &> 0.10604971913 \cdot (t^{-1}-1)^3 \exp(A(t)) \\ \text{(3.165)} \quad &> 0.10604971913 \cdot 0.21223798428 \cdot \exp(A(t)) \\ \text{(3.167)} \quad &> 0.10604971913 \cdot 0.21223798428 \cdot 1.63180974590 > 0.03672841251 \; , \end{aligned} \tag{3.169}$$

proving the lower bound in (3.162), and, for every $t \in I$,

$$
\begin{aligned}
r(t) &= \tfrac{1}{16}(3t+1)^{\frac{1}{2}}(t^{-1}-1)^3\exp(A(t)) \\
\text{(3.164)}\quad &< 0.10604971915 \cdot (t^{-1}-1)^3\exp(A(t)) \\
\text{(3.166)}\quad &< 0.10604971915 \cdot 0.21223798483 \cdot \exp(A(t)) \\
\text{(3.168)}\quad &< 0.10604971915 \cdot 0.21223798483 \cdot 1.63180974778 < 0.03672841266\ ,
\end{aligned}
\tag{3.170}
$$

proving the upper bound in (3.162). □

Lemma 126 (verified bounds for ρ)**.** *With $r(t)$ as in Definition 122 and t_0 as in Definition 108,*

$$0.03672841251 < \rho < 0.03672841266\ . \tag{3.171}$$

Proof. Since $\rho = r(t_0)$ by Definition 122, this is immediate from Lemmas 108 and 125. □

Definition 127. *We define*

(1) $p_1(\tilde{\rho},\tilde{\nu}) := \mathrm{e}^{-\tilde{\nu}}$,
(2) $p_2(\tilde{\rho},\tilde{\nu}) := \tilde{\rho}\mathrm{e}^{-\tilde{\nu}}$,
(3) $p_3(\tilde{\rho},\tilde{\nu}) := p_4(\tilde{\rho},\tilde{\nu}) := \frac{1}{2}\tilde{\rho}^2\mathrm{e}^{-\tilde{\nu}}$,
(4) $p_5(\tilde{\rho},\tilde{\nu}) := p_6(\tilde{\rho},\tilde{\nu}) := \frac{1}{2}\tilde{\rho}^3\mathrm{e}^{-\tilde{\nu}}$,
(5) $p_7(\tilde{\rho},\tilde{\nu}) := p_8(\tilde{\rho},\tilde{\nu}) := \frac{1}{6}\tilde{\rho}^3\mathrm{e}^{-\tilde{\nu}}$,
(6) $p_9(\tilde{\rho},\tilde{\nu}) := p_{10}(\tilde{\rho},\tilde{\nu}) := p_{11}(\tilde{\rho},\tilde{\nu}) := \frac{1}{2}\tilde{\rho}^4\mathrm{e}^{-\tilde{\nu}}$,
(7) $p_{12}(\tilde{\rho},\tilde{\nu}) := p_{13}(\tilde{\rho},\tilde{\nu}) := \frac{1}{4}\tilde{\rho}^4\mathrm{e}^{-\tilde{\nu}}$,
(8) $p_{14}(\tilde{\rho},\tilde{\nu}) := p_{15}(\tilde{\rho},\tilde{\nu}) := \frac{1}{6}\tilde{\rho}^4\mathrm{e}^{-\tilde{\nu}}$,
(9) $p_{16}(\tilde{\rho},\tilde{\nu}) := p_{17}(\tilde{\rho},\tilde{\nu}) := \frac{1}{8}\tilde{\rho}^4\mathrm{e}^{-\tilde{\nu}}$,
(10) $p_{18}(\tilde{\rho},\tilde{\nu}) := p_{19}(\tilde{\rho},\tilde{\nu}) := \frac{1}{24}\tilde{\rho}^4\mathrm{e}^{-\tilde{\nu}}$.

Lemma 128 (verified bounds for $\mathrm{e}^{-\nu}$)**.** *With ν as in Lemma 120,*

$$0.96325282112 < \mathrm{e}^{-\nu} < 0.96325282254\ . \tag{3.172}$$

Proof. According to Lemma 121 we have $-0.037439366735 < -\nu < -0.037439365283$, so (II).(2) in Lemma 107 is applicable and implies, by strict monotonicity of exp,

$$\mathrm{e}^{-\nu} > 1.0850694444 \cdot 10^{-11} + \sum_{0\leqslant i\leqslant 5} \tfrac{(-0.037439366735)^i}{i!} > 0.96325282112 \tag{3.173}$$

and

$$\mathrm{e}^{-\nu} < 2.1701388889 \cdot 10^{-11} + \sum_{0\leqslant i\leqslant 5} \tfrac{(-0.037439365283)^i}{i!} < 0.96325282254\ . \tag{3.174}$$

□

Lemma 129 (verified bounds for the p_i)**.** *With p_i as in Definition 127, and ρ and ν as in Lemma 120,*

(1) $0.03537874696 < p_2(\rho,\nu) < 0.03537874717$,
(2) $0.00064970260 < p_3(\rho,\nu) = p_4(\rho,\nu) < 0.00064970262$,
(3) $0.00002386254 < p_5(\rho,\nu) = p_6(\rho,\nu) < 0.00002386255$,
(4) $0.00000795418 < p_7(\rho,\nu) = p_8(\rho,\nu) < 0.00000795419$,
(5) $0.00000087643 < p_9(\rho,\nu) = p_{10}(\rho,\nu) = p_{11}(\rho,\nu) < 0.00000087644$,
(6) $0.00000043821 < p_{12}(\rho,\nu) = p_{13}(\rho,\nu) < 0.00000043822$,
(7) $0.00000029214 < p_{14}(\rho,\nu) = p_{15}(\rho,\nu) < 0.00000029215$,
(8) $0.00000021910 < p_{16}(\rho,\nu) = p_{17}(\rho,\nu) < 0.00000021911$,
(9) $0.00000007303 < p_{18}(\rho,\nu) = p_{19}(\rho,\nu) < 0.00000007304$.

Proof. To prove this, we repeatedly use Lemmas 126 and 128:

(1) $$\begin{aligned}&0.03537874696 < 0.03672841251 \cdot 0.96325282112 \\ &< p_2(\rho,\nu) < 0.03672841266 \cdot 0.96325282254 < 0.03537874717\ ,\end{aligned}$$

(2) $0.00064970260 < \frac{1}{2} \cdot (0.03672841251)^2 \cdot 0.96325282112$
$< p_3(\rho,\nu) = p_4(\rho,\nu) < \frac{1}{2} \cdot (0.03672841266)^2 \cdot 0.96325282254 < 0.00064970262$,

(3) $0.00002386254 < \frac{1}{2} \cdot (0.03672841251)^3 \cdot 0.96325282112$
$< p_5(\rho,\nu) = p_6(\rho,\nu) < \frac{1}{2} \cdot (0.03672841266)^3 \cdot 0.96325282254 < 0.00002386255$,

(4) $0.00000795418 < \frac{1}{6} \cdot (0.03672841251)^3 \cdot 0.96325282112$
$< p_7(\rho,\nu) = p_8(\rho,\nu) < \frac{1}{6} \cdot (0.03672841266)^3 \cdot 0.96325282254 < 0.00000795419$,

(5) $0.00000087643 < \frac{1}{2} \cdot (0.03672841251)^4 \cdot 0.96325282112$
$< p_9(\rho,\nu) = p_{10}(\rho,\nu) = p_{11}(\rho,\nu) < \frac{1}{2} \cdot (0.03672841266)^4 \cdot 0.96325282254 < 0.00000087644$,

(6) $0.00000043821 < \frac{1}{4} \cdot (0.03672841251)^4 \cdot 0.96325282112$
$< p_{12}(\rho,\nu) = p_{13}(\rho,\nu) < \frac{1}{4} \cdot (0.03672841266)^4 \cdot 0.96325282254 < 0.00000043822$,

(7) $0.00000029214 < \frac{1}{6} \cdot (0.03672841251)^4 \cdot 0.96325282112$
$< p_{14}(\rho,\nu) = p_{15}(\rho,\nu) < \frac{1}{6} \cdot (0.03672841266)^4 \cdot 0.96325282254 < 0.00000029215$,

(8) $0.00000021910 < \frac{1}{8} \cdot (0.03672841251)^4 \cdot 0.96325282112$
$< p_{16}(\rho,\nu) = p_{17}(\rho,\nu) < \frac{1}{8} \cdot (0.03672841266)^4 \cdot 0.96325282254 < 0.00000021911$,

(9) $0.00000007303 < \frac{1}{24} \cdot (0.03672841251)^4 \cdot 0.96325282112$
$< p_{18}(\rho,\nu) = p_{19}(\rho,\nu) < \frac{1}{24} \cdot (0.03672841266)^4 \cdot 0.96325282254 < 0.00000007304$.

□

With the following notation, $\delta_i(\rho,\nu) > 0$ is equivalent to the sequence $(p_i(\rho,\nu))_{i\in\mathbb{N}}$ satisfying the tail-bounds-term condition at i:

Definition 130 (the $\delta_i(\tilde\rho,\tilde\nu)$)**.** *For every i we define*

$$\delta_i(\tilde\rho,\tilde\nu) := p_i(\tilde\rho,\tilde\nu) - \left(1 - \sum_{1\leqslant\imath\leqslant i} p_\imath(\tilde\rho,\tilde\nu)\right) = -1 + 2p_i(\tilde\rho,\tilde\nu) + \sum_{1\leqslant\imath\leqslant i-1} p_\imath(\tilde\rho,\tilde\nu) \ . \tag{3.175}$$

Lemma 131 (verified positive lower bounds for the $\delta_i(\rho,\nu)$)**.** *For the δ_i of Definition 130, and with ρ and ν as in Lemma 120*

(1) $\delta_1(\rho,\nu) > 0.92650564224$,
(2) $\delta_2(\rho,\nu) > 0.03401031504$,
(3) $\delta_4(\rho,\nu) > 0.00058067588$,
(4) $\delta_6(\rho,\nu) > 0.00000256090$,
(5) $\delta_8(\rho,\nu) > 0.00000256090$.

Proof. Using the bounds from Lemma 129 we have the following estimates:

$$\begin{aligned} \delta_1(\rho,\nu) &= 2p_1(\rho,\nu) - 1 \\ \text{(by Lemma 128 and } p_1(\rho,\nu) = \mathrm{e}^{-\nu}) \quad &> 2 \cdot 0.96325282112 - 1 = 0.92650564224 > 0 \ , \end{aligned} \tag{3.176}$$

$$\begin{aligned} \delta_2(\rho,\nu) &\overset{(3.175)}{=} 2p_2(\rho,\nu) + p_1(\rho,\nu) - 1 \\ \text{(by Lemma 129)} \quad &> 2 \cdot 0.03537874696 + 0.96325282112 - 1 = 0.03401031504 > 0 \ , \end{aligned} \tag{3.177}$$

$$\begin{aligned} \delta_4(\rho,\nu) &\overset{(3.175)}{=} 2p_4(\rho,\nu) + p_1(\rho,\nu) + p_2(\rho,\nu) + p_3(\rho,\nu) - 1 \\ \text{(by Lemma 129)} \quad &> 2 \cdot 0.00064970260 + 0.96325282112 + 0.03537874696 + 0.00064970260 - 1 \\ &= 0.00058067588 > 0 \ , \end{aligned} \tag{3.178}$$

$$\begin{aligned} \delta_6(\rho,\nu) &\overset{(3.175)}{=} 2p_6(\rho,\nu) + p_5(\rho,\nu) + p_4(\rho,\nu) + p_3(\rho,\nu) + p_2(\rho,\nu) + p_1(\rho,\nu) - 1 \\ \text{(by Lemma 129)} \quad &> 2 \cdot 0.00002386254 + 0.00002386254 + 0.00064970260 + 0.00064970260 \\ &\quad + 0.03537874696 + 0.96325282112 - 1 \\ &= 0.00000256090 \ , \end{aligned} \tag{3.179}$$

$$
\begin{aligned}
\delta_8(\rho,\nu) &\overset{(3.175)}{=} 2p_8(\rho,\nu) + p_7(\rho,\nu) + p_6(\rho,\nu) + p_5(\rho,\nu) \\
&\quad + p_4(\rho,\nu) + p_3(\rho,\nu) + p_2(\rho,\nu) + p_1(\rho,\nu) - 1 \\
\text{(by Lemma 129)} \quad &> 2 \cdot 0.00000795418 + 0.00000795418 + 0.00002386254 + 0.00002386254 \\
&\quad + 0.00064970260 + 0.00064970260 + 0.03537874696 + 0.96325282112 - 1 \\
&= 0.00000256090 \; . \qquad (3.180)
\end{aligned}
$$

□

Let us note that the equality of the lower bounds in (3.179) and (3.180) is not an accident. The reason behind it is this:

$$
\begin{aligned}
&2p_6(\rho,\nu) + p_5(\rho,\nu) + p_4(\rho,\nu) + p_3(\rho,\nu) + p_2(\rho,\nu) + p_1(\rho,\nu) - 1 \\
=&2p_8(\rho,\nu) + p_7(\rho,\nu) + p_6(\rho,\nu) + p_5(\rho,\nu) + p_4(\rho,\nu) + p_3(\rho,\nu) + p_2(\rho,\nu) + p_1(\rho,\nu) - 1 \\
&\Longleftrightarrow \\
&p_6(\rho,\nu) = p_7(\rho,\nu) + 2p_8(\rho,\nu) \; , \qquad (3.181)
\end{aligned}
$$

and the latter is indeed true, by Definition 127. Since for each $p_i(\rho,\nu)$, the same verified lower bound is used, the equality in (3.181) implies an inequality of the lower bounds obtained by summing these lower bounds.

The following in particular proves that or every p_i corresponding to a 4-vertex planar graph, the tail bounds the term.

Lemma 132 (verified negative upper bounds for those δ_i which correspond to 4-vertex planar graphs)**.** *With δ_i as in Definition 130,*

(1) $\delta_9(\rho,\nu) < -0.00000363869$,
(2) $\delta_{10}(\rho,\nu) < -0.00000276225$,
(3) $\delta_{11}(\rho,\nu) < -0.00000188581$,
(4) $\delta_{12}(\rho,\nu) < -0.00000188581$,
(5) $\delta_{13}(\rho,\nu) < -0.00000144759$,
(6) $\delta_{14}(\rho,\nu) < -0.00000130151$,
(7) $\delta_{15}(\rho,\nu) < -0.00000100936$,
(8) $\delta_{16}(\rho,\nu) < -0.00000086329$,
(9) $\delta_{17}(\rho,\nu) < -0.00000064418$,
(10) $\delta_{18}(\rho,\nu) < -0.00000042507$,
(11) $\delta_{19}(\rho,\nu) < -0.00000064417$.

Proof. We calculate that

$$
\begin{aligned}
\delta_9(\rho,\nu) &\overset{(3.175)}{=} -1 + 2p_9(\rho,\nu) + \sum_{1\leqslant \imath\leqslant 8} p_\imath(\rho,\nu) \\
\text{(upper bounds in Lemma 129)} \quad &< -1 + 2 \cdot 0.00000087644 + 0.96325282254 + 0.03537874717 + \\
&0.00064970262 + 0.00064970262 + 0.00002386255 + \\
&0.00002386255 + 0.00000795419 + 0.00000795419 \\
&< -0.00000363869 \qquad (3.182)
\end{aligned}
$$

and

$$
\begin{aligned}
\delta_{10}(\rho,\nu) &\overset{(3.175)}{=} -1 + 2p_{10}(\rho,\nu) + \sum_{1\leqslant \imath\leqslant 9} p_\imath(\rho,\nu) \\
\text{(upper bounds in Lemma 129)} \quad &< -1 + 2 \cdot 0.00000087644 + 0.96325282254 + \\
&0.03537874717 + 0.00064970262 + 0.00064970262 + \\
&0.00002386255 + 0.00002386255 + 0.00000795419 + \\
&0.00000795419 + 0.00000087644 \\
&< -0.00000276225 \qquad (3.183)
\end{aligned}
$$

and

$$\delta_{11}(\rho,\nu) \overset{(3.175)}{=} -1 + 2p_{11}(\rho,\nu) + \sum_{1\leqslant\imath\leqslant 10} p_\imath(\rho,\nu)$$

(upper bounds in Lemma 129)
$$< -1 + 2\cdot 0.00000087644 + 0.96325282254+$$
$$0.03537874717 + 0.00064970262 + 0.00064970262+$$
$$0.00002386255 + 0.00002386255 + 0.00000795419+$$
$$0.00000795419 + 0.00000087644 + 0.00000087644 < -0.00000188581 \quad (3.184)$$

and

$$\delta_{12}(\rho,\nu) \overset{(3.175)}{=} -1 + 2p_{12}(\rho,\nu) + \sum_{1\leqslant\imath\leqslant 11} p_\imath(\rho,\nu)$$

(upper bounds in Lemma 129)
$$< -1 + 2\cdot 0.00000043822 + 0.96325282254+$$
$$0.03537874717 + 0.00064970262 + 0.00064970262+$$
$$0.00002386255 + 0.00002386255 + 0.00000795419+$$
$$0.00000795419 + 0.00000087644 + 0.00000087644 + 0.00000087644$$
$$< -0.00000188581 \quad (3.185)$$

and

$$\delta_{13}(\rho,\nu) \overset{(3.175)}{=} -1 + 2p_{13}(\rho,\nu) + \sum_{1\leqslant\imath\leqslant 12} p_\imath(\rho,\nu)$$

(upper bounds in Lemma 129)
$$< -1 + 2\cdot 0.00000043822 + 0.96325282254+$$
$$0.03537874717 + 0.00064970262 + 0.00064970262+$$
$$0.00002386255 + 0.00002386255 + 0.00000795419+$$
$$0.00000795419 + 0.00000087644 + 0.00000087644+$$
$$0.00000087644 + 0.00000043822 < -0.00000144759 \quad (3.186)$$

and

$$\delta_{14}(\rho,\nu) \overset{(3.175)}{=} -1 + 2p_{14}(\rho,\nu) + \sum_{1\leqslant\imath\leqslant 13} p_\imath(\rho,\nu)$$

(upper bounds in Lemma 129)
$$< -1 + 2\cdot 0.00000029215 + 0.96325282254+$$
$$0.03537874717 + 0.00064970262 + 0.00064970262+$$
$$0.00002386255 + 0.00002386255 + 0.00000795419+$$
$$0.00000795419 + 0.00000087644 + 0.00000087644+$$
$$0.00000087644 + 0.00000043822 + 0.00000043822 < -0.00000130151 \quad (3.187)$$

and

$$\delta_{15}(\rho,\nu) \overset{(3.175)}{=} -1 + 2p_{15}(\rho,\nu) + \sum_{1\leqslant\imath\leqslant 14} p_\imath(\rho,\nu)$$

(upper bounds in Lemma 129)

$$< -1 + 2\cdot 0.00000029215 + 0.96325282254 + 0.03537874717 + 0.00064970262 + 0.00064970262 + 0.00002386255 + 0.00002386255 + 0.00000795419 + 0.00000795419 + 0.00000087644 + 0.00000087644 + 0.00000087644 + 0.00000043822 + 0.00000043822 + 0.00000029215 < -0.00000100936 \tag{3.188}$$

and

$$\delta_{16}(\rho,\nu) \overset{(3.175)}{=} -1 + 2p_{16}(\rho,\nu) + \sum_{1\leqslant\imath\leqslant 15} p_\imath(\rho,\nu)$$

(upper bounds in Lemma 129)

$$< -1 + 2\cdot 0.00000021911 + 0.96325282254 + 0.03537874717 + 0.00064970262 + 0.00064970262 + 0.00002386255 + 0.00002386255 + 0.00000795419 + 0.00000795419 + 0.00000087644 + 0.00000087644 + 0.00000087644 + 0.00000043822 + 0.00000043822 + 0.00000029215 + 0.00000029215 < -0.00000086329 \tag{3.189}$$

and

$$\delta_{17}(\rho,\nu) \overset{(3.175)}{=} -1 + 2p_{17}(\rho,\nu) + \sum_{1\leqslant\imath\leqslant 16} p_\imath(\rho,\nu)$$

(upper bounds in Lemma 129)

$$< -1 + 2\cdot 0.00000021911 + 0.96325282254 + 0.03537874717 + 0.00064970262 + 0.00064970262 + 0.00002386255 + 0.00002386255 + 0.00000795419 + 0.00000795419 + 0.00000087644 + 0.00000087644 + 0.00000087644 + 0.00000043822 + 0.00000043822 + 0.00000029215 + 0.00000029215 + 0.00000021911 < -0.00000064418 \tag{3.190}$$

and

$$\delta_{18}(\rho,\nu) \overset{(3.175)}{=} -1 + 2p_{18}(\rho,\nu) + \sum_{1\leqslant\imath\leqslant 17} p_\imath(\rho,\nu)$$

(upper bounds in Lemma 129)

$$< -1 + 2\cdot 0.00000007304 + 0.96325282254 + 0.03537874717 + 0.00064970262 + 0.00064970262 + 0.00002386255 + 0.00002386255 + 0.00000795419 + 0.00000795419 + 0.00000087644 + 0.00000087644 + 0.00000087644 + 0.00000043822 + 0.00000043822 + 0.00000029215 + 0.00000029215 + 0.00000021911 + 0.00000021911 < -0.00000071721 \tag{3.191}$$

and

$$\delta_{19}(\rho,\nu) \overset{(3.175)}{=} -1 + 2p_{19}(\rho,\nu) + \sum_{1\leqslant \imath\leqslant 18} p_\imath(\rho,\nu)$$

(upper bounds in Lemma 129)
$$< -1 + 2\cdot 0.00000007304 + 0.96325282254$$
$$+ 0.03537874717 + 0.00064970262 + 0.00064970262+$$
$$0.00002386255 + 0.00002386255 + 0.00000795419+$$
$$0.00000795419 + 0.00000087644 + 0.00000087644+$$
$$0.00000087644 + 0.00000043822 + 0.00000043822+$$
$$0.00000029215 + 0.00000029215 + 0.00000021911+$$
$$0.00000021911 + 0.00000007304 < -0.64417 \ . \qquad (3.192)$$

□

From Lemma 101 we know that $\rho < \frac{1}{24}$, it is indeed true that for $i > 19$ we have $p_i \leqslant \frac{\rho^5}{G(\rho)} < \frac{\rho^4}{24G(\rho)} = p_{19}$, as G_i must have at least five vertices. Thus $\mathrm{G}_1,\dots,\mathrm{G}_{19}$ are the first nineteen unlabelled planar graphs when those are ordered by non-increasing value of $\mathsf{P}_{\mathrm{BP}\mathcal{P}}$.

Lemma 133. *The only $i \in \mathbb{N}$ with $p_i > \sum_{j>i} p_j$ are $i = 1,2,4,6,8$.*

Proof. Obviously, $p_i \leqslant \sum_{j>i} p_j$ whenever $p_i = p_{i+1}$. Therefore, without recourse to numerical values, we know from Definition 127 that the inequality in Lemma 133 fails for each of the indices $i \in \{3,5,7,9,10,12,14,16,18\}$. The estimates of ρ and $1/G(\rho)$ provided by Lemma 101 are sufficiently accurate (with room to spare) to determine for which other $i \leqslant 19$ the inequality holds: namely, by (1)–(5) in Lemma 131 it holds for $i \in \{1,2,4,6,8\}$. Now to complete the proof of Lemma 133, we are left with the indices $i \geqslant 20$. Let some $i \geqslant 20$ be given. Since $\mathrm{G}_1,\dots,\mathrm{G}_{19}$ from Figure 5.9 are all the isomorphism-types of planar graphs with at most four vertices, we must have $|G_i| \geqslant 5$, so the formula in Theorem 81 (with $D = G$ the exponential generating function of planar graphs) implies $p_i \leqslant \rho^5/G(\rho)$. Let $n \geqslant 6$ be the unique integer with $\frac{\rho^n}{G(\rho)} < p_i \leqslant \frac{\rho^{n-1}}{G(\rho)}$. Then $\mathsf{P}_{\mathrm{BP}\mathcal{P}}[H] = \frac{\rho^n}{|\mathrm{Aut}(H)|\cdot G(\rho)} < p_i$ for every isomorphism-type $H \in \mathcal{UP}_n$. Since $\rho < \frac{1}{24}$ by Lemma 101, we know that for every $i > 19$ we have $p_i = \mathsf{P}_{\mathrm{BP}\mathcal{P}}[\mathrm{G}_i] \leqslant$ (since $\mathrm{G}_1,\dots,\mathrm{G}_{19}$ are all isomorphism-types of planar graphs on at most four vertices, the present G_i has at least five vertices) $\leqslant \frac{\rho^5}{G(\rho)} < \frac{\rho^4}{24G(\rho)} = p_{19}$. Hence, for every $H \in \mathcal{UP}_n$ there is $j > i$ with $H = \mathrm{G}_j$ and $\sum_{j>i} p_j \geqslant \sum_{H\in\mathcal{UP}_n} \mathsf{P}_{\mathrm{BP}\mathcal{P}}[H] = \frac{\rho^n}{G(\rho)} \cdot \sum_{H\in\mathcal{UP}_n} \frac{1}{|\mathrm{Aut}(H)|} >$ (by Lemma 100, and since from Lemma 101 we know $\rho > \frac{1}{30}$) $> \frac{\rho^{n-1}}{G(\rho)} \geqslant p_i$. □

As for (limset-St.3), Lemma 133 tells us that $i_{\mathrm{m}} = 8$. Moreover, $p_1,\dots,p_{19}$ is a non-increasing ordering, as required by (limset-St.3): obviously we have $p_5 = p_6 > p_7 = p_8$ and $p_9 = p_{10} = p_{11} > p_{12} = p_{13} > p_{14} = p_{15} > p_{16} = p_{17} > p_{18} = p_{19}$. By $\rho < 1$ alone we have $p_1 > p_2 > p_3 = p_4 > p_5 = p_6$. Because of $\rho < \frac{1}{3}$ we have $p_7 = p_8 > p_9 = p_{10}$. Therefore, Corollary 95 yields the expressions in (3.38). The reason why the union in (3.38) has only five parameters despite eight p_i being involved is that the relations $p_3 = p_4$, $p_5 = p_6$ and $p_7 = p_8$ reduce the 'degrees of freedom' by three.

We now show the intervals in Theorem 99 to be pairwise disjoint. We note that for every $x > 0$,

$$|(\tfrac{1}{2}(d'-d'') + \tfrac{1}{6}(e'-e''))x^3| \geqslant \tfrac{1}{6}x^3 \qquad \text{for all } d',d'',e',e'' \in \{0,1,2\} \text{ with } (d',e') \neq (d'',e'') \ . \quad (3.193)$$

Since $x > 0$, (3.193) is equivalent to $|(\frac{1}{2}(d'-d'')+\frac{1}{6}(e'-e''))| \geqslant \frac{1}{6}$. If $d' > d''$ we have $\frac{1}{2}(d'-d'') \geqslant \frac{1}{2}$, and then $\frac{1}{2} - 2\cdot\frac{1}{6} = \frac{1}{6}$ implies that whatever the values of $e', e'' \in \{0,1,2\}$, indeed $|(\frac{1}{2}(d'-d'') + \frac{1}{6}(e'-e''))\rho^3| \geqslant \frac{1}{6}$. If on the contrary $d' \leqslant d''$, then $d' = d''$ by hypothesis implies $e' \neq e''$, hence $|e'-e''| \geqslant 1$ and the claim holds, while $d' < d''$ implies $|\frac{1}{2}(d'-d'')| \geqslant \frac{1}{2}$ and allows us to estimate $|\frac{1}{2}(d'-d'') + \frac{1}{6}(e'-e'')| \geqslant$ (in general) $\geqslant |\frac{1}{2}(d'-d'')| - |\frac{1}{6}(e'-e'')| \geqslant$ (using what we know about d', d'', e' and e'') $\geqslant \frac{1}{2} - \frac{1}{6}\cdot 2 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$, completing the proof of (3.193).

Now we make a statement using (with room to spare) the numerical value of ρ from [63, p. 310]:

$$|(a'-a'')+(b'-b'')\rho+(c'-c'')\rho^2| \geqslant \tfrac{3}{2}\rho^3 \tag{3.194}$$

for all $a', a'', b', b'' \in \{0,1\}$ and $c', c'' \in \{0,1,2\}$ such that $(a',b',c') \neq (a'',b'',c'')$, where ρ the radius of convergence of the egf of labelled planar graphs.

To prove (3.194), we note that if $a' \neq a''$, then $|(a'-a'')+(b'-b'')\rho+(c'-c'')\rho^2| \geqslant |a'-a''| - |(b'-b'')\rho+(c'-c'')\rho^2| \geqslant$ (since $|a'-a''| = 1$, $|b'-b''| \leqslant 1$, and $|c'-c''| \leqslant 2$) $\geqslant 1-\rho-2\rho^2 >$ (by Lemma 101) $> 1 - 0.037 - 2\cdot 0.037^2 = 0.960262 > 0.0000759795 = \frac{3}{2}0.037^3 > \frac{3}{2}\rho^3$. If on the contrary $a' = a''$, then $b' = b''$ via our hypotheses implies $|c'-c''| \geqslant 1$, so $|(a'-a'')+(b'-b'')\rho+(c'-c'')\rho^2| = |c'-c''|\rho^2 \geqslant \rho^2 \geqslant 0.036^2 = 0.00129 > 0.0000759795 = \frac{3}{2}0.037^3$, while the other possibility $b' \neq b''$ implies $|(a'-a'')+(b'-b'')\rho+(c'-c'')\rho^2| = |(b'-b'')\rho+(c'-c'')\rho^2| \geqslant |b'-b''|\rho - |c'-c''|\rho^2 \geqslant$ (since $|b'-b''| \geqslant 1$) $\geqslant 1\cdot\rho - 2\cdot\rho^2 > 0.036 - 2\cdot 0.037^2 = 0.033262 > 0.0000759795 = \frac{3}{2}0.037^3 >$ (by Lemma 101) $> \frac{3}{2}\rho^3$.

Now we note that, for any a', a'', b', $b'' \in \{0,1\}$ and $c', c'', d', d'', e', e'' \in \{0,1,2\}$ with $(a',b',c',d',e') \neq (a'',b'',c'',d'',e'')$, then, with ρ the radius of convergence of the exponential generating function of the set of labelled planar graphs,

$$\left|\frac{a'+b'\rho+c'\rho^2+(\frac{1}{2}d'+\frac{1}{6}e')\rho^3}{G(\rho)} - \frac{a''+b''\rho+c''\rho^2+(\frac{1}{2}d''+\frac{1}{6}e'')\rho^3}{G(\rho)}\right| \geqslant \frac{\rho^3}{6\,G(\rho)}\ . \tag{3.195}$$

To prove (3.195), we note that for any a', a'', b', $b'' \in \{0,1\}$ and $c', c'', d', d'', e', e'' \in \{0,1,2\}$, if $(a',b',c',d',e') \neq (a'',b'',c'',d'',e'')$, then for any $x \in \mathbb{R}_{>0}$,

$$|(a'-a'')+(b'-b'')x+(c'-c'')x^2+(\tfrac{1}{2}(d'-d'')+\tfrac{1}{6}(e'-e''))x^3| \geqslant \tfrac{1}{6}x^3\ . \tag{3.196}$$

The reason is that if $(a',b',c') = (a'',b'',c'')$, then $(d',e') \neq (d'',e'')$ by hypothesis and (3.193) implies (3.196), while if $(a',b',c') \neq (a'',b'',c'')$, then

$$\begin{aligned} &|(a'-a'')+(b'-b'')\rho+(c'-c'')\rho^2+(\tfrac{1}{2}(d'-d'')+\tfrac{1}{6}(e'-e''))\rho^3| \\ \text{(in general) } \geqslant &\big||(a'-a'')+(b'-b'')\rho+(c'-c'')\rho^2| - |\tfrac{1}{2}(d'-d'')+\tfrac{1}{6}(e'-e'')|\rho^3\big| \\ \text{(by (3.194)) } \geqslant &\tfrac{3}{2}\rho^3 - \big||\tfrac{1}{2}(d'-d'')+\tfrac{1}{6}(e'-e'')|\rho^3\big| \geqslant \tfrac{3}{2}\rho^3 - (\tfrac{1}{2}|d'-d''|+\tfrac{1}{6}|e'-e''|)\rho^3 \\ \geqslant &\tfrac{3}{2}\rho^3 - (\tfrac{1}{2}\cdot 2+\tfrac{1}{6}\cdot 2)\rho^3 \geqslant (\tfrac{3}{2}-\tfrac{4}{3})\rho^3 = \tfrac{1}{6}\rho^3\ , \end{aligned} \tag{3.197}$$

the penultimate estimate simply because of $d', d'', e', e'' \in \{0,1,2\}$, which again proves (3.196). From [63] we know $\rho \in \mathbb{R}_{>0}$ and $G(\rho) > 0$, hence after dividing by $G(\rho)$ we obtain (3.195).

By (3.195), we now know that any two left-endpoints of the intervals are at least $\rho^3/(6G(\rho)) = p_8$ apart. At the same time, the length of each of these intervals is $1 - (1+\rho+\rho^2+\frac{4}{3}\rho^3)/G(\rho) = \sum_{j>8} p_j < p_8$, i.e., strictly less than the minimum distance of their left-endpoints. This completes the proof that the intervals in (3.38) of Theorem 99 are disjoint, completing the proof. □

3.4 Proving logical limit laws for sentences in FO-logic w.r.t. the class of graphs embeddable on a fixed surface

McDiarmid proved a result analogous to Theorem 83 for the (non-addable) class $\mathcal{G}_S$ of all graphs embeddable on some fixed surface S under the additional assumption that $\mathcal{G}_S$ is smooth; that $\mathcal{G}_S$ *is* indeed smooth for every surface S was later established by Bender, Canfield and Richmond. Taken together, this results in:

Theorem 134 (McDiarmid [127, Theorem 3.3], with the smoothness condition removed in view of [13, Theorem 2])**.** *If S denotes a fixed surface, $\mathcal{G}_S$ the class of all graphs embeddable on S, G_n a uniformly random element of $(\mathcal{G}_S)_n$ and F_n the isomorphism type of* $\mathrm{Frag}(G_n)$*, the distribution* *of F_n for $n \to \infty$ converges in total variation to a distribution which assigns probability zero to any non-planar graph and which on planar graphs equals the Boltzmann–Poisson measure* $\mathsf{P}_{\mathrm{BP}\mathcal{P}}$ *of Theorem 81 with $\mathcal{A} := \mathcal{P}$.*

Without having to assume smoothness, McDiarmid [127] proved an analogue of Theorem 85 for graphs on surfaces:

Theorem 135 (McDiarmid; [127, Theorem 2.3])**.** *If S is a fixed surface and $\mathcal{G}_S$ the class of all graphs embeddable on S, there exists $\alpha > 0$ such that for any fixed connected rooted* planar *graph (H, u), a uniformly random element of the set of all n-vertex graphs in $\mathcal{G}_S$ a.a.s. contains $> \alpha n$ pendant copies of (H, u) with the bridge in each pendant appearance having u as one of its ends.*

Analogously to Corollary 88 one can deduce a version of Theorem 135 for uniformly random elements of the set of *connected* graphs on S:

Corollary 136. *Theorem 135 holds with 'of the set of all n-vertex graphs in $\mathcal{G}_S$' replaced by 'of the set of all connected n-vertex graphs in $\mathcal{G}_S$'.*

A proof of the following consequence of Theorem 135 can be proved analogously to Lemmma 86:

Corollary 137. *If S is a fixed surface and $\mathcal{G}_S$ the class of all graphs embeddable on S, there exists $\alpha > 0$ such that for any fixed connected rooted* planar *graph (H, u), for a uniformly random element G of the set of all n-vertex graphs in $\mathcal{G}_S$ a.a.s.* $\mathrm{Big}(G)$ *contains $> \alpha n$ pendant copies of (H, u) with the bridge in each pendant appearance having the respective copy's root as one of its ends.* □

We can now prove Theorem 30 from Chapter 1:

Proof of Theorem 30. Let S and φ be given as stated. Set $r := \mathrm{qr}(\varphi)$, and let $\mathrm{Y}_{r,\mathcal{P}}$ and R be as provided by Lemma 228 in Chapter 5 on input r and $\mathcal{A} := \mathcal{P}$, the (addable) class of planar graphs. It follows from Corollary 136 with $H := \mathrm{Y}_{r,\mathcal{P}}$ that a uniformly random element G_n of the set of *connected* elements of $(\mathcal{G}_S)_n$ for $n \to \infty$ a.a.s. satisfies Lemma 228.(lo.1), and it follows from results in [33] that it a.a.s. satisfies Lemma 228.(lo.2) with $\mathcal{A} = \mathcal{P}$. Therefore, by Lemma 228, a.a.s. $G_n \equiv_r^{\mathrm{FO}} \mathrm{Y}_{r,\mathcal{P}}$, hence the ratio in Theorem 30 indeed converges, with limit 0 if and only if *not* $\mathrm{Y}_{r,\mathcal{P}} \models \varphi$, and limit 1 if and only if $\mathrm{Y}_{r,\mathcal{P}} \models \varphi$. □

We finally come to Theorem 138, which is the more detailed version of Theorem 29 announced in Chapter 1; in spite of its non-addability, each of the graph classes $\mathcal{G}_S$ from Theorem 134 admits of the following analogue of Theorem 87.

Theorem 138 (as far as their probability limits are concerned, FO-statements about graphs on a fixed surface cannot do more than describe small planar components; joint work with T. Müller, M. Noy and A. Taraz)**.** *If S denotes some fixed surface and $\mathcal{G}_S$ the class of all graphs embeddable on S. Then for every FO-sentence φ there exists a set $\mathcal{F}(\varphi) \subseteq \mathcal{UP}$ of isomorphism-types of planar graphs such that*

$$\lim_{n\to\infty} \frac{|\{G \in (\mathcal{G}_S)_n : G \models \varphi\}|}{|(\mathcal{G}_S)_n|} = \mathsf{P}_{\mathrm{BP}}[\mathcal{F}(\varphi)] \ . \tag{3.198}$$

The set $\mathcal{F}(\varphi) \subseteq \mathcal{UP}$ does not depend on the surface S.

The main idea of our proof of Theorem 138 is to 'focus' the trivial rewriting $\frac{|\{G\in(\mathcal{G}_S)_n :\, G\models\varphi\}|}{|(\mathcal{G}_S)_n|} = \frac{|\{G\in(\mathcal{G}_S)_n :\, \mathrm{Big}(G)\sqcup\mathrm{Frag}(G)\models\varphi\}|}{|(\mathcal{G}_S)_n|}$ onto the non-giant part of the random graph, by—without changing the limit—replacing the random variable $\mathrm{Big}(G)$ with a *fixed* graph which does not depend on n, only on φ, but which is $\equiv_{\mathrm{qr}(\varphi)}^{\mathrm{MSO}}$-equivalent to $\mathrm{Big}(G)$. This is made possible by the substructure $\mathrm{Y}_{r,\mathcal{P}}$ from Lemma 228 in Chapter 5.

Proof of Theorem 138. Let S and φ be given as stated. Set $r := \mathrm{qr}(\varphi)$. Let $\mathrm{Y}_{r,\mathcal{P}}$ and R be as provided by Lemma 228 in Chapter 5, on input r and $\mathcal{A} := \mathcal{P}$, the (addable) class of planar graphs. By Corollary 137, for a uniformly random $G \in (\mathcal{G}_S)_n$ a.a.s. $\mathrm{Big}(G)$ contains a pendant copy of $\mathrm{Y}_{r,\mathcal{P}}$, i.e. $\mathrm{Big}(G)$ a.a.s. satisfies (lo.1) in Lemma 228. Moreover, it follows from results of [33] that a.a.s. every ball of radius R around any vertex of a connected $G \in (\mathcal{G}_S)_n$ is planar, i.e., $\mathrm{Big}(G)$ a.a.s. satisfies (lo.2) in Lemma 228 with $\mathcal{A} := \mathcal{P}$. So it follows from Lemma 228 that

$$\lim_{n\to\infty} \frac{|\{G \in (\mathcal{G}_S)_n \colon \mathrm{Big}(G) \equiv_r^{\mathrm{FO}} \mathrm{Y}_{r,\mathcal{P}}\}|}{|(\mathcal{G}_S)_n|} = 1 \ . \tag{3.199}$$

We now define

$$\mathcal{F}'(\varphi) := \text{set of isomorphism-types among } \textit{all} \text{ labelled graphs } H \text{ with } \mathrm{Y}_{r,\mathcal{P}} \sqcup H \models \varphi \ , \tag{3.200}$$

and, with $\mathcal{P}$ the set of planar graphs we set (not that this set does not depend on the surface S since $\mathrm{Y}_{r,\mathcal{P}}$ does not):

$$\mathcal{F}(\varphi) := \mathcal{F}'(\varphi) \cap \mathcal{P} \ . \tag{3.201}$$

With the abbreviation '$G\colon$ ' for '$G \in (\mathcal{G}_S)_n\colon$ ', we have

$$\begin{aligned}
&\{G\colon \mathrm{Big}(G) \sqcup \mathrm{Frag}(G) \models \varphi\} \cap \{G\colon \mathrm{Big}(G) \equiv_r^{\mathrm{FO}} \mathrm{Y}_{r,\mathcal{P}}\}\\
=&\{G\colon \mathrm{Big}(G) \sqcup \mathrm{Frag}(G) \models \varphi,\ \mathrm{Big}(G) \equiv_r^{\mathrm{FO}} \mathrm{Y}_{r,\mathcal{P}},\ \mathrm{Big}(G) \sqcup \mathrm{Frag}(G) \equiv_r^{\mathrm{FO}} \mathrm{Y}_{r,\mathcal{P}} \sqcup \mathrm{Frag}(G)\}\\
=&\{G\colon \mathrm{Y}_{r,\mathcal{P}} \sqcup \mathrm{Frag}(G) \models \varphi\} \cap \{G\colon \mathrm{Big}(G) \equiv_r^{\mathrm{FO}} \mathrm{Y}_{r,\mathcal{P}}\} \ .
\end{aligned} \tag{3.202}$$

where, in the first (resp. second) equality the inclusion $\supseteq$ (resp. $\subseteq$) is trivial while $\subseteq$ (resp. $\supseteq$) follows from a standard fact about $\equiv_r^{\mathrm{FO}}$ and disjoint unions ([31, Lemma 6.20 (d)]).

We can now calculate

$$\begin{aligned}
\lim_{n\to\infty} \frac{|\{G\colon G \models \varphi\}|}{|(\mathcal{G}_S)_n|} &= \lim_{n\to\infty} \frac{|\{G\colon \mathrm{Big}(G) \sqcup \mathrm{Frag}(G) \models \varphi\}|}{|(\mathcal{G}_S)_n|}\\
&= \lim_{n\to\infty} \frac{|\{G\colon \mathrm{Big}(G) \sqcup \mathrm{Frag}(G) \models \varphi\} \cap \{G\colon \mathrm{Big}(G) \equiv_r^{\mathrm{FO}} \mathrm{Y}_{r,\mathcal{P}}\}|}{|(\mathcal{G}_S)_n|}\\
&\quad + \lim_{n\to\infty} \frac{|\{G\colon \mathrm{Big}(G) \sqcup \mathrm{Frag}(G) \models \varphi\} \cap \{G\colon \mathrm{Big}(G) \not\equiv_r^{\mathrm{FO}} \mathrm{Y}_{r,\mathcal{P}}\}|}{|(\mathcal{G}_S)_n|}\\
\text{(because of (3.199))} &= \lim_{n\to\infty} \frac{|\{G\colon \mathrm{Big}(G) \sqcup \mathrm{Frag}(G) \models \varphi\} \cap \{G\colon \mathrm{Big}(G) \equiv_r^{\mathrm{FO}} \mathrm{Y}_{r,\mathcal{P}}\}|}{|(\mathcal{G}_S)_n|}\\
\text{(because of (3.202))} &= \lim_{n\to\infty} \frac{|\{G\colon \mathrm{Y}_{r,\mathcal{P}} \sqcup \mathrm{Frag}(G) \models \varphi\} \cap \{G\colon \mathrm{Big}(G) \equiv_r^{\mathrm{FO}} \mathrm{Y}_{r,\mathcal{P}}\}|}{|(\mathcal{G}_S)_n|}\\
\text{(because of (3.199))} &= \lim_{n\to\infty} \frac{|\{G\colon \mathrm{Y}_{r,\mathcal{P}} \sqcup \mathrm{Frag}(G) \models \varphi\} \cap \{G\colon \mathrm{Big}(G) \equiv_r^{\mathrm{FO}} \mathrm{Y}_{r,\mathcal{P}}\}|}{|(\mathcal{G}_S)_n|}\\
&\quad + \lim_{n\to\infty} \frac{|\{G\colon \mathrm{Y}_{r,\mathcal{P}} \sqcup \mathrm{Frag}(G) \models \varphi\} \cap \{G\colon \mathrm{Big}(G) \not\equiv_r^{\mathrm{FO}} \mathrm{Y}_{r,\mathcal{P}}\}|}{|(\mathcal{G}_S)_n|}\\
&= \lim_{n\to\infty} \frac{|\{G \in (\mathcal{G}_S)_n\colon \mathrm{Y}_{r,\mathcal{P}} \sqcup \mathrm{Frag}(G) \models \varphi\}|}{|(\mathcal{G}_S)_n|}\\
&= \lim_{n\to\infty} \frac{|\{G \in (\mathcal{G}_S)_n\colon \mathrm{Frag}(G) \in \mathcal{F}'(\varphi)\}|}{|(\mathcal{G}_S)_n|}\\
\begin{pmatrix}\text{by Theorem 134, since conver-}\\ \text{gence in total variation implies}\\ \text{convergence in distribution}\end{pmatrix} &= \mathsf{P}_{\mathrm{BP}\mathcal{P}}[\mathcal{F}'(\varphi) \cap \mathcal{P}] = \mathsf{P}_{\mathrm{BP}\mathcal{P}}[\mathcal{F}(\varphi)] \ ,
\end{aligned} \tag{3.203}$$

completing the proof. ∎

4 Sign matrices

constrain ▸ verb [with obj.] compel or force (someone) to follow a particular course of action: [...] ORIGIN Middle English: from Old French *constraindre*, from Latin *constringere* 'bind tightly together'.

Oxford Dictionary of English. Second Edition 2003.
ISBN 0-19-8613474. p. 371

This chapter contains proofs for the results introduced in Section 1.4 of Chapter 1.

4.1 Definitions specific to this chapter

Let $\mathsf{P}[\cdot]$ denote the uniform measure on $\{\pm\}^{[n]^2}$, (so that in particular Conjecture 32 in Chapter 1 reads $\mathsf{P}[\{A \in \{\pm\}^{[n]^2} : \det(A) = 0\}] \sim (\frac{1}{2} + o(1))^n$). We now introduce one of the two main protagonists of Chapter 4:

Definition 139 (the measure $\mathsf{P}_{\mathrm{lcf}}$). *For $(s,t) \in \mathbb{Z}^2_{\geqslant 2}$ and $\varnothing \subseteq I \subseteq [s-1] \times [t-1]$ let $\mathsf{P}_{\mathrm{lcf}}$ denote the* lazy coin flip measure *on $\{0,\pm\}^I$, i.e. the probability measure on $\{0,\pm\}^I$ defined by considering the values of a $B \in \{0,\pm\}^I$ as independent identically distributed random variables, each governed by the symmetric discrete distribution with values -1, 0, $+1$ and probabilities $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{4}$.*

The name of $\mathsf{P}_{\mathrm{lcf}}$ may stem from the fact that this is the distribution obtained when someone sets out to generate the entries of some $B \in \{0,\pm\}^I$ by performing $|I|$ independent fair coin flips, but there is a probability of $\frac{1}{2}$ at every single trial that out of a fleeting laziness the person decides to simply write 0 instead of flipping the coin.

The lazy coin flip measure $\mathsf{P}_{\mathrm{lcf}}$ plays a role in the recent article [26] of J. Bourgain, V. H. Vu and P. M. Wood in which the authors set the current record in a chain of improvements of upper bounds for $\mathsf{P}\,[\,\{\,A \in \{\pm\}^{[n]^2} :\ A \text{ singular}\,\}\,]$ (see Komlós [101], Kahn–Komlós–Szemerédi [94] and Tao–Vu [156] [157]):

Theorem 140 (Bourgain–Vu–Wood [26]). *For $n \to \infty$ it is true that*

$$\mathsf{P}[\{A \in \{\pm\}^{[n]^2} : \det(A) = 0\}] \leqslant (\frac{1}{\sqrt{2}} + o(1))^n \ , \tag{4.1}$$

$$\mathsf{P}_{\mathrm{lcf}}[\{B \in \{0,\pm\}^{[n]^2} : \det(B) = 0\}] \sim (\frac{1}{2} + o(1))^n \ . \tag{4.2}$$

Comments. The upper bound within $\sim$ in (4.2) is the special case $\mu = \frac{1}{2}$ of [26, Corollary 3.1, p. 567]. The lower bound within the $\sim$ is obvious: consider the event that the first column has only zero entries (the lower bound is also explicitly stated in [26, formula (7), p. 561]). The upper bound in (4.1) is the special case $S = \{\pm\}$ and $p = \frac{1}{2}$ in [26, Corollary 4.3, p. 576]. □

So Bourgain–Vu–Wood proved that the correct order of decay of $\mathsf{P}_{\mathrm{lcf}}[\{B \in \{0,\pm\}^{[n]^2} : \det(B) = 0\}]$ is $(\frac{1}{2} + o(1))^n$—which is also the conjectured one for $\mathsf{P}[\{A \in \{\pm\}^{[n]^2} : \det(A) = 0\}]$. It is this latter achievement, combined with an observation made by the present author, which motivates the approach of Chapter 4. Note that using the uniform distribution on $\{\pm\}^{[n]^2}$ is equivalent to considering the n^2 entries as i.i.d. Bernoulli variables with probability $\frac{1}{2}$. The observation is this:

When we apply one step of Chio condensation (see Definition 143) to this Bernoulli matrix, the result is a matrix whose entries are 3-wise (and 'almost' 6-wise, see Theorem 187 below) stochastically independent with $\{-2, 0, +2\}$-values which are distributed as if by the lazy coin flip measure. Since Bourgain–Vu–Wood demonstrated that for $\mathsf{P}_{\mathrm{lcf}}$-distributed entries an asymptotically correct order of decay can be proved, the observation feels like a hint at deeper connections and makes it seem imperative to investigate Chio condensation of sign matrices.
In this section we provide some tools to bind together matrices $A \in \{\pm\}^{[s]\times[t]}$, matrices $B \in \{0, \pm\}^{[s-1]\times[t-1]}$ and signed graphs.

Definition 141 (Chio[1] set)**.** *Let $(s,t) \in \mathbb{Z}^2_{\geqslant 2}$ and $I \subseteq [s] \times [t]$. Then I is called a* Chio set *if and only if $(s,t) \in I$ and for every $(i,j) \in I$ we have $(i,t) \in I$ and $(s,j) \in I$.*

Definition 142 (Chio extension[2])**.** *For every $(s,t) \in \mathbb{Z}^2_{\geqslant 2}$ and every $\varnothing \subseteq I \subseteq [s-1] \times [t-1]$,*

$$\breve{I} := \{(s,t)\} \sqcup \bigcup_{i \in \mathrm{p}_1(I)} \{(i,t)\} \sqcup \bigcup_{j \in \mathrm{p}_2(I)} \{(s,j)\} \sqcup I \quad . \tag{4.3}$$

Note that $\breve{I} \subseteq [s] \times [t]$ for every $\varnothing \subseteq I \subseteq [s-1] \times [t-1]$, in particular $\breve{\varnothing} = \{(s,t)\}$ and $([s-1] \times [t-1])\breve{} = [s] \times [t]$. Moreover, a set $I' \subseteq [s] \times [t]$ is a Chio set if and only if there exist an $I \subseteq [s-1] \times [t-1]$ with $I' = \breve{I}$.

Now we come to Chio condensation. In the special (and very common) case of $s = t$ (hence $[s] \times [t] = [n]^2$) the following definition differs from the standard convention (as is to be found in [29] and [55]) in that the entry $a_{n,n}$ instead of $a_{1,1}$ is taken to be the pivot. This seems to be more convenient for handling the indices of a Chio-condensate. There is no logical necessity for multiplying by $\frac{1}{2}$, but the author decided to keep the discussion within the realm of $\{0, \pm\}$-matrices (instead of $\{-2, 0, +2\}$-matrices).

Definition 143 (Chio condensation, $\frac{1}{2}\mathrm{C}^{\breve{I}}_{(s,t)}$)**.** *For every $(s,t) \in \mathbb{Z}^2_{\geqslant 2}$, and every $I \subseteq [s-1] \times [t-1]$ define the* Chio map with pivot $a_{s,t}$ *as*

$$\tfrac{1}{2}\mathrm{C}^{\breve{I}}_{(s,t)} \colon \{\pm\}^{\breve{I}} \longrightarrow \{0, \pm\}^{I} \ , \qquad A \longmapsto \tfrac{1}{2} \cdot \mathrm{C}_{(s,t)}(A) \ , \tag{4.4}$$

where $\mathrm{C}_{(s,t)}(A) := \big(\det\big(\begin{smallmatrix} a_{i,j} & a_{i,t} \\ a_{s,j} & a_{s,t} \end{smallmatrix}\big)\big)_{(i,j)\in I} \in \{-2, 0, +2\}^I$. An image $\mathrm{C}_{(s,t)}(A)$ of some $A \in \{\pm\}^{\breve{I}}$ is referred to as the Chio-condensate of *A.*

Definition 144 (the Chio measure $\mathsf{P}_{\mathrm{chio}}$)**.** *For every $(s,t) \in \mathbb{Z}^2_{\geqslant 2}$ and every $I \subseteq [s-1] \times [t-1]$,*

$$\mathsf{P}_{\mathrm{chio}} \colon \mathfrak{P}\big(\{0,\pm\}^I\big) \longrightarrow [0,1] \ , \qquad \mathcal{B} \longmapsto \frac{|(\frac{1}{2}\mathrm{C}^{\breve{I}}_{(s,t)})^{-1}(\mathcal{B})|}{\left|\{\pm\}^{\breve{I}}\right|} = \frac{1}{2^{|\breve{I}|}} \sum_{B \in \mathcal{B}} |(\tfrac{1}{2}\mathrm{C}^{\breve{I}}_{(s,t)})^{-1}(B)| \ . \tag{4.5}$$

Note that in the special case of $s := t := n$ and $I := [n-1]^2$, the measure $\mathsf{P}_{\mathrm{chio}}$ maps a single $B \in \{0, \pm\}^{[n-1]^2}$ to $\mathsf{P}_{\mathrm{chio}}[B] := \mathsf{P}_{\mathrm{chio}}[\{B\}] = 2^{-n^2} \cdot \big|\{A \in \{\pm\}^{[n]^2} : B = \frac{1}{2} \cdot \mathrm{C}_{(n,n)}(A)\}\big|$.

We now define two additional measures. Later we will recognise both of them as familiar ones:

[1] All three spellings Chiò, 'Chio' and 'Chió' are to be found in the literature. In [29] the authors consistently use the spelling 'Chiò', and it is said there [29, p. 790] that a copy of Chio's original paper had been at the authors' disposal. However, an original 1853 copy of [37] which the present author bought from an antiquarian bookstore in Asti, Italy gives strong circumstantial evidence in favour of the spelling 'Chio': on the title page and the inside-cover the given name 'Félix' *is* written with an accent whereas 'Chio' does not carry any accent. Moreover, the title page bears a hand-written dedication to a colleague, signed 'L'autore'. Therefore, to all appearances, Chio signed this title page, with that spelling, more than 150 years ago (though not with his last name, just with 'L'autore'). The spelling is further corroborated by [32]. There, Cauchy on several occasions consistently just writes 'M. Félix Chio', cf. [32, p. 110, pp. 112–113].

[2] Due to the spelling explained in the previous footnote, and since using the letter 'c' in its usual orientation would remind one of complementation, it was decided to use the breve accent ˘ to denote Chio extension of a set.

Definition 145 (averaged Chio measure)**.** *For every* $\varnothing \subseteq I \subseteq [n-1]^2$ *define*

$$\overline{\mathsf{P}}_{\text{chio}} \colon \mathfrak{P}(\{0,\pm\}^I) \longrightarrow [0,1] \ , \qquad \mathcal{B} \longmapsto \sum_{B \in \mathcal{B}} \frac{1}{2^{\text{supp}(B)}} \sum_{\tilde{B} \in \{0,\pm\}^I \colon \text{Supp}(\tilde{B}) = \text{Supp}(B)} \mathsf{P}_{\text{chio}}[\tilde{B}] \ . \tag{4.6}$$

Definition 146 ($|\frac{1}{2}\mathrm{C}^{\check{I}}_{(s,t)}(\cdot)|$ and $\mathsf{P}^{|\cdot|,I}_{\text{chio}}$)**.** *For every* $(s,t) \in \mathbb{Z}^2_{\geqslant 2}$ *and every* $I \subseteq [s-1] \times [t-1]$ *define a map*

$$|\tfrac{1}{2}\mathrm{C}^{\check{I}}_{(s,t)}| \colon \{\pm\}^{\check{I}} \longrightarrow \{0,1\}^I \ , \qquad A \longmapsto \tfrac{1}{2} \cdot |\mathrm{C}_{(s,t)}(A)| \ , \tag{4.7}$$

where $|\mathrm{C}_{(s,t)}(A)| := \big(|\det\left(\begin{smallmatrix} a_{i,j} & a_{i,t} \\ a_{s,j} & a_{s,t}\end{smallmatrix}\right)|\big)_{(i,j) \in I} \in \{0,2\}^I$*. Furthermore, define*

$$\mathsf{P}^{|\cdot|,I}_{\text{chio}} \colon \mathfrak{P}(\{0,1\}^I) \longrightarrow [0,1] \cap \mathbb{Q} \ , \qquad \mathcal{B} \longmapsto \frac{1}{2^{|\check{I}|}} \sum_{B \in \mathcal{B}} \big|(|\tfrac{1}{2}\mathrm{C}^{\check{I}}_{(s,t)}|)^{-1}(B)\big| \ . \tag{4.8}$$

Definition 147 (the entry-specification-events $\mathcal{E}^J_B$)**.** *For* $(s,t) \in \mathbb{Z}^2_{\geqslant 2}$*,* $\varnothing \subseteq I \subseteq J \subseteq [s-1] \times [t-1]$*, and* $B \in \{0,\pm\}^I$ *let* $\mathcal{E}^J_B := \big\{\tilde{B} \in \{0,\pm\}^J \colon \tilde{B}\mid_{\text{Dom}(B)} = B\big\}$*.*

Note that $\text{Dom}(B) = I \subseteq J = \text{Dom}(\tilde{B})$, hence $\tilde{B}\mid_{\text{Dom}(B)}$ is defined. If $\text{Dom}(B) = \varnothing$, i.e. $B = \varnothing$, then $\mathcal{E}^J_\varnothing = \{0,\pm\}^J$, and if $\text{Dom}(B) = J$, then $\mathcal{E}^J_B = \{B\}$. Furthermore, $|\mathcal{E}^J_B| = 3^{|J|-|I|}$ for arbitrary $\varnothing \subseteq I \subseteq J \subseteq [s-1] \times [t-1]$ and $B \in \{0,\pm\}^I$.

We will make use of the language of graph-theory. For the sake of specificity and ease of reference, we will explicitly name the set of auxiliary labelled bipartite graphs that we will talk about (and give it a vertex set which blends well with the matrix setting).

Definition 148. *For every* $(s,t) \in \mathbb{Z}^2_{\geqslant 2}$ *denote by* $\mathrm{BG}_{s,t}$ *the* $2^{(s-1)\cdot(t-1)}$*-element set of all bipartite graphs* $X = (V_1 \sqcup V_2, E)$ *with* $V_1 = \{(i,t) \colon 1 \leqslant i \leqslant s-1\}$ *and* $V_2 = \{(s,j) \colon 1 \leqslant j \leqslant t-1\}$*.*

There is an obvious bijection $\mathrm{BG}_{s,t} \longleftrightarrow \{0,1\}^{[s-1]\times[t-1]}$. Associating with a (partially specified) $\{0,\pm\}$-matrix the following bipartite signed graph will be helpful in our study of P_{chio}. The map $\mathrm{X}^{k,s,t}$ interprets any $B \in \{0,\pm\}^I$ as a bipartite adjacency matrix in the natural way (while ignoring the signs), and that σ takes the signs in B as a definition of a sign function.

Definition 149 (X and σ)**.** *For every* $(s,t) \in \mathbb{Z}^2_{\geqslant 2}$ *and every* $0 \leqslant k \leqslant (s-1)(t-1)$ *define*

$$\mathrm{X}^{k,s,t} \colon \bigsqcup_{I \in \binom{[s-1]\times[t-1]}{k}} \{0,\pm\}^I \longrightarrow \mathrm{BG}_{s,t}, \quad B \longmapsto \mathrm{X}^{k,s,t}_B \tag{4.9}$$

by letting vertex-set and edge-set be defined as

$$\mathrm{V}(\mathrm{X}^{k,s,t}_B) := (\text{Dom}(B))\check{} \setminus \text{Dom}(B) \setminus \{(s,t)\} = \bigcup_{i \in \mathrm{p}_1(\text{Dom}(B))} \{(i,t)\} \sqcup \bigcup_{j \in \mathrm{p}_2(\text{Dom}(B))} \{(s,j)\} \ , \tag{4.10}$$

$$\mathrm{E}(\mathrm{X}^{k,s,t}_B) := \bigsqcup_{(i,j) \in \text{Supp}(B)} \big\{\{(i,t),(s,j)\}\big\} \ . \tag{4.11}$$

Define $\sigma_B \colon \mathrm{E}(\mathrm{X}_B) \to \{\pm\}$ *by* $\sigma_B(\{(i,t),(s,j)\}) := b_{i,j} \in \{\pm\}$ *for every* $\{(i,t),(s,j)\} \in \mathrm{E}(\mathrm{X}_B)$*.*

If $k = 0$, hence $I = \varnothing$, hence $B = \varnothing$ is the empty matrix, then $\mathrm{X}^{k,s,t}_B$ is the empty graph $(\varnothing,\varnothing)$ and $\sigma_B = \varnothing$ is the empty function. Note that while for a $B \in \{0,\pm\}^I$ the set $\mathrm{V}(\mathrm{X}_B)$ depends only on $I = \text{Dom}(B)$, the set $\mathrm{E}(\mathrm{X}_B)$ depends on $\text{Supp}(B)$ and σ_B even depends on B itself.

When we take the *image* of a matrix $B \in \{0,\pm\}^I$ under $\mathrm{X}^{k,s,t}$, then usually we will know what $I \in \binom{[s-1]\times[t-1]}{k}$ we are talking about and then the superscripts k,s,t give redundant information. Whenever possible we will suppress the superscripts in such a situation and only write X_B. When we take *preimages* of a graph $X \in \mathrm{BG}_{s,t}$ under $\mathrm{X}^{k,s,t}$, however, the full notation has to be used since in general it is not possible to tell k from the labelled graph X (let alone from its isomorphism type). As an example, consider the graph $X \in \mathrm{BG}_{4,4}$ with vertex set $\{(1,4),(2,4),(3,4)\} \sqcup \{(4,1),(4,2),(4,3)\}$ and edge set $\{\{(1,4),(4,1)\},\{(2,4),(4,1)\},\{(2,4),(4,2)\},\{(1,4),(4,2)\}\}$, which is isomorphic to a 4-circuit with two additional isolated vertices. Then we have $\mathrm{X}^{5,4,4}_{B_1} = \mathrm{X}^{6,4,4}_{B_2} = X$ for $B_1 := \left(\begin{smallmatrix} 1 & 1 & \\ 1 & 1 & \\ & & 0 \end{smallmatrix}\right) \neq B_2 := \left(\begin{smallmatrix} 1 & 1 & \\ 1 & 1 & 0 \\ & 0 & \end{smallmatrix}\right)$. Here, $\mathrm{dom}(B_1) = 5 \neq 6 = \mathrm{dom}(B_2)$.

Definition 150 (ul, β_1^{ul})**.** *Let* ul *be the map which assigns a labelled graph to its isomorphism type. Let* $\beta_1^{\mathrm{ul}} \colon \mathrm{ul}(\mathrm{BG}_{n,n}) \to \mathbb{Z}_{\geqslant 0}$ *be the map which assigns an unlabelled bipartite graph G to its* 1*-dimensional Betti number* $\|G\| - |G| + 1$.

Definition 151 (${}_{\mathrm{ul}}\mathrm{X}^{k,s,t}$)**.** *For* $(s,t) \in \mathbb{Z}^2_{\geqslant 2}$ *and* $0 \leqslant k \leqslant (s-1)(t-1)$ *let* ${}_{\mathrm{ul}}\mathrm{X}^{k,s,t} := \mathrm{ul} \circ \mathrm{X}^{k,s,t}$.

If $\mathfrak{X}$ is some (verbal, pictorial, ...) description of an isomorphism type of graphs, we can now take its preimage $({}_{\mathrm{ul}}\mathrm{X}^{k,s,t})^{-1}(\mathfrak{X}) \subseteq \bigsqcup_{I \in \binom{[s-1]\times[t-1]}{k}} \{0,\pm\}^I$. To analyse how $\mathsf{P}_{\mathrm{chio}}$ and $\mathsf{P}_{\mathrm{lcf}}$ relate to one another, it is useful to have the following notations.

Definition 152 (the failure sets)**.** *For every* $k \geqslant 0$, $n \geqslant 2$, $\ell \in \mathbb{Q}_{\geqslant 0}$ *and* $p \in [0,1] \cap \mathbb{Q}$ *let*

(1) $\mathcal{F}^{\mathrm{M}}(k,n) := \{\ B \in \{0,\pm\}^I : I \in \binom{[n-1]^2}{k},\ \mathsf{P}_{\mathrm{chio}}[\mathcal{E}_B^{[n-1]^2}] \neq \mathsf{P}_{\mathrm{lcf}}[\mathcal{E}_B^{[n-1]^2}]\ \}$,
(2) $\mathcal{F}^{\mathrm{M}}_{\cdot\ell}(k,n) := \{\ B \in \mathcal{F}^{\mathrm{M}}(k,n) \colon \mathsf{P}_{\mathrm{chio}}[\mathcal{E}_B^{[n-1]^2}] = \ell \cdot \mathsf{P}_{\mathrm{lcf}}[\mathcal{E}_B^{[n-1]^2}]\ \} \subseteq \mathcal{F}^{\mathrm{M}}(k,n)$,
(3) $\mathcal{F}^{\mathrm{M}}_{=p}(k,n) := \{\ B \in \mathcal{F}^{\mathrm{M}}(k,n) \colon \mathsf{P}_{\mathrm{chio}}[\mathcal{E}_B^{[n-1]^2}] = p\ \} \subseteq \mathcal{F}^{\mathrm{M}}(k,n)$,
(4) $\mathcal{F}^{\mathrm{G}}(k,n) := {}_{\mathrm{ul}}\mathrm{X}^{k,n,n}(\mathcal{F}^{\mathrm{M}}(k,n))$, $\mathcal{F}^{\mathrm{G}}_{\cdot\ell}(k,n) := {}_{\mathrm{ul}}\mathrm{X}^{k,n,n}(\mathcal{F}^{\mathrm{M}}_{\cdot\ell}(k,n))$, *and* $\mathcal{F}^{\mathrm{G}}_{=p}(k,n) := {}_{\mathrm{ul}}\mathrm{X}^{k,n,n}(\mathcal{F}^{\mathrm{M}}_{=p}(k,n))$.

We abbreviate $\mathcal{F}^{\mathrm{M}}(k,n) := \mathcal{F}^{\mathrm{M}}(k,n,n)$*, and analogously for all the other sets just defined.*

Obviously, $\mathcal{F}^{\mathrm{M}}_{\cdot 1}(k,n) = \varnothing$ and $\mathcal{F}^{\mathrm{M}}_{\cdot 0}(k,n) = \mathcal{F}^{\mathrm{M}}_{=0}(k,n)$ for all k and n. Item (C3) in Theorem 167 will teach us that $\mathcal{F}^{\mathrm{M}}_{\cdot\ell}(k,n) = \varnothing$ for every $\ell \notin \{0\} \sqcup \{2^i \colon i \in \mathbb{Z}_{\geqslant 0}\}$ (hence in particular $\mathsf{P}_{\mathrm{chio}}[\mathcal{E}_B^{[n-1]^2}] \geqslant \mathsf{P}_{\mathrm{lcf}}[\mathcal{E}_B^{[n-1]^2}]$ for every $B \in \mathcal{F}^{\mathrm{M}}(k,n)$ with $\mathsf{P}_{\mathrm{chio}}[\mathcal{E}_B^{[n-1]^2}] > 0$).

Definition 153 (matrix-circuit, $\mathrm{Cir}(s,n)$)**.** *For every* $(s,t) \in \mathbb{Z}^2_{\geqslant 2}$ *and every* $L \subseteq [s-1] \times [t-1]$ *with even* $l := |L|$*, the set L is called a* matrix-l-circuit *if and only if* $\mathrm{X}_{\{1\}^L}$ *is a graph-theoretical* l*-circuit. Moreover,* $\mathrm{Cir}(l,s,t) := \{L \subseteq [s-1] \times [t-1] \colon |L| = l,\ L$ *is a matrix-l-circuit*$\}$ *and* $\mathrm{Cir}(l,n) := \mathrm{Cir}(l,n,n)$.

Definition 154 (($-$)-constant, ($+$)-proper vertex 2-colouring of a signed graph)**.** *For a graph* $X = (V,E)$ *and* $\sigma \in \{\pm\}^E$*, a function* $\mathrm{c} \in \{\pm\}^V$ *is called* $(\sigma, -)$-constant, $(\sigma, +)$-proper *if and only if* $\mathrm{c}(u) = \mathrm{c}(v)$ *for every* $e = uv \in \mathrm{E}(X)$ *with* $\sigma(e) = -$ *and* $\mathrm{c}(u) \neq \mathrm{c}(v)$ *for every* $e = uv \in \mathrm{E}(X)$ *with* $\sigma(e) = +$.

Definition 155 ($\mathrm{Col}(X,\sigma)$)**.** *For a graph* $X = (V,E)$ *and* $\sigma \in \{\pm\}^E$ *let* $\mathrm{Col}(X,\sigma)$ *be the set of all* $(\sigma, -)$*-constant,* $(\sigma, +)$*-proper vertex-2-colourings* $\mathrm{c} \in \{\pm\}^V$.

Definition 156 (rank-level-sets of matrices)**.** *For* $(s,t) \in \mathbb{Z}^2_{\geqslant 2}$, $0 \leqslant r \leqslant \min(s,t)$*, R an integral domain,* $U \subseteq R$ *and* $\mathcal{R} \in \mathfrak{P}(\{0,1,\ldots,\min(s,t)\})$ *let* $\mathrm{Ra}_r(U^{[s]\times[t]}) := \{A \in U^{[s]\times[t]} \colon \mathrm{rk}(A) = r\}$, $\mathrm{Ra}_{\mathcal{R}}(\{\pm\}^{[s]\times[t]}) := \bigsqcup_{r \in \mathcal{R}} \mathrm{Ra}_r(\{\pm\}^{[s]\times[t]})$ *and* $\mathrm{Ra}_{<r}(U^{[s]\times[t]}) := \mathrm{Ra}_{\{0,1,\ldots,r-1\}}(\{\pm\}^{[s]\times[t]})$.

4.2 Understanding the Chio measure

We will use the following elementary fact:

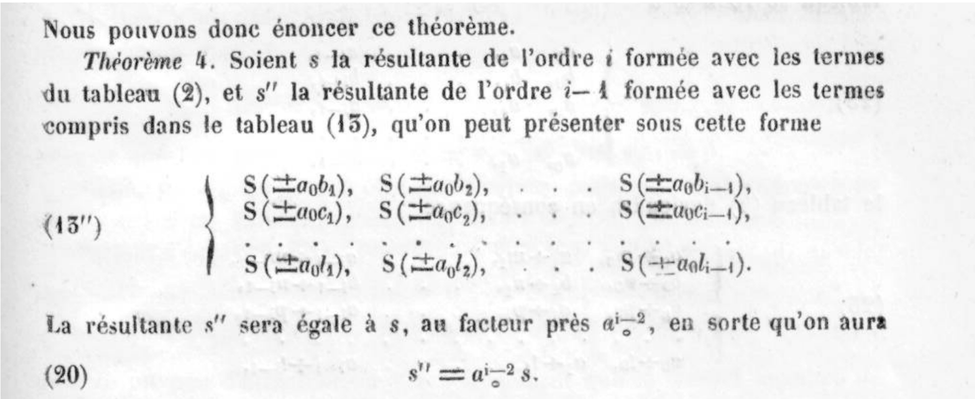
Lemma 157. $f^{-1}(f(f^{-1}(U))) = f^{-1}(U)$ *for any map* $f \colon A \to B$ *and any subset* $U \subseteq B$. $\square$

The following simple statement is essential for the approach developed in Chapter 4. More information on this identity can be found in [29, last paragraph of Section 9] and [5, Ch. 4, p. 282, Exerc. 43]. The formulation given here differs from those in [37, p. 11] and [29] in that $a_{n,n}$ instead of $a_{1,1}$ is taken to be the pivot. This seems more convenient for handling the indices of a Chio-condensate.

Lemma 158 (Chio's identity)**.** *For $n \geqslant 2$, R an integral domain and $(a_{i,j}) = A \in R^{[n]^2}$,*

$$\det\big(\mathrm{C}_{(n,n)}(A)\big) = a_{n,n}^{n-2} \cdot \det(A)\ . \tag{4.12}$$

Proof. This is stated by Chio in [37, p. 11, Théorème 4, equation (20)] and he proves it on pp. 6–11 (the notation '$\pm a_0 b_1$' employed in [37, equation (13")] is defined at the beginning of p. 6 of [37]). To contemporary eyes, this is an easy consequence of the behavior of determinants under linear transformations, cf. [55] for a direct proof of the version with pivot $a_{1,1}$. Moreover, this is a special case of *Sylvester's determinant identity.* To see this, set $k = 1$ in [29, equation (8)] to get a version of (4.12) with pivot $a_{1,1}$. Obvious modifications in the proof in [29] yield the version with pivot $a_{n,n}$. $\square$



Nous pouvons donc énoncer ce théorème.

Théorème 4. Soient s la résultante de l'ordre i formée avec les termes du tableau (2), et s'' la résultante de l'ordre $i-1$ formée avec les termes compris dans le tableau (13), qu'on peut présenter sous cette forme

$$(13'')\quad \left\{\begin{array}{llll} \mathrm{S}(\pm a_0 b_1), & \mathrm{S}(\pm a_0 b_2), & \ldots & \mathrm{S}(\pm a_0 b_{i-1}), \\ \mathrm{S}(\pm a_0 c_1), & \mathrm{S}(\pm a_0 c_2), & \ldots & \mathrm{S}(\pm a_0 c_{i-1}), \\ \ldots & \ldots & \ldots & \ldots \\ \mathrm{S}(\pm a_0 l_1), & \mathrm{S}(\pm a_0 l_2), & \ldots & \mathrm{S}(\pm a_0 l_{i-1}). \end{array}\right.$$

La résultante s'' sera égale à s, au facteur près a_0^{i-2}, en sorte qu'on aura

$$(20)\qquad s'' = a_0^{i-2}\, s.$$

Figure 4.1: M. F. Chio: *Mémoire sur les fonctions connues sous le nom De Résultantes ou de déterminans*, A. Pons et C., Turin 1853, p. 11.

The following three assertions are obviously true:

Corollary 159. *For every $A \in \{\pm\}^{[n]^2}$, $\det(A) = 0$ if and only if $\det(\frac{1}{2}\mathrm{C}_{(n,n)}(A)) = 0$.* $\square$

Lemma 160 (value of lazy coin flip measure on single matrix)**.** *For every $\varnothing \subseteq I \subseteq [n-1]^2$ and every $B \in \{0, \pm\}^I$, $\mathsf{P}_{\mathrm{lcf}}[B] = (\frac{1}{2})^{\mathrm{dom}(B)+\mathrm{supp}(B)}$.* $\square$

Lemma 161. *For any two disjoint graphs X_1 and X_2 and any two sign functions $\sigma_{X_1} \in \{\pm\}^{\mathrm{E}(X_1)}$ and $\sigma_{X_2} \in \{\pm\}^{\mathrm{E}(X_2)}$, and for every graph X obtained by a one-point wedge of X_1 and X_2 at two arbitrary vertices, the sign function $\sigma_X \in \{\pm\}^{\mathrm{E}(X)}$ obtained by uniting the maps σ_{X_1} and σ_{X_2} is balanced if and only if both (X_1, σ_{X_1}) and (X_2, σ_{X_2}) are balanced.* $\square$

The following will be needed for counting failures of equality of $\mathsf{P}_{\mathrm{chio}}$ and $\mathsf{P}_{\mathrm{lcf}}$.

Lemma 162. $|\mathrm{Cir}(2j, s, t)| = \binom{s-1}{j} \cdot \binom{t-1}{j} \cdot \frac{j!(j-1)!}{2}$ *for every $n \geqslant 2$ and every $1 \leqslant j \leqslant \min(\lfloor \frac{s}{2} \rfloor, \lfloor \frac{t}{2} \rfloor)$.*

Proof. For every $N' \in \binom{[s-1]}{j}$ and $N'' \in \binom{[t-1]}{j}$ let $\mathrm{Cir}(2j, s, t, N', N'') := \{S \in \mathrm{Cir}(2j, s, t) \colon \mathrm{p}_1(S) = N',\ \mathrm{p}_2(S) = N''\}$. Obviously, $|\mathrm{Cir}(2j, s, t)| = \sum_{N' \in \binom{[s-1]}{j}} \sum_{N'' \in \binom{[t-1]}{j}} |\mathrm{Cir}(2j, s, t, N', N'')|$. To count $\mathrm{Cir}(2j, s, t, N', N'')$, define $\mathrm{Perm}(N', N'') := \{\ P : P$ is a permutation matrix, $\mathrm{Supp}(P) \subseteq N' \times N''\ \}$. Define a binary relation $\mathfrak{R} \subseteq \mathrm{Cir}(2j, s, t, N', N'') \times \mathrm{Perm}(N', N'')$ by letting

$(S,P) \in \mathfrak{R}$ if and only if $\mathrm{Supp}(S) \supseteq \mathrm{Supp}(P)$. For every $S \in \mathrm{Cir}(2j,s,t,N',N'')$ we have $|\{P \in \mathrm{Perm}(N',N'') \colon (S,P) \in \mathfrak{R}\}| = 2$. On the other hand, for every $P \in \mathrm{Perm}(N',N'')$ we have $|\{S \in \mathrm{Cir}(2j,s,t,N',N'') \colon (S,P) \in \mathfrak{R}\}| = (j-1)!$. Therefore, $(j-1)! \cdot j! = (j-1)! \cdot |\mathrm{Perm}(N',N'')| = \sum_{P\in\mathrm{Perm}(N',N'')} |\{\ S \in \mathrm{Cir}(2j,s,t,N',N'') \colon\ (S,P) \in \mathfrak{R}\ \}| = \sum_{S\in\mathrm{Cir}(2j,s,t,N',N'')} |\{\ P \in \mathrm{Perm}(N',N'') \colon\ (S,P) \in \mathfrak{R}\ \}| = 2 \cdot |\mathrm{Cir}(2j,s,t,N',N'')|$. ☐

The following is contained in Kőnig's 1936 classic [104].

Lemma 163 (D. Kőnig)**.** *Let X be a labelled or unlabelled graph. Then:*

(Kő1) *$S_{\mathrm{bal}}(X) \neq \varnothing$ if and only if $\mathrm{Col}(X,\sigma) \neq \varnothing$,*
(Kő2) *Let $\sigma \colon \{\pm\}^E$ be arbitrary. Then $\mathrm{Col}(X,\sigma) \neq \varnothing$ if and only if $|\mathrm{Col}(X,\sigma)| = 2^{\beta_0(X)}$,*
(Kő3) *$|\{\sigma \in \{\pm\}^{\mathrm{E}(X)} \colon (X,\sigma)$ balanced$\}| = 2^{|X|-\beta_0(X)} = 2^{\text{number of edges in any spanning forest of } X}$.*

Proof. Modulo terminology a proof for (Kő1) can be found in [104, p. 152, Satz 11] (for the definition of 'p-Teilgraph' cf. [104, p. 149, Paragraph 3]). Statement (Kő1) is also proved in [75, Theorem 3]. Statement (Kő2) is implicit in the proof of [104, p. 152, Satz 14] and can easily be proved directly by induction on $|E|$. For a proof of (Kő3) cf. [104, p. 152, Satz 14]. ☐

While for a given graph $X = (V,E)$ and a given sign function $\sigma \colon E \to \{\pm\}$, the decision problem of whether (X,σ) balanced is trivially in co-NP, the less obvious fact that it is also in NP follows from (Kő1): any $(\sigma,-)$-constant, $(\sigma,+)$-proper vertex-2-colouring $\mathrm{c} \colon V \to \{\pm\}$ is a polynomially-sized certificate for (X,σ) being balanced. However, the problem is not only in the intersection of these two classes, but easily seen to be in P:

Corollary 164. *For every graph $X = (V,E)$ and every sign function $\sigma \colon E \to \{\pm\}$, the decision problem whether (X,σ) is balanced can be solved in time $O(|X| + \|X\|)$.*

Proof. By (Kő1), the question is equivalent to whether there exists a $(-)$-constant, $(+)$-proper vertex-2-colouring $\mathrm{c} \colon V \to \{\pm\}$. It is easy to see that an obvious greedy algorithm via a depth-first search on X succeeds in finding such a colouring if and only if such a colouring exists. Moreover, the algorithm requires time $O(|X| + \|X\|)$. ☐

The following simple lemma encapsulates a basic mechanism linking Chio condensation with the auxiliary graph-theoretical viewpoint. For want of topologies on source or target, 'k-fold covering' is nothing but shorthand for 'surjective map each of whose fibres has cardinality k'.

