Two-Way Relay Channel with Half-Duplex Constraint

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Diplomarbeit

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Festlegung des Themas:	Donnerstag, 19. November 2009
Anmeldung der Arbeit:	Donnerstag, 01. April 2010
Tag der mündlichen Prüfung:	Freitag, 22. Oktober 2010
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Remember why you're here is to inspire.
–RISHLOO

Abstract Two communication processes in a small network are studied. The network consists of three nodes, all having access to a common channel. Each of the network parties faces a half-duplex constraint. First a one-way scenario is reviewed where one of the nodes wants to convey a message to another node. The third node facilitates the transmission. The channel is known as the *half-duplex relay channel*. Afterwards a restricted two-way scenario is investigated where two nodes want to establish a dialog through the channel while the third node assists the bi-directional transmission. The dialog encoders are not allowed to cooperate. The channel is known as the *restricted half-duplex two-way relay channel*. For both problems upper (outer) and lower (inner) bounds on the achievable rates for discrete-memoryless channels are derived with the information theoretical approach of cut-sets, random codes and suboptimal decoders. In the first part the performance bound and possible transmission strategies that have been obtained for the full-duplex relay channel are adapted to the half-duplex constraint. The second part examines outer and inner bounds for restricted half-duplex two-way communication in the network. Different schemes (two of them new) with specific relaying strategies are analyzed and compared visually with simulations for channels with continuous Gaussian random variables. One of the core contributions is a first outer rate bound on the problem independent from a specific scheme. The outer bound established for the problem alludes to a more general transmission scheme. It contains all schemes considered before as special cases. For this scheme the achievable rates for decode-and-forward (DF) and partial-decode-and-forward (PDF) relaying are derived. Restricting to one of those strategies and fixed input distributions makes it possible to determine optimal transmission schemes with respect to the maximization of rate objectives or the minimization of cost objectives by solving a small-scale linear program.

Contents

1. Introduction

Due to the rapid evolution of a "digital society" in the last two decades the demand of data communication at high quality and rates has dramatically increased. Especially the emerging markets of selling mobile access to voice and data transmission provide challenging tasks for the engineers and researchers pushing wireless technology today. A limited amount of resources like bandwidth, power and hardware form irrevocable constraints to the design of systems satisfying the present and future request for robust data delivery at appropriate speed. Therefore, high efforts are made to increase the performance and efficiency of wireless devices and systems. The desire to theoretically understand technical communication channels and to investigate how to use them in an efficient way dates back to the first half of the twentieth century. The efforts made have led to the work [30] presenting a consistent theory of communication. This forms the foundation for many major contributions that followed. As a result today some fundamental channel models are considered to be basically understood from a theoretical perspective. The point-to-point channel is the most prominent one among those and forms the prevailing model for present wireless systems.

In the last years it has been recognized that operating wireless systems in a competitive point-to-point fashion is not optimal. Here other users act as interferers in the transmission process and are treated like noise. It has been shown that cooperative protocols can outperform such an approach. These methods have also been identified to be one of the keys to the optimal organization of more general wireless networks, e.g., ad hoc networks where many nodes form a self-configured decentralized network in order to exchange information in a certain area. One such cooperative concept is known as *relaying*. Source and destination connect over one or many intermediate nodes if isolated from each other or when facing bad channel conditions for direct communication. This can increase connectivity between users in the network. Moreover, lower energy consumption at the source can be achieved. Even in the presence of a direct path a careful design of transmission protocols has been shown to improve the communication rate [6]. The potential to further extend the efficient use of bandwidth with the innovative idea of exchanging data between two nodes in a bi-directional way over a relay [26] has attracted researchers in the last years.

Motivation Although from a theoretical viewpoint the one-way communication problem with a relay is in general an open problem, i.e., the capacity achieving coding procedure is unknown, the basic methods are considered to be understood due to the milestone [6]. In contrast the analysis of the two-way channel with a relay is far from being complete. Many works have focused on a separated channel model where the direct path between the two users is not present, e.g., [26] [24] [14] [33] [13]. Such a restriction is justified if the connectivity between network parties is a crucial problem. In a general scenario of a fully-connected wireless network, where all nodes have access to a common channel, such a model ignores the possibility of using the direct path. It can be doubted to be optimal in general to operate like in a separated model. Especially with multi-antenna nodes the direct path might play an interesting role as the *eigenmodes* of the channels bear the possibility to transmit data from one node to another separately over the direct path and the relay [21]. However, multi-antenna aspects are out of the scope of the following pages. This work attempts to study aspects of the half-duplex coding problem in a fully-connected network with relay. Simulations for Gaussian channels are limited to nodes with a single antenna.

Assumptions In order to keep the analysis tractable the following assumptions are made:

- all nodes operate in half-duplex mode,
- the transmission protocol is fixed and known by all nodes a priori,
- the input distributions are fixed for each network state configuration,
- nodes have access to all codebooks used in the network,
- all nodes are synchronized by a network clock,
- channels are time-invariant and memoryless,
- the conditional distributions of all channels are available at each node,
- the communication is not limited by delay,
- the communication is not limited by the complexity of encoders or decoders.

Throughout the thesis it is assumed that nodes in the network do not have the capability to send and receive simultaneously on the same resource, i.e., a node can not send and receive at the same time on the same frequency (*half-duplex constraint*). Either a node listens to the channel through its channel output or it talks to the channel by emitting a signal to its channel input. The particular action of an individual node will be termed *state*. A *network state* denotes a particular state configuration of all nodes. The protocol which defines the network state for each individual use of the channel is denoted *relaying scheme* or simply *scheme* and is assumed to be fixed and known by all nodes a priori. This excludes the possibility of *mode coding*, where higher rates can be achieved [17] by letting the nodes communicate additional information to one another through a coding scheme on their state, but it allows to gain rate by assigning the optimal number of channel uses to each of the network states (*time allocation*). Moreover, the input distributions for each network state are fixed. This prohibits time-sharing techniques. The restriction to half-duplex networks can be motivated by practical considerations as it reflects the ability of today's wireless technology. However, note that here the transmission process is assumed to be not limited by delay or complexity. The analysis takes for granted that each node knows the conditional distributions characterizing all channels in the network as well as all codebooks used. All this might conflict with practical arguments and therefore makes the analysis inaccurate for a real technical system. Additionally the basic results of the following pages are not limited to wireless channels.

1.1 Notation

X denotes a random variable taking values in the discrete and finite alphabet $\mathcal{X}$. $P_X(\cdot)$ is the probability distribution of X where the label X is neglected if the associated random variable becomes clear from the context. $P_{X|Y}(\cdot|\cdot)$ denotes the probability distribution of X conditioned on Y. $I(X;Y)$ symbolizes the mutual information between X and Y. A finite sequence of n elements is denoted x^n. f denotes a scalar-valued function where $\boldsymbol{f}$ denotes a vector-valued function.

2. Half-Duplex One-Way Relay Channel

In this part of the thesis the *half-duplex relay channel* is studied. The relay channel consists of a small network with three nodes. One node is considered the source, one the destination and one the relay. The source node wants to communicate a message to the destination node. The relay node, having no own message for the two other nodes, assists in the communication process with a mapping from its past channel output to its current channel input (*relaying strategy*). In order to protect the communication process against noise, error-correcting codes are used (*encoding*). The destination node estimates the message using its channel output (*decoding*). The solution to the coding problem has two aspects. One is to upper bound the achievable rates from above. The second one is to find a code which is optimal with respect to the message set size (*rate*). The achievable rates of the ultimate code would coincide with a tight upper rate bound.

Overview After a short review of related work, a formal definition of the full-duplex and the half-duplex relay channel is given. An upper and several lower single-letter bounds on the achievable rates for *discrete-memoryless* versions of the channel are derived. This is done via cut-set arguments and by studying different relaying strategies proposed in [6] with suboptimal decoding of random codes. These results can be generalized from discrete channels to the special case of channels with *continuous Gaussian* random variables. Such communication models have become popular as an approximation to the wireless scenario. Simulations for single-antenna nodes illustrate the benefits of different relaying strategies and time allocation. This chapter is intended to serve as an introduction to the basic aspects of the second part. In order to make reading easier coding proofs are outlined in the appendix.

Related Work The relay channel was introduced in [32]. The seminal work [6] presents an upper bound on the capacity of the full-duplex relay channel by introducing cut-set arguments. Moreover, different relaying strategies are presented, among them the *decode-and-forward* strategy which is shown to be capacity achieving for the *degraded relay channel* [6, Theorem 1]. A special case of the *partial-decode-compress-and-forward* strategy [6, Theorem 7], the so called *partial-decode-and-forward* strategy [18], achieves capacity for the semi-deterministic relay channel where the channel output at the relay is a deterministic function of the channel input at the source node [9]. The same holds for the relay channel with orthogonal channels from source to relay and source to destination [8]. The aspect of diversity in cooperative wireless networks is studied in [20]. In [19] multiple access channels (MAC) and broadcast channels (BC) with relays are investigated. [11] shows that the rates for the Gaussian relay channel with joint source-channel coding asymptotically achieve a cut-set bound if the number of relays grows to infinity. In contrast to the full-duplex works, [15] focuses on the wireless half-duplex relay channel and its ergodic and outage capacity.

2.1 Discrete Memoryless Half-Duplex Relay Channel

2.1.1 Channel Model

Full-Duplex Channel Model In order to introduce the channel formally the full-duplex model is shortly reviewed. Throughout this work the nodes of the network are labeled by numbers 1, 2 and 3. Node 2 always plays the roll of the relay and has no own message. In the one-way problem node 1 is considered to be the source and node 3 the destination of the communication process. W denotes the message to be transmitted from node 1 to 3. X_1 denotes the channel input at node 1,

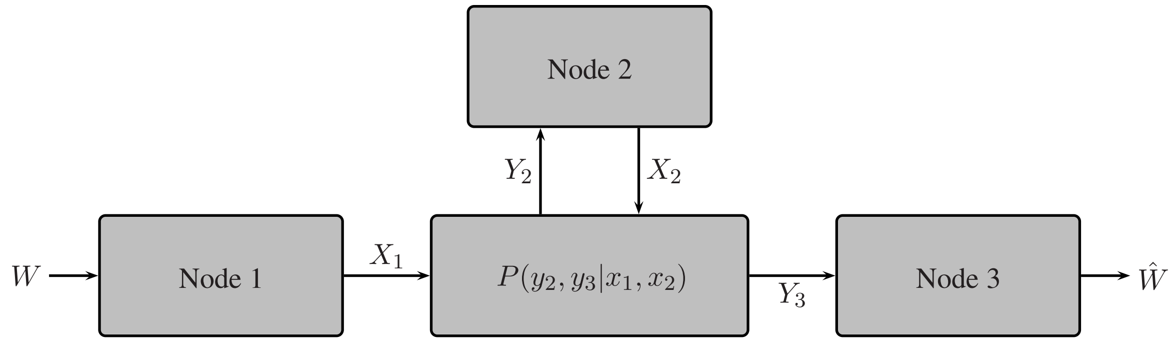


Figure 2.1: Full-Duplex Relay Channel

X_2/Y_2 the channel input/output at node 2 and Y_3 the channel output at node 3. All in- and outputs are part of a common channel. The channel is assumed to be discrete and without memory, i.e.,

$$P(y_2^n, y_3^n|x_1^n, x_2^n) = \prod_{k=1}^{n} P(y_{2,k}, y_{3,k}|x_{1,k}, x_{2,k}). \tag{2.1}$$

The conditional output distributions are assumed to be time-invariant for n channel uses

$$P(y_{2,k}, y_{3,k}|x_{1,k}, x_{2,k}) = P(y_2, y_3|x_1, x_2) \quad k = 1, \dots, n. \tag{2.2}$$

Therefore, the channel is fully characterized by its finite input and output alphabets and a conditional distribution defining the statistical dependencies between inputs and outputs

$$(\mathcal{X}_1 \times \mathcal{X}_2, P(y_2, y_3|x_1, x_2), \mathcal{Y}_2 \times \mathcal{Y}_3) \,. \tag{2.3}$$

Code A code of length n and rate R consists of a message set of size 2^{nR}

$$\mathcal{W} = \{1, 2, \dots, 2^{nR}\}, \tag{2.4}$$

an encoding function at node 1, mapping a message to a sequence of n output signals (codeword)

$$\boldsymbol{f}_1 : \mathcal{W} \to \mathcal{X}_1^n, \tag{2.5}$$

a set of relaying functions at node 2, mapping from past outputs Y_2 to the current input X_2

$$\{f_{2,k}\}_{k=1}^n \quad \text{s.t.} \quad x_{2,k} = f_{2,k}(Y_{2,1}, Y_{2,2}, \dots, Y_{2,k-1}) \tag{2.6}$$

and a decoding function at node 3 for message recovery from an output sequence y_3^n

$$g_3 : \mathcal{Y}_3^n \to \hat{\mathcal{W}}. \tag{2.7}$$

Half-Duplex Channel Model For the half-duplex channel a state variable S taking values in $\mathcal{S} : \{1, 2\}$, determining the state of node 2 (1=listen, 2=talk), is added to the channel

$$\left(\mathcal{X}_1 \times \mathcal{X}_2, P^{(s)}(y_2, y_3|x_1, x_2, s), \mathcal{Y}_2 \times \mathcal{Y}_3, \mathcal{S}\right). \tag{2.8}$$

The channel output at node 2 is inactive if $s = 2$, i.e.,

$$P^{(2)}(y_2 = \emptyset, y_3|x_1, x_2, s = 2). \tag{2.9}$$

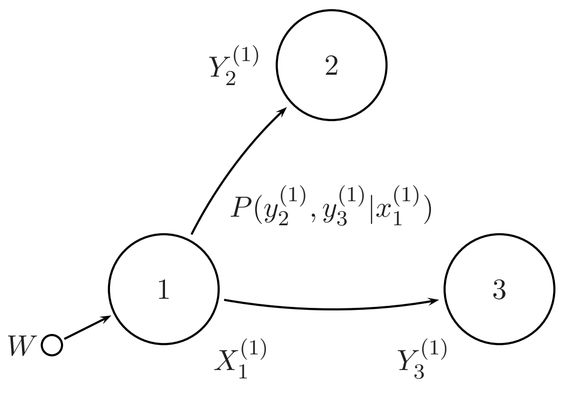
The channel input at node 2 is inactive if $s = 1$, i.e., the input distribution $P_{X_2^{(s)}}$ faces a limitation if the relay listens to the channel

$$P_{X_2^{(1)}}(\emptyset) = 1. \tag{2.10}$$

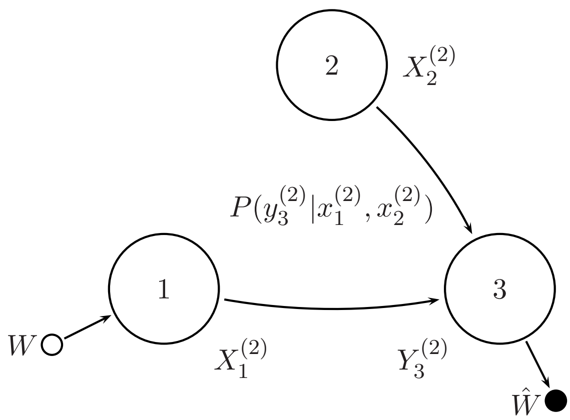
The nodes have agreed on the individual n realizations of the state variable S a priori. Node 1 always talks and node 3 always listens. The channel can be assumed to be used by two phases which are orthogonal. In the first n_1 of n channel uses node 1 talks through its channel input $X_1^{(1)}$ while nodes 2 and 3 listen with their channel outputs $Y_2^{(1)}$, $Y_3^{(1)}$. In the following n_2 of n channel uses node 1 and 2 talk through their inputs $X_1^{(2)}$, $X_2^{(2)}$ whereas node 3 listens with $Y_3^{(2)}$. Time allocation parameters are defined as $\tau_l = \frac{n_l}{n} = P_S(l)$. Equivalently the channel can be defined by

$$\left(\mathcal{X}_1^{(1)}, P(y_2^{(1)}, y_3^{(1)}|x_1^{(1)}), \mathcal{Y}_2^{(1)} \times \mathcal{Y}_3^{(1)}\right) \perp \left(\mathcal{X}_1^{(2)} \times \mathcal{X}_2^{(2)}, P(y_3^{(2)}|x_1^{(2)}, x_2^{(2)}), \mathcal{Y}_3^{(2)}\right) \tag{2.11}$$

where the duration of usage for each channel part l is provided by the time allocation parameter τ_l.



(a) First Phase (n_1 transmission slots)



(b) Second Phase (n_2 transmission slots)

Figure 2.2: Half-Duplex Relay Channel

Code A code of length n, rate R and time allocation $\boldsymbol{\tau} = \begin{bmatrix}\tau_1 & \tau_2\end{bmatrix}^T$ consists of a message set

$$\mathcal{W} = \{1, 2, \ldots, 2^{nR}\}, \tag{2.12}$$

two encoding functions and one decoding function

$$\begin{aligned}
\boldsymbol{f}_1^{(1)} &: \mathcal{W} \rightarrow \mathcal{X}_1^{n_1} \\
\boldsymbol{f}_1^{(2)} &: \mathcal{W} \rightarrow \mathcal{X}_1^{n_2} \\
g_3^{(2)} &: \mathcal{Y}_3^{n_1} \times \mathcal{Y}_3^{n_2} \rightarrow \hat{\mathcal{W}},
\end{aligned} \tag{2.13}$$

and a set of relaying functions

$$\{f_{2,k}^{(2)}\}_{k=n_1+1}^{n} \quad \text{s.t.} \quad x_{2,k} = f_{2,k}(Y_{2,1}, Y_{2,2}, \ldots, Y_{2,n_1}). \tag{2.14}$$

2.1.2 Upper Bound

To begin the analysis an upper bound on the achievable rates of the half-duplex relay channel is derived. Cut-set arguments were first developed in [6, Section 3] for the full-duplex relay channel. A generalized form for networks of any size and an arbitrary number of source-destination pairs, referred to as the *Cut-set Theorem*, can be found in [7, Theorem 15.10.1].

Theorem 2.1.1 *The rates of the half-duplex relay channel that are achievable for some joint probability distributions*

$$\begin{aligned}
P(x_1^{(1)}, y_2^{(1)}, y_3^{(1)}) &= P(x_1^{(1)})P(y_2^{(1)}, y_3^{(1)}|x_1^{(1)}) \\
P(x_1^{(2)}, x_2^{(2)}, y_3^{(2)}) &= P(x_1^{(2)}, x_2^{(2)})P(y_3^{(2)}|x_1^{(2)}, x_2^{(2)}).
\end{aligned}$$

must satisfy

$$\begin{aligned}
R \leq \min \Big\{ & \tau_1 I(X_1^{(1)}; Y_2^{(1)} Y_3^{(1)}) + \tau_2 I(X_1^{(2)}; Y_3^{(2)}|X_2^{(2)}), \\
& \tau_1 I(X_1^{(1)}; Y_3^{(1)}) + \tau_2 I(X_1^{(2)} X_2^{(2)}; Y_3^{(2)}) \Big\}
\end{aligned}$$

where $0 \leq \tau_1, \tau_2$ *and* $\tau_1 + \tau_2 \leq 1$.

Proof Consider a full-duplex network with its two cut-set partitions separating node 1 and 3. Assuming zero-error codes with the Cut-set Theorem it holds that the achievable rates of a communication from node 1 to 3 must satisfy

$$\begin{aligned}
R &\leq I(X_1; Y_2 Y_3|X_2) \\
R &\leq I(X_1 X_2; Y_3)
\end{aligned} \tag{2.15}$$

for some joint input distribution $P(x_1, x_2)$. Introducing a random state variable S known by all nodes in the network yields

$$\begin{aligned}
R &\leq I(X_1; Y_2 Y_3|X_2 S) \\
R &\leq I(X_1 X_2; Y_3|S).
\end{aligned} \tag{2.16}$$

Letting S take values in $\mathcal{S} : \{1, 2\}$ with a distribution

$$P_S(l) = \frac{n_l}{n} = \tau_l \tag{2.17}$$

and rewriting the mutual informations gives

$$\begin{aligned}
R &\leq \sum_{l=1}^{L} P_S(l) I(X_1^{(l)}; Y_2^{(l)} Y_3^{(l)} | X_2^{(l)}, S = l) \\
&= P_S(1) I(X_1^{(1)}; Y_2^{(1)} Y_3^{(1)} | X_2^{(1)} = \emptyset, S = 1) + P_S(2) I(X_1^{(2)}; Y_2^{(2)} = \emptyset, Y_3^{(2)} | X_2^{(2)}, S = 2) \\
&= \tau_1 I(X_1^{(1)}; Y_2^{(1)} Y_3^{(1)}) + \tau_2 I(X_1^{(2)}; Y_3^{(2)} | X_2^{(2)}) \\
R &\leq \sum_{l=1}^{L} P_S(l) I(X_1^{(l)} X_2^{(l)}; Y_3^{(l)} | S = l) \\
&= P_S(1) I(X_1^{(1)} X_2^{(1)} = \emptyset; Y_3^{(1)} | S = 1) + P_S(2) I(X_1^{(2)} X_2^{(2)}; Y_3^{(2)} | S = 2) \\
&= \tau_1 I(X_1^{(1)}; Y_3^{(1)}) + \tau_2 I(X_1^{(2)} X_2^{(2)}; Y_3^{(2)}) \qquad (2.18)
\end{aligned}$$

for input distributions $P(x_1^{(1)}), P(x_1^{(2)}, x_2^{(2)})$. This establishes the theorem. ■

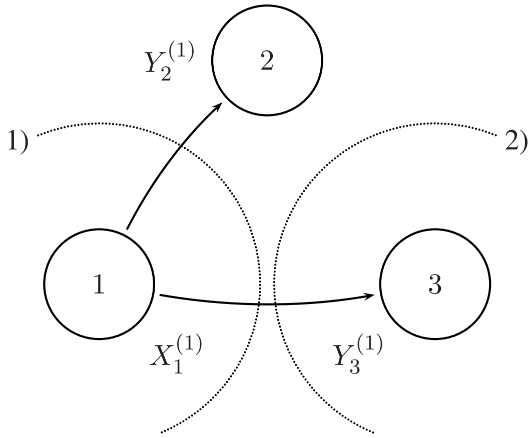
Interpretation The Cut-set Theorem states that dividing a network into two disjoint subsets Ω, Ω^c and letting the nodes in each subset cooperate without constraints, results in a point-to-point channel from Ω to Ω^c with multiple-inputs X_Ω and multiple-outputs Y_{Ω^c} limited by capacity

$$C_{P2P}(\Omega) = \max_{P_{X_\Omega X_{\Omega^c}}} I(X_\Omega; Y_{\Omega^c} | X_{\Omega^c}). \qquad (2.19)$$

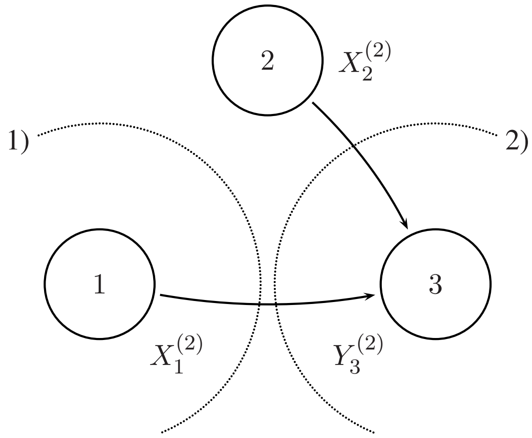
As any code imposes constraints on the cooperation between nodes, the capacity $C_{P2P}(\Omega)$ must be an upper bound for the achievable information flow from the subnetwork Ω to Ω^c. Assuming one source-destination pair and restricting the partitioning to the source node being in Ω and the destination node in Ω^c the resulting capacity $C_{P2P}(\Omega)$ provides an upper bound for achievable rates from source to destination [11]. Taking the minimum $C_{P2P}(\Omega)$ over all possible partitions Ω yields a stronger bound with a max-flow min-cut interpretation [10]. Due to the nature of the bound, results for point-to-point channels apply, e.g., available feedback will not increase the capacity $C_{P2P}(\Omega)$ [7, Section 7.12]. Note that although the cut-set bound is tight in all cases where the capacity for the relay channel could be established, e.g., [6], [9] and [11], it is believed to be lax in general [35]. Figure 2.3 shows a graphical interpretation of the Cut-set Theorem applied to the relay network with three nodes and a half-duplex constraint. In order to upper bound the size of the message set the network is divided into two disjoint subsets each containing either source or destination of the communication process. The two possible partitions are

$$\begin{aligned}
&\text{Cut 1 :} \quad && \Omega_1 = \{\text{node 1}\} \quad && \Omega_1^c = \{\text{node 2, node 3}\} \\
&\text{Cut 2 :} \quad && \Omega_2 = \{\text{node 1, node 2}\} \quad && \Omega_2^c = \{\text{node 3}\}. \qquad (2.20)
\end{aligned}$$

For the first cut node 2 and 3 and for the second cut node 1 and 2 are allowed to cooperate arbitrarily. So the first part of Theorem 2.1.1 can be interpreted as the information emitted by node 1 to node 3 while node 3 has perfect knowledge about the active channel inputs and outputs at node 2 (*receiver cooperation*). The second part denotes the information received by node 3 in both phases with perfect cooperation between node 1 and 2 (*transmitter cooperation*).



(a) First Phase (n_1 transmission slots)



(b) Second Phase (n_2 transmission slots)

Figure 2.3: Graphical interpretation: Cut-Set Theorem

2.1.3 Achievable Rates

Decode-and-Forward (Proposition 2.1.2) The first relaying strategy discussed is called *decode-and-forward* (DF). Node 2 decodes the full message that was sent by node 1 in the first phase. The input of node 2 in the second phase is chosen to be a deterministic function of the first phase message. This results in full knowledge at node 1 about the channel input of node 2. So node 1 can tradeoff the transmission of a new message to node 3 and the support of the input signal of node 2 in the second phase. An important aspect is that node 2 does not need to forward the full message from the first phase as node 3 already has side information available after listening to the channel in the first phase. The idea and a proof for the achievable rates of the full-duplex relay channel is due to [6, Theorem 1]. Note that in contrast to the full-duplex version [6] the proof here does not

Proposition 2.1.2 *All rates of the half-duplex relay channel that satisfy*

$$R \leq \min\Big\{\tau_1 I(X_1^{(1)};Y_2^{(1)}) + \tau_2 I(X_1^{(2)};Y_3^{(2)}|X_2^{(2)}),$$
$$\tau_1 I(X_1^{(1)};Y_3^{(1)}) + \tau_2 I(X_1^{(2)}X_2^{(2)};Y_3^{(2)})\Big\}$$

with $0 < \tau_1, \tau_2$ *and* $\tau_1 + \tau_2 \leq 1$ *for some joint probability distributions*

$$P(x_1^{(1)}, y_2^{(1)}, y_3^{(1)}) = P(x_1^{(1)})P(y_2^{(1)}, y_3^{(1)}|x_1^{(1)})$$
$$P(x_1^{(2)}, x_2^{(2)}, y_3^{(2)}) = P(x_2^{(2)})P(x_1^{(2)}|x_2^{(2)})P(y_3^{(2)}|x_1^{(2)}, x_2^{(2)})$$

are achievable with a decode-and-forward strategy.

Proof see A2.1.

require *block-Markov coding*. The strategy realizes transmitter cooperation resulting in the same expression as the related part of the upper bound.

Compress-and-Forward (Proposition 2.1.3) Contrary to the method above, the *compress-and-forward* strategy (CF) attempts to establish receiver cooperation. Node 2 tries to convey its channel output to node 3 by quantizing its first phase receive signal. In the second phase node 2 sends the quantization index to node 3 taking into account the side information that node 3 has after listening to the channel in the first phase. In the second phase node 1 just sends new information to node 3 as it has no deterministic knowledge about the input of node 2. Node 3 decodes from its own output and the quantized output of node 2. The idea and a proof for the achievable rates of the full-duplex relay channel is due to [6, Theorem 6]. Note the similarity to the upper bound for receiver

Proposition 2.1.3 *All rates of the half-duplex relay channel that satisfy*

$$R \leq \tau_1 I(X_1^{(1)}; \hat{Y}_2^{(1)} Y_3^{(1)}) + \tau_2 I(X_1^{(2)}; Y_3^{(2)} | X_2^{(2)})$$

subject to

$$\tau_1 I(\hat{Y}_2^{(1)}; Y_2^{(1)} | Y_3^{(1)}) \leq \tau_2 I(X_2^{(2)}; Y_3^{(2)})$$

with $0 < \tau_1, \tau_2$ *and* $\tau_1 + \tau_2 \leq 1$ *for some joint probability distributions*

$$P(x_1^{(1)}, y_2^{(1)}, \hat{y}_2^{(1)}, y_3^{(1)}) = P(x_1^{(1)}) P(y_2^{(1)}, y_3^{(1)} | x_1^{(1)}) P(\hat{y}_2^{(1)} | y_2^{(1)})$$
$$P(x_1^{(2)}, x_2^{(2)}, y_3^{(2)}) = P(x_1^{(2)}) P(x_2^{(2)}) P(y_3^{(2)} | x_1^{(2)}, x_2^{(2)})$$

are achievable with a compress-and-forward strategy.

Proof see A2.2.

cooperation. The degradation of the output $\hat{Y}_2^{(1)}$ available at node 3 is determined by the capacity of the channel between node 2 and 3 and the side information available at node 3. The problem of rate-distortion with side information is known as the *Wyner-Ziv problem* [34].

Partial-Decode-and-Forward (Proposition 2.1.4) Letting node 2 decode the full first phase message forms a hard restriction on the achievable rates of Proposition 2.1.2. The *partial-decode-and-forward* strategy (PDF) relaxes this obstacle. Node 2 decodes just a part of the message sent by node 1 in phase 1. This part is represented by the auxiliary random variable $U_1^{(1)}$. Then node 2 sends as much information as node 3 needs to decode this part together with its side information about $U_1^{(1)}$. With full knowledge about the input of node 2 in the second phase node 1 can assist this transmission or send a new message to node 3. Node 3, after having decoded the first part of the message, can subsequently decode the second part of the message of phase 1. As it is a special case of the *partial-decode-compress-and-forward strategy* (PDCF) the idea is due to [6, Theorem 7]. A proof for the achievable full-duplex rates can be found in [18, Section 9.4.1].

Partial-Decode-Compress-and-Forward (Proposition 2.1.5) The *partial-decode-compress-and-forward* strategy combines the already presented methods. Therefore, it contains all previous methods as special cases. The output at node 2 in Proposition 2.1.4 is additionally quantized and sent to node 3 in the second phase by use of superposition coding. This allows node 2 to transmit the decoded message and the quantization index in parallel while taking into consideration the

Proposition 2.1.4 *All rates of the half-duplex relay channel that satisfy*

$$R \leq \min \Big\{ \tau_1 I(U_1^{(1)}; Y_2^{(1)}) + \tau_1 I(X_1^{(1)}; Y_3^{(1)} | U_1^{(1)}) + \tau_2 I(X_1^{(2)}; Y_3^{(2)} | X_2^{(2)}),$$
$$\tau_1 I(X_1^{(1)}; Y_3^{(1)}) + \tau_2 I(X_1^{(2)} X_2^{(2)}; Y_3^{(2)}) \Big\}$$

with $0 < \tau_1, \tau_2$ *and* $\tau_1 + \tau_2 \leq 1$ *for some joint probability distributions*

$$P(u_1^{(1)}, x_1^{(1)}, y_2^{(1)}, y_3^{(1)}) = P(u_1^{(1)}) P(x_1^{(1)} | u_1^{(1)}) P(y_2^{(1)}, y_3^{(1)} | x_1^{(1)})$$
$$P(x_1^{(2)}, x_2^{(2)}, y_3^{(2)}) = P(x_2^{(2)}) P(x_1^{(2)} | x_2^{(2)}) P(y_3^{(2)} | x_1^{(2)}, x_2^{(2)})$$

are achievable with a partial-decode-and-forward strategy.

Proof see A2.3.

available side information about the two parts at node 3. In the second phase node 1 can assist the transmission of node 2 belonging to the decoded part of the message or send new information to node 3. The idea and a proof for achievable rates of the full-duplex relay channel is due to [6, Theorem 7].

Proposition 2.1.5 *All rates of the half-duplex relay channel that satisfy*

$$R \leq \min \Big\{ \tau_1 I(U_1^{(1)}; Y_2^{(1)}) + \tau_1 I(X_1^{(1)}; \hat{Y}_2^{(1)} Y_3^{(1)} | U_1^{(1)}) + \tau_2 I(X_1^{(2)}; Y_3^{(2)} | V_2^{(2)} X_2^{(2)}),$$
$$\tau_1 I(U_1^{(1)}; Y_3^{(1)}) + \tau_1 I(X_1^{(1)}; \hat{Y}_2^{(1)} Y_3^{(1)} | U_1^{(1)}) +$$
$$\tau_2 I(V_2^{(2)}; Y_3^{(2)}) + \tau_2 I(X_1^{(2)}; Y_3^{(2)} | V_2^{(2)} X_2^{(2)}) \Big\}$$

subject to

$$\tau_1 I(\hat{Y}_2^{(1)}; Y_2^{(1)} | U_1^{(1)} Y_3^{(1)}) \leq \tau_2 I(X_2^{(2)}; Y_3^{(2)} | V_2^{(2)}).$$

with $0 < \tau_1, \tau_2$ *and* $\tau_1 + \tau_2 \leq 1$ *for some joint probability distributions*

$$P(u_1^{(1)}, x_1^{(1)}, y_2^{(1)}, \hat{y}_2^{(1)}, y_3^{(1)}) = P(u_1^{(1)}) P(x_1^{(1)} | u_1^{(1)}) P(y_2^{(1)}, y_3^{(1)} | x_1^{(1)}) P(\hat{y}_2^{(1)} | y_2^{(1)}, u_1^{(1)})$$
$$P(v_2^{(2)}, x_1^{(2)}, x_2^{(2)}, y_3^{(2)}) = P(v_2^{(2)}) P(x_1^{(2)} | v_2^{(2)}) P(x_2^{(2)} | v_2^{(2)}) P(y_3^{(2)} | x_1^{(2)}, x_2^{(2)})$$

are achievable with a partial-decode-compress-and-forward strategy.

Proof see A2.4.

2.1.4 Wireline Communication

The derived strategies with decoding at the relay can be used to formulate the achievable rates on a wireline channel model. In phase l each directed link of the network from node i to j features the reliable communication of $b_{ij}^{(l)}$ bits by being used for unit time.

Proposition 2.1.6 *All rates of the half-duplex wireline relay channel that satisfy*

$$R \leq \min\left\{\tau_1 b_{12}^{(1)} + \tau_2 b_{13}^{(2)}, \tau_1 b_{13}^{(1)} + \tau_2 (b_{13}^{(2)} + b_{23}^{(2)})\right\}$$

with $0 < \tau_1, \tau_2$ *and* $\tau_1 + \tau_2 \leq 1$*, for some directed links of capacity* $b_{ij}^{(l)}$ *are achievable with a decode-and-forward strategy.*

Proposition 2.1.7 *All rates of the half-duplex wireline relay channel that satisfy*

$$R \leq \min\left\{\tau_1 (b_{12}^{(1)} + b_{13}^{(1)}) + \tau_2 b_{13}^{(2)}, \tau_1 b_{13}^{(1)} + \tau_2 (b_{13}^{(2)} + b_{23}^{(2)})\right\}$$

with $0 < \tau_1, \tau_2$ *and* $\tau_1 + \tau_2 \leq 1$*, for some directed links of capacity* $b_{ij}^{(l)}$ *are achievable with a partial-decode-and-forward strategy.*

Note that PDF coincides with the upper bound of this channel model due to orthogonal channels in the first phase as mentioned in [8].

2.2 Linear Problems

For linear rate and transmission cost objectives the rate expressions derived for different schemes have the property that optimal time allocation for fixed input distributions can be determined by solving linear programs of small size. It is shortly outlined how to formulate some relevant problems. It is assumed that a scheme and relaying strategy has been chosen, the channels have been specified by conditional distributions P_c (or densities p_c) and the input distributions are fixed to P_{in} (p_{in}). Mutual informations are denoted abstractly and can be looked up from the according proposition or the upper bound. The matrix $\boldsymbol{A}$ for the problems is determined by different constraints $\boldsymbol{a}$

$$\boldsymbol{A} = \begin{bmatrix} {\boldsymbol{a}_1}^T \\ {\boldsymbol{a}_2}^T \\ {\boldsymbol{a}_q}^T \\ \boldsymbol{a}_\tau^T \end{bmatrix} . \tag{2.21}$$

Note that the constraint $\boldsymbol{a}_q$ is just needed for propositions with quantization at the relay.

Rate Maximization The first relevant problem is the rate maximization problem (RP). This problem has relevance if the communication rate through the network should be maximized. The constraints have the form

$$\begin{aligned} {\boldsymbol{a}_1}^T &= \begin{bmatrix} 1 & -I_1^{(1)}(P_c, P_{in}) & -I_1^{(2)}(P_c, P_{in}) \end{bmatrix} \\ {\boldsymbol{a}_2}^T &= \begin{bmatrix} 1 & -I_2^{(1)}(P_c, P_{in}) & -I_2^{(2)}(P_c, P_{in}) \end{bmatrix} \\ {\boldsymbol{a}_q}^T &= \begin{bmatrix} 0 & \pm I_q^{(1)}(P_c, P_{in}) & \pm I_q^{(2)}(P_c, P_{in}) \end{bmatrix} \\ \boldsymbol{a}_\tau^T &= \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \\ \boldsymbol{b}^T &= \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} . \end{aligned} \tag{2.22}$$

The cost vector is

$$\boldsymbol{c}^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} . \tag{2.23}$$

In order to optimize the communication for the highest possible rate solve the problem

$$\begin{aligned} &\max \boldsymbol{c}^T \boldsymbol{x} \\ &s.t. \;\; \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{0} \leq \boldsymbol{x}, const(\boldsymbol{x}) \end{aligned} \tag{2.24}$$

where $const(\boldsymbol{x})$ denotes that additional constraints (optional) on $\boldsymbol{x} = \begin{bmatrix} R & \tau_1 & \tau_2 \end{bmatrix}^T$ are fulfilled.

Minimizing Transmission Cost Another problem with linear structure is the *Transmission Cost* minimization problem (TCP). Each phase l is associated with a cost linear in activation time. The cost for each phase l and unit activation time is denoted c_l. The objective is to minimize the cost for a certain fixed rate requirement R on the one-way communication through the network. The constraints have the form

$$
\begin{aligned}
{\boldsymbol{a}_1}^T &= \begin{bmatrix}-I_1^{(1)}(P_c, P_{in}) & -I_1^{(2)}(P_c, P_{in})\end{bmatrix} \\
{\boldsymbol{a}_2}^T &= \begin{bmatrix}-I_2^{(1)}(P_c, P_{in}) & -I_2^{(2)}(P_c, P_{in})\end{bmatrix} \\
{\boldsymbol{a}_q}^T &= \begin{bmatrix}\pm I_q^{(1)}(P_c, P_{in}) & \pm I_q^{(2)}(P_c, P_{in})\end{bmatrix} \\
\boldsymbol{a}_\tau^T &= \begin{bmatrix}1 & 1\end{bmatrix} \\
\boldsymbol{b}^T &= \begin{bmatrix}-R & -R & 0 & 1\end{bmatrix}
\end{aligned}
\tag{2.25}
$$

and the cost vector is

$$\boldsymbol{c}^T = \begin{bmatrix}c_1 & c_2\end{bmatrix}. \tag{2.26}$$

With $\boldsymbol{x} = \begin{bmatrix}\tau_1 & \tau_2\end{bmatrix}^T$ the optimization problem has the form

$$
\begin{aligned}
&\min \boldsymbol{c}^T\boldsymbol{x} \\
&s.t. \;\; \boldsymbol{Ax} \le \boldsymbol{b}, \boldsymbol{0} \le \boldsymbol{x}, const(\boldsymbol{x}).
\end{aligned}
\tag{2.27}
$$

If the problem has no solution the rate request R is infeasible. For the wireline model the mutual informations have to be replaced by link capacities $b_{ij}^{(l)}$.