Lemma 165. *For every $(s,t) \in \mathbb{Z}^2_{\geqslant 2}$, arbitrary $\varnothing \subseteq I \subseteq J \subseteq [s-1] \times [t-1]$, every $B \in \{0,\pm\}^I$, and with $h(I,J) := |\check{J}| - |I| - |\mathrm{p}_1(I)| - |\mathrm{p}_2(I)| \in \mathbb{Z}_{\geqslant 1}$, there exists an $2^{h(I,J)}$-fold covering*

$$\Phi \colon (\tfrac{1}{2}\mathrm{C}^{\check{J}}_{(s,t)})^{-1}(\mathcal{E}^J_B) \longrightarrow \mathrm{Col}(\mathrm{X}_B,\sigma_B)\ . \tag{4.13}$$

Proof. Let s, t, I, J and $B = (b_{i,j})_{(i,j)\in I}$ be given as stated. The claim $h(I,J) \in \mathbb{Z}_{\geqslant 1}$ is true since Definition 142 implies $|\check{J}| = 1 + |\mathrm{p}_1(J)| + |\mathrm{p}_2(J)| + |J|$ and because $J \supseteq I$ implies $|J| \geqslant |I|$, $|\mathrm{p}_1(J)| \geqslant |\mathrm{p}_1(I)|$ and $|\mathrm{p}_2(J)| \geqslant |\mathrm{p}_2(I)|$.

If $\mathrm{Col}(\mathrm{X}_B,\sigma_B) = \varnothing$, the statement of the lemma is vacuously true (there not being any point of the target for which the condition for being such a covering would have to hold). We therefore can assume that $\mathrm{Col}(\mathrm{X}_B,\sigma_B) \neq \varnothing$. We now first show that this implies $(\frac{1}{2}\mathrm{C}^{\check{J}}_{(s,t)})^{-1}(\mathcal{E}^J_B) \neq \varnothing$. Then we will construct a cover of the stated kind.

To prove $(\frac{1}{2}\mathrm{C}^{\check{J}}_{(s,t)})^{-1}(\mathcal{E}^J_B) \neq \varnothing$, choose an arbitrary $\mathrm{c} \in \mathrm{Col}(\mathrm{X}_B,\sigma_B)$ and define $A = (a_{i,j}) \in \{\pm\}^{\check{J}}$ by $a_{s,t} := +$, $a_{i,t} := \mathrm{c}((i,t))$ for every $i \in \mathrm{p}_1(J)$, $a_{s,j} := \mathrm{c}((s,j))$ for every $j \in \mathrm{p}_2(J)$. Moreover, for every $(i,j) \in J$ let $a_{i,j} := b_{i,j}$ if $b_{i,j} \neq 0$ and $a_{i,j} := \mathrm{c}((i,t)) \cdot \mathrm{c}((s,j))$ if $b_{i,j} = 0$. We now show that $\frac{1}{2}\mathrm{C}^{\check{J}}_{(s,t)}(A) \mid_{\mathrm{Dom}(B)} = B$. Let $(i,j) \in I = \mathrm{Dom}(B)$ be arbitrary. If $b_{i,j} = 0$, then $\frac{1}{2}\mathrm{C}^{\check{J}}_{(s,t)}(A)[i,j] = \frac{1}{2}(a_{i,j}a_{s,t} - a_{i,t}a_{s,j}) = \frac{1}{2}\big(a_{i,j} - \mathrm{c}((i,t))\mathrm{c}((s,j))\big) = \frac{1}{2}\big(\mathrm{c}((i,t))\mathrm{c}((s,j)) - \mathrm{c}((i,t))\mathrm{c}((s,j))\big) = 0 = b_{i,j}$. If $b_{i,j} \neq 0$, then $\frac{1}{2}\mathrm{C}^{\check{J}}_{(s,t)}(A)[i,j] = \frac{1}{2}(a_{i,j}a_{s,t} - a_{i,t}a_{s,j}) = \frac{1}{2}\big(b_{i,j} - \mathrm{c}((i,t))\mathrm{c}((s,j))\big)$. Now if $b_{i,j} = -$, then $\mathrm{c} \in \mathrm{Col}(\mathrm{X}_B, \sigma_B)$ implies $\mathrm{c}((i,t)) = \mathrm{c}((s,j))$, hence $\frac{1}{2}\mathrm{C}^{\check{J}}_{(s,t)}(A)[i,j] = \frac{1}{2}((-)-(+)) = - = b_{i,j}$, and if $b_{i,j} = +$, then $\mathrm{c} \in \mathrm{Col}(\mathrm{X}_B, \sigma_B)$ implies $\mathrm{c}((i,t)) \neq \mathrm{c}((s,j))$, hence $\frac{1}{2}\mathrm{C}^{\check{J}}_{(s,t)}(A)[i,j] = \frac{1}{2}((+) - (-)) = + = b_{i,j}$.

We have proved that $A \in (\frac{1}{2}\mathrm{C}^{\check{J}}_{(s,t)})^{-1}(\mathcal{E}^J_B)$, and therefore $(\frac{1}{2}\mathrm{C}^{\check{J}}_{(s,t)})^{-1}(\mathcal{E}^J_B) \neq \varnothing$. We can therefore define a nonempty map $\Phi\colon (\frac{1}{2}\mathrm{C}^{\check{J}}_{(s,t)})^{-1}(\mathcal{E}^J_B) \to \mathrm{Col}(\mathrm{X}_B, \sigma_B)$ as follows: for every $A = (a_{i,j})_{(i,j)\in\check{J}} \in (\frac{1}{2}\mathrm{C}^{\check{J}}_{(s,t)})^{-1}(\mathcal{E}^J_B) \subseteq \{\pm\}^{\check{J}}$ we let $\Phi(A)$ be the function $\mathrm{V}(X_B) \to \{\pm\}$ defined by $\Phi(A)((i,t)) := a_{i,t}$ and $\Phi(A)((s,j)) := a_{s,j}$ for all $i \in \mathrm{p}_1(I)$ and $j \in \mathrm{p}_2(I)$.

Claim 1. Φ is indeed a map of the stated kind, i.e. $\Phi(A) \in \mathrm{Col}(\mathrm{X}_B, \sigma_B)$. Proof: let $\{(i,t),(s,j)\} \in \mathrm{E}(X_B)$ be arbitrary. There are two cases. If $\sigma_B(\{(i,t),(s,j)\}) = -$ then $b_{i,j} = -$ by Definition 149. Moreover, since $(\frac{1}{2}\mathrm{C}^{\check{J}}_{(s,t)}(A))|_{\mathrm{Dom}(B)} = B$ by the choice of A, it follows that for every $(i,j) \in I \subseteq J$ we have $- = b_{i,j} = (\frac{1}{2}\mathrm{C}^{\check{J}}_{(s,t)}(A))[i,j] = \frac{1}{2}\cdot(a_{i,j}a_{s,t} - a_{i,t}a_{s,j})$. In view of $a_{i,j},\ a_{s,t},\ a_{i,t},\ a_{s,j} \in \{\pm\}$, this equation implies $\Phi(A)((i,t)) = a_{i,t} = a_{s,j} = \Phi(A)((s,j))$. This proves that $\Phi(A)$ is $(\sigma_B, -)$-constant. If $\sigma_B(\{(i,t),(s,j)\}) = +$ then $b_{i,j} = +$ by Definition 149. Again by the choice of A, it is true that $+ = b_{i,j} = (\frac{1}{2}\mathrm{C}^{\check{J}}_{(s,t)}(A))[i,j] = \frac{1}{2}\cdot(a_{i,j}a_{s,t} - a_{i,t}a_{s,j})$. Again in view of $a_{i,j},\ a_{s,t},\ a_{i,t},\ a_{s,j} \in \{\pm\}$, this equation implies $\Phi(A)((i,t)) = a_{i,t} \neq a_{s,j} = \Phi(A)((s,j))$. This proves that $\Phi(A)$ is $(\sigma_B, +)$-proper and hence Claim 1.

Claim 2. Φ is surjective and every fibre under Φ has cardinality $2^{h(I,J)}$. Proof: let an arbitrary $\mathrm{c} \in \mathrm{Col}(\mathrm{X}_B, \sigma_B)$ be given. We are now looking for those $A \in (\frac{1}{2}\mathrm{C}^{\check{J}}_{(s,t)})^{-1}(\mathcal{E}^J_B)$ with $\Phi(A) = \mathrm{c}$. Since the definition of Φ demands $\Phi(A)((i,t)) = a_{i,t}$ and $\Phi(A)((s,j)) = a_{s,j}$ for all (i,t) and $(s,j) \in \mathrm{V}(X_B)$, it follows that with regard to the $|\mathrm{p}_1(I)| + |\mathrm{p}_2(I)|$ different entries $a_{i,j}$ with $(i,j) \in \mathrm{V}(\mathrm{X}_B)$ we know from the outset that we have no choice but to define $a_{i,t} := \mathrm{c}((i,t))$ and $a_{s,j} := \mathrm{c}((s,j))$.

Furthermore, since A must be in $(\frac{1}{2}\mathrm{C}^{\check{J}}_{(s,t)})^{-1}(\mathcal{E}^J_B)$, there is, for every $(i,j) \in I \subseteq J$, the condition that $b_{i,j} = (\frac{1}{2}\mathrm{C}^{\check{J}}_{(s,t)}(A))[i,j] = \frac{1}{2}\cdot(\mathrm{C}^{\check{J}}_{(s,t)}(A)[i,j]) = \frac{1}{2}\cdot(a_{i,j}a_{s,t} - a_{i,t}a_{s,j}) = \frac{1}{2}(a_{i,j}a_{s,t} - \mathrm{c}((i,t))\ \mathrm{c}((s,j)))$, where in the last step we have used the information about A that we already have. Now there are three cases that can occur.

Case 1. $b_{i,j} = -$. Then by Definition 149 we have $\{(i,t),(s,j)\} \in \mathrm{E}(X_B)$ and $\sigma_B(\{(i,t),(s,j)\}) = -$. Therefore, due to the fact that $\mathrm{c} \in \mathrm{Col}(\mathrm{X}_B, \sigma_B)$ is $(\sigma_B, -)$-constant, $\mathrm{c}((i,t)) = \mathrm{c}((s,j))$. Thus, in this case, $- = \frac{1}{2}(a_{i,j}a_{s,t} - 1)$, equivalently, $a_{i,j} = -a_{s,t}$.

Case 2. $b_{i,j} = 0$. Then by Definition 149, $\{(i,t),(s,j)\} \notin \mathrm{E}(X_B)$, hence $\sigma_B(\{(i,t),(s,j)\})$ is not defined and therefore the product $\mathrm{c}((i,t))\cdot\mathrm{c}((s,j))$ in the equation $0 = \frac{1}{2}\cdot(a_{i,j}a_{s,t} - \mathrm{c}((i,t))\cdot\mathrm{c}((s,j)))$ cannot be simplified further, but the equation itself can: it is equivalent to $a_{i,j} = \mathrm{c}((i,t))\cdot\mathrm{c}((s,j))\cdot a_{s,t} \in \{\pm\}$ (where we used that $a_{s,t}^{-1} = a_{s,t}$).

Case 3. $b_{i,j} = +$. Then an entirely analogous argument as in Case 1, but this time using the $(\sigma_B, +)$-properness of c, shows that then there is the equation $a_{i,j} = a_{s,t}$.

We now know what it means to require $A \in (\frac{1}{2}\mathrm{C}^{\check{J}}_{(s,t)})^{-1}(\mathcal{E}^J_B)$ in the present situation: among the $|J|$ entries of $A = (a_{i,j}) \in \{\pm\}^J$, there are the $|\mathrm{p}_1(I)| + |\mathrm{p}_2(I)|$ 'immediately determined' entries $a_{i,j}$ which have ($i \in \mathrm{p}_1(I)$ and $j = t$) or ($i = s$ and $j \in \mathrm{p}_2(I)$), and moreover the $|I|$ different entries $a_{i,j}$ with $(i,j) \in I$ which are determined by a system $\{a_{i,j} = h_{i,j} \colon (i,j) \in I\}$ of $|I|$ equations where the right-hand sides $h_{i,j}$ are defined by the Cases 1-3 above. For the remaining $h(I,J)$ different entries $a_{i,j} \in \{\pm\}$ (note that the pivot $a_{s,t}$ is among them: it is on the right-hand side in Case 2, hence not determined by the system), the choice of their value is free; any of the $2^{h(I,J)}$ possible choices gives an $A \in (\frac{1}{2}\mathrm{C}^{\check{J}}_{(s,t)})^{-1}(\mathcal{E}^J_B)$. This proves that the cardinality of the fibre $\Phi^{-1}(\mathrm{c})$ is indeed $2^{h(I,J)}$, and in particular that Φ is surjective. Now Claim 2 and Lemma 165 are proved. □

Let us note that in the special case $I = J$, i.e. when all entries are specified, then $h(I,J) = 1$ and the statement says that there is a double cover $\Phi\colon (\frac{1}{2}\mathrm{C}^{\check{J}}_{(s,t)})^{-1}(\{B\}) \to \mathrm{Col}(\mathrm{X}_B, \sigma_B)$. This corresponds to the freedom of choosing the sign of the pivot. Now we can relate Chio condensation to balancedness:

Lemma 166. *For every $(s,t) \in \mathbb{Z}^2_{\geqslant 2}$, every $\varnothing \subseteq I \subseteq J \subseteq [s-1] \times [t-1]$ and every $B \in \{0,\pm\}^I$, the following statements are equivalent:*

(1) (X_B, σ_B) *is balanced* ,

(2) $\mathrm{Col}(\mathrm{X}_B, \sigma_B) \neq \varnothing$,

(3) $(\frac{1}{2}\mathrm{C}^{\check{J}}_{(s,t)})^{-1}(\mathcal{E}^J_B) \neq \varnothing$,

(4) $\left|(\frac{1}{2}\mathrm{C}^{\check{J}}_{(s,t)})^{-1}(\mathcal{E}^J_B)\right| = 2^{|\check{J}|-\mathrm{dom}(B)-|\mathrm{X}_B|+\beta_0(\mathrm{X}_B)}$.

Proof. Equivalence (1) ⇔ (2) is true by (Kő1) with $X := \mathrm{X}_B$. Equivalence (2) ⇔ (3) is an immediate consequence of Lemma 165 (non-emptiness of the target of a map implies non-emptiness of its source; non-emptiness of the source of a map implies non-emptiness of its target). As to (3) ⇔ (4), note that by Lemma 165, there is the equation $|(\frac{1}{2}\mathrm{C}^{\check{J}}_{(s,t)})^{-1}(\mathcal{E}^J_B)| = 2^{|\check{J}|-\mathrm{dom}(B)-|\mathrm{X}_B|} \cdot |\mathrm{Col}(\mathrm{X}_B, \sigma_B)|$, in particular since $I = \mathrm{Dom}(B)$ and $\mathrm{X}_B = \mathrm{p}_1(I) \sqcup \mathrm{p}_2(I)$. Now if (3), then $\mathrm{Col}(\mathrm{X}_B, \sigma_B) \neq \varnothing$ by the already proved equivalence (2) ⇔ (3), therefore Lemma (Kő2) implies $|\mathrm{Col}(\mathrm{X}_B, \sigma_B)| = 2^{\beta_0(\mathrm{X}_B)}$ and hence (4) is true. Conversely, if (4) is true, then this formula alone implies (3). This completes the proof of (3) ⇔ (4) and also the proof of Lemma 166. □

As an example, consider the special case $s := t := n$, $\{(1,1)\} =: I \subseteq J := [n-1]^2$, $B[(1,1)] := 0$, i.e. $\mathcal{E}^J_B$ is the event that a $\tilde{B} = (\tilde{b}_{i,j}) \in \{0,\pm\}^{[n-1]^2}$ has $\tilde{b}_{1,1} = 0$. For these data, (4) in Lemma 166 yields 2^{n^2-1}. And indeed, it is easy to convince oneself directly that there are 2^{n^2-1} possibilities to realise this event by Chio-condensates of sign matrices $A \in \{\pm\}^{[n]^2}$.
We can now characterise the Chio measure using the language of signed graphs:

Theorem 167 (graph-theoretical characterisation of the Chio measure of entry-specification events)**.** *For every $(s,t) \in \mathbb{Z}^2_{\geqslant 2}$, arbitrary $\varnothing \subseteq I \subseteq J \subseteq [s-1] \times [t-1]$ and every $B \in \{0,\pm\}^I$:*

(C1) ***positivity is determined by balancedness:***
$\mathsf{P}_{\mathrm{chio}}[\mathcal{E}^J_B] > 0$ *if and only if* (X_B, σ_B) *is balanced* ,

(C2) ***absolute value is determined by the cut space:***
$\mathsf{P}_{\mathrm{chio}}[\mathcal{E}^J_B] > 0$ *if and only if*

$$\mathsf{P}_{\mathrm{chio}}[\mathcal{E}^J_B] = (\tfrac{1}{2})^{\mathrm{dom}(B)+|\mathrm{X}_B|-\beta_0(\mathrm{X}_B)} = \frac{(\frac{1}{2})^{\mathrm{dom}(B)}}{|\mathrm{B}^1(\mathrm{X}_B;\ \mathbb{F}_2)|}\ , \tag{4.14}$$

(C3) ***relative value is determined by the cycle space:***
$\mathsf{P}_{\mathrm{chio}}[\mathcal{E}^J_B] > 0$ *if and only if*

$$\mathsf{P}_{\mathrm{chio}}[\mathcal{E}^J_B] = 2^{\beta_1(\mathrm{X}_B)} \cdot \mathsf{P}_{\mathrm{lcf}}[\mathcal{E}^J_B] = |\mathrm{Z}_1(\mathrm{X}_B;\ \mathbb{F}_2)| \cdot \mathsf{P}_{\mathrm{lcf}}[\mathcal{E}^J_B]\ . \tag{4.15}$$

Proof. As to (C1), Definition 144 implies that $\mathsf{P}_{\mathrm{chio}}[\mathcal{E}^J_B] > 0$ if and only if $(\frac{1}{2}\mathrm{C}^{\check{J}}_{(s,t)})^{-1}(\mathcal{E}^J_B) \neq \varnothing$, hence item (C1) follows from the equivalence (1) ⇔ (3) in Lemma 166.

As to (C2), by the just proved item (C1) we have $\mathsf{P}_{\mathrm{chio}}[\mathcal{E}^J_B] > 0$ if and only if (X_B, σ_B) is balanced, and by equivalence (1) ⇔ (4) in Lemma 166 this is equivalent to $\left|(\frac{1}{2}\mathrm{C}^{\check{J}}_{(s,t)})^{-1}(\mathcal{E}^J_B)\right| = 2^{|\check{J}|-\mathrm{dom}(B)-|\mathrm{X}_B|+\beta_0(\mathrm{X}_B)}$. Dividing by $2^{|\check{J}|}$ in accordance with Definition 144 gives the first equality claimed in (C2). As to the second equality, this is a reformulation not necessary for the equivalence and is true by the known formula (e.g., [66, Theorem 14.1.1]) for the dimension of the cut space of a graph, together with the obvious formula for the number of elements of a finite-dimensional vector space over a finite field.

As to (C3), this follows from (C2), Lemma 160, $\mathsf{P}_{\mathrm{lcf}}[\mathcal{E}^J_B] = \mathsf{P}_{\mathrm{lcf}}[B]$ and $\mathrm{supp}(X_B) = \|X_B\|$. The second equality in (C3) is true by definition of $\beta_1(\cdot)$ (and therefore again a reformulation not necessary for the equivalence). The proof of Theorem 167 is now complete. □

We will now derive several consequences of Theorem 167. Let us start with:

Corollary 168. *Let $(s,t) \in \mathbb{Z}^2_{\geqslant 2}$, $B \in \{0,\pm\}^{[s-1]\times[t-1]}$, $\varnothing \subseteq I_1 \subseteq J_1 \subseteq [s-1] \times [t-1]$, $\varnothing \subseteq I_2 \subseteq J_2 \subseteq [s-1] \times [t-1]$ with $|I_1| = |I_2|$, $B_1 \in \{0,\pm\}^{I_1}$ and $B_2 \in \{0,\pm\}^{I_2}$ be arbitrary. Then*

(1) $\mathcal{F}^{\mathrm{M}}(k,n) = (\beta_1 \circ \mathrm{X}^{k,n,n})^{-1}(\mathbb{Z}_{\geqslant 1})$,
(2) $B \in \mathrm{im}(\frac{1}{2}\mathrm{C}_{(s,t)} \colon \{\pm\}^{[s]\times[t]} \to \{0,\pm\}^{[s-1]\times[t-1]})$ *if and only if* $\mathsf{P}_{\mathrm{chio}}[B] = \frac{2\cdot 2^{\beta_0(\mathrm{X}_B)}}{2^{s\cdot t}}$,
(3) $\mathsf{P}_{\mathrm{chio}}[\mathcal{E}^{J_1}_{B_1}] = \mathsf{P}_{\mathrm{chio}}[\mathcal{E}^{J_2}_{B_2}]$ *if* X_{B_1} *is a one-point wedge of two components of* X_{B_2} .

Proof. As to (1), this is immediate from (C3) in Theorem 167. As to (2), this follows by setting $I := J := [s-1]\times[t-1]$ and combining the equivalence (1) $\Leftrightarrow$ (3) in Lemma 166 with (C1) $\Leftrightarrow$ (C2) in Theorem 167.

As to (3), let us first note that Lemma 161 implies that either $(\mathrm{X}_{B_1}, \sigma_{B_1})$ and $(\mathrm{X}_{B_2}, \sigma_{B_2})$ are both not balanced, or both are. If both are not balanced, then by item (C1) in Theorem 167, the claim is true in the form of $0 = 0$. If both are, then by item (C2) in Theorem 167, and using $|I_1| = |I_2|$, the equation $\mathsf{P}_{\mathrm{chio}}[\mathcal{E}^{J_1}_{B_1}] = \mathsf{P}_{\mathrm{chio}}[\mathcal{E}^{J_2}_{B_2}]$ is equivalent to $|\mathrm{X}_{B_1}| - \beta_0(\mathrm{X}_{B_1}) = |\mathrm{X}_{B_2}| - \beta_0(\mathrm{X}_{B_2})$. Since the one-point wedge product of two graphs keeps $|\cdot| - \beta_0(\cdot)$ invariant, the equation is true also in this case and the proof is complete. ☐

Corollary 169 (the lazy coin flip measure is an averaged Chio measure). $\mathsf{P}_{\mathrm{lcf}}\big[B\big] = \overline{\mathsf{P}}_{\mathrm{chio}}\big[B\big]$ *for every* $\varnothing \subseteq I \subseteq [n-1]^2$ *and every* $B \in \{0,\pm\}^I$.

Proof. It follows from Definition 149 that $\mathrm{supp}(B) = \|\mathrm{X}_B\|$ and that $\{\tilde{B} \in \{0,\pm\}^I \colon \mathrm{Supp}(\tilde{B}) = \mathrm{Supp}(B)\} = \{\tilde{B} \in \{0,\pm\}^I \colon \mathrm{X}_{\tilde{B}} = \mathrm{X}_B\}$. Moreover, by (C1) in Theorem 167, every summand with the property that $(\mathrm{X}_{\tilde{B}}, \sigma_{\tilde{B}})$ is not balanced vanishes. Thus, for every $B \in \{0,\pm\}^I$,

$$\begin{aligned}
2^{\|\mathrm{X}_B\|} \cdot \overline{\mathsf{P}}_{\mathrm{chio}}\big[B\big] &= \sum_{\substack{\text{all } \tilde{B} \in \{0,\pm\}^I \text{ with} \\ \mathrm{X}_{\tilde{B}} = \mathrm{X}_B \text{ and } (\mathrm{X}_{\tilde{B}}, \sigma_{\tilde{B}}) \text{ balanced}}} \mathsf{P}_{\mathrm{chio}}\big[\tilde{B}\big] \\
&\overset{\text{(C3)}}{=} 2^{\beta_1(\mathrm{X}_{\tilde{B}})} \cdot \mathsf{P}_{\mathrm{lcf}}\big[B\big] \cdot \big|\{\tilde{B} \in \{0,\pm\}^I \colon \mathrm{X}_{\tilde{B}} = \mathrm{X}_B \text{ and } (\mathrm{X}_{\tilde{B}}, \sigma_{\tilde{B}}) \text{ balanced}\}\big| \\
&\overset{\text{(K}\breve{\mathrm{o}}\text{3)}}{=} 2^{\beta_1(\mathrm{X}_{\tilde{B}})} \cdot \mathsf{P}_{\mathrm{lcf}}\big[B\big] \cdot 2^{|\mathrm{X}_B| - \beta_0(\mathrm{X}_B)} = 2^{\|\mathrm{X}_B\|} \cdot \mathsf{P}_{\mathrm{lcf}}\big[B\big] \ .
\end{aligned}$$
☐

Corollary 170 ($\mathsf{P}^{|\cdot|,I}_{\mathrm{chio}}$ is just the uniform distribution on $\{0,1\}^I$). *For every* $(s,t) \in \mathbb{Z}^2_{\geqslant 2}$ *and every* $\varnothing \subseteq I \subseteq [s-1]\times[t-1]$ *let* $\mathsf{P}^I_{0,1}$ *denote the uniform distribution on* $\{0,1\}^I$. *Then* $\mathsf{P}^{|\cdot|,I}_{\mathrm{chio}} = \mathsf{P}^I_{0,1}$.

Proof. This is true since for $(s,t) \in \mathbb{Z}^2_{\geqslant 2}$, $\varnothing \subseteq I \subseteq [s-1]\times[t-1]$ and $B \in \{0,1\}^I$ we have

$$\begin{aligned}
2^{|\check{I}|} \cdot \mathsf{P}^{|\cdot|,I}_{\mathrm{chio}}[B] &= \big|\{A \in \{\pm\}^{\check{I}} \colon |\tfrac{1}{2} \cdot \mathrm{C}_{(s,t)}(A)| = B\}\big| \\
&= \big|\{A \in \{\pm\}^{\check{I}} \colon \mathrm{Supp}(\tfrac{1}{2}\mathrm{C}_{(s,t)}(A)) = \mathrm{Supp}(B)\}\big| \\
\text{(using (1) } \Leftrightarrow \text{ (3) in Lemma 166)} &= \sum_{\tilde{B} \in \{0,\pm\}^I \colon \mathrm{Supp}(\tilde{B}) = \mathrm{Supp}(B),\ (\mathrm{X}_{\tilde{B}}, \sigma_{\tilde{B}}) \text{ balanced}} \big|(\tfrac{1}{2}\mathrm{C}^{\check{I}}_{(s,t)})^{-1}(\mathcal{E}^I_{\tilde{B}})\big| \\
\text{(by (4) in Lemma 166)} &= \big|\{\tilde{B} \in \{0,\pm\}^I \colon \mathrm{Supp}(\tilde{B}) = \mathrm{Supp}(B),\ \tilde{B} \text{ balanced}\}\big| \cdot 2^{|\check{I}| - \mathrm{dom}(B) - |\mathrm{X}_{\tilde{B}}| + \beta_0(\mathrm{X}_{\tilde{B}})} \\
\text{(by (K}\breve{\mathrm{o}}\text{3) in Lemma 163)} &= 2^{|\mathrm{X}_{\tilde{B}}| - \beta_0(\mathrm{X}_{\tilde{B}})} \cdot 2^{|\check{I}| - \mathrm{dom}(B) - |\mathrm{X}_{\tilde{B}}| + \beta_0(\mathrm{X}_{\tilde{B}})} = 2^{|\check{I}| - \mathrm{dom}(B)}
\end{aligned}$$
☐

Let us state the special case $s := t := n$ and $I := [n-1]^2$ in graph-theoretical language:

Corollary 171. *For random* $A \in \{\pm\}^{[n]^2}$, *the graph* $\mathrm{X}_{\frac{1}{2}\mathrm{C}_{(n,n)}(A)}$ *is a random bipartite graph with* $n-1$ *vertices in each class and each edge chosen i.i.d. with probability* $\frac{1}{2}$. ☐

Theorem 167 also teaches us how fast $\mathsf{P}_{\mathrm{chio}}$ can be computed. In both Corollary 172 and 173 the asymptotic statements are referring to $n \to \infty$ and to sequences $I = I(n)$ of index sets with the property that $|I(n)| \to \infty$ (and therefore also $|\mathrm{p}_1(I(n))| \cdot |\mathrm{p}_2(I(n))| \to \infty$) as $n \to \infty$.

Corollary 172 (complexity of computing $\mathsf{P}_{\mathrm{chio}}$). *For every* $\varnothing \subseteq I \subseteq J \subseteq [n-1]^2$ *and every* $B \in \{0,\pm\}^I$, *the value of* $\mathsf{P}_{\mathrm{chio}}[\mathcal{E}^J_B] \in \mathbb{Q}$ *can be computed exactly in time* $O(|\mathrm{p}_1(I)| + |\mathrm{p}_2(I)| + |I|) \subseteq O(|\mathrm{p}_1(I)| \cdot |\mathrm{p}_2(I)|) \subseteq O(n^2)$. *However, there does not exist a fixed algorithm computing* $\mathsf{P}_{\mathrm{chio}}[\mathcal{E}^J_B] \in \mathbb{Q}$ *exactly on arbitrary instances* $B \in \{0,\pm\}^{[n-1]^2}$ *and* $\varnothing \subseteq I \subseteq [n-1]^2$ *and taking time* $o(|\mathrm{p}_1(I)| \cdot |\mathrm{p}_2(I)|)$.

Proof. By items (C1) and (C2) in Theorem 167, to compute $\mathsf{P}_{\text{chio}}[\mathcal{E}_B^J]$ it suffices to first decide whether σ_B (which in view of Definition 149 evidently can be read in time $O(|\mathrm{p}_1(I)| \cdot |\mathrm{p}_2(I)|) \subseteq O(n^2)$) is balanced, and, if so, to compute $|\mathrm{X}_B|$ and $\beta_0(\mathrm{X}_B)$. By Corollary 164, and since the depth-first seach mentioned there also computes the numbers $|\mathrm{X}_B|$ and $\beta_0(\mathrm{X}_B)$, both tasks can be accomplished by one depth-first search in time $O(|\mathrm{X}_B| + \|\mathrm{X}_B\|) \subseteq O(|\mathrm{p}_1(I)| + |\mathrm{p}_2(I)| + |I|) \subseteq O(n^2)$. If (X_B, σ_B) is found to be not balanced, then $\mathsf{P}_{\text{chio}}[\mathcal{E}_B^J] = 0$. Otherwise, the answer is $(\frac{1}{2})^{|I|+|\mathrm{X}_B|-\beta_0(\mathrm{X}_B)}$. Since the bitlength of this dyadic fraction is $|I|+|\mathrm{X}_B|-\beta_0(\mathrm{X}_B) = |I| + \|\mathrm{X}_B\| - \beta_1(\mathrm{X}_B) \leqslant |I| + \|\mathrm{X}_B\| \leqslant 2|I| \in O(|\mathrm{p}_1(I)| + |\mathrm{p}_2(I)| + |I|)$ it is possible to write the output in the time claimed. This proves the first statement in Corollary 172.

As to the second statement, notice that any such fixed algorithm could in particular compute $\mathsf{P}_{\text{chio}}[\mathcal{E}_B^J] \in \mathbb{Q}$ exactly on those instances $B \in \{0, \pm\}^{[n-1]^2}$ and $\varnothing \subseteq I \subseteq [n-1]^2$ for which I is rectangular. But if I is rectangular, then $|I| + |\mathrm{X}_B| - \beta_0(\mathrm{X}_B) \geqslant |I| = |\mathrm{p}_1(I)| \cdot |\mathrm{p}_2(I)|$. Therefore, for these inputs, the bitlength of the dyadic fraction $(\frac{1}{2})^{|I|+|\mathrm{X}_B|-\beta_0(\mathrm{X}_B)}$ is at least $|\mathrm{p}_1(I)| \cdot |\mathrm{p}_2(I)|$. Hence for such inputs the very task of writing the output takes time $\Omega(|\mathrm{p}_1(I)| \cdot |\mathrm{p}_2(I)|)$, which precludes a running time of $o(|\mathrm{p}_1(I)| \cdot |\mathrm{p}_2(I)|) \subseteq o(n^2)$. The proof of Corollary 172 is now complete. □

A priori one might suspect that the task of merely *deciding* whether $\mathsf{P}_{\text{chio}}[B] = \mathsf{P}_{\text{lcf}}[B]$ could be accomplished much faster than the task of computing the value of $\mathsf{P}_{\text{chio}}[B]$. Theorem 167 also teaches us that this is not the case; the following is the more detailed version of Theorem 34 announced in Section 1.4 of Chapter 1:

Theorem 173 (complexity of deciding whether P_{chio} and P_{lcf} agree)**.** *For every $\varnothing \subseteq I \subseteq J \subseteq [n-1]^2$ and every $B \in \{0, \pm\}^I$, the answer to the decision problem of whether $\mathsf{P}_{\text{chio}}[\mathcal{E}_B^J] = \mathsf{P}_{\text{lcf}}[\mathcal{E}_B^J]$ can be computed in time $O(|\mathrm{p}_1(I)| \cdot |\mathrm{p}_2(I)|) \subseteq O(n^2)$. However, there does not exist a fixed algorithm (having only entry-wise access to B and no further a priori information) which decides that question on arbitrary instances $B \in \{0, \pm\}^I$ with $\varnothing \subseteq I \subseteq [n-1]^2$ in time $o(|\mathrm{p}_1(I)| \cdot |\mathrm{p}_2(I)|)$.*

Proof. Given $\varnothing \subseteq I \subseteq [n-1]^2$ and $B \in \{0, \pm\}^I$, it follows from item (C3) in Theorem 167 that the question of whether $\mathsf{P}_{\text{chio}}[\mathcal{E}_B^J] = \mathsf{P}_{\text{lcf}}[\mathcal{E}_B^J]$ is equivalent to asking whether X_B is a forest. The graph X_B can obviously be computed from B in time $O(|\mathrm{p}_1(I)| \cdot |\mathrm{p}_2(I)|) \subseteq O(n^2)$, and deciding whether X_B is a forest, i.e. whether X_B contains a circuit, can be done by a depth-first search in time $O(|\mathrm{X}_B| + \|\mathrm{X}_B\|) \subseteq O(|\mathrm{p}_1(I)| + |\mathrm{p}_2(I)| + |I|) \subseteq O(|\mathrm{p}_1(I)| \cdot |\mathrm{p}_2(I)|)$, so the first claim in Corollary 173 is proved.

As to the additional claim, suppose there were a fixed algorithm A with the stated properties. Let $\mathcal{I}$ be the set of all rectangular $\varnothing \subseteq I \subseteq [n-1]^2$. By assumption, the algorithm A is in particular capable of deciding whether $\mathsf{P}_{\text{chio}}[\mathcal{E}_B^J] = \mathsf{P}_{\text{lcf}}[\mathcal{E}_B^J]$ for each input $I \in \mathcal{I}$ and for each of them taking time $o(|\mathrm{p}_1(I)| \cdot |\mathrm{p}_2(I)|)$. However, every bipartite graph with bipartition sizes of $|\mathrm{p}_1(I)|$ and $|\mathrm{p}_2(I)|$ can be realised as a X_B with $I \in \mathcal{I}$. By item (C3) in Theorem 167 the property $\mathsf{P}_{\text{chio}}[\mathcal{E}_B^J] = \mathsf{P}_{\text{lcf}}[\mathcal{E}_B^J]$ is equivalent to X_B being a *forest*. Therefore A decides set membership for the set of all bipartite graphs which have the fixed (that is, fixed for every fixed value of n) bipartition classes $\mathrm{p}_1(I)$ and $\mathrm{p}_2(I)$ and do not contain a circuit. This set is a decreasing (i.e. closed w.r.t. deleting edges) graph property consisting of bipartite graphs only. Since all graphs in the property have the same bipartition classes $\mathrm{p}_1(I)$ and $\mathrm{p}_2(I)$ we may appeal to a theorem of A. C.-C. Yao [165, p. 518, Theorem 1] which says that every such property is evasive.[3] Hence there exists at least one $I \in \mathcal{I}$ with the property that A examines every entry of B. This takes time $\Omega(|I|) = \Omega(|\mathrm{p}_1(I)| \cdot |\mathrm{p}_2(I)|)$, the equality being true because of $|I| = |\mathrm{p}_1(I)| \cdot |\mathrm{p}_2(I)|$. This is a contradiction to the assumption about the running time of A. The proof of Corollary 173 is now complete. □

We now take a more quantitative look at the relationship between P_{chio} and P_{lcf}. It is governed by the circuits in bipartite nonforests in the auxiliary graph X_B.

[3]Due to the fact that the bipartition classes are the same for all the graphs in the property, it is not necessary to appeal to the more general theorem of E. Triesch [160, p. 266, Theorem 4] in which the assumption of *fixed* bipartition classes is no longer made.

Corollary 174 (isomorphism types for which equality of measures of entry specification events fails)**.** *With the notation from Definition 152,*

(Fa3) $\mathcal{F}^{\mathrm{G}}(k,n) = \varnothing$ *for* $0 \leqslant k \leqslant 3$, (Fa5) $\mathcal{F}^{\mathrm{G}}(5,n) = \{(t2),(t3),(t5),(t7)\}$,
(Fa4) $\mathcal{F}^{\mathrm{G}}(4,n) = \{(t1)\}$, (Fa6) $\mathcal{F}^{\mathrm{G}}(6,n) = \mathcal{F}^{\mathrm{G}}(5,n) \sqcup \{(t4)\} \sqcup \{(t6),\ldots,(t20)\}$.

Proof. By (C3) in Theorem 167 we have $\mathsf{P}_{\mathrm{chio}}[\mathcal{E}_B^{[n-1]^2}] \neq \mathsf{P}_{\mathrm{lcf}}[\mathcal{E}_B^{[n-1]^2}]$ if and only if $\beta_1(\mathrm{X}_B) > 0$. Moreover, directly from Definition 149 we have the bound $\|\mathrm{X}_B\| \leqslant |I|$. Therefore, for every k, we can get a set of candidates for membership in $\mathcal{F}^{\mathrm{G}}(k,n)$ by collecting all isomorphism types in Corollary 174 having $f_1 \leqslant k$. We then have to decide for each of these types whether it is possible to realise it as an X_B with $B \in \{0,\pm\}^I$ and $I \in \binom{[n-1]^2}{k}$.

As to (Fa3), this is true since there do not exist bipartite nonforests with three edges or less.

As to (Fa4), i.e. $k = 4$, note that the only isomorphism types in Corollary 174 with $f_1 \leqslant 4$ are (t1) and (t2). Because of $\beta_1(\mathrm{X}_B) \geqslant 1$ for every $B \in \mathcal{F}^{\mathrm{M}}(4,n)$ the set I must be a matrix-4-circuit. This implies $|\mathrm{X}_B| = 4$. Since $|(t2)| = 5$, it follows that $(t2) \notin \mathcal{F}^{\mathrm{G}}(4,n)$. Since type (t1) obviously can be realised, (Fa4) is true.

As to (Fa5), i.e. $k = 5$, note that the only isomorphism types with $f_1 \leqslant 5$ in Corollary 174 are (t1), (t2), (t3), (t5), (t6) and (t7). Since C^4 is a subgraph of each of these types, it is necessary that there be a matrix-4-circuit $S \subseteq I$. Since the sole non-matrix-circuit entry must create at least one additional vertex of X_B, type (t1) cannot occur. The type (t6) cannot occur either since there is only one position $u \in I \backslash S \in \binom{[n-1]^2}{1}$ left for us to choose freely and by the choice of u and $B[u]$ we can either create an isolated vertex in X_B or an edge intersecting the $C^4 \hookrightarrow \mathrm{X}_B$, but not both. The remaining types (t2), (t3), (t5) and (t7) can be indeed be realised, as is proved by the following examples. For all the examples let $S := \{(1,1),(1,2),(2,1),(2,2)\}$, $B\mid_S := \{-\}^S$ and $\{u\} := I\backslash S$. For (t2) take e.g. $n := 4$, $u := (2,3)$ and $B[u] := 0$. For (t3) take e.g. $n := 4$, $u := (2,3)$ and $B[u] := -$. For (t5) take e.g. $n := 4$, $u := (3,3)$ and $B[u] := 0$. For (t7) take e.g. $n := 4$, $u := (3,3)$ and $B[u] := -$. This proves (Fa5).

As to (Fa6), i.e. $k = 6$, as far as only the necessary condition $\|\mathrm{X}_B\| \leqslant |I| = k$ is concerned, all types in Lemma 230 from Chapter 5 are candidates. Type (t1) cannot be realised since $|(t1)| = 4$ but $|\mathrm{X}_B| \geqslant 5$ for every $I \in \binom{[n-1]^2}{6}$. All others can, as will now be proved by giving one example for each. In all examples again let $S := \{(1,1),(1,2),(2,1),(2,2)\}$ and $B\mid_S := \{-\}^S$. Here, $\{u,v\} := I\backslash S$. For (t2), take e.g. $n := 4$, $u := (1,3)$, $v := (2,3)$ and $B[u] := B[v] := 0$. For (t3), take e.g. $n := 4$, $u := (1,3)$, $v := (2,3)$, $B[u] := -$ and $B[v] := 0$. For (t4), take e.g. $n := 4$, $u := (1,3)$, $v := (2,3)$ and $B[u] := B[v] := -$. For (t5), take e.g. $n := 4$, $u := (1,3)$, $v := (3,3)$ and $B[u] := B[v] := 0$. For (t6), take e.g. $n := 4$, $u := (1,3)$, $v := (3,3)$ and $B[u] := -$ and $B[v] := 0$. For (t7), take e.g. $n := 4$, $u := (1,3)$, $v := (3,3)$, $B[u] := 0$ and $B[v] := -$. For (t8), take e.g. $n := 5$, $u := (1,3)$, $v := (2,4)$ and $B[u] := B[v] := -$. For (t9), take e.g. $n := 4$, $u := (1,3)$, $v := (3,2)$ and $B[u] := B[v] := -$. For (t10), take e.g. $n := 4$, $u := (1,3)$, $v := (3,3)$ and $B[u] := B[v] := -$. For (t11), take e.g. $n := 5$, $u := (1,3)$, $v := (1,4)$ and $B[u] := B[v] := -$. For (t12), we have to make an exception to our convention that $\{u,v\} = I\backslash S$ with S defined as above, and have to define the set I in its entirety. We can take, e.g., $n := 4$, $I := \{(1,1),(1,2),(2,2),(2,3),(3,1),(3,3)\} \in \binom{[n-1]^2}{k}$ and $B = \{-\}^I$. For (t13), take e.g. $n := 5$, $u := (3,3)$, $v := (3,4)$ and $B[u] := B[v] := 0$. For (t14), take e.g. $n := 5$, $u := (1,3)$, $v := (3,4)$, $B[u] := -$ and $B[v] := 0$. For (t15), take e.g. $n := 5$, $u := (3,3)$, $v := (3,4)$, $B[u] := -$ and $B[v] := 0$. For (t16), take e.g. $n := 5$, $u := (1,3)$, $v := (3,4)$ and $B[u] := B[v] := -$. For (t17), take e.g. $n := 5$, $u := (3,3)$, $v := (3,4)$ and $B[u] := B[v] := -$. For (t18), take e.g. $n := 5$, $u := (3,3)$, $v := (4,4)$ and $B[u] := B[v] := 0$. For (t19), take e.g. $n := 5$, $u := (3,3)$, $v := (4,4)$, $B[u] := 0$ and $B[v] := 1$. For (t20), take e.g. $n := 5$, $u := (3,3)$, $v := (4,4)$ and $B[u] := B[v] := -$. This proves (Fa6). The proof of Corollary 174 is now complete. □

Corollary 175 (ratios and absolute values of $\mathsf{P}_{\mathrm{Chio}}$ for up to six entry specifications)**.**

(R1) $\mathcal{F}^{\mathrm{G}}(k,n) = \mathcal{F}^{\mathrm{G}}_{\cdot 0}(k,n) = \bigsqcup_{\beta \in \mathbb{Z}_{\geqslant 1}} \mathcal{F}^{\mathrm{G}}_{\cdot 2^\beta}(k,n)$,
(R2) $\mathcal{F}^{\mathrm{M}}(k,n) = \mathcal{F}^{\mathrm{M}}_{\cdot 0}(k,n) \sqcup \bigsqcup_{\beta \in \mathbb{Z}_{\geqslant 1}} \mathcal{F}^{\mathrm{M}}_{\cdot 2^\beta}(k,n)$,
(R3) $\mathcal{F}^{\mathrm{G}}(4,n) = \mathcal{F}^{\mathrm{G}}_{\cdot 2}(4,n)$, $\mathcal{F}^{\mathrm{G}}(5,n) = \mathcal{F}^{\mathrm{G}}_{\cdot 2}(5,n)$ *and* $\mathcal{F}^{\mathrm{G}}(6,n) = \mathcal{F}^{\mathrm{G}}_{\cdot 2}(6,n) \sqcup \mathcal{F}^{\mathrm{G}}_{\cdot 4}(6,n)$,

(R4) $\mathcal{F}^{\mathrm{M}}(4,n) = \mathcal{F}^{\mathrm{M}}_{\cdot 0}(4,n) \sqcup \mathcal{F}^{\mathrm{M}}_{\cdot 2}(4,n)$, $\mathcal{F}^{\mathrm{M}}(5,n) = \mathcal{F}^{\mathrm{M}}_{\cdot 0}(5,n) \sqcup \mathcal{F}^{\mathrm{M}}_{\cdot 2}(5,n)$ *and* $\mathcal{F}^{\mathrm{M}}(6,n) = \mathcal{F}^{\mathrm{M}}_{\cdot 0}(6,n) \sqcup \mathcal{F}^{\mathrm{M}}_{\cdot 2}(6,n) \sqcup \mathcal{F}^{\mathrm{M}}_{\cdot 4}(6,n)$.

Moreover,

(A4) $\mathcal{F}^{\mathrm{G}}(4,n) = \mathcal{F}^{\mathrm{G}}_{=(\frac{1}{2})^7}(4,n)$, *with* $\mathcal{F}^{\mathrm{G}}_{=(\frac{1}{2})^7}(4,n) = \{(t1)\}$,

(A5) $\mathcal{F}^{\mathrm{G}}(5,n) = \mathcal{F}^{\mathrm{G}}_{=(\frac{1}{2})^8}(5,n) \sqcup \mathcal{F}^{\mathrm{G}}_{=(\frac{1}{2})^9}(5,n)$,
with $\mathcal{F}^{\mathrm{G}}_{=(\frac{1}{2})^8}(5,n) = \{(t2),(t5)\}$ *and* $\mathcal{F}^{\mathrm{G}}_{=(\frac{1}{2})^9}(5,n) = \{(t3),(t7)\}$,

(A6) $\mathcal{F}^{\mathrm{G}}(6,n) = \mathcal{F}^{\mathrm{G}}_{=(\frac{1}{2})^9}(6,n) \sqcup \mathcal{F}^{\mathrm{G}}_{=(\frac{1}{2})^{10}}(6,n) \sqcup \mathcal{F}^{\mathrm{G}}_{=(\frac{1}{2})^{11}}(6,n)$, *where*

(i) $\mathcal{F}^{\mathrm{G}}_{=(\frac{1}{2})^9}(6,n) = \{(t2),(t5),(t13),(t18)\}$,

(ii) $\mathcal{F}^{\mathrm{G}}_{=(\frac{1}{2})^{10}}(6,n) = \{(t3),(t4),(t6),(t7),(t14),(t15),(t19)\}$,

(iii) $\mathcal{F}^{\mathrm{G}}_{=(\frac{1}{2})^{11}}(6,n) = \{(t8),(t9),(t10),(t11),(t12),(t16),(t17),(t20)\}$.

Proof. As to (R1), let us start with $\mathcal{F}^{\mathrm{G}}(k,n) = \mathcal{F}^{\mathrm{G}}_{\cdot 0}(k,n)$. The inclusion $\supseteq$ is true directly by (4) in Definition 152. Conversely, let $\mathfrak{X} \in \mathcal{F}^{\mathrm{G}}(k,n)$. Then $\beta_1(\mathfrak{X}) \geqslant 1$ by Corollary 168.(1), hence $|\{\pm\}^{\mathrm{E}(\mathfrak{X})}\backslash S_{\mathrm{bal}}(\mathfrak{X})| = 2^{\|\mathfrak{X}\|} - 2^{|\mathfrak{X}|-\beta_0(\mathfrak{X})} = 2^{\|\mathfrak{X}\|} - 2^{\|\mathfrak{X}\|-\beta_1(\mathfrak{X})} > 0$, hence there exists $B \in \mathcal{F}^{\mathrm{M}}_{\cdot 0}(k,n)$ with $\mathfrak{X} = {}_{\mathrm{ul}}\mathrm{X}^{k,n,n}(B)$, hence $\mathfrak{X} \in {}_{\mathrm{ul}}\mathrm{X}^{k,n,n}(\mathcal{F}^{\mathrm{M}}_{\cdot 0}(k,n)) \overset{\text{Definition 152.(4)}}{=} \mathcal{F}^{\mathrm{G}}_{\cdot 0}(k,n)$, proving $\subseteq$.

As to the partition claimed in (R1), both claims follow immediately from (C3) in Theorem 167 (and the equality $\mathcal{F}^{\mathrm{G}}(k,n) = \mathcal{F}^{\mathrm{G}}_{\cdot 0}(k,n)$ is the reason why $\mathcal{F}^{\mathrm{G}}_{\cdot 0}(k,n)$ is missing in the disjoint union in (R1)). As to (R3) (respectively (R4)), this follows by combining (R1) (respectively (R2)) with (Fa4)–(Fa6) in Corollary 174. The claims (A4)–(A6) will be proved in reverse order.

As to (A6), this seems to require some calculations. However, Corollary 168.(3) can be used to reduce the amount of work to be done: if (a) and (b) are isomorphism types of graphs, let us write $(a) \twoheadrightarrow (b)$ if and only if (b) can be obtained from (a) by a single one-point wedge of two connected components of (a). Moreover, if (a) is any of the isomorphism types in $\mathcal{F}^{\mathrm{M}}(6,n)$, let us employ the abbreviation $\mathcal{E}_B := \mathcal{E}_B^{[n-1]^2}$ and let us write $\mathsf{P}_{\mathrm{chio}}[(a)]$ for the number $\mathsf{P}_{\mathrm{chio}}[\mathcal{E}_B^{[n-1]^2}]$ with B an arbitrary $B \in \{0,\pm\}^I$, $I \in \binom{[n-1]^2}{6}$, $\mathrm{X}_B \in (a)$ and (X_B, σ_B) balanced. By (C2) in Theorem 167 we know that $\mathsf{P}_{\mathrm{chio}}[(a)]$ then does indeed only depend on (a), not on the choice of such a B.

Since evidently (t18) $\twoheadrightarrow$ (t13) $\twoheadrightarrow$ (t5) $\twoheadrightarrow$ (t2), Corollary 168.(3) implies $\mathsf{P}_{\mathrm{chio}}[(t2)] = \mathsf{P}_{\mathrm{chio}}[(t5)] = \mathsf{P}_{\mathrm{chio}}[(t13)] = \mathsf{P}_{\mathrm{chio}}[(t18)]$. Since evidently (t19) $\twoheadrightarrow$ (t15) $\twoheadrightarrow$ (t6) $\twoheadrightarrow$ (t3), Corollary 168.(3) implies $\mathsf{P}_{\mathrm{chio}}[(t19)] = \mathsf{P}_{\mathrm{chio}}[(t15)] = \mathsf{P}_{\mathrm{chio}}[(t6)] = \mathsf{P}_{\mathrm{chio}}[(t3)]$. Since also (t14) $\twoheadrightarrow$ (t6), Corollary 168.(3) implies $\mathsf{P}_{\mathrm{chio}}[(t14)] = \mathsf{P}_{\mathrm{chio}}[(t6)]$. Since moreover (t7) $\twoheadrightarrow$ (t3), Corollary 168.(3) implies $\mathsf{P}_{\mathrm{chio}}[(t7)] = \mathsf{P}_{\mathrm{chio}}[(t3)]$. These equations together imply $\mathsf{P}_{\mathrm{chio}}[(t3)] = \mathsf{P}_{\mathrm{chio}}[(t6)] = \mathsf{P}_{\mathrm{chio}}[(t7)] = \mathsf{P}_{\mathrm{chio}}[(t14)] = \mathsf{P}_{\mathrm{chio}}[(t15)] = \mathsf{P}_{\mathrm{chio}}[(t19)]$. Since evidently (t20) $\twoheadrightarrow$ (t17) $\twoheadrightarrow$ (t10), Corollary 168.(3) implies $\mathsf{P}_{\mathrm{chio}}[(t20)] = \mathsf{P}_{\mathrm{chio}}[(t17)] = \mathsf{P}_{\mathrm{chio}}[(t10)]$. Since evidently (t20) $\twoheadrightarrow$ (t16) $\twoheadrightarrow$ (t10), Corollary 168.(3) implies $\mathsf{P}_{\mathrm{chio}}[(t20)] = \mathsf{P}_{\mathrm{chio}}[(t16)] = \mathsf{P}_{\mathrm{chio}}[(t10)]$. Since evidently (t17) $\twoheadrightarrow$ (t11), Corollary 168.(3) implies $\mathsf{P}_{\mathrm{chio}}[(t17)] = \mathsf{P}_{\mathrm{chio}}[(t11)]$. Since evidently (t16) $\twoheadrightarrow$ (t9), Corollary 168.(3) implies $\mathsf{P}_{\mathrm{chio}}[(t16)] = \mathsf{P}_{\mathrm{chio}}[(t9)]$. Since evidently (t16) $\twoheadrightarrow$ (t8), Corollary 168.(3) implies $\mathsf{P}_{\mathrm{chio}}[(t16)] = \mathsf{P}_{\mathrm{chio}}[(t8)]$. These equations together imply that $\mathsf{P}_{\mathrm{chio}}[(t8)] = \mathsf{P}_{\mathrm{chio}}[(t9)] = \mathsf{P}_{\mathrm{chio}}[(t10)] = \mathsf{P}_{\mathrm{chio}}[(t11)] = \mathsf{P}_{\mathrm{chio}}[(t16)] = \mathsf{P}_{\mathrm{chio}}[(t17)] = \mathsf{P}_{\mathrm{chio}}[(t20)]$.

This proves that it suffices (note that of the nineteen elements of $\mathcal{F}^{\mathrm{M}}(6,n)$ exactly $(t4)$ and $(t12)$ have not been part of one of the equality chains) to calculate only $\mathsf{P}_{\mathrm{chio}}[(t2)]$, $\mathsf{P}_{\mathrm{chio}}[(t3)]$, $\mathsf{P}_{\mathrm{chio}}[(t4)]$, $\mathsf{P}_{\mathrm{chio}}[(t8)]$ and $\mathsf{P}_{\mathrm{chio}}[(t12)]$. With the formulas in (C2) of Theorem 167 and in Lemma 160, this can be done as follows (keep in mind that, being within item (A6), $\mathrm{dom}(B) = |I| = 6$ in each calculation): If $\mathfrak{X} = (t2)$, then $|\mathfrak{X}| = 5$, $\beta_0(\mathfrak{X}) = 2$, hence $\mathsf{P}_{\mathrm{chio}}[\mathcal{E}_B] = (\frac{1}{2})^{|I|+5-2} = (\frac{1}{2})^9$.
If $\mathfrak{X} \in \{(t3),(t4)\}$, then $|\mathfrak{X}| = 5$, $\beta_0(\mathfrak{X}) = 1$, hence $\mathsf{P}_{\mathrm{chio}}[\mathcal{E}_B] = (\frac{1}{2})^{|I|+5-1} = (\frac{1}{2})^{10}$.
If $\mathfrak{X} \in \{(t8),(t12)\}$, then $|\mathfrak{X}| = 6$, $\beta_0(\mathfrak{X}) = 1$, hence $\mathsf{P}_{\mathrm{chio}}[\mathcal{E}_B] = (\frac{1}{2})^{|I|+6-1} = (\frac{1}{2})^{11}$.

As to (A5), it follows by an entirely analogous (but much shorter) argument as the one given for (A5) that it suffices to calculate only $\mathsf{P}_{\mathrm{chio}}[(t2)]$ and $\mathsf{P}_{\mathrm{chio}}[(t3)]$, and these calculations are identical to the ones made for $\mathsf{P}_{\mathrm{chio}}[(t2)]$ and $\mathsf{P}_{\mathrm{chio}}[(t3)]$ in the preceding paragraph, except that now $|I| = 5$.

As to (A4), in view of (Fa4) in Corollary 174, we only have to deal with the single type (t1) where $|(t1)| = 4$, $\beta_0(t1) = 1$ and therefore $\mathsf{P}_{\text{chio}}[(t1)] = (\frac{1}{2})^{|I|+4-1} = (\frac{1}{2})^{4+4-1} = (\frac{1}{2})^7$. $\square$

The results obtained so far can be turned into a set of instructions of how to tell the measure of $\mathsf{P}_{\text{chio}}[\mathcal{E}_B^{[n-1]^2}]$ from a given $B \in \{0,\pm\}^I$ provided that $|I| \leqslant 6$. We formulate the instructions exclusively in terms of those data, avoiding any mention of the associated signed graph (X_B,σ_B).

Corollary 176 (how to find the measure under P_{chio} of large entry-specification events)**.** *For every $\varnothing \subseteq I \subseteq [n-1]^2$ with $|I| \leqslant 6$ and every $B \in \{0,\pm\}^I$, the following instructions lead to the correct Chio measure of $\mathcal{E}_B := \mathcal{E}_B^{[n-1]^2}$:*

- (H3) *If $0 \leqslant |I| \leqslant 3$, then $\mathsf{P}_{\text{chio}}[\mathcal{E}_B] = \mathsf{P}_{\text{lcf}}[\mathcal{E}_B] = (\frac{1}{2})^{\text{dom}(B)+\text{supp}(B)}$.*
- (H4) *If $|I| = 4$, then check whether I is a matrix-4-circuit* such that *$B \in \{\pm\}^I$. If not, then $\mathsf{P}_{\text{chio}}[\mathcal{E}_B] = \mathsf{P}_{\text{lcf}}[\mathcal{E}_B] = (\frac{1}{2})^{\text{dom}(B)+\text{supp}(B)}$. If B has this property, then check whether an odd number of the four nonzero values $B[i,j]$ with $(i,j) \in I$ are $+$. If so, $\mathsf{P}_{\text{chio}}[\mathcal{E}_B] = 0$. If not, then $\mathsf{P}_{\text{chio}}[\mathcal{E}_B] = (\frac{1}{2})^7 = 2 \cdot \mathsf{P}_{\text{lcf}}[\mathcal{E}_B]$.*
- (H5) *If $|I| = 5$, then check whether there exists within I a matrix-4-circuit $S \subseteq I$* such that *$B\mid_S \in \{\pm\}^S$. If not, then $\mathsf{P}_{\text{chio}}[\mathcal{E}_B] = \mathsf{P}_{\text{lcf}}[\mathcal{E}_B] = (\frac{1}{2})^{\text{dom}(B)+\text{supp}(B)}$. If so, then check whether an odd number of the four nonzero values $B[i,j]$ with $(i,j) \in S$ are $+$. If so, $\mathsf{P}_{\text{chio}}[\mathcal{E}_B] = 0$. If not, then check whether the entry $B[i,j]$ with $(i,j) \in I\backslash S$ is zero. If it is, then $\mathsf{P}_{\text{chio}}[\mathcal{E}_B] = (\frac{1}{2})^8 = 2 \cdot \mathsf{P}_{\text{lcf}}[\mathcal{E}_B]$. If it is nonzero, then $\mathsf{P}_{\text{chio}}[\mathcal{E}_B] = (\frac{1}{2})^9 = 2 \cdot \mathsf{P}_{\text{lcf}}[\mathcal{E}_B]$.*
- (H6) *If $|I| = 6$, then check whether at least four entries in B are nonzero. If not, then $\mathsf{P}_{\text{chio}}[\mathcal{E}_B] = \mathsf{P}_{\text{lcf}}[\mathcal{E}_B] = (\frac{1}{2})^{\text{dom}(B)+\text{supp}(B)}$. If so, then check whether I contains a matrix-4-circuit $S \subseteq I$ with $B\mid_S \in \{\pm\}^S$.*
  - (i) *If I does not contain such a matrix-4-circuit S, then check whether I is a matrix-6-circuit such that $B \in \{\pm\}^I$. If not, then $\mathsf{P}_{\text{chio}}[\mathcal{E}_B] = \mathsf{P}_{\text{lcf}}[\mathcal{E}_B] = (\frac{1}{2})^{\text{dom}(B)+\text{supp}(B)}$. If so, then check whether an odd number of these six entries are $+$. If so, then $\mathsf{P}_{\text{chio}}[\mathcal{E}_B] = 0$. If not, then $\mathsf{P}_{\text{chio}}[\mathcal{E}_B] = (\frac{1}{2})^{11} = 2 \cdot \mathsf{P}_{\text{lcf}}[\mathcal{E}_B]$.*
  - (ii) *If I does indeed contain such a matrix-4-circuit, then check whether an odd number of the four nonzero values $B[i,j]$ with $(i,j) \in S$ are $+$. If so, then $\mathsf{P}_{\text{chio}}[\mathcal{E}_B] = 0$. Else, there are three further cases:*
    - (a) *If both entries indexed by $I\backslash S$ are zero, then $\mathsf{P}_{\text{chio}}[\mathcal{E}_B] = (\frac{1}{2})^9 = 2 \cdot \mathsf{P}_{\text{lcf}}[\mathcal{E}_B]$.*
    - (b) *If exactly one of the two entries indexed by $I\backslash S$ is zero, then $\mathsf{P}_{\text{chio}}[\mathcal{E}_B] = (\frac{1}{2})^{10} = 2 \cdot \mathsf{P}_{\text{lcf}}[\mathcal{E}_B]$.*
    - (c) *If both entries indexed by $I\backslash S$ are nonzero, then the positions of these two nonzero entries must be taken into account: if there do* neither *exist $1 \leqslant i < i' < i'' \leqslant n-1$ and $1 \leqslant j < j' \leqslant n-1$ such that $I = \{(i,j),\ (i',j),\ (i'',j),\ (i,j'),\ (i',j'),\ (i'',j')\}$* nor *$1 \leqslant i < i' \leqslant n-1$ and $1 \leqslant j < j' < j'' \leqslant n-1$ such that $I = \{(i,j),\ (i,j'),\ (i,j''),\ (i',j),\ (i',j'),\ (i',j'')\}$, then—whatever the B-values indexed by the two elements in $I\backslash S$ may be—you know that $\mathsf{P}_{\text{chio}}[\mathcal{E}_B] = (\frac{1}{2})^{11} = 2 \cdot \mathsf{P}_{\text{lcf}}[\mathcal{E}_B]$. Else, check whether in any one of the then existing two additional matrix-4-circuits in I the number of $+$ entries is odd. If so, then $\mathsf{P}_{\text{chio}}[\mathcal{E}_B] = 0$. If not, then $\mathsf{P}_{\text{chio}}[\mathcal{E}_B] = (\frac{1}{2})^{10} = 4 \cdot \mathsf{P}_{\text{lcf}}[\mathcal{E}_B]$.* $\square$

If $U \subseteq \mathrm{Dom}(\mathrm{X}^{k,n,n}) = \bigsqcup_{I \in \binom{[s-1]\times[t-1]}{k}} \{0,\pm\}^I$ is an arbitrary subset, then on the abstract set-theoretical level all we know is $(\mathrm{X}^{k,n,n})^{-1}(\mathrm{X}^{k,n,n}(U)) \supseteq U$. For the specific subsets $U = \mathcal{F}^{\mathrm{M}}(k,n)$, however, the inclusion is an equality:

Corollary 177. $(\mathrm{X}^{k,n,n})^{-1}(\mathrm{X}^{k,n,n}(\mathcal{F}^{\mathrm{M}}(k,n))) = \mathcal{F}^{\mathrm{M}}(k,n)$

Proof. Since $(\mathrm{X}^{k,n,n})^{-1}(\mathrm{X}^{k,n,n}(\mathcal{F}^{\mathrm{M}}(k,n))) \overset{\text{Corollary 168.(1)}}{=} (\mathrm{X}^{k,n,n})^{-1}(\mathrm{X}^{k,n,n}((\mathrm{X}^{k,n,n})^{-1}(\beta_1^{-1}(\mathbb{Z}_{\geqslant 1}))))$ $\overset{\text{Lemma 157}}{=} (\mathrm{X}^{k,n,n})^{-1}(\beta_1^{-1}(\mathbb{Z}_{\geqslant 1})) \overset{\text{again Corollary 168.(1)}}{=} \mathcal{F}^{\mathrm{M}}(k,n)$. $\square$

Corollaries 174 and 177 allow us to express the failure sets $\mathcal{F}^{\mathrm{M}}(k,n)$ as partitions indexed by isomorphism types of bipartite graphs:

Corollary 178. *For every n,*

(M4) $\mathcal{F}^{\mathrm{M}}(4,n) = ({}_{\mathrm{ul}}\mathrm{X}^{4,n,n})^{-1}(t1)$,
(M5) $\mathcal{F}^{\mathrm{M}}(5,n) = ({}_{\mathrm{ul}}\mathrm{X}^{5,n,n})^{-1}(t2) \sqcup ({}_{\mathrm{ul}}\mathrm{X}^{5,n,n})^{-1}(t3) \sqcup ({}_{\mathrm{ul}}\mathrm{X}^{5,n,n})^{-1}(t5) \sqcup ({}_{\mathrm{ul}}\mathrm{X}^{5,n,n})^{-1}(t7)$,
(M6) $\mathcal{F}^{\mathrm{M}}(6,n) = ({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t2) \sqcup ({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t3) \sqcup ({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t5) \sqcup ({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t7)$
$\sqcup ({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t4) \sqcup \bigsqcup_{6\leqslant k\leqslant 20} ({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(\mathrm{t}k)$.

Proof. In general we have $\mathcal{F}^{\mathrm{M}}(k,n) \overset{\text{Corollary 177}}{=} ({}_{\mathrm{ul}}\mathrm{X}^{k,n,n})^{-1}({}_{\mathrm{ul}}\mathrm{X}^{k,n,n}(\mathcal{F}^{\mathrm{M}}(k,n))) \overset{\text{(by Definition 152.(4))}}{=} ({}_{\mathrm{ul}}\mathrm{X}^{k,n,n})^{-1}(\mathcal{F}^{\mathrm{G}}(k,n)) \overset{\text{(for every map)}}{=} \bigsqcup_{\mathfrak{X}\in\mathcal{F}^{\mathrm{G}}(k,n)} ({}_{\mathrm{ul}}\mathrm{X}^{k,n,n})^{-1}(\mathfrak{X})$, and for the specific values $4 \leqslant k \leqslant 6$ we can use Corollary 174 to obtain the claimed partitions. □

While having the aim of explicitly determining the numbers $|({}_{\mathrm{ul}}\mathrm{X}^{k,n,n})^{-1}(\mathfrak{X})|$ for certain k and $\mathfrak{X}$ which interest us, we will start slowly: we first formulate some linear relations among $|({}_{\mathrm{ul}}\mathrm{X}^{5,n,n})^{-1}(t2)|, \ldots, |({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t20)|$ which will later serve as a check for the formulas given in Lemma 180.

Lemma 179 (linear relations among $|({}_{\mathrm{ul}}\mathrm{X}^{k,n,n})^{-1}(\mathfrak{X})|$ for $5 \leqslant k \leqslant 6$)**.**

(l1) $(3^1-1)\cdot|({}_{\mathrm{ul}}\mathrm{X}^{5,n,n})^{-1}(t2)| = |({}_{\mathrm{ul}}\mathrm{X}^{5,n,n})^{-1}(t3)|$,
(l2) $(3^1-1)\cdot|({}_{\mathrm{ul}}\mathrm{X}^{5,n,n})^{-1}(t5)| = |({}_{\mathrm{ul}}\mathrm{X}^{5,n,n})^{-1}(t7)|$,
(l3) $(3^2-1)\cdot|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t5)| = |({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t6)| + \cdots + |({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t11)|$,
(l4) $(3^2-1)\cdot|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t13)| = |({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t14)| + \cdots + |({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t17)|$,
(l5) $(3^2-1)\cdot|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t18)| = |({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t19)| + |({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t20)|$.

Proof. It follows from Definition 149 that $\mathrm{X}_B \in (t3)$ if and only if equation (4.18) is true and $B[u] \in \{\pm\}$. This implies $|({}_{\mathrm{ul}}\mathrm{X}^{5,n,n})^{-1}(t3)| = 2\cdot|({}_{\mathrm{ul}}\mathrm{X}^{5,n,n})^{-1}(t2)|$, proving (l1). It also follows from Definition 149 that $\mathrm{X}_B \in (t7)$ if and only if equation (4.19) is true and $B[u] \in \{\pm\}$. This implies $|({}_{\mathrm{ul}}\mathrm{X}^{5,n,n})^{-1}(t7)| = 2\cdot|({}_{\mathrm{ul}}\mathrm{X}^{5,n,n})^{-1}(t5)|$, proving (l2).

The isomorphism types (t5)–(t11) are all the isomorphism types of bipartite nonforests with six vertices and exactly one copy of C^4. Therefore $|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t5)| + \cdots + |({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t11)|$ is the number of all $B \in \{0,\pm\}^I$ with $I \in \binom{[n-1]^2}{6}$ such that X_B contains exactly one C^4 and $|\mathrm{X}_B| = 6$. Imagine counting these B by partitioning the set of all such B according to the copy of C^4, and for each such copy, further partitioning the B according to the mandatory $\pm$-values on the edges of the C^4, and then further partitioning according to the *positions* of the two elements of I which are not responsible for the copy of C^4. When partitioning in that way, the number of blocks of the partition obtained so far equals $|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t5)|$. The reason for this is that to realise the type $(t5)$ there is no choice for the values indexed by the positions which are not responsible for the C^4, both must be zero. In the enumeration we are currently carrying out, however, there *is* still complete freedom left on how to choose any one of the $|\{0,\pm\}^{[2]}| = 3^2$ values which can be indexed by these two positions, in other words, each of the blocks has size 3^2. Therefore $|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t5)| + \cdots + |({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t11)| = 3^2\cdot|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t5)|$, which proves (l3). Equations (l4) and (l5) are true for an entirely analogous reason. □

We will now quantify the claims in Corollary 174 by determining $|({}_{\mathrm{ul}}\mathrm{X}^{k,n,n})^{-1}(\mathfrak{X})|$ for each k and each isomorphism type $\mathfrak{X}$ mentioned there. A few comments seem in order. The behaviour of $|({}_{\mathrm{ul}}\mathrm{X}^{k,n,n})^{-1}(\mathfrak{X})|$ as a function of k for a given isomorphism type $\mathfrak{X}$ is a little subtle. For example, note that Lemma 180 tells us that

$$|({}_{\mathrm{ul}}\mathrm{X}^{5,n,n})^{-1}(t2)| > |({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t2)| \tag{4.16}$$

in spite of the fact that in the case of $|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t2)|$ we have one matrix entry more at our disposal to realise (t2). The reason for this could be summarised thus: when wanting to keep the number of isolated vertices in ${}_{\mathrm{ul}}\mathrm{X}_B$ at one, the additional matrix entry curtails our freedom more than it adds to it—after having chosen a position for one of the non-matrix-circuit-entries which ‘hides’ one of its two ‘shadows’ in one of the four shadows of the matrix-circuit-entries, we then have to position the second non-matrix-circuit-entry so as to hide *both* of its two shadows in already existing shadows, and this determines it position completely. Moreover, since (t2) is an isomorphism type in which there do not exist edges outside the 4-circuit, the non-matrix-circuit positions must index the value 0. The net result of these rigid requirements are (since in effect for $|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t2)|$ we are counting the possible 2-*sets* of non-circuit positions while for $|({}_{\mathrm{ul}}\mathrm{X}^{5,n,n})^{-1}(t2)|$ we counted the possible 1-sets of such positions) *less* possibilities. For other types it can happen that the mechanism just described is counterbalanced by the additional possibilities of indexing different values. This is the essential reason why $|({}_{\mathrm{ul}}\mathrm{X}^{5,n,n})^{-1}(t3)| = |({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t3)|$, despite (4.16) and despite the fact that the set of all *domains* in the preimages in question are the same as in (4.16), i.e.

$$\mathrm{Dom}(({}_{\mathrm{ul}}\mathrm{X}^{5,n,n})^{-1}(t2)) = \mathrm{Dom}(({}_{\mathrm{ul}}\mathrm{X}^{5,n,n})^{-1}(t3))\ , \quad \mathrm{Dom}(({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t2)) = \mathrm{Dom}(({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t3))\ . \tag{4.17}$$

Since biadjacency matrices are quite a fundamental topic, it would be of interest to treat these phenomena in more generality. The existing literature on this topic seems not to offer help in proving the following auxiliary statement:

Lemma 180 (cardinality of preimages of ${}_{\mathrm{ul}}\mathrm{X}^{k,n,n}$ on bipartite nonforests for $4 \leqslant k \leqslant 6$)**.** *The claims* (Fa4)—(Fa6) *can be quantified as follows (with* $\xi_n := 2^4 \cdot |\mathrm{Cir}(4,n)| = 2^4 \cdot \binom{n-1}{2}^2$*),*

(QFa4) *For every* $n \geqslant 3$, $|({}_{\mathrm{ul}}\mathrm{X}^{4,n,n})^{-1}(t1)| = \xi_n$.

(QFa5) *For every* $n \geqslant 3$,

(m5.t2) $|({}_{\mathrm{ul}}\mathrm{X}^{5,n,n})^{-1}(t2)| = 4 \cdot (n-3) \cdot \xi_n$ (m5.t5) $|({}_{\mathrm{ul}}\mathrm{X}^{5,n,n})^{-1}(t5)| = 1 \cdot (n-3)^2 \cdot \xi_n$

(m5.t3) $|({}_{\mathrm{ul}}\mathrm{X}^{5,n,n})^{-1}(t3)| = 8 \cdot (n-3) \cdot \xi_n$ (m5.t7) $|({}_{\mathrm{ul}}\mathrm{X}^{5,n,n})^{-1}(t7)| = 2 \cdot (n-3)^2 \cdot \xi_n$.

(QFa6) *For every* $n \geqslant 3$,

(m6.t2) $|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t2)| = 2(n-3)\xi_n$
(m6.t3) $|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t3)| = 8(n-3)\xi_n$
(m6.t4) $|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t4)| = 2^7\binom{n-1}{2}\binom{n-1}{3}$
(m6.t5) $|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t5)| = (8(n-3)^2 + 8\binom{n-3}{2})\xi_n$
(m6.t6) $|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t6)| = (24(n-3)^2 + 32\binom{n-3}{2})\xi_n$
(m6.t7) $|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t7)| = 8(n-3)^2\xi_n$
(m6.t8) $|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t8)| = 16\binom{n-3}{2}\xi_n$
(m6.t9) $|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t9)| = 16(n-3)^2\xi_n$
(m6.t10) $|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t10)| = 16(n-3)^2\xi_n$
(m6.t11) $|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t11)| = 16\binom{n-3}{2}\xi_n$
(m6.t12) $|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t12)| = 2^6|\mathrm{Cir}(6,n)|$
(m6.t13) $|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t13)| = 10(n-3)\binom{n-3}{2}\xi_n$
(m6.t14) $|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t14)| = 16(n-3)\binom{n-3}{2}\xi_n$
(m6.t15) $|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t15)| = 24(n-3)\binom{n-3}{2}\xi_n$
(m6.t16) $|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t16)| = 32(n-3)\binom{n-3}{2}\xi_n$
(m6.t17) $|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t17)| = 8(n-3)\binom{n-3}{2}\xi_n$
(m6.t18) $|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t18)| = 2\binom{n-3}{2}\binom{n-3}{2}\xi_n$
(m6.t19) $|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t19)| = 8\binom{n-3}{2}\binom{n-3}{2}\xi_n$
(m6.t20) $|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t20)| = 8\binom{n-3}{2}\binom{n-3}{2}\xi_n$.

Proof of (QFa4). We have $\mathrm{X}_B \cong C^4$ if and only if I is a matrix-4-circuit and $\mathrm{Supp}(B) = I$. By Lemma 162 there exist $\binom{n-1}{2}^2$ possible matrix-4-circuits I and for each of them there are 2^4 possibilities for a $B \in \{0, \pm\}^I$ with $\mathrm{Supp}(B) = I$. $\square$

Let us now prepare for the rest of the proof of Lemma 180 with some observations and definitions. Inspecting the isomorphism types in $\mathcal{F}^{\mathrm{G}}(6,n)\backslash\{(t4),(t12)\}$ (the types (t4) and (t12) are exceptions whose preimages are also exceptionally easy to count) we see that in each of them the graph contains exactly one C^4. We therefore know that for every $\mathfrak{X} \in \mathcal{F}^{\mathrm{G}}(6,n)$ (hence in particular for every $\mathfrak{X} \in \mathcal{F}^{\mathrm{G}}(5,n)$ since $\mathcal{F}^{\mathrm{G}}(5,n) \subseteq \mathcal{F}^{\mathrm{G}}(6,n)$ by (Fa6) in Corollary 174), and for every $I \in \binom{[n-1]^2}{6}$ it is necessary that there exists a matrix-4-circuit $S \subseteq I$ with $B\,|_S \in \{\pm\}^S$. For this there are $2^4 \cdot |\mathrm{Cir}(4,n)|$ possibilities. A priori, it could be that the number of possibilities to realise an isomorphism type depends on the choice of this necessary $S \subseteq I$. However, since we will take this S to be arbitrary in the proofs to follow, and since we will get results which do not depend on S, it follows as a byproduct that they are not, more precisely that for each $\mathfrak{X} \in \mathcal{F}^{\mathrm{G}}(6,n)\backslash\{(t12),(t4)\}$ the values of $|({}_{\mathrm{ul}}\mathrm{X}^{k,n,n})^{-1}(\mathfrak{X})|$ are equal to the product of $2^4 \cdot |\mathrm{Cir}(4,n)|$, and the number of possibilities to choose $B\,|_{I\backslash S} \in \{0,\pm\}^{I\backslash S}$ in such a way that $\mathrm{X}_B \in \mathfrak{X}$. By determining the latter number for each of the isomorphism types, we will prove all the formulas (m5.t2)—(m6.t20), except, as already mentioned, (m6.t4) and (m6.t12), which do not fit into the overall plan of the proof (in the case of (m6.t4) we would be overcounting the number of realisations since $K^{2,3}$ contains three copies of C^4) but which are easy to count directly.

Let $\prec$ denote the lexicographic ordering on $[n-1]^2$. Throughout the proof, we use the following conventions: we consider $I \supseteq S \in \binom{[n-1]^2}{4}$ and $B\,|_S \in \{\pm\}^S$ to be arbitrary. We set $\{a,b,c,d\} := S$, $a_1 := \mathrm{p}_1(a)$, $a_2 := \mathrm{p}_2(a)$ and analogously for b_1, b_2, c_1, c_2, d_1 and d_2. Since $\prec$ is a total order, we may assume $a \prec b \prec c \prec d$ which combined with the fact that S is a matrix-4-circuit implies $a_1 = b_1$, $c_1 = d_1$, $a_2 = c_2$ and $b_2 = d_2$. The cardinality of $I\backslash S$ depends on whether we are proving formulas from (QFa5) or (QFa6). In the former case we set $\{u\} := I\backslash S$, in the latter $\{u,v\} := I\backslash S$ with the assumption that $u \prec v$. Moreover, $u_1 := \mathrm{p}_1(u)$, $u_2 := \mathrm{p}_2(u)$, $v_1 := \mathrm{p}_1(v)$ and $v_2 := \mathrm{p}_2(v)$. Finally, let us abbreviate $\mathrm{p}(S) := \mathrm{p}_1(S) \sqcup \mathrm{p}_2(S)$.

Proof of (QFa5). As to (m5.t2), we start by noting that it follows directly from Definition 149 that $\mathrm{X}_B \in (t2)$ if and only if $B[u] = 0$ and

$$|\{u_1\}\backslash \mathrm{p}_1(S)| + |\{u_2\}\backslash \mathrm{p}_2(S)| = 1 \ . \tag{4.18}$$

We distinguish cases according to how (4.18) is satisfied.

- (C.(m5.t2).1) $|\{u_1\}\backslash \mathrm{p}_1(S)| = 0$, i.e. $u_1 \in \mathrm{p}_1(S)$. Then (4.18) implies that $u_2 \notin \mathrm{p}_2(S)$. Since there are 2 different u_1 with $u_1 \in \mathrm{p}_1(S)$ and for each of them there are $((n-1)-2) = (n-3)$ different u_2 with $u_2 \notin \mathrm{p}_2(S)$ it follows that if (C.(m5.t2).1), then there are $2(n-3)$ realisations of type (t2) by $B[u]$.
- (C.(m5.t2).2) $|\{u_1\}\backslash \mathrm{p}_1(S)| = 1$. This case is easily seen to be symmetric to (C.(m5.t2).1) w.r.t. swapping the subscripts 1 and 2. Therefore, if (C.(m5.t2).2), then there are also $2(n-3)$ realisations of type (t2) by $B[u]$.

It follows that there are $2(n-3) + 2(n-3) = 4(n-3)$ realisations of type (t2) by $B[u]$, proving (m5.t2). As to (m5.t3), this follows from (m5.t2) and Lemma 179.(l1).

As to (m5.t5), it follows from Definition 149 that $\mathrm{X}_B \in (t5)$ if and only if $B[u] = 0$ and

$$|\{u_1\}\backslash \mathrm{p}_1(S)| + |\{u_2\}\backslash \mathrm{p}_2(S)| = 2 \ . \tag{4.19}$$

Property (4.19) is equivalent to $u_1 \notin \mathrm{p}_1(S)$ and $u_2 \notin \mathrm{p}_2(S)$, and there are obviously $((n-1)-2)^2 = (n-3)^2$ different $u \in [n-1]^2$ satisfying this. Therefore, (m5.t5) is correct. As to (m5.t7), this follows from (m5.t5) and Lemma 179.(l2). This completes the proof of (QFa5). □

We now turn to the task of proving (QFa6). We prepare by proving four lemmas characterising the realisations of the isomorphism types (t8)–(t11) from Lemma 230 in Chapter 5.

Lemma 181. *For every $B \in \{0,\pm\}^I$ with $I \in \binom{[n-1]^2}{6}$, $I =: S \sqcup \{u,v\}$ and $\mathrm{X}_{B|_S} \cong C^4$ we have $\mathrm{X}_B \cong (t8)$ if and only if*

(P.(t8).1) $\mathrm{X}_{\{0\}^{\{u,v\}} \sqcup B|_S} \cong (t5)$,
(P.(t8).2) *$B[u] \in \{\pm\}$ and $B[v] \in \{\pm\}$,*
(P.(t8).3) $\{u_1,u_2\} \cap \{v_1,v_2\} = \varnothing$,
(P.(t8).4) *$(u_1 \in \mathrm{p}_1(S)$ and $v_1 \in \mathrm{p}_1(S))$ or $(u_2 \in \mathrm{p}_2(S)$ and $v_2 \in \mathrm{p}_2(S))$.*

Proof. First suppose that $\mathrm{X}_B \cong (t8)$. Then Definition 149 implies that both (P.(t8).1) and (P.(t8).2) are true. To prove (P.(t8).3) and (P.(t8).4), let $e \neq f \in \mathrm{E}(\mathrm{X}_B)$ denote the two edges in $\mathrm{X}_{B|_{\{u,v\}}}$, where $\{e\} := \mathrm{E}(\mathrm{X}_{B[u]}) = \big\{\{(u_1,n),(n,u_2)\}\big\}$ and $\{f\} := \mathrm{E}(\mathrm{X}_{B[v]}) = \big\{\{(v_1,n),(n,v_2)\}\big\}$. By hypothesis, $e \cap f = \varnothing$ and this implies that (P.(t8).3) is true. Moreover, again by hypothesis, both e and f intersect $\mathrm{X}_{B|_S} \cong C^4$ and the intersection set is *not* an edge of it.

If $u_1 \in \mathrm{p}_1(S)$, then there are still two possibilities for the intersection set $f \cap \mathrm{V}(\mathrm{X}_{B|_S})$, namely $f \cap \mathrm{V}(\mathrm{X}_{B|_S}) = \{(v_1,n)\}$ (equivalently, $v_1 \in \mathrm{p}_1(S)$) or $f \cap \mathrm{V}(\mathrm{X}_{B|_S}) = \{(n,v_2)\}$ (equivalently, $v_2 \in \mathrm{p}_2(S)$). It follows from Definition 149 that the vertex in the intersection set $e \cap \mathrm{V}(\mathrm{X}_{B|_S}) = \{(u_1,n)\}$ is *not* adjacent to the vertex in $f \cap \mathrm{V}(\mathrm{X}_{B|_S})$ if and only if the first possibility is true, i.e. $f \cap \mathrm{V}(\mathrm{X}_{B|_S}) = \{(v_1,n)\}$, i.e. $v_1 \in \mathrm{p}_1(S)$. This proves that the first clause of (P.(t8).4), and hence (P.(t8).4) itself, is true.

If $u_2 \in \mathrm{p}_2(S)$, then an entirely analogous argument as the one in the preceding paragraph shows that the second clause of (P.(t8).4), hence again (P.(t8).4) itself, is true. This proves that $\mathrm{X}_B \cong (t8)$ implies that (P.(t8).1)–(P.(t8).4) are true.

Conversely, assume (P.(t8).1)–(P.(t8).4). Then (P.(t8).2) implies that $\|\mathrm{X}_B\| = 6$ and (P.(t8).3) implies that the two edges in $\mathrm{E}(\mathrm{X}_{B|_{\{u,v\}}})$ do not intersect. Let e and f be defined as in the preceding proof of the other implication. It remains to show that $(e \cap \mathrm{V}(\mathrm{X}_{B|_S})) \cup (f \cap \mathrm{V}(\mathrm{X}_{B|_S})) \notin \mathrm{E}(\mathrm{X}_{B|_S})$. By definition of e, either $e \cap \mathrm{V}(\mathrm{X}_{B|_S}) = \{(u_1, n)\}$ or $e \cap \mathrm{V}(\mathrm{X}_{B|_S}) = \{(n, u_2)\}$.

In the former case we have $u_1 \in \mathrm{p}_1(S)$, hence the first clause of (P.(t8).4) implies $v_1 \in \mathrm{p}_1(S)$, hence $(v_1, n) \in f \cap \mathrm{V}(\mathrm{X}_B)$ by definition of f, hence $f \cap \mathrm{V}(\mathrm{X}_B) = \{(v_1, n)\}$ since $f \cap \mathrm{V}(\mathrm{X}_B)$ is a singleton by construction. In view of Definition 149 this implies that indeed $(e \cap \mathrm{V}(\mathrm{X}_{B|_S})) \cup (f \cap \mathrm{V}(\mathrm{X}_{B|_S})) = \{(u_1, n), (v_1, n)\} \notin \mathrm{E}(\mathrm{X}_{B|_S})$.

In the latter case we have $u_2 \in \mathrm{p}_2(S)$, hence the second clause of (P.(t8).4) implies $v_2 \in \mathrm{p}_2(S)$, hence $(n, v_2) \in f \cap \mathrm{V}(\mathrm{X}_B)$ by definition of f, hence $f \cap \mathrm{V}(\mathrm{X}_B) = \{(n, v_2)\}$ since $f \cap \mathrm{V}(\mathrm{X}_B)$ is a singleton by construction. In view of Definition 149 this implies that indeed $(e \cap \mathrm{V}(\mathrm{X}_{B|_S})) \cup (f \cap \mathrm{V}(\mathrm{X}_{B|_S})) = \{(n, u_2), (n, v_2)\} \notin \mathrm{E}(\mathrm{X}_{B|_S})$. This completes the proof that (P.(t8).1)–(P.(t8).4) imply $\mathrm{X}_B \cong (t8)$. ☐

Lemma 182. *For every* $B \in \{0, \pm\}^I$ *with* $I \in \binom{[n-1]^2}{6}$, $I =: S \sqcup \{u, v\}$ *and* $\mathrm{X}_{B|_S} \cong C^4$ *we have* $\mathrm{X}_B \cong (t9)$ *if and only if*

(P.(t9).1) $\mathrm{X}_{\{0\}^{\{u,v\}} \sqcup B|_S} \cong (t5)$,

(P.(t9).2) $B[u] \in \{\pm\}$ *and* $B[v] \in \{\pm\}$,

(P.(t9).3) $\{u_1, u_2\} \cap \{v_1, v_2\} = \varnothing$,

(P.(t9).4) *(*$u_1 \in \mathrm{p}_1(S)$ *and* $v_2 \in \mathrm{p}_2(S)$*) or (*$u_2 \in \mathrm{p}_2(S)$ *and* $v_1 \in \mathrm{p}_1(S)$*).*

Proof. First suppose that $\mathrm{X}_B \cong (t9)$. Then Definition 149 implies that both (P.(t9).1) and (P.(t9).2) are true. To prove (P.(t9).3) and (P.(t9).4), let $e \neq f \in \mathrm{E}(\mathrm{X}_B)$ denote the two edges in $\mathrm{X}_{B|_{\{u,v\}}}$, where $\{e\} := \mathrm{E}(\mathrm{X}_{B[u]}) = \big\{\{(u_1, n), (n, u_2)\}\big\}$ and $\{f\} := \mathrm{E}(\mathrm{X}_{B[v]}) = \big\{\{(v_1, n), (n, v_2)\}\big\}$. By hypothesis $e \cap f = \varnothing$, and this implies that (P.(t9).3) is true. Moreover, again by hypothesis, both e and f intersect $\mathrm{X}_{B|_S} \cong C^4$ and the intersection set is an edge of it.

If $u_1 \in \mathrm{p}_1(S)$, then there are still two possibilities for the intersection set $f \cap \mathrm{V}(\mathrm{X}_{B|_S})$, namely $f \cap \mathrm{V}(\mathrm{X}_{B|_S}) = \{(v_1, n)\}$ (equivalently, $v_1 \in \mathrm{p}_1(S)$) or $f \cap \mathrm{V}(\mathrm{X}_{B|_S}) = \{(n, v_2)\}$ (equivalently, $v_2 \in \mathrm{p}_2(S)$). It is evident from Definition 149 that the vertex in the intersection set $e \cap \mathrm{V}(\mathrm{X}_{B|_S}) = \{(u_1, n)\}$ is adjacent to the vertex in $f \cap \mathrm{V}(\mathrm{X}_{B|_S})$ if and only if the second possibility is true, i.e. $f \cap \mathrm{V}(\mathrm{X}_{B|_S}) = \{(n, v_2)\}$, i.e. $v_2 \in \mathrm{p}_2(S)$. This proves that the first clause of (P.(t9).4), and hence (P.(t9).4) itself, is true.

If $u_2 \in \mathrm{p}_2(S)$, an entirely analogous argument as the one in the preceding paragraph shows that then the second clause of (P.(t9).4), hence again (P.(t9).4) itself is true. This completes the proof that $\mathrm{X}_B \cong (t9)$ implies properties (P.(t9).1)–(P.(t9).4).

Conversely, suppose that (P.(t9).1)–(P.(t9).4) are true. Then (P.(t9).2) implies that $\|\mathrm{X}_B\| = 6$ and (P.(t9).3) implies that the two edges in $\mathrm{E}(\mathrm{X}_{B|_{\{u,v\}}})$ do not intersect. Let e and f be defined as in the preceding proof of the other implication. It remains to show that $(e \cap \mathrm{V}(\mathrm{X}_{B|_S})) \cup (f \cap \mathrm{V}(\mathrm{X}_{B|_S})) \in \mathrm{E}(\mathrm{X}_{B|_S})$. By definition of e, either $e \cap \mathrm{V}(\mathrm{X}_{B|_S}) = \{(u_1, n)\}$ or $e \cap \mathrm{V}(\mathrm{X}_{B|_S}) = \{(n, u_2)\}$.

In the former case we have $u_1 \in \mathrm{p}_1(S)$, hence the first clause of (P.(t9).4) implies that $v_2 \in \mathrm{p}_2(S)$, hence $(n, v_2) \in f \cap \mathrm{V}(\mathrm{X}_{B|_S})$ by definition of f, hence $f \cap \mathrm{V}(\mathrm{X}_{B|_S}) = \{(n, v_2)\}$ since $f \cap \mathrm{V}(\mathrm{X}_{B|_S})$ is a singleton by construction. In view of Definition 149 this implies that indeed $(e \cap \mathrm{V}(\mathrm{X}_{B|_S})) \cup (f \cap \mathrm{V}(\mathrm{X}_{B|_S})) = \{(u_1, n), (n, v_2)\} \in \mathrm{E}(\mathrm{X}_{B|_S})$.

In the latter case we have $u_2 \in \mathrm{p}_2(S)$, hence the second clause of (P.(t9).4) implies that $v_1 \in \mathrm{p}_1(S)$, hence $(v_1, n) \in f \cap \mathrm{V}(\mathrm{X}_{B|_S})$ by definition of f, hence $f \cap \mathrm{V}(\mathrm{X}_{B|_S}) = \{(v_1, n)\}$ since $f \cap \mathrm{V}(\mathrm{X}_{B|_S})$ is a singleton by construction. In view of Definition 149 this implies that indeed $(e \cap \mathrm{V}(\mathrm{X}_{B|_S})) \cup (f \cap \mathrm{V}(\mathrm{X}_{B|_S})) = \{(n, u_2), (v_1, n)\} \in \mathrm{E}(\mathrm{X}_{B|_S})$. This completes the proof that (P.(t9).1)–(P.(t9).4) imply $\mathrm{X}_B \cong (t9)$. ☐

Lemma 183. *For every* $B \in \{0, \pm\}^I$ *with* $I \in \binom{[n-1]^2}{6}$, $I =: S \sqcup \{u, v\}$ *and* $\mathrm{X}_{B|_S} \cong C^4$ *we have* $\mathrm{X}_B \cong (t10)$ *if and only if*

(P.(t10).1) $\mathrm{X}_{\{0\}^{\{u,v\}} \sqcup B|_S} \cong (t5)$,

(P.(t10).2) $B[u] \in \{\pm\}$ *and* $B[v] \in \{\pm\}$,

(P.(t10).3) $\{u_1, u_2\} \cap \{v_1, v_2\} \neq \varnothing$,

(P.(t10).4) $\{u_1, u_2\} \cap \mathrm{p}(S) = \varnothing$ *or* $\{v_1, v_2\} \cap \mathrm{p}(S) = \varnothing$.

Proof. First suppose that $X_B \cong (t10)$. Then Definition 149 implies that both (P.(t10).1) and (P.(t10).2) are true. To prove (P.(t10).3) and (P.(t10).4), let $e \in E(X_B)$ denote the unique edge which does not intersect $X_{B|_S} \cong C^4$, and let $f \in E(X_B)$ denote the unique edge which intersects $X_{B|_S} \cong C^4$. In view of Definition 149,

$$(e = \{(u_1, n), (n, u_2)\} \text{ and } f = \{(v_1, n), (n, v_2)\}) \text{ or } (e = \{(v_1, n), (n, v_2)\} \text{ and } f = \{(u_1, n), (n, u_2)\}). \tag{4.20}$$

By definition of e and f we have $e \cap f \neq \varnothing$, hence whatever of the two clauses of (4.20) is true, either $u_1 = v_1$ or $u_2 = v_2$. Therefore property (P.(t10).3) is true. By definition of e, if $e = \{(u_1, n), (n, u_2)\}$, then $u_1 \notin p_1(S)$ and $u_2 \notin p_2(S)$, hence the first clause of (P.(t10).4) is true, and if $e = \{(v_1, n), (n, v_2)\}$, then $v_1 \notin p_1(S)$ and $v_2 \notin p_2(S)$, hence then the second clause of (P.(t10).4) is true.

Conversely, suppose that properties (P.(t10).1)–(P.(t10).4) are true. Then (P.(t10).2) implies $\|X_B\| = 6$ and (P.(t10).3) implies that the two edges corresponding to $u \neq v$ intersect. It remains to prove that the 2-path consisting of these edges intersects $X_{B|_S} \cong C^4$ with one of its endvertices. To see this, note first that (P.(t10).1) implies (4.23) which combined with $u \neq v$ implies that

$$\{u_1, u_2, v_1, v_2\} \cap p(S) \neq \varnothing \ . \tag{4.21}$$

Moreover, we know from (4.20) together with $e \cap f \neq \varnothing$ that $u_1 = v_1$ or $u_2 = v_2$.

If $u_1 = v_1$, then (4.21) cannot be true by virtue of $u_1 = v_1 \in p_1(S)$ since then (P.(t10).4) would become false. Therefore, if $u_1 = v_1$, then $u_1 = v_1 \notin p_1(S)$, and (4.21) implies $\{u_2, v_2\} \cap p_2(S) \neq \varnothing$. It is impossible that $\{u_2, v_2\} \subseteq p_2(S)$ for this combined with $u_1 = v_1$ would imply $|\{u_1, v_1\} \backslash p_1(S)| + |\{u_2, v_2\} \backslash p_2(S)| = 1$, hence contradict (4.23). Therefore, $|\{u_2, v_2\} \cap p_2(S)| = 1$, and since $u_1 = v_1$ implies $u_2 \neq v_2$, this is what we wanted to prove: exactly one of the two endvertices $(n, u_2), (n, v_2) \in V(X_B)$ intersects $X_{B|_S} \cong C^4$.

If $u_2 = v_2$, then an entirely analogous argument as in the preceding paragraph shows that exactly one of the two endvertices $(u_1, n), (v_1, n) \in V(X_B)$ intersects the $X_{B|_S} \cong C^4$.

The proof that properties (P.(t10).1)–(P.(t10).4) imply $X_B \cong (t10)$ is now complete. □

Lemma 184. *For every $B \in \{0, \pm\}^I$ with $I \in \binom{[n-1]^2}{6}$, $I =: S \sqcup \{u, v\}$ and $X_{B|_S} \cong C^4$ we have $X_B \cong (t11)$ if and only if*

(P.(t11).1) $X_{\{0\}^{\{u,v\}} \sqcup B|_S} \cong (t5)$,
(P.(t11).2) $B[u] \in \{\pm\}$ *and* $B[v] \in \{\pm\}$,
(P.(t11).3) $\{u_1, u_2\} \cap \{v_1, v_2\} \neq \varnothing$,
(P.(t11).4) $\{u_1, u_2\} \cap p(S) \neq \varnothing$ *and* $\{v_1, v_2\} \cap p(S) \neq \varnothing$.

Proof. First suppose that $X_B \cong (t11)$. Then Definition 149 implies that both (P.(t11).1) and (P.(t11).2) are true. To prove (P.(t11).3) and (P.(t11).4), let $e \neq f \in E(X_B)$ denote the two edges in $E(X_B)$ forming the 2-path which intersects $X_{B|_S} \cong C^4$ with its inner vertex. As in the proof of Lemma 183, we know that (4.20) is true. By definition of e and f we have $e \cap f \neq \varnothing$, hence whatever of the two clauses of (4.20) is true, either $u_1 = v_1$ or $u_2 = v_2$. Therefore property (P.(t11).3) is true. By definition of e and f, both e and f intersect $X_{B|_S} \cong C^4$. If the first clause in (4.20) is true then e intersecting $X_{B|_S} \cong C^4$ is equivalent to ($u_1 \in p_1(S)$ or $u_2 \in p_2(S)$) and f intersecting $X_{B|_S} \cong C^4$ is equivalent to ($v_1 \in p_1(S)$ or $v_2 \in p_2(S)$). Then (P.(t11).4) is indeed true. If the second clause in (4.20) is true, interchanging 'u' and 'v' in the preceding sentence shows that then (P.(t11).4) is true as well. This completes the proof that $X_B \cong (t11)$ implies (P.(t11).1)–(P.(t11).4).

Conversely, suppose that properties (P.(t11).1)–(P.(t11).4) are true. Then (P.(t11).2) implies $\|X_B\| = 6$ and (P.(t11).3) implies that the two edges corresponding to $u \neq v$ intersect. It remains to prove that the 2-path consisting of these edges intersects $X_{B|_S} \cong C^4$ with its inner vertex. Similar to the proof of Lemma 183 we know that (4.21) and that $u_1 = v_1$ or $u_2 = v_2$.

If $u_1 = v_1$, then $u_2 \in p_2(S)$ is impossible since this together with $u \neq v$ would imply $u_1 = v_1 \notin p_1(S)$ which due to the second clause of (P.(t11).4) would imply $v_2 \in p_2(S)$; but $u_2 \in p_2(S)$, $u_1 = v_1$ and $v_2 \in p_2(S)$ combined imply $|\{u_1, v_1\} \backslash p_1(S)| + |\{u_2, v_2\} \backslash p_2(S)| = 1$, a contradiction to (4.23). For an entirely analogous reason $v_2 \in p_2(S)$ is impossible, too. Since both $u_2 \notin p_2(S)$ and $v_2 \notin p_2(S)$, it follows from (4.21) that $u_1 = v_1 \in p_1(S)$. This is what we wanted to prove: the common vertex $(u_1, n) = (v_1, n)$ of e and f (i.e. the inner vertex of the 2-path formed by e and f) is also the unique vertex of intersection with $\mathrm{X}_{B|_S} \cong C^4$.

If $u_2 = v_2$, then an entirely analogous argumentation as in the preceding paragraph shows that the common vertex $(n, u_2) = (n, v_2)$ of e and f (i.e. the inner vertex of the 2-path which is formed by e and f) is also the unique vertex of intersection with $\mathrm{X}_{B|_S} \cong C^4$.

The proof that properties (P.(t11).1)–(P.(t11).4) imply $\mathrm{X}_B \cong (t11)$ is now complete. □

Proof of (QFa6). As to (m6.t2), it follows from Definition 149 that $\mathrm{X}_B \cong (t2)$ if and only if

$$|\{u_1, v_1\} \backslash \mathrm{p}_1(S)| + |\{u_2, v_2\} \backslash \mathrm{p}_2(S)| = 1 \ . \tag{4.22}$$

(C.(m6.t2).1) $|\{u_1, v_1\} \backslash \mathrm{p}_1(S)| = 0$. Then (4.22) implies that $|\{u_2, v_2\} \backslash \mathrm{p}_2(S)| = 1$ which is equivalent to (4.28). The property defining Case 1 is equivalent to $\{u_1, v_1\} \subseteq \mathrm{p}_1(S)$. Property (4.28) implies two cases:

(1) $u_2 = v_2$ and $\{u_2, v_2\} \cap \mathrm{p}_2(S) = \varnothing$. Then $u_2 = v_2$ and $u \prec v$ imply that $u_1 < v_1$. This together with $\{u_1, v_1\} \subseteq \mathrm{p}_1(S)$ implies $u_1 = a_1 = c_1$ and $v_1 = b_1 = d_1$. Therefore, it is $u_2 = v_2$ alone which determines the two pairs u and v. The property $u_2 = v_2$ and $\{u_2, v_2\} \cap \mathrm{p}_2(S) = \varnothing$ is equivalent to $u_2 = v_2 \notin \mathrm{p}_2(S)$. It follows that if (C.(m6.t2).1).(1), then there are $(n-1)-2$ realisations of type (t2) by u and v.

(2) $u_2 \neq v_2$ and $|\{u_2, v_2\} \cap \mathrm{p}_2(S)| = 1$. Then either $u_2 \in \mathrm{p}_2(S)$ or $v_2 \in \mathrm{p}_2(S)$. If $u_2 \in \mathrm{p}_2(S)$, then because of $\{u_1, v_1\} \subseteq \mathrm{p}_1(S)$ it follows that $u \in \{a, b, c, d\}$, a contradiction to $I \backslash S = \{u, v\}$. Similarly, if $v_2 \in \mathrm{p}_2(S)$, then the same contradiction arises with regard to v. Therefore, the case (C.(m6.t2).1).(2) cannot occur.

It follows that if (C.(m6.t2).1), then there are exactly $(n-1)-2$ realisations of type (t2) by $B\,|_{\{u,v\}}$.

(C.(m6.t2).2) $|\{u_1, v_1\} \backslash \mathrm{p}_1(S)| = 1$. Then (4.22) implies $|\{u_2, v_2\} \backslash \mathrm{p}_2(S)| = 0$ which is equivalent to $\{u_2, v_2\} \subseteq \mathrm{p}_2(S)$. The property defining (C.(m6.t2).2) is equivalent to (4.26). Now an argument entirely analogous to the one given for (C.(m6.t2).1) shows that if (C.(m6.t2).2), then there are exactly $(n-1)-2$ realisations of type (t2) by $B\,|_{\{u,v\}}$.

It follows that there are exactly $2 \cdot ((n-1)-2)$ different $I \backslash S = \{u, v\}$ with $\mathrm{X}_B \cong (t2)$. This completes the proof of (m6.t2).

As to (m6.t3), notice that a necessary condition for type (t3) is that $|\mathrm{V}(\mathrm{X}_B) \backslash \mathrm{V}(\mathrm{X}_{B|_S})| = 1$. Therefore the set of all suitable $I \in \binom{[n-1]^2}{6}$ is a subset (possibly nonproper) of those which are suitable for type (t2). We may therefore reexamine the analysis carried out for (m6.t2) and in each of the cases count the number of $B \in \{0, \pm\}^I$ with $\mathrm{X}_B \cong (t3)$.

If (C.(m6.t2).1).(1), then properties $u_1 = a_1 = c_1$ and $v_1 = b_1 = d_1$ show that both u and v have the property that if one of them indexes a nonzero value of B, then there is an edge intersecting $\mathrm{X}_{B|_S} \cong C^4$. Since otherwise we would have $K^{2,3}$, *exactly* one of them must be nonzero. This implies exactly 4 possibilities to realise type (t3) for each of the $(n-1)-2$ realisations of type (t2) which were offered in (C.(m6.t2).1).(1). Therefore, if (C.(m6.t2).1).(1), then there are exactly $4 \cdot ((n-1)-2)$ realisations of type (t3) by $B\,|_{I \backslash S}$. Since the case (C.(m6.t2).1).(2) is as impossible now as it was back then, it follows that this is also the number of realisations for the entire (C.(m6.t2).1).

If (C.(m6.t2).2), then by interchanging the subscripts 1 and 2 we may use the same analysis as for the case (C.(m6.t2).1) to reach the conclusion that there are exactly $4 \cdot ((n-1)-2)$ realisations of type (t3) by $B\,|_{\{u,v\}} = B\,|_{I \backslash S}$. It follows that there are are exactly $4 \cdot ((n-1)-2) + 4 \cdot ((n-1)-2) = 8 \cdot (n-3)$ realisations of type (t3) by $B\,|_{I \backslash S}$. This proves (m6.t3).

As to (m6.t4), it is evident that the number of possibilities to realise a $K^{2,3}$ is $2 \cdot 2^6 \cdot \binom{n-1}{2} \cdot \binom{n-1}{3}$, the first factor accouting for the two possibilities of either choosing two of the first indices and three of the last, or vice versa.

As to (m6.t5), note that $\|(t5)\| = 4$, hence it is necessary that $B[u] = B[v] = 0$. Therefore, the number of $B\,|_{\{u,v\}} \in \{0, \pm\}^{I\setminus S}$ with $\mathrm{X}_B \cong (t5)$ equals the number of $\{u,v\} \in \binom{[n-1]^2\setminus S}{2}$ such that

$$|\{u_1, v_1\}\backslash \mathrm{p}_1(S)| + |\{u_2, v_2\}\backslash \mathrm{p}_2(S)| = 2\ . \tag{4.23}$$

We now distinguish cases according to how (4.23) is satisfied.

(C.(m6.t5).1) $|\{u_1, v_1\}\backslash \mathrm{p}_1(S)| = 0$. Then (4.23) implies $|\{u_2, v_2\}\backslash \mathrm{p}_2(S)| = 2$, which is equivalent to (4.25). The property defining Case (C.(m6.t5).1) is equivalent to $\{u_1, v_1\} \subseteq \mathrm{p}_1(S)$. There are now two further cases:

(1) $u_1 = v_1$. Then there are exactly 2 possible set inclusions $\{u_1, v_1\} = \{u_1\} \subseteq \mathrm{p}_1(S) = \{a_1, c_1\}$. For each of them, there are exactly $\binom{(n-1)-2}{2}$ different sets $\{u_2, v_2\}$ with property (4.25). Since $u_1 = v_1$ and $u \prec v$ imply $u_2 < v_2$, each of these sets determines the two pairs u and v. Therefore, if (C.(m6.t5).1).(1), then there are exactly $2 \cdot \binom{(n-1)-2}{2}$ realisations of type (t5) by $B\,|_{\{u,v\}}$.

(2) $u_1 \neq v_1$. Then $u \prec u$ implies $u_1 < u_1$. Now there is exactly one possible set inclusion $\{u_1, v_1\} \subseteq \mathrm{p}_1(S) = \{a_1, c_1\}$. When this inclusion holds, there are exactly $\binom{(n-1)-2}{2}$ different sets $\{u_2, v_2\}$ with property (4.25). Each of them can be realised in exactly 2 ways by u and v, either by $u_2 < v_2$ or by $v_2 < u_2$. Therefore, if (C.(m6.t5).1).(2), then there are again (with a qualitatively different reason for the factor 2) exactly $2 \cdot \binom{(n-1)-2}{2}$ realisations of type (t5) by $B\,|_{\{u,v\}}$.

It follows that if (C.(m6.t5).1), then there are exactly $4 \cdot \binom{(n-1)-2}{2}$ different realisations of type (t5) by $B\,|_{\{u,v\}}$.

(C.(m6.t5).2) $|\{u_1, v_1\}\backslash \mathrm{p}_1(S)| = 1$. Then (4.23) implies $|\{u_2, v_2\}\backslash \mathrm{p}_2(S)| = 1$. Hence, in the present situation, the equations (4.26) and (4.28) are simultaneously true. There are now two further cases depending on the manner in which (4.26) is true:

(1) $u_1 = v_1$ and $\{u_1, v_1\} \cap \mathrm{p}_1(S) = \varnothing$. There are $((n-1)-2)$ possibilities for this. For each of them the clause $u_2 = v_2$ and $\{u_2, v_2\} \cap \mathrm{p}_1(S) = \varnothing$ in (4.28) cannot be true because it would imply $u = v$ and therefore for each of them $u_2 \neq v_2$ and $|\{u_2, v_2\} \cap \mathrm{p}_2(S)| = 1$ must be true. In the following, keep in mind that $u_1 = v_1$ and $u \prec v$ implies $u_2 < v_2$. If $u_2 = a_2 = c_2$, then $v_2 \neq b_2 = d_2$, hence there are exactly $n-1-a_2-1$ different v_2, and therefore as many different realisations of type (t5) by u and v. If $u_2 = b_2 = d_2$, then there are exactly $n-1-b_2$ different v_2, and therefore as many different realisations of type (t5) by u and v. If $v_2 = a_2 = c_2$, then there are exactly $a_2 - 1$ different u_2, and therefore as many different realisations of type (t5) by u and v. If $v_2 = b_2 = d_2$, then $u_2 \neq a_2 = c_2$, hence there are exactly $b_2 - 1 - 1$ different u_2 and therefore as many different realisations of type (t5) by u and v.
Therefore, if (C.(m6.t5).2).(1), then there are $\big((n-1-a_2-1) + (n-1-b_2) + (a_2-1) + (b_2-1-1)\big) \cdot ((n-1)-2) = 2 \cdot ((n-1)-2)^2$ realisations of type (t5) by $B\,|_{I\setminus S} = B\,|_{\{u,v\}}$.

(2) $u_1 \neq v_1$ and $|\{u_1, v_1\} \cap \mathrm{p}_1(S)| = 1$. Then $u \prec v$ implies $u_1 < v_1$ and there are two further cases:

(1) $|\{u_1, v_1\} \cap \mathrm{p}_1(S)| = 1$ is true due to $u_1 \in \mathrm{p}_1(S) = \{a_1, c_1\}$. Then $v_1 \notin \{a_1, c_1\} = \mathrm{p}_1(S)$ and there are two further cases:

(1) $u_1 = a_1 = b_1$. Since $u_1 < v_1$ and $v_1 \notin \{a_1, c_1\}$, there are then exactly $n-1-a_1-1$ different v_1. For each of them, there are two cases. If the first clause of (4.28) is true, then there are exactly $(n-1)-2$ different such $u_2 = v_2$, and each such $u_2 = v_2$ *determines* the two pairs u and

v. Therefore in this case there are exactly $(n-1-a_1-1)\cdot((n-1)-2)$ realisations of type (t5) by $B\,|_{\{u,v\}}$. If the second clause of (4.28) is true, then since $u_1 = a_1 = b_1$ combined with $u \neq a$ and $u \neq b$ implies $u_2 \notin \{a_2 = c_2, b_2 = d_2\} = \mathrm{p}_2(S)$, it follows that $|\{u_2, v_2\} \cap \mathrm{p}_2(S)| = 1$ is true as $v_2 \in \mathrm{p}_2(S)$. Therefore in this case there are exactly $(n-1)-2$ different u_2 and exactly 2 different v_2 for each of them and hence exactly $(n-1-a_1-1)\cdot 2\cdot((n-1)-2)$ realisations of type (t5) by $B\,|_{\{u,v\}}$.

Therefore, if (C.(m6.t5).2).(2).(1).(1), then there are exactly $3\cdot(n-1-a_1-1)\cdot((n-1)-2)$ different realisations of type (t5) by $B\,|_{\{u,v\}}$.

(2) $u_1 = c_1 = d_1$. Since $u_1 < v_1$, there are then exactly $n-1-c_1$ different v_1. For each of them, there are two cases. If the first clause of (4.28) is true, then there are exactly $(n-1-c_1)\cdot((n-1)-2)$ realisations of type (t5) by $B\,|_{\{u,v\}}$. If the second clause of (4.28) is true, then since $u_1 = c_1 = d_1$ combined with $u \neq c$ and $u \neq d$ implies $u_2 \notin \{a_2 = c_2, b_2 = d_2\} = \mathrm{p}_2(S)$, it follows that $|\{u_2, v_2\} \cap \mathrm{p}_2(S)| = 1$ is true as $v_2 \in \mathrm{p}_2(S)$. Therefore in this case there are exactly $(n-1)-2$ different u_2 and for each of them exactly 2 different v_2, thus exactly $(n-1-c_1)\cdot 2\cdot((n-1)-2)$ realisations of type (t5) by $B\,|_{\{u,v\}}$.

Therefore, if (C.(m6.t5).2).(2).(1).(2), then there are exactly $3\cdot(n-1-c_1)\cdot((n-1)-2)$ realisations of type (t5) by $B\,|_{\{u,v\}}$.

It follows that if (C.(m6.t5).2).(2).(1), then there are exactly $3\cdot(n-1-a_1-1)\cdot((n-1)-2)+3\cdot(n-1-c_1)\cdot((n-1)-2) = 3\cdot(2n-a_1-c_1-3)\cdot((n-1)-2)$ different realisations of type (t5) by $B\,|_{\{u,v\}}$.

(2) $|\{u_1, v_1\} \cap \mathrm{p}_1(S)| = 1$ is true due to $v_1 \in \mathrm{p}_1(S) = \{a_1, c_1\}$. Then $u_1 \notin \{a_1, c_1\}$ and there are two further cases:

(1) $v_1 = a_1 = b_1$. Since $u_1 < v_1$, there are then exactly $a_1 - 1$ different u_1. For each of them, there are two cases. If the first clause of (4.28) is true, then there are exactly $(n-1)-2$ different $u_2 = v_2$ and each such $u_2 = v_2$ determines the pair (u, v). Hence in this case there are exactly $(a_1-1)\cdot((n-1)-2)$ realisations of type (t5) by $B\,|_{\{u,v\}}$. If the second clause of (4.28) is true, then since $v_1 = a_1 = b_1$ combined with $v \neq a$ and $v \neq b$ implies $v_2 \notin \{a_2 = c_2, b_2 = d_2\} = \mathrm{p}_2(S)$, we know that $|\{u_2, v_2\} \cap \mathrm{p}_2(S)| = 1$ must be true as $u_2 \in \mathrm{p}_2(S)$, hence there are 2 different u_2 and for each of them exactly $(n-1)-2$ different v_2, hence in this case there are exactly $(a_1-1)\cdot 2\cdot((n-1)-2)$ realisations of type (t5) by $B\,|_{\{u,v\}}$.

Therefore, if (C.(m6.t5).2).(2).(2).(1), then there are exactly $3\cdot(a_1-1)\cdot((n-1)-2)$ different realisations of type (t5) by $B\,|_{\{u,v\}}$.

(2) $v_1 = c_1 = d_1$. Since $u_1 < v_1$ and $u_1 \notin \{a_1, c_1\}$, there are then exactly $c_1 - 1 - 1$ different u_1. For each of them, there are two cases. If the first clause of (4.28) is true, then there are exactly $(c_1-1-1)\cdot((n-1)-2)$ realisations of type (t5) by $B\,|_{\{u,v\}}$. If the second clause of (4.28) is true, then since $v_1 = c_1 = d_1$ combined with $v \neq c$ and $v \neq d$ implies $v_2 \notin \{a_2 = c_2, b_2 = d_2\} = \mathrm{p}_2(S)$, it follows that $|\{u_2, v_2\} \cap \mathrm{p}_2(S)| = 1$ is true as $u_2 \in \mathrm{p}_2(S)$. Therefore in this case there are exactly $(n-1)-2$ different v_2 and for each of them exactly 2 different u_2, hence exactly $(c_1-1-1)\cdot 2\cdot((n-1)-2)$ realisations of type (t5) by $B\,|_{\{u,v\}}$.

Therefore, if (C.(m6.t5).2).(2).(2).(2), then there are exactly $3\cdot(c_1-1-1)\cdot((n-1)-2)$ realisations of type (t5) by $B\,|_{\{u,v\}}$.

It follows that if (C.(m6.t5).2).(2).(2), then there are exactly $3\cdot(a_1-1)\cdot((n-1)-2)+3\cdot(c_1-1-1)\cdot((n-1)-2)=3\cdot(a_1+c_1-3)\cdot((n-1)-2)$ realisations of type (t5) by $B\mid_{\{u,v\}}$.

It follows that if (C.(m6.t5).2).(2), then there are exactly $3\cdot(2n-a_1-c_1-3)\cdot((n-1)-2)+3\cdot(a_1+c_1-3)\cdot((n-1)-2)=6\cdot((n-1)-2)^2$ realisations of type (t5) by $B\mid_{\{u,v\}}$.

It follows that if (C.(m6.t5).2), then there are exactly $2\cdot((n-1)-2)^2+6\cdot((n-1)-2)^2=8\cdot((n-1)-2)^2$ realisations of type (t5) by $B\mid_{\{u,v\}}$.

(C.(m6.t5).3) $|\{u_1,v_1\}\backslash \mathrm{p}_1(S)|=2$. This is equivalent to (4.27). Furthermore, (4.23) implies $|\ \{u_2,v_2\}\setminus \mathrm{p}_2(S)\ |=0$, which is equivalent to $\{u_2,v_2\}\subseteq \mathrm{p}_2(S)$. Swapping the subscripts 1 and 2 in the analysis of (C.(m6.t5).1) shows that if (C.(m6.t5).3), then there are exactly $4\cdot\binom{(n-1)-2}{2}$ realisations of type (t5) by $B\mid_{\{u,v\}}$.

Thus for the fixed S, there are exactly $4\cdot\binom{(n-1)-2}{2}+8\cdot((n-1)-2)^2+4\cdot\binom{(n-1)-2}{2}=8\cdot(n-3)^2+8\cdot\binom{n-3}{2}$ different $I\backslash S=\{u,v\}$ with $\mathrm{X}_B\cong(t5)$. This completes the proof of (m6.t5).

As to (m6.t6) and (m6.t7), let us first note that both for (t6) and for (t7) a necessary condition is that X_B contain exactly two vertices not in $\mathrm{X}_{B|_S}\cong C^4$. Therefore, the set of all $I\backslash S=\{u,v\}$ with $\mathrm{X}_B\cong(t6)$ is a subset of the set of those $\{u,v\}$ with $\mathrm{X}_{\{0\}^{\{u,v\}}}\cong(t5)$, and likewise for (t7). We may therefore determine both (m6.t6) and (m6.t7) by a single reexamination of the analysis given for type (t5). In each of the cases which we distinguished there we now have to count the number of those $B\mid_{\{u,v\}}\in\{0,\pm\}^{\{u,v\}}$ with $\mathrm{X}_B\cong(t6)$ and also of those $B\mid_{\{u,v\}}\in\{0,\pm\}^{\{u,v\}}$ with $\mathrm{X}_B\cong(t7)$. We can prepare for this as follows. Consider the properties

(i1) $(\{u_1,u_2\}\cap \mathrm{p}(S)\neq\varnothing$ and $\{v_1,v_2\}\cap \mathrm{p}(S)=\varnothing)$ or $(\{u_1,u_2\}\cap \mathrm{p}(S)=\varnothing$ and $\{v_1,v_2\}\cap \mathrm{p}(S)\neq\varnothing)$,
(i2) $(\{u_1,u_2\}\cap \mathrm{p}(S)\neq\varnothing$ and $\{v_1,v_2\}\cap \mathrm{p}(S)\neq\varnothing)$.

In each of the cases to be reexamined, these properties alone determine how many $B\mid_{\{u,v\}}\in\{0,\pm\}^{\{u,v\}}$ realise (t6) or (t7). Let us first focus on (t6). In (i1), each of the two clauses of that disjunction has the property that if it is true, then there are exactly 2 possibilities for a $B\mid_{\{u,v\}}$ with $\mathrm{X}_B\cong(t6)$. For the first clause these are ($B[u]\in\{\pm\}$ and $B[v]=0$), for the second clause ($B[u]=0$ and $B[v]\in\{\pm\}$). Moreover, the disjunction is evidently exclusive. Therefore, if property (i1) is true, then the number of $B\mid_{\{u,v\}}\in\{0,\pm\}^{\{u,v\}}$ with $\mathrm{X}_B\cong(t6)$ is exactly 2-times as large as the number of $\{u,v\}\in\binom{[n-1]^2}{2}$ with $\mathrm{X}_{\{0\}^{\{u,v\}}}\cong(t5)$ which was determined in the proof of (m6.t5). If property (i2) is true, then there are exactly 4 possibilities for a $B\mid_{\{u,v\}}$ with $\mathrm{X}_B\cong(t6)$: ($B[u]\in\{\pm\}$ and $B[v]=0$) or ($B[u]=0$ and $B[v]\in\{\pm\}$). Therefore, if property (i2) is true, then there are exactly 4-times as many realisations of isomorphism type (t6) by $B\mid_{I\backslash S}=B\mid_{\{u,v\}}$ as there had been for type (t5).

Let us now turn to (t7). In case (i1) there are (just as for type (t6)) exactly 2 possibilities for a $B\mid_{\{u,v\}}$ with $\mathrm{X}_B\cong(t7)$. This time, these are ($B[u]=0$ and $B[v]\in\{\pm\}$) for the first clause of (i1), and ($B[u]\in\{\pm\}$ and $B[v]=0$) for the second clause. Again, due to the mutual exclusiveness of the clauses, it follows that whenever case (i1) is true (no matter by way of which clause), there are exactly 2-times as many $B\mid_{\{u,v\}}\in\{0,\pm\}^{\{u,v\}}$ with $\mathrm{X}_B\cong(t7)$ as there are $\{u,v\}\in\binom{[n-1]^2}{2}$ with $\mathrm{X}_{\{0\}^{\{u,v\}}}\cong(t5)$. Concerning property (i2), however, there is a genuine difference: when this property is true, there is *no* possibility to choose $B\mid_{\{u,v\}}$ so as to create exactly one edge disjoint from the $\mathrm{X}_{B|_S}\cong C^4$. We can now begin inspecting the cases.

If (C.(m6.t5).1), then the inclusion $\{u_1,v_1\}\subseteq \mathrm{p}_1(S)$ alone, no matter whether $u_1=v_1$ or not, implies that property (i2) is true and without going any deeper we know that there are exactly $4\cdot 4\cdot\binom{(n-1)-2}{2}=16\cdot\binom{(n-1)-2}{2}$ realisations of type (t6) and 0 realisations of type (t7) by $B\mid_{\{u,v\}}$.

If (C.(m6.t13).2), we have to descend one level deeper. If (C.(m6.t13).2).(1) then it is known that $u_1=v_1$, $\{u_1,v_1\}\cap \mathrm{p}_1(S)=\varnothing$, $u_2<v_2$ and $|\{u_2,v_2\}\cap \mathrm{p}_2(S)|=1$ and obviously this implies that property (i1) is true. Therefore without having to reexamine further subcases we know that if (C.(m6.t13).2).(1), then there are exactly $2\cdot 2\cdot((n-1)-2)^2=4\cdot((n-1)-2)^2$ realisations of type (t6) and also exactly $2\cdot 2\cdot((n-1)-2)^2=4\cdot((n-1)-2)^2$ realisations of type (t7) by $B\mid_{\{u,v\}}$.

If (C.(m6.t13).2).(2), however, then we have to go deeper still. Although we then already know that $u_1 < v_1$ and $|\{u_1, v_1\} \cap \mathrm{p}_1(S)| = 1$, and therefore know that

$$\{u_1, u_2\} \cap \mathrm{p}(S) \neq \varnothing \text{ or } \{v_1, v_2\} \cap \mathrm{p}(S) \neq \varnothing \quad , \tag{4.24}$$

at the present stage of our knowledge this latter property is compatible with both (i1) and (i2) (i.e., we do not know yet whether the 'or' in (4.24) is true as an 'and'). The reason is that we do not yet have any knowledge about u_2 and v_2. Therefore, neither descending down to (C.(m6.t13).2).(2).(1) nor to (C.(m6.t13).2).(2).(1).(1) is sufficient for us to know whether (i1) or (i2) is true. We therefore have to go all the way down to the two (anonymous) subcases of maximal depth within the case (C.(m6.t13).2).(2).(1).(1). In the first of the two subcases we know that $\{u_2, v_2\} \cap \mathrm{p}_2(S) = \varnothing$ and combining this with our knowledge of $|\{u_1, v_1\} \cap \mathrm{p}_1(S)| = 1$ we may conclude that exactly one of the two clauses in (4.24), and hence property (i1) is true. Therefore, in the present subcase there are exactly $2 \cdot (n-1-a_1-1) \cdot ((n-1)-2)$ realisations of type (t6) and also $2 \cdot (n-1-a_1-1) \cdot ((n-1)-2)$ realisations of type (t7) by $B\ |_{\{u,v\}}$.

In the second of the two subcases we know that $u_2 \neq v_2$ and $|\{u_2, v_2\} \cap \mathrm{p}_2(S)| = 1$. Recall that at present we also know that $u_1 < v_1$ and $|\{u_1, v_1\} \cap \mathrm{p}_1(S)| = 1$. Keeping in mind the fact that because of $u \notin S$ at most one projection of u can be contained in $\mathrm{p}(S)$ (and the analogous fact about v), we may argue that if $|\{u_1, v_1\} \cap \mathrm{p}_1(S)| = 1$ is true as ($u_1 \in \mathrm{p}_1(S)$ and $v_1 \notin \mathrm{p}_1(S)$), then $|\{u_2, v_2\} \cap \mathrm{p}_2(S)| = 1$ must be true as ($u_2 \notin \mathrm{p}_2(S)$ and $v_2 \in \mathrm{p}_2(S)$), and if $|\{u_1, v_1\} \cap \mathrm{p}_1(S)| = 1$ is true as ($u_1 \notin \mathrm{p}_1(S)$ and $v_1 \in \mathrm{p}_1(S)$), then $|\{u_2, v_2\} \cap \mathrm{p}_2(S)| = 1$ must be true as ($u_2 \in \mathrm{p}_2(S)$ and $v_2 \notin \mathrm{p}_2(S)$). Since in both cases both clauses of (4.24) are true, it follows that (i2) is true. Therefore in the present subcase there are exactly $4 \cdot (n-1-a_1-1) \cdot 2 \cdot ((n-1)-2) = 8 \cdot (n-1-a_1-1) \cdot ((n-1)-2)$ realisations of type (t6) and 0 realisations of type (t7) by $B\ |_{\{u,v\}}$. Adding up our findings, it follows that if (C.(m6.t13).2).(2).(1).(1), then there are exactly $2 \cdot (n-1-a_1-1) \cdot ((n-1)-2) + 8 \cdot (n-1-a_1-1) \cdot ((n-1)-2) = 10 \cdot (n-1-a_1-1) \cdot ((n-1)-2)$ realisations of type (t6) but merely $2 \cdot (n-1-a_1-1) \cdot ((n-1)-2)$ realisations of type (t7) by $B\ |_{\{u,v\}}$.

The case (C.(m6.t13).2).(2).(1).(2) is again not sufficient for us to know whether (i1) or (i2) is true and we again have to consider its anonymous subcases. In the first of them, an argument entirely analogous to the one given for the first subcase of (C.(m6.t13).2).(2).(1).(1) proves that then property (i1) is true and therefore we know that there are exactly $2 \cdot (n-1-c_1) \cdot ((n-1)-2)$ realisations of type (t6) and also exactly $2 \cdot (n-1-c_1) \cdot ((n-1)-2)$ realisations of type (t7) by $B\ |_{\{u,v\}}$. In the second of them, analogously to the second subcase of (C.(m6.t13).2).(2).(1).(1) proves that then property (i2) is true and therefore there are exactly $4 \cdot (n-1-c_1) \cdot 2 \cdot ((n-1)-2) = 8 \cdot (n-1-c_1) \cdot ((n-1)-2)$ realisations of type (t6) and 0 realisations of type (t7) by $B\ |_{\{u,v\}}$. It follows that if (C.(m6.t13).2).(2).(1).(2), then there are exactly $2 \cdot (n-1-c_1) \cdot ((n-1)-2) + 8 \cdot (n-1-c_1) \cdot ((n-1)-2) = 10 \cdot (n-1-c_1) \cdot ((n-1)-2)$ realisations of type (t6) but only $2 \cdot (n-1-c_1) \cdot ((n-1)-2)$ realisations of type (t7) by $B\ |_{\{u,v\}}$.

It now follows that if (C.(m6.t13).2).(2).(1), then there are exactly $10 \cdot (n-1-a_1-1) \cdot ((n-1)-2) + 10 \cdot (n-1-c_1) \cdot ((n-1)-2) = 10 \cdot (2n-a_1-c_1-3) \cdot ((n-1)-2)$ realisations of type (t6) but only $2 \cdot (2n-a_1-c_1-3) \cdot ((n-1)-2)$ of type (t7) by $B\ |_{\{u,v\}}$.

The case (C.(m6.t13).2).(2).(2) will now be treated analogously to (C.(m6.t13).2).(2).(1).

In the first subcase of (C.(m6.t13).2).(2).(2).(1) we find that property (i1) is true and therefore there are exactly $2 \cdot (a_1-1) \cdot ((n-1)-2)$ realisations of type (t6) and also $2 \cdot (a_1-1) \cdot ((n-1)-2)$ realisations of type (t7) by $B\ |_{\{u,v\}}$. In the second subcase of (C.(m6.t13).2).(2).(2).(1) we find that property (i2) is true and therefore there are exactly $4 \cdot (a_1-1) \cdot 2 \cdot ((n-1)-2) = 8 \cdot (a_1-1) \cdot ((n-1)-2)$ realisations of type (t6) but 0 realisations of type (t7) by $B\ |_{\{u,v\}}$. Therefore, if (C.(m6.t13).2).(2).(2).(1), then there are exactly $2 \cdot (a_1-1) \cdot ((n-1)-2) + 8 \cdot (a_1-1) \cdot ((n-1)-2) = 10 \cdot (a_1-1) \cdot ((n-1)-2)$ realisations of type (t6) but only $2 \cdot (a_1-1) \cdot ((n-1)-2)$ realisations of type (t7) by $B\ |_{\{u,v\}}$.

In the first subcase of (C.(m6.t13).2).(2).(2).(2) we conclude that property (i1) is true and therefore there exist exactly $2 \cdot (c_1-1-1) \cdot ((n-1)-2)$ realisations of type (t6) and also $2 \cdot (c_1-1-1) \cdot ((n-1)-2)$ realisations of type (t7) by $B\ |_{\{u,v\}}$. In the second subcase of

(C.(m6.t13).2).(2).(2).(2) we conclude that property (i2) is true and therefore there are exactly $4 \cdot (c_1 - 1 - 1) \cdot 2 \cdot ((n-1) - 2) = 8 \cdot (c_1 - 1 - 1) \cdot ((n-1) - 2)$ realisations of type (t6) and 0 realisations of type (t7) by $B\,|_{\{u,v\}}$. Therefore, if (C.(m6.t13).2).(2).(2).(2), then there are exactly $2 \cdot (c_1 - 1 - 1) \cdot ((n-1) - 2) + 8 \cdot (c_1 - 1 - 1) \cdot ((n-1) - 2) = 10 \cdot (c_1 - 1 - 1) \cdot ((n-1) - 2)$ realisations of type (t6) but only $2 \cdot (c_1 - 1 - 1) \cdot ((n-1) - 2)$ realisations of type (t7) by $B\,|_{\{u,v\}}$.

It now follows that if (C.(m6.t13).2).(2).(2), then there are exactly $10 \cdot (a_1 - 1) \cdot ((n-1) - 2) + 10 \cdot (c_1 - 1 - 1) \cdot ((n-1) - 2) = 10 \cdot (a_1 + c_1 - 3) \cdot ((n-1) - 2)$ realisations of type (t6) but merely $2 \cdot (a_1 - 1) \cdot ((n-1) - 2) + 2 \cdot (c_1 - 1 - 1) \cdot ((n-1) - 2) = 2 \cdot (a_1 + c_1 - 3) \cdot ((n-1) - 2)$ realisations of type (t7) by $B\,|_{\{u,v\}}$. Moreover we may now conclude that if (C.(m6.t13).2).(2), then there are exactly $10 \cdot (2n - a_1 - c_1 - 3) \cdot ((n-1) - 2) + 10 \cdot (a_1 + c_1 - 3) \cdot ((n-1) - 2) = 20 \cdot ((n-1) - 2)^2$ realisations of type (t6) but only $2 \cdot (2n - a_1 - c_1 - 3) \cdot ((n-1) - 2) + 2 \cdot (a_1 + c_1 - 3) \cdot ((n-1) - 2) = 4 \cdot ((n-1) - 2)^2$ realisations of type (t7) by $B\,|_{\{u,v\}}$. Finally we can conclude that if (C.(m6.t13).2), then there are exactly $4 \cdot ((n-1) - 2)^2 + 20 \cdot ((n-1) - 2)^2 = 24 \cdot ((n-1) - 2)^2$ realisations of type (t6) but only $4 \cdot ((n-1) - 2)^2 + 4 \cdot ((n-1) - 2)^2 = 8 \cdot ((n-1) - 2)^2$ realisations of type (t7) by $B\,|_{\{u,v\}}$.

If (C.(m6.t5).3), then the inclusion $\{u_2, v_2\} \subseteq \mathrm{p}_2(S)$ alone implies that property (i2) is true and therefore there are exactly $4 \cdot 4 \cdot \binom{(n-1)-2}{2} = 16 \cdot \binom{(n-1)-2}{2}$ realisations of type (t6) but 0 realisations of type (t7) by $B\,|_{\{u,v\}}$.

Summing up, it follows that for each fixed S there are exactly $2 \cdot 16 \cdot \binom{(n-1)-2}{2} + 24 \cdot ((n-1)-2)^2 = 24 \cdot (n-3)^2 + 32 \cdot \binom{n-3}{2}$ realisations of type (t6) but only $8 \cdot ((n-1) - 2)^2$ realisations of type (t7) by $B\,|_{\{u,v\}}$. This completes the proof of both (m6.t6) and (m6.t7).

We can now turn to counting the realisations of (t8)–(t11), i.e. to proving (m6.t8)–(m6.t11)

By (P.(t8).1), (P.(t9).1), (P.(t10).1) and (P.(t11).1), for each of the four types (t8), (t9), (t10) and (t11) it is necessary that $\mathrm{X}_{\{0\}^{u,v} \sqcup B|_S} \cong (t5)$. We may therefore determine each of the four functions (m6.t8), (m6.t9), (m6.t10), (m6.t11) in the course of one reexamination of the proof of (m6.t5). We consider each of the cases in turn, each time descending just deep enough until we are able to decide which of the four isomorphism types (t8), (t9), (t10) can be realised in that case.

Since by (P.(t8).2), (P.(t9).2), (P.(t10).2) and (P.(t11).2) the property $B[u] \in \{\pm\}$ and $B[v] \in \{\pm\}$ is necessary for each of the four types, the positions u and v alone, not $B\,|_{\{u,v\}}$ itself, decide about which type can be realised. Therefore, if a decision is reached about which of the four types can be realised in a case, then we obtain the number of realisations by multiplying the number of realisations of type (t5) in that particular case by 4.

We now reexamine (C.(m6.t5).1). If (C.(m6.t5).1), then the property $|\{u_1, v_1\} \backslash \mathrm{p}_1(S)| = 0$ makes (P.(t10).4) impossible, hence in the entire case (C.(m6.t5).1) the type (t10) is impossible.

If (C.(m6.t5).1).(1), then we have $u_1 = v_1$, which makes both (P.(t9).3) and (P.(t8).3) impossible. The only type remaining is (t11) (and all the properties (P.(t11).1)–(P.(t11).4) are satisfied). It follows that if (C.(m6.t5).1).(1), then there are exactly $8 \cdot \binom{(n-1)-2}{2}$ realisations of type (t11) by $B\,|_{\{u,v\}}$.

If (C.(m6.t5).1).(2), then we have $u_1 \neq v_1$. Since we also have $u_2 \neq v_2$ throughout (C.(m6.t5).1), this makes (P.(t11).3) impossible. Moreover, since throughout (C.(m6.t5).1) we also have (4.25), in particular $\{u_2, v_2\} \cap \mathrm{p}_2(S) = \varnothing$, it follows that (P.(t9).4) is impossible. The only type remaining is (t8) (and all the properties (P.(t8).1)–(P.(t8).4) are indeed satisfied; notice in particular that in (P.(t8).4) the first of the two mutually exclusive clauses is true). If follows that if (C.(m6.t5).1).(2), then there are exactly $8 \cdot \binom{(n-1)-2}{2}$ realisations of type (t8) by $B\,|_{\{u,v\}}$. This completes our reexamination of (C.(m6.t5).1).

We now reexamine (C.(m6.t5).2). The information defining (C.(m6.t5).2) is by itself not yet sufficient to rule out any of the four types (t8)–(t11). The information defining (C.(m6.t5).2).(1), more specificly $u_1 = v_1$, makes both (P.(t8).3) and (P.(t9).3) impossible, still leaving two types. We argued in (C.(m6.t5).2).(1) that we have $u_2 \neq v_2$ and $|\{u_2, v_2\} \cap \mathrm{p}_2(S)| = 1$ in this case. If the latter is true as $u_2 \in \mathrm{p}_2(S)$, then $v_2 \notin \mathrm{p}_2(S)$, and combining this information with $v_1 \notin \mathrm{p}_1(S)$ (which we know since we are in (C.(m6.t5).2).(1)) makes the second clause of the conjunction (P.(t11).4) impossible. If on the other hand it is true as $v_2 \in \mathrm{p}_2(S)$, then $u_2 \notin \mathrm{p}_2(S)$, and combining this with $u_1 \notin \mathrm{p}_1(S)$ (which again we know since we are in (C.(m6.t5).2).(1)) makes (P.(t11).4) impossible (this time, the first clause). This rules out type (t11). The only type remaining is (t10) (and all the properties (P.(t10).1)–(P.(t10).4) are indeed satisfied; note that due to $|\{u_2, v_2\} \cap \mathrm{p}_2(S)| = 1$ the two clauses of (P.(t10).4) are mutually exclusive, the second being true if $u_2 \in \mathrm{p}_2(S)$ and the first if $v_2 \in \mathrm{p}_2(S)$). It follows that in case (1) of (C.(m6.t5).2) there are exactly $8 \cdot ((n-1)-2)^2$ realisations of type (t10) by $B\,|_{\{u,v\}}$.

The information defining (C.(m6.t5).2).(2) is not enough to rule out any of the four types (t8)–(t11), and descending one level deeper to (C.(m6.t5).2).(2).(1) does not change this.

If (C.(m6.t5).2).(2).(1).(1), then the decision still cannot be made and depends on the (anonymous) subcases which we distinguished in that case, namely whether the first or the second clause of (4.28) is true:

If (C.(m6.t5).2).(2).(1).(1), and the first clause of (4.28) is true, then in particular we know that $v_1 \notin \mathrm{p}_1(S)$ and $u_2 = v_2 \notin \mathrm{p}_2(S)$. The latter contradicts (P.(t8).3) and (P.(t9).3). Moreover, $v_1 \notin \mathrm{p}_1(S)$ and $v_2 \notin \mathrm{p}_2(S)$ combined render the second clause of the conjunction (P.(t11).4) false. Note that for each of three discarded types we used in whole or in part the information $u_2 = v_2 \notin \mathrm{p}_2(S)$, which defines the present subcase. Hence deferring any decision about the types for so long was necessary. We are now left with only the type (t10) (and indeed the properties (P.(t10).1)–(P.(t10).4) are all satisfied). Since within (C.(m6.t5).2).(2).(1).(1) we found that in this situation there are exactly $(n-1-a_1-1) \cdot ((n-1)-2)$ realisations of type (t5) by $B\,|_{\{u,v\}}$, it follows that here there are exactly $4 \cdot (n-1-a_1-1) \cdot ((n-1)-2)$ realisations of type (t10).

If (C.(m6.t5).2).(2).(1).(1) and the second clause of (4.28) is true, then we know that $u_1 \neq v_1$, $u_1 \in \mathrm{p}_1(S)$, $v_1 \notin \mathrm{p}_1(S)$, $u_2 \neq v_2$, $u_2 \notin \mathrm{p}_2(S)$ and $v_2 \in \mathrm{p}_2(S)$. We can now rule out three types: properties $v_1 \notin \mathrm{p}_1(S)$ and $u_2 \notin \mathrm{p}_2(S)$ combined render both clauses of the disjunction (P.(t8).4) false. Properties $u_1 \in \mathrm{p}_1(S)$ and $v_2 \in \mathrm{p}_2(S)$ combined render both clauses of the disjunction (P.(t10).4) false. Properties $u_1 \neq v_1$ and $u_2 \neq v_2$ proves (P.(t11).3) to be false. Again note that in all three decisions we used the information defining the present subcase. The only type remaining now is (t9), and indeed all properties (P.(t9).1)–(P.(t9).4) are satisfied (in (P.(t9).4) only the first clause of the disjunction). Since in (C.(m6.t5).2).(2).(1).(1) we found that in this situation there are exactly $(n-1-a_1-1) \cdot 2 \cdot ((n-1)-2)$ realisations of type (t5) by $B\,|_{\{u,v\}}$, it follows that here there are exactly $8 \cdot (n-1-a_1-1) \cdot ((n-1)-2)$ realisations of type (t9).

If (C.(m6.t5).2).(2).(1).(2), then again we have to distinguish whether the first or the second clause of (4.28) is true to reach a conclusion:

If (C.(m6.t5).2).(2).(1).(2) and the first clause of (4.28) is true, then again we in particular know that $v_1 \notin \mathrm{p}_1(S)$ and $u_2 = v_2 \notin \mathrm{p}_2(S)$, and therefore an argument analogous to the one given for the first subcase of (C.(m6.t5).2).(2).(1).(1) shows that in the present situation there are exactly $4 \cdot (n-1-c_1) \cdot ((n-1)-2)$ realisations of type (t10) by $B\,|_{\{u,v\}}$ and no other type possible.

If (C.(m6.t5).2).(2).(1).(2) and the second clause of (4.28) is true, then again we know that $u_1 \neq v_1$, $u_1 \in \mathrm{p}_1(S)$, $v_1 \notin \mathrm{p}_1(S)$ and $u_2 \neq v_2$, but this time we have $u_2 \notin \mathrm{p}_2(S)$ and $v_2 \in \mathrm{p}_2(S)$. We can now rule out three types: properties $v_1 \notin \mathrm{p}_1(S)$ and $u_2 \notin \mathrm{p}_2(S)$ combined render both clauses of the disjunction (P.(t8).4) false. Properties $u_1 \in \mathrm{p}_1(S)$ and $v_2 \in \mathrm{p}_2(S)$ combined render both clauses of the disjunction (P.(t10).4) false. Properties $u_1 \neq v_1$ and $u_2 \neq v_2$ contradict (P.(t11).3). What remains is type (t9), and all properties (P.(t9).1)–(P.(t9).4) are satisfied (in (P.(t9).4) only the first clause of the disjunction). Since in (C.(m6.t5).2).(2).(1).(2) we found that in this situation there are exactly $2 \cdot (n-1-c_1) \cdot ((n-1)-2)$ realisations of type (t5) by $B\,|_{\{u,v\}}$, it follows that here there are exactly $8 \cdot (n-1-c_1) \cdot ((n-1)-2)$ realisations of type (t9).

The next case to reexamine is (C.(m6.t5).2).(2).(2) which again does not give enough information to decide about the types.

If (C.(m6.t5).2).(2).(2).(1), then the decision still cannot be made and once more depends on the nameless subcases that were distinguished in that case, namely whether the first or the second clause of (4.28) is true:

If (C.(m6.t5).2).(2).(2).(1), and the first clause of 4.28 is true, then we know that $u_1 \neq v_1$, $u_1 \notin \mathrm{p}_1(S)$, $v_1 \in \mathrm{p}_1(S)$ and $u_2 = v_2 \notin \mathrm{p}_2(S)$. The latter, more specificly $u_2 = v_2$, contradicts both (P.(t8).3) and (P.(t9).3). Combining $u_1 \notin \mathrm{p}_1(S)$ and $u_2 \notin \mathrm{p}_2(S)$ proves the first clause of the conjunction (P.(t11).4) to be false. The only type remaining is (t10), and all properties (P.(t10).1)–

(P.(t10).4) are indeed satisfied (with only the first clause of the disjunction (P.(t10).4) being true). Since in (C.(m6.t5).2).(2).(2).(1) we found that there are exactly $(a_1 - 1) \cdot ((n-1) - 2)$ realisations of type (t5) by $B\mid_{\{u,v\}}$, it follows that in the present situation there are exactly $4 \cdot (a_1 - 1) \cdot ((n-1) - 2)$ realisations of type (t10).

If (C.(m6.t5).2).(2).(2).(1), and the second clause of (4.28) is true, then we know that $u_1 \neq v_1$, $u_1 \notin \mathrm{p}_1(S)$, $v_1 \in \mathrm{p}_1(S)$, $u_2 \in \mathrm{p}_2(S)$ and $v_2 \notin \mathrm{p}_2(S)$. Since then $u_2 \neq v_2$ and $u_1 \neq v_1$, both (P.(t10).3) and (P.(t11).3) are impossible. Moreover, combining $u_1 \notin \mathrm{p}_1(S)$ and $v_2 \notin \mathrm{p}_2(S)$ shows that both clauses of the disjunction (P.(t8).4). The remaining type is (t9) and all the properties (P.(t9).1)–(P.(t9).4) are indeed satisfied (as to the disjunction (P.(t9).4), only its second clause is true). Since in (C.(m6.t5).2).(2).(2).(1) we found exactly $2 \cdot (a_1 - 1) \cdot ((n-1) - 2)$ realisations of type (t5) by $B\mid_{\{u,v\}}$, it follows that right now there are exactly $8 \cdot (a_1 - 1) \cdot ((n-1) - 2)$ realisations of type (t9).

If (C.(m6.t5).2).(2).(2).(2), then one more time we have to distinguish the anonymous subcases. If (C.(m6.t5).2).(2).(2).(2), and the first clause of (4.28) is true, then we know $u_1 \neq v_1$, $u_1 \notin \mathrm{p}_1(S)$, $v_1 \in \mathrm{p}_1(S)$ and $u_2 = v_2 \notin \mathrm{p}_2(S)$, with $u_2 = v_2$ ruling out both (t8) and (t9). Moreover, combining $u_1 \notin \mathrm{p}_1(S)$ with $u_2 \notin \mathrm{p}_2(S)$ proves the first clause of the conjunction (P.(t11).4) to be false. Again, for each decision the information in the first clause of (4.28) was used. Now only (t10) is left and indeed all properties (P.(t10).1)–(P.(t10).4) are true (with the disjunction (P.(t10).4) satisfied only by way of its second clause). Since in (C.(m6.t5).2).(2).(2).(2) we found that in the first subcase there are exactly $(c_1 - 2) \cdot ((n-1) - 2)$ realisations of type (t5) by $B\mid_{\{u,v\}}$, it follows for our present situation that there are exactly $4 \cdot (c_1 - 2) \cdot ((n-1) - 2)$ realisations of (t10) by $B\mid_{\{u,v\}}$.

If (C.(m6.t5).2).(2).(2).(2), and the second clause of (4.28) is true, then we know that $u_1 \neq v_1$, $u_1 \notin \mathrm{p}_1(S)$, $v_1 \in \mathrm{p}_1(S)$ and $u_2 \in \mathrm{p}_2(S)$ and $v_2 \notin \mathrm{p}_2(S)$. Since then $u_2 \neq v_2$ it follows $\{u_1, u_2\} \cap \{v_1, v_2\} = \varnothing$, contradicting both (P.(t10).3) and (P.(t11).3). Combining $u_1 \notin \mathrm{p}_1(S)$ and $v_2 \notin \mathrm{p}_2(S)$ we see that both clauses of the disjunction (P.(t8).4) are false. We are left with type (t9) and indeed, all properties (P.(t9).1)–(P.(t9).4) are satisfied (for the disjunction (P.(t9).4) it is only the second clause, which is). Since in (C.(m6.t5).2).(2).(2).(2) we found that there exist exactly $2 \cdot (c_1 - 2) \cdot ((n-1) - 2)$ realisations of type (t5) by $B\mid_{\{u,v\}}$, it follows that in the present situation there are exactly $8 \cdot (c_1 - 2) \cdot ((n-1) - 2)$ realisations of type (t9) by $B\mid_{\{u,v\}}$.

The next case to reexamine is (C.(m6.t5).3). This case is symmetric to (C.(m6.t5).1) under swapping the subscripts 1 and 2. Therefore, we can analyse its subcases by reexaming the analysis of (C.(m6.t5).1) with this swap in mind. First of all, if (C.(m6.t5).3), then $\{u_2, v_2\} \subseteq \mathrm{p}_2(S)$, and this renders both clauses of the disjunction (P.(t10).4) false.

Reading (C.(m6.t5).1).(1) this way implies that we know $u_2 = v_2 \in \mathrm{p}_2(S)$, and $u_2 = v_2$ contradicts both (P.(t8).3) and (P.(t9).3). The only type remaining is (t11) and indeed, all properties (P.(t11).1)–(P.(t11).4) are satisfied. Since in (C.(m6.t5).1).(1) we found that there are exactly $2 \cdot \binom{(n-1)-2}{2}$ realisations of type (t5) by $B\mid_{\{u,v\}}$, it follows that here there exist exactly $8 \cdot \binom{(n-1)-2}{2}$ realisations of type (t11) by $B\mid_{\{u,v\}}$.

Reading (C.(m6.t5).1).(2) this way implies that we know $u_2 \neq v_2$, $\{u_2, v_2\} \cap \mathrm{p}_2(S) = \varnothing$, $u_1 \neq v_1$ and $\{u_1, v_1\} \subseteq \mathrm{p}_1(S)$. The properties $u_1 \neq v_1$ and $u_2 \neq v_2$ taken together contradict both (P.(t10).3) and (P.(t11).3). The property $\{u_2, v_2\} \cap \mathrm{p}_2(S) = \varnothing$ alone renders both clauses of the disjunction (P.(t9).4) false. What we are left with is type (t8) and indeed all properties (P.(t8).1)–(P.(t8).4) are satisfied. Since in (C.(m6.t5).1).(2) we found that there are exactly $2 \cdot \binom{(n-1)-2}{2}$ realisations of type (t5) by $B\mid_{\{u,v\}}$, it follows by symmetry that here, too, there exist exactly $8 \cdot \binom{(n-1)-2}{2}$ realisations of type (t8) by $B\mid_{\{u,v\}}$.

We have now completed reexamining the analysis of (m6.t5) and we may now add up (separately for each of the four types (t8)–(t11) the number of realisations we found during the reexamination.

For (t8) we found exactly $8 \cdot \binom{(n-1)-2}{2} + 8 \cdot \binom{(n-1)-2}{2} = 16 \cdot \binom{(n-1)-2}{2}$ realisations by $B\mid_{\{u,v\}}$. Now (m6.t8) is proved. For (t9) we found exactly $8 \cdot (n - 1 - a_1 - 1) \cdot ((n-1) - 2) + 8 \cdot (n - 1 - c_1) \cdot ((n-1) - 2) + 8 \cdot (a_1 - 1) \cdot ((n-1) - 2) + 8 \cdot (c_1 - 2) \cdot ((n-1) - 2) = 16 \cdot ((n-1) - 2)^2$ realisations by $B\mid_{\{u,v\}}$. Now (m6.t9) is proved. For (t10) we found exactly $8 \cdot ((n-1) - 2)^2 + 4 \cdot (n - 1 - a_1 - 1) \cdot ((n-1) - 2) + 4 \cdot (n - 1 - c_1) \cdot ((n-1) - 2) + 4 \cdot (a_1 - 1) \cdot ((n-1) - 2) + 4 \cdot (c_1 - 2) \cdot ((n-1) - 2) =$

$8\cdot((n-1)-2)^2+8\cdot((n-1)-2)^2=16\cdot((n-1)-2)^2$ realisations by $B\,|_{\{u,v\}}$. Now (m6.t10) is proved. For (t11) we found exactly $8\cdot\binom{(n-1)-2}{2}+8\cdot\binom{(n-1)-2}{2}=16\cdot\binom{(n-1)-2}{2}$ realisations by $B\,|_{\{u,v\}}$. Now (m6.t11) is proved.

As to (m6.t13), let us first note that Definition 149 implies:

Lemma 185. *For every $B\in\{0,\pm\}^I$ with $I\in\binom{[n-1]^2}{6}$, $I=:S\sqcup\{u,v\}$ and $\mathrm{X}_{B|_S}\cong C^4$ we have $\mathrm{X}_B\cong(t13)$ if and only if*

(P.(t13).1) $B[u]=B[v]=0$, (P.(t13).2) $|\{u_1,v_1\}\backslash \mathrm{p}_1(S)|+|\{u_2,v_2\}\backslash \mathrm{p}_2(S)|=3$.

We now distinguish cases according to how (P.(t13).2) is satisfied.

(C.(m6.t13).1) $|\{u_1,v_1\}\backslash \mathrm{p}_1(S)|=0$. Then $|\{u_2,v_2\}\backslash \mathrm{p}_2(S)|=3$ by (P.(t13).2), which is impossible. Hence Case 1 does not occur.

(C.(m6.t13).2) $|\{u_1,v_1\}\backslash \mathrm{p}_1(S)|=1$. Then $|\{u_2,v_2\}\backslash \mathrm{p}_2(S)|=2$ by (P.(t13).2), equivalently,

$$u_2\neq v_2 \text{ and } \{u_2,v_2\}\cap \mathrm{p}_2(S)=\varnothing\ . \tag{4.25}$$

Since the condition defining (C.(m6.t13).2) is equivalent to

$$(u_1=v_1 \text{ and } \{u_1,v_1\}\cap \mathrm{p}_1(S)=\varnothing) \text{ or } (u_1\neq v_1 \text{ and } |\{u_1,v_1\}\cap \mathrm{p}_1(S)|=1)\quad , \tag{4.26}$$

there are two further cases.

(1) $u_1=v_1$ and $\{u_1,v_1\}\cap \mathrm{p}_1(S)=\varnothing$. Then there are exactly $(n-1)-2$ such $u_1=v_1$. Combining (4.25) with $u\prec v$ it follows that $u_2<v_2$, therefore in the present case each of the $\binom{(n-1)-2}{2}$ different sets $\{u_2,v_2\}$ satisfying (4.25) determines the two pairs u and v. Therefore there are exactly $((n-1)-2)\cdot\binom{(n-1)-2}{2}$ realisations of (1).

(2) $u_1\neq v_1$ and $|\{u_1,v_1\}\cap \mathrm{p}_1(S)|=1$. From $u_1\neq v_1$ and the assumption $u\prec v$ it follows that $u_1<v_1$. By (4.25), $u_2<v_2$ or $v_2<u_2$, but nothing more is known about u_2 and v_2. Therefore, both possibilities must be taken into account. Because of $\mathrm{p}_1(S)=\{a_1,b_1,c_1,d_1\}=\{a_1,c_1\}$ and $u_1<v_1$ there are exactly four possibilities for $|\{u_1,v_1\}\cap \mathrm{p}_1(S)|=1$ to be true:

  (1) $u_1=a_1=b_1$. Then because of $u_1<v_1$ and $v_1\neq c_1=d_1$ it follows that there are exactly $n-1-a_1-1$ different v_1 with $v_1\notin \mathrm{p}_1(S)$ in this case. For each of them, there exist exactly $\binom{(n-1)-2}{2}$ different *sets* $\{u_2,v_2\}$ satisfying (4.25). Now the two pairs u and v are not determined by them: each of the sets can be realised in exactly two ways, both by $u_2<v_2$ and by $v_2<u_2$. Therefore, there are exactly $(n-1-a_1-1)\cdot 2\cdot\binom{(n-1)-2}{2}$ realisations of (1) by u and v.

  (2) $u_1=c_1=d_1$. Then because of $u_1<v_1$ it follows that there are exactly $n-1-c_1$ different v_1 with $v_1\notin \mathrm{p}_1(S)$. As in the preceding case, for each of these v_1 there exist exactly $\binom{(n-1)-2}{2}$ different sets $\{u_2,v_2\}$ satisfying (4.25), hence exactly $2\cdot\binom{(n-1)-2}{2}$ different u and v. Therefore there exist exactly $(n-1-c_1)\cdot 2\cdot\binom{(n-1)-2}{2}$ different realisations of type (t13) by $B\,|_{\{u,v\}}$.

  (3) $v_1=a_1=b_1$. Then because of $u_1<v_1$ it follows that there are exactly a_1-1 different u_1 with $u_1\notin \mathrm{p}_1(S)$ in this case. For the same reasons as in the preceding two cases we know that here there exist exactly $(a_1-1)\cdot 2\cdot\binom{(n-1)-2}{2}$ different realisations of type (t13) by $B\,|_{\{u,v\}}$.

  (4) $v_1=c_1=d_1$. Then because of $u_1<v_1$ and $u_1\neq a_1=b_1$ it follows that there are c_1-1-1 different u_1 with $u_1\notin \mathrm{p}_1(S)$ in this case. For the same reasons as in the preceding three cases we know that here there exist exactly $(c_1-1-1)\cdot 2\cdot\binom{(n-1)-2}{2}$ different realisations of type (t13) by $B\,|_{\{u,v\}}$.

It follows that if (C.(m6.t13).2).(2), then there exist exactly $\big((n-1-a_1-1)+(n-1-c_1)+(a_1-1)+(c_1-1-1)\big)\cdot 2\cdot\binom{(n-1)-2}{2} = 4\cdot((n-1)-2)\cdot\binom{(n-1)-2}{2}$ different realisations of type (t13) by $B\,|_{\{u,v\}}$.

It follows that if (C.(m6.t13).2), then there are exactly $((n-1)-2+4\cdot((n-1)-2))\cdot\binom{(n-1)-2}{2} = 5\cdot((n-1)-2)\cdot\binom{(n-1)-2}{2}$ realisations of type (t13) by $B\,|_{\{u,v\}}$.

(C.(m6.t13).3) $|\{u_1, v_1\}\backslash \mathrm{p}_1(S)| = 2$. This is equivalent to

$$u_1 \neq v_1 \text{ and } \{u_1, v_1\} \cap \mathrm{p}_1(X) = \varnothing\ . \tag{4.27}$$

Equation (P.(t13).2) implies $|\{u_2, v_2\}\backslash \mathrm{p}_2(S)| = 1$, which is equivalent to

$$(u_2 = v_2 \text{ and } \{u_2, v_2\} \cap \mathrm{p}_2(S) = \varnothing) \text{ or } (u_2 \neq v_2 \text{ and } |\{u_2, v_2\} \cap \mathrm{p}_2(S)| = 1)\ . \tag{4.28}$$

By swapping the subscripts 1 and 2 in the argument given for Case 2 it now follows that if (C.(m6.t13).3), then there are exactly $5\cdot((n-1)-2)\cdot\binom{(n-1)-2}{2}$ different realisations of type (t13) by $B\,|_{\{u,v\}}$.

(C.(m6.t13).4) $|\{u_1, v_1\}\backslash \mathrm{p}_1(S)| = 3$. This is impossible, hence (C.(m6.t13).4) does not occur.