2.3 Gaussian Half-Duplex Relay Channel

In order to illustrate and compare the results from the previous section the expressions for the special case of continuous random variables will be derived here and optimized with respect to maximum rate. In the previous section an upper bound and different achievable rates have been established for the half-duplex relay channel. All of them were derived under the assumption of a memoryless channel with discrete random variables and unconstrained active input distributions. Fortunately, it can be assumed that all achievable rate expressions in Section 2.1.3 can be generalized to continuous variables with Gaussian distributions [19, Remark 28, 30]. The upper bound is valid for any discrete or continuous version of the channel [7, Section 9.2]. Nevertheless, note that it is not discussed here if Gaussian distributions maximize mutual informations under a power constraint for the studied channel and strategies.

In order to use this section also for the two-way problem the model is denoted in a more general form. The message W_{ij} is to be sent from node i to j, $\forall i, j = 1, 2, 3$. It is assumed that n channel uses are divided into L non-overlapping transmission phases each occupying n_l of the n channel uses. In each phase only one network state is used. The input sequence in the l-th transmission phase at the i-th network party is denoted by $\boldsymbol{X}_i^{(l)} \in \mathbb{C}^{n_l \times 1}$ with entries being complex Gaussian with a per symbol power constraint

$$\mathrm{E}\left[\left|X_{i,k}^{(l)}\right|^2\right] \leq P_i^{(l)} \qquad k = 1, \ldots, n_l. \tag{2.28}$$

Each channel input sequence $\boldsymbol{X}_i^{(l)}$ is composed of a linear combination of M encoding functions $\boldsymbol{f}_{im}^{(l)}(W_{ij}) : \mathcal{W}_{ij} \rightarrow \mathcal{F}_{im}^{n_l}$ with independent entries normalized to unit variance $f_{im,k}^{(l)} \sim \mathcal{N}_{\mathbb{C}}(0, 1)$. Each encoding function is weighted by an amplification term $\sqrt{P_{im}^{(l)}}$. The power allocation coefficients $P_{im}^{(l)}$ have to be chosen such that the power constraint on the according output is met. The notation of encoding functions depending on messages W_{ij} is used in order to illustrate the task of each encoding function. For example

$$\boldsymbol{X}_2^{(2)} = \sqrt{\beta P_2^{(2)}}\boldsymbol{f}_{21}^{(2)}(W_{13}) + \sqrt{(1-\beta)P_2^{(2)}}\boldsymbol{f}_{22}^{(2)}(Q_{23}) \qquad \beta \in [0, 1] \tag{2.29}$$

denotes the input sequence $\boldsymbol{X}_2^{(2)}$ in phase 2 at node 2 formed by a superposition of two encoding functions with the parameter β controlling the power allocation. One of the functions is associated with the propagation of message W_{13} and the other with the transmission of the quantization index Q_{23}. All channels follow an additive model with static channel coefficients. Consequently, the channel output sequence in phase l at the j-th party of the network is

$$\boldsymbol{Y}_j^{(l)} = \sum_{\mathcal{I}} h_{ij}\boldsymbol{X}_i^{(l)} + \boldsymbol{Z}_j^{(l)} \qquad \boldsymbol{Y}_j^{(l)}, \boldsymbol{X}_i^{(l)}, \boldsymbol{Z}_j^{(l)} \in \mathbb{C}^{n_l \times 1} \tag{2.30}$$

where $h_{ij} \in \mathbb{C}$ denotes the constant coefficient of the directed channel from node i to j. Additive noise is independent, zero mean and normalized to unit variance $Z_{j,k}^{(l)} \sim \mathcal{N}_{\mathbb{C}}(0, 1)$. As the nodes are assumed to have full knowledge about all channel coefficients h_{ij} (amplitude and phase) in the network, they are able to adapt their code rates, signaling, power and time allocation perfectly with respect to the objective agreed on.

The channel output sequences for the one-way strategies are

$$\begin{aligned}
\boldsymbol{Y}_2^{(1)} &= h_{12}\boldsymbol{X}_1^{(1)} + \boldsymbol{Z}_2^{(1)} \\
\boldsymbol{Y}_3^{(1)} &= h_{13}\boldsymbol{X}_1^{(1)} + \boldsymbol{Z}_3^{(1)} \\
\boldsymbol{Y}_3^{(2)} &= h_{13}\boldsymbol{X}_1^{(2)} + h_{23}\boldsymbol{X}_2^{(2)} + \boldsymbol{Z}_3^{(2)}.
\end{aligned} \tag{2.31}$$

The messages W_{13} and Q_{23} are abbreviated W, Q. After the appropriate choice of input parameterization fulfilling the input distribution restrictions of the propositions and the chosen power constraint, the achievable rate expressions follow by straightforward calculations after replacing mutual informations in the expressions of Sections 2.1.2 and 2.1.3 by the differential entropies of scalar complex Gaussian variables.

2.3.1 Upper Bound and Achievable Rates

Upper Bound For the upper bound expression the channel input sequences are given by

$$\begin{aligned}
\boldsymbol{X}_1^{(1)} &= \sqrt{P_1^{(1)}}\boldsymbol{f}_{11}^{(1)}(W) \\
\boldsymbol{X}_1^{(2)} &= \underbrace{\sqrt{\beta P_1^{(2)}}\boldsymbol{f}_{21}^{(2)}(W)}_{\sqrt{\beta P_1^{(2)}/P_2^{(2)}}\boldsymbol{X}_2^{(2)}} + \sqrt{(1-\beta)P_1^{(2)}}\boldsymbol{f}_{11}^{(2)}(W) \qquad \beta \in [0,1] \\
\boldsymbol{X}_2^{(2)} &= \sqrt{P_2^{(2)}}\boldsymbol{f}_{21}^{(2)}(W).
\end{aligned} \tag{2.32}$$

The same encoding function $\boldsymbol{f}_{21}^{(2)}(W)$ is used at node 1 and 2 in order to realize the statistical dependence $p(x_1^{(2)}, x_2^{(2)})$. Varying β changes the dependence between the random input sequences $\boldsymbol{X}_1^{(2)}$ and $\boldsymbol{X}_2^{(2)}$. The according equation for the upper bound is

$$\begin{aligned}
R \leq \min\Big\{&\tau_1 \log\left(1 + (|h_{12}|^2 + |h_{13}|^2)P_1^{(1)}\right) + \tau_2 \log\left(1 + |h_{13}|^2 (1-\beta)P_1^{(2)}\right), \\
&\tau_1 \log\left(1 + |h_{13}|^2 P_1^{(1)}\right) + \tau_2 \log\left(1 + |h_{13}|^2 P_1^{(2)} + |h_{23}|^2 P_2^{(2)} + 2|h_{13}h_{23}|\sqrt{\beta P_1^{(2)} P_2^{(2)}}\right)\Big\}.
\end{aligned}$$

Decode-and-Forward The same input parameterization yields an achievable rate

$$\begin{aligned}
R_{\mathrm{DF}} \leq \min\Big\{&\tau_1 \log\left(1 + |h_{12}|^2 P_1^{(1)}\right) + \tau_2 \log\left(1 + |h_{13}|^2 (1-\beta)P_1^{(2)}\right), \\
&\tau_1 \log\left(1 + |h_{13}|^2 P_1^{(1)}\right) + \tau_2 \log\left(1 + |h_{13}|^2 P_1^{(2)} + |h_{23}|^2 P_2^{(2)} + 2|h_{13}h_{23}|\sqrt{\beta P_1^{(2)} P_2^{(2)}}\right)\Big\}
\end{aligned}$$

for the decode-and-forward strategy. The parameter $\beta \in [0,1]$ controls the amount of power invested by node 1 in order to assist the input signal of node 2. Equivalently one can say that node 1 and 2 agree to cooperate by a tradeoff maximizing node 3 receive signal power (beamforming) and new information node 1 can send to node 3 over the direct path. Note that both bounds have to be optimized over β and $\boldsymbol{\tau}$ while maximal powers $P_1^{(1)}, P_1^{(2)}, P_2^{(2)}$ are used. For fixed β the problem of finding optimal time allocation $\boldsymbol{\tau}^\star$ is a simple *linear program* [2].

Compress-and-Forward For the compress-and-forward strategy a different input parameterization is needed as node 1 can not assist node 2 in transmission to node 3, i.e., $p(x_1^{(2)}, x_2^{(2)}) = p(x_1^{(2)})p(x_2^{(2)})$. The input sequences are modeled as

$$\begin{aligned}\boldsymbol{X}_1^{(1)} &= \sqrt{P_1^{(1)}}\boldsymbol{f}_{11}^{(1)}(W)\\ \boldsymbol{X}_1^{(2)} &= \sqrt{P_1^{(2)}}\boldsymbol{f}_{11}^{(2)}(W)\\ \boldsymbol{X}_2^{(2)} &= \sqrt{P_2^{(2)}}\boldsymbol{f}_{21}^{(2)}(Q)\end{aligned} \tag{2.33}$$

where the variable Q denotes the resulting quantization codebook index at node 2. According to the quantization channel $p(\hat{y}_2^{(1)}|y_2^{(1)})$ the quantized version of the channel output sequence $\boldsymbol{Y}_2^{(1)}$ is assumed to have the form

$$\hat{\boldsymbol{Y}}_2^{(1)} = \boldsymbol{Y}_2^{(1)} + \hat{\boldsymbol{Z}}_2^{(1)} \tag{2.34}$$

with independent quantization noise $\hat{Z}_{2,k}^{(1)} \sim \mathcal{N}_{\mathbb{C}}(0, \hat{\sigma}^2)$. The rates can not exceed

$$R_{\text{CF}} \leq \tau_1 \log\left(1 + |h_{13}|^2 P_1^{(1)} + \frac{|h_{12}|^2 P_1^{(1)}}{1+\hat{\sigma}^2}\right) + \tau_2 \log\left(1 + |h_{13}|^2 P_1^{(2)}\right)$$

subject to

$$\tau_1 \log\left(1 + \frac{1}{\hat{\sigma}^2}\left(1 + \frac{|h_{12}|^2 P_1^{(1)}}{1 + |h_{13}|^2 P_1^{(1)}}\right)\right) \leq \tau_2 \log\left(1 + \frac{|h_{23}|^2 P_2^{(2)}}{1 + |h_{13}|^2 P_1^{(2)}}\right). \tag{2.35}$$

Partial-Decode-and-Forward The partial-decode-and-forward strategy allows node 2 to decode only a part of the channel input of node 1. This part was represented by an auxiliary random variable $U_1^{(1)}$ in Proposition 2.1.4 forming the cloud centers of superposition coding for the channel input $X_1^{(1)}$. Therefore, the Gaussian input sequence at node 1 in the first phase is divided into two encoding functions $\boldsymbol{f}_{11}^{(1)}(W)$ associated with the auxiliary and independent $\boldsymbol{f}_{12}^{(1)}(W)$, i.e., $p(u_1^{(1)}, x_1^{(1)}) = p(u_1^{(1)})p(x_1^{(1)}|u_1^{(1)})$. The power allocation parameter β controls the power allocation to the message decoded at node 2. In the second phase γ assigns the powers between the signal coherent with node 2 and the signal related to new information. The input sequences are

$$\begin{aligned}\boldsymbol{X}_1^{(1)} &= \underbrace{\sqrt{\beta P_1^{(1)}}\boldsymbol{f}_{11}^{(1)}(W)}_{\boldsymbol{U}_1^{(1)}} + \sqrt{(1-\beta)P_1^{(1)}}\boldsymbol{f}_{12}^{(1)}(W) \qquad && \beta \in [0,1]\\ \boldsymbol{X}_1^{(2)} &= \underbrace{\sqrt{\gamma P_1^{(2)}}\boldsymbol{f}_{21}^{(2)}(W)}_{\sqrt{\gamma P_1^{(2)}/P_2^{(2)}}\boldsymbol{X}_2^{(2)}} + \sqrt{(1-\gamma)P_1^{(2)}}\boldsymbol{f}_{11}^{(2)}(W) \qquad && \gamma \in [0,1]\\ \boldsymbol{X}_2^{(2)} &= \sqrt{P_2^{(2)}}\boldsymbol{f}_{21}^{(2)}(W)\end{aligned} \tag{2.36}$$

and yield the lower achievable rate bound

$$R_{\mathrm{PDF}} \leq \min\left\{\tau_1 \log\left(\frac{\left(1+|h_{12}|^2 P_1^{(1)}\right)\left(1+|h_{13}|^2(1-\beta)P_1^{(1)}\right)}{1+|h_{12}|^2(1-\beta)P_1^{(1)}}\right)+\tau_2 \log\left(1+|h_{13}|^2(1-\gamma)P_1^{(2)}\right),\right.$$
$$\left.\tau_1 \log\left(1+|h_{13}|^2 P_1^{(1)}\right)+\tau_2 \log\left(1+|h_{13}|^2 P_1^{(2)}+|h_{23}|^2 P_2^{(2)}+2\,|h_{13}h_{23}|\sqrt{\gamma P_1^{(2)} P_2^{(2)}}\right)\right\}.$$

Assuming this strategy to be able to outperform the decode-and-forward strategy

$$\begin{aligned}\frac{\left(1+|h_{12}|^2 P_1^{(1)}\right)\left(1+|h_{13}|^2(1-\beta)P_1^{(1)}\right)}{1+|h_{12}|^2(1-\beta)P_1^{(1)}} &> 1+|h_{12}|^2 P_1^{(1)}\\ 1+|h_{13}|^2(1-\beta)P_1^{(1)} &> 1+|h_{12}|^2(1-\beta)P_1^{(1)}\\ |h_{13}|^2 &> |h_{12}|^2\end{aligned} \tag{2.37}$$

shows that this requires the channel between node 1 and 3 to be stronger than the channel between node 1 and 2. Under these conditions the parameter β that maximizes the rate expression is observed to be $\beta = 0$, resulting in turning off the relay. Therefore, the partial-decode-and-forward can be considered to be a "clever" decode-and-forward strategy for the scalar channel studied here. If decoding at the relay restricts the size of the message set, the relay is turned off. However, simulations will show that the simpler decode-and-forward strategy with optimal time allocation features the same advantage for the considered channel by just using one phase, i.e., choosing $\tau_1 \to 0$.

Partial-Decode-Compress-and-Forward In contrast to the method before, the partial-decode-compress-and-forward strategy exploits the possibility to additionally quantize the channel output at node 2. Therefore, the channel input sequence $\boldsymbol{X}_1^{(1)}$ is split into two encoding functions. Encoding function $\boldsymbol{f}_{11}^{(1)}(W)$ represents the part that will be decoded by the relay and $\boldsymbol{f}_{12}^{(1)}(W)$ represents a superimposed independent signal that will only be decoded by the destination. The power allocation parameter β controls the tradeoff between the two parts. The output sequence $\boldsymbol{X}_2^{(2)}$ is divided into the encoding function $\boldsymbol{f}_{21}^{(2)}$ related to the propagation of the decoded message part and $\boldsymbol{f}_{22}^{(2)}$ related to the transmission of the quantization index. Node 1 assists the transmission of node 2 related to the decoded message part and sends new information directly to node 3 where γ denotes the tradeoff between both tasks

$$\begin{aligned}\boldsymbol{X}_1^{(1)} &= \underbrace{\sqrt{\beta P_1^{(1)}}\boldsymbol{f}_{11}^{(1)}(W)}_{\boldsymbol{U}_1^{(1)}}+\sqrt{(1-\beta)P_1^{(1)}}\boldsymbol{f}_{12}^{(1)}(W) && \beta\in[0,1]\\ \boldsymbol{X}_1^{(2)} &= \underbrace{\sqrt{\gamma P_1^{(2)}}\boldsymbol{f}_{21}^{(2)}(W)}_{\sqrt{\gamma P_1^{(2)}/\delta P_2^{(2)}}\boldsymbol{V}_2^{(2)}}+\sqrt{(1-\gamma)P_1^{(2)}}\boldsymbol{f}_{11}^{(2)}(W) && \gamma\in[0,1]\\ \boldsymbol{X}_2^{(2)} &= \underbrace{\sqrt{\delta P_2^{(2)}}\boldsymbol{f}_{21}^{(2)}(W)}_{\boldsymbol{V}_2^{(2)}}+\sqrt{(1-\delta)P_2^{(2)}}\boldsymbol{f}_{22}^{(2)}(Q) && \delta\in[0,1].\end{aligned} \tag{2.38}$$

The output sequence of the quantizer of node 2 has the same form like in the compress-and-forward strategy,

$$\hat{\boldsymbol{Y}}_2^{(1)} = \boldsymbol{Y}_2^{(1)} + \hat{\boldsymbol{Z}}_2^{(1)} \tag{2.39}$$

with $\hat{Z}_{2,k}^{(1)} \sim \mathcal{N}_{\mathbb{C}}(0, \hat{\sigma}^2)$ being the quantization noise. The rate bound is given by

$$\begin{aligned} R_{\text{PDCF}} \leq \min \Bigg\{ & \tau_1 \log \left(\frac{1 + |h_{12}|^2 P_1^{(1)}}{1 + |h_{12}|^2 (1-\beta) P_1^{(1)}} \right), \tau_1 \log \left(\frac{1 + |h_{13}|^2 P_1^{(1)}}{1 + |h_{13}|^2 (1-\beta) P_1^{(1)}} \right) \\ & + \tau_2 \log \left(\frac{1 + |h_{13}|^2 P_1^{(2)} + |h_{23}|^2 P_2^{(2)} + 2 |h_{13} h_{23}| \sqrt{\gamma \delta P_1^{(2)} P_2^{(2)}}}{1 + |h_{13}|^2 (1-\gamma) P_1^{(2)} + |h_{23}|^2 (1-\delta) P_2^{(2)}} \right) \Bigg\} \\ & + \tau_1 \log \left(1 + \left(|h_{13}|^2 + \frac{|h_{12}|^2}{1 + \hat{\sigma}^2} \right) (1-\beta) P_1^{(1)} \right) + \tau_2 \log \left(1 + |h_{13}|^2 (1-\gamma) P_1^{(2)} \right) \end{aligned}$$

subject to

$$\tau_1 \log \left(1 + \frac{1}{\hat{\sigma}^2} \left(1 + \frac{|h_{12}|^2 (1-\beta) P_1^{(1)}}{1 + |h_{13}|^2 (1-\beta) P_1^{(1)}} \right) \right) \leq \tau_2 \log \left(1 + \frac{|h_{23}|^2 (1-\delta) P_2^{(2)}}{1 + |h_{13}|^2 (1-\gamma) P_1^{(2)}} \right). \tag{2.40}$$

2.3.2 Simulations

Line/Plane Network For example scenarios the line and plane network are introduced. These network models arrange nodes on the (x, y)-plane. Node 1 is fixed at the origin $\begin{bmatrix}0 & 0\end{bmatrix}^T$ and node 3 at $\begin{bmatrix}1 & 0\end{bmatrix}^T$. For the line network node 2 can be moved to all positions $\begin{bmatrix}x & 0\end{bmatrix}^T$ with $x \in (-\infty; \infty)$. For the plane network node 2 is allowed to take any position $\begin{bmatrix}x & y\end{bmatrix}^T$ with $x, y \in (-\infty; \infty)$. The channel coefficients are considered to be determined by the distances between nodes

$$h_{ij} = 1/d_{ij}^{\frac{\alpha}{2}}, \tag{2.41}$$

and the path loss exponent α. Note that this model is not accurate if node 2 is near node 1 or 3 due to the far-field assumption on the path-loss. Power parameters $P_i^{(l)}$ need to be interpreted as the *signal-to-noise-ratio* (SNR_{ij}) at a node j resulting from node i if located at unit distance to each other and communicating in a simple point-to-point fashion. The simulations done here as example use $P_i^{(l)} = 10$ and $\alpha = 3$. So reducing the distance between two nodes i and j by a factor of 2 increases the receive SNR_{ij} at node j caused by node i by a factor of 8 (~ 9 dB).