It follows that for every fixed S there are exactly $10\cdot((n-1)-2)\cdot\binom{(n-1)-2}{2}$ possibilities to position the two zeros indexed by $I\backslash S$ such that $\mathrm{X}_B \cong (t13)$. This completes the proof of (m6.t13).

As to (m6.t14)–(m6.t17) we begin by noting that for each of the four isomorphism types (t14)–(t17), a necessary condition is that $|\mathrm{V}(\mathrm{X}_B)\backslash \mathrm{V}(\mathrm{X}_{B|_S})| = 3$. We can therefore prove (m6.t14)–(m6.t17) during one reexamination of the proof of (m6.t13). Since (C.(m6.t13).1) and (C.(m6.t13).4) are impossible, we only have to consider (C.(m6.t13).2) and (C.(m6.t13).3). If (C.(m6.t13).2), we know (4.25) but this is not sufficient to rule out any of the types (t14)–(t17).

If (C.(m6.t13).2).(1) we know that

$$u_1 = v_1,\ \{u_1, v_1\} \cap \mathrm{p}_1(S) = \varnothing,\ u_2 \neq v_2,\ \{u_2, v_2\} \cap \mathrm{p}_2(S) = \varnothing\ , \tag{4.29}$$

and will now consider the consequences of this for (m6.t14)–(m6.t17).

(1) Concerning contributions to (m6.t14), note that properties $\{u_1, v_1\}\cap \mathrm{p}_1(S) = \varnothing$ and $\{u_2, v_2\}\cap \mathrm{p}_2(S) = \varnothing$ make an edge intersecting $\mathrm{X}_{B|_S} \cong C^4$ impossible, hence the case (C.(m6.t13).2).(1) does not contribute[4] to $|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t14)|$.
(2) Concerning contributions to (m6.t15), note that (4.29) implies that $\mathrm{X}_B \cong (t15)$ if and only if either ($B[u] \in \{\pm\}$ and $B[v] = 0$) or ($B[u] = 0$ and $B[v] \in \{\pm\}$). Each of these clauses corresponds to 2 different B. It follows that if (C.(m6.t13).2).(1), then there are 4-times as many realisations of type (t15) as there are of type (t13). Therefore, if (C.(m6.t13).2).(1), there are exactly $4\cdot((n-1)-2)\cdot\binom{(n-1)-2}{2}$ realisations of type (t15) by $B\,|_{\{u,v\}}$.
(3) Concerning contributions to (m6.t16), since properties $\{u_1, v_1\} \cap \mathrm{p}_1(S) = \varnothing$ and $\{u_2, v_2\} \cap \mathrm{p}_2(S) = \varnothing$ make an edge intersecting $\mathrm{X}_{B|_S} \cong C^4$ impossible, the case (C.(m6.t13).2).(1) does not contribute to (m6.t16).
(4) Concerning contributions to (m6.t17), we see from (4.29) that $\mathrm{X}_B \cong (t17)$ if and only if ($B[u] \in \{\pm\}$ and $B[v] \in \{\pm\}$), and there are 4 different $B\,|_{\{u,v\}} \in \{0,\pm\}^{\{u,v\}}$ satisfying this. Therefore, if (C.(m6.t13).2).(1), there are exactly $4\cdot((n-1)-2)\cdot\binom{(n-1)-2}{2}$ realisations of type (t17) by $B\,|_{\{u,v\}}$.

If (C.(m6.t13).2).(2), then we know

$$u_1 \neq v_1,\ |\{u_1, v_1\} \cap \mathrm{p}_1(S)| = 1,\ u_2 \neq v_2,\ \{u_2, v_2\} \cap \mathrm{p}_2(S) = \varnothing\ . \tag{4.30}$$

and will now consider the consequences of this for (m6.t14)–(m6.t17).

[4] The fact that neither (C.(m6.t13).2).(1) nor the corresponding subcase of (C.(m6.t13).3) (which due to symmetry was not spelled out in the proof of (m6.t13) and therefore does not have a name) contribute to $|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t14)|$ is a reason why $|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t14)|$ is larger but not twice as large as $|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t13)|$ even though in the cases where (t14) *can* be realised the number of realisations is twice as large as for (t13).

(1) Concerning contributions to (m6.t14), we can argue as follows: If $|\{u_1, v_1\} \cap \mathrm{p}_1(S)| = 1$ is true as $u_1 \in \mathrm{p}_1(S)$, then there are exactly two $B\ |_{\{u,v\}} \in \{0, \pm\}^{\{u,v\}}$ with $\mathrm{X}_B \cong (t14)$, namely those which satisfy ($B[u] \in \{\pm\}$ and $B[v] = 0$). If it is true as $v_1 \in \mathrm{p}_1(S)$, then again there are exactly two such $B\ |_{\{u,v\}}$, namely those which satisfy ($B[u] = 0$ and $B[v] \in \{\pm\}$). It follows that without having to reexamine the subcases (C.(m6.t13).2).(2).(1)–(C.(m6.t13).2).(2).(4) we know that there are twice as many realisations of (t14) by $B\ |_{\{u,v\}}$ in the case (C.(m6.t13).2).(2) than of (m6.t13). Therefore, if (C.(m6.t13).2).(2), then there are exactly $8 \cdot ((n-1)-2) \cdot \binom{(n-1)-2}{2}$ realisations of (t14) by $B\ |_{\{u,v\}}$.

(2) Concerning contributions to (m6.t15), we have to distinguish in what way property $|\{u_1, v_1\} \cap \mathrm{p}_1(S)| = 1$ in (4.30) is satisfied. If $u_1 \in \mathrm{p}_1(S)$ but $v_1 \notin \mathrm{p}_1(S)$, then $\mathrm{X}_B \cong (m6.t15)$ if and only if $B[u] = 0$ and $B[v] \in \{\pm\}$, hence in this case there exist 2 different $B\ |_{\{u,v\}}$ with $\mathrm{X}_B \cong (m6.t15)$. If $u_1 \notin \mathrm{p}_1(S)$ but $v_1 \in \mathrm{p}_1(S)$, then $\mathrm{X}_B \cong (m6.t15)$ if and only if $B[u] \in \{\pm\}$ and $B[v] = 0$, hence in this case there again exist 2 different $B\ |_{\{u,v\}}$ with $\mathrm{X}_B \cong (m6.t15)$. It follows that if (C.(m6.t13).2).(2), then there there are 2-times as many realisations of (t15) than of (t13) by $B\ |_{\{u,v\}}$. Therefore, if (C.(m6.t13).2).(2), then there are exactly $2 \cdot 4 \cdot ((n-1)-2) \cdot \binom{(n-1)-2}{2} = 8 \cdot (n-3) \cdot \binom{n-3}{2}$ realisations of (t15) by $B\ |_{\{u,v\}}$.

(3) Concerning contributions to (m6.t16), note that no matter how (4.30) is satisfied, we have $\mathrm{X}_B \cong (m6.t16)$ if and only if ($B[u] \in \{\pm\}$ and $B[v] \in \{\pm\}$). Hence, if (C.(m6.t13).2).(2), then there are 4-times as many realisations of (m6.t16) than there are of (m6.t13), that is, if (C.(m6.t13).2).(2), then there are exactly $4 \cdot 4 \cdot ((n-1)-2) \cdot \binom{(n-1)-2}{2} = 16 \cdot (n-3) \cdot \binom{n-3}{2}$ realisations of type (m6.t16) by $B\ |_{\{u,v\}}$.

(4) Concerning contributions to (m6.t17), note that (4.30) says that $u_1 \neq v_1$ and $u_2 \neq v_2$, and this makes it impossible to create a 2-path outside of $\mathrm{X}_{B|_S} \cong C^4$. Therefore, if (C.(m6.t13).2).(2), there is no contribution to (m6.t17).

We now take stock of what we found in the subcases (C.(m6.t13).2).(1) and (C.(m6.t13).2).(2) in order to know what the entire case (C.(m6.t13).2) contributes to (m6.t14)–(m6.t17).

Since (C.(m6.t13).2).(1) did not contribute to $|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t14)|$ but (C.(m6.t13).2).(2) did contribute $8 \cdot (n-3) \cdot \binom{n-3}{2}$, it follows that if (C.(m6.t13).2), then there are exactly $8 \cdot (n-3) \cdot \binom{n-3}{2}$ realisations of type (t14) by $B\ |_{\{u,v\}}$.

Since (C.(m6.t13).2).(1) contributed $4 \cdot (n-3) \cdot \binom{n-3}{2}$ to $|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t15)|$ while (C.(m6.t13).2).(2) contributed $8 \cdot (n-3) \cdot \binom{n-3}{2}$, it follows that if (C.(m6.t13).2), then there are exactly $12 \cdot (n-3) \cdot \binom{n-3}{2}$ realisations of type (t15) by $B\ |_{\{u,v\}}$.

Since (C.(m6.t13).2).(1) did not contribute to $|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t16)|$ but (C.(m6.t13).2).(2) did contribute $16 \cdot (n-3) \cdot \binom{n-3}{2}$, it follows that if (C.(m6.t13).2), then there are exactly $16 \cdot (n-3) \cdot \binom{n-3}{2}$ realisations of type (t16) by $B\ |_{\{u,v\}}$.

Since (C.(m6.t13).2).(1) contributed $4 \cdot (n-3) \cdot \binom{n-3}{2}$ to $|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t17)|$ while (C.(m6.t13).2).(2) did not contribute anything, it follows that if (C.(m6.t13).2), then there are exactly $4 \cdot (n-3) \cdot \binom{n-3}{2}$ realisations of type (t17) by $B\ |_{\{u,v\}}$.

Since the case (C.(m6.t13).3) is symmetric to the case (C.(m6.t13).2) via interchanging the subscripts 1 and 2, we will get the same contributions to (m6.t14)–(m6.t17) as in the case (C.(m6.t13).2). We therefore have to double each of the four results found for (C.(m6.t13).2) to get the correct numbers of realisations of types (t14)–(t17). This proves (m6.t14)–(m6.t17).

As to (m6.t18), it suffices to note that the two values of $B\ |_{\{u,v\}}$ are determined: since there does not exist an edge outside $\mathrm{X}_{B|_S} \cong C^4$, they both must be zero. Therefore (m6.t18) is the number of $\{u, v\} \in \binom{[n-1]^2}{2}$ such that $\mathrm{X}_{B|_{S \sqcup \{0\}^{\{u,v\}}}} \cong (t18)$. By definition of S the latter is equivalent to saying that $\mathrm{X}_{B|_{S \sqcup \{0\}^{\{u,v\}}}}$ has exactly eight vertices. It follows from Definition 149 that this is the case if and only if simultaneously

$$|\{u_1, v_1\} \backslash \mathrm{p}_1(S)| = 2 \quad \text{and} \quad |\{u_2, v_2\} \backslash \mathrm{p}_2(S)| = 2\ . \tag{4.31}$$

Due to $u_1 < v_1$, the number of $\{u_1, v_1\} \subseteq [n-1]$ with $|\{u_1, v_1\} \backslash \mathrm{p}_1(S)| = 2$ is $\binom{n-3}{2}$. Since we only assume $u < v$ and hence both $u_2 < v_2$ and $u_2 > v_2$ are possible, for each of these $\binom{n-3}{2}$ different $\{u_1, v_1\}$ there are $2 \cdot \binom{n-3}{2}$ different $\{u_2, v_2\}$ with $|\{u_2, v_2\} \backslash \mathrm{p}_2(S)| = 2$. This proves (m6.t18).

As to (m6.t19) and (m6.t20), note that for both ismorphism types (t19) and (t20) it is necessary that $X_{B|_{S \sqcup \{0\}^{\{u,v\}}}} \cong (t18)$. We can therefore prove (m6.t19) and (m6.t20) by reexamining the proof of (m6.t18). In whatever way (4.31) is satisfied, there are exactly 4 different $B\ |_{\{u,v\}}$ with $\mathrm{X}_B \cong (t19)$, namely those satisfying

$$(B[u] \in \{\pm\} \text{ and } B[v] = 0) \quad \text{or} \quad (B[u] = 0 \text{ and } B[v] \in \{\pm\}) \ , \tag{4.32}$$

but there are also 4 different $B\ |_{\{u,v\}}$ with $\mathrm{X}_B \cong (t20)$, namely those satisfying

$$B[u] \in \{\pm\} \text{ and } B[v] \in \{\pm\} \ . \tag{4.33}$$

This proves both $|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t19)| = |({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t20)| = 4 \cdot |({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t18)|$, and therefore both (m6.t19) and (m6.t20). The proof of (QFa6) is now complete. $\square$

The relations (l1)–(l5) in Lemma 179 give us a plausibility check (i.e. necessary conditions) for the explicit formulas $|({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t2)|, \ldots, |({}_{\mathrm{ul}}\mathrm{X}^{6,n,n})^{-1}(t20)|$ that we found in (m5.t2)–(m6.t20). For brevity let $x := n-3$ and $y := \binom{n-3}{2}$. Then, indeed, the explicit formulas that we found in (QFa5) and (QFa6) pass the test: the formulas in (QFa5) evidently satisfy (l1) and (l2). Moreover, since $(3^2-1)\cdot\big(8x^2+8y\big) = 24x^2+32y+8x^2+16y+16x^2+16x^2+16y$, the formulas (m6.t5)–(m6.t11) satisfy (l3). Since $(3^2-1) \cdot 10xy = 16xy + 24xy + 32xy + 8xy$, the formulas in (m6.t13)–(m6.t17) satisfy (l4). Since $(3^2-1) \cdot 2y^2 = 8y^2 + 8y^2$, the formulas in (m6.t18)–(m6.t20) satisfy (l5).

4.2.1 Counting failures of equality of $\mathsf{P}_{\mathrm{chio}}$ and $\mathsf{P}_{\mathrm{lcf}}$

While determining an absolute cardinality $|({}_{\mathrm{ul}}\mathrm{X}^{k,n,n})^{-1}(\mathfrak{X})|$ seems to necessitate work specifically depending on the isomorphism type $\mathfrak{X}$, the ratio of all *balanced* matrix realisations to *all* realisations is easy to compute: it is determined by $\dim \mathrm{Z}_1(\mathfrak{X};\mathbb{F}_2)$ alone. This is the content of (E1) in the following lemma:

Lemma 186. *For every $(s,t) \in \mathbb{Z}^2_{\geqslant 2}$, every $0 \leqslant k \leqslant (s-1)(t-1)$, every unlabelled bipartite graph $\mathfrak{X}$ and every $\beta \in \mathbb{Z}_{\geqslant 1}$,*

(E1) $|\{B \in ({}_{\mathrm{ul}}\mathrm{X}^{k,s,t})^{-1}(\mathfrak{X})\colon (\mathrm{X}_B, \sigma_B) \text{ balanced}\}| = (\tfrac{1}{2})^{\beta_1(\mathfrak{X})} \cdot |({}_{\mathrm{ul}}\mathrm{X}^{k,s,t})^{-1}(\mathfrak{X})|$,

(E2) $|\mathcal{F}^{\mathrm{M}}_{.2^\beta}(k,s,t)| = \sum_{\mathfrak{X} \in \mathrm{im}({}_{\mathrm{ul}}\mathrm{X}^{k,s,t})\colon \beta_1(\mathfrak{X}) = \beta} (\tfrac{1}{2})^{\beta} \cdot \big|\big({}_{\mathrm{ul}}\mathrm{X}^{k,s,t}\big)^{-1}(\mathfrak{X})\big|$,

(E3) $|\mathcal{F}^{\mathrm{M}}_{.0}(k,s,t)| = \sum_{\mathfrak{X} \in \mathrm{im}({}_{\mathrm{ul}}\mathrm{X}^{k,s,t})\colon \beta_1(\mathfrak{X}) \geqslant 1} \big(1 - (\tfrac{1}{2})^{\beta_1(\mathfrak{X})}\big) \cdot |({}_{\mathrm{ul}}\mathrm{X}^{k,s,t})^{-1}(\mathfrak{X})|$.

Proof. If $\mathcal{M}$ is a set of matrices, let us define $\mathrm{Dom}(\mathcal{M}) := \{\mathrm{Dom}(B)\colon B \in \mathcal{M}\}$ and $\mathrm{Supp}(\mathcal{M}) := \{\mathrm{Supp}(B)\colon B \in \mathcal{M}\}$. Moreover, if S is a set, $\mathcal{S} \subseteq \mathfrak{P}(S)$ a set of subsets and $U \in \mathfrak{P}(S)$ a subset, then $U \cap \mathcal{S} := \{U \cap S\colon S \in \mathcal{S}\}$. Using these notations, we can prove (E1) by the following calculation: for every unlabelled $\mathfrak{X}$ we have $|\{\ B \in ({}_{\mathrm{ul}}\mathrm{X}^{k,s,t})^{-1}\ (\mathfrak{X})\colon\ (\mathrm{X}_B, \sigma_B) \text{ balanced } \}| = \sum_{J \in \mathrm{Supp}(({}_{\mathrm{ul}}\mathrm{X}^{k,s,t})^{-1}(\mathfrak{X}))} |\{\ B \in ({}_{\mathrm{ul}}\mathrm{X}^{k,s,t})^{-1}(\mathfrak{X})\colon\ (\mathrm{X}_B, \sigma_B) \text{ balanced, } \mathrm{Supp}(B) = J\ \}|$ $=$ ((Kő3) in Lemma 163) $= \sum_{J \in \mathrm{Supp}(({}_{\mathrm{ul}}\mathrm{X}^{k,s,t})^{-1}(\mathfrak{X}))} 2^{|J|-\beta_1(\mathfrak{X})} = (\tfrac{1}{2})^{\beta_1(\mathfrak{X})} \sum_{I \in \mathrm{Dom}(({}_{\mathrm{ul}}\mathrm{X}^{k,s,t})^{-1}(\mathfrak{X}))} \sum_{J \in I \cap \mathrm{Supp}(({}_{\mathrm{ul}}\mathrm{X}^{k,s,t})^{-1}(\mathfrak{X}))} 2^{|J|} =$ (directly from the definitions) $= (\tfrac{1}{2})^{\beta_1(\mathfrak{X})}\ |({}_{\mathrm{ul}}\mathrm{X}^{k,s,t})^{-1}(\mathfrak{X})|$. As to (E2), this is true since $|\mathcal{F}^{\mathrm{M}}_{.2^\beta}(k,s,t)| =$ ((C3) in Theorem 167) $= \sum_{\mathfrak{X} \in \mathrm{im}({}_{\mathrm{ul}}\mathrm{X}^{k,s,t})\colon \beta_1(\mathfrak{X}) = \beta} |\{\ B \in ({}_{\mathrm{ul}}\mathrm{X}^{k,s,t})^{-1}(\mathfrak{X})\colon\ (\mathrm{X}_B, \sigma_B) \text{ balanced } \}| =$ (by (E1)) $= \sum_{\mathfrak{X} \in \mathrm{im}({}_{\mathrm{ul}}\mathrm{X}^{k,s,t})\colon \beta_1(\mathfrak{X}) = \beta} (\tfrac{1}{2})^{\beta} \cdot \big|\big({}_{\mathrm{ul}}\mathrm{X}^{k,s,t}\big)^{-1}(\mathfrak{X})\big|$. As to (E3), note that $|\mathcal{F}^{\mathrm{M}}_{.0}(k,s,t)| =$ ((C1) in Theorem 167) $= \sum_{\mathfrak{X} \in \mathrm{im}({}_{\mathrm{ul}}\mathrm{X}^{k,s,t})\colon \beta_1(\mathfrak{X}) \geqslant 1} |\{\ B \in ({}_{\mathrm{ul}}\mathrm{X}^{k,s,t})^{-1}\ (\mathfrak{X})\colon\ (\mathrm{X}_B, \sigma_B) \text{ not balanced } \}| =$ (using (E1)) $= \sum_{\mathfrak{X} \in \mathrm{im}({}_{\mathrm{ul}}\mathrm{X}^{k,s,t})\colon \beta_1(\mathfrak{X}) \geqslant 1} \big(1 - (\tfrac{1}{2})^{\beta_1(\mathfrak{X})}\big) \cdot |({}_{\mathrm{ul}}\mathrm{X}^{k,s,t})^{-1}(\mathfrak{X})|$. $\square$

The fewer the number $\mathrm{dom}(B)$ of entries specified, the larger an entry-specification event $\mathcal{E}_B^{[n-1]^2}$ is (as a set). Any two probability measures by definition agree on the largest possible event, the entire sample space. The following theorem explores to what extent $\mathsf{P}_{\mathrm{chio}}$ and $\mathsf{P}_{\mathrm{lcf}}$ agree on successively smaller entry-specification events, descending down to sets defined by six specifications.

Theorem 187 (number of exceptions to equality of $\mathsf{P}_{\mathrm{chio}}$ and $\mathsf{P}_{\mathrm{lcf}}$ on large entry-specification events)**.** *In the following statements let* $\varnothing \subseteq I \subseteq [n-1]^2$*,* $B \in \{0,\pm\}^I$ *and* $\mathcal{E}_B := \mathcal{E}_B^{[n-1]^2}$*.*

(Ex3) *For each of the* $\sum_{0\leqslant k\leqslant 3} 3^k \cdot \binom{(n-1)^2}{k} \sim \frac{9}{2}\cdot n^6$ *possible events* $\mathcal{E}_B$ *with* $0 \leqslant \mathrm{dom}(B) \leqslant 3$*, it is true that* $\mathsf{P}_{\mathrm{chio}}[\mathcal{E}_B] = \mathsf{P}_{\mathrm{lcf}}[\mathcal{E}_B] = (\frac{1}{2})^{\mathrm{dom}(B)+\mathrm{supp}(B)}$*.*

(Ex4) *Among the* $3^4\cdot\binom{(n-1)^2}{4} \sim \frac{27}{8}\cdot n^8$ *possible events* $\mathcal{E}_B$ *with* $\mathrm{dom}(B) = 4$*, there are precisely* $|\mathcal{F}^{\mathrm{M}}(4,n)| = 2^4\cdot\binom{n-1}{2}\cdot\binom{n-1}{2} \sim 4\cdot n^4$ *events for which* $\mathsf{P}_{\mathrm{chio}}[\mathcal{E}_B] = \mathsf{P}_{\mathrm{lcf}}[\mathcal{E}_B]$ *does not hold. Of these, we have* $\frac{|\mathcal{F}^{\mathrm{M}}_{\cdot 0}(4,n)|}{|\mathcal{F}^{\mathrm{M}}(4,n)|} = \frac{|\mathcal{F}^{\mathrm{M}}_{\cdot 2}(4,n)|}{|\mathcal{F}^{\mathrm{M}}(4,n)|} = \frac{1}{2}$*.*

(Ex5) *Among the* $3^5\cdot\binom{(n-1)^2}{5} \sim \frac{81}{40}\cdot n^{10}$ *different events* $\mathcal{E}_B$ *with* $\mathrm{dom}(B) = 5$*, there are precisely* $|\mathcal{F}^{\mathrm{M}}(5,n)| = 48\cdot((n-1)^2-4)\cdot\binom{n-1}{2}\cdot\binom{n-1}{2} \sim 12\cdot n^6$ *events for which* $\mathsf{P}_{\mathrm{chio}}[\mathcal{E}_B] = \mathsf{P}_{\mathrm{lcf}}[\mathcal{E}_B]$ *does not hold. Of these, we have* $\frac{|\mathcal{F}^{\mathrm{M}}_{\cdot 0}(5,n)|}{|\mathcal{F}^{\mathrm{M}}(5,n)|} = \frac{|\mathcal{F}^{\mathrm{M}}_{\cdot 2}(5,n)|}{|\mathcal{F}^{\mathrm{M}}(5,n)|} = \frac{1}{2}$*.*

(Ex6) *Among the* $3^6\cdot\binom{(n-1)^2}{6} \sim \frac{81}{80}n^{12}$ *different events* $\mathcal{E}_B$ *with* $\mathrm{dom}(B) = 6$*, there are precisely* $|\mathcal{F}^{\mathrm{M}}(6,n)| = 18n^8 - 180n^7 + \frac{1868}{3}n^6 - \frac{2176}{3}n^5 - \frac{754}{3}n^4 + \frac{428}{3}n^3 + \frac{8144}{3}n^2 - \frac{11536}{3}n + 1504 \sim 18n^8$ *events for which* $\mathsf{P}_{\mathrm{chio}}[\mathcal{E}_B] = \mathsf{P}_{\mathrm{lcf}}[\mathcal{E}_B]$ *does not hold. Of these, we have* $|\mathcal{F}^{\mathrm{M}}_{\cdot 0}(6,n)| = 9n^8 - 90n^7 + \frac{934}{3}n^6 - 360n^5 - \frac{449}{3}n^4 + 154n^3 + \frac{3664}{3}n^2 - 1816n + 720 \sim 9n^8$*,* $|\mathcal{F}^{\mathrm{M}}_{\cdot 2}(6,n)| = 9n^8 - 90n^7 + \frac{934}{3}n^6 - 368n^5 - \frac{233}{3}n^4 - 94n^3 + \frac{4888}{3}n^2 - 2136n + 816 \sim 9n^8$*, and* $|\mathcal{F}^{\mathrm{M}}_{\cdot 4}(6,n)| = \frac{8}{3}n^5 - 24n^4 + \frac{248}{3}n^3 - 136n^2 + \frac{320}{3}n - 32 \sim \frac{8}{3}n^5$*, hence in particular* $\lim_{n\to\infty}\frac{|\mathcal{F}^{\mathrm{M}}_{\cdot 0}(6,n)|}{|\mathcal{F}^{\mathrm{M}}(6,n)|} = \lim_{n\to\infty}\frac{|\mathcal{F}^{\mathrm{M}}_{\cdot 2}(6,n)|}{|\mathcal{F}^{\mathrm{M}}(6,n)|} = \frac{1}{2}$ *and* $\lim_{n\to\infty}\frac{|\mathcal{F}^{\mathrm{M}}_{\cdot 4}(6,n)|}{|\mathcal{F}^{\mathrm{M}}(6,n)|} = 0$*.*

Proof. The total numbers of entry specification events mentioned at the beginning of (Ex4)–(Ex6), and all the asymptotic equalities are easily checked, so we do not have to say more about them.

As to (Ex3), this follows immediately from (Fa3) in Corollary 174. As to (Ex4), the claimed value of $|\mathcal{F}^{\mathrm{M}}(4,n)|$ is true by (M4) in Corollary 178 combined with Lemma 162 and the obvious fact that $|({}_{\mathrm{ul}}\mathrm{X}^{4,n,n})^{-1}(t1)| = 2^4\cdot|\mathrm{Cir}(4,n)|$. The claimed ratios can be deduced as follows: note that $\{\mathfrak{X} \in \mathrm{im}({}_{\mathrm{ul}}\mathrm{X}^{4,n,n})\colon \beta_1(\mathfrak{X}) \geqslant 1\} = \{(t1)\}$ by Corollary 174, hence $|\mathcal{F}^{\mathrm{M}}_{\cdot 0}(4,n)| =$ (by (E3), because of $\mathcal{F}^{\mathrm{M}}_{\cdot 0}(4,n) = \mathcal{F}^{\mathrm{M}}_{\cdot 0}(4,n,n)$) $= \left(1-(\frac{1}{2})^{\beta_1(t1)}\right)\cdot|({}_{\mathrm{ul}}\mathrm{X}^{4,n,n})^{-1}(t1)| = \frac{1}{2}\cdot|({}_{\mathrm{ul}}\mathrm{X}^{4,n,n})^{-1}(t1)| =$ (by (M4) in Corollary 178) $= \frac{1}{2}\cdot|\mathcal{F}^{\mathrm{M}}(4,n)|$, which together with the equation $\mathcal{F}^{\mathrm{M}}(4,n) = \mathcal{F}^{\mathrm{M}}_{\cdot 0}(4,n)\sqcup\mathcal{F}^{\mathrm{M}}_{\cdot 2}(4,n)$ from (R4) in Corollary 175 proves both $|\mathcal{F}^{\mathrm{M}}_{\cdot 0}(4,n)|/|\mathcal{F}^{\mathrm{M}}(4,n)| = \frac{1}{2}$ and $|\mathcal{F}^{\mathrm{M}}_{\cdot 2}(4,n)|/|\mathcal{F}^{\mathrm{M}}(4,n)| = \frac{1}{2}$.

As to (Ex5), the number stated first is obvious. The claimed value of $|\mathcal{F}^{\mathrm{M}}(5,n)|$ can be deduced as follows: by (C3) in Theorem 167 we have $\mathsf{P}_{\mathrm{chio}}[\mathcal{E}_B] \neq \mathsf{P}_{\mathrm{lcf}}[\mathcal{E}_B]$ if and only if $\beta_1(\mathrm{X}_B) > 0$. Since due to Definition 149 we have $0 \leqslant \|\mathrm{X}_B\| \leqslant \mathrm{dom}(B) = |I| = 5$, it is easy to see that $\beta_1(\mathrm{X}_B) \leqslant 1$. Therefore $\mathsf{P}_{\mathrm{chio}}[\mathcal{E}_B] \neq \mathsf{P}_{\mathrm{lcf}}[\mathcal{E}_B]$ if and only if $C^4 \hookrightarrow \mathrm{X}_B$. The latter property is equivalent to the existence of a matrix-4-circuit $S \subseteq I$ with $S \subseteq \mathrm{Supp}(B)$. Note that every $I \in \binom{[n-1]^2}{5}$ contains at most one matrix-4-circuit. Therefore, the number of all $B \in \{0,\pm\}^I$ with $I \in \binom{[n-1]^2}{5}$ and $\mathsf{P}_{\mathrm{chio}}[\mathcal{E}_B] \neq \mathsf{P}_{\mathrm{lcf}}[\mathcal{E}_B]$ is equal to the number of all matrix-4-circuits $S \in \binom{[n-1]^2}{4}$ with $S \subseteq \mathrm{Supp}(B)$, multiplied by the number of possibilities to choose an arbitrary position $u \in [n-1]^2\backslash S$ and an arbitrary $B[u] \in \{0,\pm\}$, i.e. $2^4\cdot\binom{n-1}{2}^2\cdot 3\cdot((n-1)^2-4) = 48\cdot((n-1)^2-4)\cdot\binom{n-1}{2}^2$. This proves the second claim in (Ex5). The claimed ratios can be deduced as follows: note that $\{\mathfrak{X} \in \mathrm{im}({}_{\mathrm{ul}}\mathrm{X}^{5,n,n})\colon \beta_1(\mathfrak{X}) \geqslant 1\} = \{(t2),(t3),(t5),(t7)\}$ by Corollary 174, hence $|\mathcal{F}^{\mathrm{M}}_{\cdot 0}(5,n)| =$ (by (E3)) $= \left(1-(\frac{1}{2})^{\beta_1(t2)}\right)\cdot|({}_{\mathrm{ul}}\mathrm{X}^{5,n,n})^{-1}(t2)| + \left(1-(\frac{1}{2})^{\beta_1(t3)}\right)\cdot|({}_{\mathrm{ul}}\mathrm{X}^{5,n,n})^{-1}(t3)| + \left(1-(\frac{1}{2})^{\beta_1(t5)}\right)\cdot|({}_{\mathrm{ul}}\mathrm{X}^{5,n,n})^{-1}(t5)| + \left(1-(\frac{1}{2})^{\beta_1(t7)}\right)\cdot|({}_{\mathrm{ul}}\mathrm{X}^{5,n,n})^{-1}(t7)| = \frac{1}{2}\ \big(\ |({}_{\mathrm{ul}}\mathrm{X}^{5,n,n})^{-1}(t2)| + |({}_{\mathrm{ul}}\mathrm{X}^{5,n,n})^{-1}(t3)| + |({}_{\mathrm{ul}}\mathrm{X}^{5,n,n})^{-1}(t5)| + |({}_{\mathrm{ul}}\mathrm{X}^{5,n,n})^{-1}(t7)|\ \big) =$ (by (M5) in Corollary 178) $= \frac{1}{2}\cdot|\mathcal{F}^{\mathrm{M}}(5,n)|$, which together with the equation $\mathcal{F}^{\mathrm{M}}(5,n) = \mathcal{F}^{\mathrm{M}}_{\cdot 0}(5,n)\sqcup\mathcal{F}^{\mathrm{M}}_{\cdot 2}(5,n)$ from (R4) in Corollary 175 proves both $|\mathcal{F}^{\mathrm{M}}_{\cdot 0}(5,n)|/|\mathcal{F}^{\mathrm{M}}(5,n)| = \frac{1}{2}$ and $|\mathcal{F}^{\mathrm{M}}_{\cdot 2}(5,n)|/|\mathcal{F}^{\mathrm{M}}(5,n)| = \frac{1}{2}$.

As to (Ex6), the claimed value of $|\mathcal{F}^{\mathrm{M}}(6,n)|$ can be deduced as follows: using (Fa6) in Corollary 174, and inspecting the list of isomorphism types in Lemma 230 from Chapter 5, we know that equality of the measures fails if and only if $C^4 \hookrightarrow \mathrm{X}_B$ or $C^6 \hookrightarrow \mathrm{X}_B$. To count the events for which this is true let us define $h_{C^6}(n) := |\{B \in \{0,\pm\}^I\colon I \in \binom{[n-1]^2}{6},\quad C^6 \hookrightarrow \mathrm{X}_B\}|$, $h_{K^{2,3}}(n) := |\{B \in \{0,\pm\}^I\colon I \in \binom{[n-1]^2}{6},\quad K^{2,3} \hookrightarrow \mathrm{X}_B\}|$ and $h_{C^4,\neg K^{2,3}}(n) := |\{B \in \{0,\pm\}^I\colon I \in \binom{[n-1]^2}{6},\quad C^4 \hookrightarrow$

$\mathrm{X}_B,\quad K^{2,3} \not\hookrightarrow \mathrm{X}_B\}|$. Since for a graph X with at most six edges the properties $C^6 \hookrightarrow X$, $K^{2,3} \hookrightarrow X$, and $C^4 \hookrightarrow X$ but $K^{2,3} \not\hookrightarrow X$ are mutually exclusive, it follows that

$$|\mathcal{F}^{\mathrm{M}}(6,n)| = h_{C^6}(n) + h_{C^4,\ \neg K^{2,3}}(n) + h_{K^{2,3}}(n)\ . \tag{4.34}$$

Lemma 162 implies that $h_{C^6}(n) = 2^6 \cdot 3! \cdot \binom{n-1}{3}^2$, the factor of 2^6 accounting for the fact that the property $C^6 \hookrightarrow \mathrm{X}_B$ is indifferent to the choice of the six signs in B. Moreover, evidently, $h_{K^{2,3}}(n) = 2 \cdot 2^6 \cdot \binom{n-1}{3} \cdot \binom{n-1}{2}$, where the first 2 accounts for the two possibilities $|\mathrm{p}_1(I)| = 2$ and $|\mathrm{p}_2(I)| = 3$ or $|\mathrm{p}_1(I)| = 3$ and $|\mathrm{p}_2(I)| = 2$.

In order to compute $h_{C^4,\ \neg K^{2,3}}(n)$, we will employ a simple inclusion-exclusion-argument: for any of the $\binom{n-1}{2}^2$ choices $1 \leqslant i_1 < i_2 \leqslant n-1$ and $1 \leqslant j_1 < j_2 \leqslant n-1$ for the position of a matrix-4-circuit $S = \{(i_1,j_1),(i_1,j_2),(i_2,j_1),(i_2,j_2)\}$, the number of distinct $I \subseteq [n-1]^2$ with $|I| = 6$ such that B contains a matrix-4-circuit *at least* at the positions (i_1,j_1), (i_1,j_2), (i_2,j_1) and (i_2,j_2) is $2^4 \cdot 3^2 \cdot \binom{(n-1)^2-4}{2}$, where the factor 2^4 accounts for the mandatory $(\pm)$-values of the 4-circuit entries, the factor 3^2 accounts for the arbitrary $\{0,\pm\}$-values of the two non-4-circuit-entries and the factor $\binom{(n-1)^2-4}{2}$ accounts for the arbitrary positions of the two non-4-circuit-entries in $[n-1]^2 \backslash S$. Summing this expression over all $\binom{n-1}{2}^2$ possible choices of S, we obtain $h_{\geqslant}(n) := 2^4 \cdot \binom{n-1}{2}^2 \cdot 3^2 \cdot \binom{(n-1)^2-4}{2}$. But this is not $h_{C^4,\ \neg K^{2,3}}(n)$ yet: from the list in Lemma 230 on p. 209 we see that there is precisely one type which contains more than one copy of C^4, namely $K^{2,3}$, which contains exactly three copies. Therefore, in $h_{\geqslant}(n)$ every I with $\mathrm{X}_B \cong K^{2,3}$ has been counted exactly three times, and we did not overcount any of the realisations of the other isomorphism types. Thus, in order to arrive at $h_{C^4,\ \neg K^{2,3}}(n)$, we have to subtract three times $h_{K^{2,3}}(n)$. Hence, according to [87],

$$h_{C^4,\ \neg K^{2,3}}(n) = h_{\geqslant}(n) - 3 \cdot h_{K^{2,3}}(n) = 18n^8 - 180n^7 + 612n^6 - 608n^5 - 774n^4 + 1348n^3 + 1200n^2 - 2864n + 1248. \tag{4.35}$$

Substituting this into (4.34), one indeed arrives at the claimed value of $|\mathcal{F}^{\mathrm{M}}(6,n)|$.

As to $|\mathcal{F}^{\mathrm{M}}_{\cdot 0}(6,n)|$, we can use (E2) in Lemma 186 to calculate $|\mathcal{F}^{\mathrm{M}}_{\cdot 0}(6,n)| = \sum_{\mathfrak{X} \in \mathrm{im}({}_{\mathrm{ul}}\mathrm{X}^{6,n,n}):\ \beta_1(\mathfrak{X})=1} (1-(\frac{1}{2})^1) \cdot |(\mathrm{X}^{6,n,n})^{-1}(\mathfrak{X})| + \sum_{\mathfrak{X} \in \mathrm{im}({}_{\mathrm{ul}}\mathrm{X}^{6,n,n}):\ \beta_1(\mathfrak{X})=2} (1-(\frac{1}{2})^2) \cdot |(\mathrm{X}^{6,n,n})^{-1}(\mathfrak{X})| =$ (from the list in Lemma 230) $= \frac{1}{2} \cdot \sum_{\mathfrak{X} \in \{(t2),\ldots,(t20)\} \backslash \{(t4)\}} |(\mathrm{X}^{6,n,n})^{-1}(\mathfrak{X})| + \frac{3}{4} \cdot |(\mathrm{X}^{6,n,n})^{-1}(t4)| = \frac{1}{2} \cdot (h_{C^4,\neg K^{2,3}}(n) + h_{C^6}(n)) + \frac{3}{4} \cdot h_{K^{2,3}}(n)$, and it can be checked that this equals the claimed value of $|\mathcal{F}^{\mathrm{M}}_{\cdot 0}(6,n)|$.

Similarly, $|\mathcal{F}^{\mathrm{M}}_{\cdot 2}(6,n)| = \sum_{\mathfrak{X} \in \mathrm{im}({}_{\mathrm{ul}}\mathrm{X}^{6,n,n}):\ \beta_1(\mathfrak{X})=1} (\frac{1}{2})^1 \cdot |(\mathrm{X}^{6,n,n})^{-1}(\mathfrak{X})| =$ (from the list in Lemma 230) $= (\frac{1}{2}) \cdot \sum_{\mathfrak{X} \in \{(t2),\ldots,(t20)\} \backslash \{(t4)\}} |(\mathrm{X}^{6,n,n})^{-1}(\mathfrak{X})| = \frac{1}{2}\ (\ h_{C^4,\neg K^{2,3}}(n) + h_{C^6}(n)\ )$, and it can be checked that this is equal to the value of $|\mathcal{F}^{\mathrm{M}}_{\cdot 2}(6,n)|$ which is claimed in (Ex6). Finally, $|\mathcal{F}^{\mathrm{M}}_{\cdot 4}(6,n)| = \sum_{\mathfrak{X} \in \mathrm{im}({}_{\mathrm{ul}}\mathrm{X}^{6,n,n}):\ \beta_1(\mathfrak{X})=2)} (\frac{1}{2})^2 \cdot |(\mathrm{X}^{6,n,n})^{-1}(\mathfrak{X})| =$ (from the list in Lemma 230) $= (\frac{1}{4}) \cdot |(\mathrm{X}^{6,n,n})^{-1}(t4)| = \frac{1}{4} \cdot h_{K^{2,3}}$, and this equals the value of $|\mathcal{F}^{\mathrm{M}}_{\cdot 4}(6,n)|$ claimed in (Ex6). $\square$

4.2.1.1 An alternative check of the formulas counting the number of failures from Theorem 187

We did not need Lemma 180 in our proof of Theorem 187. It can, nevertheless, provide additional security since via Corollary 178 and Lemma 180 one may take an inclusion-exclusion-free (but, all told, much more laborious) alternative route to the claimed values of $|\mathcal{F}^{\mathrm{M}}(5,n)|$ and $|\mathcal{F}^{\mathrm{M}}(6,n)|$. As to the claimed value of $|\mathcal{F}^{\mathrm{M}}(5,n)|$, by (M5) in Corollary 178 combined with Lemma 180, we have $|\mathcal{F}^{\mathrm{M}}(5,n)| = (\text{m5.t2}) + (\text{m5.t3}) + (\text{m5.t5}) + (\text{m5.t7}) = 2^4 \cdot \binom{n-1}{2}^2 \cdot \big(4 \cdot (n-3) + 8 \cdot (n-3) + 1 \cdot (n-3)^2 + 2 \cdot (n-3)^2\big) = 48 \cdot ((n-1)^2 - 4) \cdot \binom{n-1}{2} \cdot \binom{n-1}{2}$.

As to the claimed value of $|\mathcal{F}^{\mathrm{M}}(6,n)|$, by (M6) in Corollary 178, the function $|\mathcal{F}^{\mathrm{M}}(6,n)|$ equals the sum of the nineteen functions which were found in (m6.t2)–(m6.t20) of (QFa6) in Lemma 180, and one can check (e.g., with [87]) that indeed $|\mathcal{F}^{\mathrm{M}}(6,n)| = \sum_{2 \leqslant k \leqslant 20} (\text{m6.t}k) = 18n^8 - 180n^7 + \frac{1868}{3}n^6 - \frac{2176}{3}n^5 - \frac{754}{3}n^4 + \frac{428}{3}n^3 - \frac{8144}{3}n^2 - \frac{11536}{3}n + 1504$.

Incidentally, let us note that by summing all functions in (m6.t2)–(m6.t20) except (m6.t4) and (m6.t13) (i.e., by summing seventeen functions) one may also check the equation $h_{C^4,\ \neg K^{2,3}}(n) =$

$h_{\geqslant}(n) - 3 \cdot h_{K^{2,3}}(n) = 18n^8 - 180n^7 + 612n^6 - 608n^5 - 774n^4 + 1348n^3 + 1200n^2 - 2864n + 1248$ claimed in the proof above.

4.2.1.2 Quantitatively dominant graph-theoretical reasons for $\mathsf{P}_{\text{chio}} \neq \mathsf{P}_{\text{lcf}}$

Lemma 180 tells us that of the $|\mathcal{F}^{\mathrm{M}}(6,n)| \in \Omega_{n\to\infty}(n^8)$ six-element-entry-specifications which cause non-agreement of P_{chio} and P_{lcf}, most of the failures are concentrated at only three out of the nineteen isomorphism types in (QFa6): only the types (t18), (t19) and (t20) have a preimage under ${}_{\text{ul}}\mathrm{X}^{6,n,n}$ which is of size $\Omega_{n\to\infty}(n^8)$. The quantitative domination of these isomorphism types is, however, a rather slow one in that

$$\frac{\sum_{2\leqslant k\leqslant 17}|({}_{\text{ul}}\mathrm{X}^{6,n,n})^{-1}(\mathrm{t}k)|}{|({}_{\text{ul}}\mathrm{X}^{6,n,n})^{-1}(t18)| + |({}_{\text{ul}}\mathrm{X}^{6,n,n})^{-1}(t19)| + |({}_{\text{ul}}\mathrm{X}^{6,n,n})^{-1}(t20)|} \in \Theta(n^{-1}) \ .$$

4.2.1.3 Estimate of the number of failures of equality of P_{chio} and P_{lcf} for events $\mathcal{E}_B$ with $B \in \{0,\pm\}^I$ and $I \in \binom{[n-1]^2}{k}$ and k general

Proposition 188 (for fixed k the measures P_{chio} and P_{lcf} agree for almost all entry-specifications)**.** *For every fixed $k \geqslant 1$ we have $|\mathcal{F}^{\mathrm{M}}(k,n)| \ / \ |\{B \in \{0,\pm\}^I,\ I \in \binom{[n-1]^2}{k}\}| \in \mathcal{O}_{n\to\infty}(n^{-2})$.*

Proof. We will estimate numerator and denominator of this fraction separately. The denominator is equal to $3^k \cdot \binom{(n-1)^2}{k} \in \Omega_{n\to\infty}(n^{2k})$. Moreover, a very rough estimate suffices to obtain a bound on the numerator which nevertheless is sufficiently small to prove that the ratio vanishes: $|\mathcal{F}^{\mathrm{M}}(k,n)| \overset{\text{Theorem 167.(C3)}}{=} |\{\ B \in \{0,\pm\}^I:\ I \in \binom{[n-1]^2}{k},\ B \text{ contains a matrix-circuit }\}| = \big|\ \bigcup_{1\leqslant j\leqslant\lfloor\frac{k}{2}\rfloor} \bigcup_{L\in\mathrm{Cir}(2j,n)} \{\ B \in \{0,\pm\}^I:\ I \in \binom{[n-1]^2}{k},\ L \subseteq \mathrm{Supp}(B)\ \}\ \big| \leqslant \sum_{1\leqslant j\leqslant\lfloor\frac{k}{2}\rfloor} \sum_{L\in\mathrm{Cir}(2j,n)} |\{\ B \in \{0,\pm\}^I:\ I \in \binom{[n-1]^2}{k},\ L \subseteq \mathrm{Supp}(B)\ \}| = \sum_{1\leqslant j\leqslant\lfloor\frac{k}{2}\rfloor} 2^{2j} \cdot 3^{k-2j} \cdot \binom{(n-1)^2-2j}{k-2j} \cdot |\mathrm{Cir}(2j,n)| \overset{\text{Lemma 162}}{=} \sum_{1\leqslant j\leqslant\lfloor\frac{k}{2}\rfloor} 2^{2j} \cdot 3^{k-2j} \cdot \binom{(n-1)^2-2j}{k-2j} \cdot \binom{n-1}{j}^2 \cdot \frac{j!(j-1)!}{2} \in \sum_{1\leqslant j\leqslant\lfloor\frac{k}{2}\rfloor} \mathcal{O}_{n\to\infty}(1) \cdot \mathcal{O}_{n\to\infty}(n^{2k-4j}) \cdot \mathcal{O}_{n\to\infty}(n^{2j}) \cdot \mathcal{O}_{n\to\infty}(1) \subseteq \sum_{1\leqslant j\leqslant\lfloor\frac{k}{2}\rfloor} \mathcal{O}_{n\to\infty}(n^{2k-2j}) \subseteq \mathcal{O}_{n\to\infty}(n^{2k-2})$. □

4.3 Connection to counting singular $\{\pm\}$-matrices

4.3.1 Basic connections

The Lemmas 189 and 190, which are consequences of Chio's identity in Lemma 158, are the basic reason why the measure P_{chio} is relevant for the study of singular $\pm$-matrices.

Lemma 189 (Chio condensation affects rank to the least possible degree)**.** *For every integral domain R, every $(s,t) \in \mathbb{Z}^2_{\geqslant 2}$ and every $A \in R^{[s]\times[t]}$ with $a_{s,t} \neq 0$ we have* $\mathrm{rk}(\frac{1}{2}\mathrm{C}_{(s,t)}(A)) = \mathrm{rk}(A) - 1$.

Proof. If $\mathrm{rk}(A) = 1$, then obviously $\mathrm{C}_{(s,t)}(A) = \{0\}^{[s-1]\times[t-1]}$ and the claim is true. We may therefore assume that $r := \mathrm{rk}(A) \geqslant 2$. By the equality of rank and determinantal rank over integral domains (cf. e.g., [5, Corollary 2.29(2)]) there exists $S \in \binom{[s]}{r}$ and $T \in \binom{[t]}{r}$ such that $\det(A\ |_{S\times T}) \neq 0$. If $s \notin S$, then by temporarily passing to the field of fractions of R we may appeal to Steinitz' exchange lemma for vector spaces to prove the existence of at least one $i_0 \in S$ such that $\det\big(A\ |_{((S\backslash\{i_0\})\sqcup\{s\})\times T}\big) \neq 0$. Analogously for $t \notin T$. Therefore we may assume that $s \in S$ and $t \in T$. Hence $S \times T = ((S\backslash\{s\}) \times (T\backslash\{t\}))^{\check{}}$ and therefore $\mathrm{C}_{(s,t)}(A\ |_{S\times T})$ is defined. By Lemma 158 we know that $\det(\mathrm{C}_{(s,t)}(A\ |_{S\times T})) = a_{s,t}^{r-2} \cdot \det(A\ |_{S\times T}) \neq 0$, the latter since R is an integral domain and $a_{s,t} \neq 0$ by assumption. Since $\mathrm{C}_{(s,t)}(A\ |_{S\times T}) = \mathrm{C}_{(s,t)}(A)\ |_{(S\backslash\{s\})\times(T\backslash\{t\})} \in R^{(r-1)\times(r-1)}$, and by the equality of rank and determinantal rank, this implies $\mathrm{rk}(\mathrm{C}_{(s,t)}(A)) \geqslant r - 1$. On the other hand we also have $\mathrm{rk}(\mathrm{C}_{(s,t)}(A)) \leqslant r-1$. To see this, it suffices to note that every $r \times r$ submatrix of $\mathrm{C}_{(s,t)}(A)$ is the Chio-condensate of an $(r+1)\times(r+1)$ submatrix of A, hence by Chio's identity a nonvanishing $r \times r$ minor of $\mathrm{C}_{(s,t)}(A)$ would imply a nonvanishing $(r+1)\times(r+1)$ minor of A, contrary to the assumption of $\mathrm{rk}(A) = r$. □

Lemma 190. $\mathsf{P}\big[\mathrm{Ra}_r(\{\pm\}^{[s]\times[t]})\big] = \mathsf{P}_{\mathrm{chio}}\big[\mathrm{Ra}_{r-1}(\{0,\pm\}^{[s-1]\times[t-1]})\big]$ *for every* $(s,t) \in \mathbb{Z}^2_{\geqslant 2}$ *and* $1 \leqslant r \leqslant \min(s,t)$.

Proof. This follows from the calculation $\mathsf{P}\,\big[\,\mathrm{Ra}_r(\,\{\pm\}^{[s]\times[t]})\,\big] = \frac{1}{2^{s\cdot t}}\,|\{\,A \in \{\pm\}^{[s]\times[t]}\colon\ \mathrm{rk}(A) = r\,\}| \overset{\text{Lemma (189)}}{=} \frac{1}{2^{s\cdot t}} \sum_{B\in\{0,\pm\}^{[s-1]\times[t-1]}\colon\,\mathrm{rk}(B)=r-1} |(\,\frac{1}{2}\mathrm{C}_{(s,t)}\,)^{-1}\,(B)\,| \overset{\text{Definition 4.5}}{=} \mathsf{P}_{\mathrm{chio}}\,[\,\mathrm{Ra}_{r-1}(\,\{\,0, \pm\,\}^{[s-1]\times[t-1]}\,)\,]$. Note, incidentally, that with the third equality sign, many zero-summands are introduced. □

Corollary 191. $\mathsf{P}\big[\mathrm{Ra}_{\mathcal{R}}(\{\pm\}^{[s]\times[t]})\big] = \mathsf{P}_{\mathrm{chio}}\big[\mathrm{Ra}_{\mathcal{R}-1}(\{0,\pm\}^{[s-1]\times[t-1]})\big]$ *for every* $(s,t) \in \mathbb{Z}^2_{\geqslant 2}$ *and every* $\mathcal{R} \in \mathfrak{P}([\min(s,t)] \sqcup \{0\})$*, and with* $\mathfrak{R} - 1 := \{l-1\colon l \in \mathfrak{R}\}$.

Proof. Immediate from Lemma 190 and Definition 156. □

Corollary 192. $\mathsf{P}_{\mathrm{chio}}[\mathrm{Ra}_{<n-1}(\{0,\pm\}^{[n-1]^2})] \leqslant (1/\sqrt{2}+o(1))^n$ *for* $n \to \infty$.

Proof. By combining Lemma 191 with (4.1) in Theorem 140. □

4.3.2 Sign functions which are both singular and balanced

Definition 193 ($G_{s,t}$)**.** *If* $(s,t) \in \mathbb{Z}^2_{\geqslant 2}$*, we define* $G_{s,t} := \big(\bigoplus_{1\leqslant i\leqslant s-1}\mathbb{F}_2\big) \oplus \big(\bigoplus_{1\leqslant j\leqslant t-1}\mathbb{F}_2\big)$.

We will use the following group actions. Informally, α_X is the action by switching signs of edges simultaneously in all 'stars' centered at those vertices for which g has nonzero components.

Definition 194. *If* $(s,t) \in \mathbb{Z}^2_{\geqslant 2}$*, we define the group action* $\alpha_{s,t}\colon G_{s,t} \longrightarrow \mathrm{Sym}(\{0,\pm\}^{[s-1]\times[t-1]})$, $((g_i)_{i\in[s-1]},(g_j)_{j\in[t-1]}) \longmapsto \big(\substack{\{0,\pm\}^{[s-1]\times[t-1]}\to\{0,\pm\}^{[s-1]\times[t-1]} \\ (b_{i,j})_{(i,j)\in[s-1]\times[t-1]}\mapsto(-1)^{g_i}\cdot(-1)^{g_j}\cdot b_{i,j}}\big)$*. For every* $X \in \mathrm{BG}_{s,t}$ *define the group action* $\alpha_X\colon G_{s,t} \longrightarrow \mathrm{Sym}(\{\pm\}^{\mathrm{E}(X)})$ *which is defined by* $(\alpha_X(g)(\sigma))(e) := (-1)^{g_i}\cdot(-1)^{g_j}\cdot\sigma(e)$ *for every* $\sigma \in \{\pm\}^{\mathrm{E}(X)}$ *and every* $e = \{(i,t),(s,j)\} \in \mathrm{E}(X)$.

Note that neither $\alpha_{s,t}$ nor α_X are faithful group actions. More precisely, not only is both $\ker(\alpha_{s,t})$ and $\ker(\alpha_X)$ a 2-element set, but $\alpha_{s,t}$ and α_X are both double-covers onto their images. We could construct a faithful action by making an arbitrary choice of a $\iota \in [s-1] \cup [t-1]$ and then refraining from switching at this index (analogously, by making an arbitrary choice of a star in X and then refraining from switching that particular star).

Balancedness is a global property of an edge-signing which is already determined by the signing of a sparse substructure; an arbitrary spanning tree:

Lemma 195 (rigidity of balanced edge signings)**.** *For every connected graph* X *and every spanning tree* T *of* X*, there is a bijection* $\{\pm\}^{\mathrm{E}(T)} \leftrightarrow \{\sigma \in \{\pm\}^{\mathrm{E}(X)}\colon (X,\sigma)$ *balanced*$\}$.

Sketch of proof. Since the balancedness-preserving sign of every edge $e \in \mathrm{E}(X)\backslash\mathrm{E}(T)$ is determined by the unique circuit in $\mathrm{E}(T) \cup \{e\}$, for every given $\sigma \in \{\pm\}^{\mathrm{E}(T)}$, there is at most one balanced extension of σ, i.e. at most one $\tilde{\sigma} \in \{\pm\}^{\mathrm{E}(X)}$ with $\sigma = \tilde{\sigma}\,|_{\mathrm{E}(T)}$ and $(X,\tilde{\sigma})$ balanced. Moreover, this extension can be constructed in the obvious 'greedy' way by successively adding in the elements of $\mathrm{E}(X)\backslash\mathrm{E}(T)$ in an arbitrary order while at each step of the construction choosing the sign of the added edge so as to avoid non-balanced circuits. That this is indeed possible can be proved by an induction on the number $|\mathrm{E}(X)\backslash\mathrm{E}(T)|$ of edges to be added. A key observation (routine to prove and known since at least [75, Theorem 2]) is that at each step of the construction, for each pair of vertices either *all* paths in the partially constructed graphs with these two vertices as endvertices have sign $(-)$ or *all* such paths have sign $(+)$, so the greedy construction never stalls. □

Lemma 196. *For every graph X the restriction $\mathrm{im}(\alpha_X)\,|_{S_{\mathrm{bal}}(X)}$ is a transitive permutation group on $S_{\mathrm{bal}}(X)$.*

Sketch of proof. One way to look at this is as 'making use of the rigidity of balanced signings': we can choose an arbitrary spanning tree T_i for each connected component X_i of X, then show that $\mathrm{im}(\alpha_X)\,|_{S_{\mathrm{bal}}(X)}$ is transitive on the set $\{\pm\}^{\mathrm{E}(T_i)}$ of all edge-signings of T_i, and then appeal to Lemma 195 which says that this transitivity already implies transitivity on the full set $S_{\mathrm{bal}}(X)$. $\square$

Given a $\{0,1\}$-matrix, it can be possible to increase its $\mathbb{Z}$-rank by choosing signs for the entries. If we require the signed matrix to be *balanced*, however, the rank must stay the same. This follows quickly from the graph-theoretical considerations above:

Proposition 197 (balanced signings of a $\{0,1\}$-matrix have equal rank)**.** *Let $B \in \{0,1\}^{[s-1]\times[t-1]}$. Let $\tilde{B}$ be an arbitrary 'balanced signing of B', i.e. $\tilde{B} \in \{0,\pm\}^{[s-1]\times[t-1]}$, $\mathrm{Supp}(\tilde{B}) = \mathrm{Supp}(B)$ and $(\mathrm{X}_{\tilde{B}}, \sigma_{\tilde{B}}) = (\mathrm{X}_B, \sigma_{\tilde{B}})$ is a balanced signed graph. Then $\mathrm{rk}(\tilde{B}) = \mathrm{rk}(B)$.*

Proof. Since both (X_B, σ_B) and $(\mathrm{X}_{\tilde{B}}, \sigma_{\tilde{B}})$ are balanced, by Lemma 196 there exists $g \in G_{s,t}$ such that $\alpha_X(g)(\sigma_{\tilde{B}}) = \sigma_B$. In view of Definition 194 and Definition 149, this implies $\alpha_{s,t}(g)(\tilde{B}) = B$. Since $\alpha_{s,t}$ obviously keeps the rank invariant, the claim is proved. $\square$

We will now use the knowledge established so far to analyse the tempting 'absolute' route of using Corollary 191 and then partitioning according to isomorphism type of the associated bipartite graph. The conclusion is that this will lead us onto a well-beaten path (counting singular $\{0,1\}$-matrices):

Proposition 198 (on rank-level-sets, the Chio measure agrees with the uniform measure after forgetting the signs)**.** *Let $(s,t) \in \mathbb{Z}^2_{\geqslant 2}$ and $\mathcal{R} \in \mathfrak{P}(\{1,\dots,\min(s,t)\})$. Then*

$$\mathsf{P}_{\mathrm{chio}}[\mathrm{Ra}_{\mathcal{R}}(\{0,\pm\}^{[s-1]\times[t-1]})] = \mathsf{P}[\mathrm{Ra}_{\mathcal{R}}(\{0,1\}^{[s-1]\times[t-1]})]\ . \tag{4.36}$$

Proof. This follows from the calculation

$$\begin{aligned}
\mathsf{P}_{\mathrm{chio}}[\mathrm{Ra}_{\mathcal{R}}(\{0,\pm\}^{[s-1]\times[t-1]})] &\overset{\text{((C1) in Theorem 167)}}{=} \mathsf{P}_{\mathrm{chio}}\Big[\Big\{B \in \{0,\pm\}^{[s-1]\times[t-1]} \colon {\scriptstyle \mathrm{rk}(B)\,\in\,\mathcal{R},\ (\mathrm{X}_B,\,\sigma_B)\text{ balanced}}\Big\}\Big] \\
&= \sum_{\mathfrak{X} \in \mathrm{ul}(\mathrm{BG}_{s,t})} \mathsf{P}_{\mathrm{chio}}\Big[\Big\{B \in \{0,\pm\}^{[s-1]\times[t-1]} \colon {\scriptstyle \mathrm{rk}(B)\,\in\,\mathcal{R},\ \mathrm{X}_B\,\cong\,\mathfrak{X},\ (\mathrm{X}_B,\,\sigma_B)\text{ balanced}}\Big\}\Big] \\
\text{(by (2) in Corollary 168)} &= \sum_{\mathfrak{X} \in \mathrm{ul}(\mathrm{BG}_{s,t})} 2^{-st+\beta_0(\mathfrak{X})+1} \cdot \Big|\Big\{\, B \in \{0,\pm\}^{[s-1]\times[t-1]} \colon {\scriptstyle \mathrm{rk}(B)\,\in\,\mathcal{R},\ \mathrm{X}_B\,\cong\,\mathfrak{X},\ (\mathrm{X}_B,\,\sigma_B)\text{ balanced}}\Big\}\Big| \\
\text{(by (Kő3) in Lemma 163 and Proposition 197)} &= \sum_{\mathfrak{X} \in \mathrm{ul}(\mathrm{BG}_{s,t})} 2^{|\mathfrak{X}|-st+1} \cdot \big|\{\, B \in \{0,1\}^{[s-1]\times[t-1]} \colon \mathrm{rk}(B) \in \mathcal{R},\ \mathrm{X}_B \cong \mathfrak{X}\,\}\big| \\
\text{(since } |\mathfrak{X}| \text{ is equal to } (s-1)+(t-1) \text{ for every } \mathfrak{X} \in \mathrm{ul}(\mathrm{BG}_{s,t})) &= \sum_{\mathfrak{X} \in \mathrm{ul}(\mathrm{BG}_{s,t})} 2^{-(s-1)(t-1)} \cdot \big|\{\, B \in \{0,1\}^{[s-1]\times[t-1]} \colon \mathrm{rk}(B) \in \mathcal{R},\ \mathrm{X}_B \cong \mathfrak{X}\,\}\big| \\
&= (\tfrac{1}{2})^{(s-1)(t-1)} \cdot \big|\{B \in \{0,1\}^{[s-1]\times[t-1]} \colon \mathrm{rk}(B) \in \mathcal{R}\}\big| \\
&= \mathsf{P}[\mathrm{Ra}_{\mathcal{R}}(\{0,1\}^{[s-1]\times[t-1]})]\ .
\end{aligned}$$

The proof of Proposition 198 is now complete. $\square$

It should be noted that the equality $\mathsf{P}[\mathrm{Ra}_{<n}(\{\pm\}^{[n]^2})] = \mathsf{P}[\mathrm{Ra}_{<n-1}(\{0,1\}^{[n-1]^2})]$, which follows by combining Corollary 191 with Proposition 198, seems well-known (the author does not have an explicit reference corroborating this, but there are publications in which this is implicit (e.g. [163]).

4.3.3 A relative point of view

It appears to promise progress on Conjecture 32 (which is (S1) in the following equivalence) to use the theorem of Bourgain–Vu–Wood to take a more relative point of view:

Proposition 199 (relative formulations of Conjecture 32)**.** *The following statements are equivalent:*

(S1) $\mathsf{P}\big[\mathrm{Ra}_{<n}(\{\pm\}^{[n]^2})\big] \leqslant (\frac{1}{2} + o_{n\to\infty}(1))^n$

(S2) $\mathsf{P}_{\mathrm{chio}}\big[\mathrm{Ra}_{<n-1}(\{0,\pm\}^{[n-1]^2})\big] \leqslant (\frac{1}{2} + o_{n\to\infty}(1)) \cdot \mathsf{P}_{\mathrm{lcf}}\big[\mathrm{Ra}_{<n-1}(\{0,\pm\}^{[n-1]^2})\big]$

(S3) $\sum_{B' \in \mathrm{Ra}_{<n-1}(\{0,\pm\}^{[n-1]^2})} \mathsf{P}_{\mathrm{chio}}[B'] \leqslant \big(\frac{1}{2} + o_{n\to\infty}(1)\big) \cdot \sum_{B'' \in \mathrm{Ra}_{<n-1}(\{0,\pm\}^{[n-1]^2})} \mathsf{P}_{\mathrm{lcf}}\big[B''\big]$

(S4) $|\{B' \in \{0,1\}^{[n-1]^2} : \mathrm{rk}(B') < n-1\}| \leqslant (\frac{1}{2} + o_{n\to\infty}(1)) \cdot \sum_{B\in\{0,1\}^{[n-1]^2}} (\frac{1}{2})^{\mathrm{supp}(B)} \cdot \left|\left\{\begin{matrix} B'' \in \{0,\pm\}^{[n-1]^2} : \\ \mathrm{Supp}(B'') = \mathrm{Supp}(B), \\ \mathrm{rk}(B'') < n-1 \end{matrix}\right\}\right|$

Proof. As to the equivalence (S1) $\Leftrightarrow$ (S2), if (S1), then $\mathsf{P}_{\mathrm{chio}}\big[\mathrm{Ra}_{<n-1}(\{0,\pm\}^{[n-1]^2})\big] \overset{\text{Corollary (191)}}{=} \mathsf{P}\big[\mathrm{Ra}_{<n}(\{\pm\}^{[n]^2})\big] \overset{(\text{S1})}{\leqslant} (\frac{1}{2}+o_{n\to\infty}(1))^n = (\frac{1}{2}+o_{n\to\infty}(1))\cdot (\frac{1}{2}+o_{n\to\infty}(1))^{n-1} \overset{\text{Theorem 140}}{\sim} (\frac{1}{2}+o_{n\to\infty}(1)) \cdot \mathsf{P}_{\mathrm{lcf}}\big[\mathrm{Ra}_{<n-1}(\{0,\pm\}^{[n-1]^2})\big]$, which is (S2). As to the converse, (S2) implies $\mathsf{P}\big[\mathrm{Ra}_{<n}(\{\pm\}^{[n]^2})\big] \overset{\text{Corollary (191)}}{=} \mathsf{P}_{\mathrm{chio}}\big[\mathrm{Ra}_{<n-1}(\{0,\pm\}^{[n-1]^2})\big] \overset{(\text{S2})}{\leqslant} (\frac{1}{2} + o_{n\to\infty}(1)) \cdot \mathsf{P}_{\mathrm{lcf}}\big[\mathrm{Ra}_{<n-1}(\{0,\pm\}^{[n-1]^2})\big] \overset{\text{Theorem 140}}{\sim} (\frac{1}{2} + o_{n\to\infty}(1))\cdot (\frac{1}{2} + o_{n\to\infty}(1))^{n-1} = (\frac{1}{2} + o_{n\to\infty}(1))^n$, which is (S1). The equivalence (S2) $\Leftrightarrow$ (S3) is obvious. As to (S3) $\Leftrightarrow$ (S4), note that (S3) $\overset{\text{Proposition 198}}{\Leftrightarrow} \sum [\![B' \in \{0,1\}^{[n-1]^2} :\ \mathrm{rk}(B') < n-1]\!]\ \mathsf{P}[B'] \leqslant \big(\frac{1}{2}+o_{n\to\infty}(1)\big) \cdot \sum [\![B'' \in \mathrm{Ra}_{<n-1}(\{0,\pm\}^{[n-1]^2})]\!]\ \mathsf{P}_{\mathrm{lcf}}\ \big[\ B''\ \big] \Leftrightarrow |\{ B' \in \{0,1\}^{[n-1]^2} :\ \mathrm{rk}(B') < n-1 \}| \leqslant (\frac{1}{2}+o_{n\to\infty}(1)) \cdot \sum_{B''\in\{0,\pm\}^{[n-1]^2} :\, \mathrm{rk}(B')<n-1} (\frac{1}{2})^{\mathrm{supp}(B'')} \Leftrightarrow |\{ B' \in \{0,1\}^{[n-1]^2} :\ \mathrm{rk}(B') < n-1 \}| \leqslant (\frac{1}{2}+o_{n\to\infty}(1)) \cdot \sum_{B\in\{0,1\}^{[n-1]^2}} \sum [\![B'' \in \{0,\pm\}^{[n-1]^2} :\ \mathrm{Supp}(B'') = \mathrm{Supp}(B), \mathrm{rk}(B'') < n-1]\!]\ (\frac{1}{2})^{\mathrm{supp}(B'')} \Leftrightarrow |\{ B' \in \{0,1\}^{[n-1]^2} :\ \mathrm{rk}(B') < n-1 \}| \leqslant (\frac{1}{2} + o_{n\to\infty}(1)) \cdot \sum_{B\in\{0,1\}^{[n-1]^2}} (\frac{1}{2})^{\mathrm{supp}(B)} \cdot |\{ B'' \in \{0,\pm\}^{[n-1]^2} :\ \mathrm{Supp}(B'') = \mathrm{Supp}(B), \mathrm{rk}(B'') < n-1 \}| \Leftrightarrow$ (S4). $\square$

Note the 'relativising' effect of having two sums over the same index set on either side of a (conjectured) inequality: thanks to commutativity of addition one may go about pitting (collections of) unequally indexed summands on both sides of (S3) against one another, in the hope of finding a rearrangement that allows one to prove the inequality without any a priori knowledge about the size of the index set of the sums. Of course, *if* (S1) is true, *then* the inequality is true for *every* permutation of the summands but the point is that this is not known and that it would suffice to prove the existence of only one suitable rearrangement of the summands to prove (or disprove) Conjecture (S1).

4.3.3.1 The inequality (S2) fails 'locally' on the entry-specification events

Let us remark that in view of the formula (C3) in Theorem 167 we find ourselves in the following situation: while (S2), which speaks about the $\mathsf{P}_{\mathrm{chio}}$-measure of the event $\mathrm{Ra}_{<n-1}(\{0,\pm\}^{[n-1]^2})$, might be true, it cannot possibly be true in a non-trivial way (left-hand side nonzero) on any of the entry-specification events.

Already in Corollary 176 we have seen examples that the inequality (S2) can fail when the event $\mathrm{Ra}_{<n-1}(\{0,\pm\}^{[n-1]^2})$ is replaced by other events—the failure seeming more likely and more severe as the events get smaller. We will now see that (S2) fails arbitrarily badly on every atom of the measure space we are dealing with (i.e. a singleton event $\{B\}$ with $B \in \{0,\pm\}^{[n-1]^2}$ and $\mathsf{P}_{\mathrm{chio}}[B] > 0$). For such events the ratio of $\mathsf{P}_{\mathrm{chio}}$ and $\mathsf{P}_{\mathrm{lcf}}$ diverges as quickly as 2^{n^2} when $n \to \infty$, while (S2) asserts a bounded ratio as $n \to \infty$.

By (C3) in Theorem 167, we know $\mathsf{P}_{\mathrm{chio}}[\mathcal{E}_B^J]/\mathsf{P}_{\mathrm{lcf}}[\mathcal{E}_B^J] = 2^{\beta_1(\mathrm{X}_B)}$. Therefore, to determine the maximum of $\mathsf{P}_{\mathrm{chio}}[\mathcal{E}_B^J]/\mathsf{P}_{\mathrm{lcf}}[\mathcal{E}_B^J]$ over all entry specification events $\mathcal{E}_B^J$ it suffices to determine the maximum of $\beta_1(\mathrm{X}_B)$ over all bipartite graphs $\mathrm{X}_B \in \mathrm{BG}_{n,n}$ with $B \in \{0,\pm\}^{[n-1]^2}$. The Betti number $\beta_1(X) = \|X\| - |X| + \beta_0(X)$ as a function of $X \in \mathrm{BG}_{n,n}$ attains a unique maximum at $X =$

$K^{n-1,n-1}$. The corresponding value is $(n-1)^2-2(n-1)+1=(n-2)^2$. Since $K^{n-1,n-1}$ can indeed occur as X_B with $B\in\mathrm{Ra}_{<n-1}(\{0,\pm\}^{[n-1]^2})\cap\mathrm{im}(\frac{1}{2}\mathrm{C}_{(n,n)}\colon\ \{\pm\}^{[n]^2}\to\{0,\pm\}^{[n-1]^2})$, it follows that for every fixed n, the maximum of $\mathsf{P}_{\mathrm{chio}}[\mathcal{E}_B^J]/\mathsf{P}_{\mathrm{lcf}}[\mathcal{E}_B^J]$ over all $\mathcal{E}_B^J$ with $\varnothing\neq I\subseteq J\subseteq[n-1]^2$ and $B\in\{0,\pm\}^I\cap\mathrm{im}(\frac{1}{2}\mathrm{C}_{(n,n)}\colon\{\pm\}^{[n]^2}\to\{0,\pm\}^{[n-1]^2})$ is $2^{(n-2)^2}\sim 2^{n^2}$. Since Proposition 197 implies that every $B\in\{0,\pm\}^I\cap\mathrm{im}(\frac{1}{2}\mathrm{C}_{(n,n)}\colon\{\pm\}^{[n]^2}\to\{0,\pm\}^{[n-1]^2})$ which realizes the maximum, i.e. $\mathrm{X}_B\cong K^{n-1,n-1}$, has rank 1, it follows that $2^{(n-2)^2}$ is also the maximum of $\mathsf{P}_{\mathrm{chio}}[\mathcal{E}_B^J]/\mathsf{P}_{\mathrm{lcf}}[\mathcal{E}_B^J]$ over all $\mathcal{E}_B^J$ with $\varnothing\neq I\subseteq J\subseteq[n-1]^2$ and $B\in\mathrm{Ra}_{<n-1}(\{0,\pm\}^{[n-1]^2})\cap\mathrm{im}(\frac{1}{2}\mathrm{C}_{(n,n)}\colon\{\pm\}^{[n]^2}\to\{0,\pm\}^{[n-1]^2})$.

4.3.3.2 The extent of failure of (S2) on Chio-condensates of a random $A\in\{\pm\}^{[n]^2}$

Let us have a quick informal look at the typical value of the ratio $\mathsf{P}_{\mathrm{chio}}[B]$ and $\mathsf{P}_{\mathrm{lcf}}[B]$ for $B\in\{0,\pm\}^{[n-1]^2}$ which are of the form $B=\mathrm{C}_{(n,n)}(A)$ with $A\in\{\pm\}^{[n]^2}$ chosen uniformly at random. Of course, such a B is (by Theorem 140 and Lemma 189) asymptotically almost surely *not* an element of $\mathrm{Ra}_{<n-1}(\{0,\pm\}^{[n-1]^2})$.

By Corollary 171, for $A\in\{\pm\}^{[n]^2}$ chosen uniformly at random, the graph X_B is a random bipartite graph with partition classes of $n-1$ vertices on either side and having i.i.d. edges with probability $\frac{1}{2}$. Such a random bipartite graph is connected a.a.s., i.e., $\beta_0(\mathrm{X}_B)=1$. As to $\|\mathrm{X}_B\|$, a standard argument using Chernoff's bound shows that for every $\epsilon>0$ a.a.s. (and approaching 1 exponentially fast) $(\frac{1}{2}-\epsilon)\cdot(n-1)^2\leqslant\|\mathrm{X}_B\|\leqslant(\frac{1}{2}+\epsilon)\cdot(n-1)^2$. For simplicity let us pretend that $\|\mathrm{X}_B\|=\frac{1}{2}(n-1)^2$ exactly. Then $\mathsf{P}_{\mathrm{chio}}[B]/\mathsf{P}_{\mathrm{lcf}}[B]=2^{\beta_1(\mathrm{X}_B)}=2^{\|\mathrm{X}_B\|-|\mathrm{X}_B|+\beta_0(\mathrm{X}_B)}=2^{\frac{1}{2}n^2-3n+\frac{3}{2}}\sim 2^{\frac{1}{2}n^2}=\sqrt{2^{n^2}}\to\infty$ as $n\to\infty$, and we have learned that the worst-case failure-ratio of (S2) found in 4.3.3.1 arises roughly by squaring the failure-ratio for a $B=\frac{1}{2}\mathrm{C}_{(n,n)}(A)$ with $A\in\{\pm\}^{[n]^2}$ random.

4.4 Concluding questions

Let us close with three questions:

4.4.1 More accurate estimations?

Note that Theorem 187.(Ex4)—(Ex6) teaches us that, asymptotically, the ratio $|\mathcal{F}^{\mathrm{M}}(k,n)|\ /\ |\{B\in\{0,\pm\}^I,\ I\in\binom{[n-1]^2}{k}\}|$ for $k\in\{4,5,6\}$ takes values $4n^4/\frac{27}{8}n^8=\frac{32}{27}\ n^{-4}\in[\ 1.1n^{-4},\ 1.2n^{-4}\ ]$, $12n^6/\frac{81}{40}n^{10}=\frac{480}{81}\ n^{-4}\in[\ 5.9n^{-4},\ 6.0n^{-4}\ ]$ and $18n^8/\frac{81}{80}n^{12}=\frac{1440}{81}n^{-4}\in[\ 17.7n^{-4},\ 17.8n^{-4}\ ]$ which are all—although of course growing with k—vanishing with the same speed $\mathcal{O}_{n\to\infty}(n^{-4})$. Hence the rough bound in Proposition 188 is not an asymptotically tight one. This of course raises two questions: *Is it true that* $|\mathcal{F}^{\mathrm{M}}(k,n)|\ /\ |\{B\in\{0,\pm\}^I,\ I\in\binom{[n-1]^2}{k}\}|\in\mathcal{O}_{n\to\infty}(n^{-4})$ *for every fixed k?* Moreover, note that while Proposition 188 shows that $|\mathcal{F}^{\mathrm{M}}(k,n)|\ /\ |\{B\in\{0,\pm\}^I,\ I\in\binom{[n-1]^2}{k}\}|$ vanishes as $n\to\infty$ for every fixed k, the discussion in Section 4.3.3.1 shows that for the extreme case of $k=(n-1)^2$, i.e. if the entire matrix $B\in\{0,\pm\}^{[n-1]^2}$ is specified, $\mathsf{P}_{\mathrm{chio}}[\mathcal{E}_B^{[n-1]^2}]\neq\mathsf{P}_{\mathrm{lcf}}[\mathcal{E}_B^{[n-1]^2}]$ for the vast majority of $B\in\{0,\pm\}^{[n-1]}$. In between these two extremes, i.e. almost sure agreement as opposed to almost sure non-agreement of $\mathsf{P}_{\mathrm{chio}}$ and $\mathsf{P}_{\mathrm{lcf}}$, there should be a tipping point. This raises the question: *For what order of growth $k=k(n)$ does* $|\mathcal{F}^{\mathrm{M}}(k,n)|\ /\ |\{B\in\{0,\pm\}^I,\ I\in\binom{[n-1]^2}{k}\}|$ *first become bounded away from zero? And for what order does it first tilt in favour of the non-agreement events?*

4.4.2 What to make of the k-wise independence?

Note that Theorem 187 in particular says that one application of the Chio-map $\frac{1}{2}\mathrm{C}_{(n,n)}$ to a $\{\pm\}$-valued $n\times n$-matrix yields a $\{0,\pm\}$-matrix whose entries are distributed as $\frac{1}{4},\frac{1}{2},\frac{1}{4}$ but are merely 3-wise stochastically independent (in the sense of [67, Definition 2.4, p. 209]) (and 'almost' k-wise, the ‘almost’ quantified exactly for $k \in \{4, 5, 6\}$ in Theorem 187 and quantified roughly for general k in Proposition 188). How partial independence relates to full independence is still a subject of current research (see e.g. [67]), a keyword being ‘k-wise independence’. The author wonders whether the theory of k-wise independence has any bearing on the present problem. In particular, can the proof of Bourgain–Vu–Wood for $\mathsf{P}_{\mathrm{lcf}}$ be deconstructed and somehow reassembled for $\mathsf{P}_{\mathrm{chio}}$, aided by knowledge about k-wise independence?

4.4.3 Hidden connections to the Guralnick–Maróti-theorem?

If $\sigma\colon G \to \mathrm{Aut}_K(V)$ is a representation of a group G on a K-vector space V, then for every $g \in G$ let $\mathrm{Fix}_V(g)$ denote the *fixed-point space of* g, i.e. the K-linear subspace $\{v \in V : \sigma(g)(v) = v\}$.

In recent times there have been advances ([153], [88], [28]) concerning the problem of bounding averages of dimensions of fixed-point spaces by a fraction of the dimension of the representation, leading to a full proof (and in more general form) by R. M. Guralnick and A. Maróti [72] of a 1966 conjecture of P. M. Neumann:

Theorem 200 (Guralnick–Maróti [72, Theorem 1.1])**.** *For every finite group G with smallest prime factor of $|G|$ denoted by p, every field K, every finite-dimensional K-vector space V, every homomorphism $\sigma\colon G \to \mathrm{Aut}_K(V)$, and every normal subgroup N of G which does not have a trivial composition factor on V,*

$$\frac{1}{|N|} \sum_{\tilde{g} \in N \cdot g} \dim_K(\mathrm{Fix}_V(\tilde{g})) \leqslant \frac{1}{p} \, \dim_K(V) \ . \tag{4.37}$$

Although the resemblance is likely to be merely superficial, the author cannot help being intrigued by the similarity of this inequality to (S3) on p. 188, together with the fact that both in (S3) and in (4.37) there can be zero-summands on the left-hand side. Moreover, when trying to combine Chio condensation of sign matrices with group actions, one gets the impression that groups of even order (i.e. $p = 2$) play a natural role. One goal along these lines is to discover a vector space avatar of the lazy coin flip measure. Via Theorem 167 the author found the following formula (which is of course easy to check directly): for every $B \in \{0, \pm\}^{[n-1]^2}$ we have

$$\mathsf{P}_{\mathrm{lcf}}[B] = (\tfrac{1}{2})^{(n-1)^2} \cdot (\tfrac{1}{2})^{\dim_{\mathbb{F}_2}(\mathrm{B}^1(\mathrm{X}_B;\, \mathbb{F}_2) \oplus \mathrm{Z}_1(\mathrm{X}_B;\, \mathbb{F}_2))} \ . \tag{4.38}$$

This suggests studying group actions on the direct sum $\mathrm{B}^1(\mathrm{X}_{\frac{1}{2}\mathrm{C}_{(n,n)}(A)}; \mathbb{F}_2) \oplus \mathrm{Z}_1(\mathrm{X}_{\frac{1}{2}\mathrm{C}_{(n,n)}(A)}; \mathbb{F}_2)$. A sensible first choice are those actions which are induced by the standard $|\det(\cdot)|$-preserving (hence intransitive) group actions on $\{\pm\}^{[n]^2}$, e.g. transposing, permuting rows or columns, and flipping all signs of a row or a column (a merit of those actions is that they commute with Chio condensation).

5 Definitions, prerequisites and auxiliary substructures used in this thesis

This chapter does more than only give notation and definitions: in accordance with the organising principle of this thesis, Section 5.3 describes all substantial auxiliary structures used in the other chapters, both those explicitly defined (such as the graphs $\mathrm{C}_n^{2^-}$ from Definition 214) and those implicitly proved to exist (such as the graphs in Lemma 226). In particular, the present chapter contains some statements and proofs.

5.1 Some basic definitions

Here we give some basic definitions used in this thesis.

We use the notations $[n] := \{1, \ldots, n\}$, $[n]_0 := \{0, 1, \ldots, n\}$ and $|\cdot|$ for the cardinality of a set, 'a.a.s.' as an abbreviation for 'asymptotically almost surely', $\mathrm{Poi}(\mu)$ for the Poisson distribution with parameter μ, $\mathbb{Z}/a := \mathbb{Z}/a\mathbb{Z}$, $- := -1$, $+ := +1$, $\{\pm\} := \{-1, +1\}$, $\{0, \pm\} := \{-1, 0, +1\}$, $\mathrm{Ra}_{<n}(\{\pm\}^{[n]^2}) := \{A \in \{\pm\}^{[n]^2} \colon \det(A) = 0\}$. For $A = (a_{i,j})_{(i,j)\in[n]^2} \in \{\pm\}^{[n]^2}$, any $(i,j) \in [n-1]^2$ and any $\varnothing \subseteq I \subseteq [n-1]^2$ let $A[i,j] := a_{i,j}$ and $A[I] := (a_{i,j})_{(i,j)\in I}$, hence in particular $A[\varnothing]$ is the empty function and $A\big[[n]^2\big] = A$. By $\mathfrak{P}(X)$ we denote the power set of a set X. For a cartesian product $M \times N$ of two sets M and N let $\mathrm{p}_1 \colon M \times N \to M$ be the projection onto the first, and $\mathrm{p}_2 \colon M \times N \to N$ the projection onto the second factor. If M and N are finite and $\varnothing \subseteq I \subseteq M \times N$ is some subset, then I is called *rectangular* if and only if $|I| = |\mathrm{p}_1(I)| \cdot |\mathrm{p}_2(I)|$.

In Chapter 4 we view functions as sets and matrices as functions. If D is a set and $f \colon D \to \mathbb{Z}$ is a function let us write $D =: \mathrm{Dom}(f) \supseteq \mathrm{Supp}(f) := \{d \in D \colon f(d) \neq 0\}$ for its domain and support, and let us employ the abbreviations $|\mathrm{Dom}(f)| =: \mathrm{dom}(f) \geqslant \mathrm{supp}(f) := |\mathrm{Supp}(f)|$. We have $\mathrm{Dom}(\varnothing) = \mathrm{Supp}(\varnothing) = \varnothing$ and therefore $\mathrm{dom}(\varnothing) = \mathrm{supp}(\varnothing) = 0$. If $U \subseteq \mathbb{Z}$, $\varnothing \subseteq I \subseteq [s-1] \times [t-1]$ and $B \in U^I$, then we have $[s-1] \times [t-1] \supseteq I = \mathrm{Dom}(B) \supseteq \mathrm{Supp}(B) = \{(i,j) \in [s-1] \times [t-1] \colon B[(i,j)] \neq 0\}$. The all-ones-matrix with domain $D \subseteq [s] \times [t]$ is denotes $\{1\}^D$. For a matrix $M = (m_{i,j})_{(i,j)\in I} \in \mathbb{Q}^I$ and a $q \in \mathbb{Q}$ we define, as usual, $q \cdot M := (q \cdot m_{i,j})_{(i,j)\in I}$. The symbol $\sqcup$ denotes a set union $\cup$ and at the same time makes the claim that the union is disjoint. The term *rank* of matrix has its usual meaning (and we will only use it in the context of integral domains, so that row-rank, column-rank and determinantal rank are all the same). For a set S, the group of all permutations of S is denoted by $\mathrm{Sym}(S)$.

We adopt the conventions that a 2-set $\{v', v''\}$ can be abbreviated as $v'v''$, and that $\sqcup$ indicates a union of sets together with the claim that the sets are disjoint. The word 'graph' without any further qualifications means 'finite simple undirected graph' (i.e. 'finite 1-dimensional simplicial complex'). If G is a graph, $\mathrm{V}(G)$ denotes its vertex set, $\mathrm{E}(G)$ its edge set, $|G| := |\mathrm{V}(G)|$ and $\|G\| := |\mathrm{E}(G)|$. By *language of finite graphs* we mean a fixed formalisation of finite simple undirected graphs with the symbols ',', '(', ')', 'v_0', 'v_1', ..., '{', '}', and '$\sim$', the latter having the semantics of meaning 'adjacent'. I.e., in this language one can write down (sets of) graphs, but one cannot expressly write numbers and other symbols. (One can of course try to *encode* all sorts of things as strings made of the above symbols, but this usually takes much more space than the usual notations, which is what makes complexity-theoretic questions about that language interesting).

The *f-vector* of a graph G (using a standard term from the theory of simplicial complexes) is defined to be the vector $(|G|, \|G\|) \in \mathbb{Z}^2_{\geqslant 0}$. If $\mathcal{G}$ denotes some set of graphs, $\mathcal{G}_n := \{G \in \mathcal{G} \colon |G| = n\}$. If G is a graph, $v \in \mathrm{V}(G)$ and $r \in \mathbb{Z}_{\geqslant 0}$, then $\mathrm{B}_G(v, \leqslant r)$, denotes the *ball of radius r around v* i.e. the *set* of all vertices of G having graph-theoretic distance (length of shortest path) at most

r from v. The subgraph of G induced by the set of all vertices of graph distance at most r from u is written $G[\mathrm{B}_G(v, \leqslant r)]$. If G and H are graphs, then $H \hookrightarrow G$ means that there exists an injective graph homomorphism $H \to G$ (hence there is a subgraph of G isomorphic to H). A path of *length* (i.e. number of its edges) ℓ will be denoted by P_ℓ and a circuit of length ℓ by C_ℓ. As in [20] we reserve the word 'cycle' for the elements of $\mathrm{Z}_1(G; \mathbb{F}_2)$ and use the term 'circuit' for '2-regular connected graph'. A *Hamilton-circuit* of a graph is a circuit containing all of its vertices. If C is a circuit with $\mathrm{V}(C) = \{v_0, v_1, v_2, \ldots, v_{\ell-1}\}$ and $\mathrm{E}(C) = \{v_0v_1, v_1v_2, \ldots, v_{\ell-1}v_0\}$, then we abbreviate $v_0v_1v_2 \ldots v_{\ell-1}v_0 := \mathrm{E}(C)$. A *cubic* (resp. *quartic*) graph is a graph all of whose vertices have degree 3 (resp. 4).

A subgraph H of a graph G is called *non-separating* if and only if the graph $G - H := (\mathrm{V}(G)\backslash\mathrm{V}(H), \mathrm{E}(G)\backslash\{e \in \mathrm{E}(G) \colon e \cap \mathrm{V}(H) \neq \varnothing\})$ is connected. A circuit C in a graph G is called *non-separating induced* if and only if C is non-separating and C has no chords in G (i.e. $\{e \in \mathrm{E}(G) \colon e \subseteq \mathrm{V}(C)\} = \mathrm{E}(C)$). We write $\mathrm{c}_e \in \mathbb{F}_2^{\mathrm{E}(G)}$ for the unique map with $\mathrm{c}_e(e) = 1 \in \mathbb{F}_2$ and $\mathrm{c}_e(e') = 0 \in \mathbb{F}_2$ for every $e \neq e' \in \mathrm{E}(G)$. The *edge space* of a graph (cf. [50, p. 23]) of G, denoted $\mathrm{C}_1(G; \mathbb{F}_2)$, is the $\mathbb{F}_2$-linear span of $\{\mathrm{c}_e \colon e \in \mathrm{E}(G)\}$. The *group of* 1*-dimensional chains* of a graph G, i.e. the free abelian group generated by its edges, is denoted $\mathrm{C}_1(G)$. By $\mathcal{H}(G)$ we denote the set of Hamilton circuits in a graph G, and by $\vec{\mathcal{H}}(G)$ the set of all simple flows on G with support equal to a Hamilton-circuit of G.

For any set $\mathcal{M}$ of circuits in G we say that '$\mathcal{M}$ generates $\mathrm{Z}_1(G; \mathbb{F}_2)$' if and only if $\{\mathrm{c}_C \colon C \in \mathcal{M}\}$ is an $\mathbb{F}_2$-generating system of $\mathrm{Z}_1(G; \mathbb{F}_2)$, where c_C is defined as the element of $\mathrm{C}_1(G; \mathbb{F}_2)$ with support equal to $\mathrm{E}(C)$. A bipartite graph is called *balanced* if and only if its bipartition classes have equal size. If G and H are graphs, we denote by $G \,\square\, H$ the *cartesian product* of G and H (cf. e.g. [86, Section 1.4]). If G is a graph, then we write $\mathrm{N}_G(v) := \{w \in \mathrm{V}(G) \colon \{v, w\} \in \mathrm{E}(G)\}$ for every $v \in \mathrm{V}(G)$, $\delta(G) := \min_{v \in \mathrm{V}(G)} |\mathrm{N}_G(v)|$ (called *minimum degree* of G), and $\Delta(G) := \max_{v \in \mathrm{V}(G)} |\mathrm{N}_G(v)|$ (called *maximum degree* of G). By *k-connected* we mean the standard graph-theoretical notion of being 'vertex-k-connected' (cf. [50, Section 1.4]). A *bridge* in a graph is an edge whose deletion increases the number of connected components. (Cf. the term 'bridge-addable' from Definition 203.) A *Cayley graph* is a graph G which can be realised as follows: there exists a group Γ and a subset $S \subseteq \Gamma$ with $S^{-1} := \{s^{-1} \colon s \in S\} = S$, such that $\mathrm{V}(G) = \Gamma$ and $\mathrm{E}(G) = \{\ \{g_1, g_2\} \colon\ g_1g_2^{-1} \in S\ \}$. (Because of $(g_1g_2^{-1})^{-1} = g_2g_1^{-1}$ and $S^{-1} = S$, the latter condition is well-defined, even though $g_1g_2^{-1} \neq g_2g_1^{-1}$ is of course possible.)

The *cycle space of* X (i.e. 1-dimensional cycle group with $\mathbb{F}_2$-coefficients in the sense of simplicial homology theory) will be denoted by $\mathrm{Z}_1(X; \mathbb{F}_2)$ and the *coboundary space of* X by $\mathrm{B}^1(X;\ \mathbb{F}_2)$ (this is the 1-dimensional coboundary group with $\mathbb{F}_2$-coefficients in the sense of simplicial cohomology theory; a synonym is 'cut space of X'). Let $\beta_0(X)$ denote the number of connected components of a graph X and $\beta_1(X) := \dim_{\mathbb{F}_2}\ Z_1(X; \mathbb{F}_2)$ the first Betti number (a synonym in the graph-theoretical literature is 'cyclomatic number' [150]). We will (without further notification) use the 1-dimensional case of the alternating sum relation between the ranks of the chain groups and the ranks of the homology groups of a free chain complex, i.e. $\beta_1(X) - \beta_0(X) = f_1(X) - f_0(X)$ for every graph X. For any two disjoint graphs X_1 and X_2, the graph obtained by identifying exactly one vertex of X_1 with exactly one vertex of X_2 is called the *(one-point) wedge of X_1 and X_2* and denoted by $X_1 \vee X_2$. This is the standard wedge product of pointed topological spaces (but only vertices of a graph are allowed as basepoints); a synonym within the graph-theoretical literature is 'coalescence' [71, p. 140].

In Chapter 4, we use the notion of *signed graphs* (see [167] for a comprehensive overview). It is customary in signed graph theory to work with multigraphs (i.e. finite 1-dimensional CW-complexes) for reasons of higher flexibility in proofs and applications. However, in this thesis, all we will need are signed *simple* graphs, i.e. for us a signed graph (X, σ) will simply consist of a graph $X = (V, E)$ together with an arbitrary *sign function* $\sigma \colon E \to \{\pm\}$. We call *$(+)$-edge* (resp. $(-)$-edge) every $e \in \mathrm{E}(X)$ with $\sigma(e) = +$ (resp. $\sigma(e) = -$). Define $(+)$-paths (resp. $(-)$-paths) as paths all of whose edges are $(+)$-edges (resp. $(-)$-edges). For emphasizing the sign function we employ the notation '$(\sigma,\ +)$-edge'. If (X, σ) is a signed graph let $f_1^{(-)}(X, \sigma) := |\{e \in \mathrm{E}(X) \colon \sigma(e) = -\}|$

denote the number of $(\sigma, -)$-edges in it. A signed graph (X, σ) is called *balanced*[1] if and only if $f_1^{(-)}(C, \sigma)$ is even for every circuit C of X. We will denote the set of all balanced signings of X by $S_{\mathrm{bal}}(X) := \{\sigma \in \{\pm\}^{\mathrm{E}(X)} : (X, \sigma) \text{ balanced}\}$.

The *square* H^2 of a graph H is the graph obtained from H by adding an edge between any two vertices having distance two in H. A graph H has *bandwidth at most b* if and only if there exists a bijection $\mathrm{b}\colon \mathrm{V}(H) \to \{1, \ldots, |H|\}$ such that if $vv' \in \mathrm{E}(H)$, then $|\mathrm{b}(v) - \mathrm{b}(v')| \leqslant b$; any such bijection b is called a *bandwidth-b-labelling of H*. Moreover, if H is a graph, $\mathrm{b}\colon \mathrm{V}(H) \to \{1, \ldots, |H|\}$ is a bijection and if $(c_1, c_2) \in \mathbb{Z}^2_{\geqslant 1}$ and $\rho \in \mathbb{Z}_{\geqslant 1}$, then a map $h\colon \mathrm{V}(H) \to \{0, \ldots, \rho\}$ is called (c_1, c_2)*-zero-free w.r.t.* b (cf. [24, p. 178]) if and only if for for every $v' \in \mathrm{V}(H)$ there exists a $v'' \in \mathrm{b}^{-1}\big(\{\mathrm{b}(v'), \mathrm{b}(v') + 1, \ldots, \min(|H|, \mathrm{b}(v') + c_1)\}\big)$ such that $h(v''') \neq 0$ for every $v''' \in \mathrm{b}^{-1}\big(\{\mathrm{b}(v''), \mathrm{b}(v'') + 1, \ldots, \min(|H|, \mathrm{b}(v'') + c_2)\}\big)$.

We use *FO-logic* (first order logic) and *MSO-logic* (monadic second order logic) of graphs, in the standard sense fully defined in e.g. [54, p. 4 for FO, p. 38 for MSO]. In brief, formulas in FO-logic of graphs are well-formed strings made from the alphabet $\{x_i : i \in \mathbb{N}\} \cup \{\neg, \vee, \wedge, \exists, =, (,), \sim\}$, the symbols having as the usual semantics 'vertex name', 'not', 'or', 'and', 'there exists', 'now open a region to be interpreted first', 'now close a region to be interpreted first', 'is adjacent to'. In MSO, an additional set of variables $\{X_i : i \in \mathbb{N}\}$, disjoint from all the other symbols, is available, whose semantics are that they stand for vertex-subsets, and moreover the symbol '$\in$' can be used, whose semantics are 'is an element of'. One must not use numerals or arithmetic symbols in any of these languages, nor can one express the cardinality of a set.[2] Such limitations of MSO are what that make FO- or MSO-restricted questions about the definability or complexity of graph-properties interesting, in particular when that graph property is *defined* by algebraic props (like e.g. the property $\mathrm{Bas}\mathcal{C}_{|\cdot|}$ from Definition 204.(15) which probably is not definable in MSO-logic of graphs, but this is not proved). There is a subset of MSO-logic called *existential monadic second order logic* (synonyms: 'monadic NP' or 'monadic Σ^1_1-formulas'), or *EMSO*-logic for short, in which existentially-quantified set-variables are allowed, while universal quantifiers speaking about sets are forbidden. A *sentence* is a formula without free variables. We use the notion of *quantifier rank* $\mathrm{qr}(\varphi)$ of a formula φ in FO- or MSO-logic, in the standard sense of, e.g., [54, p. 7]. Informally, it is the maximum nesting-depth the quantifiers $\exists$ and $\forall$ reached inside the formula φ. In the literature there exists the synonym '*quantifier depth*' for 'quantifier rank'.

We say that a rooted connected graph (H, w) *is a pendant copy of G* (a more descriptive phrase would be '*H appears with its root at the end of a bridge*') if G contains an induced subgraph $\widetilde{H} \subseteq G$ isomorphic to H, and there is exactly one edge between $\widetilde{H}$ and $G - \widetilde{H}$, this edge being incident with the root w.

Following McDiarmid [128, p. 586], we denote by $\mathrm{Big}(G)$ the largest component of a graph G, where (unlikely) ties are broken by (say) taking the lexicographically first among the components of the largest order (i.e., we look at the labels of the vertices and take the component in which the smallest label occurs). Moreover, the *fragment* of G is defined as $\mathrm{Frag}(G) := G - \mathrm{Big}(G)$, i.e., as the union of all connected components of G other than $\mathrm{Big}(G)$.

A *lattice* is a free abelian subgroup with rank d of the group $\mathbb{R}^d$ with addition. Following [155, p. 695], an element $z \in \mathrm{Z}_1(G)$ is called *simple flow* if and only if all its coordinates are in $\{0, \pm\}$ and $\mathrm{Supp}(z)$ is a graph-theoretical circuit. (Synonyms in the literature: in the terminology of [15, Exercise 4i], simple flows are called *primitive flows*; in another terminology (cf. e.g. [44, p. 6]), simple flows are the *elementary vectors* of the lattice $\mathrm{Z}_1(G)$.) A *Hamilton-flow* on G is defined as a *simple* flow whose support is a Hamilton-circuit of G.

The flow lattice of a graph G is denoted $\mathrm{Z}_1(G)$. Let us take the time to explicitly point out that for a graph G, the notations $\mathcal{F}(G)$ from [68, p. 733], and $\Gamma(G)$ from [155, p. 691], and $\mathrm{Z}_1(G)$ from algebraic topology (see e.g. [139]), denote one and the same object:

[1] The use of this term seems to have been initiated in [75]. The notion itself was already studied over seventy years ago by D. Kőnig [104, p. 149, Paragraph 3] under the name '*p*-Teilgraph'.

[2] But one can e.g. easily express 'S is the empty set' (e.g. as '$\neg(\exists x \in S)$') and 'S equals the universe' (e.g. as '$\forall x\colon x \in S$') in MSO-logic.

Remark 201 ($\mathcal{F}(G) = \Lambda(G) = \mathrm{Z}_1(G)$)**.** *If G is a finite simple graph, $\mathcal{F}(G)$ has its meaning from [68, p. 733] and $\Lambda(G)$ its meaning from [155, p. 691], then $\mathcal{F}(G) = \Lambda(G) = \mathrm{Z}_1(G)$, essentially in the sense of equality of sets.* □

We use the notation $\mathrm{Z}_1(G)$, since it appears to be the most traditional and, even today, most standard[3] choice. Elements of $\mathrm{Z}_1(G)$ are often called *cycles* or *circulations* (cf. e.g. [10, p. 133]). Calling an element of $\mathrm{Z}_1(G)$ a 'flow' follows [9] and [68], is consistent with *all* of the literature in that every circulation is a flow, yet *in*consistent with *parts* of the literature (such as [10]) in that some authors take 'flow' to mean an arbitrary assignment of values to the edges, without requiring the flow-condition, and accord the name 'circulation' only to those which satisfy it. The present author decided to let the literature closest to the main concerns of the thesis (like [9] [155] [68] [45]) gauge the terminology and stick to 'flow lattice', and in particular to not try start a new polysyllabic usage 'circulation lattice'. In this thesis 'flow' is a synonym for 'circulation'. For every graph G, the abelian group $\mathrm{Z}_1(G)$ has rank $\|G\| - |G| + 1$, i.e., there exist $\|G\| - |G| + 1$ distinct cycles (i.e., elements of $\mathrm{Z}_1(G)$), which form a $\mathbb{Z}$-module-basis of $\mathrm{Z}_1(G)$. It is known (e.g. [50, Proposition 1.9.1]) that for every graph G the flow lattice $\mathrm{Z}_1(G)$ is generated by those simple-flows whose support is an *induced* circuit.

Lemma 202 ($\mathrm{Z}_1(G)$ is generated by simple flows; cf. [155, Lemma 9] specialised to graphs)**.** *For every $\mathbb{Z}$-basis $\mathcal{B} \subseteq \mathrm{Z}_1(G)$ of $\mathrm{Z}_1(G)$, every $b \in \mathcal{B}$ is a simple flow. In particular, for every graph G the $\mathbb{Z}$-linear span of the set of all simple flows in $\mathrm{Z}_1(G)$ equals $\mathrm{Z}_1(G)$.* □

In the study of sets of graphs with certain (closure-)properties, it is customary to call an infinite set of finite graphs w.r.t. some fixed language, closed under graph isomorphism, and possibly satisfying certain additional properties a *class of graphs* (a synonym is: *graph property*). This is not a reference to the notion of 'class' from set-theory. When we use the phrase 'set of all graphs' it is understood that we mean 'set of all graphs with respect to some fixed language', which then is indeed a set. A graph property $\mathcal{G}$ is called *monotone increasing* if and only if for every $G \in \mathcal{G}$, adding to G an arbitrary edge again results in an element of $\mathcal{G}$. A graph property $\mathcal{G}$ consisting of bipartite graphs only is called a *monotone increasing property of bipartite graphs* if and only if for every $G \in \mathcal{G}$, adding to G any edge not creating an odd circuit again leaves in an element of $\mathcal{G}$.

A class $\mathcal{G}$ of graphs is *minor-closed* if every minor of a graph in $\mathcal{G}$ is also in $\mathcal{G}$. By the graph minor theorem, every minor-closed class is characterised by not containing any of a finite set of forbidden graphs as a minor.

A *toroidal graph* is a graph admitting an embedding into the *torus surface*. A *projective graph* is a graph admitting an embedding into the *projective plane*.

Definition 203 (addable, bridge-addable, decomposable class of graphs)**.** *If $\mathcal{G}$ is a set of graphs, then $\mathcal{G}$ is called*

(1) decomposable[4] *iff* $[\ G \in \mathcal{G}\ ] \Leftrightarrow [\ G' \in \mathcal{G}$ *for every connected component* G' *of* $G\ ]$,
(2) bridge-addable*, cf. [129] (synonym:* weakly addable*) iff for any $G \in \mathcal{G}$ the graph obtained by joining any two vertices in two connected components of G by a new edge is again in G* ,
(3) addable *iff it is both decomposable and bridge-addable* ,

[3]According to [151, p. 922], already in the manuscript for Hausdorff's 1933 lecture course [78] the notation $\mathcal{Z}_k$ is used for the group of k-dimensional cycles. Moreover, in the classic 1934 textbook of Seifert and Threlfall the authors speak of the 'lattice of all closed k-chains' and denote it by $\mathfrak{Z}$ (the letter Z in Fraktur type). In [117, p. 105], too, the letter Z is used for the cycle group, without a subscript 1, though. Modern examples using the letter 'Z' are [139], and [134, p. 35], which uses '$Z(G)$' for our $\mathrm{Z}_1(G)$.

[4]It might be instructive to note the following: in the literature relevant for this thesis, most of the time only *minor-closed* decomposable classes are studied. Once one has assumed minor-closedness, the term 'decomposable minor-closed' becomes somewhat of a pleonastic misnomer, as the direction '⇒' is implied by minor-closedness alone, so being *de*composable in the literal sense then makes no new demand on the structure. Then, '⇐' in Definition 203.(1) is the genuine requirement, so '*com*posable minor-closed class' would be shorter and more logical language. In the future, it might be a good idea to consider changing the terminology from 'decomposable class of graphs' into the shorter and more compatible 'composable class of graphs'.

(4) small *iff there exist* $d > 0$ *and* n_0 *such that* $|\{G \in \mathcal{C} \colon |G| = n\}| \leqslant d^n n!$ *for every* $n \geqslant n_0$,
(5) smooth *iff there exists* $\xi \in \mathbb{R}$ *with* $\frac{n|\mathcal{G}_{n-1}|}{|\mathcal{G}_n|} \xrightarrow{n\to\infty} \xi$.

Planar graphs constitute an addable class of graphs, but graphs embeddable on a surface other than the sphere may not. (For example, a 5-clique is embeddable on the torus, but, as a special case of the general additivity of the genus of a graph [134, Theorem 4.4.2], the vertex-disjoint union of two 5-cliques is not). Other examples of addable classes are outerplanar graphs, series-parallel graphs, graphs with bounded tree-width, and graphs with given 3-connected components.

To succinctly formulate properties of auxiliary substructures, we introduce the following technical definitions of graph classes (the phrasing 'set of all graphs' is, of course, to be understood w.r.t. some fixed language):

Definition 204. *If* $\mathrm{Z}_1(G)$ *denotes the flow lattice and* $\mathrm{Z}_1(G;\mathbb{F}_2)$ *the cycle space of a graph* G*, and if* A *is any finitely-generated abelian group,* $\mathfrak{L}$ *any map from graphs to subsets of* $\mathbb{Z}_{\geqslant 1}$*,* $\mathfrak{L} - 1 := \{l-1 \colon l \in \mathfrak{L}\}$ *and for every* $\xi \in \mathbb{Z}_{\geqslant 0}$*, we define*

(1) *a graph* G *to be* $\mathfrak{L}$-path-connected *(if* $\mathfrak{L} = \{|\cdot| - 1\}$ *we speak of being* Hamilton-connected*) if and only if for every* $\{v, w\} \in \binom{\mathrm{V}(G)}{2}$ *there exists in* G *at least one* v*-*w*-path having its length in the set* $\mathfrak{L}(G)$ *(we denote the set of all such graphs by* $\mathcal{CO}_{\mathfrak{L}}$*)* ,
(2) *a variant of* $\mathcal{CO}_{\mathfrak{L}}$ *for bipartite graphs: adopting a by now widespread usage dating back to [154], a bipartite graph* G *will be called* $\mathfrak{L}$-laceable *(if* $\mathfrak{L} = \{|\cdot| - 1\}$ *also* Hamilton-laceable*) if and only if, for any two* $v, w \in \mathrm{V}(G)$ *not in the same bipartition class, there is at least one* v*-*w*-path having its length in the set* $\mathfrak{L}(G)$ *(we denote the set of all such graphs by* $\mathcal{LA}_{\mathfrak{L}}$*)* ,
(3) *for a graph* G *the set* $\mathcal{C}_{\mathfrak{L}}(G)$ *as the set of all graph-theoretical circuits in* G *whose length is in* $\mathfrak{L}(G)$ *(in particular,* $\mathcal{C}_{|G|}(G) = \mathcal{H}(G)$*)* ,
(4) *for a graph* G *the set* $\vec{\mathcal{C}}_{\mathfrak{L}}(G)$ *as the set of all those* $z \in \mathrm{Z}_1(G)$ *which are simple flows, and moreover have the length of their support(-circuit) contained in the set* $\mathfrak{L}(G)$ *(in particular,* $\vec{\mathcal{C}}_{|\cdot|}(G) = \vec{\mathcal{H}}(G)$ *is the set of all simple flows with support a Hamilton-circuit),*
(5) $\mathrm{Cd}_\xi\mathcal{C}_{\mathfrak{L}}$ *as the set of all graphs* G *with* $\dim_{\mathbb{F}_2}\big(\langle \mathcal{C}_{\mathfrak{L}}(G)\rangle_{\mathbb{F}_2}\big) = \beta_1(G) - \xi$*, with* $\mathcal{C}_{\mathfrak{L}}$ *as in* (3)*; in the case* $\xi = 0$ *and* $\mathcal{L} = \{|\cdot|\}$*, the graph* G *is said to be* Hamilton-generated ,
(6) $\mathrm{Cd}_\xi\mathcal{C}_{\mathfrak{L}}^-$ *as the set of all graphs* G *for which there exists some* $z^- \in \mathcal{C}_{\mathfrak{L}-1\cup\mathfrak{L}}(G)$ *such that* $\dim_{\mathbb{F}_2}$ $(\langle \{z^-\} \cup \mathcal{C}_{\mathfrak{L}}(G)\rangle_{\mathbb{F}_2}) = \beta_1(G) - \xi$*, with* $\mathcal{C}_{\mathfrak{L}}$ *as in* (3)*; then* G *is said to be* ξ-almost-$\mathfrak{L}$-generated*; in the case* $\xi = 0$ *and* $\mathcal{L} = \{|\cdot|\}$*, the graph* G *is said to be* almost-Hamilton-generated ,
(7) $\mathrm{Quo}_A\mathcal{C}_{\mathfrak{L}}$ *as the set of all graphs* G *with* $\mathrm{Z}_1(G)/\langle \vec{\mathcal{C}}_{\mathfrak{L}}(G)\rangle_{\mathbb{Z}} \cong A$*; if* $A = \{0\}$ *and* $\mathfrak{L} = \{|\cdot|\}$*, then* G *is said to have* Hamilton-generated flow lattice ,
(8) $\mathrm{Quo}_A\mathcal{C}_{\mathfrak{L}}^-$ *as the set of all graphs* G *for which there exists some* $z^- \in \mathcal{C}_{\mathfrak{L}-1\cup\mathfrak{L}}(G)$ *such that* $\mathrm{Z}_1(G)/\langle \{z^-\} \cup \vec{\mathcal{C}}_{\mathfrak{L}}(G)\rangle_{\mathbb{Z}} \cong A$*; if* $A = \{0\}$ *and* $\mathfrak{L} = \{|\cdot|\}$*,* G *is said to have* almost-Hamilton-generated flow lattice ,
(9) $\mathcal{M}_{\mathfrak{L}}^{\mathbb{Z}\mathrm{Bas}^-} := \mathrm{Bas}\mathcal{C}_{\mathfrak{L}}^- \cap \mathcal{CO}_{\mathfrak{L}}$,
(10) $\mathrm{bCd}_\xi\mathcal{C}_{\mathfrak{L}} \subseteq \mathrm{Cd}_\xi\mathcal{C}_{\mathfrak{L}}$ *as the set of all the* bipartite *elements of* $\mathrm{Cd}_\xi\mathcal{C}_{\mathfrak{L}}$,
(11) $\mathrm{bQuo}_A\mathcal{C}_{\mathfrak{L}} \subseteq \mathrm{Quo}_A\mathcal{C}_{\mathfrak{L}}$ *as the set of all the* bipartite *elements of* $\mathrm{Quo}_A\mathcal{C}_{\mathfrak{L}}$,
(12) $\mathcal{M}_{\mathfrak{L},\xi} := \mathrm{Cd}_\xi\mathcal{C}_{\mathfrak{L}} \cap \mathcal{CO}_{\mathfrak{L}-1}$ *and* $\mathrm{b}\mathcal{M}_{\mathfrak{L},\xi} := \mathrm{bCd}_\xi\mathcal{C}_{\mathfrak{L}} \cap \mathcal{LA}_{\mathfrak{L}-1}$,
(13) $\mathcal{M}_{\mathfrak{L},\xi}{}^- := \mathrm{Cd}_\xi\mathcal{C}_{\mathfrak{L}}{}^- \cap \mathcal{CO}_{\mathfrak{L}-1}$ *and* $\mathrm{b}\mathcal{M}_{\mathfrak{L},\xi}{}^- := \mathrm{bCd}_\xi\mathcal{C}_{\mathfrak{L}}{}^- \cap \mathcal{LA}_{\mathfrak{L}-1}$,
(14) $\mathcal{M}_{\mathfrak{L},A}^{\mathbb{Z}} := \mathrm{Quo}_A\mathcal{C}_{\mathfrak{L}} \cap \mathcal{CO}_{\mathfrak{L}-1}$ *and* $\mathrm{b}\mathcal{M}_{\mathfrak{L},A}^{\mathbb{Z}} := \mathrm{bQuo}_A\mathcal{C}_{\mathfrak{L}} \cap \mathcal{LA}_{\mathfrak{L}-1}$,
(15) $\mathrm{Bas}\mathcal{C}_{\mathfrak{L}}$ *as the set of all graphs* G *for which there is a basis* $\mathcal{B}$ *of the abelian group* $\mathrm{Z}_1(G)$ *with* $z \in \vec{\mathcal{C}}_{\mathfrak{L}}(G)$ *for every* $z \in \mathcal{B}$*; then* $\mathrm{Z}_1(G)$ *is called* $\mathcal{L}$*-based (and* Hamilton-based *if* $\mathcal{L} = \{|\cdot|\}$*)* ,
(16) $\mathrm{Bas}\mathcal{C}_{\mathfrak{L}}{}^-$ *as the set of all graphs* G *for which there is a basis* $\mathcal{B}$ *of* $\mathrm{Z}_1(G)$ *such that there exists* $z^- \in \mathcal{B}$ *with* $z^- \in \vec{\mathcal{C}}_{(\mathfrak{L}-1)\cup\mathfrak{L}}(G)$ *and* $z \in \vec{\mathcal{C}}_{\mathfrak{L}}(G)$ *for every* $z \in \mathcal{B}\backslash\{z^-\}$*; then* $\mathrm{Z}_1(G)$ *is called* almost-$\mathcal{L}$*-based (and* almost-Hamilton-based *if* $\mathcal{L} = \{|\cdot|\}$*)* ,
(17) $\mathrm{bBas}\mathcal{C}_{\mathfrak{L}}$ *as the set of all bipartite elements of* $\mathrm{Bas}\mathcal{C}_{\mathfrak{L}}$ *from* (15) ,
(18) $\mathrm{bBas}\mathcal{C}_{\mathfrak{L}}^-$ *as the set of all bipartite elements of* $\mathrm{Bas}\mathcal{C}_{\mathfrak{L}}^-$ *from* (16) ,
(19) $\mathcal{M}_{\mathfrak{L}}^{\mathbb{Z}\mathrm{Bas}} := \mathrm{Bas}\mathcal{C}_{\mathfrak{L}} \cap \mathcal{CO}_{\mathfrak{L}-1}$,
(20) $\mathrm{b}\mathcal{M}_{\mathfrak{L}}^{\mathbb{Z}\mathrm{Bas}} := \mathrm{bBas}\mathcal{C}_{\mathfrak{L}} \cap \mathcal{LA}_{\mathfrak{L}-1}$,

(21) $\mathcal{M}^{\beta_0=1}_{\mathrm{B}\nsubseteq\mathrm{G}}$ *as the set of all* connected *graphs* G
for which there exists at least one generating set of $\mathrm{Z}_1(G)$ *not containing any basis* ,
(22) $\mathrm{b}\mathcal{M}^{\beta_0=1}_{\mathrm{B}\nsubseteq\mathrm{G}}$ *as the set of all* bipartite *elements of* $\mathcal{M}^{\beta_0=1}_{\mathrm{B}\nsubseteq\mathrm{G}}$.

In case that $\mathcal{L}$ *takes only singletons as values, e.g.* $\mathfrak{L} = (G \mapsto \{|G|\})$ *(which for us is the most common case) we sometimes write the subscript like in* $\mathcal{C}_{|\cdot|}$ *or* $\mathcal{M}_{|\cdot|,0}$ *instead of* $\mathcal{C}_{\{|\cdot|\}}$ *or* $\mathcal{M}_{\{|\cdot|\},0}$.

We hasten to confirm that $\vec{\mathcal{C}}_{\mathfrak{L}}(G)$ from (4) contains *both* of the two arbitrary orientations of each circuit $C \in \mathcal{C}(G)$ (or in language more akin to $\mathrm{Z}_1(G)$, the set of cycles $\vec{\mathcal{C}}_{\mathfrak{L}}(G) \subseteq \mathrm{Z}_1(G)$ is closed w.r.t. multiplication by -1); it would be a purposeless complication to rule this out. Moreover, the use of the function $\mathfrak{L}$ in Definition 204 was designed to be general enough for the study of the generative properties of long, yet possibly non-Hamilton circuits, but in this thesis the function $\mathfrak{L} = (G \mapsto \{|G|\})$ is of course the main interest. Logically, it is redundant to use the notation Cd_ξ alongside Quo_A, since, with the notations from (5) and (7), both $\mathrm{Cd}_\xi\mathcal{C}_{\mathfrak{L}} = \mathrm{Quo}_{\oplus^\xi\ \mathbb{Z}/2\mathbb{Z}}\mathcal{C}_{\mathfrak{L}}$ and $\mathcal{M}_{\mathfrak{L},\xi} = \mathcal{M}_{\mathfrak{L},\oplus^\xi\ \mathbb{Z}/2\mathbb{Z}}$; the notation $\mathrm{Cd}_\xi\mathcal{C}_{\mathfrak{L}}$ from [82] is retained, however, for brevity (with the minor change of writing ‘Cd’ instead of ‘cd’, to emphasise that these are sets.)

We use the notion in Definition 204.(6) only as ‘0-almost-$\{|\cdot|\}$-generated’, but for systematic studies of Conjecture 9 in Chapter 1 it appears to be useful to have this general definition. By making the decision to allow $z^- \in \vec{\mathcal{C}}_{\mathfrak{L}\cup(\mathcal{L}-1)}$ in (16) of Definition 204, i.e., to allow the support of the exceptional generator z' to have length in $\mathcal{L}$, we make sure that $\mathrm{Bas}\mathcal{C}_{\mathfrak{L}} \subseteq \mathrm{Bas}\mathcal{C}^-_{\mathfrak{L}}$, which conforms to the usual convention of having a weakened notion subsume the stronger one. Without that convention, i.e. if we would insist on ‘almost-$\mathfrak{L}$-based’ to mean that there *must* be one basis element of length one less than a value in $\mathfrak{L}(G)$, then ‘almost-$\mathfrak{L}$-based flow lattice’ would not be a relaxation of ‘$\mathfrak{L}$-based flow lattice’ but a rather artificial, separate property.

The condition in (5) is equivalent to $\dim_{\mathbb{F}_2}\big(\mathrm{Z}_1(G;\mathbb{F}_2)/\langle\mathcal{C}_{\mathfrak{L}(G)}(G)\rangle_{\mathbb{F}_2}\big) = \xi$, in other words, $\mathrm{Cd}_\xi\mathcal{C}_{\mathfrak{L}}(G)$ is the set of all graphs for which $\langle\mathcal{C}_{\mathfrak{L}(G)}(G)\rangle_{\mathbb{F}_2}$ has codimension ξ in $\mathrm{Z}_1(G;\mathbb{F}_2)$. In particular, $\mathrm{Cd}_0\mathcal{C}_{|\cdot|}(G)$ is the set of all graphs whose cycle space is generated by their Hamilton circuits.

5.2 Some general mathematical prerequisites

We use the following standard terms, often only very cursorily, which we do not define in the thesis: adjacency matrix, automorphism group of a graph, basis of an abelian group, binomial random graph, bipartite, c-colourable, c-chromatic, coNP, chromatic number of a graph, elementary divisors of a matrix, embedding of a graph in a surface, exponential generating function (*egf* for short) of a set of finite sets w.r.t. some size-function, forest, free abelian group, planar graph, graph, graph isomorphism, group, group action, incidence matrix, induced subgraph, index of a subgroup, invariant factors of a submodule w.r.t. some containing module, minor of a graph, module, NP, NP-hard, projective plane, simplicial complex, simplicial homology, Smith Normal Form, subgraph, surface, torus.

We again and again make use of the standard fact that the invariant factors of the $\mathbb{Z}$-linear span of the rows of an incidence matrix $A \in \mathbb{Z}^{s\times t}$, w.r.t. the containing group $\mathbb{Z}^t$, are given by the elementary divisors of that incidence matrix. (Using the row-space is inessential; most of the time we just happen to consider incidence-matrices in which the columns are indexed by edges and the rows by circuits.) A reference for this is e.g. [25, §4 in Chapter VII]. As a reminder, let us state what we need of this here: if $A \in \mathbb{Z}^{s\times t}$, and if $r := \mathrm{rank}_{\mathbb{Z}}(\mathrm{rowspan}_{\mathbb{Z}}(A)) = t$, then $\mathbb{Z}^t/\mathrm{rowspan}_{\mathbb{Z}}(A)$ is isomorphic as an abelian group to $\mathbb{Z}/s_r \oplus \mathbb{Z}/s_{r-1} \oplus \cdots \oplus \mathbb{Z}/s_1$ with $s_i = d_i/d_{i-1}$, where $d_0 := 1$ and d_i for every $i \geqslant 1$ is the greatest common divisor of all $i \times i$ minors of A. The d_i are called *determinantal divisors* of A. The s_i are called *elementary divisors* of A, and a synonym for that is *invariant factors* of the submodule $\mathrm{rowspan}_{\mathbb{Z}}(A)$ w.r.t. the module $\mathbb{Z}^t$.

5.3 Local structures used in this thesis

This section contains the definitions of auxiliary substructures used in this thesis.

The following definitions[5] play a central role in the proofs of the results about constructing $\mathbb{F}_2$-cycles as symmetric differences of Hamilton-circuits:

Definition 205 (Bipartite cyclic ladder)**.** *For* $r \in \mathbb{Z}_{\geqslant 3}$ *let* CL_r *be the bipartite graph with* $\mathrm{V}(\mathrm{CL}_r) := \{a_0, \dots, a_{r-1}\} \sqcup \{b_0, \dots, b_{r-1}\}$ *and* $\mathrm{E}(\mathrm{CL}_r) := \bigsqcup_{i=0}^{r-1} \{a_i b_{i-1}\} \sqcup \bigsqcup_{i=0}^{r-1} \{a_i b_i\} \sqcup \bigsqcup_{i=0}^{r-1} \{a_i b_{i+1}\}$.

Whereas for (I4) our use of Theorem 40 dictates employing $\mathrm{C}^2_{|\cdot|}$ as the auxiliary subgraph, there are choices to be made as to what subgraph to employ from the set of spanning subgraphs offered by the Theorems 38 and 39. We will choose to use the following graphs (in Definition 205 let $b_r := b_0$):

Definition 206 (prism, Möbius ladder)**.** *For every* $n \geqslant 3$ *and* $r \geqslant 3$ *let (where* $v_n := v_0$*,* $x_r := x_0$ *and* $y_r := y_0$*) the* prism Pr_r *be defined by* $\mathrm{V}(\mathrm{Pr}_r) := \{x_0, \dots, x_{r-1}, y_0, \dots, y_{r-1}\}$ *and* $\mathrm{E}(\mathrm{Pr}_r) := \bigsqcup_{i=0}^{r-1} \{\ x_i x_{i+1}\ \} \sqcup \bigsqcup_{i=0}^{r-1} \{\ y_i y_{i+1}\ \} \sqcup \bigsqcup_{i=0}^{r-1} \{\ x_i y_i\ \}$*, and the* Möbius ladder M_r *be defined by* $\mathrm{V}(\mathrm{M}_r) := \mathrm{V}(\mathrm{Pr}_r)$ *and* $\mathrm{E}(\mathrm{M}_r) := \big(\ \mathrm{E}(\mathrm{Pr}_r) \setminus \{\ x_{r-1}x_0,\ y_{r-1}y_0\ \}\ \big) \sqcup \{\ x_0 y_{r-1},\ y_0 x_{r-1}\ \}$.

Definition 207 ($\mathrm{Pr}_r^{\boxtimes}$ and $\mathrm{M}_r^{\boxtimes}$)**.** *For every* $r \geqslant 3$ *let* $\mathrm{Pr}_r^{\boxtimes}$ *be defined by* $\mathrm{V}(\mathrm{Pr}_r^{\boxtimes}) := \mathrm{V}(\mathrm{Pr}_r) \sqcup \{z\}$*, with* z *some new element, and* $\mathrm{E}(\mathrm{Pr}_r^{\boxtimes}) := \mathrm{E}(\mathrm{Pr}_r) \sqcup \{\ zx_0,\ zy_0,\ zx_1,\ zy_1\ \}$*. Let* $\mathrm{M}_r^{\boxtimes}$ *be defined by* $\mathrm{V}(\mathrm{M}_r^{\boxtimes}) := \mathrm{V}(\mathrm{Pr}_r^{\boxtimes})$ *and* $\mathrm{E}(\mathrm{M}_r^{\boxtimes}) := (\ \mathrm{E}(\mathrm{Pr}_r^{\boxtimes}) \setminus \{\ x_{r-1}x_0,\ y_{r-1}y_0\ \}\ ) \sqcup \{\ x_0 y_{r-1},\ y_0 x_{r-1}\ \}$.

Definition 208 ($\mathrm{Pr}_r^{\boxminus}$ and $\mathrm{M}_r^{\boxminus}$)**.** *For every* $r \geqslant 3$ *let* $\mathrm{Pr}_r^{\boxminus}$ *be defined by* $\mathrm{V}(\mathrm{Pr}_r^{\boxminus}) := \mathrm{V}(\mathrm{Pr}_r) \sqcup \{z', z''\}$ *with* z' *and* z'' *two new elements, and let* $\mathrm{E}(\mathrm{Pr}_r^{\boxminus}) := \mathrm{E}(\mathrm{Pr}_r) \sqcup \{\ x_0 z',\ y_0 z',\ x_0 z'',\ x_1 z'',\ y_1 z'',\ z' z''\ \}$*. Let* $\mathrm{M}_r^{\boxminus}$ *be defined by* $\mathrm{V}(\mathrm{M}_r^{\boxminus}) := \mathrm{V}(\mathrm{Pr}_r^{\boxminus})$ *and then let* $\mathrm{E}(\mathrm{M}_r^{\boxminus}) := (\ \mathrm{E}(\mathrm{Pr}_r^{\boxminus}) \setminus \{\ x_{r-1}x_0,\ y_{r-1}y_0\ \}\ ) \sqcup \{\ x_0 y_{r-1},\ y_0 x_{r-1}\ \}$.

Definition 209. *For every odd* $r \geqslant 5$ *we define the sets of edge sets*

$$\text{(M.}\boxtimes\text{.ES.1)}\quad \mathcal{CB}^{(1)}_{\mathrm{M}_r^{\boxtimes}} := \left\{\begin{array}{lcc} C_{\mathrm{od},r,1} & := & zy_1x_1x_2y_2y_3 \dots y_{r-2}x_{r-2}x_{r-1}y_{r-1}x_0y_0z\ , \\ C_{\mathrm{od},r,2} & := & zx_1x_2y_2y_3 \dots y_{r-2}x_{r-2}x_{r-1}y_{r-1}x_0y_0y_1z\ , \\ C_{\mathrm{od},r,3} & := & zx_1y_1y_2x_2x_3 \dots y_{r-2}y_{r-1}x_{r-1}y_0x_0z\ , \\ C_{\mathrm{od},r,4} & := & zx_0x_1y_1y_2 \dots x_{r-3}x_{r-2}y_{r-2}y_{r-1}x_{r-1}y_0z\ , \\ C_{\mathrm{od},r,5} & := & zy_1y_2x_2x_3 \dots y_{r-2}y_{r-1}x_{r-1}y_0x_0x_1z \end{array}\right\},$$

$$\text{(M.}\boxtimes\text{.ES.2)}\quad \mathcal{CB}^{(2)}_{\mathrm{M}_r^{\boxtimes}} := \left\{\begin{array}{ccc} C^{x_1y_1}_{\mathrm{od},r} & := & zx_0y_{r-1}y_{r-2} \dots y_2x_2x_3 \dots x_{r-1}y_0y_1x_1z\ , \\ C^{x_2y_2}_{\mathrm{od},r} & := & zx_0y_{r-1}y_{r-2} \dots y_3x_3x_4 \dots x_{r-1}y_0y_1y_2x_2x_1z\ , \\ & \vdots & \\ C^{x_{r-2}y_{r-2}}_{\mathrm{od},r} & := & zx_0y_{r-1}x_{r-1}y_0y_1 \dots y_{r-2}x_{r-2}x_{r-3} \dots x_1z\ , \\ C^{x_{r-1}y_{r-1}}_{\mathrm{od},r} & := & zx_0x_1 \dots x_{r-1}y_{r-1}y_{r-2} \dots y_0z \end{array}\right\}.$$

Let us note that $C^{x_{r-1}y_{r-1}}_{\mathrm{od},r}$ does not conform to the pattern to be found in $C^{x_1y_1}_{\mathrm{od},r}, \dots, C^{x_{r-2}y_{r-2}}_{\mathrm{od},r}$.

Definition 210. *For every even* $r \geqslant 4$ *we define*[6] *the sets of edge sets*

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[5] Let us note that in [82, Definitions 7–9], when defining $\mathrm{E}(\mathrm{M}_r)$, $\mathrm{E}(\mathrm{M}_r^{\boxtimes})$ and $\mathrm{E}(\mathrm{M}_r^{\boxminus})$, the edges being removed from $\mathrm{E}(\mathrm{Pr}_r)$, $\mathrm{E}(\mathrm{Pr}_r^{\boxtimes})$, $\mathrm{E}(\mathrm{Pr}_r^{\boxminus})$, were written '$x_0x_{r-1}$' and '$y_0y_{r-1}$'. This is exactly equivalent to the present defintion (due to our convention that xy is a notation for the 2-set $\{x, y\}$). In the present chapter, though, orientations of edges will become significant. Therefore, it was decided to emphasise the fact that in Definition 206 the edge $x_0x_{r-1} = x_{r-1}x_0$ appears in the form '$x_{r-1}x_0$', by using this notation already in Definitions 206 and 208. The arbitrary convention to orient $\{x_0, x_{r-1}\}$ that way agrees with the convention used to set up the incidence matrix (2.18) for the flows in Figure 2.3.

[6] In [82, p. 508] the set $C_{\mathrm{ev},r,4}$ from Definition 210, while exactly the same as the set $C_{\mathrm{ev},r,4}$ used in this thesis, is given in the briefer form $C_{\mathrm{ev},r,4} := zx_0x_1y_1y_2 \dots y_{r-3}y_{r-2}x_{r-2}x_{r-1}y_{r-1}y_0z$. This is not a mistake, yet could be considered a little misleading since the string $y_1y_2 \dots y_{r-3}y_{r-2}$ might have one believe that between y_2 and y_{r-3} there do not appear any x_i's. However, in [82, p. 508] it becomes unambiguously clear from the context that it is $C_{\mathrm{ev},r,4} := zx_0x_1y_1y_2x_2x_3y_3 \dots y_{r-3}y_{r-2}x_{r-2}x_{r-1}y_{r-1}y_0z$ what is meant.

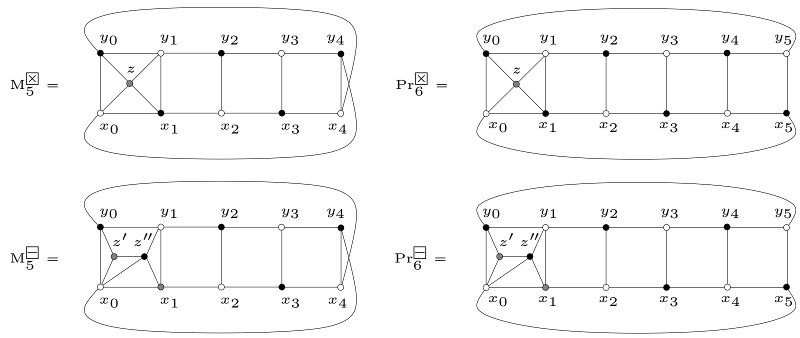

Figure 5.1: The graphs $\mathrm{M}_r^{\boxtimes}$ and $\mathrm{M}_r^{\boxminus}$ for odd r, and $\mathrm{Pr}_r^{\boxtimes}$ and $\mathrm{Pr}_r^{\boxminus}$ for even r play a key role in the proof. Figure 5.1 shows $\mathrm{M}_5^{\boxtimes}$, $\mathrm{M}_5^{\boxtimes}$, $\mathrm{Pr}_6^{\boxtimes}$ and $\mathrm{Pr}_6^{\boxminus}$. These are bounded-degree, bounded-bandwidth and 3-chromatic graphs admitting a 3-colouring with a constant-sized third colour class. The bandwidth-theorem of Böttcher, Schacht and Taraz, in its full form [24, Theorem 2], is sufficiently general to guarantee the existence of embeddings of these graphs as *spanning* subgraphs into graphs G with $\delta(G) \geqslant (\frac{1}{2}+\gamma)|G|$. If $\mathrm{M}_r^{\boxtimes}$ or $\mathrm{Pr}_r^{\boxtimes}$ spanningly embed into G, this implies that $\mathrm{Z}_1(G;\mathbb{F}_2)$ is generated by Hamilton circuits. If $\mathrm{M}_r^{\boxminus}$ or $\mathrm{Pr}_r^{\boxminus}$ spanningly embed into G, this implies that $\mathrm{Z}_1(G;\mathbb{F}_2)$ is generated by the circuits having lengths in $\{|G|-1,|G|\}$. If the edge x_0z'' were omitted from $\mathrm{M}_r^{\boxminus}$ or $\mathrm{Pr}_r^{\boxminus}$, the remaining graph could no longer serve the purpose these graphs have in this thesis.

$$\text{(P.}\boxtimes\text{.ES.1)}\quad \mathcal{CB}^{(1)}_{\mathrm{Pr}_r^{\boxtimes}} := \left\{\begin{array}{lcc} C_{\mathrm{ev},r,1} & := & zy_1x_1x_2y_2y_3\dots x_{r-2}y_{r-2}y_{r-1}x_{r-1}x_0y_0z\ , \\ C_{\mathrm{ev},r,2} & := & zx_1x_2y_2y_3\dots x_{r-2}y_{r-2}y_{r-1}x_{r-1}x_0y_0y_1z\ , \\ C_{\mathrm{ev},r,3} & := & zx_1y_1y_2x_2x_3\dots x_{r-2}x_{r-1}y_{r-1}y_0x_0z\ , \\ C_{\mathrm{ev},r,4} & := & zx_0x_1y_1y_2x_2x_3y_3\dots y_{r-3}y_{r-2}x_{r-2}x_{r-1}y_{r-1}y_0z\ , \\ C_{\mathrm{ev},r,5} & := & zy_1y_2x_2x_3\dots x_{r-2}x_{r-1}y_{r-1}y_0x_0x_1z \end{array}\right\},$$

$$\text{(P.}\boxtimes\text{.ES.2)}\quad \mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxtimes}} := \left\{\begin{array}{lcc} C^{x_1y_1}_{\mathrm{ev},r} & := & zx_0x_{r-1}x_{r-2}\dots x_2y_2y_3\dots y_{r-1}y_0y_1x_1z\ , \\ C^{x_2y_2}_{\mathrm{ev},r} & := & zx_0x_{r-1}x_{r-2}\dots x_3y_3y_4\dots y_{r-1}y_0y_1y_2x_2x_1z\ , \\ & \vdots & \\ C^{x_{r-2}y_{r-2}}_{\mathrm{ev},r} & := & zx_0x_{r-1}y_{r-1}y_0y_1\dots y_{r-2}x_{r-2}x_{r-3}\dots x_1z\ , \\ C^{x_{r-1}y_{r-1}}_{\mathrm{ev},r} & := & zx_0x_1\dots x_{r-1}y_{r-1}y_{r-2}\dots y_0z \end{array}\right\}.$$

Definition 211. *For every even $r \geqslant 4$ we define the sets of edge sets*

$$\text{(P.}\boxminus\text{.ES.1)}\quad \mathcal{CB}^{(1)}_{\mathrm{Pr}_r^{\boxminus}} := \left\{\begin{array}{lcc} C_{\boxminus,\mathrm{ev},r,1} & := & z'x_0z''x_1x_2\dots x_{r-1}y_{r-1}y_{r-2}\dots y_0z'\ , \\ C_{\boxminus,\mathrm{ev},r,2} & := & z'z''x_0x_{r-1}x_{r-2}\dots x_1y_1y_2\dots y_{r-1}y_0z'\ , \\ C_{\boxminus,\mathrm{ev},r,3} & := & z'x_0z''x_1y_1y_2x_2x_3\dots x_{r-2}x_{r-1}y_{r-1}y_0z'\ , \\ C_{\boxminus,\mathrm{ev},r,4} & := & z'z''x_1x_2\dots x_{r-1}y_{r-1}y_{r-2}\dots y_0x_0z'\ , \\ C_{\boxminus,\mathrm{ev},r,5} & := & z'x_0x_{r-1}y_{r-1}y_{r-2}x_{r-2}x_{r-3}\dots x_2x_1z''y_1y_0z' \end{array}\right\},$$

$$\text{(P.}\boxminus\text{.ES.2)}\quad \mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxminus}} := \left\{\begin{array}{lcc} C^{x_1y_1}_{\boxminus,\mathrm{ev},r} & := & z'x_0x_{r-1}x_{r-2}\dots x_2y_2y_3\dots y_{r-1}y_0y_1x_1z''z'\ , \\ C^{x_2y_2}_{\boxminus,\mathrm{ev},r} & := & z'x_0x_{r-1}x_{r-2}\dots x_3y_3y_4\dots y_{r-1}y_0y_1y_2x_2x_1z''z'\ , \\ & \vdots & \\ C^{x_{r-2}y_{r-2}}_{\boxminus,\mathrm{ev},r} & := & z'x_0x_{r-1}y_{r-1}y_0y_1\dots y_{r-2}x_{r-2}x_{r-3}\dots x_1z''z'\ , \\ C^{x_{r-1}y_{r-1}}_{\boxminus,\mathrm{ev},r} & := & z'z''x_0x_1\dots x_{r-1}y_{r-1}y_{r-2}\dots y_0z' \end{array}\right\}.$$

Definition 212 (the graph $\mathrm{CE}_{(I1)}$)**.** *Let* $\mathrm{CE}_{(I1)}$ *denote the seven-vertex graph with* $\mathrm{V}(\mathrm{CE}_{(I1)}) := \{v_1,v_2,v_3,v_4,v_5,v_6,v_7\}$ *and* $\mathrm{E}(\mathrm{CE}_{(I1)}) := \{\ \{v_1,v_4\},\ \{v_1,v_6\},\ \{v_1,v_7\},\ \{v_2,v_4\},\ \{v_2,v_5\},\ \{v_2,v_7\},\ \{v_3,v_4\},\ \{v_3,v_5\},\ \{v_3,v_6\},\ \{v_5,v_6\},\ \{v_5,v_7\},\ \{v_6,v_7\}\ \}$. *(This is the graph underlying Figure 2.1.)*

Definition 213 (the graph $\mathrm{X}_9^{\triangle}$)**.** *We denote by* $\mathrm{X}_9^{\triangle}$ *the graph with vertex-set* $\mathrm{V}(\mathrm{X}_9^{\triangle}) := [9]$ *and edge-set* $\mathrm{E}(\mathrm{X}_9^{\triangle}) := \{\ \{1,2\},\ \{1,5\},\ \{1,6\},\ \{1,9\},\ \{2,3\},\ \{2,7\},\ \{3,4\},\ \{3,9\},\ \{4,5\},\ \{4,8\},\ \{5,6\},\ \{6,7\},\ \{7,8\},\ \{8,9\}\ \}$.

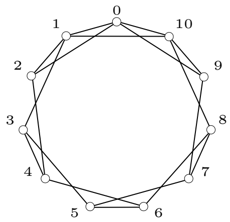

Figure 5.2: This is the graph $\mathrm{C}_n^{2^-}$ for $n = 11$. By Proposition 69, the abelian groups $\mathrm{Z}_1(\mathrm{C}_n^{2^-})$ are Hamilton-based for every $n \geqslant 11$ with $n \equiv 3 \pmod 4$. Compared with the seed graphs $\mathrm{Pr}_r^{\boxtimes}$ and $\mathrm{M}_r^{\boxtimes}$ from [82] the graph $\mathrm{C}_n^{2^-}$ is slightly more economical in that it has only three vertices of degree four, as opposed to five such vertices in $\mathrm{M}_r^{\boxtimes}$ and $\mathrm{Pr}_r^{\boxtimes}$ (cf. [82, Fig. 1]). It is impossible to use the seed graphs $\mathrm{M}_r^{\boxtimes}$ and $\mathrm{Pr}_r^{\boxtimes}$ from [82, p. 508]) to prove Conjecture 79 via the monotonicity argument and the embedding technology of [92] and [107]: they contain the 4-wheel W_4, for which $\|\mathrm{W}_4\|/|\mathrm{W}_4| = \frac{8}{5} > 1$, hence $\max\ \{\ \|H'\|\ /\ |H'|\colon\ H' \subseteq H,\ \|H'\| \geqslant 1\} > 1$ for both $H \in \{\mathrm{M}_r^{\boxtimes}, \mathrm{Pr}_r^{\boxtimes}\}$, so these graphs cannot be guaranteed via a multi-exposure argument as in [107], if $p \geqslant n^{-\frac{2}{3}+\varepsilon}$. In the first exposure step, we would have to guarantee a subgraph isomorphic to W_4, and because of $-\frac{2}{3} < -\frac{5}{8}$ this is not possible. The graph $\mathrm{C}_n^{2^-}$, though, contains only W_4^- as its densest subgraph, which *can* be guaranteed as a subgraph of $G(n, n^{-2/3+\varepsilon})$, as $\mathrm{d}(\mathrm{W}_4) = \frac{7}{5} < \frac{3}{2}$. Unfortunately, for $n \equiv 1 \pmod 4$, the flow lattice $\mathrm{Z}_1(\mathrm{C}_n^{2^-})$ to all appearances (cf. Conjecture 73) is *not* Hamilton-based, not even Hamilton-generated. (This negative fact is not formally proved in this thesis, but has carefully been checked empirically in the two special cases $n = 13$ and $n = 17$). This is the reason why in case of $n \equiv 1 \pmod 4$ one *has* to employ slightly different auxiliary graphs. (When working with random graphs $G(n, n^{-2/3+\varepsilon})$, using these sparsest-possible seed graphs is 'more' necessary than when working with a hypothesis of $\delta(G) \geqslant (\frac{1}{2} + \gamma)|G|$, for which the bandwidth theorem would allow using denser seed graphs). A proof of Conjecture 79 is work in progress and is not to be found in this thesis.

Definition 214 (the graph $\mathrm{C}_n^{2^-}$, shown for $n = 11$ in Figure 5.2)**.** *For every odd $n \geqslant 7$ we denote by $\mathrm{C}_n^{2^-}$ the graph with vertex set $\{0, 1, \ldots, n-1\}$ and edge set $\{\{i, i+1\}, \{i, i+2\}\colon i \in \mathbb{Z}/n\} \setminus \{\{2j, 2j+1\}\colon j \in [\frac{1}{2}(n-3)]\}$.*

Definition 215 ($\mathrm{C}_n^{2^{--}}$)**.** *If $n \geqslant 7$ is odd, $\mathrm{C}_n^{2^{--}}$ denotes the graph obtained by deleting the edge $\{n-1, 0\}$ in the graph $\mathrm{C}_n^{2^-}$ from Definition 214.*

5.3.1 A suitable seed graph for $n \equiv 2 \pmod 4$

Definition 216 ($\mathrm{M}_r^{\boxbslash}$)**.** *For every odd $r \geqslant 5$ let $\mathrm{M}_r^{\boxbslash}$ denote the graph with vertex-set $\{0, 1, \ldots, 2r-1\}$ and edge-set $\{\{i, i+1\}\colon i \in \{0, 1, \ldots, 2r-1\}\} \sqcup \{\{0, r+1\}\} \sqcup \{\{1, r\}\} \sqcup \{\{i, i+r\}\colon i \in \{2, 3, \ldots, r\}\}$.*

The graph $\mathrm{M}_9^{\boxbslash}$ is shown in Figure 5.6. Ongoing work of the author strongly suggests that $\mathrm{M}_r^{\boxbslash}$ is (at most one edge away from) a sparsest-possible infinite family of auxiliary substructures suitable for proving Conjecture 3.(I.2).

A proof of the following is left out of this thesis due to time- and space-constraints. In particular, a proof necessitates writing down a formal proof of the claim $\mathrm{M}_r^{\boxbslash} \in \mathrm{Bas}\mathcal{C}_{|\cdot|}{}^{-}$, which takes comparable effort as Section 2.2.3.3 in Chapter 2:

Conjecture 217 (sparsest-possible seed graphs for proving a graph to be almost-Hamilton-based)**.** *The set $\{\ \mathrm{M}_r^{\boxbslash}\colon$ odd $r \geqslant 5\ \}$, with $\mathrm{M}_r^{\boxbslash}$ the graph from Definition 216 (see Figure 5.7 for the cases $r = 5, 7, 9$), is a set of suitable seed-graphs for proving Conjecture 3.(I.2), via Theorem 38, i.e., $\mathrm{M}_r^{\boxbslash} \in \mathrm{Bas}\mathcal{C}_{|\cdot|}{}^{-}$ (cf. Definition 204) for every $\beta > 0$ there exists $r_0 \in \mathbb{N}_{\mathrm{odd}}$ such that $\mathrm{bw}(\mathrm{M}_r^{\boxbslash}) \leqslant \beta \cdot |\mathrm{M}_r^{\boxbslash}|$ for all $r_0 \leqslant r \in \mathbb{N}_{\mathrm{odd}}$, $\Delta(\mathrm{M}_r^{\boxbslash}) \leqslant 4$ and for every $\beta > 0$ and every r there exists a*

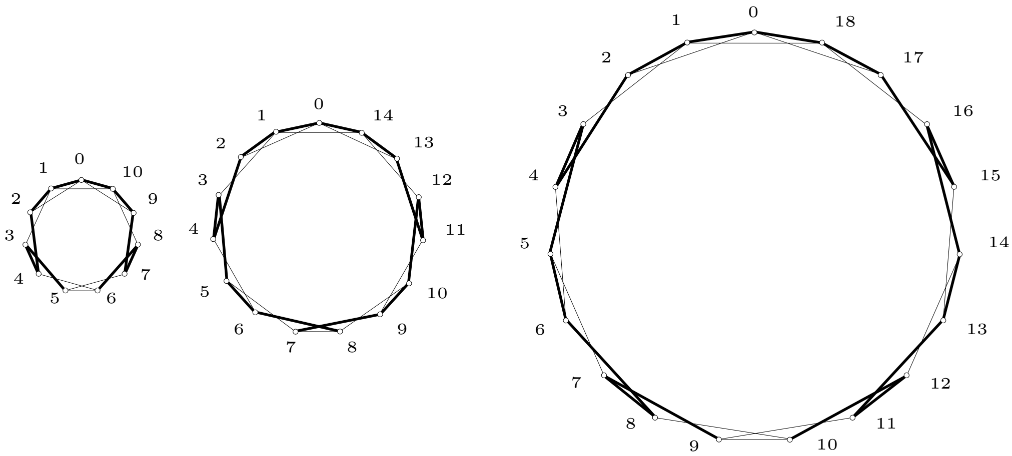

Figure 5.3: These are, in the cases $n = 11, 15, 19$, the spanning trees T_n (spanning *paths*, in fact) of the graph $\mathrm{C}_n^{2^-}$ from Definition 218. They define the fundamental flows employed for proving that $\mathcal{B}_n$ from (2.49) in Proposition 69 is a generating set. In some regards, this choice of spanning tree seems to be particularly advantageous for proving $\mathcal{B}_n$ to be a generating set of $\mathrm{Z}(\mathrm{C}_n^{2^-})$ by way of constructing each fundamental flow of T_n as a $\mathbb{Z}$-linear combination of Hamilton-flows from $\mathcal{B}_n$. In particular, with these T_n, the entries of the matrix describing the change of basis have magnitude at most two (cf. (2.89) and (2.92) for the special cases $n = 11$ and $n = 19$). Alas, and this can be seen in Figure 5.3, there is some dependence of the (otherwise rather convenient) spanning trees T_n on the remainder of the odd n modulo 8; this remainder determines whether the paths end with the sequence $i_n - 2, i_n - 3, i_n - 1$ or $i_n - 3, i_n - 2, i_n$, where $i_n := \frac{1}{2}(n+1)$. This, even when concentrating on the case $n \equiv 3 \pmod 4$, necessitates additional case-analysis. The author did not find spanning trees which would keep the matrix of the change of basis manageably pattern-rich *and* work uniformly for every $n \equiv 3 \pmod 4$. This is one reason why this thesis focuses on giving a complete proof for the case $n \equiv 3 \pmod 8$, an otherwise inessential condition. Such divisibility issues are the price for using sparsest-possible auxiliary substructures; the generality of the bandwidth-theorem from [24] would allow to circumvent them by settling for slightly denser substructures as seed graphs. It was decided not to do so, so as to have the technical work become a multi-purpose endeavour: the graphs $\mathrm{C}_n^{2^-}$ are slated to serve in proving Hamilton-basedness of flow lattices of graphs under other hypotheses than $\delta(G) \geqslant (\frac{1}{2} + \gamma)n$ in the future, in particular to serve in giving a complete proof of Conjecture 79 on p. 114.

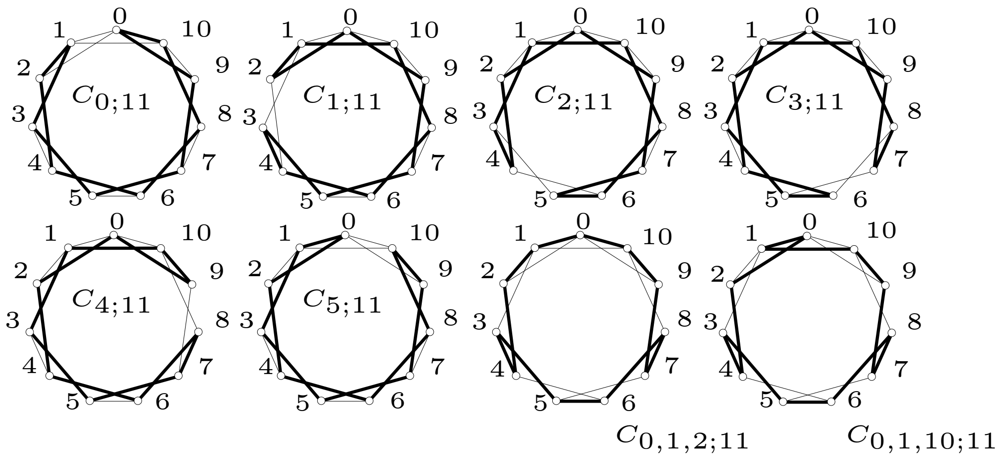

Figure 5.4: The Hamilton circuits underlying the Hamilton-flow-basis from Proposition 69, in the case $n = 11$. Arbitrarily choosing one of the two orientations for each of these eight circuits yields a basis for the abelian group $\mathrm{Z}_1(\mathrm{C}_{11}^{2^-})$ consisting only of simple Hamilton-flows. The incidence matrix of one such basis is (2.89) in Chapter 2.

proper 3*-colouring* $\mathrm{c}\colon \mathrm{V}(\mathrm{M}_r^{\boxslash}) \to \{0,1,2\}$ *with constant-sized third colour-class, which in particular is* $\bigl(8 \cdot 2 \cdot \beta \cdot |\mathrm{M}_r^{\boxslash}|, 4 \cdot 2 \cdot \beta \cdot |\mathrm{M}_r^{\boxslash}|\bigr)$*-zero-free.*

5.3.2 Some more local substructures used in this thesis

The following set of spanning trees has a central role in the certification of the Hamilton-flow-basis $\mathcal{B}_n$ from Section 2.2.3.3 of Chapter 2:

Definition 218 (the spanning paths T_n of $\mathrm{C}_n^{2^-}$; see Figure 5.3 for the cases $n = 11, 15, 19$)**.** *For every* $n \equiv 3 \pmod 8$ *let* T_n *denote the graph with vertex-set* $\mathbb{Z}/n$ *and edge-set defined as follows: let 'repeat* (a_1, a_2, a_3, a_4) *for* t *interations' be shorthand for 'if* v *is the the current vertex, then go to vertex* $v + a_1$*, then go to* $v + a_1 + a_2$*, then to* $v + a_1 + a_2 + a_3$*, and then to* $v + a_1 + a_2 + a_3 + a_4$*; then repeat this,* t *times in total. Then the edges of* T_n *are the precisely those edges traversed when carrying out the following: for every* n *with* $n \equiv 3 \pmod 8$*, the following sequence of vertices is a Hamilton-path (in particular, a spanning tree) of* $\mathrm{C}_n^{2^-}$*:*
start at vertex $\frac{1}{2}(n+1)$*, then repeat* $(+2, -1, +2, +1)$ *for* $\frac{1}{8}(n-3)$ *iterations to reach* $n-1$*, then traverse the path* $n-1, 0, 1$*, then repeat* $(+1, +2, -1, +2)$ *for* $\frac{1}{8}(n-3)$ *iterations to reach* $\frac{1}{2}(n-1)$*.*

The following spanning substructures define (up to orienting them) a Hamilton-flow-basis of $\mathrm{C}_n^{2^-}$ (cf. Proposition 69 in Chapter 2):

Definition 219 (the Hamilton-circuits $C_{0,1,2;n}$, $C_{0,1,n-1;n}$, $C_{i;n}$)**.** *If* $n \geqslant 11$ *and* $n \equiv 3 \pmod 4$*, and with*

$$E_n := \bigsqcup_{j \in \{2+4k\colon\, k \in \{0,1,\ldots,\frac{1}{4}(n-7)\}\}} \{\{j, j+2\}, \{j+2, j+1\}, \{j+1, j+3\}, \{j+3, j+4\}\} \ , \qquad (5.1)$$

we denote by $C_{0,1,2;n}$*,* $C_{0,1,n-1;n}$ *and* $C_{i;n}$ *the Hamilton-circuits of* $\mathrm{C}_n^{2^-}$ *defined by*

(1) $\mathrm{E}(C_{0,1,2;n}) := \{\{0,1\}, \{0, n-1\}, \{1,2\}\} \sqcup E_n$,
(2) $\mathrm{E}(C_{0,1,n-1;n}) := \{\{0,1\}, \{0,2\}, \{1, n-1\}\} \sqcup E_n$,
(3) $\mathrm{E}(C_{i;n}) := \{\{2i+1, 2i+2\}\} \sqcup \{\{2i-1, 2i\}\} \sqcup \bigsqcup_{j \in \mathbb{Z}/n \setminus \{2i, 2i+1\}} \{\{j-1, j+1\}\}$.

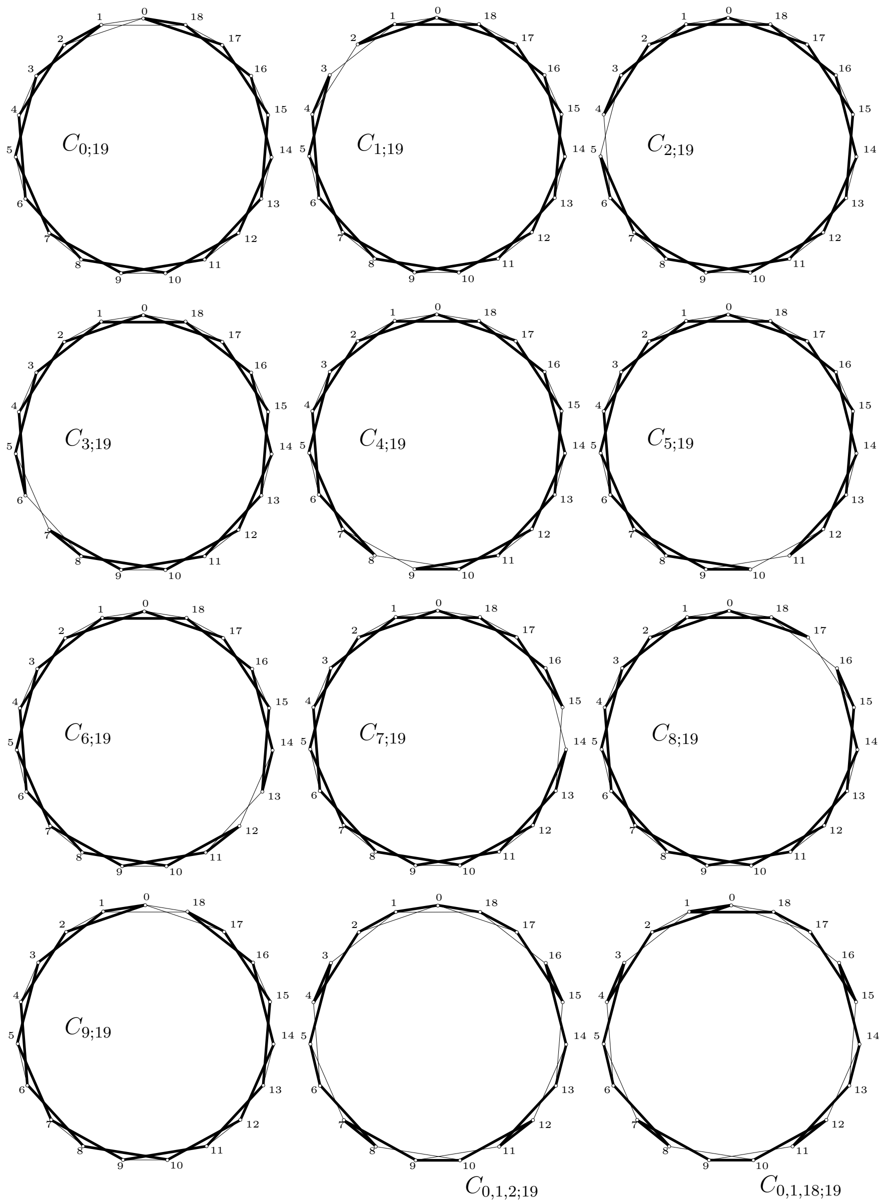

Figure 5.5: Choosing any one of the two orientations for each of these eight circuits yields a *basis* for the abelian group $Z_1(C_{19}^{2^-})$ consisting only of Hamilton-circuit-supported flows. The incidence matrix of one such basis is in (2.91) in Chapter 2. The underlying graph is $C_n^{2^-}$ from Definition 214, in the case $n = 19$.

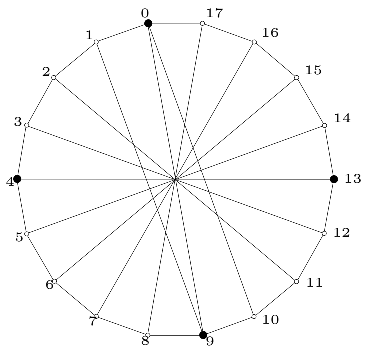

Figure 5.6: The graph $\mathrm{M}_r^{\boxbslash}$ from Definition 216 for $r = 9$. It is obtained from the 9-rung Möbius-ladder M_9 by removing the edge $\{1, 10\}$ and then adding the two edges $\{0, 10\}$ and $\{1, 9\}$. According to ongoing work of the author, these graphs are (sparsest-possible) suitable seed graphs for proving Conjecture 3.(I.2) about *almost*-Hamilton-based flow lattices (cf. Definition 204.(16)). Due to the focus on giving a complete proof of Theorem 4, this is not proved in this thesis. The graphs $\mathrm{M}_r^{\boxbslash}$—their degree-sequence being $(4^{\times 2}, 3^{\times(n-2)})$—have smallest-possible number of edges among all almost-Hamilton-based suitable seed graphs on the same even number of vertices. Being suitable seed graphs for using the bandwidth-theorem, the graphs $\mathrm{M}_r^{\boxbslash}$ in particular have low-bandwidth, as witnessed by e.g. the existence of $(4, 1/2)$-separators in the sense of [23]. Here, in the case $r = 9$, such a separator is indicated by bold vertices. The Möbius ladders M_r from Definition 206 themselves have one edge less than $\mathrm{M}_r^{\boxbslash}$, but are *not* suitable as a seed graph for proving that even-order G with $|G| \equiv 2 \pmod 4$ and $\delta(G) \geqslant (\frac{1}{2} + \gamma)|G|$ have $\mathrm{Z}_1(G)$ almost-Hamilton-based: while for odd r we do have $|\mathrm{M}_4| \equiv 2 \pmod 4$, the graph M_r is then bipartite, hence not Hamilton-connected, hence not a suitable seed graph for the almost-Hamilton-based property $\mathrm{Bas}\mathcal{C}^{-}_{\{|\cdot|\}}$.