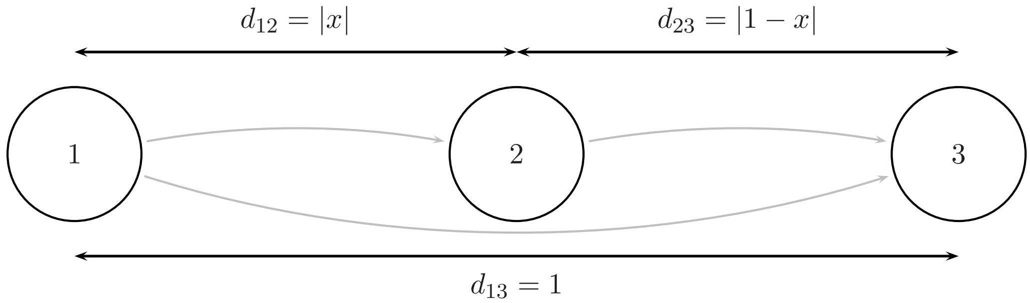


Figure 2.4: Line Network

Equal Time Allocation Figure 2.5 shows the achievable rates in the line network with the basic relaying strategies DF and CF. Time is allocated equaly to both transmission phases $\boldsymbol{\tau} = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\end{bmatrix}^T$. Decode-and-forward (DF) achieves capacity if node 2 is near the source node. In the extreme case where $x = 0$ the system performs half the time as single-input-single-output (SISO) system between node 1 and node 3 with the rate

$$R_{\text{SISO}} \leq \frac{1}{2} \log\left(1 + P_1^{(1)}\right). \tag{2.42}$$

The channel between node 1 and 2 features unlimited capacity allowing the two nodes to agree on arbitrary transmitter cooperation for the second phase. The rest of the time the network becomes a multiple-input-single-output (MISO) system

$$R_{\text{MISO}} \leq \frac{1}{2} \log\left(1 + P_1^{(2)} + P_2^{(2)} + 2\sqrt{P_1^{(2)} P_2^{(2)}}\right) \tag{2.43}$$

where node 1 and 2 send together in full coherence ($\beta = 1$) to node 3. The additional term

$$2\sqrt{P_1^{(2)} P_2^{(2)}} \tag{2.44}$$

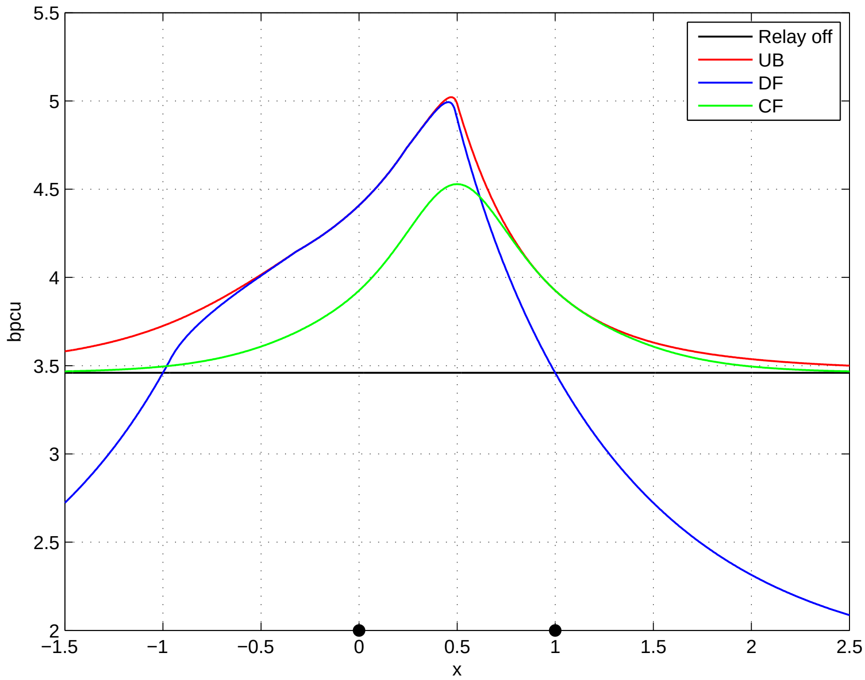


Figure 2.5: Line Network with $\boldsymbol{\tau}^T = \left[\frac{1}{2}, \frac{1}{2}\right]$, $P_i^{(l)} = 10$, $\alpha = 3$

results from both nodes sending signals adding constructively at node 3 (*beamforming gain*).

Due to the ability to convey a good quality estimate of $Y_2^{(2)}$ to node 3, compress-and-forward (CF) performs good if node 2 is in the vicinity of the destination node. With node 2 being located at the extreme point $x = 1$ the system turns into a single-input-multiple-output system (SIMO) in the first phase with rate

$$R_{SIMO} \leq \frac{1}{2} \log \left(1 + 2P_1^{(1)}\right) \tag{2.45}$$

followed by a second phase where CF performs like the above-mentioned SISO channel. So for the extreme cases one of the methods realizes perfect cooperation for half the transmission time.

For DF it can be observed that the rates fall below the transmission without relay if the channel gain $|h_{12}|^2$ between node 1 and 2 is lower than the gain of the direct channel $|h_{13}|^2$. Figure 2.6 shows the achievable rates in the line network for the advanced relaying methods. The partial-decode-and-forward strategy (PDF) shows the predicted behavior by turning off the relay if the connection between node 1 and 3 is stronger than the channel between node 1 and 2. Consequently, PDF is in any situation at least as good as the direct transmission. The partial-decode-compress-and-forward strategy (PDCF) performs a hard decision for the best transmission strategy among DF, CF and PDF. Unfortunately, an additional PDCF rate gain due to the possible combination of PDF and CF is not observed in the example.

Time Allocation Figure 2.7 compares the rates of the line network with and without optimized time allocation. A rate gain can be observed for CF if node 2 is located around the source or the destination node. For DF the rate is higher around the source node and in the area between half

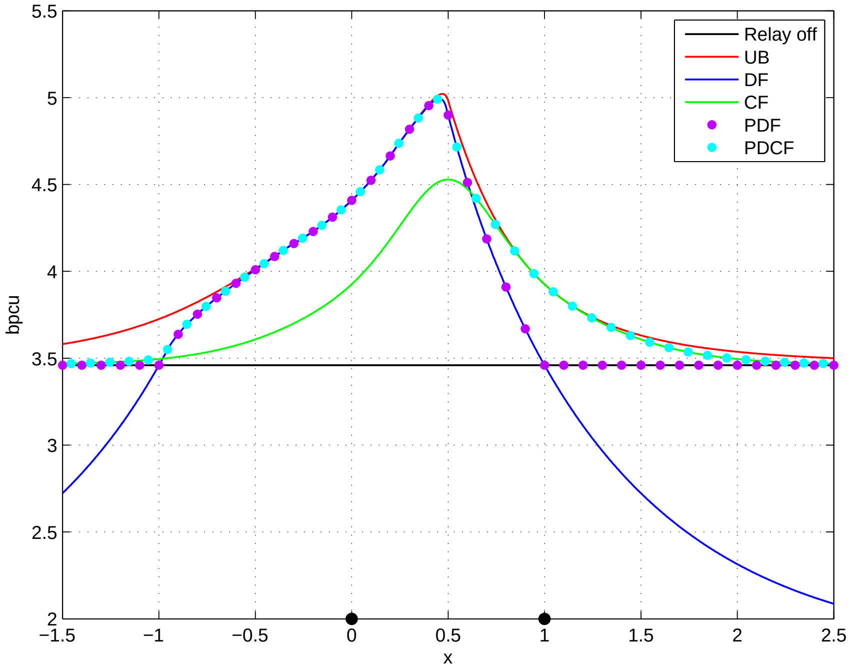


Figure 2.6: Line Network with $\boldsymbol{\tau}^T = [\frac{1}{2}, \frac{1}{2}]$, $P_i^{(l)} = 10$, $\alpha = 3$

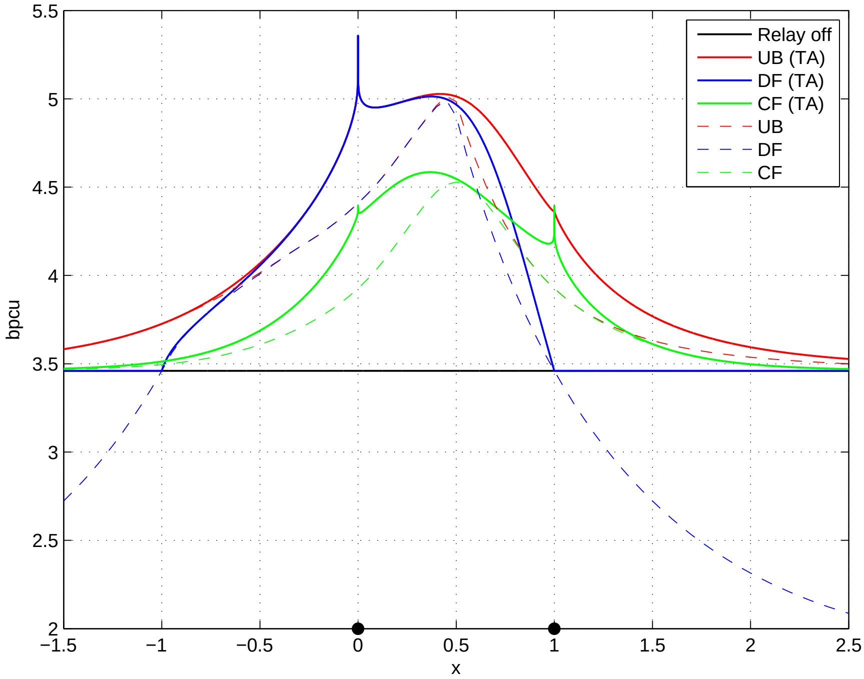


Figure 2.7: Time Allocation, Line Network with $P_i^{(l)} = 10$, $\alpha = 3$

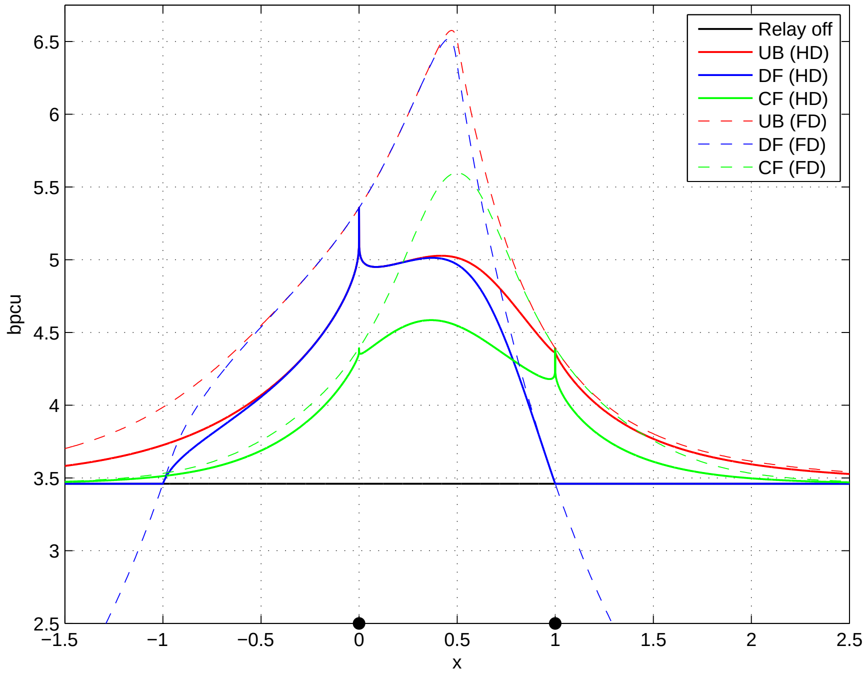


Figure 2.8: HD vs. FD, Line Network with $P_i^{(l)} = 10$, $\alpha = 3$

the way and node 3. Also for the case where the rate between node 1 and 2 is smaller than the rate of the direct link DF benefits from time allocation with using just one phase $\tau_1 \to 0$. Therefore, the relay is disconnected from the transmission process. The peaks at the extreme points of the network result from time-allocation which offers the possibility to enforce a full-duplex system. The comparison between half-duplex and full-duplex rates is depicted in Figure 2.8.

Plane Network Some results for a plane network with the same parameterization are presented in Figures 2.9-2.12. Figure 2.9 shows that the achievable rate gain compared to direct transmission without relay is up to 45 percent for the example. Figures 2.10 and 2.11 show the rate gain for DF and CF with optimal time allocation in contrast to equal time slots. Note that although the gain of up to 10 percent in some areas seems to be low it comes without cost as all nodes are assumed to have full knowledge about the channel coefficients. Figure 2.12 shows the relation of the achievable rates to the upper bound.

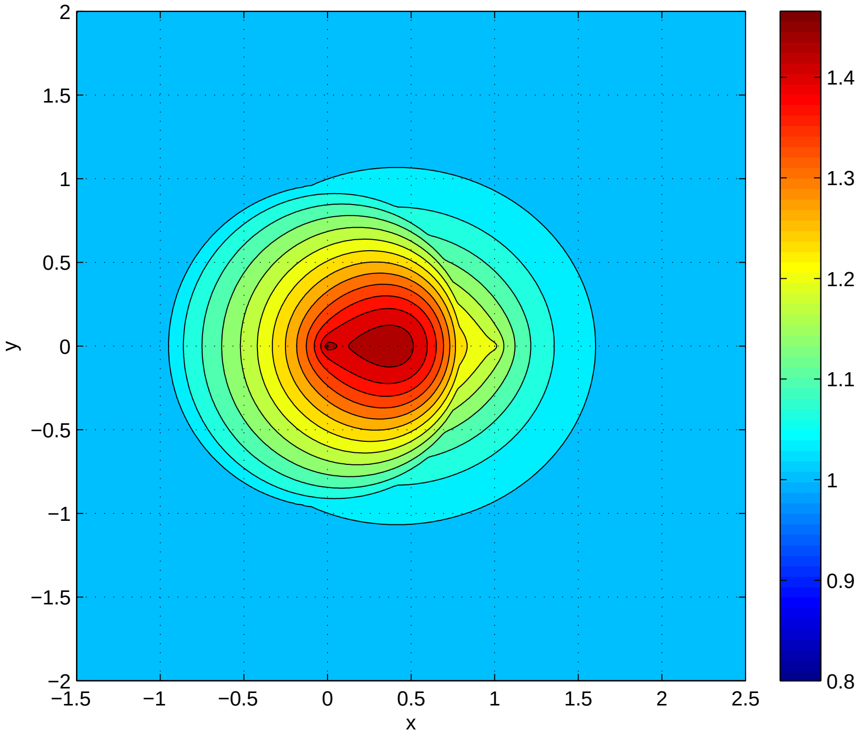


Figure 2.9: Achievable Rate Gain, Plane Network with $P_i^{(l)} = 10$, $\alpha = 3$

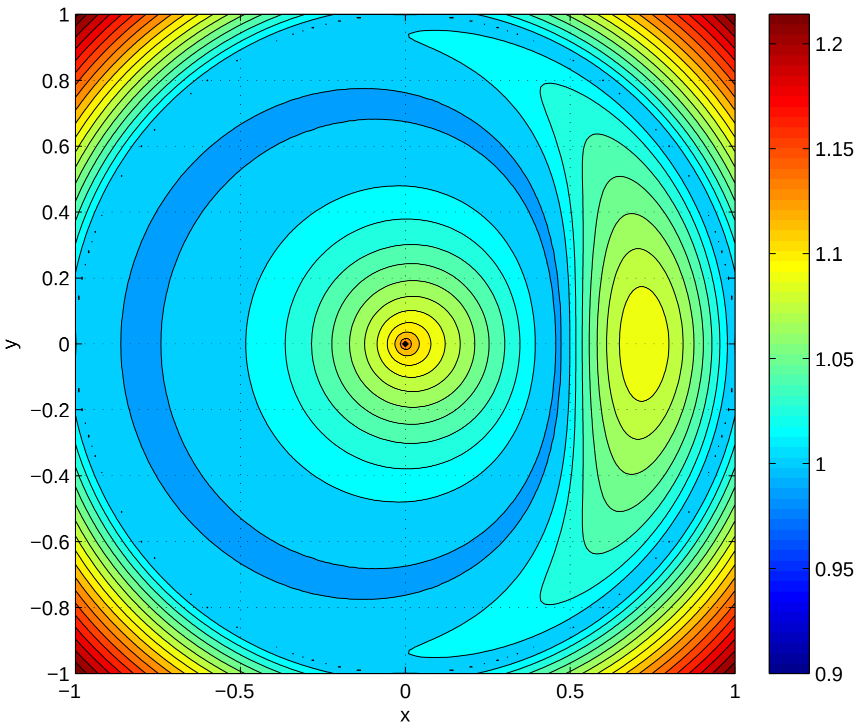


Figure 2.10: Rate Gain for DF with TA, Plane Network with $P_i^{(l)} = 10$, $\alpha = 3$

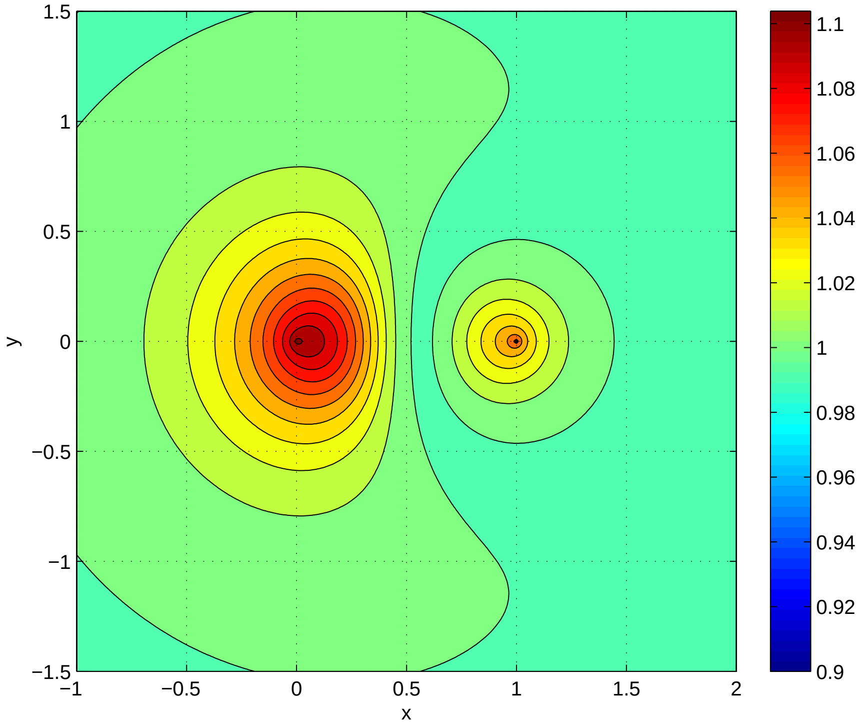


Figure 2.11: Rate Gain for CF with TA, Plane Network with $P_i^{(l)} = 10$, $\alpha = 3$