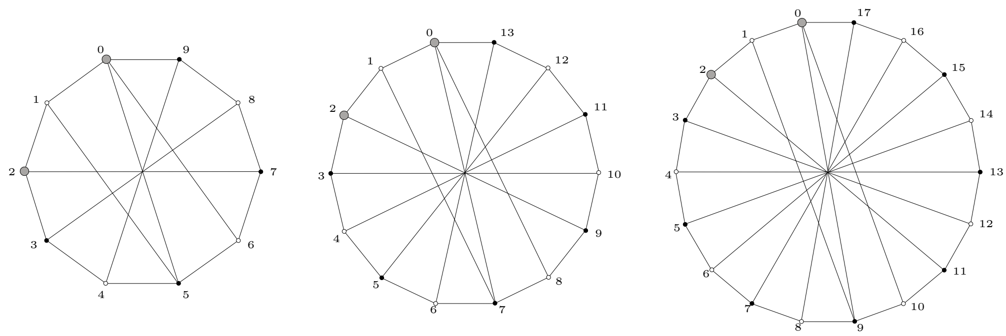

Figure 5.7: Examples for proper 3-colourings of $\mathrm{M}_r^{\boxbslash}$ from Definition 216, with smallest-possible third colour-classes among all proper 3-colourings. The existence of such colourings is one of the reasons why the graphs $\mathrm{M}_r^{\boxbslash}$ are suitable seed graphs for proving Conjecture 3.(I.2) via the monotonicity argument from Section 2.2.2 and the bandwidth theorem from [24, Theorem 2].

Let us note that in Definition 219, we have $|\mathrm{E}(C_{0,1,2;n})| = |\mathrm{E}(C_{0,1,n-1;n})| = 3 + 4 \cdot (1 + \frac{1}{4}(n-7)) = n$ and $|\mathrm{E}(C_{i;n})| = 2 + (n-2) = n$, as is necessary for Hamilton-circuits.

The following is the formal definition of the flows in Figure 1.1 in Chapter 1:

Definition 220 (the simple flows $\vec{F}_{i,j;r}$)**.** *On the graph* W_6 *with vertex-set* $[6]$ *and edge-set* $\{\ \{1,2\},\ \{1,3\},\ \{1,4\},\ \{1,5\},\ \{1,6\},\ \{1,7\},\ \{2,3\},\ \{2,7\},\ \{3,4\},\ \{4,5\},\ \{5,6\},\ \{6,7\}\ \}$ *we define seven simple flows via the following incidence-matrix:*

$$\begin{array}{lcccccccccccc} \mathrm{E}(\mathrm{W}_6) & 1,2 & 1,3 & 1,4 & 1,5 & 1,6 & 1,7 & 2,3 & 2,7 & 3,4 & 4,5 & 5,6 & 6,7 \\ \vec{F}_{2,3;6} & + & - & 0 & 0 & 0 & 0 & 0 & + & - & - & - & - \\ \vec{F}_{3,4;6} & 0 & + & - & 0 & 0 & 0 & - & + & 0 & - & - & - \\ \vec{F}_{3,5;6} & 0 & + & 0 & - & 0 & 0 & - & + & 0 & 0 & - & - \\ \vec{F}_{4,6;6} & 0 & 0 & + & 0 & - & 0 & - & + & - & 0 & 0 & - \\ \vec{F}_{5,7;6} & 0 & 0 & 0 & + & 0 & - & - & + & - & - & 0 & 0 \\ \vec{F}_{2,6;6} & + & 0 & 0 & 0 & - & 0 & + & 0 & + & + & + & 0 \\ \vec{F}_{3,7;6} & 0 & + & 0 & 0 & 0 & - & 0 & 0 & + & + & + & + \end{array} \ . \tag{5.2}$$

Definition 221 ($\mathrm{X}^{\mathrm{hg}}_{\neg\mathrm{hb}}$, shown in Figure 2.2)**.** *We denote by* $\mathrm{X}^{\mathrm{hg}}_{\neg\mathrm{hb}}$ *the graph with vertex-set* $\{1,\dots,13\}$ *and edge-set* $\{\ \{1,2\},\ \{1,7\},\ \{1,13\},\ \{2,3\},\ \{2,13\},\ \{3,4\},\ \{3,5\},\ \{4,5\},\ \{4,8\},\ \{5,6\},\ \{5,12\},\ \{6,7\},\ \{6,10\},\ \{6,11\},\ \{7,8\},\ \{8,9\},\ \{9,10\},\ \{9,13\},\ \{10,11\},\ \{11,12\},\ \{12,13\}\ \}$.

Definition 222 ($\mathrm{K}^{\hat{s},s-1}$)**.** *For every* $s \geqslant 3$ *we define* $\mathrm{K}^{\hat{s},s-1}$ *to be the graph obtained from the complete bipartite graph with classes* $\{1,3,5,\dots,2s-1\}$ *and* $\{2,4,\dots,2s-2\}$ *by adding the vertex* 0 *and the two edges* $\{0,1\}$ *and* $\{0,2s-1\}$.

Definition 223 (the graph G underlying Figure 1.2.)**.** *Let* G *be the graph with* $\mathrm{V}(\mathrm{G}) := \{v_1,\dots,v_7\}$ *and* $\mathrm{E}(\mathrm{G}) := \big\{\ v_1v_2,\ v_1v_3,\ v_1v_6,\ v_1v_7,\ v_2v_3,\ v_2v_6,\ v_2v_7,\ v_3v_4,\ v_3v_5,\ v_4v_5,\ v_4v_6,\ v_4v_7,\ v_5v_6,\ v_5v_7\ \big\}$.

5.3.3 Substructures which decide about Ehrenfeucht-Fraïssé-equivalence

For two graphs G and H, the notation $G \equiv^{\mathrm{MSO}}_k H$ (resp. $G \equiv^{\mathrm{FO}}_k H$) means that every MSO-sentence (resp. FO-sentence) φ with $\mathrm{qr}(\varphi) \leqslant k$ is either satisfied by both G and H, or false in both. We adopt the usual convention of not reflecting the type of the underlying structure in the symbol $\equiv^{\mathrm{MSO}}_k$, in particular we write $\equiv^{\mathrm{MSO}}_k$ both between graphs and rooted graphs. We consider it understood that for rooted graphs the partial isomorphisms involved in the definition of $\equiv^{\mathrm{MSO}}_k$ must respect the unary relation (i.e. the root), too. Both $\equiv^{\mathrm{MSO}}_k$ and $\equiv^{\mathrm{FO}}_k$ are equivalence relations on the set of all graphs w.r.t. a fixed language. Moreover, we will use the following fact:

Lemma 224. *For every* $k \in \mathbb{N}$*, each of the equivalence relations* $\equiv^{\mathrm{MSO}}_k$ *and* $\equiv^{\mathrm{FO}}_k$ *has only finitely-many equivalence classes.* □

A proof of Lemma 224 can be found in e.g. [54, Proposition 3.1.3].

The following is similar to a standard fact in finite model theory about disjoint unions (for example, since identifying at a single vertex essentially behaves like disjoint union as far as the rooted Ehrenfeucht-Fraïssé-game[7] is concerned, [31, Corollary 6.25] is a reference for this, with L taken to be the language of graphs, $n = r$, $m(r) = N_{\mathsf{L}}(0,n)$, $S_i = S^{\star}_j = G$, $I = [m_a]$, $J = [m_b]$ for all $(i,j) \in I \times J$); one possible proof proceeds by describing a winning-strategy for Duplicator and will not be repeated here:

Lemma 225 (multiplicity of copies eventually becomes MSO-indistinguishable w.r.t. constant quantifier rank)**.** *For every quantifier-rank* $r \in \mathbb{N}$ *there exists* $m = m(r) \in \mathbb{N}$ *such that for every graph* G *and every* $(m_a, m_b) \in \mathbb{N}^2$*: if both* $m_a,\ m_b \geqslant m$*, and if* A *(resp.* B*) is the graph obtained from* m_a *(resp.* m_b*) copies of* G *by identifying the roots, then* $A \equiv^{\mathrm{MSO}}_r B$*.* □

[7] Ehrenfeucht-Fraïssé-games are a standard proof-technique in finite model theory, see e.g. [54, p. 18 for the FO-variant, p. 38 for the MSO-variant].

The following is inspired by a similar argumentation of McColm [126, pp. 336–339]: we will (non-uniquely) define constant-sized substructures $\mathrm{X}_{r,\mathcal{G}}$ whose existence within a large graph as a pendant copy (in the sense from p. 193 in Section 5.1) implies the global conclusion of the large graph being indistinguishable from those small graphs w.r.t. MSO-formulas of quantifier-rank r:

Lemma 226 ($\mathrm{X}_{r,\mathcal{A}}$; a local substructure whose existence clinches $\equiv_r^{\mathrm{MSO}}$-equivalence among arbitrary members of an addable class)**.** *For every quantifier-rank $r \geqslant 0$ and every addable, minor-closed class $\mathcal{A}$ of graphs there exists a (not uniquely determined) graph $\mathrm{X}_{r,\mathcal{A}} \in \mathcal{A}$ with the following property: if a* connected *$G \in \mathcal{A}$ contains $\mathrm{X}_{r,\mathcal{A}}$ as a pendant copy, then $G \equiv_r^{\mathrm{MSO}} \mathrm{X}_{r,\mathcal{A}}$.*

Proof. Let $\mathcal{C} \subseteq \mathcal{A}$ denote the subset of all connected elements of $\mathcal{A}$. Using a notation of [34, p. 9], we denote by $\mathcal{C}_\circ$ the set of all rooted graphs that can be obtained from $\mathcal{C}$, i.e. for every $G \in \mathcal{C}$ there are in $\mathcal{C}_\circ$ exactly $|G|$ elements of the form (G, v) with $v \in \mathrm{V}(G)$.

The equivalence relation $\equiv_r^{\mathrm{MSO}}$ has finitely-many equivalence classes on any relational structure with finite arity (more precisely, in the terminology of [54], on any τ-structure), in particular it has finitely-many equivalence classes on the set of rooted graphs, hence in particular it partitions our set $\mathcal{C}_\circ$ of rooted graphs into a finite number $\ell = \ell(r) \in \mathbb{N}$ of equivalence classes $\mathcal{C}_\circ^{(1)} \sqcup \cdots \sqcup \mathcal{C}_\circ^{(\ell)} = \mathcal{C}_\circ$. Select one arbitrary representative $(G_i, x_i) \in \mathcal{C}_\circ^{(i)}$ for every $i \in [\ell]$. Let $m = m(r) \in \mathbb{N}$ denote the number from Lemma 225. For every $i \in [\ell]$ create m copies $(G_i^{(1)}, x_i^{(1)}), \ldots, (G_i^{(m)}, x_i^{(m)})$ of (G_i, x_i), then identify all $m \cdot \ell$ roots. The (not uniquely determined) rooted graph thus obtained we denote by $(\mathrm{X}_{r,\mathcal{A}}, x)$. Thus,

$$\mathrm{X}_{r,\mathcal{A}} = \bigvee_{1 \leqslant i \leqslant \ell} \left(G_i^{(1)} \vee \cdots \vee G_i^{(m)} \right) . \tag{5.3}$$

We note that the graph $\mathrm{X}_{r,\mathcal{A}}$ is indeed in $\mathcal{A}$: the class $\mathcal{A}$ contains the one-vertex graph, just as any minor-closed set of graphs. Now consider the disjoint union of one copy of the one-vertex graph and all the copies involved in the construction of $\mathrm{X}_{r,\mathcal{A}} \in \mathcal{A}$. Since all those copies are in $\mathcal{A}$, by definition of 'addable', the disjoint union is in $\mathcal{A}$, too. By Definition 203, being addable also implies that the graph obtained by inserting an edge from the one-vertex graph to each of the various copies is again in $\mathcal{A}$. Finally, since $\mathcal{A}$ is minor-closed, contracting each of the newly-added edges leaves an element of $\mathcal{A}$, which is $\mathrm{X}_{r,\mathcal{A}}$.

We now show that $\mathrm{X}_{r,\mathcal{A}} \in \mathcal{A}$ has the property claimed in Lemma 226. Let an arbitrary connected $G \in \mathcal{A}$ be given, and suppose $\mathrm{X}_{r,\mathcal{A}}$ exists in G as a pendant copy. We will re-use the notation $\mathrm{X}_{r,\mathcal{A}}$ to also mean the pendant copy of $\mathrm{X}_{r,\mathcal{A}}$ in G, in particular $\mathrm{X}_{r,\mathcal{A}} \subseteq G$ and $x \in \mathrm{V}(G_i^{(j)}) \subseteq \mathrm{V}(G)$ is the other end of the bridge connecting $\mathrm{X}_{r,\mathcal{A}}$ to $G - \mathrm{X}_{r,\mathcal{A}}$. We now define

$$G^- := G - \bigcup_{(i,j) \in [\ell] \times [m]} (G_i^{(j)} - x) . \tag{5.4}$$

Then $G^- \in \mathcal{A}$ since it is a minor of G. Moreover, G^- is connected. (Let us note that, as a consequence of the definition of 'pendant copy', G^- contains x as a degree-one vertex.) Although the vertex x at which $\mathrm{X}_{r,\mathcal{A}}$ appears in G has been handed down to us, by definition of $\mathcal{C}_\circ$ we know that the rooted graph (G^-, x) is in $\mathcal{C}_\circ$. Thus, by construction, there exists $i_0 \in [\ell]$ with

$$(G_{i_0}, x) \equiv_r^{\mathrm{MSO}} (G^-, x) . \tag{5.5}$$

We write '$\vee$' to denote identification of x-containing graphs at a vertex x. By choice of $m = m(r)$,

$$G_{i_0}^{(1)} \vee \cdots \vee G_{i_0}^{(m)} \quad \equiv_r^{\mathrm{MSO}} \quad G^- \vee (G_{i_0}^{(1)} \vee \cdots \vee G_{i_0}^{(m)}) . \tag{5.6}$$

It is easy to prove via Ehrenfeucht-Fraïssé-games (and, essentially, a known fact, cf. e.g. [31, Lemma 6.20 (d)], take $\mathbf{S}_1^\star = \mathbf{S}_1$, iterate the statement, and use that identification at one vertex behaves essentially like disjoint union from the perspective of Ehrenfeucht-Fraïssé-games on rooted graphs) that replacing one of the constituent graphs in a graph made from several graphs identified at one vertex by some $\equiv_r^{\mathrm{MSO}}$-equivalent graph results in a graph which is again $\equiv_r^{\mathrm{MSO}}$-equivalent to the original one; this, and (5.6), explains the third step in

$$\begin{aligned} G &= G^- \vee \bigvee_{1\leqslant i\leqslant \ell} \left(G_i^{(1)} \vee \cdots \vee G_i^{(m)}\right) \\ &= \left(G^- \vee (G_{i_0}^{(1)} \vee \cdots \vee G_{i_0}^{(m)})\right) \vee \bigvee_{i\in[\ell]\setminus\{i_0\}} \left(G_i^{(1)} \vee \cdots \vee G_i^{(m)}\right) \\ &\equiv_r^{\mathrm{MSO}} \left(G_{i_0}^{(1)} \vee \cdots \vee G_{i_0}^{(m)}\right) \vee \bigvee_{i\in[\ell]\setminus\{i_0\}} \left(G_i^{(1)} \vee \cdots \vee G_i^{(m)}\right) \\ &= \bigvee_{i\in[\ell]} \left(G_i^{(1)} \vee \cdots \vee G_i^{(m)}\right) \overset{(5.3)}{=} \mathrm{X}_{r,\mathcal{A}} \ , \end{aligned} \tag{5.7}$$

which completes the proof of Lemma 226. □

In spite of non-addability of the set $\mathcal{G}_S$ of graphs on a surface S, it admits of an analogue of Lemma 226, provided we restrict ourselves to FO-logic. In fact, the only property of $\mathcal{G}_S$ that we will need for this analogue is *local planarity* of large random elements of $\mathcal{G}_S$.

We will now formulate the last statement (see Lemma 228 below) about global consequences of local structure in this thesis. Although we will only use it only in a context of local *planarity* (cf. our proof of Theorems 138 in Chapter 3), i.e. $\mathcal{A} = \mathcal{P}$, its proof merely needs all neighbourhoods to be in some fixed addable class, so we formulate it more generally. An essential tool in proving Lemma 228 is the following fundamental theorem of Gaifman on locality of first-order formulas:

Theorem 227 (Gaifman's theorem; cf. [60, p. 109] [54, Theorem 2.5.1], where it is proved for more general relational structures than graphs)**.** *Every sentence φ in the first-order logic of graphs (FO for short) is logically equivalent to some boolean combination of a finite number of FO-sentences, each of the form*

$$\exists x_1,\ldots,x_s\colon \left(\bigwedge_{1\leqslant i\leqslant s} \psi^{\mathrm{B}(x_i,\leqslant R)}(x_i)\right) \wedge \left(\bigwedge_{1\leqslant i<j\leqslant s} \mathrm{dist}(x_i,x_j) > 2R\right) \ , \tag{5.8}$$

where $\mathrm{dist}(x_i,x_j) > 2R$ *is an abbreviation for some FO formula equivalent to 'the distance between x_i and x_j is larger than $2R$' and each $\psi^{\mathrm{B}(x_i,\leqslant R)}(x_i)$ is an FO formula which, when interpreted in a graph G via a variable-assignment $\alpha\colon \{x_1,\ldots,x_s\} \to \mathrm{V}(G)$, has its truth-value completely determined by the graph induced by the ball of radius R around $\alpha(x_i)$.*

Both the numbers R and s, and the formulas ψ depend in a very complicated way on the given sentence φ (cf. e.g. [46] [97]).

Lemma 228 ($\mathrm{Y}_{r,\mathcal{A}}$; a local substructure whose pendant existence in a connected locally-$\mathcal{A}$-graph clinches $\equiv_r^{\mathrm{FO}}$-equivalence)**.** *For every quantifier-rank $r \geqslant 0$ and any addable minor-closed class $\mathcal{A}$ of graphs there exists a (not uniquely determined) graph $\mathrm{Y}_{r,\mathcal{A}} \in \mathcal{A}$ and a number $R = R(r) > 0$ with the following property. If G is a* connected *graph with*

(lo.1) *G contains $\mathrm{Y}_{r,\mathcal{A}}$ as a pendant copy ,*
(lo.2) *$G[\mathrm{B}_G(v,\leqslant R)] \in \mathcal{A}$ for every $v \in \mathrm{V}(G)$,*

then $G \equiv_r^{\mathrm{FO}} \mathrm{Y}_{r,\mathcal{A}}$.

Proof. Let r and $\mathcal{A}$ be given. By Theorem 227, each of the finitely-many equivalence-classes of FO-sentences with quantifier-rank $\leqslant r$ contains an FO-sentence β which is a boolean combination of sentences of the form (5.8). For each such equivalence class we choose one such β, and denote by $\mathcal{B} = \{\varphi_1,\ldots,\varphi_m\}$ the set of the sentences of the form (5.8) to be found within the various β, i.e.,

$$\varphi_i := \exists x_{i,1},\ldots,x_{i,s_i}\colon \left(\bigwedge_{1\leqslant a\leqslant s_i} \psi_i^{\mathrm{B}(x_{i,a},\leqslant R_i)}(x_{i,a})\right) \wedge \left(\bigwedge_{1\leqslant a<b\leqslant s_i} \mathrm{dist}(x_{i,a},x_{i,b}) > 2R_i\right) \tag{5.9}$$

for every $1 \leqslant i \leqslant m$. We set $R := R(r) := \max(R_1, \ldots, R_m)$ and

$$r' := \max(\mathrm{qr}(\varphi_1), \ldots, \mathrm{qr}(\varphi_m)) \ .$$

Now we choose any of the graphs $\mathrm{X}_{r',\mathcal{A}} \in \mathcal{A}$ guaranteed by Lemma 226, and attach to it (which vertex of $\mathrm{X}_{r',\mathcal{A}}$ we attach it to does not matter) a new path of length R. The resulting (far from uniquely determined) graph we call $\mathrm{Y}_{r,\mathcal{A}}$. We have $\mathrm{Y}_{r,\mathcal{A}} \in \mathcal{A}$ because of addability, and since $\mathcal{A}$, like any nontrivial addable minor-closed class of graphs, contains every path. We now prove that $\mathrm{Y}_{r,\mathcal{A}}$ has the property claimed in Lemma 228.

Let any connected graph G with (lo.1) and (lo.2) be given. By definition, the claim $G \equiv_r^{\mathrm{FO}} \mathrm{Y}_{r,\mathcal{A}}$, that we have to prove, means

$$G \models \varphi \Leftrightarrow \mathrm{Y}_{r,\mathcal{A}} \models \varphi \quad \text{for every FO-sentence } \varphi \text{ with } \mathrm{qr}(\varphi) \leqslant r \ . \tag{5.10}$$

We will show that

$$G \models \varphi_i \Leftrightarrow \mathrm{Y}_{r,\mathcal{A}} \models \varphi_i \qquad \text{for every } i \in [m] \ , \tag{5.11}$$

for every φ_i. Since every given FO-sentence φ with $\mathrm{qr}(\varphi) \leqslant r$ is equivalent to a boolean combination of the φ_i with $i \in [m]$, this then proves (5.10). We distinguish two cases. Let $i \in [m]$ be arbitrary. We abbreviate

$$\sigma_i := \exists x, y \colon \psi_i^{\mathrm{B}(x,\leqslant R_i)}(x) \wedge (\mathrm{dist}(x,y) = R_i) \tag{5.12}$$

and will make use of

$$\varphi_i \quad \Longrightarrow \quad \sigma_i \ . \tag{5.13}$$

Case 0. The sentence σ_i is not satisfied by *any* connected graph in $\mathcal{A}$. Then in particular the graph $\mathrm{Y}_{r,\mathcal{A}} \in \mathcal{A}$ just constructed does not satisfy σ_i. We now argue that not only does $\mathrm{Y}_{r,\mathcal{A}}$ not satisfy σ_i, but it does not satisfy φ_i from (5.9) either: on account of the path we attached when contructing $\mathrm{Y}_{r,\mathcal{A}}$, for every $u \in \mathrm{V}(\mathrm{Y}_{r,\mathcal{A}})$ there is at least one $v \in \mathrm{V}(\mathrm{Y}_{r,\mathcal{A}})$ with $\mathrm{dist}_{\mathrm{Y}_{r,\mathcal{A}}}(u,v) \geqslant R_i$, hence also with $\mathrm{dist}_{\mathrm{Y}_{r,\mathcal{A}}}(u,v) = R_i$. Thus, the reason for $\mathrm{Y}_{r,\mathcal{A}}$ not satisfying σ_i must be that it does not satisfy the sub-sentence $\exists x, y \colon \psi_i^{\mathrm{B}(x,\leqslant R_i)}(x)$ of σ_i. Therefore, $\mathrm{Y}_{r,\mathcal{A}}$ does not satisfy the stronger sentence $\exists x_{i,1}, \ldots, x_{i,s_i} \colon \left(\bigwedge_{1 \leqslant a \leqslant s_i} \psi^{\mathrm{B}(x_{i,a},\leqslant R_i)}(x_{i,a})\right)$ either, so indeed $\mathrm{Y}_{r,\mathcal{A}}$ does not satisfy φ_i. Thus, to prove (5.11), we now have to show that G does not satisfy φ_i either. Aiming at a contradiction, assume $G \models \varphi_i$. Then $G \models \sigma_i$ by (5.13), hence there exists at least one σ_i-satisfying assignment $\alpha \colon \{x, y\} \to \mathrm{V}(G)$, i.e. $G \models \sigma_i[\alpha]$ (in the notation of [54, p. 6]). By (5.12), and abbreviating $v_x := \alpha(x)$ and $v_y := \alpha(y)$, this means

$$\text{there exist } v_x, v_y \in \mathrm{V}(G) \text{ such that } \mathrm{dist}_G(v_x, v_y) = R_i \tag{5.14}$$

and

$$G \models \psi_i^{\mathrm{B}(v_x,\leqslant R_i)} \ . \tag{5.15}$$

Since G is connected, so is $G[\mathrm{B}(v_x, \leqslant R_i)]$; together with (5.14) this implies

$$\mathrm{dist}_{G[\mathrm{B}(v_x,\leqslant R_i)]}(v_x, v_y) = R_i \ . \tag{5.16}$$

By definition of $\psi_i^{\mathrm{B}(v_x,\leqslant R_i)}$ (cf. e.g. [54, p. 31]), (5.15) is equivalent to $G[\mathrm{B}(v_x, \leqslant R_i)] \models \psi_i$, and the notation allows us to equivalently write this as

$$G[\mathrm{B}(v_x, \leqslant R_i)] \models \psi_i^{\mathrm{B}(v_x,\leqslant R_i)} \ . \tag{5.17}$$

Now comes the time of (lo.2): from (5.16) and (5.17) it follows that $G[\mathrm{B}(v_x, \leqslant R_i)] \models \sigma_i$, which, since by (lo.2) we know the connected graph $G[\mathrm{B}(v_x, \leqslant R_i)]$ to be in $\mathcal{A}$, contradicts the property defining Case 0. Therefore, it is impossible that $G \models \varphi_i$. We have thus shown that (5.11) indeed holds in Case 0 (with both sides of the equivalence being false).

Case 1. This is the negation of Case 0: there exists at least one connected $H \in \mathcal{A}$ with $H \models \sigma_i$. Then we consider any such H and any σ_i-satisfying assignment $\alpha \colon \{x, y\} \to \mathrm{V}(H)$. We define H' to be the following graph: we start with a new vertex w and connect w to the free end of the path we attached when constructing $\mathrm{Y}_{r,\mathcal{A}}$. Then we create s_i disjoint copies $H_1, \ldots, H_{s_i}$ of H, and for every $j \in [s_i]$ let $\alpha_j \colon \{x, y\} \to \mathrm{V}(H_j)$ be the assignment corresponding to the σ_i-satisfying assignment α, i.e., (in the notation from [54, p. 6]),

$$H_j \vDash \sigma_i[\alpha_j] \quad \text{for every } j \in [s_i] \ . \tag{5.18}$$

For every $j \in [s_i]$ we connect $\alpha_j(y) \in H_j$ by a new edge to w. The resulting graph we define to be H'. Since H, hence each of the H_i, and also $\mathrm{Y}_{r,\mathcal{A}}$, all are elements of the addable class $\mathcal{A}$, we have $H' \in \mathcal{A}$. We now construct a φ_i-satisfying assignment $\alpha' \colon \{x_{i,1}, \ldots, x_{i,s_i}\} \to \mathrm{V}(H')$: choose isomorphisms $\Phi_a \colon H \to H_j \subseteq H'$ and set $\alpha'(x_{i,a}) := \Phi_a(\alpha(x))$, for every $a \in [s_i]$.

Since a graph-isomorphism in particular preserves graph-theoretical length, for every $a \in [s_i]$ we know

$$\begin{aligned} \mathrm{dist}_{H'}(\Phi_a(\alpha(x)), \Phi_a(\alpha(y))) &= \mathrm{dist}_H(\alpha(x), \alpha(y)) \\ (\text{since } \alpha \text{ is a } \sigma_i\text{-satisfying assignment}) &= R_i \ . \end{aligned} \tag{5.19}$$

We can now calculate as follows, for any $1 \leqslant a < b \leqslant s_i$ (as to the first equality, by construction, the triangle-inequality holds with equality for each of these four pairs of points):

$$\begin{aligned} \mathrm{dist}_{H'}(\alpha'(x_{i,a}), \alpha'(x_{i,b})) &= \mathrm{dist}_{H'}(\alpha'(x_{i,a}), \Phi_a(\alpha(y))) \\ &\quad + \mathrm{dist}_{H'}(\Phi_a(\alpha(y)), w) + \mathrm{dist}_{H'}(w, \Phi_b(\alpha(y))) \\ &\quad + \mathrm{dist}_{H'}(\Phi_b(\alpha(y)), \alpha'(x_{i,b})) \\ &= \mathrm{dist}_{H'}(\Phi_a(\alpha(x)), \Phi_a(\alpha(y))) \\ &\quad + \mathrm{dist}_{H'}(\Phi_a(\alpha(y)), w) + \mathrm{dist}_{H'}(w, \Phi_b(\alpha(y))) \\ &\quad + \mathrm{dist}_{H'}(\Phi_b(\alpha(y)), \Phi_b(\alpha(x))) \\ &\overset{(5.19)}{=} R_i + 1 + 1 + R_i = 2R_i + 2 > 2R_i \ . \end{aligned} \tag{5.20}$$

Moreover, by construction and since $\alpha \colon \{x, y\} \to \mathrm{V}(H)$ is an assignment satisfying $\psi_i^{\mathrm{B}(x,\leqslant R_i)}(x)$,

$$H' \vDash \left(\bigwedge_{1 \leqslant a \leqslant s_i} \psi_i^{\mathrm{B}(x_{i,a},\leqslant R_i)}(x_{i,a}) \right) [\alpha'] \ . \tag{5.21}$$

From (5.20) and (5.21) it follows that indeed $H' \vDash \varphi_i[\alpha']$, hence

$$H' \vDash \varphi_i \ . \tag{5.22}$$

At this point we know

(1) $H' \equiv_{r'}^{\mathrm{MSO}} \mathrm{X}_{r',\mathcal{A}}$, (2) $\mathrm{Y}_{r,\mathcal{A}} \equiv_{r'}^{\mathrm{MSO}} \mathrm{X}_{r',\mathcal{A}}$,

both guaranteed by Lemma 226, the reason for (1) being that the connected graph $H' \in \mathcal{A}$ contains $\mathrm{X}_{r',\mathcal{A}}$ as a pendant copy, while the reason for (2) is that the connected graph $\mathrm{Y}_{r,\mathcal{A}}$ contains $\mathrm{X}_{r',\mathcal{A}}$ as a pendant copy. From (1) and (2) it follows by transitivity of the relation $\equiv_{r'}^{\mathrm{MSO}}$ that

$$H' \equiv_{r'}^{\mathrm{MSO}} \mathrm{Y}_{r,\mathcal{A}} \ . \tag{5.23}$$

By (5.22), $\mathrm{qr}(\varphi_i) = r_i \leqslant r'$, $\varphi_i \in \mathrm{FO} \subseteq \mathrm{MSO}$, and (5.23), it follows that

$$\mathrm{Y}_{r,\mathcal{A}} \vDash \varphi_i \ . \tag{5.24}$$

From (lo.1) we know that G contains $\mathrm{Y}_{r,\mathcal{A}}$, and therefore $\mathrm{X}_{r',\mathcal{A}}$, as a pendant copy, so $G \equiv_{r'}^{\mathrm{MSO}}$ (by Lemma 226 and choice of r') $\equiv_{r'}^{\mathrm{MSO}} \mathrm{X}_{r',\mathcal{A}} \equiv_{r'}^{\mathrm{MSO}}$ (by (2)) $\equiv_{r'}^{\mathrm{MSO}} \mathrm{Y}_{r,\mathcal{A}}$. Therefore, (5.24) implies $G \vDash \varphi_i$. Thus, we have shown that (5.11) indeed holds in Case 1, too (with both sides being true). As already mentioned, this proves (5.10), and completes the proof of Lemma 228. □

Definition 229 (the substructures $\mathrm{F}_1, \ldots, \mathrm{F}_9$ and $\mathrm{G}_1, \ldots, \mathrm{G}_{19}$)**.** *Let* $\mathrm{F}_1, \ldots, \mathrm{F}_9$ *denote the isomorphism types of forests defined by Figure 5.8. Let* $\mathrm{G}_1, \ldots, \mathrm{G}_{19}$ *denote the isomorphism types of planar graphs defined by Figure 5.9.*

The following enumeration of possible substructures in the graph $\mathrm{X}_{\frac{1}{2}\mathrm{C}_{(n,n)}(A)}$ is used in Chapter 4 to graph-theoretically characterise how $\mathsf{P}_{\mathrm{chio}}$ and $\mathsf{P}_{\mathrm{lcf}}$ relate to one another. The fact that we stop the enumeration at the f-vector $(|\cdot|, \|\cdot\|) = (f_0, f_1) = (8, 6)$ of the graphs, even though there exist bipartite nonforests with $(f_0, f_1) = (8, 7)$, is explained by the use made of this enumeration in Chapter 4: we will only be concerned with bipartite nonforests having up to six edges.

Lemma 230 (bipartite nonforests ordered by their f-vectors)**.** *The isomorphism types of bipartite nonforests, ordered lexicographically by their* f*-vectors up to* $(f_0, f_1) = (8, 6)$*, are:*

(t1) $= C^4$
(t2) $=$ *disjoint union of* C^4 *and one isolated vertex*
(t3) $= C^4$ *intersecting one edge*
(t4) $= K^{2,3}$
(t5) $=$ *disjoint union of* C^4 *and two isolated vertices*
(t6) $= C^4$ *intersecting one edge, and one extra isolated vertex*
(t7) $=$ *disjoint union of* C^4 *and an isolated edge*
(t8) $= C^4$ *intersecting two disjoint edges, the intersection set no edge of* C^4
(t9) $= C^4$ *intersecting two disjoint edges, the intersection set an edge of* C^4
(t10) $= C^4$ *intersecting a 2-path in an endvertex*
(t11) $= C^4$ *intersecting a 2-path in its inner vertex*
(t12) $= C^6$
(t13) $=$ *disjoint union of* C^4 *and three isolated vertices*
(t14) $= C^4$ *intersecting one edge, and two extra isolated vertices*
(t15) $=$ *disjoint union of* C^4 *and an edge, and one extra isolated vertex*
(t16) $= C^4$ *intersecting one edge, and one extra isolated edge*
(t17) $=$ *disjoint union of* C^4 *and a 2-path*
(t18) $=$ *disjoint union of* C^4 *and four isolated vertices*
(t19) $=$ *disjoint union of* C^4 *and an edge and two extra isolated vertices*
(t20) $=$ *disjoint union of* C^4 *and two disjoint edges*

Proof. Easy to check since the graphs are required to be bipartite and have $f_1 \leqslant 6$. $\square$

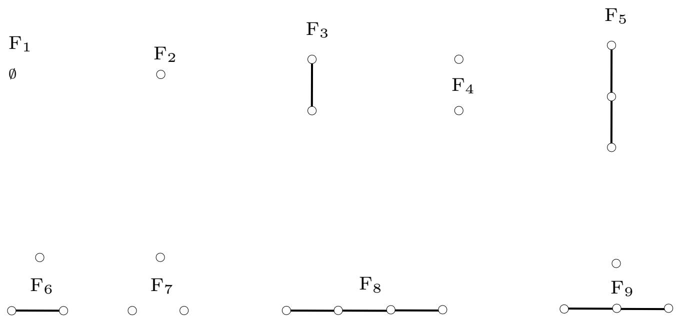

Figure 5.8: The forests $\mathrm{F}_1, \ldots, \mathrm{F}_9$ used in the proof of Theorem 97 in Chapter 3. These are the substructures whose possible appearance as a small component in a random forest is the local structural cause of the global piece of information that is the set of probability limits of MSO-statements about forests. The ordering is according to non-increasing probability w.r.t. the Boltzmann–Poisson measure from [128].

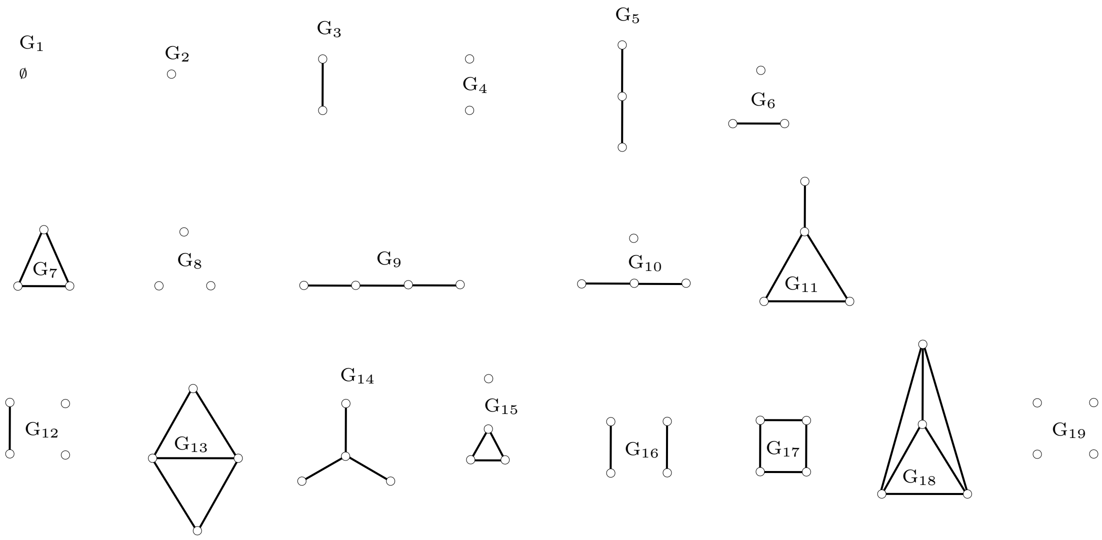

Figure 5.9: The planar graphs $\mathrm{G}_1, \ldots, \mathrm{G}_{19}$ used in the proof of Theorem 99 in Chapter 3. These are the substructures whose possible appearance as $\mathrm{Frag}(G)$ of a large random planar graph G is the local structural cause of the global piece of information that is the set of probability limits of MSO-statements about planar graphs. The ordering is according to non-increasing probability w.r.t. the *Boltzmann–Poisson measure* on the isomorphism types in a decomposable graph class that was provided by C. McDiarmid (cf. Theorem 81 on p. 116). For example, G_{14} is strictly more likely to occur (as the part of a large random planar graph *outside* its giant component) than G_{16}, but e.g. G_{14} and G_{15}, both having the same order and precisely six automorphisms, are exactly equally likely to so appear: it is equally likely that the part $\mathrm{Frag}(G)$ of a large uniformly random planar graph outside its giant component is isomorphic to G_{14} as that it is isomorphic to G_{15}.

List of some symbols

$\models$	is a model of, satisfies
$\{\pm\}$	$= \{-1, +1\}$
$\{0, \pm\}$	$= \{-1, 0, +1\}$
$\lvert\cdot\rvert$	cardinality of a set, number of vertices of a graph
$\lVert G \rVert$	number of edges of a graph G
$\equiv_r^{\mathrm{FO}}$	equivalence w.r.t. r-rounds of the FO-Duplicator-Spoiler-game, cf. [54, Corollary 2.2.9]
$\equiv_r^{\mathrm{MSO}}$	equivalence w.r.t. r-rounds of the MSO-Duplicator-Spoiler-game, cf. [54, p. 38]
$\sqcup$	union of sets and simultaneously the claim that the union is disjoint
$\wedge$	wedge product in the exterior algebra of a module
$[n]$	$\{1, \ldots, n\}$
$[n]_0$	$\{0, 1, \ldots, n\}$
$\mathbb{Z}/n$	$\mathbb{Z}/n\mathbb{Z}$
$=_n$	equality of integers modulo n
$\langle \cdot, \cdot \rangle$	standard inner product on the group of 1-chains $\mathrm{C}_1(G)$ of a graph G
$[\,\cdot\,]_k$	$= \langle\, k-1 \wedge k+1, \cdot\, \rangle$, p. 62
$A[i,j]$	$A[i,j] := a_{i,j}$ if $A = (a_{i,j})_{(i,j)\in[s]\times[t]}$
$\mathrm{Aut}(G)$	group of automorphisms of G
$\mathrm{Bas}\mathcal{C}_{\mathfrak{L}}$	set of graphs with $\mathfrak{L}$-based flow lattice, p. 195
$\mathrm{Bas}\mathcal{C}_{\lvert\cdot\rvert}$	set of graphs with Hamilton-based flow lattice, p. 195
$\mathrm{Bas}\mathcal{C}_{\mathfrak{L}}{}^{-}$	set of graphs with almost $\mathfrak{L}$-based flow lattice, p. 195
$\mathrm{bCd}_{\xi}\mathcal{C}_{\mathfrak{L}}$	p. 195
$\mathrm{B}_G(v, \leqslant r)$	p. 191
$\beta_0 = 1$	a notation for marking a set as containing connected objects only, not using a word but a reference to the 0-th Betti number
$\mathrm{Big}(G)$	largest component of the graph G, ties broken according to some rule, p. 193

$\mathrm{b}\mathcal{M}^{\beta_0=0}_{\mathrm{B}\not\subseteq\mathrm{G}}$	p. 195
$\mathrm{bQuo}_A\mathcal{C}_{\mathfrak{L}}$	p. 195
$\mathrm{Cd}_{\mathfrak{l}}\mathcal{C}_{\mathfrak{L}}$	p. 195
$\mathrm{Cd}_{\mathfrak{l}}\mathcal{C}_{\mathfrak{L}}{}^{-}$	p. 195
Cir	p. 156
$\mathcal{C}_{\mathfrak{L}}(G)$	set of all graph-theoretical circuits with length in $\mathfrak{L}(G)$, p. 195
$\vec{\mathcal{C}}_{\mathfrak{L}}(G)$	set of all simple flows with length in $\mathfrak{L}(G)$, p. 195
$\mathrm{C}_1(G)$	p. 192
$\mathrm{C}_1(G;\mathbb{F}_2)$	1-dimensional chain group with $\mathbb{F}_2$ coefficients of the graph G (synonym: edge-space of the graph G); p. 192
CL_r	bipartite cyclic ladder with r-rungs; p. 197
C_n^2	p. 28
$\mathrm{C}_n^{2^-}$	p. 199
$\mathcal{CO}_{\mathfrak{L}}$	set of $\mathfrak{L}$-path-connected graphs; p. 195
$\mathrm{Col}(X,\sigma)$	set of all $(-)$-constant, $(+)$-proper vertex-2-colourings of a signed graph (X,σ); p. 156
$\frac{1}{2}\mathrm{C}^{\breve{I}}_{(s,t)}$	Chio map with pivot $a_{s,t}$, p. 154
$\delta(G)$	minimum vertex-degree of a graph G
$\Delta(G)$	maximum vertex-degree of a graph G
$\mathrm{Dom}(A)$	domain of a matrix A (when A is viewed as a function on an index-set), p. 191
$\mathrm{dom}(A)$	$\lvert\mathrm{Dom}(A)\rvert$, p. 191
$\mathcal{E}^J_B$	entry-specification-event with specifications in $\mathrm{Dom}(B)\subseteq J\subseteq[s-1]\times[t-1]$, p. 155
$\mathrm{f}_T(u,v)$	fundamental flow w.r.t. the spanning tree T and adding the edge uv, p. 55
$\mathcal{F}^{\mathrm{M}}(k,n)$	set of entry-specification matrices for which equality of $\mathsf{P}_{\mathrm{chio}}$ and $\mathsf{P}_{\mathrm{lcf}}$ fails, w.r.t. k specifications, p. 156
$\mathrm{Frag}(G)$	union of all non-largest components of the graph G, ties broken according to some rule, p. 193
$\mathcal{G}_n$	set of all n-vertex elements in a class of graphs $\mathcal{G}$
$\mathcal{G}_S$	class of all graphs which admit an embedding into the surface S
$G_{n,p}=G(n,p)$	binomial random graph on n vertices with edge-probability p
$\mathcal{H}(G)$	set of all Hamilton-circuits in G; p. 192
$\vec{\mathcal{H}}(G)$	set of all Hamilton-flows in G; p. 192
$\breve{I}$	Chio extension of a subset $I\subseteq[s-1]\times[t-1]$, making it compatible with Chio condensation; p. 154

$\mathcal{LA}_{\mathfrak{L}}$	set of $\mathfrak{L}$-path-connected graphs in the bipartite sense; p. 195
$\mathbb{L}_{\mathcal{A},\mathrm{FO}}$	set of all density limits of FO-statements about graphs from the class $\mathcal{A}$; p. 119
$\mathbb{L}_{\mathcal{A},\mathrm{MSO}}$	set of all density limits of MSO-statements about graphs from the class $\mathcal{A}$; p. 119
$\mathcal{M}^{\beta_0=1}_{\mathrm{B}\nsubseteq\mathrm{G}}$	p. 196
$\mathcal{M}^{\mathbb{Z}}_{\mathfrak{L},A}$	p. 195
$\mathcal{M}^{\mathbb{Z}\mathrm{Bas}}_{\mathfrak{L}}$	monotonised subset of $\mathrm{Bas}\mathcal{C}_{\mathfrak{L}}$ obtained by intersecting with $\mathcal{CO}_{\mathfrak{L}}$, p. 195
$\mathcal{M}_{\mathfrak{L},\xi}$	p. 195
$\mathcal{M}_{\mathfrak{L},\xi}{}^{-}$	p. 195
$\mathcal{M}^{\mathbb{Z}\mathrm{Bas}^-}_{\mathfrak{L}}$	monotonised subset of $\mathrm{Bas}\mathcal{C}_{\mathfrak{L}}{}^{-}$ obtained by intersecting with $\mathcal{CO}_{\mathfrak{L}}$, p. 195
$\mathsf{P}_{\mathrm{BP}\mathcal{A}}$	Boltzmann–Poisson measure on the set of isomorphism types in the addable class $\mathcal{A}$, p. 116
$\mathsf{P}_{\mathrm{chio}}$	Chio measure on $\{0,\pm\}^{[s]\times[t]}$, p. 154
$\mathsf{P}_{\mathrm{lcf}}$	lazy coin flip measure on $\{0,\pm\}^{[s]\times[t]}$, p. 153
$\mathrm{Poi}(\lambda)$	the Poisson-distribution with parameter λ
qr	quantifier rank, p. 193
$\mathrm{Quo}_A\mathcal{C}_{\mathfrak{L}}$	p. 195
$\mathrm{Quo}_A\mathcal{C}_{\mathfrak{L}}{}^{-}$	p. 195
$\mathrm{Ra}_{<r}$	set of matrices with rank less than r, p. 156
σ_B	edge-signing of $\mathrm{X}^{k,s,t}_B$ defined by $B \in \{0,\pm\}^I$, p. 155
Σ_n	an auxiliary abbreviation in the proof that $\mathrm{C}^{2^-}_n$ has Hamilton-based flow lattice, p. 54
$\mathrm{supp}(A)$	cardinality of support of a matrix A (viewed as a function on an index-set), p. 191
$\mathrm{Supp}(A)$	support of a matrix A (viewed as a function on an index-set), p. 191
$\mathcal{UG}$	the set of all isomorphism types in a set $\mathcal{G}$ of graphs
$\mathrm{X}^{k,s,t}_B$	auxiliary bipartite graph associated to $B \in \{0,\pm\}^I$, p. 155
$_{\mathrm{ul}}\mathrm{X}^{k,s,t}(B)$	isomorphism type of $\mathrm{X}^{k,s,t}_B$, p. 156
$\mathrm{X}_{r,\mathcal{A}}$	any of the graphs from Lemma 226 on p. 205, substructures which when they exist in any graph $G \in \mathcal{A}$ are $\equiv^{\mathrm{MSO}}_r$-equivalent to the entire ambient graph
$\mathbb{Z}$	the integers
$\mathrm{Z}_1(G)$	flow lattice (synonym: lattice of integral flows) of a graph G; p. 2
$\mathrm{Z}_1(G;\mathbb{F}_2)$	mod-2-cycle space of a graph G in the usual sense

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Appendix

Grund für diesen Anhang

Laut §6, (1), Satz 4 der für diese Dissertation maßgeblichen Promotionsordnung der TUM (vom 1. August 2001 in der neunten Änderungssatzung) wird verlangt (Zitat): '*Die zur Publikation angenommenen und im Druck erschienenen Veröffentlichungen sind der Dissertation als Appendix beizufügen.*'

Die beiden Eigenschaften 'angenommen' und 'im Druck erschienen' treffen im Falle dieser Dissertation genau auf die Vorveröffentlichung [82] zu. Nur aus diesem Grund füge ich hier die publizierte Arbeit [82], die in der vorliegenden Dissertation verarbeitet und erheblich weitergeführt worden ist, als Anhang bei, genau so, wie sie im Druck erschienen ist. *Das ist vom Verlag ausdrücklich erlaubt*, obwohl ich das copyright an der finalen Version nicht habe: siehe die Aufzählung der 'Author Use'-Rechte im weiter unten ebenfalls beigefügten screenshot vom 11. Juni 2014 (insbesondere 'final published article' und 'Inclusion in a thesis or dissertation').

Für Leser der vorliegenden Dissertation besteht wohl kaum Anlass, den Anhang zu lesen, da die Arbeit ihn enthält.

Reason for this appendix (translation of the German above)

By §6, (1), Satz 4 of TUM's doctoral degree regulations (the version which applies to this dissertation is the one of August 1, 2001, ninth revision) the following is required (my translation): *accepted publications which have appeared in print are to be included into the dissertation as an appendix.*

The two properties 'accepted' and 'appeared in print' apply to exactly one of my publications relevant to this thesis: [82]. It is for this reason only that I append the paper [82] to this thesis, as it appears in print. In the present dissertation, it has been incorporated and significantly extended. *This is expressly allowed by the publisher*, even though I do not have the copyright for the final published version: see the enumeration of the 'Author Use'-rights in the screenshot of June 11, 2014 appended below (in particular, 'final published article' and 'Inclusion in a thesis or dissertation').

For readers of the present thesis there probably is no reason for reading the appendix, as the thesis contains it.

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European Journal of Combinatorics 36 (2014) 503–530

Contents lists available at ScienceDirect

European Journal of Combinatorics

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On prisms, Möbius ladders and the cycle space of dense graphs

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ARTICLE INFO

Article history:
Received 22 December 2011
Accepted 14 September 2013

ABSTRACT

For a graph G, let $|G|$ denote its number of vertices, $\delta(G)$ its minimum degree and $Z_1(G; \mathbb{F}_2)$ its cycle space. Call a graph *Hamilton-generated* if and only if every cycle in G is a symmetric difference of some Hamilton circuits of G. The main purpose of this paper is to prove: for every $\gamma > 0$ there exists $n_0 \in \mathbb{Z}$ such that for every graph G with $|G| \geqslant n_0$ vertices,

(1) if $\delta(G) \geqslant (\frac{1}{2} + \gamma)|G|$ and $|G|$ is odd, then G is Hamilton-generated,
(2) if $\delta(G) \geqslant (\frac{1}{2} + \gamma)|G|$ and $|G|$ is even, then the set of all Hamilton circuits of G generates a codimension-one subspace of $Z_1(G; \mathbb{F}_2)$ and the set of all circuits of G having length either $|G| - 1$ or $|G|$ generates all of $Z_1(G; \mathbb{F}_2)$,
(3) if $\delta(G) \geqslant (\frac{1}{4} + \gamma)|G|$ and G is balanced bipartite, then G is Hamilton-generated.

All these degree-conditions are essentially best-possible. The implications in (1) and (2) give an asymptotic affirmative answer to a special case of an open conjecture which according to [I.B.-A. Hartman, Long cycles generate the cycle space of a graph, European J. Combin. 4 (3) (1983) 237–246] originates with A. Bondy.

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1. Introduction

There exist investigations in which the set underlying a finite-dimensional vector space is not forgotten, but made to play a central part. One such investigation was begun thirty years ago by

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Hartman: when does the cycle-space $Z_1(G; \mathbb{F}_2)$ of a graph G admit an $\mathbb{F}_2$-basis consisting of *long* graph-theoretical circuits only? In [32, Theorem 1] Hartman proved that—barring the sole exception of G being a complete graph with an even number of vertices—for every 2-connected finite graph G, the set of all circuits of length *at least* $\delta(G) + 1$ generates $Z_1(G; \mathbb{F}_2)$.

The lower the minimum degree $\delta(G)$, the larger the set of cycle-lengths one has to allow in order to be guaranteed a generating system by Hartman's theorem. In particular, statements guaranteeing a generating system consisting entirely of *Hamilton circuits* remain almost inaccessible via this theorem: one has to set $\delta(G) := |G| - 1$, hence $G \cong K^{|G|}$, and what remains of Hartman's general theorem is a rather special (albeit still non-obvious) statement about the complete graph.

The property of $Z_1(G; \mathbb{F}_2)$ being generated by the Hamilton circuits of G seems to have been first studied by Alspach, Locke and Witte [5]. They proved that G has the property if G is a connected Cayley graph on a finite abelian group and is either bipartite or has odd order (these hypotheses being mutually exclusive for connected Cayley graphs on finite abelian groups).

Here, we will for the first time prove minimum degree conditions guaranteeing this property. We will accomplish this by way of a two-layered strategy which first uses theorems from extremal graph theory to prove the existence of certain spanning subgraphs that help transfer the property to the host graph in a second step. The main purpose of the present paper is to prove the following previously unknown implications.

Theorem 1 (*Sufficient conditions for a cycle space generated by Hamilton circuits; the case (I3) had already been announced in [12]*)**.** *Call a graph* Hamilton-generated *if and only if every cycle in G is a symmetric difference of some number of Hamilton circuits of G. For every $\gamma > 0$ there exists $n_0 \in \mathbb{Z}$ such that for every graph G with $|G| \geqslant n_0$, the following is true:*

(I1) *if $\delta(G) \geqslant (\frac{1}{2} + \gamma)|G|$ and $|G|$ is odd, then G is Hamilton-generated,*
(I2) *if $\delta(G) \geqslant (\frac{1}{2} + \gamma)|G|$ and $|G|$ is even, then the set of all Hamilton-circuits of G generates a codimension-one subspace of $Z_1(G; \mathbb{F}_2)$ and the set of all circuits of G with lengths either $|G| - 1$ or $|G|$ generates all of $Z_1(G; \mathbb{F}_2)$,*
(I3) *if $\delta(G) \geqslant (\frac{1}{4} + \gamma)|G|$ and G is balanced bipartite, then G is Hamilton-generated,*
(I4) *if in* (I1) *and* (I2) *the condition '$\delta(G) \geqslant (\frac{1}{2}+\gamma)|G|$' is replaced by '$\delta(G) \geqslant \frac{2}{3}|G|$', then without further change to* (I1) *or* (I2) *it suffices to take $n_0 := 2 \cdot 10^8$.*

Implication (I1) *becomes false if '$(\frac{1}{2} + \gamma)|G|$' is replaced by '$\lfloor \frac{|G|}{2} \rfloor$ and G Hamilton-connected'. Implication* (I3) *becomes false if '$(\frac{1}{4} + \gamma)|G|$' is replaced by '$\frac{1}{4}|G|$ and G Hamiltonian'.*

In (I1), the hypothesis of odd $|G|$ is necessary: as a consequence of Mantel's theorem, every G with $\delta(G) \geqslant \lfloor |G|/2 \rfloor + 1$ contains a triangle T. If $|G|$ is even, the vector with support T cannot be an $\mathbb{F}_2$-linear combination of the (even-length) Hamilton circuits.

The hypotheses of (I1) imply that G is Hamilton-connected. Hamilton-connectedness by itself, however, does not imply Hamilton-generatedness (see (i) and (ii) in Section 4.1).

A purely combinatorial way of phrasing (I1) and (I3) is to say that 'every circuit in G can be constructed as a symmetric difference of some Hamilton circuits of G'. In this variant phrasing, talking about graph-theoretical circuits does not lose any generality since for any graph G and any cycle $c \in Z_1(G; \mathbb{F}_2)$, the support $\mathrm{Supp}(c)$ is an edge-disjoint union of graph-theoretical circuits [24, Proposition 1.9.2]. Let us note in passing that this generalises to locally-finite *infinite* graphs [26, Theorem 7.2, equivalence (i) $\Leftrightarrow$ (iii)], and that it has been given a precise sense for arbitrary *compact metric spaces* [30]. Linear-algebraic properties of Hamilton circuits *in infinite graphs* (cf. [16])—i.e. the role of infinite Hamilton circles vis-à-vis the cycle space (in the sense of [23,25–27])—is an unexplored research topic.

A purely combinatorial way of phrasing the first conclusion in (I2) is to say that 'every *even* circuit in G can be constructed as a symmetric difference of Hamilton circuits of G'. This equivalent rephrasing is possible since $\delta(G) \geqslant (\frac{1}{2} + \gamma)|G|$ implies that G is not bipartite (in a bipartite graph, codimension one of $\langle \mathcal{H}(G) \rangle_{\mathbb{F}_2}$ in $Z_1(G; \mathbb{F}_2)$ would not be equivalent to this combinatorial statement). For the second conclusion in (I2), a rephrasing is '*every* circuit in G can be constructed as a symmetric difference of circuits with lengths $|G|-1$ or $|G|$' (and, by the first comment at the end of Section 3.2, a *single* circuit of length $|G|-1$ suffices here).

Theorem 1, the main result of the present paper, adds to the growing corpus of knowledge about the following phenomenon: when studying the set of Hamilton circuits as a function of the minimum degree $\delta(G)$, it pours if it rains—slightly below a sufficient threshold there still exist graphs which do not have *any* Hamilton circuit, slightly above the threshold suddenly every graph contains not merely one but rather a plethora of Hamilton circuits satisfying many *additional requirements*. This line of investigation appears to begin with Nash-Williams' proof [52, Theorem 2] [53, Theorem 3] that for every graph G with $\delta(G) \geqslant \frac{1}{2}|G|$ there exists not only one (Dirac's theorem [28, Theorem 3][24, Theorem 10.1.1]) but at least $\lfloor \frac{5}{224} n \rfloor$ edge-disjoint Hamilton circuits. For sufficiently large graphs G with $\delta(G)$ a little larger than $\frac{1}{2}|G|$, Nash-Williams' theorem was improved by Christofides, Kühn and Osthus [19, Theorem 2] to a guarantee that there are at least $\frac{1}{8}n$ edge-disjoint Hamilton circuits—this being an asymptotically best-possible result in view of examples [52, p. 818] which show that in graphs G with $\delta(G) \geqslant \frac{1}{2}|G|$ and having a slightly irregular degree sequence, the number of edge-disjoint Hamilton circuits is bounded by $\frac{1}{8}n$. More can be achieved if besides a high minimum-degree, additional requirements are imposed on the host graph. Two aspects of this are (1) a regular degree sequence, and (2) a random host graph.

As to (1), if the host graph is required to be regular in advance, a still unsettled conjecture of Jackson [35, p. 13, l. 17] posits that a d-regular graph with $d \geqslant \frac{|G|-1}{2}$ actually realizes the obvious upper bound $\lfloor \frac{1}{2} d \rfloor$ for the number of edge-disjoint Hamilton circuits. Christofides, Kühn and Osthus proved a theorem which in a sense comes arbitrarily close to the conjecture [19, Theorem 5]. This has recently been further improved by Kühn and Osthus [44, Theorem 1.3].

As to (2), Frieze and Krivelevich conjectured [29, p. 222] that for *any* $0 \leqslant p_n \leqslant 1$ an Erdős–Rényi random graph $\mathbb{G}_{n,p_n}$ asymptotically almost-surely attains the a priori maximum of $\lfloor \delta/2 \rfloor$ edge-disjoint Hamilton-circuits, which they proved [29, Theorem 1] for $p_n \leqslant (1+o(1))\frac{\log n}{n}$. In [39, Theorem 2] Knox, Kühn and Osthus proved the conjecture for a class of functions p_n that sweeps a huge portion of the range $\frac{\log n}{n} \ll p_n \ll 1$. A remaining gap (starting at $\frac{\log n}{n}$) in the probability range heretofore covered was closed by Krivelevich and Samotij [41]. As a consequence of a recent breakthrough of Kühn and Osthus [43], using their notion of *robust outexpanders*, Krivelevich and Frieze's conjecture has now been proved completely [44, Theorem 1.10 and Section 5.2]. The question about the number of edge-disjoint copies of Hamilton circuits has been refined by means of a function related to spanning subgraphs [44, Theorem 1.5] [42, Theorem 3].

One way to look at these results is as providing 'extremely orthogonal' (i.e. no additive cancellation is involved in the vanishing of the standard bilinear form) sets of Hamilton circuits. As they stand, these theorems are far from providing 'orthogonal' Hamilton-circuit-bases for $\mathrm{Z}_1(G;\mathbb{F}_2)$: at the relevant minimum degrees, the dimension of $\mathrm{Z}_1(G;\mathbb{F}_2)$ is much higher than $\delta(G)/2$ (roughly, one has $\dim_{\mathbb{F}_2} \mathrm{Z}_1(G;\mathbb{F}_2) \in \Theta_{|G|\to\infty}(\delta(G)^2)$), so the sets of mutually disjoint Hamilton circuits are—while 'very' orthogonal—far from being generating sets of $\mathrm{Z}_1(G;\mathbb{F}_2)$. Yet it does not seem unlikely that the above-mentioned theorems can be extended in a more algebraic vein by devising generalizations of 'edge-disjoint' (e.g. 'size of the intersection of the supports even') and thus be made to resonate with results like Theorem 1.

Further context for Theorem 1 is provided by Table 1, and, in particular, by the following open conjecture (thirty years ago, Locke proved [47, Theorem 2 and Corollary 4] that Bondy's conjecture is true under the *additional* assumption of 'G non-Hamiltonian or $|G| \geqslant 4d-5$').

Conjecture 2 (*Bondy 1979; [32, p. 246] [47, Conjecture 1] [48, p. 256] [49, Conjecture 1] [7, Conjecture A] [2, p. 21] [3, p. 12]*)**.** *If $d \in \mathbb{Z}$, in every vertex-3-connected graph G with $|G| \geqslant 2d$ and $\delta(G) \geqslant d$, the set of all circuits of length at least $2d-1$ is an $\mathbb{F}_2$-generating system of $\mathrm{Z}_1(G;\mathbb{F}_2)$.*

The present paper gives an asymptotic answer for two special cases of Conjecture 2: if $\gamma > 0$, $|G|$ is sufficiently large, and '$\delta(G) \geqslant d$' is replaced by '$\delta(G) \geqslant (1+\gamma)d$', then (I2) in Theorem 1 says that if '$|G| \geqslant 2d$' holds as '$|G| = 2d$', Bondy's conclusion is true, and if '$|G| \geqslant 2d$' holds as '$|G| = 2d+1$', then (I1) in Theorem 1 says that of the three lengths $|G|-2$, $|G|-1$ and $|G|$ which Bondy allows as lengths

Table 1
Some aspects of Hamilton circuits in graphs with high minimum degree.

Aspects of Hamilton circuits	Literature
Efficient algorithms for finding a copy	[11, Section 4], [55]
Number of all copies	[56,21,20]
Number of mutually edge-disjoint copies	[52,53,42]
Host graph random	[9,38,8,39,45,41]
Linear algebraic properties, recombining them into shorter circuits	This paper

of generating circuits, $|G|$ alone is enough. It seems likely that with the techniques of this paper it will be possible to make further inroads towards the full Conjecture 2.

Structure of the paper

In Section 2 we develop a plan for proving Theorem 1, in the process introducing all the auxiliary statements that we will later draw upon. In Section 3, the plan is carried out in detail, in particular by giving proofs for all the auxiliary statements. Section 4 surveys the literature relevant to Theorem 1 and mentions open problems.

2. Outline of, and preparations for, the proof of Theorem 1

We adopt the conventions that a 2-set $\{v', v''\}$ can be abbreviated as $v'v''$, and that $\sqcup$ means $\cup$ and at the same time claims that the union is disjoint. By 'graph' we will mean 'finite simple undirected graph'. If G and H are graphs, then $H \hookrightarrow G$ means that there exists an injective graph homomorphism $H \to G$ (hence there is a subgraph of G isomorphic to H). A path of *length* (i.e. number of its edges) ℓ will be denoted by P_ℓ and a circuit of length ℓ by C_ℓ. As in [10] we reserve the word 'cycle' for the elements of $\mathrm{Z}_1(G; \mathbb{F}_2)$ and use the term 'circuit' for '2-regular connected graph'. For a graph G we will write $\mathrm{V}(G)$ for its vertex set, $\mathrm{E}(G)$ for its edge set, $|G| := |\mathrm{V}(G)|$ and $\|G\| := |\mathrm{E}(G)|$. If C is a circuit with $\mathrm{V}(C) = \{v_0, v_1, v_2, \ldots, v_{\ell-1}\}$ and $\mathrm{E}(C) = \{v_0v_1, v_1v_2, \ldots, v_{\ell-1}v_0\}$, then we abbreviate $v_0v_1v_2\cdots v_{\ell-1}v_0 := \mathrm{E}(C)$. A subgraph H of a graph G is called *non-separating* if and only if the graph $G - H := (\mathrm{V}(G) \setminus \mathrm{V}(H), \mathrm{E}(G) \setminus \{e \in \mathrm{E}(G): e \cap \mathrm{V}(H) \neq \varnothing\})$ is connected. A circuit C in a graph G is called *non-separating induced* if and only if C is non-separating and C has no chords in G (i.e. $\{e \in \mathrm{E}(G): e \subseteq \mathrm{V}(C)\} = \mathrm{E}(C)$). We write $\mathrm{c}_e \in (\mathbb{F}_2)^{\mathrm{E}(G)}$ for the unique map with $\mathrm{c}_e(e) = 1 \in \mathbb{F}_2$ and $\mathrm{c}_e(e') = 0 \in \mathbb{F}_2$ for every $e \neq e' \in \mathrm{E}(G)$. The *edge space* of a graph (cf. [24, p. 23]) of G, denoted $\mathrm{C}_1(G; \mathbb{F}_2)$, is the $\mathbb{F}_2$-linear span of $\{\mathrm{c}_e: e \in \mathrm{E}(G)\}$. The *cycle space* of a graph, denoted $\mathrm{Z}_1(G; \mathbb{F}_2)$, is the $\mathbb{F}_2$-linear span of all circuits in G. It is a vector space over $\mathbb{F}_2$ with $\dim_{\mathbb{F}_2} \mathrm{Z}_1(G; \mathbb{F}_2) = \|G\| - |G| + 1 =: \beta_1(G)$. The notation $\mathcal{H}(G)$ denotes the set of Hamilton circuits in G. For any set $\mathcal{M}$ of circuits in G we say that '$\mathcal{M}$ generates $\mathrm{Z}_1(G; \mathbb{F}_2)$' if and only if $\{\mathrm{c}_C: C \in \mathcal{M}\}$ is an $\mathbb{F}_2$-generating system of $\mathrm{Z}_1(G; \mathbb{F}_2)$, where c_C is defined as the element of $\mathrm{C}_1(G; \mathbb{F}_2)$ with support equal to $\mathrm{E}(C)$. A bipartite graph is called *balanced* if and only if its bipartition classes have equal size. If G and H are graphs, we denote by $G \,\square\, H$ the *Cartesian product* of G and H (see e.g. [34, Section 1.4]). If G is a graph, then we write $\mathrm{N}_G(v) := \{w \in \mathrm{V}(G): \{v, w\} \in \mathrm{E}(G)\}$ for every $v \in \mathrm{V}(G)$, $\delta(G) := \min_{v\in\mathrm{V}(G)} |\mathrm{N}_G(v)|$ (called *minimum degree* of G), and $\Delta(G) := \max_{v\in\mathrm{V}(G)} |\mathrm{N}_G(v)|$ (called *maximum degree* of G). By *k-connected* we mean the standard graph-theoretical notion of being 'vertex-k-connected' (cf. [24, Section 1.4]).

2.1. Plan of the proof of Theorem 1

The proof of Theorem 1 will be broken into the following steps (the strategy is the same for (I1)–(I4), but the auxiliary spanning subgraphs used are different):

(St1) Prove the existence of suitably chosen spanning subgraphs $H \hookrightarrow G$; for (I1) and (I2) by using Theorem 3, for (I3) by using Theorem 4, and for (I4) by using Theorem 5. These graphs H serve as 'rebar' in the construction performed in step (St3); they help to confer the desired properties to the host graph G.

(St2) Prove that in each case the subgraph H itself has its cycle space generated by its Hamilton circuits, and moreover that H is Hamilton-connected.[1]

(St3) By adapting a lemma of Locke [46, Lemma 1] argue that the properties proved in (St2) transfer from the subgraph H to the host graph G, thereby proving Theorem 1.

We now explain (St1)–(St3) in more detail.

2.1.1. Explanation of step (St1)

The theorems mentioned in (St1) are the following. As to terminology, the *square* H^2 of a graph H is the graph obtained from H by adding an edge between any two vertices having distance two in H. A graph H has *bandwidth at most b* if and only if there exists a bijection $\mathrm{b}\colon \mathrm{V}(H) \to \{1, \dots, |H|\}$ such that if $vv' \in \mathrm{E}(H)$, then $|\mathrm{b}(v) - \mathrm{b}(v')| \leqslant b$; any such bijection b is called a *bandwidth-b-labelling of H*. Moreover, if H is a graph, $\mathrm{b}\colon \mathrm{V}(H) \to \{1, \dots, |H|\}$ is a bijection and if $(c_1, c_2) \in \mathbb{Z}^2_{\geqslant 1}$ and $\rho \in \mathbb{Z}_{\geqslant 1}$, then a map $h\colon \mathrm{V}(H) \to \{0, \dots, \rho\}$ is called (c_1, c_2)*-zero-free w.r.t.* b (cf. [14, p. 178]) if and only if for every $v' \in \mathrm{V}(H)$ there exists a $v'' \in \mathrm{b}^{-1}\big(\{\mathrm{b}(v'), \mathrm{b}(v') + 1, \dots, \min(|H|, \mathrm{b}(v') + c_1)\}\big)$ such that $h(v''') \neq 0$ for every $v''' \in \mathrm{b}^{-1}\big(\{\mathrm{b}(v''), \mathrm{b}(v'') + 1, \dots, \min(|H|, \mathrm{b}(v'') + c_2)\}\big)$. As a tool for proving Theorem 1 we use the following.

Theorem 3 (*Böttcher–Schacht–Taraz [14, Theorem 2]*)**.** *For every $\gamma > 0$ and arbitrary $\rho \in \mathbb{Z}_{\geqslant 2}$ and $\Delta \in \mathbb{Z}_{\geqslant 2}$ there exist numbers $\beta = \beta(\gamma, \Delta) > 0$ and $n_0 = n_0(\gamma, \Delta)$ such that the following is true: for every graph G with $|G| \geqslant n_0$ and $\delta(G) \geqslant (\frac{\rho-1}{\rho} + \gamma)|G|$, and for every graph H having $|G| = |H|$, $\Delta(H) \leqslant \Delta$ and $\mathrm{bw}(H) \leqslant \beta|H|$, and admitting a bandwidth-$\beta|H|$-labelling* $\mathrm{b}\colon \mathrm{V}(H) \to \{1, \dots, |H|\}$ *and a $(\rho + 1)$-colouring $h\colon \mathrm{V}(H) \to \{0, 1, \dots, \rho\}$ which is $\big(8\rho\beta|H|, 4\rho\beta|H|\big)$-zero-free w.r.t.* b *and has $|h^{-1}(0)| \leqslant \beta|H|$, there is an embedding $H \hookrightarrow G$.* □

Theorem 4 (*Böttcher–Heinig–Taraz [12, Theorem 3]*)**.** *For every $\gamma > 0$ and every $\Delta \in \mathbb{Z}$ there exist numbers $\beta = \beta(\gamma, \Delta) > 0$ and $n_0 = n_0(\gamma, \Delta) \in \mathbb{Z}$ such that the following is true: for every balanced bipartite graph G with $|G| \geqslant n_0$ and $\delta(G) \geqslant (\frac{1}{4} + \gamma)|G|$, and for every balanced bipartite graph H with $|H| = |G|$, $\Delta(H) \leqslant \Delta$ and $\mathrm{bw}(H) \leqslant \beta|H|$, there is an embedding $H \hookrightarrow G$.* □

Moreover, the lower bound of terrestrial magnitude that is provided in (I4) depends on a recent theorem of Châu, DeBiasio and Kierstead (who say [17, p. 17, Section 5, l. 5] that by optimizing their proof one may not push the bound further down than to about $n_0 = 10^5$, but who express optimism as to the possibility of getting rid of the lower bound on $|\cdot|$ altogether).

Theorem 5 (*Komlós–Sárközy–Szemerédi [40, Theorem 1], Jamshed [36, Chapter 3]; explicit lower bound on $|G|$ proved by Châu–DeBiasio–Kierstead [17, Theorem 7]*)**.** *For every graph G with $|G| \geqslant 2 \cdot 10^8$ and $\delta(G) \geqslant \frac{2}{3}|G|$ there exists an embedding $\mathrm{C}^2_{|G|} \hookrightarrow G$.* □

Whereas for (I4) our use of Theorem 5 dictates employing $\mathrm{C}^2_{|\cdot|}$ as the auxiliary subgraph, there are choices to be made as to what subgraph to employ from the set of spanning subgraphs offered by Theorems 3 and 4. We will choose to use the following graphs (in Definition 6 let $b_r := b_0$).

Definition 6 (*Bipartite cyclic ladder*)**.** For $r \in \mathbb{Z}_{\geqslant 3}$ let CL_r be the bipartite graph with $\mathrm{V}(\mathrm{CL}_r) := \{a_0, \dots, a_{r-1}\} \sqcup \{b_0, \dots, b_{r-1}\}$ and $\mathrm{E}(\mathrm{CL}_r) := \bigsqcup_{i=0}^{r-1}\{a_i b_{i-1}\} \sqcup \bigsqcup_{i=0}^{r-1}\{a_i b_i\} \sqcup \bigsqcup_{i=0}^{r-1}\{a_i b_{i+1}\}$.

Definition 7 (*Prism, Möbius ladder*)**.** For every $n \geqslant 3$ and $r \geqslant 3$ let (where $v_n := v_0, x_r := x_0$ and $y_r := y_0$) the *prism* Pr_r be defined by $\mathrm{V}(\mathrm{Pr}_r) := \{x_0, \dots, x_{r-1}, y_0, \dots, y_{r-1}\}$ and $\mathrm{E}(\mathrm{Pr}_r) :=$

[1] The weaker property 'any two *non-adjacent* vertices are connected by a Hamilton path' would suffice here, but we will work with the better-known property of being Hamilton-connected.

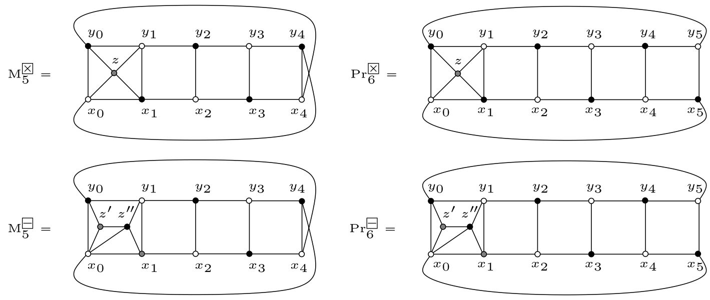


Fig. 1. The graphs $\mathrm{M}_r^{\boxtimes}$ and $\mathrm{M}_r^{\boxminus}$ for odd r, and $\mathrm{Pr}_r^{\boxtimes}$ and $\mathrm{Pr}_r^{\boxminus}$ for even r play a key role in the proof. These are bounded-degree, bounded-bandwidth and 3-chromatic graphs admitting a 3-colouring with a constant-sized third colour class. The bandwidth theorem of Böttcher, Schacht and Taraz, in its full form [14, Theorem 2], is sufficiently general to guarantee the existence of embeddings of these graphs as *spanning* subgraphs into graphs G with $\delta(G) \geqslant (\frac{1}{2}+\gamma)|G|$. If $\mathrm{M}_r^{\boxtimes}$ or $\mathrm{Pr}_r^{\boxtimes}$ spanningly embed into G, this implies that $\mathrm{Z}_1(G;\mathbb{F}_2)$ is generated by Hamilton circuits. If $\mathrm{M}_r^{\boxminus}$ or $\mathrm{Pr}_r^{\boxminus}$ spanningly embed into G, this implies that $\mathrm{Z}_1(G;\mathbb{F}_2)$ is generated by the circuits having lengths in $\{|G|-1, |G|\}$. If the edge x_0z'' were omitted from $\mathrm{M}_r^{\boxminus}$ or $\mathrm{Pr}_r^{\boxminus}$, the remaining graph could no longer serve the purpose these graphs have in the present paper.

$\bigsqcup_{i=0}^{r-1}\{x_ix_{i+1}\} \sqcup \bigsqcup_{i=0}^{r-1}\{y_iy_{i+1}\} \sqcup \bigsqcup_{i=0}^{r-1}\{x_iy_i\}$, and the *Möbius ladder* M_r be defined by $\mathrm{V}(\mathrm{M}_r) := \mathrm{V}(\mathrm{Pr}_r)$ and $\mathrm{E}(\mathrm{M}_r) := \big(\mathrm{E}(\mathrm{Pr}_r) \setminus \{x_0x_{r-1},\ y_0y_{r-1}\}\big) \sqcup \{x_0y_{r-1},\ y_0x_{r-1}\}$.

Definition 8 ($\mathrm{Pr}_r^{\boxtimes}$ *and* $\mathrm{M}_r^{\boxtimes}$)**.** For every $r \geqslant 3$ let $\mathrm{Pr}_r^{\boxtimes}$ be defined by $\mathrm{V}(\mathrm{Pr}_r^{\boxtimes}) := \mathrm{V}(\mathrm{Pr}_r) \sqcup \{z\}$, with z some new element, and $\mathrm{E}(\mathrm{Pr}_r^{\boxtimes}) := \mathrm{E}(\mathrm{Pr}_r) \sqcup \{zx_0,\ zy_0,\ zx_1,\ zy_1\}$. Let $\mathrm{M}_r^{\boxtimes}$ be defined by $\mathrm{V}(\mathrm{M}_r^{\boxtimes}) := \mathrm{V}(\mathrm{Pr}_r^{\boxtimes})$ and $\mathrm{E}(\mathrm{M}_r^{\boxtimes}) := (\mathrm{E}(\mathrm{Pr}_r^{\boxtimes}) \setminus \{x_0x_{r-1},\ y_0y_{r-1}\}) \sqcup \{x_0y_{r-1},\ y_0x_{r-1}\}$.

Definition 9 ($\mathrm{Pr}_r^{\boxminus}$ *and* $\mathrm{M}_r^{\boxminus}$)**.** For every $r \geqslant 3$ let $\mathrm{Pr}_r^{\boxminus}$ be defined by $\mathrm{V}(\mathrm{Pr}_r^{\boxminus}) := \mathrm{V}(\mathrm{Pr}_r) \sqcup \{z', z''\}$ with z' and z'' two new elements, $\mathrm{E}(\mathrm{Pr}_r^{\boxminus}) := \mathrm{E}(\mathrm{Pr}_r) \sqcup \{x_0z',\ y_0z',\ x_0z'',\ x_1z'',\ y_1z'',\ z'z''\}$. Let $\mathrm{M}_r^{\boxminus}$ be defined by $\mathrm{V}(\mathrm{M}_r^{\boxminus}) := \mathrm{V}(\mathrm{Pr}_r^{\boxminus})$ and $\mathrm{E}(\mathrm{M}_r^{\boxminus}) := \big(\mathrm{E}(\mathrm{Pr}_r^{\boxminus}) \setminus \{x_0x_{r-1}, y_0y_{r-1}\}\big) \sqcup \{x_0y_{r-1}, y_0x_{r-1}\}$.

Justifying that CL_r is indeed one of the subgraphs guaranteed by Theorem 4 will pose no difficulty and can be done uniformly for every $r \in \mathbb{Z}_{\geqslant 3}$. Matters are being complicated by parity issues when it comes to step (St2). We will later make essential use of the following sets (for each of the circuits C in these sets, the reader may use Fig. 1 to visualize C).

Definition 10. For every even $r \geqslant 4$ we define the sets of edge sets

$$(\mathrm{P}.\boxtimes.\mathrm{ES}.1)\ \ \mathcal{CB}^{(1)}_{\mathrm{Pr}_r^{\boxtimes}} := \left\{\begin{array}{l} C_{\mathrm{ev},r,1} := zy_1x_1x_2y_2y_3\cdots x_{r-2}y_{r-2}y_{r-1}x_{r-1}x_0y_0z,\\ C_{\mathrm{ev},r,2} := zx_1x_2y_2y_3\cdots x_{r-2}y_{r-2}y_{r-1}x_{r-1}x_0y_0y_1z,\\ C_{\mathrm{ev},r,3} := zx_1y_1y_2x_2x_3\cdots x_{r-2}x_{r-1}y_{r-1}y_0x_0z,\\ C_{\mathrm{ev},r,4} := zx_0x_1y_1y_2\cdots y_{r-3}y_{r-2}x_{r-2}x_{r-1}y_{r-1}y_0z,\\ C_{\mathrm{ev},r,5} := zy_1y_2x_2x_3\cdots x_{r-2}x_{r-1}y_{r-1}y_0x_0x_1z \end{array}\right\},$$

$$(\mathrm{P}.\boxtimes.\mathrm{ES}.2)\ \ \mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxtimes}} := \left\{\begin{array}{c} C^{x_1y_1}_{\mathrm{ev},r} := zx_0x_{r-1}x_{r-2}\cdots x_2y_2y_3\cdots y_{r-1}y_0y_1x_1z,\\ C^{x_2y_2}_{\mathrm{ev},r} := zx_0x_{r-1}x_{r-2}\cdots x_3y_3y_4\cdots y_{r-1}y_0y_1y_2x_2x_1z,\\ \vdots\\ C^{x_{r-2}y_{r-2}}_{\mathrm{ev},r} := zx_0x_{r-1}y_{r-1}y_0y_1\cdots y_{r-2}x_{r-2}x_{r-3}\cdots x_1z,\\ C^{x_{r-1}y_{r-1}}_{\mathrm{ev},r} := zx_0x_1\cdots x_{r-1}y_{r-1}y_{r-2}\cdots y_0z \end{array}\right\}.$$

Let us note that $C^{x_{r-1}y_{r-1}}_{\mathrm{ev},r}$ does not follow the pattern to be found in $C^{x_1y_1}_{\mathrm{ev},r}, \ldots, C^{x_{r-2}y_{r-2}}_{\mathrm{ev},r}$.