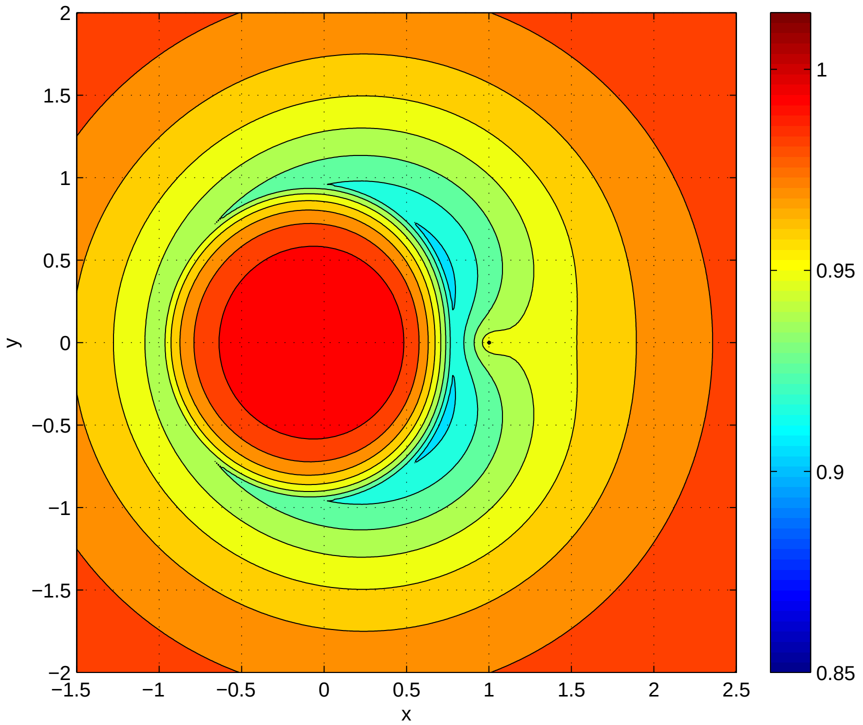


Figure 2.12: Achievable Rates/Upper Bound, Plane Network with $P_i^{(l)} = 10$, $\alpha = 3$

3. Half-Duplex Two-Way Relay Channel

The second part of the thesis focuses on the *restricted half-duplex two-way relay channel*. Nodes 1 and 3 want to exchange independent messages through the network, while node 2 is able to help in the communication process. The two-way channel is restricted in the sense that nodes 1 and 3 choose their channel inputs just in dependence of the individual messages they want to send. They can not use their past receive signals in order to pick the optimal current channel inputs. Therefore, the two dialog encoders are not allowed to establish a cooperation over their past receive signals. As in the one-way problem all nodes operate in half-duplex mode. In contrast to the one-way scenario now a combinatorial aspect joins the coding problem as many different schemes are possible. Literature has focused on a two-phase relaying scheme (2P-MA-BC) [26] where in the first phase nodes 1 and 3 send simultaneously to node 2 in a multiple-access (MA) fashion. In the second phase node 2 broadcasts (BC) to nodes 1 and 3. The restriction to such a scheme is frequently justified by the assumption of a *separated two-way relay channel* [13] where nodes 1 and 3 have no direct connection. Recently, a second scheme with three phases (3P-BC) [25] [28] [16] has been proposed where nodes 1 and 3 send to node 2 one after another. In a last step node 2 sends to both dialog parties. This scheme can benefit from the orthogonal transmission to the relay and the use of the direct path. Further, a four phase scheme (without outer bound) has been proposed by adding a multiple access phase to the 3P-BC scheme [28]. Surprisingly, to the best of the author's knowledge, other schemes have not been considered nor an attempt has been made in order to define the problem and reveal all its possibilities. Moreover, none of the works found verifies and compares results in the presence of a problem outer bound. The two outer bounds presented in literature are associated with one of the schemes mentioned above and can therefore not serve as outer approximations of the achievable rate region of a fully-connected model as this region is a limiting property of the channel itself and not of a specific way the channel is used.

Wireline Example The benefit of a two-way relay channel can be seen by the following example. Assume a wireline network of two nodes, both facing a half-duplex constraint. For simplicity all links support one reliable bit per channel use. Nodes 1 and 3 want to exchange messages (bits). The obvious way to communicate is to let the nodes send sequentially to each other (see Figure 3.1). The two-way channel allows to transmit two bits within two steps. Therefore, the two-way

Figure 3.1: Two-way channel (TWC, 1.0 bps / 1.0 lpb / 1.0 npb)

transmission is carried out at a rate of 1.0 bits per step ($R_{TW} = R_{13} + R_{31} = 1.0$ bps). Assume now that a direct connection is not possible. Extend the network to a third node (relay) connected to both dialog nodes through separate links. This makes it possible to exchange messages over the relay. A straightforward half-duplex protocol is given by four steps depicted in Figure 3.2. Each node sends sequentially to the relay and the relay sends to both nodes one after another. It is possible to exchange two bits in four steps ($R_{TW} = 0.5$ bps). Figure 3.3 shows that combining the two steps towards the relay to a multiple-acess phase and the two steps from the relay to a broadcast phase [26] makes it possible to exchange two bits in two steps ($R_{TW} = 1.0$ bps). Note that this scheme allows to recover the rate of the direct two-way communication under a connectivity problem.

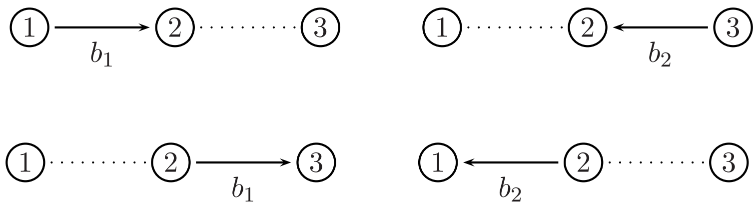


Figure 3.2: Two-way relaying in four steps (4P with DF, 0.5 bps / 2.0 lpb / 2.0 npb)

Figure 3.3: Two-way relaying in two steps (2P-MA-BC with DF, 1.0 bps / 2.0 lpb / 1.5 npb)

Fully-Connected Wireline Example The example is now extended to a fully-connected network. Each node can connect to all other nodes in the network through separated links. The half-duplex constraint on the nodes remains. TWC and the 2P-MA-BC scheme achieve the same rates as before. Consider the three-step scheme sketched in Figure 3.4. A multiple-acess phase is followed by two transmissions from the relay to one of the dialog nodes. The dialog nodes use the possibility to transmit a new bit while node 2 sends to their dialog counterpart. This scheme, here referred to as 3P-MA, makes it possible to exchange four bits within three steps ($R_{TW} = 1.33$ bps). Another possibility of a three-phase scheme is the 3P-BC scheme. Here nodes 1 and 3 send

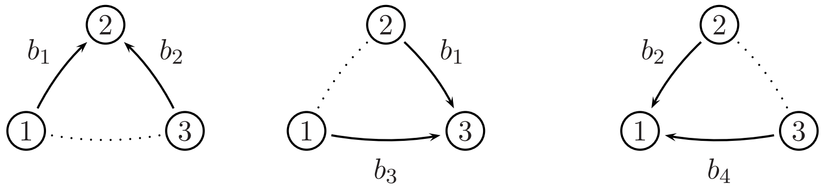


Figure 3.4: Two-way relaying in three steps (3P-MA with DF, 1.33 bps / 1.5 lpb / 1.5 npb)

to node 2 one after another. Node 2 broadcasts the received bits. If DF relaying is used (see Figure 3.5) the communication is limited by the source-relay links in the first two phases. By definition the relay must decode the full messages send by the two dialog nodes. Only two bits can be exchanged in three steps ($R_{TW} = 0.67$ bps). Instead if a PDF strategy is used (see Figure 3.6) the relay is allowed to decode only parts of the messages sent. It is now possible for nodes 1 and 3 to emit an additional bit to their dialog partner. This allows to exchange four bits in three steps ($R_{TW} = 1.33$ bps). Using four steps (see Figure 3.7) even allows to exchange six bits in four steps ($R_{TW} = 1.5$ bps). This obviously outperforms the direct two-way communication.

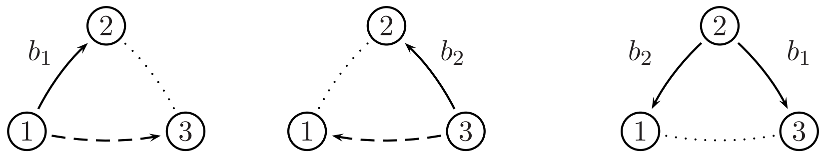


Figure 3.5: Two-way relaying in three steps (3P-BC with DF, 0.67 bps / 2.0 lpb / 1.5 npb)

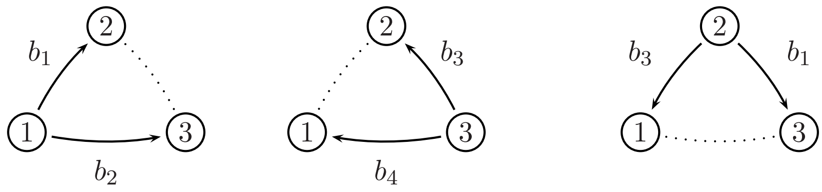


Figure 3.6: Two-way relaying in three steps (3P-BC with PDF, 1.33 bps / 1.5 lpb / 0.75 npb)

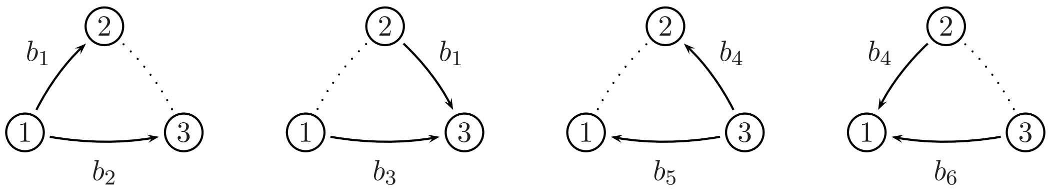


Figure 3.7: Two-way relaying in four steps (4P-OWRC with PDF, 1.5 bps / 1.33 lpb / 1.0 npb)

Communication Cost Another important aspect of the example is the cost related to the communication process. Unit cost can be associated with the number of used links per bit (lpb). It is observed that the 2P-MA-BC scheme consumes 2.0 lpb. The 3P-MA DF and 3P-BC PDF schemes only require 1.5 lpb. The four step scheme reduces the cost to 1.33 lpb. None of the schemes can outperform the two-way channel operating at 1.0 lpb. Alternatively, unit cost can be related to the activation of a node. For such a cost model the benchmark lies at one node-activation per bit (1.0 npb) for the direct two-way channel. The 3P-BC scheme with PDF for example operates at a lower cost of 0.75 npb. Even the 4P-OWRC scheme achieves the benchmark of 1.0 npb while providing higher rates.

Wireless Channels The above wireline example neglects the properties of wireless channels, i.e., channels supporting asymmetric rates, statistical dependence of links resulting in broadcasting, superposition or interference. For single-antenna networks the previous one-way analysis has shown that the PDF strategy can not outperform the DF strategy for the interesting cases. In order to maximize the rate in the half-duplex relay channel the source needs to maximize the sum-rate of data emitted in the first phase. As scalar channels can be statistically ordered the source invests all its power on the signal decoded by the node connected with a "stronger" channel. The PDF strategy turns to a DF strategy or disconnects the relay from the transmission process. This will be different in multiple-input-multiple-output (MIMO) systems due to the structure of the channel. The case of orthogonal channels will be unlikely but with precoding a situation similar to the example can be enforced by exploiting the structure of the channel matrices.

Coding Another subtle aspect that is hidden by the abstraction and symmetry of the example is the interaction between the encoders and decoders of the separated links. Here each link represents an independent chain of an encoder, a channel and a decoder together supporting the transmission of one reliable bit. It should have become clear with the one-way scenario that this might not be optimal. A decoder can wait with decoding until having received side information over a different link.

Overview However, the given example shows that two-way relaying schemes might offer possibilities to enhance the performance of the communication in a fully-connected half-duplex network with three nodes. Therefore, they deserve a precise survey. As no analysis of two-way protocols with PDF could be found and the analysis of strategies like DF, CF and PDCF seems to be incomplete for the underlying channel model a rigorous information theoretical approach through general channels must be carried out before focusing on special channels like the wireless ones. After a short summary of related work, the next pages will give a possible definition of the *restricted half-duplex two-way relaying problem* followed by an outer bound on the achievable rates. In order to understand some of the possibilities and mechanisms of half-duplex two-way communication with a relay the two schemes, 2P-MA-BC and 3P-BC, found in literature are revisited. Subsequently, a new three-phase scheme (3P-MA) is proposed where nodes 1 and 3 send together to node 2. Two orthogonal transmission phases from the relay, each intended for one of the nodes, follow. This scheme makes it possible to assist the relay in the two last phases by one of the dialog nodes or to send new additional information. A second new scheme with four phases (4P-OWRC) is proposed. It separates the two-way channel into two subsequent one-way relay channels. Finally, a general scheme with six phases and relaying strategies limited to DF and PDF relaying is established. For fixed input distributions the time allocation solution for the DF protocol is argued to provide the optimal decode-and-forward transmission scheme with respect to the maximization of any reasonable rate objective. Like in the first part, simulations for channels with scalar Gaussian random variables are used to visually compare the performance of the schemes. All schemes are analyzed by the problem outer bound, their individual performance outer bound and inner bounds derived with different relaying strategies. The main contribution is the derivation of a problem outer bound providing an ultimate performance benchmark and the proposal of a new transmission scheme taking into account all possible network state configurations of the channel.

Related Work The two-way communication problem was introduced in [31]. The work [26] established the idea of exchanging messages with two phases in a bi-directional way over a relay using amplify-and-forward (AF) and DF strategies. [25] proposes the 3P-BC scheme and derives the achievable rates with network coding. In [23] the broadcast phase of the 2P-MA-BC scheme is examined in detail and the capacity achieving coding scheme is derived if the relay has available both dialog messages. Using a CF strategy for the two-phase two-way relay channel is investigated by [29]. The recent work [16] investigates the 3P-BC scheme in detail. This scheme is also investigated in [28] where additionally a four phase scheme is proposed. Full-duplex works are rare. [27] studies the restricted full-duplex two-way relay channel while [13] focuses on a separated full-duplex model. [1] proposes a deterministic approach to approximate the capacity of the two-way relay channel.

3.1 Discrete Memoryless Half-Duplex Two-Way Relay Channel

3.1.1 Channel Models

Full-Duplex Model Like for the one-way problem a full-duplex model is reviewed. The studied network consists of three nodes labeled by $i = 1, 2, 3$ each equipped with an input X_i and an output Y_i to a common channel. The channel is time-invariant, discrete and memoryless. Therefore the two-way relay channel can be defined by

$$(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3, P(y_1, y_2, y_3|x_1, x_2, x_3), \mathcal{Y}_1 \times \mathcal{Y}_2 \times \mathcal{Y}_3) \tag{3.1}$$

where $\mathcal{X}_i$ and $\mathcal{Y}_i$ are finite and discrete input and output alphabets. W_{13} denotes the message to be transmitted from node 1 to node 3. W_{31} is to be transmitted from node 3 to node 1.

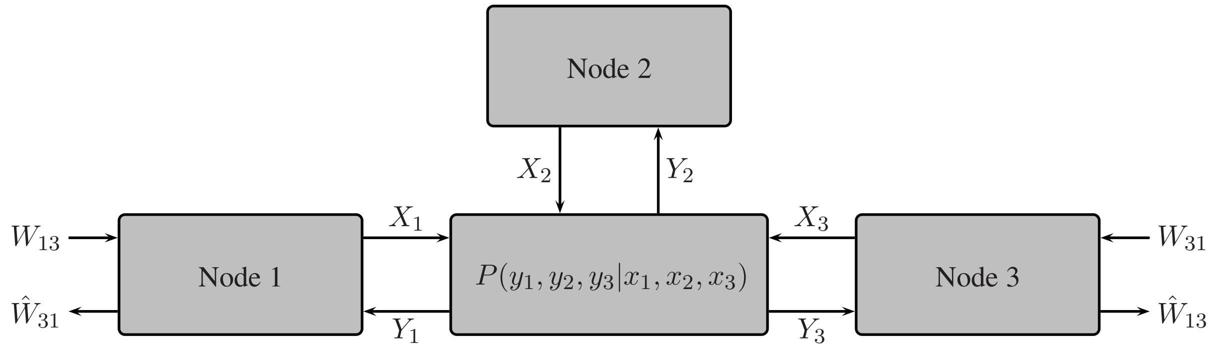


Figure 3.8: Full-Duplex Two-Way Relay Channel

Restricted Codes A restricted code of length n and rate $\boldsymbol{R}^T = \begin{bmatrix} R_{13} & R_{31} \end{bmatrix}$ consists of two message sets

$$\begin{aligned} \mathcal{W}_{13} &= \{1, 2, ..., 2^{nR_{13}}\} \\ \mathcal{W}_{31} &= \{1, 2, ..., 2^{nR_{31}}\}, \end{aligned} \tag{3.2}$$

two encoding functions for node 1 and 3

$$\begin{aligned} \boldsymbol{f}_1 &: \mathcal{W}_{13} \to \mathcal{X}_1^n \\ \boldsymbol{f}_3 &: \mathcal{W}_{31} \to \mathcal{X}_3^n, \end{aligned} \tag{3.3}$$

a set of relaying functions

$$\{f_{2,k}\}_{k=1}^n \quad \text{s.t.} \quad x_{2,k} = f_{2,k}(Y_{2,1}, Y_{2,2}, \ldots, Y_{2,k-1}) \tag{3.4}$$

and two decoding functions

$$\begin{aligned} g_1 &: \mathcal{Y}_1^n \times \mathcal{W}_{13} \to \mathcal{W}_{31} \\ g_3 &: \mathcal{Y}_3^n \times \mathcal{W}_{31} \to \mathcal{W}_{13}\,. \end{aligned} \tag{3.5}$$

The code is restricted to the encoding functions being independent of past receive signals.

Half-Duplex Model For the half-duplex problem it can be observed that $2^3 = 8$ network states are in general possible. Each of the three nodes can take two states, listen or talk. It can be assumed that the two network states where all nodes talk or listen will not contribute to a positive information flow. The relevant number of network states reduces to $2^3 - 2 = 6$. Within three of the states one of the nodes sends to two other nodes while for the other states two nodes send simultaneously to one node. Figure 3.9 depicts the six relevant *elementary network states*. The

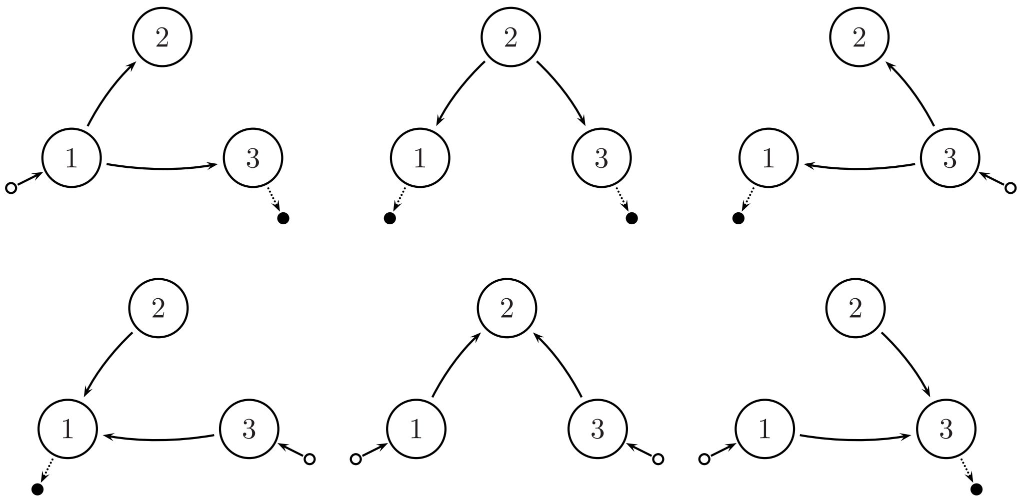


Figure 3.9: Elemetary Network States

question arises which of the states in conjunction with which code should be used in order to maximize a certain objective as for example *sum-rate*. It is also apparently not clear if the order of the states has an effect on the problem.

Here the channel is defined with finite input and output alphabets $\mathcal{Y}_i, \mathcal{X}_i$ as

$$\left(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3, P^{(s)}(y_1, y_2, y_3 | x_1, x_2, x_3, s), \mathcal{Y}_1 \times \mathcal{Y}_2 \times \mathcal{Y}_3, \mathcal{S}\right) \tag{3.6}$$

with a network state variable $S := \begin{bmatrix} S_1 & S_2 & S_3 \end{bmatrix}^T$, $s_i \in \{0, 1\}$ imposing following restrictions on the output values and input distributions

$$\begin{aligned} y_i &= 0 && \text{if} \quad s_i = 1 \\ P_{X_i^{(s)}}(0) &= 1 && \text{if} \quad s_i = 0 \qquad \forall i = 1, 2, 3. \end{aligned} \tag{3.7}$$

On the following pages different realizations of S will be labeled by numbers $l = 1, \ldots, L$ instead of a three element binary vector. For each scheme the labeling will be defined individually.

Restricted Codes Consider a scheme S^n and a choice of L network states. S_k takes values in $\mathcal{S} : \{1, \ldots, L\}$ and determines the individual network state l for the k-th of n channel uses. n_l denotes the number of occurrences of the l-th network state in n channel uses. Time allocation is defined as the ratio of n_l to n

$$\tau_l = \frac{n_l}{n}, \qquad \forall l \tag{3.8}$$

with

$$
\begin{aligned}
&0 < \tau_l \\
&\sum_{l=1}^{L} \tau_l \leq 1.
\end{aligned}
\tag{3.9}
$$

A code of length n and rate $\boldsymbol{R}^T = \begin{bmatrix} R_{13} & R_{31} \end{bmatrix}$ consists of a fixed scheme, two message sets

$$
\begin{aligned}
\mathcal{W}_{13} &= \{1, 2, ..., 2^{nR_{13}}\} \\
\mathcal{W}_{31} &= \{1, 2, ..., 2^{nR_{31}}\},
\end{aligned}
\tag{3.10}
$$

$2 \times L$ encoding functions

$$
\begin{aligned}
&\boldsymbol{f}_1^{(l)} : \mathcal{W}_{13} \rightarrow \mathcal{X}_1^{n_l} \\
&\boldsymbol{f}_3^{(l)} : \mathcal{W}_{31} \rightarrow \mathcal{X}_3^{n_l},
\end{aligned}
\tag{3.11}
$$

a set of relaying functions

$$
\{f_{2,k}\}_{k=1}^{n} \quad \text{s.t.} \quad x_{2,k} = f_{2,k}(S_k, Y_{2,1}, Y_{2,2}, \ldots, Y_{2,k-1}) \tag{3.12}
$$

and two decoding functions

$$
\begin{aligned}
g_1 &: \mathcal{Y}_1^n \times \mathcal{W}_{13} \rightarrow \mathcal{W}_{31} \\
g_3 &: \mathcal{Y}_3^n \times \mathcal{W}_{31} \rightarrow \mathcal{W}_{13} .
\end{aligned}
\tag{3.13}
$$

The code is restricted as the encoding functions are independent of past receive signals.

3.1.2 Problem Outer Bound

The Cut-set Theorem is applied to outer bound the achievable rates of the restricted two-way relaying problem with half-duplex constraint. For any fixed scheme which uses less than the six relevant network states an individual performance outer bound can be derived without further proof by setting the occurrence of the unused network states to zero.

Theorem 3.1.1 *All rate pairs of the half-duplex two-way relay channel that are achievable for some joint probability distributions*

$$
\begin{aligned}
P(x_1^{(1)}, y_2^{(1)}, y_3^{(1)}) &= P(x_1^{(1)})P(y_2^{(1)}, y_3^{(1)}|x_1^{(1)}) \\
P(x_3^{(2)}, y_1^{(2)}, y_2^{(2)}) &= P(x_3^{(2)})P(y_1^{(2)}, y_2^{(2)}|x_3^{(2)}) \\
P(x_1^{(3)}, x_3^{(3)}, y_2^{(3)}) &= P(x_1^{(3)})P(x_3^{(3)})P(y_2^{(3)}|x_1^{(3)}, x_3^{(3)}) \\
P(x_2^{(4)}, y_1^{(4)}, y_3^{(4)}) &= P(x_2^{(4)})P(y_1^{(4)}, y_3^{(4)}|x_2^{(4)}) \\
P(x_2^{(5)}, x_3^{(5)}, y_1^{(5)}) &= P(x_2^{(5)}, x_3^{(5)})P(y_1^{(5)}|x_2^{(5)}, x_3^{(5)}) \\
P(x_1^{(6)}, x_2^{(6)}, y_3^{(6)}) &= P(x_1^{(6)}, x_2^{(6)})P(y_3^{(6)}|x_1^{(6)}, x_2^{(6)})
\end{aligned}
$$

must satisfy

$$R_{13} \le \min\Big\{\tau_1 I(X_1^{(1)};Y_2^{(1)}Y_3^{(1)}) + \tau_3 I(X_1^{(3)};Y_2^{(3)}|X_3^{(3)}) + \tau_6 I(X_1^{(6)};Y_3^{(6)}|X_2^{(6)}),$$
$$\tau_1 I(X_1^{(1)};Y_3^{(1)}) + \tau_4 I(X_2^{(4)};Y_3^{(4)}) + \tau_6 I(X_1^{(6)}X_2^{(6)};Y_3^{(6)})\Big\}$$
$$R_{31} \le \min\Big\{\tau_2 I(X_3^{(2)};Y_1^{(2)}Y_2^{(2)}) + \tau_3 I(X_3^{(3)};Y_2^{(3)}|X_1^{(3)}) + \tau_5 I(X_3^{(5)};Y_1^{(5)}|X_2^{(5)}),$$
$$\tau_2 I(X_3^{(2)};Y_1^{(2)}) + \tau_4 I(X_2^{(4)};Y_1^{(4)}) + \tau_5 I(X_2^{(5)}X_3^{(5)};Y_1^{(5)})\Big\}$$

where $0 \le \tau_l$ *and* $\sum_{l=1}^{6} \tau_l \le 1$.

Proof Consider a full-duplex three-node network and the two possible cut-set partitions Ω_1, Ω_2 separating nodes 1 and 3. Under the assumption of zero-error codes it holds with the cut-set theorem [7, Theorem 15.10.1] that the rates are outer bounded by

$$\begin{aligned}\Omega_1 :& R_{13} \le I(X_1;Y_2Y_3|X_2X_3)\\ & R_{31} \le I(X_2X_3;Y_1|X_1)\\ \Omega_2 :& R_{13} \le I(X_1X_2;Y_3|X_3)\\ & R_{31} \le I(X_3;Y_1Y_2|X_1X_2)\end{aligned} \tag{3.14}$$

for some joint input distribution $P(x_1, x_2, x_3)$. As here the encoders are not allowed to cooperate the marginal $P(x_1, x_3)$ is restricted to distributions that factorize $P(x_1)P(x_3)$. Introducing a state variable S, taking values in $\mathcal{S} : \{1, \dots, L\}$ and distributed according to

$$P_S(l) = \frac{n_l}{n} = \tau_l \qquad l = 1, \dots, L \tag{3.15}$$

yields

$$\begin{aligned}R_{13} &\le I(X_1;Y_2Y_3|X_2X_3S)\\ R_{13} &\le I(X_1X_2;Y_3|X_3S)\\ R_{31} &\le I(X_3;Y_1Y_2|X_1X_2S)\\ R_{31} &\le I(X_2X_3;Y_1|X_1S).\end{aligned} \tag{3.16}$$

Equivalently,

$$\begin{aligned}R_{13} &\le \sum_{l=1}^{L} P_S(l) I(X_1^{(l)};Y_2^{(l)}Y_3^{(l)}|X_2^{(l)}X_3^{(l)}S = l)\\ R_{13} &\le \sum_{l=1}^{L} P_S(l) I(X_1^{(l)}X_2^{(l)};Y_3^{(l)}|X_3^{(l)}S = l)\\ R_{31} &\le \sum_{l=1}^{L} P_S(l) I(X_3^{(l)};Y_1^{(l)}Y_2^{(l)}|X_1^{(l)}X_2^{(l)}S = l)\\ R_{31} &\le \sum_{l=1}^{L} P_S(l) I(X_2^{(l)}X_3^{(l)};Y_1^{(l)}|X_1^{(l)}S = l).\end{aligned} \tag{3.17}$$

Agreeing to use $L = 6$ elementary network states defined by

$$\begin{aligned}
l = 1 &: s_1 = 1, s_2 = 0, s_3 = 0 \\
l = 2 &: s_1 = 0, s_2 = 0, s_3 = 1 \\
l = 3 &: s_1 = 1, s_2 = 0, s_3 = 1 \\
l = 4 &: s_1 = 0, s_2 = 1, s_3 = 0 \\
l = 5 &: s_1 = 0, s_2 = 1, s_3 = 1 \\
l = 6 &: s_1 = 1, s_2 = 1, s_3 = 0
\end{aligned} \tag{3.18}$$

and reformulating the mutual informations like in the proof of Theorem 2.1.1 establishes the theorem above. ■

3.2 Schemes, Outer Bounds and Achievable Rates

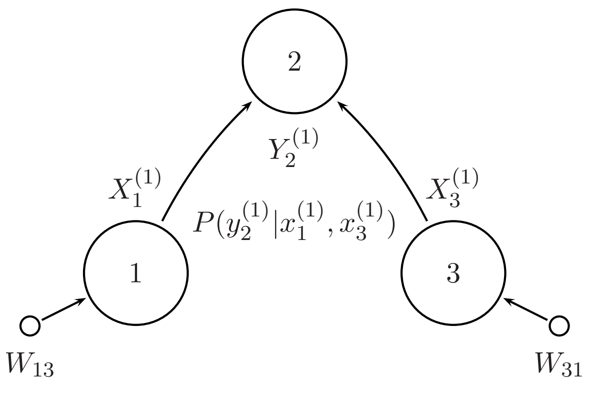
This section is intended to study the performance of different two-way schemes in conjunction with a variety of relaying strategies. The idea is to understand the possibilities of the problem in order to derive inner bounds in the most general form possible. Therefore, different individual schemes, some proposed by literature, some new, are analyzed. The insides to the coding problem gained from the one-way problem are used and adapted. Guided by the problem outer bound a scheme using all six elementary network states is suggested at the end of this section. For this scheme inner bounds on the achievable rates are derived for strategies with decoding at the relay. Short comments are given to all expressions in order to point out the main aspects of the underlying coding proof. The individual proofs have been moved to the appendix.

3.2.1 2P-MA-BC Scheme

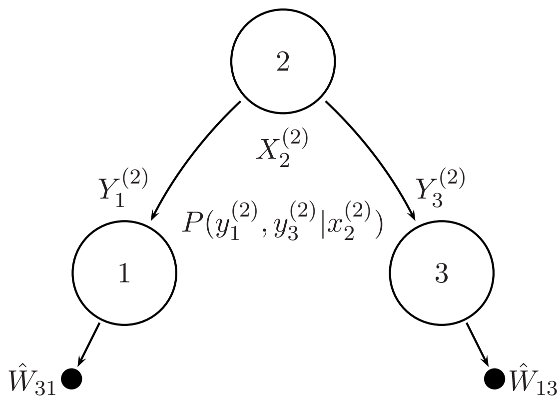
First a well-studied scheme is considered where only two elementary network states are used in the communication process. In the first phase nodes 1 and 3 send in parallel to node 2. In a second phase node 2 broadcasts to both dialog nodes. The scheme is frequently motivated by a scenario where no positive rate can be established in the dialog of nodes 1 and 3 without the help of node 2. In the presence of a direct connection the scheme ignores the extended amount of possibilities. The two network states for this scheme are defined as

$$\begin{aligned} l = 1 &: s_1 = 1, s_2 = 0, s_3 = 1 \\ l = 2 &: s_1 = 0, s_2 = 1, s_3 = 0. \end{aligned} \tag{3.19}$$

Figure 3.10 shows the basic parts of the scheme.



(a) First Phase (n_1 transmission slots)



(b) Second Phase (n_2 transmission slots)

Figure 3.10: HD-TW Relay Channel: 2P-MA-BC Scheme

Outer Bound An outer bound for the achievable rates with this scheme can be derived as a corollary of the problem outer bound. It can be observed that meeting the first part of the outer bound would require to decode each message from the output at node 2 while having full knowledge of the other dialog message. This is the result of the outer bound expressions being derived from two network partitions, each allowing one of the dialog nodes and node 2 to cooperate without restrictions. The second part requires to establish in parallel two point-to-point channels from node 2 to 1 and 3 with one codebook.

Corollary 3.2.1 *All rate pairs of the half-duplex two-way relay channel that are achievable with a 2P-MA-BC scheme and some joint probability distributions*

$$P(x_1^{(1)}, x_3^{(1)}, y_2^{(1)}) = P(x_1^{(1)})P(x_3^{(1)})P(y_2^{(1)}|x_1^{(1)}, x_3^{(1)})$$
$$P(x_2^{(2)}, y_1^{(2)}, y_3^{(2)}) = P(x_2^{(2)})P(y_1^{(2)}, y_3^{(2)}|x_2^{(2)})$$

must satisfy

$$R_{13} \leq \min\left\{\tau_1 I(X_1^{(1)}; Y_2^{(1)}|X_3^{(1)}), \tau_2 I(X_2^{(2)}; Y_3^{(2)})\right\}$$
$$R_{31} \leq \min\left\{\tau_1 I(X_3^{(1)}; Y_2^{(1)}|X_1^{(1)}), \tau_2 I(X_2^{(2)}; Y_1^{(2)})\right\}$$

where $0 \leq \tau_l$ *and* $\tau_1 + \tau_2 \leq 1$.

Decode-and-Forward With decode-and-forward both messages are decoded at node 2 after the first phase. This turns the phase into a multiple-acess channel (MAC). Instead of using the suboptimal approach of superposition coding in order to send the two messages to each dialog node [26], node 2 uses one codebook for both receivers [23]. The codewords are labeled by a two dimensional array resulting in each word having a label with x and y coordinate. The two messages decoded at node 2 determine the two label-coordinates of the codeword sent in the second phase. Node 1 knows its own message and therefore one of the label-coordinates of the codeword sent by node 2. In order to decode node 1 needs only to check all codewords in the unknown dimension with respect to typicality. Node 3 proceeds in the same way.

Proposition 3.2.2 *All rate pairs of the half-duplex two-way relay channel that satisfy*

$$R_{13} \leq \min\left\{\tau_1 I(X_1^{(1)}; Y_2^{(1)}|X_3^{(1)}), \tau_2 I(X_2^{(2)}; Y_3^{(2)})\right\}$$
$$R_{31} \leq \min\left\{\tau_1 I(X_3^{(1)}; Y_2^{(1)}|X_1^{(1)}), \tau_2 I(X_2^{(2)}; Y_1^{(2)})\right\}$$
$$R_{13} + R_{31} \leq \tau_1 I(X_1^{(1)} X_3^{(1)}; Y_2^{(1)})$$

with $0 < \tau_l$ *and* $\tau_1 + \tau_2 \leq 1$, *for some joint probability distributions*

$$P(x_1^{(1)}, x_3^{(1)}, y_2^{(1)}) = P(x_1^{(1)})P(x_3^{(1)})P(y_2^{(1)}|x_1^{(1)}, x_3^{(1)})$$
$$P(x_2^{(2)} y_1^{(2)}, y_3^{(2)}) = P(x_2^{(2)})P(y_1^{(2)}, y_3^{(2)}|x_2^{(2)})$$

are achievable with a 2P-MA-BC scheme and a decode-and-forward strategy.

Proof see A3.1.

As a result, the same rate expressions as for the outer bound are obtained by the proof of achievability with an additional sum-rate constraint resulting from the MAC in the first phase.

Note that there is no partial-decode-and-forward strategy for this scheme since all information has to be exchanged over the relay.

Compress-and-Forward (2LQ) As decoding both messages at node 2 is limited by the interference caused by one of the two transmitters and therefore is in general suboptimal, the compress-and-forward strategy applies quantization to the output of node 2 and tries to convey this estimation to the dialog nodes. As multicasting one quantized version to both receivers would be limited by the channel to the "weaker" node here the two layer approach found in [13] and [28] is used. The output at node 2 is first quantized by a "coarse" quantizer. The coarse quantization index determines the choice of a subsequent quantizer used for refinement. The coarse quantization is multicasted to both receivers while the refinement is delivered to the "stronger" receiver via superposition coding. As a result, for channels supporting asymmetric rates, one of the dialog nodes has an estimate of the channel output at node 2 with higher quality than his dialog partner. Both receivers can use their input signal jointly with the quantized outputs at node 2 to decode the message of their dialog partner. Therefore, the strategy is limited by the quantization noise at node 2.

Proposition 3.2.3 *All rate pairs of the half-duplex two-way relay channel that satisfy*

$$R_{13} \leq \tau_1 I(X_1^{(1)}; \hat{Y}_{21}^{(1)} | X_3^{(1)})$$
$$R_{31} \leq \tau_1 I(X_3^{(1)}; \hat{Y}_{21}^{(1)} \hat{Y}_{22}^{(1)} | X_1^{(1)})$$

subject to

$$\tau_1 I(\hat{Y}_{21}^{(1)}; Y_2^{(1)} | X_3^{(1)}) \leq \tau_2 I(U_2^{(2)}; Y_3^{(2)})$$
$$\tau_1 I(\hat{Y}_{21}^{(1)}; Y_2^{(1)} | X_1^{(1)}) \leq \tau_2 I(U_2^{(2)}; Y_1^{(2)})$$
$$\tau_1 I(\hat{Y}_{22}^{(1)}; Y_2^{(1)} | X_1^{(1)} \hat{Y}_{21}^{(1)}) \leq \tau_2 I(X_2^{(2)}; Y_1^{(2)} | U_2^{(2)})$$

with $0 < \tau_l$ *and* $\tau_1 + \tau_2 \leq 1$*, for some joint probability distributions*

$$P(x_1^{(1)}, x_3^{(1)}, y_2^{(1)}, \hat{y}_{21}^{(1)}, \hat{y}_{22}^{(1)}) = P(x_1^{(1)}) P(x_3^{(1)}) P(y_2^{(1)} | x_1^{(1)}, x_3^{(1)}) P(\hat{y}_{21}^{(1)} | y_2^{(1)}) P(\hat{y}_{22}^{(1)} | \hat{y}_{21}^{(1)}, y_2^{(1)})$$
$$P(u_2^{(2)}, x_2^{(2)}, y_1^{(2)}, y_3^{(2)}) = P(u_2^{(2)}) P(x_2^{(2)} | u_2^{(2)}) P(y_1^{(2)}, y_3^{(2)} | x_2^{(2)})$$

are achievable with a 2P-MA-BC scheme and a two-layer compress-and-forward strategy.

Proof see A3.2.

A second proposition can be established by interchanging the roles of node 1 and 3.

Partial-Decode-Compress-and-Forward (2LQ) Partial-decode-compress-and-forward combines both strategies mentioned before. Only parts of the messages sent by nodes 1 and 3 are decoded by node 2. Additionally the channel output at node 2 is quantized with the two-layer quantization approach. Decoded messages and quantization indices are delivered to both receivers where the "stronger" receiver additionally receives a quantization refinement via superposition coding.