Definition 11. For every odd $r \geqslant 5$ we define the sets of edge sets

$$(\mathrm{M}.\boxtimes.\mathrm{ES}.1)\ \ \mathcal{CB}^{(1)}_{\mathrm{M}_r^{\boxtimes}} := \left\{\begin{array}{l} C_{\mathrm{od},r,1} := zy_1x_1x_2y_2y_3\cdots y_{r-2}x_{r-2}x_{r-1}y_{r-1}x_0y_0z,\\ C_{\mathrm{od},r,2} := zx_1x_2y_2y_3\cdots y_{r-2}x_{r-2}x_{r-1}y_{r-1}x_0y_0y_1z,\\ C_{\mathrm{od},r,3} := zx_1y_1y_2x_2x_3\cdots y_{r-2}y_{r-1}x_{r-1}y_0x_0z,\\ C_{\mathrm{od},r,4} := zx_0x_1y_1y_2\cdots x_{r-3}x_{r-2}y_{r-2}y_{r-1}x_{r-1}y_0z,\\ C_{\mathrm{od},r,5} := zy_1y_2x_2x_3\cdots y_{r-2}y_{r-1}x_{r-1}y_0x_0x_1z \end{array}\right\},$$

$$(\mathrm{M}.\boxtimes.\mathrm{ES}.2)\ \mathcal{CB}^{(2)}_{\mathrm{M}_r^{\boxtimes}} := \left\{\begin{array}{r} C^{x_1y_1}_{\mathrm{od},r} := zx_0y_{r-1}y_{r-2}\cdots y_2x_2x_3\cdots x_{r-1}y_0y_1x_1z, \\ C^{x_2y_2}_{\mathrm{od},r} := zx_0y_{r-1}y_{r-2}\cdots y_3x_3x_4\cdots x_{r-1}y_0y_1y_2x_2x_1z, \\ \vdots \\ C^{x_{r-2}y_{r-2}}_{\mathrm{od},r} := zx_0y_{r-1}x_{r-1}y_0y_1\cdots y_{r-2}x_{r-2}x_{r-3}\cdots x_1z, \\ C^{x_{r-1}y_{r-1}}_{\mathrm{od},r} := zx_0x_1\cdots x_{r-1}y_{r-1}y_{r-2}\cdots y_0z \end{array}\right\}.$$

Let us note that $C^{x_{r-1}y_{r-1}}_{\mathrm{od},r}$ does not conform to the pattern to be found in $C^{x_1y_1}_{\mathrm{od},r}, \ldots, C^{x_{r-2}y_{r-2}}_{\mathrm{od},r}$.

Definition 12. For every even $r \geqslant 4$ we define the sets of edge sets

$$(\mathrm{P}.\boxminus.\mathrm{ES}.1)\ \mathcal{CB}^{(1)}_{\mathrm{Pr}_r^{\boxminus}} := \left\{\begin{array}{l} C_{\boxminus,\mathrm{ev},r,1} := z'x_0z''x_1x_2\cdots x_{r-1}y_{r-1}y_{r-2}\cdots y_0z', \\ C_{\boxminus,\mathrm{ev},r,2} := z'z''x_0x_{r-1}x_{r-2}\cdots x_1y_1y_2\cdots y_{r-1}y_0z', \\ C_{\boxminus,\mathrm{ev},r,3} := z'x_0z''x_1y_1y_2x_2x_3\cdots x_{r-2}x_{r-1}y_{r-1}y_0z', \\ C_{\boxminus,\mathrm{ev},r,4} := z'z''x_1x_2\cdots x_{r-1}y_{r-1}y_{r-2}\cdots y_0x_0z', \\ C_{\boxminus,\mathrm{ev},r,5} := z'x_0x_{r-1}y_{r-1}y_{r-2}x_{r-2}x_{r-3}\cdots x_2x_1z''y_1y_0z' \end{array}\right\},$$

$$(\mathrm{P}.\boxminus.\mathrm{ES}.2)\ \mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxminus}} := \left\{\begin{array}{r} C^{x_1y_1}_{\boxminus,\mathrm{ev},r} := z'x_0x_{r-1}x_{r-2}\cdots x_2y_2y_3\cdots y_{r-1}y_0y_1x_1z''z', \\ C^{x_2y_2}_{\boxminus,\mathrm{ev},r} := z'x_0x_{r-1}x_{r-2}\cdots x_3y_3y_4\cdots y_{r-1}y_0y_1y_2x_2x_1z''z', \\ \vdots \\ C^{x_{r-2}y_{r-2}}_{\boxminus,\mathrm{ev},r} := z'x_0x_{r-1}y_{r-1}y_0y_1\cdots y_{r-2}x_{r-2}x_{r-3}\cdots x_1z''z', \\ C^{x_{r-1}y_{r-1}}_{\boxminus,\mathrm{ev},r} := z'z''x_0x_1\cdots x_{r-1}y_{r-1}y_{r-2}\cdots y_0z' \end{array}\right\}.$$

Definition 13. For every odd $r \geqslant 5$ we define the sets of edge sets

$$(\mathrm{M}.\boxminus.\mathrm{ES}.1)\ \mathcal{CB}^{(1)}_{\mathrm{M}_r^{\boxminus}} := \left\{\begin{array}{l} C_{\boxminus,\mathrm{od},r,1} := z'x_0z''x_1x_2\cdots x_{r-1}y_{r-1}y_{r-2}\cdots y_0z' = C_{\boxminus,\mathrm{ev},r,1}, \\ C_{\boxminus,\mathrm{od},r,2} := z'z''x_0y_{r-1}y_{r-2}\cdots y_1x_1x_2\cdots x_{r-1}x_0z', \\ C_{\boxminus,\mathrm{od},r,3} := z'x_0z''x_1y_1y_2x_2x_3y_3\cdots y_{r-2}y_{r-1}x_{r-1}x_0z', \\ C_{\boxminus,\mathrm{od},r,4} := z'z''x_1x_2\cdots x_{r-1}y_{r-1}y_{r-2}\cdots y_0x_0z' = C_{\boxminus,\mathrm{ev},r,4}, \\ C_{\boxminus,\mathrm{od},r,5} := z'x_0y_{r-1}x_{r-1}x_{r-2}y_{r-2}y_{r-3}\cdots x_2x_1z''y_1y_0z' \end{array}\right\},$$

$$(\mathrm{M}.\boxminus.\mathrm{ES}.2)\ \mathcal{CB}^{(2)}_{\mathrm{M}_r^{\boxminus}} := \left\{\begin{array}{r} C^{x_1y_1}_{\boxminus,\mathrm{od},r} := z'x_0y_{r-1}y_{r-2}\cdots y_2x_2x_3\cdots x_{r-1}y_0y_1x_1z''z', \\ C^{x_2y_2}_{\boxminus,\mathrm{od},r} := z'x_0y_{r-1}y_{r-2}\cdots y_3x_3x_4\cdots x_{r-1}y_0y_1y_2x_2x_1z''z', \\ \vdots \\ C^{x_{r-2}y_{r-2}}_{\boxminus,\mathrm{od},r} := z'x_0y_{r-1}x_{r-1}y_0y_1\cdots y_{r-2}x_{r-2}x_{r-3}\cdots x_1z''z', \\ C^{x_{r-1}y_{r-1}}_{\boxminus,\mathrm{od},r} := z'z''x_0x_1\cdots x_{r-1}y_{r-1}y_{r-2}\cdots y_0z' = C^{x_{r-1}y_{r-1}}_{\boxminus,\mathrm{ev},r} \end{array}\right\}.$$

2.1.2. Explanation of step (St2)

If $\mathcal{A}$ is a finite abelian group in additive notation, and $0 \notin S \subseteq \mathcal{A}$ has the property that $-S := \{-s: s \in S\} = S$, then we write $\langle S\rangle := \sum_{s\in S}\mathbb{Z}s$ for the abelian group generated by S and define a graph $G := \mathrm{Cay}(\langle S\rangle; S)$ by $\mathrm{V}(G) := \langle S\rangle$ and $\{a, b\} \in \mathrm{E}(G) :\Leftrightarrow a - b \in S$, called the *Cayley graph* associated to $\mathcal{A}$ and S. The following theorem of Chen and Quimpo has proved to be fertile for the theory of Cayley graphs on finite abelian groups.

Theorem 14 (*Chen–Quimpo; [18, Theorem 4] gives the non-bipartite case.*[2]). *For every finite abelian group $\mathcal{A}$ and every $S \subseteq \mathcal{A}$ with $-S = S$ and $|S| \geqslant 3$ the graph $G = \mathrm{Cay}(\langle S\rangle; S)$ is Hamilton-connected in case G is not bipartite, and Hamilton-laceable in case G is bipartite.* □

We will use the following theorem of Alspach, Locke and Witte which appears to be the first result in the literature dealing with linear algebraic properties of Hamilton circuits (as to terminology, a graph G is called a *prism over the graph H* if and only if $G \cong H \,\square\, \mathrm{P}_1$).

[2] The bipartite case appears to be susceptible to analogous arguments as in [18]. The author does not know of any published proof of the bipartite case. Nevertheless, it is mentioned in [6, Theorem 1.4], [4, Theorem 1.7], [51, Introductory Remarks and Proposition 2.1] and [50, Proposition 3]. Moreover, what little we need of the general bipartite case, namely Lemma 17.(a14), can be easily shown directly.

Theorem 15 (*Alspach–Locke–Witte [5, Theorem 2.1 and Corollary 2.3]*). *For every finite abelian group $\mathcal{A}$ and every $0 \notin S \subseteq \mathcal{A}$ with $-S = S$ the graph $G := \mathrm{Cay}(\langle S\rangle; S)$ has the following properties:*

(1) *if G is bipartite, then $\mathcal{H}(G)$ generates $\mathrm{Z}_1(G; \mathbb{F}_2)$,*
(2) *if $|G| = |\langle S\rangle|$ is odd, then $\mathcal{H}(G)$ generates $\mathrm{Z}_1(G; \mathbb{F}_2)$,*
(3) *if $|G| = |\langle S\rangle|$ is even and G is not bipartite and not a prism over any circuit of odd length, then $\dim_{\mathbb{F}_2}\big(\mathrm{Z}_1(G; \mathbb{F}_2)/\langle\mathcal{H}(G)\rangle_{\mathbb{F}_2}\big) = 1$.* □

To succinctly formulate properties of the auxiliary substructures, we introduce notation.

Definition 16. Let $\mathfrak{L}$ be a map from graphs to subsets of $\mathbb{Z}_{\geqslant 1}$, let $\mathfrak{L}-1 := \{l-1 : l \in \mathfrak{L}\}$ and let $\xi \in \mathbb{Z}_{\geqslant 0}$. We define

(1) a graph G to be $\mathfrak{L}$-*path-connected* (if $\mathfrak{L} = \{|\cdot|-1\}$ we speak of being *Hamilton-connected*) if and only if for every $\{v, w\} \in \binom{\mathrm{V}(G)}{2}$ there exists in G at least one v–w-path having its length in the set $\mathfrak{L}(G)$ (we denote the collection of all such graphs by $\mathcal{CO}_{\mathfrak{L}}$),
(2) a variant of $\mathcal{CO}_{\mathfrak{L}}$ for bipartite graphs: adopting a by now widespread usage dating back at least to work of Simmons [57], a bipartite graph G will be called $\mathfrak{L}$-*laceable* (if $\mathfrak{L} = \{|\cdot|-1\}$ also *Hamilton-laceable*) if and only if for any two $v, w \in \mathrm{V}(G)$ not in the same bipartition class there exists at least one v–w-path having its length in the set $\mathfrak{L}(G)$ (we denote the collection of all such graphs by $\mathcal{LA}_{\mathfrak{L}}$),
(3) for a graph G the set $\mathcal{C}_{\mathfrak{L}}(G)$ as the set of all graph-theoretical circuits in G whose length is an element of $\mathfrak{L}$. (In particular, $\mathcal{C}_{\{|G|\}}(G) = \mathcal{H}(G)$.)
(4) $\mathrm{cd}_\xi\mathcal{C}_{\mathfrak{L}}$ as the collection of graphs G with $\dim_{\mathbb{F}_2}\big(\langle\mathcal{C}_{\mathfrak{L}}(G)\rangle_{\mathbb{F}_2}\big) = \beta_1(G) - \xi$,
(5) $\mathrm{bcd}_\xi\mathcal{C}_{\mathfrak{L}} \subseteq \mathrm{cd}_\xi\mathcal{C}_{\mathfrak{L}}$ as the collection of all the *bipartite* elements of $\mathrm{cd}_\xi\mathcal{C}_{\mathfrak{L}}$,
(6) $\mathcal{M}_{\mathfrak{L},\xi} := \mathrm{cd}_\xi\mathcal{C}_{\mathfrak{L}} \cap \mathcal{CO}_{\mathfrak{L}-1}$ and $\mathrm{b}\mathcal{M}_{\mathfrak{L},\xi} := \mathrm{bcd}_\xi\mathcal{C}_{\mathfrak{L}} \cap \mathcal{LA}_{\mathfrak{L}-1}$.

The condition in (4) is equivalent to $\dim_{\mathbb{F}_2}\big(\mathrm{Z}_1(G; \mathbb{F}_2)/\langle\mathcal{C}_{\mathfrak{L}(G)}(G)\rangle_{\mathbb{F}_2}\big) = \xi$, in other words, $\mathrm{cd}_\xi\mathcal{C}_{\mathfrak{L}}(G)$ is the set of all graphs for which $\langle\mathcal{C}_{\mathfrak{L}(G)}(G)\rangle_{\mathbb{F}_2}$ has codimension ξ in $\mathrm{Z}_1(G; \mathbb{F}_2)$. In particular $\mathrm{cd}_0\mathcal{C}_{\{|\cdot|\}}(G)$ is the set of all graphs whose cycle space is generated by the set of their Hamilton circuits. We will now formulate all the properties of the auxiliary spanning substructures that we use in the proof.

Lemma 17 (*Properties of the auxiliary structures*). *For every $n \geqslant 5$ and every $r \in \mathbb{Z}_{\geqslant 4}$,*

(a1) $\mathrm{C}_n^2 \cong \mathrm{Cay}(\mathbb{Z}/n; \{1, 2, n-2, n-1\})$,
(a2) C_n^2 *is not a prism over a graph (i.e. there does not exist H with $\mathrm{C}_n^2 \cong H \,\square\, \mathrm{P}_1$),*
(a3) *if n is even, then $\mathrm{C}_n^2 \in \mathcal{M}_{\{|\cdot|\},1}$,*
(a4) *if n is odd, then $\mathrm{C}_n^2 \in \mathcal{M}_{\{|\cdot|\},0}$,*
(a5) *if n is even, then $\mathrm{C}_n^2 \in \mathcal{M}_{\{|\cdot|-1,|\cdot|\},0}$,*
(a6) $\mathrm{Pr}_r \cong \mathrm{Cay}(\mathbb{F}_2 \oplus \mathbb{Z}/r; \{(1, 0), (0, 1), (0, r-1)\})$,
(a7) $\mathrm{M}_r \cong \mathrm{Cay}(\mathbb{Z}/(2r); \{1, r, 2r-1\})$,
(a8) *if r is even, then $\mathrm{Pr}_r \in \mathcal{LA}_{|\cdot|-1}$,*
(a9) *if r is odd, then $\mathrm{M}_r \in \mathcal{LA}_{|\cdot|-1}$,*
(a10) *if r is even, then $\mathrm{Pr}_r \in \mathrm{b}\mathcal{M}_{\{|\cdot|\},0}$,*
(a11) *if r is odd, then $\mathrm{M}_r \in \mathrm{b}\mathcal{M}_{\{|\cdot|\},0}$,*
(a12) *if r is even, then $\mathrm{CL}_r \cong \mathrm{Pr}_r$,*
(a13) *if r is odd, then $\mathrm{CL}_r \cong \mathrm{M}_r$,*
(a14) $\mathrm{CL}_r \in \mathcal{LA}_{|\cdot|-1}$,
(a15) $\mathrm{CL}_r \in \mathrm{b}\mathcal{M}_{\{|\cdot|\},0}$,
(a16) *if r is even, then $\mathrm{Pr}_r^{\boxtimes} \in \mathcal{CO}_{\{|\cdot|-1\}}$,*
(a17) *if r is odd, then $\mathrm{M}_r^{\boxtimes} \in \mathcal{CO}_{\{|\cdot|-1\}}$,*
(a18) *if r is even, then $\mathrm{Pr}_r^{\boxminus} \in \mathcal{CO}_{\{|\cdot|-1\}}$,*
(a19) *if r is odd, then $\mathrm{M}_r^{\boxminus} \in \mathcal{CO}_{\{|\cdot|-1\}}$,*
(a20) *concerning $\mathrm{Pr}_r^{\boxtimes}$ and $\mathrm{Pr}_r^{\boxminus}$ for even r, and concerning $\mathrm{M}_r^{\boxtimes}$ and $\mathrm{M}_r^{\boxminus}$ for odd r, the set $\{\mathrm{c}_C : C \in \mathcal{CB}_G^{(1)}\}$ is a linearly independent subset of $\mathrm{Z}_1(G; \mathbb{F}_2)$ for all $G \in \{\mathrm{Pr}_r^{\boxtimes}, \mathrm{Pr}_r^{\boxminus}, \mathrm{M}_r^{\boxtimes}, \mathrm{M}_r^{\boxminus}\}$,*

(a21) *concerning* $\mathrm{Pr}_r^{\boxtimes}$ *and* $\mathrm{Pr}_r^{\boxminus}$ *for even* r, *and concerning* $\mathrm{M}_r^{\boxtimes}$ *and* $\mathrm{M}_r^{\boxminus}$ *for odd* r, *the set* $\{\mathrm{c}_C : C \in \mathcal{C}\mathcal{B}_G^{(2)}\}$ *is a linearly independent subset of* $\mathrm{Z}_1(G; \mathbb{F}_2)$ *for all* $G \in \{\mathrm{Pr}_r^{\boxtimes}, \mathrm{Pr}_r^{\boxminus}, \mathrm{M}_r^{\boxtimes}, \mathrm{M}_r^{\boxminus}\}$,

(a22) *concerning* $\mathrm{Pr}_r^{\boxtimes}$ *and* $\mathrm{Pr}_r^{\boxminus}$ *for even* $r \geqslant 4$, *and concerning* $\mathrm{M}_r^{\boxtimes}$ *and* $\mathrm{M}_r^{\boxminus}$ *for odd* $r \geqslant 5$, *the sum* $\left\langle \mathcal{C}\mathcal{B}_G^{(1)} \right\rangle_{\mathbb{F}_2} + \left\langle \mathcal{C}\mathcal{B}_G^{(2)} \right\rangle_{\mathbb{F}_2} \subseteq \mathrm{C}_1(G; \mathbb{F}_2)$ *is direct for all* $G \in \{\mathrm{Pr}_r^{\boxtimes}, \mathrm{Pr}_r^{\boxminus}, \mathrm{M}_r^{\boxtimes}, \mathrm{M}_r^{\boxminus}\}$,

(a23) *concerning* $\mathrm{Pr}_r^{\boxtimes}$ *and* $\mathrm{Pr}_r^{\boxminus}$ *for even* r, *and concerning* $\mathrm{M}_r^{\boxtimes}$ *and* $\mathrm{M}_r^{\boxminus}$ *for odd* r,

- ($\boxtimes$.(0)) $\langle \mathcal{H}(\mathrm{Pr}_r^{\boxtimes}) \rangle_{\mathbb{F}_2} = \mathrm{Z}_1(\mathrm{Pr}_r^{\boxtimes}; \mathbb{F}_2)$,
- ($\boxtimes$.(1)) $\langle \mathcal{H}(\mathrm{M}_r^{\boxtimes}) \rangle_{\mathbb{F}_2} = \mathrm{Z}_1(\mathrm{M}_r^{\boxtimes}; \mathbb{F}_2)$,
- ($\boxminus$.(0)) $\dim_{\mathbb{F}_2}\big(\mathrm{Z}_1(\mathrm{Pr}_r^{\boxminus}; \mathbb{F}_2) / \langle \mathcal{H}(\mathrm{Pr}_r^{\boxminus}) \rangle_{\mathbb{F}_2}\big) = 1$,
- ($\boxminus$.(1)) $\dim_{\mathbb{F}_2}\big(\mathrm{Z}_1(\mathrm{M}_r^{\boxminus}; \mathbb{F}_2) / \langle \mathcal{H}(\mathrm{M}_r^{\boxminus}) \rangle_{\mathbb{F}_2}\big) = 1$,
  - ($\boxminus$.$|\cdot| - 1$.(0)) $\langle \mathcal{C}_{\{|\cdot|-1,|\cdot|\}}(\mathrm{Pr}_r^{\boxminus}) \rangle_{\mathbb{F}_2} = \mathrm{Z}_1(\mathrm{Pr}_r^{\boxminus}; \mathbb{F}_2)$,
  - ($\boxminus$.$|\cdot| - 1$.(1)) $\langle \mathcal{C}_{\{|\cdot|-1,|\cdot|\}}(\mathrm{M}_r^{\boxminus}) \rangle_{\mathbb{F}_2} = \mathrm{Z}_1(\mathrm{M}_r^{\boxminus}; \mathbb{F}_2)$,

(a24) *if* r *is even, then* $\mathrm{Pr}_r^{\boxtimes} \in \mathcal{M}_{\{|\cdot|\},0}$,

(a25) *if* r *is odd, then* $\mathrm{M}_r^{\boxtimes} \in \mathcal{M}_{\{|\cdot|\},0}$,

(a26) *if* r *is even, then* $\mathrm{Pr}_r^{\boxminus} \in \mathcal{M}_{\{|\cdot|\},1}$,

(a27) *if* r *is odd, then* $\mathrm{M}_r^{\boxminus} \in \mathcal{M}_{\{|\cdot|\},1}$,

(a28) *if* r *is even, then* $\mathrm{Pr}_r^{\boxminus} \in \mathcal{M}_{\{|\cdot|-1,|\cdot|\},0}$,

(a29) *if* r *is odd, then* $\mathrm{M}_r^{\boxminus} \in \mathcal{M}_{\{|\cdot|-1,|\cdot|\},0}$,

(a30) *for every* $\beta > 0$ *there exists* $n_0 = n_0(\beta) \in \mathbb{Z}$ *such that—in case of* $\mathrm{Pr}_r^{\boxtimes}$ *and* $\mathrm{Pr}_r^{\boxminus}$ *for even* r *while in case of* $\mathrm{M}_r^{\boxtimes}$ *and* $\mathrm{M}_r^{\boxminus}$ *for odd* r*—if* $H \in \{C_n^2, \mathrm{CL}_r, \mathrm{Pr}_r^{\boxtimes}, \mathrm{Pr}_r^{\boxminus}, \mathrm{M}_r^{\boxtimes}, \mathrm{M}_r^{\boxminus}\}$ *and* $|H| \geqslant n_0$, *the following is true: the bandwidth satisfies* $\mathrm{bw}(H) \leqslant \beta \cdot |H|$, *and moreover for each* $H \in \{\mathrm{Pr}_r^{\boxtimes}, \mathrm{Pr}_r^{\boxminus}, \mathrm{M}_r^{\boxtimes}, \mathrm{M}_r^{\boxminus}\}$ *there exists a bijection* $\mathrm{b}_H : \mathrm{V}(H) \to \{1, \ldots, |H|\}$ *and a map* $\mathrm{h}_H : \mathrm{V}(H) \to \{0, 1, 2\}$ *such that* b_H *is a bandwidth-*$\beta|H|$*-labelling and* h_H *a* 3*-colouring of* H, *and* h_H *has* $|\mathrm{h}_H^{-1}(0)| \leqslant \beta|H|$ *and is* $\big(8 \cdot 2 \cdot \beta \cdot |H|, 4 \cdot 2 \cdot \beta \cdot |H|\big)$*-zero-free w.r.t.* b_H.

There are arbitrary choices to be made when proving Lemma 17. Let us especially mention that there are three different feasible strategies for proving (a15):

(A1) Realize CL_r as a Cayley graph on a finite abelian group. Then cite a theorem of Alspach, Locke and Witte which implies that $\mathrm{Z}_1(\mathrm{CL}_r; \mathbb{F}_2)$ is generated by Hamilton circuits.

(A2) Determine the full set of non-separating induced circuits of CL_r, then realize every single such circuit as an $\mathbb{F}_2$-sum of Hamilton circuits of CL_r and then appeal to a theorem of Tutte [58, Statement (2.5)] [24, Theorem 3.2.3] which states that in a 3-connected graph G the cycle space $\mathrm{Z}_1(G; \mathbb{F}_2)$ is generated by the set of all non-separating induced circuits.

(A3) Exhibit sufficiently many explicit Hamilton circuits of CL_r so that after choosing some basis the matrix of these circuits has $\mathbb{F}_2$-rank equal to $\dim_{\mathbb{F}_2} \mathrm{Z}_1(\mathrm{CL}_r; \mathbb{F}_2)$. It then follows that $\mathrm{Z}_1(\mathrm{CL}_r; \mathbb{F}_2) = \langle \mathcal{H}(\mathrm{CL}_r) \rangle_{\mathbb{F}_2}$, since in a vector space, a maximal linearly independent subset is a generating system.

Each of (A1)–(A3) demands attention to the parity of r, for despite a superficial similarity, the sets of circuits in CL_r for odd and even r turn out to be quite different. A positive way to look at this is as helping to decide which of (A1)–(A3) to choose. While each argument can be used for each parity of r, there are some reasons to choose (A2) for even r. The reason is a *trade-off between being a circulant graph (i.e. a Cayley graph on a finite cyclic group) and being a planar graph:* if r is even, then it can be shown that CL_r is not isomorphic to any Cayley graph on a cyclic group, whereas when r is odd, CL_r *is* a circulant graph. In return, CL_r is *planar* if and only if r is even, and this facilitates (A2): when it comes to proving that no non-separating induced circuits of CL_r have been overlooked, the planarity of CL_r for even r opens up a shortcut via a theorem of Kelmans [37, p. 264] (a hypothetical overlooked non-separating induced circuit implies an edge contained in more than two such circuits, contradicting Kelmans' theorem). For odd r, however, the non-planarity of CL_r (easy to prove via Kuratowski's theorem, cf. [31, p. 494]), makes this shortcut disappear. For these reasons, (A2) takes considerably more work when r is odd than when r is even, and we will not make any use of it. In the proofs in Section 3.2 we will opt for the shortest route, i.e. (A1). Argument (A3), the most arbitrary of all three (usually there is no overriding justification for choosing a particular set of linearly-independent

Hamilton circuits except that it works) will be used for proving (a23), i.e. for dealing with the rather ad-hoc auxiliary structures $\mathrm{Pr}_r^{\boxtimes}$, $\mathrm{Pr}_r^{\boxminus}$, $\mathrm{M}_r^{\boxtimes}$ and $\mathrm{M}_r^{\boxminus}$.

2.1.3. Explanation of (St3)

A set of graphs is called a *graph property* if and only if it is fixed (as a set) by graph isomorphisms. A graph property $\mathfrak{G}$ is called *monotone increasing* if and only if for every $G \in \mathfrak{G}$, adding to G an arbitrary edge again results in an element of $\mathfrak{G}$. A graph property $\mathfrak{G}$ consisting of bipartite graphs only is called a *monotone increasing property of bipartite graphs* if and only if for every $G \in \mathfrak{G}$, adding to G an arbitrary edge which does not create an odd circuit again results in an element of $\mathfrak{G}$.

Lemma 18. *For any function $\mathfrak{L}$ mapping graphs to subsets of $\mathbb{Z}_{\geqslant 1}$ and any $\xi \in \mathbb{Z}_{\geqslant 0}$,*

(1) *the set $\mathcal{M}_{\mathfrak{L},\xi}$ is a monotone increasing graph property,*
(2) *the set $\mathrm{b}\mathcal{M}_{\mathfrak{L},\xi}$ is a monotone increasing property of bipartite graphs.*

Lemma 18 can serve to elevate theorems guaranteeing the existence of spanning subgraphs with a certain property to theorems guaranteeing this property for the entire ambient graph.

Corollary 19 (*Lifting properties from spanning subgraphs to host graphs*)**.** *Let $\mathfrak{L}$ be a function mapping graphs to subsets of $\mathbb{Z}_{\geqslant 1}$, let $\xi \in \mathbb{Z}_{\geqslant 0}$, let $\mathfrak{G}$ be a set of graphs and let $\mathrm{b}\mathfrak{G}$ be a set of bipartite graphs. Then:*

(1) $\begin{pmatrix} \textit{if } G \in \mathfrak{G},\ \textit{then } \exists H \in \mathcal{M}_{\mathfrak{L},\xi} \textit{ with} \\ |H| = |G| \textit{ and } H \hookrightarrow G \end{pmatrix} \Longrightarrow \big(\textit{if } G \in \mathfrak{G},\ \textit{then } G \in \mathcal{M}_{\mathfrak{L},\xi}\big),$

(2) $\begin{pmatrix} \textit{if } G \in \mathrm{b}\mathfrak{G},\ \textit{then } \exists H \in \mathrm{b}\mathcal{M}_{\mathfrak{L},\xi} \textit{ with} \\ |H| = |G| \textit{ and } H \hookrightarrow G \end{pmatrix} \Longrightarrow \big(\textit{if } G \in \mathrm{b}\mathfrak{G},\ \textit{then } G \in \mathrm{b}\mathcal{M}_{\mathfrak{L},\xi}\big).$ $\square$

Lemma 18 is what makes (St3) of the argument tick. It is very similar to a lemma of Locke [46, Lemma 1], but we will re-prove Lemma 18 in Section 3.2, for two reasons: first, Locke's assumption of 2-connectedness and the attendant appeal to Menger's theorem [46, p. 253, last line] was appropriate while being concerned with a (possibly small) subgraph of special nature within a larger 2-connected graph. But it seems out of place when dealing with *spanning* subgraphs. It feels more to the point to explicitly name a one-dimensional direct summand which is acquired as a result of the added edge. Second, we will need a version of Locke's lemma especially phrased for bipartite graphs, and this is not to be found in (but easily obtained from) [46].

For Lemma 18, we need a simple lemma about vector spaces. If K is a field, M a K-vector space and $\mathcal{B} \subseteq M$ a K-linear subspace of M, then for every $v \in M$ we write $(\lambda_{\mathcal{B},v,b})_{b\in\mathcal{B}} \in K^{\mathcal{B}}$ for the unique element of $K^{\mathcal{B}}$ with $v = \sum_{b\in\mathcal{B}} \lambda_{\mathcal{B},v,b}\, b$. Moreover, $\mathrm{Supp}_{\mathcal{B}}(v) := \{b \in \mathcal{B} : \lambda_{\mathcal{B},v,b} \neq 0\} \subseteq \mathcal{B}$. We can now formulate the algebraic mechanism underlying Lemma 18.

Lemma 20. *If K is a field, M is a finite-dimensional K-vector space, $\mathcal{B} \subseteq M$ a K-basis of M, $b_0 \in \mathcal{B}$ an arbitrary element, $U \subseteq M$ an arbitrary K-linear subspace, and $u_0 \in U$ an arbitrary element with $\lambda_{\mathcal{B},u_0,b_0} \in K \setminus \{0\}$,*

$$U = \langle\{u \in U : b_0 \notin \mathrm{Supp}_{\mathcal{B}}(u)\}\rangle_K \oplus \langle u_0\rangle_K. \tag{1}$$

Let us emphasise that the 'monotonising' intersection

$$\begin{aligned}\mathcal{M}_{|\cdot|,0} &= \mathrm{cd}_0\mathcal{C}_{|\cdot|} \cap \mathcal{C}\mathcal{O}_{|\cdot|-1} \\ &= \{\text{graphs whose } \mathbb{F}_2\text{-span of Hamilton-circuits equals the cycle space}\} \\ &\quad \cap \{\text{Hamilton-connected graphs}\}\end{aligned}$$

from Definition 16, which is an essential tool in the argument, is a non-trivial intersection in the sense that it is not just the intersection of a set with a subset; neither of the two intersectands is contained in the other (as witnessed by e.g. circuit graphs for one, and the example $\mathrm{CE}_{(11)}$ for the other non-inclusion).

3. Proofs

3.1. Proofs of the main results

3.1.1. Proofs of the implications in Theorem 1

As to (I1), let $\gamma > 0$ be given and invoke Theorem 3 with this γ, $\rho := 2$ and $\Delta := 4$ to get a $\beta > 0$ and an n_0, here denoted by n_0', with the property stated there. Give this β to Lemma 17.(a30) to get an $n_0 = n_0(\beta)$, here denoted by n_0'', with the properties stated there. We now argue that with $n_0 := \max(n_0', n_0'')$ the claim in (I1) is true. Let $\mathfrak{G}$ be the set of all graphs G with odd $|G| \geqslant n_0$ and $\delta(G) \geqslant (\frac{1}{2} + \gamma)|G|$. Let $G \in \mathfrak{G}$ be arbitrary, $r := \frac{1}{2}(|G| - 1)$ and $H := \mathrm{Pr}_r^{\boxtimes}$ in case $|G| \equiv 1 \pmod 4$, resp. $H := \mathrm{M}_r^{\boxtimes}$ in case $|G| \equiv 3 \pmod 4$. Then $H \in \mathcal{M}_{\{|\cdot|\},0}$ in view of Lemmas 17.(a24) and 17.(a25), moreover $|H| = |G|$ and also $H \hookrightarrow G$ since $\Delta(H) = 4 \leqslant \Delta$ and Lemma 17.(a30) in the case '$H = \mathrm{Pr}_r^{\boxtimes}$' (resp. '$H = \mathrm{M}_r^{\boxtimes}$') allows us to apply Theorem 3—with the $\gamma, \rho, \Delta, \beta, n_0$ we already fixed—to the graphs G and H. Therefore, by Corollary 19.(1) it follows that $G \in \mathcal{M}_{\{|\cdot|\},0}$, in particular $G \in \mathrm{cd}_0\mathcal{C}_{\{|\cdot|\}}$, as claimed in (I1).

As to (I2), if throughout the preceding paragraph we replace '(I1)' by '(I2)', 'odd' by 'even', '$r := \frac{1}{2}(|G| - 1)$' by '$r := \frac{1}{2}|G|$', '$\mathrm{Pr}_r^{\boxtimes}$' by '$\mathrm{Pr}_r^{\boxminus}$', '$\mathrm{M}_r^{\boxtimes}$' by '$\mathrm{M}_r^{\boxminus}$', '$\mathcal{M}_{\{|\cdot|\},0}$' by '$\mathcal{M}_{\{|\cdot|\},1}$', ' Lemma 17.(a24)' by ' Lemma 17.(a26)', ' Lemma 17.(a25)' by ' Lemma 17.(a27)', '$\Delta(H) = 4$' by '$\Delta(H) = 5$', and '$\mathrm{cd}_0\mathcal{C}_{|\cdot|}$' by '$\mathrm{cd}_1\mathcal{C}_{|\cdot|}$', then we obtain a proof of the codimension-one-statement in (I2). Moreover, if in these replacement instructions we replace '$\mathcal{M}_{\{|\cdot|\},1}$' by '$\mathcal{M}_{\{|\cdot|-1,|\cdot|\},0}$', ' Lemma 17.(a26)' by ' Lemma 17.(a28)', and ' Lemma 17.(a27)' by ' Lemma 17.(a29)', and then apply the new instructions once more to the first paragraph, we obtain a proof of the second claim in (I2).

As to (I3), let $\gamma > 0$ be given and invoke Theorem 4 with this γ and $\Delta := 3$ to get a $\beta > 0$ and an n_0, here denoted by n_0', with the property stated there. Give this β to Lemma 17.(a30) to get an $n_0 = n_0(\beta)$, here denoted by n_0'', with the properties stated there. We now argue that with $n_0 := \max(n_0', n_0'')$ the claim in (I3) is true. Let $\mathrm{b}\mathfrak{G}$ be the set of all balanced bipartite graphs G with $|G| \geqslant n_0$ and $\delta(G) \geqslant (\frac{1}{4} + \gamma)|G|$. Let $G \in \mathfrak{G}$ be arbitrary and set $r := \frac{1}{2}|G|$ and $H := \mathrm{CL}_r$. Then $H \in \mathrm{b}\mathcal{M}_{\{|\cdot|\},0}$ in view of Lemma 17.(a15), moreover $|H| = |G|$ and also $H \hookrightarrow G$ since $\Delta(H) = 3 \leqslant \Delta$ and Lemma 17.(a30) in the case $H = \mathrm{CL}_r$ allows us to apply Theorem 4—with the $\gamma, \rho, \Delta, \beta, n_0$ we already fixed—to the graphs G and H. Therefore, by Corollary 19.(2) it follows that $G \in \mathrm{b}\mathcal{M}_{\{|\cdot|\},0}$, in particular $G \in \mathrm{bcd}_0\mathcal{C}_{\{|\cdot|\}}$, which is what is claimed in (I3).

As to (I4), let $\mathfrak{G}$ be the set of all graphs G with $|G| \geqslant 2 \cdot 10^8$ and $\delta(G) \geqslant \frac{2}{3}|G|$. Let $G \in \mathfrak{G}$ be arbitrary. Then Theorem 5 guarantees that $\mathrm{C}_{|G|}^2 \hookrightarrow G$. If $|G|$ is odd, then by combining Corollary 19.(1) and Lemma 17.(a4), it follows that $G \in \mathcal{M}_{\{|\cdot|\},0}$, in particular $G \in \mathrm{cd}_0\mathcal{C}_{\{|\cdot|\}}$, which proves (I4) in the case of odd $|\cdot|$. If $|G|$ is even, then (I4) follows by combining Corollary 19.(1) with Lemma 17.(a3), resp. Lemma 17.(a5). All the implications in Theorem 1 have now been proved.

3.1.2. Proof of the claim about weakening the hypothesis of (I1) in Theorem 1

Let $\mathrm{CE}_{(\mathrm{I1})}$ denote the seven-vertex graph with $\mathrm{V}(\mathrm{CE}_{(\mathrm{I1})}) := \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and $\mathrm{E}(\mathrm{CE}_{(\mathrm{I1})}) := \{v_1v_4, v_1v_6, v_1v_7, v_2v_4, v_2v_5, v_2v_7, v_3v_4, v_3v_5, v_3v_6, v_5v_6, v_5v_7, v_6v_7\}$. (This example, crucial for the topic, is illustrated in Fig. 2. Hamilton-connected, yet not Hamilton-generated graphs appear to be an unexplored topic for structural graph theory. In the House of Graphs database [15] this currently is the graph with name 'Self Dual Graph 3'.) Then $\frac{1}{2}|\mathrm{CE}_{(\mathrm{I1})}| = 3.5 \not\leqslant 3 = \delta(\mathrm{CE}_{(\mathrm{I1})})$, i.e. $\mathrm{CE}_{(\mathrm{I1})}$ barely misses the Dirac threshold. The graph $\mathrm{CE}_{(\mathrm{I1})}$ has odd number of vertices, is 3-vertex-connected, pancyclic (i.e. contains at least one circuit of each of all possible lengths $3, \ldots, |G|$), and is Hamilton-connected (by, e.g. [59, Theorem 1.2]: the only independent set with three vertices is $\{v_1, v_2, v_3\}$, and for this vertex-set the criterion [59, Theorem 1.2] holds, as $\deg(v_1) + \deg(v_2) + \deg(v_3) - |\mathrm{N}(v_1) \cap \mathrm{N}(v_2) \cap \mathrm{N}(v_3)| = \deg(v_1) + \deg(v_2) + \deg(v_3) - |\{v_4\}| = 3 + 3 + 3 - 1 = 8 \geqslant 7 + 1 = |\mathrm{CE}_{(\mathrm{I1})}| + 1$). Therefore the following fact (which proves the claim made in Theorem 1 about weakening (I1)) also shows that the open question (Q3) in Section 4 can easily acquire a negative answer if its hypotheses are slightly weakened.

Proposition 21. $\dim_{\mathbb{F}_2}\big(\mathrm{Z}_1(\mathrm{CE}_{(\mathrm{I1})}; \mathbb{F}_2)/\langle\mathcal{H}(\mathrm{CE}_{(\mathrm{I1})})\rangle_{\mathbb{F}_2}\big) = 1$.

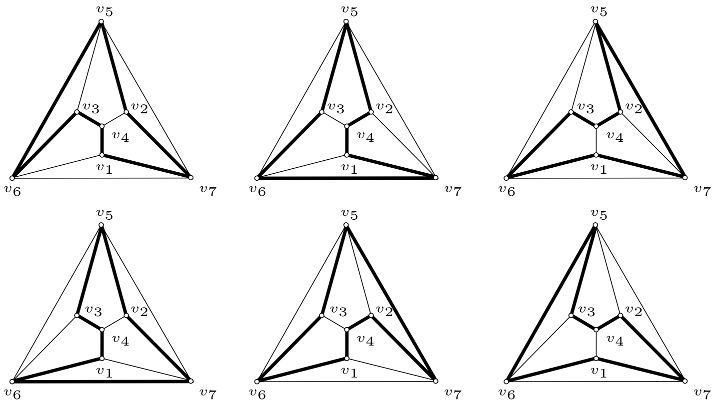


Fig. 2. A counterexample which proves that a graph having several properties which intuitively may seem conducive to the property of having its cycle space generated by Hamilton circuits, can nevertheless fail to have it: the graph $\mathrm{CE}_{(\mathrm{I1})}$ underlying Fig. 2 has an odd number of vertices, is 3-vertex-connected, is pancyclic, and it is Hamilton-connected. And yet it has its cycle space not generated by its Hamilton circuits (it can be checked that the six shown in the figure are all that $\mathrm{CE}_{(\mathrm{I1})}$ has, and there is a non-trivial $\mathbb{F}_2$-linear relation among them). Note that $\mathrm{CE}_{(\mathrm{I1})}$ fails the Dirac condition (barely), hence is not a counterexample to Question (Q1) in Section 4.1.

Proof. The smallness of $\mathrm{CE}_{(\mathrm{I1})}$ makes it easy to check that $\mathcal{H}(\mathrm{CE}_{(\mathrm{I1})})$ consists precisely of the six circuits (shown in Fig. 2) $C_1 := v_1v_7v_2v_5v_6v_3v_4v_1$, $C_2 := v_1v_7v_6v_3v_5v_2v_4v_1$, $C_3 := v_1v_7v_5v_2v_4v_3v_6v_1$, $C_4 := v_1v_6v_7v_2v_5v_3v_4v_1$, $C_5 := v_1v_6v_3v_5v_7v_2v_4v_1$, $C_6 := v_1v_6v_5v_3v_4v_2v_7v_1$. If the standard basis of $\mathrm{C}_1(\mathrm{CE}_{(\mathrm{I1})};\mathbb{F}_2)$ is labelled $\mathrm{e}_1 := \mathrm{c}_{v_1v_4}$, $\mathrm{e}_2 := \mathrm{c}_{v_1v_6}$, $\mathrm{e}_3 := \mathrm{c}_{v_1v_7}$, $\mathrm{e}_4 := \mathrm{c}_{v_2v_4}$, $\mathrm{e}_5 := \mathrm{c}_{v_2v_5}$, $\mathrm{e}_6 := \mathrm{c}_{v_2v_7}$, $\mathrm{e}_7 := \mathrm{c}_{v_3v_4}$, $\mathrm{e}_8 := \mathrm{c}_{v_3v_5}$, $\mathrm{e}_9 := \mathrm{c}_{v_3v_6}$, $\mathrm{e}_{10} := \mathrm{c}_{v_5v_6}$, $\mathrm{e}_{11} := \mathrm{c}_{v_5v_7}$, $\mathrm{e}_{12} := \mathrm{c}_{v_6v_7}$, then w.r.t. to this basis the Hamilton circuits $C_1, \ldots, C_6$ give rise to the matrix shown in (2), which has $\mathbb{F}_2$-rank 5.

$$
\begin{array}{c|cccccc}
 & C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \\
\hline
\mathrm{e}_1 & 1 & 1 & 0 & 1 & 1 & 0 \\
\mathrm{e}_2 & 0 & 0 & 1 & 1 & 1 & 1 \\
\mathrm{e}_3 & 1 & 1 & 1 & 0 & 0 & 1 \\
\mathrm{e}_4 & 0 & 1 & 1 & 0 & 1 & 1 \\
\mathrm{e}_5 & 1 & 1 & 1 & 1 & 0 & 0 \\
\mathrm{e}_6 & 1 & 0 & 0 & 1 & 1 & 1 \\
\mathrm{e}_7 & 1 & 0 & 1 & 1 & 0 & 1 \\
\mathrm{e}_8 & 0 & 1 & 0 & 1 & 1 & 1 \\
\mathrm{e}_9 & 1 & 1 & 1 & 0 & 1 & 0 \\
\mathrm{e}_{10} & 1 & 0 & 0 & 0 & 0 & 1 \\
\mathrm{e}_{11} & 0 & 0 & 1 & 0 & 1 & 0 \\
\mathrm{e}_{12} & 0 & 1 & 0 & 1 & 0 & 0
\end{array}
\tag{2}
$$

Therefore $\left\langle \mathcal{H}(\mathrm{CE}_{(\mathrm{I1})}) \right\rangle_{\mathbb{F}_2}$ is a 5-dimensional subspace of $\mathrm{Z}_1(\mathrm{CE}_{(\mathrm{I1})};\mathbb{F}_2)$, which has dimension $\beta_1(\mathrm{CE}_{(\mathrm{I1})}) = \|\mathrm{CE}_{(\mathrm{I1})}\| - |\mathrm{CE}_{(\mathrm{I1})}| + 1 = 12 - 7 + 1 = 6$. This proves Proposition 21. $\square$

3.1.3. Proof of the claim about weakening the hypothesis of (I3) in Theorem 1

Let $\mathrm{CE}_{(\mathrm{I3})}$ denote the balanced bipartite graph with $\mathrm{V}(\mathrm{CE}_{(\mathrm{I3})}) := \{v_1, \ldots, v_6\} \sqcup \{v_7, \ldots, v_{12}\}$ (bipartition classes indicated) and $\mathrm{E}(\mathrm{CE}_{(\mathrm{I3})}) := \{v_1v_7, v_1v_8, v_1v_9, v_1v_{12}, v_2v_7, v_2v_8, v_2v_9, v_3v_7, v_3v_8, v_3v_9, v_4v_9, v_4v_{10}, v_4v_{11}, v_5v_{10}, v_5v_{11}, v_5v_{12}, v_6v_{10}, v_6v_{11}, v_6v_{12}\}$. (This is the graph in Fig. 3.) Then $\frac{1}{4}|\mathrm{CE}_{(\mathrm{I3})}| = \delta(\mathrm{CE}_{(\mathrm{I3})}) = 3$ and $\mathrm{CE}_{(\mathrm{I3})}$ is Hamiltonian. We will now prove by a short argument that $\langle \mathcal{H}(\mathrm{CE}_{(\mathrm{I3})}) \rangle_{\mathbb{F}_2}$ has *at least* codimension one in $\mathrm{Z}_1(\mathrm{CE}_{(\mathrm{I3})};\mathbb{F}_2)$, which is enough to establish $\mathrm{CE}_{(\mathrm{I3})}$ as a counterexample of the claimed kind. (By determining all 16 Hamilton circuits of $\mathrm{CE}_{(\mathrm{I3})}$ and subsequently computing the $\mathbb{F}_2$-rank of a 12 by 16 matrix with zero–one entries it is possible to show that $\dim_{\mathbb{F}_2} \langle \mathcal{H}(\mathrm{CE}_{(\mathrm{I3})}) \rangle_{\mathbb{F}_2} = 7 = \dim_{\mathbb{F}_2} \mathrm{Z}_1(\mathrm{CE}_{(\mathrm{I3})};\mathbb{F}_2) - 1$, i.e. the codimension is equal to 1.)

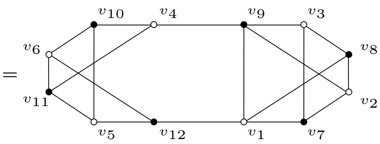
$\mathrm{CE}_{(\mathrm{I3})} =$ 

Fig. 3. A counterexample which proves that if in (I3) the hypothesis '$\delta(G) \geqslant (\frac{1}{4} + \gamma)|G|$' is weakened to '$\delta(G) \geqslant \frac{1}{4}|G|$ and G Hamiltonian' the implication becomes false: the graph $\mathrm{CE}_{(\mathrm{I3})}$ has $\delta = 3 = \frac{1}{4}|\mathrm{CE}_{(\mathrm{I3})}|$ and is Hamiltonian, yet $\langle \mathscr{H}(\cdot)\rangle_{\mathbb{F}_2}$ has codimension one in $\mathrm{Z}_1(\cdot; \mathbb{F}_2)$. If the edge $\{v_1\ v_9\}$ were omitted, we *would* have $\langle \mathscr{H}(\cdot)\rangle_{\mathbb{F}_2} = \mathrm{Z}_1(\cdot; \mathbb{F}_2)$, hence the resulting graph $\mathrm{CE}_{(\mathrm{I3})} - \{v_1, v_9\}$ would—while still satisfying the weakened hypotheses with respect to which $\mathrm{CE}_{(\mathrm{I3})}$ *is* a counterexample—cease to be a counterexample. (This does not contradict the fact that 'Hamilton-laceable and Hamilton-generated' is a monotone property of bipartite graphs: $\mathrm{CE}_{(\mathrm{I3})} - \{v_1, v_9\}$ is not Hamilton-laceable.) The author could not find a counterexample showing that (I3) would become false were '$\delta(G) \geqslant (\frac{1}{4} + \gamma)|G|$' weakened only to '$\delta(G) \geqslant \frac{1}{4}|G|$ and G Hamilton-laceable'.

Proposition 22. $\dim_{\mathbb{F}_2}\big(\mathrm{Z}_1(\mathrm{CE}_{(\mathrm{I3})}; \mathbb{F}_2)/\langle \mathscr{H}(\mathrm{CE}_{(\mathrm{I3})})\rangle_{\mathbb{F}_2}\big) \geqslant 1.$

Proof. It is enough to make the following simple observation: since $\{v_1, v_9\}$ is a separator of $\mathrm{CE}_{(\mathrm{I3})}$, the edge $\{v_1, v_9\}$ cannot be an edge of any Hamilton circuit of $\mathrm{CE}_{(\mathrm{I3})}$. Therefore the set of all Hamilton circuits of $\mathrm{CE}_{(\mathrm{I3})}$ *equals* the set of all Hamilton circuits of the graph $\mathrm{CE}_{(\mathrm{I3})} - \{v_1, v_9\}$ obtained after deleting $\{v_1, v_9\}$ from the edge set of $\mathrm{CE}_{(\mathrm{I3})}$. This in particular implies the first equality in the calculation $\dim_{\mathbb{F}_2}\langle \mathscr{H}(\mathrm{CE}_{(\mathrm{I3})})\rangle_{\mathbb{F}_2} = \dim_{\mathbb{F}_2}\langle \mathscr{H}(\mathrm{CE}_{(\mathrm{I3})} - \{v_1, v_9\})\rangle_{\mathbb{F}_2} \leqslant$ (since the dimension of a subspace of a vector space is bounded by the dimension of the latter's dimension) $\leqslant \dim_{\mathbb{F}_2} \mathrm{Z}_1(\mathrm{CE}_{(\mathrm{I3})} - \{v_1, v_9\}; \mathbb{F}_2) =$ (by the Euler–Poincaré relation) $= \dim_{\mathbb{F}_2} \mathrm{Z}_1(\mathrm{CE}_{(\mathrm{I3})}; \mathbb{F}_2) - 1$, proving Proposition 22. □

3.2. *Proofs of the auxiliary results*

Proof of Lemma 18. First note that for both $\mathcal{M}_{\mathfrak{L},\xi}$ and $\mathrm{b}\mathcal{M}_{\mathfrak{L},\xi}$, it is obvious that the sets are fixed (as sets) under any graph isomorphism, i.e. both are graph properties.

As to the monotonicity claim in (1), if $\mathcal{M}_{\mathfrak{L},\xi} = \varnothing$, the claim is vacuously true. Otherwise, let $G \in \mathcal{M}_{\mathfrak{L},\xi}$ be an arbitrary element and let $e \in \binom{\mathrm{V}(G)}{2} \setminus \mathrm{E}(G)$ be arbitrary. We will use the abbreviation $G + e := (\mathrm{V}(G), \mathrm{E}(G) \sqcup \{e\})$. We have to prove $G + e \in \mathcal{M}_{\mathfrak{L},\xi}$. Trivially, $G + e \in \mathcal{CO}_{\mathfrak{L}-1}$. What has to be justified is that $G + e \in \mathrm{cd}_{\xi}\,\mathcal{C}_{\mathfrak{L}}$. Since $G \in \mathcal{CO}_{\mathfrak{L}-1}$, there exists in G a path P with length in $\{l - 1 : l \in \mathfrak{L}\}$ linking the endvertices of e and we have $e \notin \mathrm{E}(P)$ since $e \notin \mathrm{E}(G)$. Choose any such P. We now use Lemma 20 twice: let $R := \mathbb{F}_2$, $M := \mathrm{C}_1(G + e; \mathbb{F}_2)$, $\mathcal{B} := \{\mathrm{c}_{\tilde{e}} : \tilde{e} \in \mathrm{E}(G + e)\}$ (the standard basis of $\mathrm{C}_1(G + e; \mathbb{F}_2)$) and $b_0 := e$. Since (with $\{u, v\} := e$) the circuit $C := uPvu$ satisfies both $C \in \mathcal{C}_{\mathfrak{L}}(G + e)$ and $C \in \mathrm{Z}_1(G + e; \mathbb{F}_2)$, it follows that whether we define $U := \langle \mathcal{C}_{\mathfrak{L}}(G + e)\rangle_{\mathbb{F}_2}$ or $U := \mathrm{Z}_1(G + e; \mathbb{F}_2)$, in both cases we have $u_0 := \mathrm{c}_C \in U$, and therefore Lemma 20 gives us

(ds1) $\langle \mathcal{C}_{\mathfrak{L}}(G + e)\rangle_{\mathbb{F}_2} = \langle \mathcal{C}_{\mathfrak{L}}(G)\rangle_{\mathbb{F}_2} \oplus \langle \mathrm{c}_C\rangle_{\mathbb{F}_2}$,
(ds2) $\mathrm{Z}_1(G + e; \mathbb{F}_2) = \mathrm{Z}_1(G; \mathbb{F}_2) \oplus \langle \mathrm{c}_C\rangle_{\mathbb{F}_2}$.

The direct sum decompositions (ds1) and (ds2) imply $\dim_{\mathbb{F}_2}\big(\mathrm{Z}_1(G + e; \mathbb{F}_2)/\langle \mathcal{C}_{\mathfrak{L}}(G + e)\rangle_{\mathbb{F}_2}\big) = \dim_{\mathbb{F}_2}\big(\mathrm{Z}_1(G; \mathbb{F}_2)/\langle \mathcal{C}_{\mathfrak{L}}(G)\rangle_{\mathbb{F}_2}\big) = \xi$ and therefore $G + e \in \mathrm{cd}_{\xi}\,\mathcal{C}_{\mathfrak{L}}$, completing the proof of statement (1). As to (2), it suffices to note that the proof of (1) may be repeated to yield a proof of (2), the only change required being to restrict e to be an edge whose addition keeps the graph bipartite and to replace '$\mathcal{CO}_{\mathfrak{L}-1}$' by '$\mathcal{LA}_{\mathfrak{L}-1}$'. □

Proof of Lemma 20. The sum is obviously direct: $b_0 \in \mathrm{Supp}_{\mathcal{B}}(u_0)$ while $b_0 \notin \mathrm{Supp}_{\mathcal{B}}(v)$ for every $v \in \langle\{u \in U : b_0 \notin \mathrm{Supp}_{\mathcal{B}}(u)\}\rangle_K$, hence the intersection of the summands is $\{0\}$. What is to be justified is that $U \subseteq \langle\{u \in U : b_0 \notin \mathrm{Supp}_{\mathcal{B}}(u)\}\rangle_K + \langle u_0\rangle_K$. So let $v \in U$ be arbitrary and let $\mathcal{E} \in \binom{U}{\dim_K(U)}$ denote an arbitrary finite K-basis of U. Let $\mathcal{E}_0 := \{e \in \mathcal{E} : b_0 \in \mathrm{Supp}_{\mathcal{B}}(e)\}$. Then $\lambda_{\mathcal{B},\cdot,b_0}\left(\sum_{e \in \mathcal{E}\setminus\mathcal{E}_0} \lambda_{\mathcal{E},v,e}\, e + \left(\sum_{e \in \mathcal{E}_0} \lambda_{\mathcal{E},v,e}\, \left(e - \lambda_{\mathcal{B},e,b_0}\, (\lambda_{\mathcal{B},u_0,b_0})^{-1}\, u_0\right)\right)\right) = 0$, by linearity of the coefficient-map $\lambda_{\mathcal{B},\cdot,b_0}$ defined before Lemma 20. So b_0 is not an element of the $\mathrm{Supp}_{\mathcal{B}}(\cdot)$ of

$$\sum_{e\in\mathcal{E}\setminus\mathcal{E}_0}\lambda_{\mathcal{E},v,e}\,e+\left(\sum_{e\in\mathcal{E}_0}\lambda_{\mathcal{E},v,e}\,(e-\lambda_{\mathcal{B},e,b_0}(\lambda_{\mathcal{B},u_0,b_0})^{-1}u_0)\right)$$
$$=v-\left((\lambda_{\mathcal{B},u_0,b_0})^{-1}\sum_{e\in\mathcal{E}_0}\lambda_{\mathcal{E},v,e}\,\lambda_{\mathcal{B},e,b_0}\right)u_0. \tag{3}$$

Thus, writing $v=\big(v-\big((\lambda_{\mathcal{B},u_0,b_0})^{-1}\sum_{e\in\mathcal{E}_0}\lambda_{\mathcal{E},v,e}\,\lambda_{\mathcal{B},e,b_0}\big)\,u_0\big)+\big((\lambda_{\mathcal{B},u_0,b_0})^{-1}\sum_{e\in\mathcal{E}_0}\lambda_{\mathcal{E},v,e}\,\lambda_{\mathcal{B},e,b_0}\big)\,u_0$ shows that $v\in\langle\{u\in U\colon b_0\notin\mathrm{Supp}_{\mathcal{B}}(u)\}\rangle_R+\langle u_0\rangle_R$, completing the proof of $U\subseteq\langle\{u\in U\colon b_0\notin\mathrm{Supp}_{\mathcal{B}}(u)\}\rangle_R\oplus\langle u_0\rangle_R$. $\square$

The above proof of Lemma 20 does not work if the assumption of M being finitely generated is dropped: while U then still admits a basis, there is no reason why $\mathcal{E}_0$ should be a finite set, so the sums in (3) may not be defined. Within the unexplored realm of linear-algebraic properties of Hamilton circles in infinite graphs, this obstacle to naïvely adapting the monotonicity argument might be a good point to start.

Proof of Lemma 17. As to (a1), an easy verification shows that the map $\{v_0,\dots,v_{n-1}\}\to\mathbb{Z}/n$, $v_i\mapsto i$ is a graph isomorphism $\mathrm{C}_n^2\to\mathrm{Cay}(\mathbb{Z}/n;\{1,2,n-2,n-1\})$. (Both for this verification and for the ones required in (a6), (a7), (a12) and (a13), it is recommendable to use an obvious and known [33, Section 1.5, first paragraph] characterization of graph isomorphisms: *every injective graph homomorphism between two graphs with equal* $|\cdot|$ *and* $\|\cdot\|$ *is a graph isomorphism*. This relieves one of the responsibility to explicitly show that non-edges are mapped to non-edges.)

As to (a2), the definition of $\square$ implies that for every graph G, every vertex of the graph $G\,\square\,\mathrm{P}_1$ has odd degree. But for every $n\geqslant 5$ the graph C_n^2 is regular with vertex degree four.

As to (a3) and (a4), first note that C_n^2 is non-bipartite, for both parities of n, and therefore (a1) and Theorem 14 combined imply that $\mathrm{C}_n^2\in\mathcal{CO}_{\{|\cdot|\}}$, for every n. It remains to justify that $\mathrm{C}_n^2\in\mathrm{cd}_1\mathcal{C}_{\{|\cdot|\}}$ for even n, resp. $\mathrm{C}_n^2\in\mathrm{cd}_0\mathcal{C}_{\{|\cdot|\}}$ for odd n. Both these statements follows from combining (a1) and (a2) with Theorem 15.(2) and Theorem 15.(3).

As to (a5), first note that C_n^2 does indeed contain circuits of length $|\mathrm{C}_n^2|-1$ (in fact, $|\mathrm{C}_n^2|$ different ones), and then arbitrarily choose one such circuit C. Since n is even, C has odd length, and therefore $\mathrm{c}_C\notin\langle\mathcal{H}(\mathrm{C}_n^2)\rangle_{\mathbb{F}_2}$. Moreover, $\dim_{\mathbb{F}_2}\langle\mathcal{H}(\mathrm{C}_n^2)\rangle_{\mathbb{F}_2}=\dim_{\mathbb{F}_2}\mathrm{Z}_1(\mathrm{C}_n^2;\mathbb{F}_2)-1$ by (a3), hence $\dim_{\mathbb{F}_2}\langle\{\mathrm{c}_C\}\sqcup\mathcal{H}(\mathrm{C}_n^2)\rangle_{\mathbb{F}_2}\geqslant\dim_{\mathbb{F}_2}\mathrm{Z}_1(\mathrm{C}_n^2;\mathbb{F}_2)$ and due to $\langle\{\mathrm{c}_C\}\sqcup\mathcal{H}(\mathrm{C}_n^2)\rangle_{\mathbb{F}_2}$ being a $\mathbb{F}_2$-linear subspace of $\mathrm{Z}_1(\mathrm{C}_n^2;\mathbb{F}_2)$, this must hold with equality, proving (a5).

As to (a6), an easy verification shows that the map $\{x_0,\dots,x_{r-1},\ y_0,\dots,y_{r-1}\}\to\mathbb{F}_2\oplus\mathbb{Z}/r$, $x_i\mapsto(0,i)$, $y_i\mapsto(1,i)$ is a graph isomorphism $\mathrm{Pr}_r\to\mathrm{Cay}(\mathbb{F}_2\oplus\mathbb{Z}/r;\{(1,0),(0,1),(0,r-1)\})$.

As to (a7), an easy verification shows that the map $\mathrm{V}(\mathrm{M}_r)=\{x_0,\dots,x_{r-1},\ y_0,\dots,y_{r-1}\}\to\mathbb{Z}/(2r)$, $x_i\mapsto i$, $y_i\mapsto i+r$ is a graph isomorphism $\mathrm{M}_r\to\mathrm{Cay}\,(\mathbb{Z}/(2r);\{1,r,2r-1\})$.

As to (a8), it is easy to check that r being even implies that Pr_r is bipartite. Therefore (a8) follows from (a6) combined with Theorem 14. Moreover, (a8) is straightforward to prove directly.

As to (a9), it is easy to check that r being odd implies that M_r is bipartite. Therefore (a9) follows from (a7) combined with Theorem 14. Moreover, (a9) is straightforward to prove directly.

As to (a10), it is easy to check that r being even implies that Pr_r is bipartite. Therefore, combining (a6) with Theorem 14 yields that $\mathrm{Pr}_r\in\mathcal{LA}_{\{|\cdot|-1\}}$, and combining (a6) with Theorem 15.(1) yields $\mathrm{Pr}_r\in\mathrm{cd}_0\mathcal{C}_{\{|\cdot|\}}$, completing the proof of (a10).

As to (a11), it is easy to check that r being odd implies that M_r is bipartite. Therefore, combining (a7) with Theorem 14 yields that $\mathrm{M}_r\in\mathcal{LA}_{\{|\cdot|-1\}}$, and combining (a7) with Theorem 15.(1) yields $\mathrm{M}_r\in\mathrm{cd}_0\mathcal{C}_{\{|\cdot|\}}$, completing the proof of (a11).

As to (a12) and (a13), an easy verification shows that the map $\mathrm{V}(\mathrm{CL}_r)\to\mathrm{V}(\mathrm{Pr}_r)=\mathrm{V}(\mathrm{M}_r)$ defined by $a_i\mapsto x_i$ for every even $0\leqslant i\leqslant r-1$, $a_i\mapsto y_i$ for every odd $0\leqslant i\leqslant r-1$, $b_i\mapsto y_i$ for every even $0\leqslant i\leqslant r-1$, $b_i\mapsto x_i$ for every odd $0\leqslant i\leqslant r-1$, is a graph isomorphism $\mathrm{CL}_r\to\mathrm{Pr}_r$ for every even $r\geqslant 4$ and a graph isomorphism $\mathrm{CL}_r\to\mathrm{M}_r$ for every odd $r\geqslant 4$.

As to (a14), this follows by combining (a8) and (a9) with (a12) and (a13).

As to (a15), this follows by combining (a10) and (a11) with (a12) and (a13).

As to (a16) and (a17), the literature apparently does not contain a sufficient criterion for Hamilton-connectedness which would apply to either $\mathrm{Pr}_r^{\boxtimes}$ or $\mathrm{M}_r^{\boxtimes}$. Therefore a direct proof by distinguishing cases and providing explicit Hamilton paths appears to be unavoidable. Let $\{v, w\} \subseteq \mathrm{V}(\mathrm{M}_r^{\boxtimes}) = \mathrm{V}(\mathrm{Pr}_r^{\boxtimes})$ be arbitrary distinct vertices.

We will repeatedly reduce the work to be done by making use of symmetries. The automorphism group of both $\mathrm{Pr}_r^{\boxtimes}$ and $\mathrm{M}_r^{\boxtimes}$ is the group generated by the two unique homomorphic extensions of the maps $\left(\substack{\{z,x_0,y_0,x_1,y_1\}\to\{z,x_0,y_0,x_1,y_1\}\\ z\mapsto z,\ x_0\leftrightarrow y_0,\ x_1\leftrightarrow y_1}\right)$ and $\left(\substack{\{z,x_0,y_0,x_1,y_1\}\to\{z,x_0,y_0,x_1,y_1\}\\ z\mapsto z,\ x_0\leftrightarrow x_1,\ y_0\leftrightarrow y_1}\right)$ to all of $\mathrm{V}(\mathrm{Pr}_r^{\boxtimes}) = \mathrm{V}(\mathrm{M}_r^{\boxtimes})$ (thus both $\mathrm{Aut}(\mathrm{Pr}_r^{\boxtimes})$ and $\mathrm{Aut}(\mathrm{M}_r^{\boxtimes})$ are isomorphic to the Klein four-group $\mathbb{F}_2 \oplus \mathbb{F}_2$). These extensions are involutions on $\mathrm{V}(\mathrm{Pr}_r^{\boxtimes}) = \mathrm{V}(\mathrm{M}_r^{\boxtimes})$ and will be denoted by Ψ_{xy} (the map $z \mapsto z$ and $x_i \leftrightarrow y_i$ for every $0 \leqslant i \leqslant r-1$) and Ψ_{xx} (the map $z \mapsto z$ and, for $u \in \{x, y\}$, by $u_1 \leftrightarrow u_0, u_2 \leftrightarrow u_{r-1}$, $u_3 \leftrightarrow u_{r-2}, \ldots, u_{\lfloor \frac{r+1}{2} \rfloor} \leftrightarrow u_{\lceil \frac{r+1}{2} \rceil}$). Both Ψ_{xy} and Ψ_{xx} are automorphisms of both $\mathrm{M}_r^{\boxtimes}$ (for every $r \geqslant 5$) and $\mathrm{Pr}_r^{\boxtimes}$ (for every $r \geqslant 4$).

Case 1. $z \in \{v, w\}$. In the absence of information distinguishing v from w we may assume $z = v$.

Case 1.1. $w \in \{x_0, y_0, x_1, y_1\}$. Since $\mathrm{Aut}(\mathrm{Pr}_r^{\boxtimes})$ acts transitively on the set $\{x_0, y_0, x_1, y_1\}$ while keeping z fixed, we may assume that $w = x_0$. Then $x_0x_1 \cdots x_{r-1}y_{r-1}y_{r-2} \cdots y_1y_0z$ in both $\mathrm{Pr}_r^{\boxtimes}$ and $\mathrm{M}_r^{\boxtimes}$ is Hamilton path linking v and w. This proves both (a16) and (a17) in the Case 1.1.

Case 1.2. $w \notin \{x_0, y_0, x_1, y_1\}$. Due to Ψ_{xy} we may assume that $w = x_i$ with $2 \leqslant i \leqslant r-1$. Now consider the expressions:

(Pr.1.2.(0)) $x_iy_iy_{i+1}x_{i+1}x_{i+2}y_{i+2} \cdots y_{r-2}y_{r-1}x_{r-1}x_0x_1x_2 \cdots x_{i-1}y_{i-1}y_{i-2}y_{i-3} \cdots y_0z$,
(Pr.1.2.(1)) $x_iy_iy_{i+1}x_{i+1}x_{i+2}y_{i+2} \cdots x_{r-2}x_{r-1}y_{r-1}y_0y_1y_2 \cdots y_{i-1}x_{i-1}x_{i-2}x_{i-3} \cdots x_0z$,
(M.1.2.(0)) $x_iy_iy_{i+1}x_{i+1}x_{i+2}y_{i+2} \cdots x_{r-2}x_{r-1}y_{r-1}x_0x_1x_2 \cdots x_{i-1}y_{i-1}y_{i-2}y_{i-3} \cdots y_0z$,
(M.1.2.(1)) $x_iy_iy_{i+1}x_{i+1}x_{i+2}y_{i+2} \cdots y_{r-2}y_{r-1}x_{r-1}y_0y_1y_2 \cdots y_{i-1}x_{i-1}x_{i-2}x_{i-3} \cdots x_0z$.

If i is even, then (Pr.1.2.(0)), and if i is odd then (Pr.1.2.(1)) is a Hamilton path of Pr_r linking v and w, for every even $r \geqslant 4$. If i is even, then (M.1.2.(0)), and if i is odd then (M.1.2.(1)) is a Hamilton path of M_r linking v and w, for every odd $r \geqslant 5$. This proves both (a16) and (a17) in the Case 1.2.

Case 2. $z \notin \{v, w\}$.

Case 2.1. $\{v, w\} \subseteq \{x_0, \ldots, x_{r-1}\}$ or $\{v, w\} \subseteq \{y_0, \ldots, y_{r-1}\}$. In view of Φ_{xy} we may assume that $\{v, w\} \subseteq \{x_0, \ldots, x_{r-1}\}$.

Case 2.1.1. $\{v, w\} \cap \{x_0, x_1\} \neq \varnothing$. In the absence of information distinguishing v from w we may assume that $v \in \{x_0, x_1\}$. In view of the transitivity of both $\mathrm{Aut}(\mathrm{Pr}_r^{\boxtimes})$ and $\mathrm{Aut}(\mathrm{M}_r^{\boxtimes})$ on $\{x_0, x_1, y_0, y_1\}$ we may further assume that $v = x_0$. Then $w = x_i$ for some $i \in [1, r-1]$. We can now reduce the claim we are currently proving to claims about a Cartesian product of the form $\mathrm{P}_1 \,\square\, \mathrm{P}_l$ (for some l) which is obtained after deleting certain vertices. The reduction is made possible by making—depending on the parity of the i in x_i—the right choice of a 3-path or a 4-path within the graph induced by $\{z, x_0, x_1, y_0, y_1\}$.

If i is even (hence in particular $i \geqslant 2$), then starting out with the 4-path $x_0y_0zx_1y_1$ leaves us facing the task of connecting y_2 with x_i (which lies in the opposite colour class compared to y_2) via a Hamilton path of the graph remaining after deletion of $\{x_0, y_0, x_1, y_1, z\}$. This remaining graph is—regardless of whether we are currently speaking about $\mathrm{M}_r^{\boxtimes}$ or $\mathrm{Pr}_r^{\boxtimes}$—isomorphic to the Cartesian product $\mathrm{P}_2 \,\square\, \mathrm{P}_{r-3}$, of which the vertex y_2 is a 'corner vertex' in the sense of [18, Section 2]. Therefore this task *can* be accomplished according to [18, Lemma 1].

If on the contrary i is odd, then starting out with the 3-path $x_0zy_0y_1$ leaves us facing the task of connecting y_1 with x_i (which lies in the opposite colour class compared to y_1) by a Hamilton path of the graph remaining after deletion of $\{x_0, y_0, z\}$. This remaining graph is—regardless of whether we are currently speaking about $\mathrm{M}_r^{\boxtimes}$ or $\mathrm{Pr}_r^{\boxtimes}$—isomorphic to the Cartesian product $\mathrm{P}_2 \,\square\, \mathrm{P}_{r-2}$, of which the vertex '$y_1$' is a corner vertex. Therefore this task, too, *can* be accomplished according to [18, Lemma 1]. This proves both (a16) and (a17) in the Case 2.1.1.

Case 2.1.2. $\{v, w\} \cap \{x_0, x_1\} = \varnothing$. Then $v = x_i$ and $w = x_j$ for some $\{i, j\} \in \binom{\{2,3,\ldots,r-1\}}{2}$. In the absence of information distinguishing v from w we may assume that $2 \leqslant i < j \leqslant r-1$.

Now consider the expressions

(Pr.2.1.2.(1)) $x_ix_{i+1}\cdots x_{j-1}y_{j-1}y_{j-2}\cdots y_{i-1}x_{i-1}x_{i-2}y_{i-2}y_{i-3}\cdots x_2y_2y_1x_1zy_0x_0x_{r-1}y_{r-1}y_{r-2}\cdots x_{j+1}y_{j+1}y_jx_j$,

(Pr.2.1.2.(2)) $x_ix_{i+1}\cdots x_{j-1}y_{j-1}y_{j-2}\cdots y_{i-1}x_{i-1}x_{i-2}y_{i-2}y_{i-3}\cdots x_2y_2y_1x_1zx_0y_0y_{r-1}x_{r-1}x_{r-2}\cdots x_{j+1}y_{j+1}y_jx_j$,

(Pr.2.1.2.(3)) $x_ix_{i+1}\cdots x_{j-1}y_{j-1}y_{j-2}\cdots y_{i-1}x_{i-1}x_{i-2}y_{i-2}y_{i-3}\cdots y_2x_2x_1y_1zy_0x_0x_{r-1}y_{r-1}y_{r-2}\cdots x_{j+1}y_{j+1}y_jx_j$,

(Pr.2.1.2.(4)) $x_ix_{i+1}\cdots x_{j-1}y_{j-1}y_{j-2}\cdots y_{i-1}x_{i-1}x_{i-2}y_{i-2}y_{i-3}\cdots y_2x_2x_1y_1zx_0y_0y_{r-1}x_{r-1}x_{r-2}\cdots x_{j+1}y_{j+1}y_jx_j$

and

(M.2.1.2.(1)) $x_ix_{i+1}\cdots x_{j-1}y_{j-1}y_{j-2}\cdots y_{i-1}x_{i-1}x_{i-2}y_{i-2}y_{i-3}\cdots x_2y_2y_1x_1zy_0x_0y_{r-1}x_{r-1}x_{r-2}\cdots x_{j+1}y_{j+1}y_jx_j$,

(M.2.1.2.(2)) $x_ix_{i+1}\cdots x_{j-1}y_{j-1}y_{j-2}\cdots y_{i-1}x_{i-1}x_{i-2}y_{i-2}y_{i-3}\cdots x_2y_2y_1x_1zx_0y_0x_{r-1}y_{r-1}y_{r-2}\cdots x_{j+1}y_{j+1}y_jx_j$,

(M.2.1.2.(3)) $x_ix_{i+1}\cdots x_{j-1}y_{j-1}y_{j-2}\cdots y_{i-1}x_{i-1}x_{i-2}y_{i-2}y_{i-3}\cdots y_2x_2x_1y_1zy_0x_0y_{r-1}x_{r-1}x_{r-2}\cdots x_{j+1}y_{j+1}y_jx_j$,

(M.2.1.2.(4)) $x_ix_{i+1}\cdots x_{j-1}y_{j-1}y_{j-2}\cdots y_{i-1}x_{i-1}x_{i-2}y_{i-2}y_{i-3}\cdots y_2x_2x_1y_1zx_0y_0x_{r-1}y_{r-1}y_{r-2}\cdots x_{j+1}y_{j+1}y_jx_j$.

If i is even and j is even, then (Pr.2.1.2.(1)) for even r is a Hamilton path of $\mathrm{Pr}_r^{\boxtimes}$ linking v and w and (M.2.1.2.(1)) for odd r is one of $\mathrm{M}_r^{\boxtimes}$, while if i is even and j is odd, then (Pr.2.1.2.(2)) for even r is a Hamilton path of $\mathrm{Pr}_r^{\boxtimes}$ linking v and w and (M.2.1.2.(2)) for odd r is one of $\mathrm{M}_r^{\boxtimes}$, while if i is odd and j is even, then (Pr.2.1.2.(3)) for even r is a Hamilton path of $\mathrm{Pr}_r^{\boxtimes}$ linking v and w and (M.2.1.2.(3)) for odd r is one of $\mathrm{M}_r^{\boxtimes}$, while if i is odd and j is odd, then (Pr.2.1.2.(4)) for even r is a Hamilton path of $\mathrm{Pr}_r^{\boxtimes}$ linking v and w and (M.2.1.2.(4)) for odd r is one of $\mathrm{M}_r^{\boxtimes}$. This proves both (a16) and (a17) in the Case 2.1.2.

Case 2.2. $\{v, w\} \cap \{x_0, \ldots, x_{r-1}\} \neq \varnothing$ and $\{v, w\} \cap \{y_0, \ldots, y_{r-1}\} \neq \varnothing$. Since we are within Case 2 we know that $\{v, w\} \subseteq \{x_0, \ldots, x_{r-1}\} \sqcup \{y_0, \ldots, y_{r-1}\}$. Therefore the statement defining Case 2.2 is the negation of the one defining Case 2.1. Due to Φ_{xy} we may assume $v = x_i$ with $0 \leqslant i \leqslant r-1$ and $w = y_j$ with $0 \leqslant j \leqslant r-1$. Due to Φ_{xx} we may further assume that $i \leqslant j$.

Case 2.2.1. $i \in \{0, 1\}$. Not only do both $\mathrm{Aut}(\mathrm{Pr}_r^{\boxtimes})$ and $\mathrm{Aut}(\mathrm{M}_r^{\boxtimes})$ act transitively on $\{x_0, x_1, y_0, y_1\}$, but it is possible to use this symmetry while still preserving the assumption $i \leqslant j$ that we already made: namely, if $i = 1$, hence $v = x_1$ and $w = y_j$ with $1 = i \leqslant j$, then $\Psi_{xx}(v) = x_0$ and $\Psi_{xx}(w) = y_{r+1-i}$ (with $y_r := y_0$) and still $0 = i \leqslant j = r + 1 - i$. Therefore we may further assume that $i = 0$, i.e. $v = x_0$. Now consider the expressions

(Pr.2.2.1.(0)) $x_0zx_1x_2\cdots x_{j+1}y_{j+1}y_{j+2}x_{j+2}\cdots x_{r-2}x_{r-1}y_{r-1}y_0\cdots y_{j-1}y_j$,

(Pr.2.2.1.(1)) $x_0x_{r-1}x_{r-2}\cdots x_{j+1}y_{j+1}y_{j+2}\cdots y_{r-1}y_0zx_1y_1y_2x_2\cdots x_{j-1}x_jy_j$.

(M.2.2.1.(0)) $x_0zx_1x_2\cdots x_{j+1}y_{j+1}y_{j+2}x_{j+2}\cdots y_{r-2}y_{r-1}x_{r-1}y_0y_1\cdots y_j$,

(M.2.2.1.(1)) $x_0y_{r-1}x_{r-1}x_{r-2}y_{r-2}\cdots x_jx_{j+1}\cdots x_1zy_0y_1\cdots y_j$.

If j is even, then (Pr.2.2.1.(0)), and if j is odd then (Pr.2.2.1.(1)) is a Hamilton path of $\mathrm{Pr}_r^{\boxtimes}$ linking v and w, for every even $r \geqslant 4$. If j is even, then (M.2.2.1.(0)), and if j is odd then (M.2.2.1.(1)) is a Hamilton path of $\mathrm{M}_r^{\boxtimes}$ linking v and w, for every odd $r \geqslant 4$. This proves (a16) in the Case 2.2.1.

Case 2.2.2. $i \notin \{0, 1\}$. Now consider the expressions

(Pr.2.2.2.(0)) $x_ix_{i+1}\cdots x_{j+1}y_{j+1}y_{j+2}x_{j+2}\cdots x_{r-2}x_{r-1}y_{r-1}y_0x_0zx_1y_1y_2x_2x_3y_3\cdots x_{i-2}x_{i-1}y_{i-1}y_iy_{i+1}\cdots y_j$,

(Pr.2.2.2.(1)) $x_ix_{i+1}\cdots x_{j+1}y_{j+1}y_{j+2}x_{j+2}\cdots y_{r-2}y_{r-1}x_{r-1}x_0y_0zx_1y_1y_2x_2x_3y_3\cdots x_{i-2}x_{i-1}y_{i-1}y_iy_{i+1}\cdots y_j$.

(M.2.2.2.(0)) $x_ix_{i+1}\cdots x_{j+1}y_{j+1}y_{j+2}x_{j+2}\cdots y_{r-2}y_{r-1}x_{r-1}y_0x_0zx_1y_1y_2x_2x_3y_3\cdots x_{i-2}x_{i-1}y_{i-1}y_iy_{i+1}\cdots y_j$,

(M.2.2.2.(1)) $x_ix_{i+1}\cdots x_{j+1}y_{j+1}y_{j+2}x_{j+2}\cdots x_{r-2}x_{r-1}y_{r-1}x_0y_0zx_1y_1y_2x_2x_3y_3\cdots x_{i-2}x_{i-1}y_{i-1}y_iy_{i+1}\cdots y_j$.

Since the automorphism Ψ_{xx} changes the parity of the index of an x_i, and since (as explained in Case 2.2.1) the relation $i \leqslant j$ is preserved by Ψ_{xx}, we may assume that i is even.

If j is even, (Pr.2.2.2.(0)), and if j is odd, (Pr.2.2.2.(1)) is a Hamilton path of $\mathrm{Pr}_r^{\boxtimes}$ linking v and w, for every even $r \geqslant 4$. If j is even, then (M.2.2.2.(0)), and if j is odd then (M.2.2.2.(1)) is a Hamilton path of $\mathrm{M}_r^{\boxtimes}$ linking v and w, for every odd $r \geqslant 5$, completing the Case 2.2.2.

Since at each level of the case distinction the property defining the preceding level was partitioned into mutually exclusive properties, both (a16) and (a17) have now been proved.

As to (a18) and (a19), let $\{v, w\} \subseteq \mathrm{V}(\mathrm{Pr}_r^{\boxminus})$ be arbitrary distinct vertices. For most of the instances of the property of being Hamilton-connected it is possible to deduce the Hamilton-connectedness of $\mathrm{Pr}_r^{\boxminus}$ and $\mathrm{M}_r^{\boxminus}$ from (the proof of) (a16) in Lemma 17: if $\{v, w\} \cap \{z', z''\} = \varnothing$, then we have $\{v, w\} \subseteq \mathrm{V}(\mathrm{Pr}_r) \setminus \{z\}$ and therefore each Hamilton path P in Pr_r or M_r linking v and w contains z as a vertex of degree two. This implies that P can be extended to a Hamilton path in $\mathrm{Pr}_r^{\boxminus}$ linking v and w.

If on the contrary $\{v, w\} \cap \{z', z''\} \neq \varnothing$, then there are subcases: if $\{v, w\} = \{z', z''\}$, then $z'x_0y_0y_1 \cdots y_{r-1}x_{r-1}x_{r-2} \cdots x_1z''$ is—in Pr_r and in M_r as well—a Hamilton path linking v and w.

We are left with the case $|\{v, w\} \cap \{z', z''\}| = 1$. In the absence of information distinguishing v from w we may assume that $v \in \{z', z''\}$ and $w \notin \{z', z''\}$. One may treat this case, too, by re-using Hamilton paths in Pr_r or M_r, but now it can make a difference (for the extendability) *how* such a Hamilton path looks like around the 'special' subgraph induced on the vertices $\{z, x_0, y_0, x_1, y_1\}$ and it therefore seems quicker to treat this case directly. Since the property '$v \in \{z', z''\}$ and $w \notin \{z', z''\}$', at face value, still comprises several cases, we should reduce their number via automorphisms. However—essentially due to x_0z'' and the unique degree-5-vertex x_0 caused by it—both $\mathrm{Aut}(\mathrm{Pr}_r^{\boxminus})$ and $\mathrm{Aut}(\mathrm{M}_r^{\boxminus})$ are trivial. But since Hamilton-connectedness is a monotone graph property, it suffices to prove that $\mathrm{Pr}_r^{\boxminus,-} := \mathrm{Pr}_r^{\boxminus} - x_0z''$ and $\mathrm{M}_r^{\boxminus,-} := \mathrm{M}_r^{\boxminus} - x_0z''$ are Hamilton-connected, and these graphs *do* have symmetries again, essentially the same as $\mathrm{Pr}_r^{\boxtimes}$ and $\mathrm{M}_r^{\boxtimes}$.

The automorphism group of both $\mathrm{Pr}_r^{\boxminus,-}$ and $\mathrm{M}_r^{\boxminus,-}$ is the group generated by the two unique homomorphic extensions of $\left(\substack{\{z',z'',x_0,y_0,x_1,y_1\} \to \{z',z'',x_0,y_0,x_1,y_1\} \\ z' \mapsto z',\ z'' \mapsto z'',\ x_0 \leftrightarrow y_0,\ x_1 \leftrightarrow y_1}\right)$ and $\left(\substack{\{z',z'',x_0,y_0,x_1,y_1\} \to \{z',z'',x_0,y_0,x_1,y_1\} \\ z' \leftrightarrow z'',\ x_0 \leftrightarrow x_1,\ y_0 \leftrightarrow y_1}\right)$ to all of $\mathrm{V}(\mathrm{Pr}_r^{\boxminus,-}) = \mathrm{V}(\mathrm{M}_r^{\boxminus,-})$ (thus both $\mathrm{Aut}(\mathrm{Pr}_r^{\boxminus,-})$ and $\mathrm{Aut}(\mathrm{M}_r^{\boxminus,-})$ are isomorphic to the Klein four-group $\mathbb{F}_2 \oplus \mathbb{F}_2$). These extensions are involutions on $\mathrm{V}(\mathrm{Pr}_r^{\boxminus,-}) = \mathrm{V}(\mathrm{M}_r^{\boxminus,-})$ and will be denoted by Ξ_{xy} (the map with $z' \mapsto z', z'' \mapsto z''$ and $x_i \leftrightarrow y_i$ for every $0 \leqslant i \leqslant r-1$) and Ξ_{xx} (the map with $z' \leftrightarrow z''$ and, for $u \in \{x, y\}$, $u_1 \leftrightarrow u_0, u_2 \leftrightarrow u_{r-1}, u_3 \leftrightarrow u_{r-2}, \ldots, u_{\lfloor \frac{r+1}{2} \rfloor} \leftrightarrow u_{\lceil \frac{r+1}{2} \rceil}$). Both Ξ_{xy} and Ξ_{xx} are automorphisms of both $\mathrm{M}_r^{\boxminus,-}$ (for every $r \geqslant 5$) and $\mathrm{Pr}_r^{\boxminus,-}$ (for every $r \geqslant 4$).

Since Ξ_{xx} interchanges z' and z'', we may assume that $v = z'$. Then there are two cases left: $w \in \{x_0, y_0, x_1, y_1\}$ and its negation $w \in \{x_2, y_2, x_3, y_3, \ldots, x_{r-1}, y_{r-1}\}$ (keep in mind that we already assumed $w \notin \{z', z''\}$ and therefore this indeed is the negation).

Case 1. $w \in \{x_0, y_0, x_1, y_1\}$. Then since Ξ_{xy} maps $x_0 \leftrightarrow y_0$ and $x_1 \leftrightarrow y_1$ while keeping z' fixed, we may assume that $w \in \{x_0, x_1\}$ and are left with two cases.

Case 1.1. If $w = x_0$, then $z'y_0y_1z''x_1x_2y_2y_3x_3 \cdots y_{r-2}y_{r-1}x_{r-1}x_0$ is a Hamilton path linking v and w in $\mathrm{Pr}_r^{\boxminus,-}$ for every even $r \geqslant 4$, and $z'y_0y_1z''x_1x_2y_2y_3x_3 \cdots x_{r-2}x_{r-1}y_{r-1}x_0$ is one in $\mathrm{M}_r^{\boxminus,-}$ for every odd $r \geqslant 5$.

Case 1.2. If $w = x_1$, then $z'x_0y_0y_{r-1}x_{r-1}x_{r-2}y_{r-2}y_{r-3}x_{r-3} \cdots y_2y_1z''x_1$ is a Hamilton path linking v and w in $\mathrm{Pr}_r^{\boxminus,-}$ for every even $r \geqslant 4$, and $z'x_0y_0x_{r-1}y_{r-1}y_{r-2}x_{r-2}x_{r-3}y_{r-3} \cdots y_2y_1z''x_1$ is one in $\mathrm{M}_r^{\boxminus,-}$ for every odd $r \geqslant 5$.

Case 2. $w \in \{x_2, y_2, x_3, y_3, \ldots, x_{r-1}, y_{r-1}\}$. Then since Ξ_{xy} interchanges the sets $\{x_0, \ldots, x_{r-1}\}$ and $\{y_0, \ldots, y_{r-1}\}$ while fixing z', we may assume that $w = x_i$ with $2 \leqslant i \leqslant r-1$. If $i \geqslant 3$, then $z'x_0y_0y_1z''x_1x_2y_2y_3 \cdots y_{r-1}x_{r-1}x_{r-2} \cdots x_i$ is—regardless of whether i is odd or even—a Hamilton path linking v and w in both $\mathrm{Pr}_r^{\boxminus,-}$ and $\mathrm{M}_r^{\boxminus,-}$. In the case that $i = 2$, the path $z'y_0x_0x_{r-1}y_{r-1}y_{r-2}x_{r-2}x_{r-3} \cdots x_3y_3y_2y_1z''x_1x_2$ is a Hamilton path linking v and w in $\mathrm{Pr}_r^{\boxminus,-}$, and $z'y_0x_0y_{r-1}x_{r-1}x_{r-2}y_{r-2}y_{r-3} \cdots x_3y_3y_2y_1z''x_1x_2$ is one in $\mathrm{M}_r^{\boxminus,-}$, completing Case 2, and also the proof of both (a18) and (a19).

As to (a20) in the case $G = \mathrm{Pr}_r^{\boxtimes}$, for every even $r \geqslant 4$, the (5×5)-minor indexed by x_0y_0, x_1y_1, zx_1, zy_1, y_0y_{r-1} of the $\left(\| \mathrm{Pr}_r^{\boxtimes} \| \times 5\right)$-matrix which represents the elements of $\{\mathrm{c}_C : C \in \mathcal{CB}^{(1)}_{\mathrm{Pr}_r^{\boxtimes}}\}$ as elements of $\mathrm{C}_1(\mathrm{Pr}_r^{\boxtimes};\mathbb{F}_2) \supseteq \mathrm{Z}_1(\mathrm{Pr}_r^{\boxtimes};\mathbb{F}_2)$ w.r.t. the standard basis of $\mathrm{C}_1(\mathrm{Pr}_r^{\boxtimes};\mathbb{F}_2)$, is the one shown in (4).

$$\begin{array}{cccccc} & C_{\mathrm{ev},r,1} & C_{\mathrm{ev},r,2} & C_{\mathrm{ev},r,3} & C_{\mathrm{ev},r,4} & C_{\mathrm{ev},r,5} \\ x_0y_0 & 1 & 1 & 1 & 0 & 1 \\ x_1y_1 & 1 & 0 & 1 & 1 & 0 \\ zx_1 & 0 & 1 & 1 & 0 & 1 \\ zy_1 & 1 & 1 & 0 & 0 & 1 \\ y_0y_{r-1} & 0 & 0 & 1 & 1 & 1 \end{array} \tag{4}$$

The matrix in (4) is a nonsingular element of $(\mathbb{F}_2)^{[5]^2}$, its inverse being $\begin{pmatrix} 1&0&1&0&0\\ 0&1&0&1&1\\ 1&0&0&1&0\\ 0&1&1&1&0\\ 1&1&1&0&1 \end{pmatrix} \in (\mathbb{F}_2)^{[5]^2}$.

The existence of one such minor by itself proves (a20) in the case $G = \mathrm{Pr}_r^{\boxtimes}$. As to (a20) in the case $G = \mathrm{M}_r^{\boxtimes}$, for every odd $r \geqslant 5$, the (5×5)-minor indexed by $x_0y_0, x_1y_1, zx_1, zy_1, x_0y_{r-1}$ of the $\big(\|\mathrm{M}_r^{\boxtimes}\| \times 5\big)$-matrix which represents the elements of $\{\mathrm{c}_C\colon C \in \mathcal{CB}^{(1)}_{\mathrm{M}_r^{\boxtimes}}\}$ as elements of $\mathrm{C}_1(\mathrm{M}_r^{\boxtimes};\mathbb{F}_2) \supseteq \mathrm{Z}_1(\mathrm{M}_r^{\boxtimes};\mathbb{F}_2)$ w.r.t. the standard basis of $\mathrm{C}_1(\mathrm{M}_r^{\boxtimes};\mathbb{F}_2)$, is the one shown in (5).

$$\begin{array}{cccccc} & C_{\mathrm{od},r,1} & C_{\mathrm{od},r,2} & C_{\mathrm{od},r,3} & C_{\mathrm{od},r,4} & C_{\mathrm{od},r,5} \\ x_0y_0 & 1 & 1 & 1 & 0 & 1 \\ x_1y_1 & 1 & 0 & 1 & 1 & 0 \\ zx_1 & 0 & 1 & 1 & 0 & 1 \\ zy_1 & 1 & 1 & 0 & 0 & 1 \\ x_0y_{r-1} & 1 & 1 & 0 & 0 & 0 \end{array} \tag{5}$$

The matrix in (5) is a nonsingular element of $(\mathbb{F}_2)^{[5]^2}$, its inverse being $\begin{pmatrix} 1&0&1&0&0\\ 1&0&1&0&1\\ 1&0&0&1&0\\ 0&1&1&1&0\\ 0&0&0&1&1 \end{pmatrix} \in (\mathbb{F}_2)^{[5]^2}$.

The existence of one such minor by itself proves (a20) in the case $G = \mathrm{M}_r^{\boxtimes}$. As to (a20) in the case $G = \mathrm{Pr}_r^{\boxminus}$, for every even $r \geqslant 4$ the (5×5)-minor indexed by $x_0y_0, x_1y_1, z'x_0, z''y_1$ and x_0x_{r-1} of the $\big(\|\mathrm{Pr}_r^{\boxminus}\| \times 5\big)$-matrix which represents the elements of $\{\mathrm{c}_C\colon C \in \mathcal{CB}^{(1)}_{\mathrm{Pr}_r^{\boxminus}}\}$ as elements of $\mathrm{C}_1(\mathrm{Pr}_r^{\boxminus};\mathbb{F}_2) \subseteq \mathrm{Z}_1(\mathrm{Pr}_r^{\boxminus};\mathbb{F}_2)$ w.r.t. the standard basis of $\mathrm{C}_1(\mathrm{Pr}_r^{\boxminus};\mathbb{F}_2)$, is the one shown in (6).

$$\begin{array}{cccccc} & C_{\boxminus,\mathrm{ev},r,1} & C_{\boxminus,\mathrm{ev},r,2} & C_{\boxminus,\mathrm{ev},r,3} & C_{\boxminus,\mathrm{ev},r,4} & C_{\boxminus,\mathrm{ev},r,5} \\ x_0y_0 & 0 & 0 & 0 & 1 & 0 \\ x_1y_1 & 0 & 1 & 1 & 0 & 0 \\ z'x_0 & 1 & 0 & 1 & 1 & 1 \\ z''y_1 & 0 & 0 & 0 & 0 & 1 \\ x_0x_{r-1} & 0 & 1 & 0 & 0 & 1 \end{array} \tag{6}$$

The matrix in (6) is a nonsingular element of $(\mathbb{F}_2)^{[5]^2}$ with inverse $\begin{pmatrix} 1&1&1&0&1\\ 0&0&0&1&1\\ 0&1&0&1&1\\ 1&0&0&0&0\\ 0&0&0&1&0 \end{pmatrix}$. The existence of one such minor by itself proves (a20) in the case $G = \mathrm{Pr}_r^{\boxminus}$. As to (a20) in the case $G = \mathrm{M}_r^{\boxminus}$, due to the similar definitions in (P.⊟.ES.2) and (M.⊟.ES.2), it suffices to note that if in the preceding paragraph '$\mathrm{Pr}_r^{\boxminus}$' is replaced by '$\mathrm{M}_r^{\boxminus}$', 'even $r \geqslant 4$' by 'odd $r \geqslant 5$' and 'x_0x_{r-1}' by 'x_0y_{r-1}', then the matrix obtained is exactly the one in (6). This completes the proof of (a20) in its entirety.

As to (a21) in the case $G = \mathrm{Pr}_r^{\boxtimes}$, for every even $r \geqslant 4$, the $\big((r-1)\times(r-1)\big)$-minor indexed by $x_1y_1, x_2y_2, \ldots, x_{r-1}y_{r-1}$ of the $\big(\|\mathrm{Pr}_r^{\boxtimes}\| \times (r-1)\big)$-matrix which represents the elements of $\{\mathrm{c}_C\colon C \in \mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxtimes}}\}$ as elements of $\mathrm{C}_1(\mathrm{Pr}_r^{\boxtimes};\mathbb{F}_2) \supseteq \mathrm{Z}_1(\mathrm{Pr}_r^{\boxtimes};\mathbb{F}_2)$ w.r.t. the standard basis of $\mathrm{C}_1(\mathrm{Pr}_r^{\boxtimes};\mathbb{F}_2)$, is the element A of $(\mathbb{F}_2)^{[r-1]^2}$ which is defined by $A\big[x_1y_1, C^{x_1y_1}_{\mathrm{ev},r}\big] := 1$, $A\big[x_iy_i, C^{x_jy_j}_{\mathrm{ev},r}\big] := 1$ for every

$(i,j) \in \bigsqcup_{2\leqslant\iota\leqslant r-1}\{(\iota,\iota-1),(\iota,\iota)\}$ and $A\big[x_iy_i, C_{\mathrm{ev},r}^{x_jy_j}\big] := 0$ for every other $(i,j) \in \{1,\ldots,r-1\}^2$. This is a band matrix which in particular is 'lower' triangular with its main diagonal filled entirely with ones, hence nonsingular. The existence of one such minor alone implies the claim in the case $G = \mathrm{Pr}_r^{\boxtimes}$. As to the case $G = \mathrm{Pr}_r^{\boxminus}$, due to the similar definition of $\mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxminus}}$ compared to $\mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxtimes}}$, a proof in this case is obtained if in the first paragraph '$\mathrm{Pr}_r^{\boxtimes}$' is replaced by '$\mathrm{Pr}_r^{\boxminus}$', '$C_{\mathrm{ev},r}^{x_1y_1}$' by '$C_{\boxminus,\mathrm{ev},r}^{x_1y_1}$' and '$C_{\mathrm{ev},r}^{x_iy_i}$' by '$C_{\boxminus,\mathrm{ev},r}^{x_iy_i}$'. As to (a21) in the cases $G = \mathrm{M}_r^{\boxtimes}$ (respectively, $G = \mathrm{M}_r^{\boxminus}$), due to the similar definition of $\mathcal{CB}^{(2)}_{\mathrm{M}_r^{\boxtimes}}$ compared to $\mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxtimes}}$, a proof of these two cases is obtained if in the first paragraph 'even $r \geqslant 4$' is replaced by 'odd $r \geqslant 5$', '$\mathrm{Pr}_r^{\boxtimes}$' by '$\mathrm{M}_r^{\boxtimes}$' (respectively, '$\mathrm{M}_r^{\boxminus}$'), '$C_{\mathrm{ev},r}^{x_1y_1}$' by '$C_{\mathrm{od},r}^{x_1y_1}$' (respectively, '$C_{\boxminus,\mathrm{od},r}^{x_1y_1}$'), and '$C_{\mathrm{ev},r}^{x_iy_i}$' by '$C_{\mathrm{od},r}^{x_iy_i}$' (respectively, '$C_{\boxminus,\mathrm{od},r}^{x_iy_i}$'). This completes the proof of (a21) in its entirety.

As to (a22) in the case $G = \mathrm{Pr}_r^{\boxtimes}$, for an arbitrary even $r \geqslant 4$ let $c \in \left\langle \mathcal{CB}^{(1)}_{\mathrm{Pr}_r^{\boxtimes}}\right\rangle_{\mathbb{F}_2} \cap \left\langle \mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxtimes}}\right\rangle_{\mathbb{F}_2}$ be arbitrary. Then there exist $(\lambda^{(1)}) \in (\mathbb{F}_2)^{[5]}$ and $(\lambda^{(2)}) \in (\mathbb{F}_2)^{[r-1]}$ such that

(⊠.Su 1) $c = \sum_{1\leqslant i\leqslant 5} \lambda_i^{(1)}\, \mathrm{c}_{C_{\mathrm{ev},r,i}}$,
(⊠.Su 2) $c = \sum_{1\leqslant i\leqslant r-1} \lambda_i^{(2)}\, \mathrm{c}_{C_{\mathrm{ev},r}^{x_iy_i}}$,

where c_M for some set of edges M denotes the element $c \in \mathrm{C}_1(\mathrm{Pr}_r^{\boxtimes};\mathbb{F}_2)$ with $\mathrm{Supp}(c) = M$. We now show by contradiction that $\lambda_1^{(2)} = \cdots = \lambda_{r-1}^{(2)} = 0$, hence $\left\langle \mathcal{CB}^{(1)}_{\mathrm{Pr}_r^{\boxtimes}}\right\rangle_{\mathbb{F}_2} \cap \left\langle \mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxtimes}}\right\rangle_{\mathbb{F}_2} = \{0\}$. To this end, we make the assumption that, on the contrary,

$$\lambda_i^{(2)} = 1 \quad \text{(for at least one } 1 \leqslant i \leqslant r-1). \tag{7}$$

Drawing on the facts (straightforward to check using the definitions (P.⊠.ES.1) and (P.⊠.ES.2)),

(F1) $\{x_2y_2, x_3y_3, \ldots, x_{r-1}y_{r-1}\} = C_{\mathrm{ev},r,1} \cap C_{\mathrm{ev},r,2} \cap C_{\mathrm{ev},r,3} \cap C_{\mathrm{ev},r,4} \cap C_{\mathrm{ev},r,5}$,
(F2) $x_0x_{r-1} \in C_{\mathrm{ev},r,1} \cap C_{\mathrm{ev},r,2}$, $\quad x_0x_{r-1} \notin C_{\mathrm{ev},r,3} \cup C_{\mathrm{ev},r,4} \cup C_{\mathrm{ev},r,5}$,
(F3) $y_0y_{r-1} \notin C_{\mathrm{ev},r,1} \cup C_{\mathrm{ev},r,2}$, $\quad y_0y_{r-1} \in C_{\mathrm{ev},r,3} \cap C_{\mathrm{ev},r,4} \cap C_{\mathrm{ev},r,5}$,
(F4) $\{x_2y_2, x_3y_3, \ldots, x_{r-1}y_{r-1}\} \cap C_{\mathrm{ev},r}^{x_iy_i} \neq \varnothing$ for every $1 \leqslant i \leqslant r-1$,
(F5) $\{i \in \{1,2,\ldots,r-1\}: x_1y_1 \in C_{\mathrm{ev},r}^{x_iy_i}\} = \{1\}$,
(F6) $\{i \in \{1,\ldots,r-1\}: zx_1 \in C_{\mathrm{ev},r}^{x_iy_i}\} = \{1,\ldots,r-2\}$,
(F7) $\{\iota \in \{1,2,\ldots,r-1\}: x_iy_i \in C_{\mathrm{ev},r}^{x_\iota y_\iota}\} = \{i-1, i\}$ for every $2 \leqslant i \leqslant r-1$,
(F8) $\{zy_1, x_0y_0\} \cap C_{\mathrm{ev},r}^{x_iy_i} = \varnothing$ for every $1 \leqslant i \leqslant r-1$,
(F9) $\{x_0x_{r-1}, y_0y_{r-1}\} \subseteq C_{\mathrm{ev},r}^{x_iy_i}$ for every $1 \leqslant i \leqslant r-2$,
(F10) $\{x_0x_{r-1}, y_0y_{r-1}\} \cap C_{\mathrm{ev},r}^{x_{r-1}y_{r-1}} = \varnothing$,

we can now reason as follows, distinguishing whether $x_2y_2 \in \mathrm{Supp}(c)$ or not.

Case 1. $x_2y_2 \in \mathrm{Supp}(c)$. Then (⊠.Su 1) together with (F1) implies that $|\{i \in \{1,\ldots,5\}: \lambda_i^{(1)} = 1\}|$ is odd, and this implies that exactly one of the two numbers $|\{i \in \{1,2\}: \lambda_i^{(1)} = 1\}|$ and $|\{i \in \{3,4,5\}: \lambda_i^{(1)} = 1\}|$ is odd, which combined with (⊠.Su 1), (F2) and (F3) implies that $|\{x_0x_{r-1}, y_0y_{r-1}\} \cap \mathrm{Supp}(c)| = 1$. But this contradicts (⊠.Su 2), (F9) and (F10), which when taken together imply that $|\{x_0x_{r-1}, y_0y_{r-1}\} \cap \mathrm{Supp}(c)| \in \{0,2\} \not\ni 1$. This contradiction proves that Case 1 cannot occur (and we have not used our assumption (7) to arrive at this conclusion).

Case 2. $x_2y_2 \notin \mathrm{Supp}(c)$. From this we deduce

(Co 1) $zy_1 \notin \mathrm{Supp}(c)$,
(Co 2) $|\{i \in \{1,\ldots,5\}: \lambda_i^{(1)} = 1\}|$ is even,
(Co 3) $\{x_2y_2, x_3y_3, \ldots, x_{r-1}y_{r-1}\} \cap \mathrm{Supp}(c) = \varnothing$,
(Co 4) $\lambda_1^{(2)} = \cdots = \lambda_{r-1}^{(2)} = 1$,
(Co 5) $\{x_0x_{r-1}, y_0y_{r-1}\} \cap \mathrm{Supp}(c) = \varnothing$,

(Co 6) $zx_1 \notin \mathrm{Supp}(c)$,
(Co 7) $x_1y_1 \in \mathrm{Supp}(c)$,
(Co 8) $x_0y_0 \notin \mathrm{Supp}(c)$.

These claims can be justified thus: (Co 1) follows from (⊠.Su 2) and (F8). (Co 2) follows from combining $x_2y_2 \notin \mathrm{Supp}(c)$ with (⊠.Su 1) and (F1). (Co 3) follows from (Co 2), (⊠.Su 1) and (F1). (Co 4) follows from (Co 3), (⊠.Su 2), (F4) and (F7), together with our assumption (7). (At this instance we have learned that in (7)—if it is true—the existential quantifier must necessarily hold as a universal quantifier.) (Co 5) follows from (Co 4), (⊠.Su 2), (F9), (F10) and the evenness of $r-2$. (Co 6) follows from (Co 4), (F6), and the evenness of $r-2 = |\{1,\dots,r-2\}|$. (Co 7) follows from (⊠.Su 2) and (F5). (Co 8) follows from (⊠.Su 2) and (F8).

Now from (Co 5) combined with (F2) and (F3), it follows that (Co 2) cannot be true with both $n_{1,2} := |\{i \in \{1,2\}: \lambda_i^{(1)} = 1\}|$ and $n_{3,4,5} := |\{i \in \{3,4,5\}: \lambda_i^{(1)} = 1\}|$ being odd. Therefore both $n_{1,2}$ and $n_{3,4,5}$ must be even. To finish the proof, we use the abbreviations $S_{1,2} := \mathrm{Supp}(\lambda_1^{(1)} \cdot \mathrm{c}_{C_{\mathrm{ev},r,1}} + \lambda_2^{(1)} \cdot \mathrm{c}_{C_{\mathrm{ev},r,2}})$ and $S_{3,4,5} := \mathrm{Supp}(\lambda_3^{(1)} \cdot \mathrm{c}_{C_{\mathrm{ev},r,3}} + \lambda_4^{(1)} \cdot \mathrm{c}_{C_{\mathrm{ev},r,4}} + \lambda_5^{(1)} \cdot \mathrm{c}_{C_{\mathrm{ev},r,5}})$, with which we have

$$\mathrm{Supp}(c) = S_{1,2} \mathbin{\triangle} S_{3,4,5} \quad \text{(symmetric difference)}, \tag{8}$$

and distinguish cases according to the value of $n_{1,2} \in \{0,2\}$.

Case 2.1. $n_{1,2} = 0$. Then in particular $x_1y_1 \notin S_{1,2}$, $zx_1 \notin S_{1,2}$ and $zy_1 \notin S_{1,2}$.

Case 2.1.1. $n_{3,4,5} = 0$. This implies that $S_{3,4,5} = \varnothing$, and this together with $x_1y_1 \notin S_{1,2}$ and (8) in particular implies $x_1y_1 \notin \mathrm{Supp}(c)$, contradicting (Co 7) and proving Case 2.1.1 to be impossible.

Case 2.1.2. $n_{3,4,5} = 2$. Let us distinguish whether $\lambda_5^{(1)} \in \mathbb{F}_2$ is 0 or 1 (the motivation for this being that $zy_1 \notin S_{1,2}$ and among $C_{\mathrm{ev},r,3}$, $C_{\mathrm{ev},r,4}$, $C_{\mathrm{ev},r,5}$ only $C_{\mathrm{ev},r,5}$ contains zy_1, making it possible to draw a conclusion from the value of $\lambda_5^{(1)}$). If $\lambda_5^{(1)} = 1$, then $zy_1 \in \mathrm{Supp}(\lambda_5^{(1)} \cdot \mathrm{c}_{C_{\mathrm{ev},r,5}})$ and moreover exactly one of $\lambda_3^{(1)}$ and $\lambda_4^{(1)}$ is $= 1$. Whichever it is, due to $zy_1 \notin \mathrm{Supp}(\lambda_3^{(1)} \cdot \mathrm{c}_{C_{\mathrm{ev},r,3}})$ and $zy_1 \notin \mathrm{Supp}(\lambda_4^{(1)} \cdot \mathrm{c}_{C_{\mathrm{ev},r,4}})$ it follows that $zy_1 \in S_{3,4,5}$, which combined with $zy_1 \notin S_{1,2}$ and (8) implies $zy_1 \in \mathrm{Supp}(c)$, contradicting (Co 1) and proving $\lambda_5^{(1)} = 1$ to be impossible. If on the contrary $\lambda_5^{(1)} = 0$, then $\lambda_3^{(1)} = \lambda_4^{(1)} = 1$ and it follows that $zx_1 \in S_{3,4,5}$. Being within Case 2.1 we know that $zx_1 \notin S_{1,2}$, hence in view of (8) we may conclude that $zx_1 \in \mathrm{Supp}(c)$, contradicting (Co 6), proving Case 2.1.2, and therefore Case 2.1 as a whole, to be impossible.

Case 2.2. $n_{1,2} = 2$. This implies $x_0y_0 \notin S_{1,2}$, $x_1y_1 \in S_{1,2}$ and $zx_1 \in S_{1,2}$. Again it remains to consider the possibilities for $n_{3,4,5} \in \{0,1,2,3\}$ to be even.

Case 2.2.1. $n_{3,4,5} = 0$. Then $S_{3,4,5} = \varnothing$, and this together with $zx_1 \in S_{1,2}$ and (8) in particular implies $zx_1 \in \mathrm{Supp}(c)$, contradicting (Co 6) and proving Case 2.2.1 to be impossible.

Case 2.2.2. $n_{3,4,5} = 2$. Again we analyse this case by distinguishing whether $\lambda_5^{(1)} \in \mathbb{F}_2$ is 0 or 1. If $\lambda_5^{(1)} = 1$, then exactly one of $\lambda_3^{(1)}$ and $\lambda_4^{(1)}$ is 1 and, whichever it is, it follows that $x_1y_1 \in S_{3,4,5}$. Being within Case 2.2. we know $x_1y_1 \in S_{1,2}$, hence in view of (8) it follows that $x_1y_1 \notin \mathrm{Supp}(c)$, contradicting (Co 7) and proving $\lambda_5^{(1)} = 1$ to be impossible. If on the contrary $\lambda_5^{(1)} = 0$, then $\lambda_3^{(1)} = \lambda_4^{(1)} = 1$ and it follows that $x_0y_0 \in S_{3,4,5}$. Being within Case 2.2 we know that $x_0y_0 \in S_{1,2}$ which in view of (8) implies $x_0y_0 \in \mathrm{Supp}(c)$, contradicting (Co 8) and proving $\lambda_5^{(1)} = 0$ to be impossible. This proves Case 2.2.2, and therefore also Case 2.2 and the entire Case 2, to be impossible. Since the mutually exclusive Cases 1 and 2 both lead to contradictions, the assumption (7) is false, completing the proof of (a22) for $G = \mathrm{Pr}_r^{\boxtimes}$.

As to (a22) in the case $G = \mathrm{M}_r^{\boxtimes}$, the proof given for the case $G = \mathrm{Pr}_r^{\boxtimes}$ can be repeated with the appropriate minor changes to obtain a proof in the case $G = \mathrm{M}_r^{\boxtimes}$, these changes being the following: first of all, the statements (F1)–(F10) have been chosen in such a way that each of (F1)–(F10) becomes a true statement about the set $\mathcal{CB}^{(2)}_{\mathrm{M}_r^{\boxtimes}}$ if exactly the following changes are made in (F1)–(F10): 'ev' is to be replaced by 'od', 'x_0x_{r-1}' is to be replaced by 'x_0y_{r-1}' (all occurrences, i.e. in (F2), in (F9) and in (F10)), 'y_0y_{r-1}' is to be replaced by 'y_0x_{r-1}' (all occurrences, i.e. in (F3), in (F9) and in (F10)). With the references to (F1)–(F10) now referring to the statements thus modified, the only thing to be done in the entire remaining proof of the case $G = \mathrm{Pr}_r^{\boxtimes}$ (in order to arrive at a proof of the case $G = \mathrm{M}_r^{\boxtimes}$) is to replace '$x_0x_{r-1}$' by '$x_0y_{r-1}$' and '$y_0y_{r-1}$' by '$y_0x_{r-1}$' at all three occurrences of these edges (twice in Case 1, once in (Co 5)), and moreover to replace 'ev' by 'od'. This completes the proof of (a22) for $G = \mathrm{M}_r^{\boxtimes}$.

As to (a22) in the case $G = \mathrm{Pr}_r^{\boxminus}$, for an arbitrary even $r \geqslant 4$ let $c \in \left\langle \mathcal{CB}^{(1)}_{\mathrm{Pr}_r^{\boxminus}} \right\rangle_{\mathbb{F}_2} \cap \left\langle \mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxminus}} \right\rangle_{\mathbb{F}_2}$ be arbitrary. Then there are $(\lambda^{(1)}) \in (\mathbb{F}_2)^{[5]}$ and $(\lambda^{(2)}) \in (\mathbb{F}_2)^{[r-1]}$ such that

($\boxminus$.Su 1) $c = \sum_{1 \leqslant i \leqslant 5} \lambda_i^{(1)} \cdot \mathsf{c}_{C_{\boxminus,\mathrm{ev},r,i}}$,
($\boxminus$.Su 2) $c = \sum_{1 \leqslant i \leqslant r-1} \lambda_i^{(2)} \cdot \mathsf{c}_{C^{x_i y_i}_{\boxminus,\mathrm{ev},r}}$,

where C_M for some set of edges M denotes the unique element $c \in \mathsf{C}_1(\mathrm{Pr}_r^{\boxminus}; \mathbb{F}_2)$ with $\mathrm{Supp}(c) = M$. We will show directly (this time we will not have any use for making the assumption (7)) that $c = 0$, hence $\left\langle \mathcal{CB}^{(1)}_{\mathrm{Pr}_r^{\boxminus}} \right\rangle_{\mathbb{F}_2} \cap \left\langle \mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxminus}} \right\rangle_{\mathbb{F}_2} = \{0\}$. We can now use the evident facts

($\boxminus$.F1) $z'z'' \in \bigcap_{1 \leqslant i \leqslant r-1} C^{x_i y_i}_{\boxminus,\mathrm{ev},r}$,
($\boxminus$.F2) $\{x_0 x_{r-1}, y_0 y_{r-1}\} \subseteq C^{x_i y_i}_{\boxminus,\mathrm{ev},r}$ for every $1 \leqslant i \leqslant r-2$,
($\boxminus$.F3) $z'z'' \notin C_{\boxminus,\mathrm{ev},r,1}, z'z'' \in C_{\boxminus,\mathrm{ev},r,2}, z'z'' \notin C_{\boxminus,\mathrm{ev},r,3}, z'z'' \in C_{\boxminus,\mathrm{ev},r,4}, z'z'' \notin C_{\boxminus,\mathrm{ev},r,5}$,
($\boxminus$.F4) for even $r \geqslant 4$, the only circuit among the circuits in $\mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxminus}}$ to contain $x_0 z''$ is $C^{x_{r-1} y_{r-1}}_{\boxminus,\mathrm{ev},r}$,
($\boxminus$.F5) for even $r \geqslant 4$, the only circuit among the circuits in $\mathcal{CB}^{(1)}_{\mathrm{Pr}_r^{\boxminus}} \sqcup \mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxminus}}$ to contain $y_1 z''$ is $C_{\boxminus,\mathrm{ev},r,5}$,
($\boxminus$.F6) for even $r \geqslant 4$, the only circuit among the circuits in $\mathcal{CB}^{(1)}_{\mathrm{Pr}_r^{\boxminus}} \sqcup \mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxminus}}$ to contain $x_0 y_0$ is $C_{\boxminus,\mathrm{ev},r,4}$,
($\boxminus$.F7) for even $r \geqslant 4$, the only circuits among the circuits in $\mathcal{CB}^{(1)}_{\mathrm{Pr}_r^{\boxminus}} \sqcup \mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxminus}}$ to contain an odd number of the two edges $x_0 x_{r-1}$ and $y_0 y_{r-1}$ are the two circuits $C_{\boxminus,\mathrm{ev},r,3}$ and $C_{\boxminus,\mathrm{ev},r,5}$,

to argue as follows. First of all, we immediately conclude that

($\boxminus$.Co 1) $\lambda_4^{(1)} = 0$ because of ($\boxminus$.Su 1) and ($\boxminus$.Su 2) combined with ($\boxminus$.F6),
($\boxminus$.Co 2) $\lambda_5^{(1)} = 0$ because of ($\boxminus$.Su 1) and ($\boxminus$.Su 2) combined with ($\boxminus$.F5).

Case 1. $|\{i \in \{1, \ldots, r-1\} \colon \lambda_i^{(2)} = 1\}|$ is odd. Then ($\boxminus$.Su 2) together with ($\boxminus$.F1) implies $z'z'' \in \mathrm{Supp}(c)$. Therefore, and because of ($\boxminus$.F3), it follows that exactly one of $\lambda_2^{(1)}$ and $\lambda_4^{(1)}$ is equal to 1, hence $\lambda_2^{(1)} = 1$ because of ($\boxminus$.Co 1). Now let us consider $\lambda_3^{(1)}$. It cannot be true that $\lambda_3^{(1)} = 1$, since then ($\boxminus$.F7) implies $\lambda_5^{(1)} = 1$, contradicting ($\boxminus$.Co 2). Thus we may assume that $\lambda_3^{(1)} = 0$. This implies $x_1 y_1 \in \mathrm{Supp}(c)$ due to ($\boxminus$.Su 1), $\lambda_2^{(1)} = 1$, ($\boxminus$.Co 1) and the fact that for every even $r \geqslant 4$, the only circuits among the circuits in $\mathcal{CB}^{(1)}_{\mathrm{Pr}_r^{\boxminus}}$ to contain $x_1 y_1$ are $C_{\boxminus,\mathrm{ev},r,2}$ and $C_{\boxminus,\mathrm{ev},r,3}$. Among the coefficients $\lambda_i^{(1)}$, $1 \leqslant i \leqslant 5$, only the value of $\lambda_1^{(1)}$ is not yet known to us.

Case 1.1. $\lambda_1^{(1)} = 0$. Then $z'y_0 \in C_{\boxminus,\mathrm{ev},r,2}$, $\lambda_2^{(1)} = 1$ and $\lambda_1^{(1)} = \lambda_3^{(1)} = \lambda_4^{(1)} = \lambda_5^{(1)} = 0$ together with ($\boxminus$.Su 1) imply that $z'y_0 \in \mathrm{Supp}(c)$. Since for every even $r \geqslant 4$, the only circuit among the circuits in $\mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxminus}}$ to contain $y_0 z'$ is $C^{x_{r-1} y_{r-1}}_{\boxminus,\mathrm{ev},r}$, from $z'y_0 \in \mathrm{Supp}(c)$ it follows that $\lambda_{r-1}^{(2)} = 1$. Being within Case 1, this implies that $|\{i \in \{1, \ldots, r-2\} \colon \lambda_i^{(2)} = 1\}|$ is even, which by ($\boxminus$.F2) implies that $\{x_0 x_{r-1}, y_0 y_{r-1}\} \cap \mathrm{Supp}(c) = \varnothing$; but $\{x_0 x_{r-1}, y_0 y_{r-1}\} \subseteq C_{\boxminus,\mathrm{ev},r,2}$ together with ($\boxminus$.Su 1), $\lambda_1^{(1)} = \lambda_3^{(1)} = \lambda_4^{(1)} = \lambda_5^{(1)} = 0$ and $\lambda_2^{(1)} = 1$ implies that, on the contrary, $\{x_0 x_{r-1}, y_0 y_{r-1}\} \subseteq \mathrm{Supp}(c)$. This contradiction proves Case 1.1 to be impossible.

Case 1.2. $\lambda_1^{(1)} = 1$. Then $\lambda_3^{(1)} = \lambda_4^{(1)} = \lambda_5^{(1)} = 0, \lambda_1^{(1)} = \lambda_2^{(1)} = 1$ and ($\boxminus$.Su 1) together imply $x_0 z'' \notin \mathrm{Supp}(c)$. Because of ($\boxminus$.F4), this implies $\lambda_{r-1}^{(2)} = 0$. Being within Case 1, it follows that $|\{i \in \{1, \ldots, r-2\} \colon \lambda_i^{(2)} = 1\}|$ is even, hence ($\boxminus$.F2) together with ($\boxminus$.Su 2) implies that $\{x_0 x_{r-1}, y_0 y_{r-1}\} \cap \mathrm{Supp}(c) = \varnothing$; but $\lambda_3^{(1)} = \lambda_4^{(1)} = \lambda_5^{(1)} = 0, \lambda_1^{(1)} = \lambda_2^{(1)} = 1$, and ($\boxminus$.Su 2), together with the facts that $\{x_0 x_{r-1}, y_{r-1}\} \cap C_{\boxminus,r,1} = \varnothing$ and $\{x_0 x_{r-1}, y_{r-1}\} \subseteq C_{\boxminus,r,2}$ imply $\{x_0 x_{r-1}, y_0 y_{r-1}\} \subseteq \mathrm{Supp}(c)$, contradiction. Therefore Case 1.2 is impossible, too.

This proves the entire Case 1 to be impossible.

Case 2. $|\{i \in \{1, \ldots, r-1\} \colon \lambda_i^{(2)} = 1\}|$ is even. Then ($\boxminus$.Su 2) together with ($\boxminus$.F1) imply $z'z'' \notin \mathrm{Supp}(c)$, hence in view of ($\boxminus$.F3) it follows that either $\lambda_2^{(1)} = \lambda_4^{(1)} = 0$ or $\lambda_2^{(1)} = \lambda_4^{(1)} = 1$, the latter being impossible because of ($\boxminus$.Co 1). Therefore, $\lambda_2^{(1)} = \lambda_4^{(1)} = 0$.

Case 2.1. $\lambda_3^{(1)} = 1$. This, together with (⊟.Su 1), (⊟.Su 2), (⊟.F7) and the fact that every $C \in \{C_{\boxminus,\mathrm{ev},r}^{x_iy_i} : 1 \leqslant i \leqslant r-1\}$ contains an even number of the edges x_0x_{r-1} and y_0y_{r-1}, implies that we must have $\lambda_5^{(1)} = 1$, contradicting (⊟.Co 2).

Case 2.2. $\lambda_3^{(1)} = 0$. Then (⊟.Su 1), $\lambda_2^{(1)} = 0$ and the fact that $C_{\boxminus,r,2}$ and $C_{\boxminus,r,3}$ are the only circuits among $C_{\boxminus,r,1}, \ldots, C_{\boxminus,r,5}$ to contain x_1y_1 imply that $x_1y_1 \notin \mathrm{Supp}(c)$. Hence from (⊟.Su 2), together with the fact that for every even $r \geqslant 4$, the only circuit among the circuits in $\mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxminus}}$ to contain x_1y_1 is $C_{\boxminus,\mathrm{ev},r}^{x_1y_1}$, it follows that $\lambda_1^{(2)} = 0$. Now let us consider $\lambda_{r-1}^{(2)}$. If we would have $\lambda_{r-1}^{(2)} = 1$, then—being within Case 2—the number $|\{i \in \{2, \ldots, r-2\} : \lambda_i^{(2)} = 1\}|$ is odd, hence $\{x_0x_{r-1}, y_0y_{r-1}\} \subseteq \mathrm{Supp}(c)$ by (⊟.Su 2) and (⊟.F2); but this contradicts (⊟.Su 1), $\lambda_2^{(1)} = \lambda_3^{(1)} = 0$, $\{x_0x_{r-1}, y_0y_{r-1}\} \cap C_{\boxminus,r,1} = \varnothing$, $\{x_0x_{r-1}, y_0y_{r-1}\} \cap C_{\boxminus,r,4} = \varnothing$ and $\{x_0x_{r-1}, y_0y_{r-1}\} \cap C_{\boxminus,r,5} = \{x_0x_{r-1}\}$, which when taken together imply $\{x_0x_{r-1}, y_0y_{r-1}\} \cap \mathrm{Supp}(c) \in \left\{\varnothing, \{x_0, x_{r-1}\}\right\}$. Therefore we may assume $\lambda_{r-1}^{(2)} = 0$. Then—being within Case 2—the number $|\{i \in \{2, \ldots, r-2\} : \lambda_i^{(2)} = 1\}|$ is even, hence (⊟.Su 2) and (⊟.F2) imply that $\{x_0x_{r-1}, y_0y_{r-1}\} \cap \mathrm{Supp}(c) = \varnothing$. Since among $C_{\boxminus,r,1}, \ldots, C_{\boxminus,r,5}$ only $C_{\boxminus,r,5}$ contains x_0x_{r-1}, this implies $\lambda_5^{(1)} = 0$. We now know that $\lambda_2^{(2)} = \lambda_3^{(2)} = \lambda_4^{(2)} = \lambda_5^{(2)} = 0$. Therefore, if we would have $\lambda_1^{(1)} = 1$, then $x_1z'' \in \mathrm{Supp}(c)$, contradicting the fact that (⊟.Su 2), $\lambda_{r-1}^{(2)} = 0$, the evenness of $|\{i \in \{2, \ldots, r-2\} : \lambda_i^{(2)} = 1\}|$ and the property $x_1z'' \in C_{\boxminus,\mathrm{ev},r}^{x_iy_i}$ for every $1 \leqslant i \leqslant r-2$ together imply $x_1z'' \notin \mathrm{Supp}(c)$. Thus, $\lambda_1^{(1)} = \lambda_2^{(2)} = \lambda_3^{(2)} = \lambda_4^{(2)} = \lambda_5^{(2)} = 0$, hence $c = 0$ by (⊟.Su 1), completing the proof of $\left\langle \mathcal{CB}^{(1)}_{\mathrm{Pr}_r^{\boxminus}} \right\rangle_{\mathbb{F}_2} \cap \left\langle \mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxminus}} \right\rangle_{\mathbb{F}_2} = \{0\}$ in Case 2. This completes the proof of (a22) in the case $G = \mathrm{Pr}_r^{\boxminus}$. As to (a22) in the case $G = \mathrm{M}_r^{\boxminus}$, again the proof of the case $G = \mathrm{Pr}_r^{\boxminus}$ can be repeated with the necessary small changes, namely: throughout, 'Pr_r' is to be replaced by 'M_r', 'ev' by 'od', 'x_0x_{r-1}' by 'x_0y_{r-1}', and 'y_0y_{r-1}' by 'y_0x_{r-1}'. Afterwards, (⊟.F1)–(⊟.F2) are still true and the proof given for the case $G = \mathrm{Pr}_r^{\boxminus}$ has become a proof for the case $G = \mathrm{M}_r^{\boxminus}$. The proof of Lemma (a22) is now complete.

As to (a23).(⊠.(0)), note that $\dim_{\mathbb{F}_2} \mathrm{Z}_1(\mathrm{Pr}_r^{\boxtimes}; \mathbb{F}_2) = (3r+4) - (2r+1) + 1 = r+4$, and that (a20), (a21) and (a22) in the case $G = \mathrm{Pr}_r^{\boxtimes}$ together imply that for even $r \geqslant 4$ we have $\dim_{\mathbb{F}_2}\left(\left\langle \mathcal{CB}^{(1)}_{\mathrm{Pr}_r^{\boxtimes}} \right\rangle_{\mathbb{F}_2} + \left\langle \mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxtimes}} \right\rangle_{\mathbb{F}_2}\right) = r+4$. Therefore the set $\left\langle \mathcal{CB}^{(1)}_{\mathrm{Pr}_r^{\boxtimes}} \right\rangle_{\mathbb{F}_2} + \left\langle \mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxtimes}} \right\rangle_{\mathbb{F}_2}$ is a $\mathbb{F}_2$-linear subspace of $\mathrm{Z}_1(\mathrm{Pr}_r^{\boxtimes}; \mathbb{F}_2)$ having the same dimension as the ambient space. In a vector space this implies equality as a set. This proves (⊠.(0)). An entirely analogous argument proves (a23).(⊠.(1)).

As to (a23).(⊟.(0)), note that $\dim_{\mathbb{F}_2} \mathrm{Z}_1(\mathrm{Pr}_r^{\boxminus}; \mathbb{F}_2) = (3r+6) - (2r+2) + 1 = r+5$ and that (a20), (a21) and (a22) in the case $G = \mathrm{Pr}_r^{\boxtimes}$ together imply that for even $r \geqslant 4$ we have $\dim_{\mathbb{F}_2}\left(\left\langle \mathcal{CB}^{(1)}_{\mathrm{Pr}_r^{\boxtimes}} \right\rangle_{\mathbb{F}_2} + \left\langle \mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxtimes}} \right\rangle_{\mathbb{F}_2}\right) = r+4$. Since $\dim_K(V/U) = \dim_K(V) - \dim_K(U)$ for finite-dimensional K-vector spaces $U \subseteq V$, this implies (⊟.(0)). An entirely analogous argument proves (a23).(⊟.(1)).

As to (a23).(⊟.$|\cdot|-1$.(0)), this claim follows quickly from (⊟.(0)): it suffices to note that in $\mathrm{Pr}_r^{\boxminus}$ there actually exists a circuit of length $|\cdot|-1$. Since $|\mathrm{Pr}_r^{\boxminus}| = |\mathrm{M}_r^{\boxminus}| = r+4$ is even for even r, and since the support of the sum of two circuits of even length is an edge-disjoint union of circuits of even length, any circuit of length $|\cdot|-1$ in $\mathrm{Pr}_r^{\boxminus}$ is not contained in $\left\langle \mathcal{CB}^{(1)}_{\mathrm{Pr}_r^{\boxtimes}} \right\rangle_{\mathbb{F}_2} + \left\langle \mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxtimes}} \right\rangle_{\mathbb{F}_2}$, hence after adding this circuit to the set $\mathcal{CB}^{(1)}_{\mathrm{Pr}_r^{\boxtimes}} \sqcup \mathcal{CB}^{(2)}_{\mathrm{Pr}_r^{\boxtimes}}$, the $\mathbb{F}_2$-linear span has dimension $(r+4)+1 = r+5 = \dim_{\mathbb{F}_2} \mathrm{Z}_1(\mathrm{Pr}_r^{\boxminus}; \mathbb{F}_2)$, proving (⊟.$|\cdot|-1$.(0)), since finite-dimensional vector spaces do not contain proper subspaces of the same dimension. An entirely analogous argumentation proves (a23).(⊟.$|\cdot|-1$.(1)), this time using (⊟.(1)).

We have now proved (a24)–(a29): property (a24) follows from (⊠.(0)) (which is equivalent to $\mathrm{Pr}_r^{\boxtimes} \in \mathrm{cd}_0\mathcal{C}_{\{|\cdot|\}}$), (a16) and Definition 16.(6); property (a25) follows from (⊠.(1)) (which is equivalent to $\mathrm{M}_r^{\boxtimes} \in \mathrm{cd}_0\mathcal{C}_{\{|\cdot|\}}$), (a17) and Definition 16.(6); property (a26) follows from (⊟.(0)) (which is equivalent to $\mathrm{Pr}_r^{\boxminus} \in \mathrm{cd}_1\mathcal{C}_{\{|\cdot|\}}$), (a18) and Definition 16.(6); property (a27) follows from (⊟.(1)) (which is equivalent to $\mathrm{M}_r^{\boxminus} \in \mathrm{cd}_1\mathcal{C}_{\{|\cdot|\}}$), (a19) and Definition 16.(6); property (a28) follows from (⊟.$|\cdot|-1$.(0))

(which is equivalent to $\mathrm{Pr}_r^{\boxminus} \in \mathrm{cd}_0\mathcal{C}_{\{|\cdot|-1,|\cdot|\}}$), (a18) and Definition 16.(6); property (a29) follows from ($\boxminus$.$|\cdot|-1$.(1)) (which is equivalent to $\mathrm{M}_r^{\boxminus} \in \mathrm{cd}_0\mathcal{C}_{\{|\cdot|-1,|\cdot|\}}$), (a19) and Definition 16.(6).

As to (a30), the bandwidth of any of C_n^2, CL_r, $\mathrm{Pr}_r^{\boxtimes}$, $\mathrm{Pr}_r^{\boxminus}$, $\mathrm{M}_r^{\boxtimes}$ and $\mathrm{M}_r^{\boxminus}$ is constant, i.e. does not grow with r or n. Therefore (a30) is true in stronger form than is stated here. Since knowing the exact bandwidths would profit us nothing given the proof technology that is available at present, knowing the statement (a30) is enough. To prove it, we employ a general characterization [13, Theorem 8] of low-bandwidth graphs due to Böttcher, Pruessmann, Taraz and Würfl. This characterization allows us to prove the smallness of the bandwidth for each of the rather different graphs C_n^2, CL_r, $\mathrm{Pr}_r^{\boxtimes}$, $\mathrm{Pr}_r^{\boxminus}$, $\mathrm{M}_r^{\boxtimes}$ and $\mathrm{M}_r^{\boxminus}$ without any close attention to the specifics of these graphs—simply by exhibiting small separators: in C_n^2 there does not exist any edge between the two sets $A := \{0, 1, \ldots, \lfloor \frac{n}{2} \rfloor - 2\}$ and $B := \{\lfloor \frac{n}{2} \rfloor + 1, \ldots, n-3\}$, and since both $|A|$ and $|B|$ are $\leqslant \frac{2}{3}|C_n^2|$, the existence of the separator $S := \{\lfloor \frac{n}{2} \rfloor - 1, \lfloor \frac{n}{2} \rfloor, n-2, n-1\}$ implies that the separation number (in the sense of [13, Definition 2]) of C_n^2 is at most 4. The claim (a30) in the case of $G = \mathrm{CL}_r$ now follows by [13, Theorem 8, equivalence (2) $\Leftrightarrow$ (4)]. To prove the case $G = \mathrm{CL}_r$ of (a30), in the first sentence of this paragraph use '$A := \bigsqcup_{1 \leqslant i \leqslant \lfloor \frac{r}{2} \rfloor - 1}\{a_i, b_i\}$', '$B := \bigsqcup_{\lfloor \frac{r}{2} \rfloor + 1 \leqslant i \leqslant r-1}\{a_i, b_i\}$' and '$S := \{a_0, b_0, a_{\lfloor \frac{r}{2} \rfloor}, b_{\lfloor \frac{r}{2} \rfloor}\}$'. To prove the cases $G \in \{\mathrm{Pr}_r^{\boxtimes}, \mathrm{M}_r^{\boxtimes}\}$ of (a30), in the first sentence of this paragraph use '$A := \{z\} \sqcup \bigsqcup_{1 \leqslant i \leqslant \lfloor \frac{r}{2} \rfloor - 1}\{x_i, y_i\}$', '$B := \bigsqcup_{\lfloor \frac{r}{2} \rfloor + 1 \leqslant i \leqslant r-1}\{x_i, y_i\}$' and '$S := \{a_0, b_0, a_{\lfloor \frac{r}{2} \rfloor}, b_{\lfloor \frac{r}{2} \rfloor}\}$'. To prove the cases $G \in \{\mathrm{Pr}_r^{\boxminus}, \mathrm{M}_r^{\boxminus}\}$ of (a30), use B and S as in the preceding sentence but '$A := \{z', z''\} \sqcup \bigsqcup_{1 \leqslant i \leqslant \lfloor \frac{r}{2} \rfloor - 1}\{x_i, y_i\}$'. This proves the statement about the bandwidth in (a30), for every $H \in \{C_n^2, \mathrm{CL}_r, \mathrm{Pr}_r^{\boxtimes}, \mathrm{Pr}_r^{\boxminus}, \mathrm{M}_r^{\boxtimes}, \mathrm{M}_r^{\boxminus}\}$.

As to the additional claims concerning $H \in \{\mathrm{Pr}_r^{\boxtimes}, \mathrm{Pr}_r^{\boxminus}, \mathrm{M}_r^{\boxtimes}, \mathrm{M}_r^{\boxminus}\}$, we explicitly give suitable maps b_H and h_H (thus for $\mathrm{Pr}_r^{\boxtimes}$, $\mathrm{Pr}_r^{\boxminus}$, $\mathrm{M}_r^{\boxtimes}$, $\mathrm{M}_r^{\boxminus}$ giving another proof of the small bandwidth).

As to $H = \mathrm{Pr}_r^{\boxtimes}$, for every even $r \geqslant 4$, the map b_H defined by $z \mapsto 1$, $x_0 \mapsto 2$ $x_i \mapsto 4i$ for $1 \leqslant i \leqslant \lfloor \frac{r}{2} \rfloor$, $x_i \mapsto 4(r-i)+2$ for $\lfloor \frac{r}{2} \rfloor + 1 \leqslant i \leqslant r-1$, $y_0 \mapsto 3$, $y_i \mapsto 4i+1$ for $1 \leqslant i \leqslant \lfloor \frac{r}{2} \rfloor$, and $y_i \mapsto 4(r-i)+3$ for $\lfloor \frac{r}{2} \rfloor + 1 \leqslant i \leqslant r-1$ is a bandwidth-4-labelling of $\mathrm{Pr}_r^{\boxtimes}$. Moreover, the map h_H defined by $z \mapsto 0$, $x_i \mapsto 1$ and $y_i \mapsto 2$ for even $0 \leqslant i \leqslant r-1$, $x_i \mapsto 2$ and $y_i \mapsto 1$ for odd $0 \leqslant i \leqslant r-1$, is a 3-colouring of $\mathrm{Pr}_r^{\boxtimes}$ which for every r large enough to have simultaneously $\beta|H| = \beta(2r+1) \geqslant 1 = |\mathrm{h}_H^{-1}(0)|$ and $8 \cdot 2 \cdot \beta \cdot |H| = 16\beta(2r+1) \geqslant 2$ obviously satisfies the requirement in Theorem 3 of being $(8 \cdot 2 \cdot \beta \cdot |H|, 4 \cdot 2 \cdot \beta \cdot |H|)$-zero-free w.r.t. b_H and having $|\mathrm{h}_H^{-1}(0)| \leqslant \beta|H|$. This proves (a30) for $H = \mathrm{Pr}_r^{\boxtimes}$.

As to $H = \mathrm{M}_r^{\boxtimes}$, the same map b_H that was defined at the beginning of the preceding paragraph is (this being the reason for having used $\lfloor \cdot \rfloor$ despite even r) a bandwidth-5-labelling of $\mathrm{M}_r^{\boxtimes}$ (which has bandwidth 4, by the way), for every odd $r \geqslant 5$. Likewise, the same map h_H defined in the preceding paragraph is a 3-colouring of $\mathrm{M}_r^{\boxtimes}$ for which concerning $|\mathrm{h}_H^{-1}(0)|$ and zero-freeness w.r.t. b_H exactly the same can be said as in the previous paragraph. This proves (a30) for $H = \mathrm{M}_r^{\boxtimes}$.

As to $H = \mathrm{Pr}_r^{\boxminus}$, for every even $r \geqslant 4$, the map b_H defined by $z' \mapsto 1$, $z'' \mapsto 2$, $x_0 \mapsto 3$, $y_0 \mapsto 4$, $x_i \mapsto 4i+1$ and $y_i \mapsto 4i+2$ for $1 \leqslant i \leqslant \lfloor \frac{r}{2} \rfloor$, $x_i \mapsto 4(r-i)+3$ and $y_i \mapsto 4(r-i)+4$ for $\lfloor \frac{r}{2} \rfloor + 1 \leqslant i \leqslant r-1$ is a bandwidth-5-labelling of $\mathrm{Pr}_r^{\boxminus}$. Moreover, the map h_H defined by $z' \mapsto 2$, $z'' \mapsto 0$, $x_0 \mapsto 1$, $y_0 \mapsto 2$, $x_1 \mapsto 0$, $y_1 \mapsto 1$, $x_i \mapsto 1$ and $y_i \mapsto 2$ for even $2 \leqslant i \leqslant r-1$, and $x_i \mapsto 2$ and $y_i \mapsto 1$ for odd $2 \leqslant i \leqslant r-1$ is a 3-colouring of $\mathrm{Pr}_r^{\boxminus}$. In view of $|\mathrm{h}_H^{-1}(0)| = 2$ and in particular in view of the fact that $\mathrm{b}(\mathrm{h}_H^{-1}(0)) = \{2, 5\}$ for *every* even $r \geqslant 4$ (i.e. the distance along the bandwidth-5-labelling of the two 0-labelled vertices is constantly 3, i.e. independent of $|H|$), it is obvious that h_H is $(8 \cdot 2 \cdot \beta \cdot |H|, 4 \cdot 2 \cdot \beta \cdot |H|)$-zero-free w.r.t. b_H, provided that r is large enough to have $4 \cdot 2 \cdot \beta \cdot |H| = 8\beta(2r+2) \geqslant 5$ (when testing the zero-freeness-property for the vertex $z' = \mathrm{b}_H^{-1}(1)$, we have to make five steps forward in order to have a zero-free interval ahead of us—but this is also the highest number of necessary repositioning steps we can encounter). If r is large enough to have $\beta|H| = \beta(2r+2) \geqslant 2 = |\mathrm{h}_H^{-1}(0)|$, too, then both requirements about h_H are met. This completes the proof of (a30) in the case $H = \mathrm{Pr}_r^{\boxminus}$.

As to $H = \mathrm{M}_r^{\boxminus}$, replace '$\mathrm{M}_r^{\boxtimes}$' by '$\mathrm{M}_r^{\boxminus}$' throughout the paragraph before the last (and delete the comment about bandwidth equal to 4) in order to arrive at a proof of (a30) in the case $H = \mathrm{M}_r^{\boxminus}$.

Since n_0 can be chosen large enough to simultaneously satisfy the finitely many (and only β-dependent) requirements on r encountered in the above cases, we have now proved (a30) (where the n_0 is promised *before* the choice $H \in \{\mathrm{C}_n^2, \mathrm{CL}_r, \mathrm{Pr}_r^{\boxtimes}, \mathrm{Pr}_r^{\boxminus}, \mathrm{M}_r^{\boxtimes}, \mathrm{M}_r^{\boxminus}\}$ is made) in its entirety. $\square$

Let us close Section 3.2 with two comments. First, our proof of ($\boxminus$.$|\cdot|-1$.(0)) shows that out of the generating set $\mathcal{C}_{\{|\cdot|-1,|\cdot|\}}(\mathrm{Pr}_r^{\boxminus})$ it suffices to use only *one* circuit having the length $|\cdot|-1$. The same is true for $\mathcal{C}_{\{|\cdot|-1,|\cdot|\}}(\mathrm{Pr}_r^{\boxminus})$. Since the monotonicity-argument used for proving Theorem 1 keeps adding Hamilton circuits to the current generating system—but never adds a circuit of length $|\cdot|-1$ to it—this also implies that in Theorem 1.(I2), a single circuit of length $|\cdot|-1$ suffices in a generating set. Second, with $\mathrm{Pr}_r^{\boxminus,-} := \mathrm{Pr}_r^{\boxminus} - x_0z''$ and $\mathrm{M}_r^{\boxminus,-} := \mathrm{M}_r^{\boxminus} - x_0z''$, the study of the special cases $r=4$ and $r=6$ strongly suggests that for every even $r \geqslant 4$,

($\boxminus$, $-$.(0)) $\dim_{\mathbb{F}_2}\big(\mathrm{Z}_1(\mathrm{Pr}_r^{\boxminus,-};\mathbb{F}_2)/\langle\mathcal{H}(\mathrm{Pr}_r^{\boxminus})\rangle_{\mathbb{F}_2}\big) = 2$,
($\boxminus$, $-$.(0)) $\dim_{\mathbb{F}_2}\big(\mathrm{Z}_1(\mathrm{M}_r^{\boxminus,-};\mathbb{F}_2)/\langle\mathcal{H}(\mathrm{M}_r^{\boxminus})\rangle_{\mathbb{F}_2}\big) = 2$,

but we will not prove this in this paper. The statements ($\boxminus$, $-$.(0)) and ($\boxminus$, $-$.(1)), if true in general, provide a justification for employing the symmetry-destroying edge x_0z'': because of these two codimensions, the graphs $\mathrm{Pr}_r^{\boxminus,-}$ and $\mathrm{M}_r^{\boxminus,-}$—while spanning—are unsuitable as auxiliary substructures for proving (I2) in Theorem 1; for when adding an edge, the codimension of the span of Hamilton circuits within the cycle space can at most stay the same, never decrease.

4. Concluding remarks

4.1. Two open questions and the state of contemporary knowledge

Theorem 1 invites further improvements (e.g. eliminating the lower bound on $|G|$, proving non-asymptotic minimum-degree thresholds, and finding an infinite set of counter-examples disproving the strengthened implications for all $|\cdot|$, instead of only for $|G|=7$ and $|G|=12$ as was done in Sections 3.1.2 and 3.1.3 above). In particular, the following questions are still open:

(Q1) Does (I1) remain true when $(\frac{1}{2}+\gamma)|G|$ is lowered to the Dirac threshold $\frac{1}{2}|G|$?
(Q2) Does (I3) remain true when $(\frac{1}{4}+\gamma)|G|$ is lowered to $\delta(G) \geqslant \frac{1}{4}|G|+1$?

The road we took to (I1) and (I3) suggests the following open questions about spanning subgraphs:

(Q3) Let G be a graph with $|G|$ odd and $\delta(G) \geqslant \frac{1}{2}|G|$. Does it follow that $H \hookrightarrow G$, with $H := \mathrm{Pr}_r^{\boxtimes}$ if $r := \frac{1}{2}(|G|-1)$ is even, and $H := \mathrm{M}_r^{\boxtimes}$ if $r := \frac{1}{2}(|G|-1)$ is odd?
(Q4) Let G be balanced bipartite with $\delta(G) \geqslant \frac{1}{4}|G|+1$. Does it follow that $\mathrm{CL}_{\frac{1}{2}|G|} \hookrightarrow G$?

An affirmative answer to (Q3) implies an affirmative answer to (Q1).
An affirmative answer to (Q4) implies an affirmative answer to (Q2).

The two latter implications hold because of the argument summarized in (St1)–(St3) above. The graphs $\mathrm{Pr}_r^{\boxtimes}$ and $\mathrm{M}_r^{\boxtimes}$ from (Q3) are visualized in Fig. 1.

As to (Q4), it should be noted that a theorem of Czygrinow and Kierstead [22, Theorem 1] comes tantalizingly close: if G is a sufficiently large balanced bipartite graph, then $\delta(G) \geqslant \frac{1}{4}|G|+1$ implies that G contains a spanning copy of the *non*-cyclic ladder NCL_r (defined as CL_r with the two edges $\{a_{r-1},b_0\}$ and $\{a_0,b_{r-1}\}$ removed). Alas, this small defect is enough to render this spanning subgraph unsuitable for serving as an auxiliary substructure in the same way CL_r did above: while the non-cyclic ladder still is Hamilton-laceable, the loss of the two edges causes a drastic drop in the dimension of $\langle\mathcal{H}(\cdot)\rangle_{\mathbb{F}_2}$: whereas $\mathrm{CL}_r \in \mathrm{cd}_0\mathcal{C}_{\{|\cdot|\}}$ by (a15), it can be checked that NCL_r contains only *one* Hamilton circuit, hence $\mathrm{NCL}_r \in \mathrm{cd}_{\beta_1(\mathrm{NCL}_r)-1}\mathcal{C}_{\{|\cdot|\}}$.

We now briefly survey the literature relevant to Question (Q1), an affirmative answer to which would be a nice strengthening of Dirac's theorem. In the pursuit of Question (Q1), one should simultaneously keep in mind the following two facts:

(i) every graph G with $|G|$ odd and $\delta(G) \geqslant \frac{1}{2}|G|$ is Hamilton-connected,
(ii) Hamilton-connectedness by itself does not imply Hamilton-generatedness.

Here, (i) is an immediate corollary of a theorem of O. Ore (owing to oddness of $|G| =: n$, it follows from $\delta(G) \geqslant n/2$ that in [54, Theorem 3.1] $\rho(u)+\rho(v) \geqslant n+1$ for any two non-adjacent vertices u and v). Moreover, (ii) is proved by the example $\mathrm{CE}_{(\mathrm{I1})}$ in Section 3.1.2.

Question (Q1) seems not to have been explicitly asked in the literature. There is, however, the aforementioned Conjecture 2, which according to [47, Ref. 1] [49, Ref. 3] dates back to 1979 and apparently is still open. For $n := |G| = 2d$, Conjecture 2 asks for a generating system consisting of Hamilton circuits together with all circuits shorter by one. For the case of even $n = 2d$, these additional circuits are clearly necessary, but the point of Question (Q1) is that for odd $n := 2d + 1$ it seems quite possible to make do solely with Hamilton circuits (instead of the three lengths $2d - 1$, $2d$ and $2d + 1 = |G|$ allowed by Bondy's conjecture), all the more so as Theorem 1 of the present paper gives an asymptotic affirmative answer to (Q1). The only papers explicitly addressing Bondy's conjecture apparently are [32,47–49,7,2,3]. We will briefly consider each of them. In [32, p. 246], Conjecture 2 is merely mentioned at the end as a related open conjecture. In [47, Theorem 2 and Corollary 4] it is proved that for every $d \in \mathbb{Z}$, if G is a 3-connected graph with $\delta(G) \geqslant d$ which is either non-Hamiltonian or has $|G| \geqslant 4d - 5$, then $\mathrm{Z}_1(G; \mathbb{F}_2)$ is generated by its circuits of length at least $2d - 1$ (note that if $|G| \geqslant 4d - 5$, the conclusion in Bondy's conjecture is far from generatedness by Hamilton circuits). The paper [48] does not have the cycle space as its main concern but announces the results of [47] at the very end. Moreover, the concern of [49,3] is if and when there are inclusions $\mathcal{CO}_{\mathfrak{L}'} \subseteq \mathrm{cd}_0\mathcal{C}_{\mathfrak{L}''}$ for different sets of lengths $\mathfrak{L}'$ and $\mathfrak{L}''$; the paper does not deal with minimum-degree conditions and Conjecture 2 is merely mentioned in passing [49, p. 77] [3, p. 12]. As to [7], it can be proved that [7] does not answer (Q1).

Theorem 23 (*Barovich–Locke [7, Theorem 2.2]*)**.** *Let $d \in \mathbb{Z}$, let G be a finite Hamiltonian graph, let G be* 3*-connected, $\delta(G) \geqslant d$ and $|G| \geqslant 2d - 1$. If $|G| \in \{9, \ldots, 4d - 8\}$, and if there exists at least one $v \in \mathrm{V}(G)$ such that $G - v$ is* not *Hamiltonian, and if another condition holds (which is irrelevant here), then $\mathrm{Z}_1(G; \mathbb{F}_2)$ is generated by the set of all circuits of length at least $2d - 1$.*

The point to be made is that if $|G|$ is odd and $\delta(G) \geqslant \lceil \frac{|G|}{2} \rceil$, and if the theorem of Barovich–Locke is to yield generatedness by Hamilton circuits, then necessarily we must set $2d - 1 = |G|$. While this automatically makes the hypothesis $|G| \in \{9, \ldots, 4d-8\}$ true, and while $\delta(G) \geqslant \lceil \frac{|G|}{2} \rceil$ ensures that G is Hamiltonian and also that G is 3-connected, the remaining hypothesis of Theorem 23 *cannot possibly be true* in the setting of Question (Q1): for every $v \in \mathrm{V}(G)$ we have $\delta(G - v) \geqslant \delta(G) - 1 \geqslant$ (since $\delta(G)$ is an integer) $\geqslant \lceil \frac{1}{2}|G| \rceil - 1 = \frac{|G|}{2} - \frac{1}{2} = \frac{1}{2}|G - v|$, hence $G - v$ is still Hamiltonian by Dirac's theorem. Hence Theorem 23, as it stands, does not answer Question (Q1). Furthermore, in [2] the phrase "in the presence of a long cycle every k-path-connected graph is $(k + 1)$-generated" [2, Introduction, last paragraph] cannot be construed so as to answer Question (Q1): each of the slightly different ways in which this phrase is made precise by the authors (cf. [2, Corollary 5, Lemmas 9 and 10]) involves additional assumptions one of which always is that there exists a circuit of length $2k-2$ or $2k-3$. The existence of such a circuit implies that '$(k+1)$-generated' is far from meaning 'generated by Hamilton circuits'.

4.2. A positive example for Question (Q1)

We will now analyse a small yet relevant example which is a positive instance for Question (Q1). It provides an explicit illustration for how a minimum degree just barely satisfying the Dirac condition can endow a *non*-Cayley graph with the property of having its cycle space generated by its Hamilton circuits.

Definition 24 (*The graph* G *underlying* Fig. 4)**.** Let G be the graph with $\mathrm{V(G)} := \{v_1, \ldots, v_7\}$ and $\mathrm{E(G)} := \{v_1v_2, v_1v_3, v_1v_6, v_1v_7, v_2v_3, v_2v_6, v_2v_7, v_3v_4, v_3v_5, v_4v_5, v_4v_6, v_4v_7, v_5v_6, v_5v_7\}$ (see Fig. 5).

Obviously G satisfies the hypotheses in Question (Q1) (barely so), and $\dim_{\mathbb{F}_2}(\mathrm{G}; \mathbb{F}_2) = \beta_1(\mathrm{G}) = \|\mathrm{G}\| - |\mathrm{G}| + 1 = 14 - 7 + 1 = 8$. Furthermore, because of the following fact we cannot prove that G is a positive instance for Question (Q1) just by appealing to Theorem 15.(2).

Proposition 25. *The graph* G *is not a Cayley graph.*

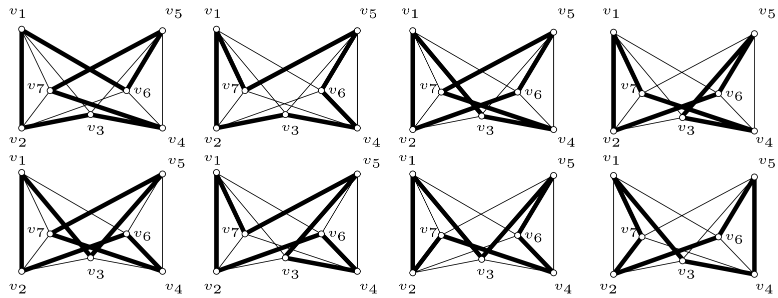


Fig. 4. An example of a $\mathbb{F}_2$-basis for $Z_1(G;\mathbb{F}_2)$ consisting only of Hamilton circuits in a situation where the underlying graph G is not a Cayley graph and presumably owes its being Hamilton-generated to the Dirac condition (which it satisfies just barely).

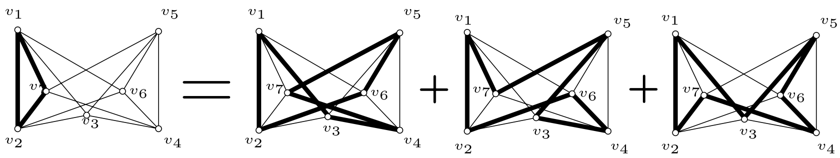


Fig. 5. An example of a realization of a 3-circuit in terms of the Hamilton circuit basis from Fig. 4.

Proof. While provable elementarily, let us give a high-context proof of this: the order $|G| = 7$ being prime, the only possible underlying group is $\mathbb{Z}/7$ with addition. Suppose that G were a Cayley graph on $\mathbb{Z}/7$. Since the spectrum of the adjacency matrix of G is $(4, 1, -1, -1, 0, 0, -3)$, the graph G would then be a quartic connected Cayley graph on an abelian group having only integer adjacency-eigenvalues. But this would contradict a classification theorem due to Abdollahi and Vatandoost [1, Theorem 1.1] according to which the set of all orders of such graphs is a finite set which does not contain 7. $\square$

Proposition 26 (G *is Hamilton-generated*). $\langle \mathcal{H}(G)\rangle_{\mathbb{F}_2} = Z_1(G;\mathbb{F}_2)$.

Proof. We give an $\mathbb{F}_2$-basis (shown in Fig. 4) for $Z_1(G;\mathbb{F}_2)$ consisting of Hamilton circuits only. Let $C_1^G := v_1v_2v_3v_4v_7v_5v_6v_1$, $C_2^G := v_1v_2v_3v_4v_6v_5v_7v_1$, $C_3^G := v_1v_2v_6v_5v_7v_4v_3v_1$, $C_4^G := v_1v_2v_6v_5v_3v_4v_7v_1$, $C_5^G := v_1v_2v_6v_4v_7v_5v_3v_1$, $C_6^G := v_1v_2v_6v_4v_3v_5v_7v_1$, $C_7^G := v_1v_2v_7v_4v_6v_5v_3v_1$, $C_8^G := v_1v_7v_2v_6v_5v_4v_3v_1$. W.r.t. the standard basis of $C_1(G;\mathbb{F}_2)$ the circuits $C_1^G, \ldots, C_8^G$ give rise to the matrix shown in (9), which has $\mathbb{F}_2$-rank equal to $8 = \dim_{\mathbb{F}_2}(Z_1(G;\mathbb{F}_2))$.

	C_1^G	C_2^G	C_3^G	C_4^G	C_5^G	C_6^G	C_7^G	C_8^G
v_1v_2	1	1	1	1	1	1	1	0
v_1v_3	0	0	1	0	1	0	1	1
v_1v_6	1	0	0	0	0	0	0	0
v_1v_7	0	1	0	1	0	1	0	1
v_2v_3	1	1	0	0	0	0	0	0
v_2v_6	0	0	1	1	1	1	0	1
v_2v_7	0	0	0	0	0	0	1	1
v_3v_4	1	1	1	1	0	1	0	1
v_3v_5	0	0	0	1	1	1	1	0
v_4v_5	0	0	0	0	0	0	0	1
v_4v_6	0	1	0	0	1	1	1	0
v_4v_7	1	0	1	1	1	0	1	0
v_5v_6	1	1	1	1	0	0	1	1
v_5v_7	1	1	1	0	1	1	0	0

(9)

Therefore the $\mathbb{F}_2$-span of $C_1^G, \ldots, C_8^G$ is an 8-dimensional subspace of the 8-dimensional $\mathbb{F}_2$-vector space $Z_1(G; \mathbb{F}_2)$, hence is *equal* to $Z_1(G; \mathbb{F}_2)$, completing the proof of Proposition 26. □

Acknowledgements

The author thanks the anonymous referees for comments that improved the presentation.

The author was partially supported by DFG grant TA 309/2-2, and the ENB graduate program TopMath. The author gratefully acknowledges the support of TUM Graduate School's Thematic Graduate Center TopMath at Technische Universität München. A large part of the present work was done while the author was partially supported by a scholarship from the Max Weber-Programm Bayern.

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