Proposition 3.2.4 *All rate pairs of the half-duplex two-way relay channel that satisfy*

$$\begin{aligned}
R_{13} &\le \min\left\{\tau_1 I(U_1^{(1)}; Y_2^{(1)}|U_3^{(1)}), \tau_2 I(U_2^{(2)}; Y_3^{(2)})\right\} + \tau_1 I(X_1^{(1)}; \hat{Y}_{21}^{(1)}|U_1^{(1)}U_3^{(1)}X_3^{(1)}) \\
R_{31} &\le \min\left\{\tau_1 I(U_3^{(1)}; Y_2^{(1)}|U_1^{(1)}), \tau_2 I(U_2^{(2)}; Y_1^{(2)})\right\} + \tau_1 I(X_3^{(1)}; \hat{Y}_{21}^{(1)}\hat{Y}_{22}^{(1)}|U_1^{(1)}U_3^{(1)}X_1^{(1)}) \\
R_{13} + R_{31} &\le \tau_1 I(U_1^{(1)}U_3^{(1)}; Y_2^{(1)}) + \tau_1 I(X_1^{(1)}; \hat{Y}_{21}^{(1)}|U_1^{(1)}U_3^{(1)}X_3^{(1)}) \\
&\quad + \tau_1 I(X_3^{(1)}; \hat{Y}_{21}^{(1)}\hat{Y}_{22}^{(1)}|U_1^{(1)}U_3^{(1)}X_1^{(1)})
\end{aligned}$$

subject to

$$\begin{aligned}
\tau_1 I(\hat{Y}_{21}^{(1)}; Y_2^{(1)}|U_1^{(1)}U_3^{(1)}X_1^{(1)}) &\le \tau_2 I(V_2^{(2)}; Y_1^{(2)}|U_2^{(2)}) \\
\tau_1 I(\hat{Y}_{21}^{(1)}; Y_2^{(1)}|U_1^{(1)}U_3^{(1)}X_3^{(1)}) &\le \tau_2 I(V_2^{(2)}; Y_3^{(2)}|U_2^{(2)}) \\
\tau_1 I(\hat{Y}_{22}^{(1)}; Y_2^{(1)}|\hat{Y}_{21}^{(1)}U_1^{(1)}U_3^{(1)}X_1^{(1)}) &\le \tau_2 I(X_2^{(2)}; Y_1^{(2)}|U_2^{(2)}V_2^{(2)})
\end{aligned}$$

with $0 < \tau_l$ *and* $\tau_1 + \tau_2 \le 1$, *for some joint probability distributions*

$$\begin{aligned}
P(u_1^{(1)}, u_3^{(1)}, x_1^{(1)}, x_3^{(1)}, y_2^{(1)}, \hat{y}_{21}^{(1)}, \hat{y}_{22}^{(1)}) &= P(u_1^{(1)})P(u_3^{(1)})P(x_1^{(1)}|u_1^{(1)})P(x_3^{(1)}|u_3^{(1)})P(y_2^{(1)}|x_1^{(1)}, x_3^{(1)}) \\
&\quad \cdot P(\hat{y}_{21}^{(1)}|u_1^{(1)}, u_3^{(1)}, y_2^{(1)})P(\hat{y}_{22}^{(1)}|u_1^{(1)}, u_3^{(1)}, \hat{y}_{21}^{(1)}, y_2^{(1)}) \\
P(u_2^{(2)}, v_2^{(2)}, x_2^{(2)}, y_1^{(2)}, y_3^{(2)}) &= P(u_2^{(2)})P(v_2^{(2)}|u_2^{(2)})P(x_2^{(2)}|v_2^{(2)}, u_2^{(2)})P(y_1^{(2)}, y_3^{(2)}|x_2^{(2)})
\end{aligned}$$

are achievable with a 2P-MAC-BC scheme and a two-layer partial-decode-compress-and-forward strategy.

Proof see A3.3

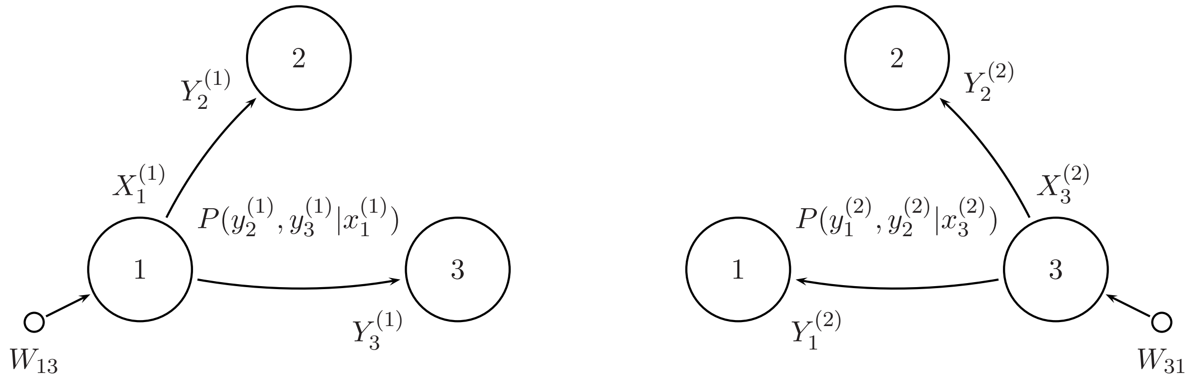
A second proposition can be established by interchanging the roles of node 1 and 3.

3.2.2 3P-BC Scheme

As observed with the 2P-MA-BC scheme the parallel communication of nodes 1 and 3 to node 2 can be suboptimal due to mutual interference. Also the direct path (in general present) is ignored by the two-phase scheme. Therefore, the idea of a 3P-BC scheme [25] is to orthogonalize the transmission from the dialog nodes to node 2. Nodes 1 and 3 send sequentially to node 2 which broadcasts to both dialog nodes in a third phase. The scheme also makes it possible for nodes 1 and 3 to obtain side information by listening to their dialog partner through the direct channel in the first or second phase. The required three network states are defined by

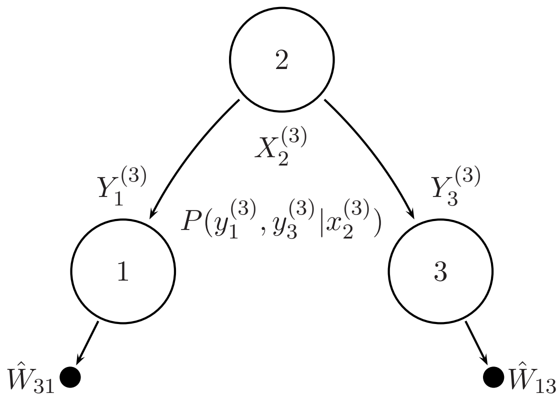
$$\begin{aligned} l = 1 &: s_1 = 1, s_2 = 0, s_3 = 0 \\ l = 2 &: s_1 = 0, s_2 = 0, s_3 = 1 \\ l = 3 &: s_1 = 0, s_2 = 1, s_3 = 0. \end{aligned} \tag{3.20}$$

Figure 3.11 visualizes the scheme and its basic parts.



(a) First Phase (n_1 transmission slots)

(b) Second Phase (n_2 transmission slots)



(c) Third Phase (n_3 transmission slots)

Figure 3.11: HD-TW Relay Channel: 3P-BC Scheme

Outer Bound An outer bound is derived as a corollary of the problem outer bound without further proof by setting the appropriate 3 elementary network states to a duration of zero.

Corollary 3.2.5 *All rate pairs of the half-duplex two-way relay channel that are achievable with a 3P-BC scheme for some joint probability distributions*

$$P(x_1^{(1)}, y_2^{(1)}, y_3^{(1)}) = P(x_1^{(1)})P(y_2^{(1)}, y_3^{(1)}|x_1^{(1)})$$
$$P(x_3^{(2)}, y_1^{(2)}, y_2^{(2)}) = P(x_3^{(2)})P(y_1^{(2)}, y_2^{(2)}|x_3^{(2)})$$
$$P(x_2^{(3)}, y_1^{(3)}, y_3^{(3)}) = P(x_2^{(3)})P(y_1^{(3)}, y_3^{(3)}|x_2^{(3)})$$

must satisfy

$$R_{13} \leq \min\left\{\tau_1 I(X_1^{(1)}; Y_2^{(1)} Y_3^{(1)}), \tau_1 I(X_1^{(1)}; Y_3^{(1)}) + \tau_3 I(X_2^{(3)}; Y_3^{(3)})\right\}$$
$$R_{31} \leq \min\left\{\tau_2 I(X_3^{(2)}; Y_1^{(2)} Y_2^{(2)}), \tau_2 I(X_3^{(2)}; Y_1^{(2)}) + \tau_3 I(X_2^{(3)}; Y_1^{(3)})\right\}$$

where $0 \leq \tau_l$ *and* $\tau_1 + \tau_2 + \tau_3 \leq 1$.

It will be seen that meeting this bound is basically a problem of the terms on the left, the broadcast bounds. They require the decoding of the messages at nodes 1 and 3 from their channel outputs jointly with full knowledge of the output at node 2.

Decode-and-Forward/Partial-Decode-and-Forward With these strategies the messages of nodes 1 and 3 are fully or partially decoded by node 2 after the first and second transmission phases. In the third phase node 2 sends like in the 2P-MA-BC scheme with a codebook indexed in two dimensions. The difference to 2P-MA-BC lies in the fact that both nodes have already side information available. This allows node 2 to send just the amount of information that is needed at both nodes in order to resolve the messages with side information.

Proposition 3.2.6 *All rate pairs of the half-duplex two-way relay channel that satisfy*

$$R_{13} \leq \min\left\{\tau_1 I(X_1^{(1)}; Y_2^{(1)}), \tau_1 I(X_1^{(1)}; Y_3^{(1)}) + \tau_3 I(X_2^{(3)}; Y_3^{(3)})\right\}$$
$$R_{31} \leq \min\left\{\tau_2 I(X_3^{(2)}; Y_2^{(2)}), \tau_2 I(X_3^{(2)}; Y_1^{(2)}) + \tau_3 I(X_2^{(3)}; Y_1^{(3)})\right\}$$

with $0 < \tau_l$ *and* $\tau_1 + \tau_2 + \tau_3 \leq 1$, *for some joint probability distributions*

$$P(x_1^{(1)}, y_2^{(1)}, y_3^{(1)}) = P(x_1^{(1)})P(y_2^{(1)}, y_3^{(1)}|x_1^{(1)})$$
$$P(x_3^{(2)}, y_1^{(2)}, y_2^{(2)}) = P(x_3^{(2)})P(y_1^{(2)}, y_2^{(2)}|x_3^{(2)})$$
$$P(x_2^{(3)}, y_1^{(3)}, y_3^{(3)}) = P(x_2^{(3)})P(y_1^{(3)}, y_3^{(3)}|x_2^{(3)})$$

are achievable with a 3P-BC scheme and a decode-and-forward strategy.

Proof A4.1

Proposition 3.2.7 *All rate pairs of the half-duplex two-way relay channel that satisfy*

$$R_{13} \leq \min\left\{\tau_1 I(U_1^{(1)}; Y_2^{(1)}) + \tau_1 I(X_1^{(1)}; Y_3^{(1)} | U_1^{(1)}), \tau_1 I(X_1^{(1)}; Y_3^{(1)}) + \tau_3 I(X_2^{(3)}; Y_3^{(3)})\right\}$$
$$R_{31} \leq \min\left\{\tau_2 I(U_3^{(2)}; Y_2^{(2)}) + \tau_2 I(X_3^{(2)}; Y_1^{(2)} | U_3^{(2)}), \tau_2 I(X_3^{(2)}; Y_1^{(2)}) + \tau_3 I(X_2^{(3)}; Y_1^{(3)})\right\}$$

with $0 < \tau_l$ *and* $\tau_1 + \tau_2 + \tau_3 \leq 1$, *for some joint probability distributions*

$$P(u_1^{(1)}, x_1^{(1)}, y_2^{(1)}, y_3^{(1)}) = P(u_1^{(1)}) P(x_1^{(1)} | u_1^{(1)}) P(y_2^{(1)}, y_3^{(1)} | x_1^{(1)})$$
$$P(u_3^{(2)}, x_3^{(2)}, y_1^{(2)}, y_2^{(2)}) = P(u_3^{(2)}) P(x_3^{(2)} | u_3^{(2)}) P(y_1^{(2)}, y_2^{(2)} | x_3^{(2)})$$
$$P(x_2^{(3)}, y_1^{(3)}, y_3^{(3)}) = P(x_2^{(3)}) P(y_1^{(3)}, y_3^{(3)} | x_2^{(3)})$$

are achievable with a 3P-BC scheme and a partial-decode-and-forward strategy.

Proof A4.2

Compress-and-Forward Decoding the messages at node 2 can restrict the rates of the transmission scheme. Compress-and-forward quantizes the channel outputs at node 2 after each of the first two transmission phases. The two quantization indices are sent to nodes 1 and 3 like in a broadcast channel. As each index is intended only for one of the receive nodes the transmission is protected by a Gel'fand-Pinsker encoder [12].

Proposition 3.2.8 *All rate pairs of the half-duplex two-way relay channel that satisfy*

$$R_{13} \leq \tau_1 I(X_1^{(1)}; \hat{Y}_2^{(1)} Y_3^{(1)})$$
$$R_{31} \leq \tau_2 I(X_3^{(2)}; \hat{Y}_2^{(2)} Y_1^{(2)})$$

subject to

$$\tau_1 I(\hat{Y}_2^{(1)}; Y_2^{(1)} | Y_3^{(1)}) \leq \tau_3 I(U_{22}^{(3)}; Y_3^{(3)}) - \tau_3 (1 - \kappa) I(U_{21}^{(3)}; U_{22}^{(3)})$$
$$\tau_2 I(\hat{Y}_2^{(2)}; Y_2^{(2)} | Y_1^{(2)}) \leq \tau_3 I(U_{21}^{(3)}; Y_1^{(3)}) - \tau_3 \kappa I(U_{21}^{(3)}; U_{22}^{(3)})$$

with $0 \leq \kappa \leq 1$, $0 < \tau_l$ *and* $\tau_1 + \tau_2 + \tau_3 \leq 1$, *for some joint probability distributions*

$$P(x_1^{(1)}, y_2^{(1)}, y_3^{(1)}) = P(x_1^{(1)}) P(y_2^{(1)}, y_3^{(1)} | x_1^{(1)}) P(\hat{y}_2^{(1)} | y_2^{(1)})$$
$$P(x_3^{(2)}, y_1^{(2)}, y_2^{(2)}) = P(x_3^{(2)}) P(y_1^{(2)}, y_2^{(2)} | x_3^{(2)}) P(\hat{y}_2^{(2)} | y_2^{(2)})$$
$$P(u_{21}^{(3)}, u_{22}^{(3)}, x_2^{(3)}, y_1^{(3)}, y_3^{(3)}) = P(u_{21}^{(3)}, u_{22}^{(3)}) P(x_2^{(3)} | u_{21}^{(3)}, u_{22}^{(3)}) P(y_1^{(3)}, y_3^{(3)} | x_2^{(3)})$$

are achievable with a 3P-BC scheme and a compress-and-forward strategy.

Proof A4.3

Note that this strategy does not use all the dependencies in the system. In the broadcast phase each of the dialog nodes has some side-information available about the message that is intended for the other receiver through their emitted signal of the first phase and second phase. This is not exploited with the proposed strategy. An attempt to overcome this can be found in [16].

Partial-Decode-Compress-and-Forward This strategy combines the PDF and the CF strategies. Parts of the messages of nodes 1 and 3 are decoded. Additionally the channel outputs at node 2 are quantized. In the third step Marton's broadcast scheme [22] is used to convey the decoded parts as a common message and the quantization indices as private messages to both receivers.

Proposition 3.2.9 *All rate pairs of the half-duplex two-way relay channel that satisfy*

$$R_{13} \leq \min\left\{\tau_1 I(U_1^{(1)}; Y_3^{(1)}) + \tau_3 I(U_2^{(3)}; Y_3^{(3)}), \tau_1 I(U_1^{(1)}; Y_2^{(1)})\right\} + \tau_1 I(X_1^{(1)}; \hat{Y}_2^{(1)} Y_3^{(1)} | U_1^{(1)})$$
$$R_{31} \leq \min\left\{\tau_2 I(U_3^{(2)}; Y_1^{(2)}) + \tau_3 I(U_2^{(3)}; Y_1^{(3)}), \tau_2 I(U_3^{(2)}; Y_2^{(2)})\right\} + \tau_2 I(X_3^{(2)}; \hat{Y}_2^{(2)} Y_1^{(2)} | U_3^{(2)})$$

subject to

$$\tau_1 I(\hat{Y}_2^{(1)}; Y_2^{(1)} | Y_3^{(1)}, U_1^{(1)}) \leq \tau_3 I(V_{22}^{(3)}; Y_3^{(3)} | U_2^{(3)}) - \tau_3 (1-\kappa) I(V_{21}^{(3)}; V_{22}^{(3)} | U_2^{(3)})$$
$$\tau_2 I(\hat{Y}_2^{(2)}; Y_2^{(2)} | Y_1^{(2)}, U_3^{(2)}) \leq \tau_3 I(V_{21}^{(3)}; Y_1^{(3)} | U_2^{(3)}) - \tau_3 \kappa I(V_{21}^{(3)}; V_{22}^{(3)} | U_2^{(3)})$$

with $0 \leq \kappa \leq 1$ *and with* $0 < \tau_l$ *and* $\tau_1 + \tau_2 + \tau_3 \leq 1$*, for some joint probability distributions*

$$P(u_1^{(1)}, x_1^{(1)}, y_2^{(1)}, y_3^{(1)}) = P(u_1^{(1)}) P(x_1^{(1)} | u_1^{(1)}) P(y_2^{(1)}, y_3^{(1)} | x_1^{(1)}) P(\hat{y}_2^{(1)} | y_2^{(1)}, u_1^{(1)})$$
$$P(u_3^{(2)}, x_3^{(2)}, y_1^{(2)}, y_2^{(2)}) = P(u_3^{(2)}) P(x_3^{(2)} | u_3^{(2)}) P(y_1^{(2)}, y_2^{(2)} | x_3^{(2)}) P(\hat{y}_2^{(2)} | y_2^{(2)}, u_3^{(2)})$$
$$P(u_2^{(3)}, v_{21}^{(3)}, v_{22}^{(3)}, x_2^{(3)}, y_1^{(3)}, y_3^{(3)}) = P(u_2^{(3)}) P(v_{21}^{(3)}, v_{22}^{(3)} | u_2^{(3)}) P(x_2^{(3)} | v_{21}^{(3)}, v_{22}^{(3)}, u_2^{(3)}) P(y_1^{(3)}, y_3^{(3)} | x_2^{(3)})$$

are achievable with a 3P-BC scheme and a partial-decode-compress-and-forward strategy.

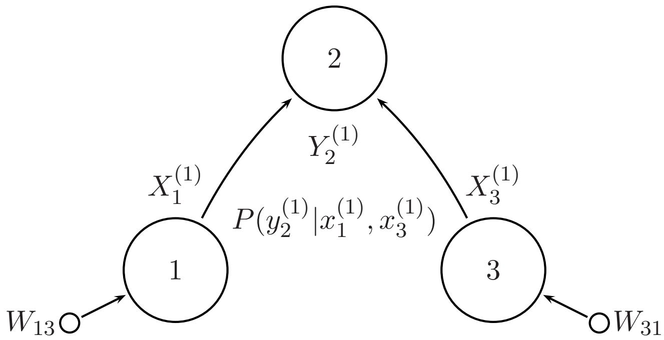
Proof A4.4

3.2.3 3P-MA Scheme

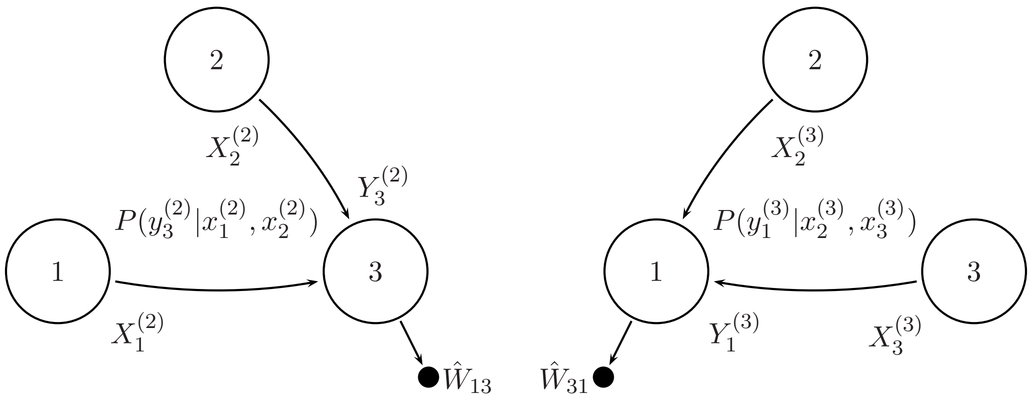
The 3P-MA scheme uses the first phase as multiple-acess phase from nodes 1 and 3 to node 2. The transmission from node 2 to the nodes 1 and 3 is orthogonal. Figure 3.12 shows that this makes it possible for one of the nodes to assist the transmission of node 2 in the last two phases or to send new information to the dialog partner over the direct path. This scheme has, to the best of the author's knowledge, not been studied yet in literature. The $L = 3$ used network states are defined by

$$\begin{aligned} l = 1 &: s_1 = 1, s_2 = 0, s_3 = 1 \\ l = 3 &: s_1 = 1, s_2 = 1, s_3 = 0 \\ l = 2 &: s_1 = 0, s_2 = 1, s_3 = 1 \end{aligned} \tag{3.21}$$

and depicted in Figure 3.12.



(a) First Phase (n_1 transmission slots)



(b) Second Phase (n_2 transmission slots) (c) Third Phase (n_3 transmission slots)

Figure 3.12: HD-TW Relay Channel: 3P-MA Scheme

Outer Bound An outer bound is derived as a corollary of the problem outer bound, without further proof by leaving the appropriate elementaries unused.

Corollary 3.2.10 *All rate pairs of the half-duplex two-way relay channel that are achievable with a 3P-MA scheme for some joint probability distributions*

$$\begin{aligned}
P(x_1^{(1)}, x_3^{(1)}, y_2^{(1)}) &= P(x_1^{(1)})P(x_3^{(1)})P(y_2^{(1)}|x_1^{(1)}, x_3^{(1)}) \\
P(x_1^{(2)}, x_2^{(2)}, y_3^{(2)}) &= P(x_1^{(2)}, x_2^{(2)})P(y_3^{(2)}|x_1^{(2)}, x_2^{(2)}) \\
P(x_2^{(3)}, x_3^{(3)}, y_1^{(3)}) &= P(x_2^{(3)}, x_3^{(3)})P(y_1^{(3)}|x_2^{(3)}, x_3^{(3)})
\end{aligned}$$

must satisfy

$$\begin{aligned}
R_{13} &\leq \min\left\{\tau_1 I(X_1^{(1)}; Y_2^{(1)}|X_3^{(1)}) + \tau_2 I(X_1^{(2)}; Y_3^{(2)}|X_2^{(2)}), \tau_2 I(X_1^{(2)} X_2^{(2)}; Y_3^{(2)})\right\} \\
R_{31} &\leq \min\left\{\tau_1 I(X_3^{(1)}; Y_2^{(1)}|X_1^{(1)}) + \tau_3 I(X_3^{(3)}; Y_1^{(3)}|X_2^{(3)}), \tau_3 I(X_2^{(3)} X_3^{(3)}; Y_1^{(3)})\right\}
\end{aligned}$$

where $0 \leq \tau_l$ *and* $\tau_1 + \tau_2 + \tau_3 \leq 1$.

Decode-and-Forward Decoding the two messages of nodes 1 and 3 at node 2 turns the first phase into a multiple-acess channel (MAC). Therefore, the achievable rates face a sum-rate constraint. In the second and third phase one of the nodes can choose to help node 2 as the output is determined by the according first phase message or can send new data to its dialog partner.

Proposition 3.2.11 *All rate pairs of the half-duplex two-way relay channel that satisfy*

$$\begin{aligned}
R_{13} &\leq \min\left\{\tau_1 I(X_1^{(1)}; Y_2^{(1)}|X_3^{(1)}) + \tau_2 I(X_1^{(2)}; Y_3^{(2)}|X_2^{(2)}), \tau_2 I(X_1^{(2)} X_2^{(2)}; Y_3^{(2)})\right\} \\
R_{31} &\leq \min\left\{\tau_1 I(X_3^{(1)}; Y_2^{(1)}|X_1^{(1)}) + \tau_3 I(X_3^{(3)}; Y_1^{(3)}|X_2^{(3)}), \tau_3 I(X_2^{(3)} X_3^{(3)}; Y_1^{(3)})\right\} \\
R_{13} + R_{31} &\leq \tau_1 I(X_1^{(1)} X_3^{(1)}; Y_2^{(1)}) + \tau_2 I(X_1^{(2)}; Y_3^{(2)}|X_2^{(2)}) + \tau_3 I(X_3^{(3)}; Y_1^{(3)}|X_2^{(3)})
\end{aligned}$$

with $0 < \tau_l$ *and* $\tau_1 + \tau_2 + \tau_3 \leq 1$, *for some joint probability distributions*

$$\begin{aligned}
P(x_1^{(1)}, x_3^{(1)}, y_2^{(1)}) &= P(x_1^{(1)})P(x_3^{(1)})P(y_2^{(1)}|x_1^{(1)}, x_3^{(1)}) \\
P(x_1^{(2)}, x_2^{(2)}, y_3^{(2)}) &= P(x_2^{(2)})P(x_1^{(2)}|x_2^{(2)})P(y_3^{(2)}|x_1^{(2)}, x_2^{(2)}) \\
P(x_2^{(3)}, x_3^{(3)}, y_1^{(3)}) &= P(x_2^{(3)})P(x_3^{(3)}|x_2^{(3)})P(y_1^{(3)}|x_2^{(3)}, x_3^{(3)})
\end{aligned}$$

are achievable with a 3P-MA scheme and a decode-and-forward strategy.

Proof see A5.1.

Compress-and-Forward Like in the 2P-MA-BC scheme decoding at node 2 results in a sum-rate constraint. Quantization can be used to avoid decoding at node 2. The strategy uses two independent quantizations of the first phase channel output at node 2 in order to adapt to the channels from node 2 to both receivers. As the quantization indices can not be determined at both dialog nodes the second and third phase is used to send new data over the direct path. The first phase messages are decoded at the dialog nodes jointly from one of the estimates and their first phase input sequence.

Proposition 3.2.12 *All rate pairs of the half-duplex two-way relay channel that satisfy*

$$R_{13} \le \tau_1 I(X_1^{(1)};\hat{Y}_{21}^{(1)}|X_3^{(1)}) + \tau_2 I(X_1^{(2)};Y_3^{(2)}|X_2^{(2)})$$
$$R_{31} \le \tau_1 I(X_3^{(1)};\hat{Y}_{22}^{(1)}|X_1^{(1)}) + \tau_3 I(X_3^{(3)};Y_1^{(3)}|X_2^{(3)})$$

subject to

$$\tau_1 I(\hat{Y}_{21}^{(1)};Y_2^{(1)}|X_3^{(1)}) \le \tau_2 I(X_2^{(2)};Y_3^{(2)})$$
$$\tau_1 I(\hat{Y}_{22}^{(1)};Y_2^{(1)}|X_1^{(1)}) \le \tau_3 I(X_2^{(3)};Y_1^{(3)})$$

with $0 < \tau_l$ *and* $\tau_1 + \tau_2 + \tau_3 \le 1$*, for some joint probability distributions*

$$P(x_1^{(1)},x_3^{(1)},y_2^{(1)},\hat{y}_{21}^{(1)},\hat{y}_{22}^{(1)}) = P(x_1^{(1)})P(x_3^{(1)})P(y_2^{(1)}|x_1^{(1)},x_3^{(1)})P(\hat{y}_{21}^{(1)}|y_2^{(1)})P(\hat{y}_{22}^{(1)}|y_2^{(1)})$$
$$P(x_1^{(2)},x_2^{(2)},y_3^{(2)}) = P(x_1^{(2)})P(x_2^{(2)})P(y_3^{(2)}|x_1^{(2)},x_2^{(2)})$$
$$P(x_2^{(3)},x_3^{(3)},y_1^{(3)}) = P(x_2^{(3)})P(x_3^{(3)})P(y_1^{(3)}|x_2^{(3)},x_3^{(3)})$$

are achievable with a 3P-MA scheme and a compress-and-forward strategy.

Proof see A5.2.

Note that a partial-decode-and-forward strategy is not considered as all information of the first phase has to be exchanged via the relay.

Partial-Decode-Compress-and-Forward The strategy combines the two methods beforementioned. Parts of the messages received at node 2 in the multiple-acess phase are decoded and the channel output is quantized. Two independent quantization-books are used at node 2 in order to adapt to the individual channels to both dialog nodes. These can partially assist the transmission signals of node 2 in the last two phases and send new data over the direct path.

Proposition 3.2.13 *All rate pairs of the half-duplex two-way relay channel that satisfy*

$$\begin{aligned}
R_{13} \le \min\left\{\tau_1 I(U_1^{(1)};Y_2^{(1)}|U_3^{(1)}), \tau_2 I(U_2^{(2)};Y_3^{(2)})\right\} &+ \tau_1 I(X_1^{(1)};\hat{Y}_{21}^{(1)}|U_1^{(1)}U_3^{(1)}X_3^{(1)}) + \\
&\tau_2 I(X_1^{(2)};Y_3^{(2)}|X_2^{(2)}U_2^{(2)}) \\
R_{31} \le \min\left\{\tau_1 I(U_3^{(1)};Y_2^{(1)}|U_1^{(1)}), \tau_3 I(U_2^{(3)};Y_1^{(3)})\right\} &+ \tau_1 I(X_3^{(1)};\hat{Y}_{22}^{(1)}|U_1^{(1)}U_3^{(1)}X_1^{(1)}) + \\
&\tau_3 I(X_3^{(3)};Y_1^{(3)}|X_2^{(3)}U_2^{(3)}) \\
R_{13} + R_{31} \le &\tau_1 I(U_1^{(1)}U_3^{(1)};Y_2^{(1)}) + \tau_1 I(X_1^{(1)};\hat{Y}_{21}^{(1)}|U_1^{(1)}U_3^{(1)}X_3^{(1)}) + \\
&\tau_1 I(X_3^{(1)};\hat{Y}_{22}^{(1)}|U_1^{(1)}U_3^{(1)}X_1^{(1)}) + \tau_2 I(X_1^{(2)};Y_3^{(2)}|X_2^{(2)}U_2^{(2)}) + \\
&\tau_3 I(X_3^{(3)};Y_1^{(3)}|X_2^{(3)}U_2^{(3)})
\end{aligned}$$

subject to

$$\tau_1 I(\hat{Y}_{21}^{(1)}; Y_2^{(1)} | U_1^{(1)} U_3^{(1)} X_3^{(1)}) \leq \tau_2 I(X_2^{(2)}; Y_3^{(2)} | U_2^{(2)})$$
$$\tau_1 I(\hat{Y}_{22}^{(1)}; Y_2^{(1)} | U_1^{(1)} U_3^{(1)} X_1^{(1)}) \leq \tau_3 I(X_2^{(3)}; Y_1^{(3)} | U_2^{(3)})$$

with $0 < \tau_l$ *and* $\tau_1 + \tau_2 + \tau_3 \leq 1$, *for some joint probability distributions*

$$\begin{aligned} P(u_1^{(1)}, u_3^{(1)}, x_1^{(1)}, x_3^{(1)}, y_2^{(1)}, \hat{y}_{21}^{(1)}, \hat{y}_{22}^{(1)}) &= P(u_1^{(1)}) P(u_3^{(1)}) P(x_1^{(1)} | u_1^{(1)}) P(x_3^{(1)} | u_3^{(1)}) P(y_2^{(1)} | x_1^{(1)}, x_3^{(1)}) \\ &\quad \cdot P(\hat{y}_{21}^{(1)} | y_2^{(1)}, u_1^{(1)}, u_3^{(1)}) P(\hat{y}_{22}^{(1)} | y_2^{(1)}, u_1^{(1)}, u_3^{(1)}) \\ P(u_2^{(2)}, x_1^{(2)}, x_2^{(2)}, y_3^{(2)}) &= P(u_2^{(2)}) P(x_1^{(2)} | u_2^{(2)}) P(x_2^{(2)} | u_2^{(2)}) P(y_3^{(2)} | x_1^{(2)}, x_2^{(2)}) \\ P(u_2^{(3)}, x_2^{(3)}, x_3^{(3)}, y_1^{(3)}) &= P(u_2^{(3)}) P(x_2^{(3)} | u_2^{(3)}) P(x_3^{(3)} | u_2^{(3)}) P(y_1^{(3)} | x_2^{(3)}, x_3^{(3)}) \end{aligned}$$

are achievable with a 3P-MA scheme and a partial-decode-compress-and-forward strategy.

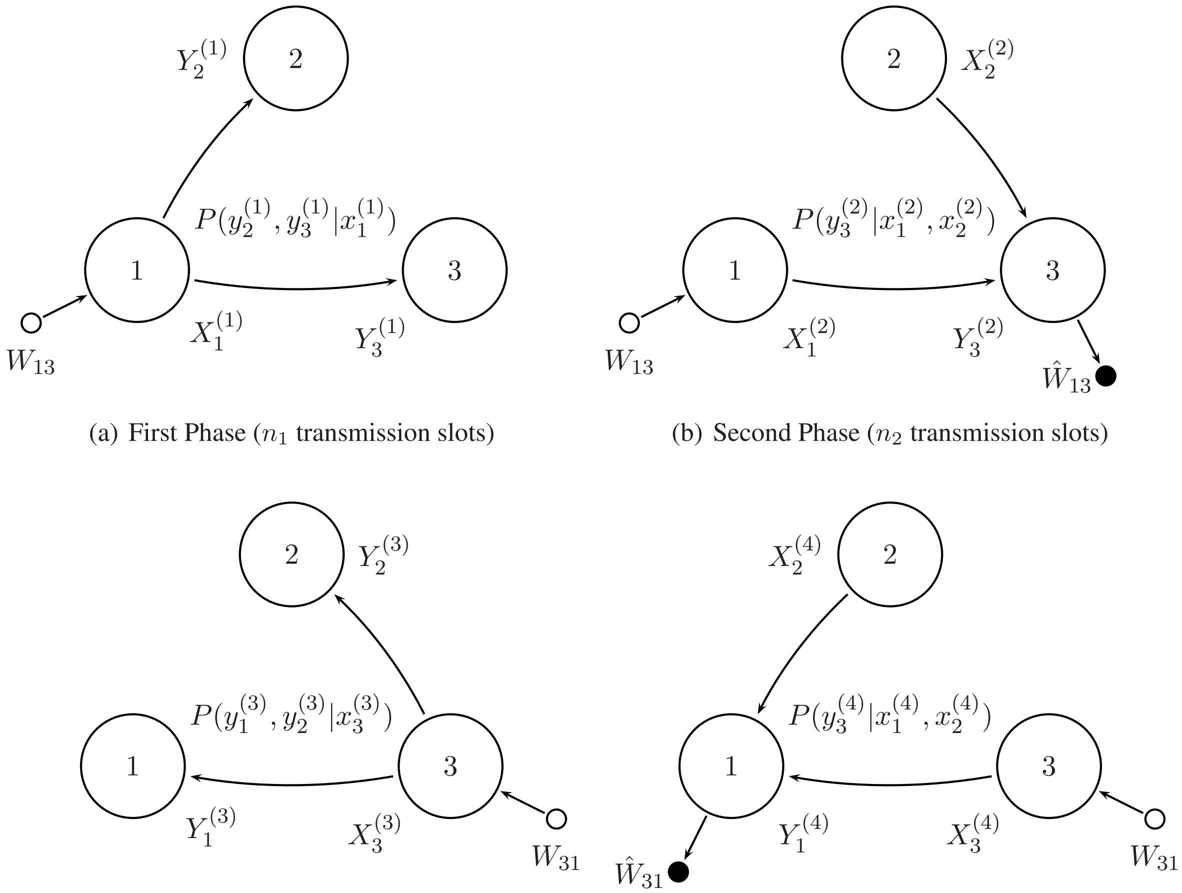
Proof see A5.3.

3.2.4 4P-OWRC Scheme

For this scheme the transmission is divided into four phases. The according elementaries are used such that the channel separates into two subsequent one-way relay channels. To the best of the author's knowledge, such a two-way scheme has not been studied yet in literature. The used network states are defined as

$$\begin{aligned} l = 1 &: s_1 = 1, s_2 = 0, s_3 = 0 \\ l = 2 &: s_1 = 1, s_2 = 1, s_3 = 0 \\ l = 3 &: s_1 = 0, s_2 = 0, s_3 = 1 \\ l = 4 &: s_1 = 0, s_2 = 1, s_3 = 1. \end{aligned} \tag{3.22}$$

The outer bound follows as a corollary from the problem outer bound. The achievable rates follow directly from the first part of the work and are therefore stated without further comments.



(a) First Phase (n_1 transmission slots)

(b) Second Phase (n_2 transmission slots)

(c) Third Phase (n_3 transmission slots)

(d) Fourth Phase (n_4 transmission slots)

Figure 3.13: Half-Duplex Two-Way Relay Channel: 4P-OWRC Scheme

Outer Bound

Corollary 3.2.14 *All rate pairs of the half-duplex two-way relay channel that are achievable with a 4P-OWRC scheme for some joint probability distributions*

$$
\begin{aligned}
P(x_1^{(1)}, y_2^{(1)}, y_3^{(1)}) &= P(x_1^{(1)}) P(y_2^{(1)}, y_3^{(1)} | x_1^{(1)}) \\
P(x_1^{(2)}, x_2^{(2)}, y_3^{(2)}) &= P(x_1^{(2)}, x_2^{(2)}) P(y_3^{(2)} | x_1^{(2)}, x_2^{(2)}) \\
P(x_3^{(3)}, y_2^{(3)}, y_1^{(3)}) &= P(x_3^{(3)}) P(y_2^{(3)}, y_1^{(3)} | x_3^{(3)}) \\
P(x_3^{(4)}, x_2^{(4)}, y_1^{(4)}) &= P(x_2^{(4)}, x_3^{(4)}) P(y_1^{(4)} | x_2^{(4)}, x_3^{(4)})
\end{aligned}
$$

must satisfy

$$
\begin{aligned}
R_{13} \leq \min \Big\{ &\tau_1 I(X_1^{(1)}; Y_2^{(1)} Y_3^{(1)}) + \tau_2 I(X_1^{(2)}; Y_3^{(2)} | X_2^{(2)}), \\
&\tau_1 I(X_1^{(1)}; Y_3^{(1)}) + \tau_2 I(X_1^{(2)} X_2^{(2)}; Y_3^{(2)}) \Big\} \\
R_{31} \leq \min \Big\{ &\tau_3 I(X_3^{(3)}; Y_2^{(3)} Y_1^{(3)}) + \tau_4 I(X_3^{(4)}; Y_1^{(4)} | X_2^{(4)}), \\
&\tau_3 I(X_3^{(3)}; Y_1^{(3)}) + \tau_4 I(X_3^{(4)} X_2^{(4)}; Y_1^{(4)}) \Big\}
\end{aligned}
$$

where $0 \leq \tau_l$ *and* $\tau_1 + \tau_2 + \tau_3 + \tau_4 \leq 1$.

Decode-and-Forward

Proposition 3.2.15 *All rate pairs of the half-duplex two-way relay channel that satisfy*

$$
\begin{aligned}
R_{13} &\leq \min \Big\{ \tau_1 I(X_1^{(1)}; Y_2^{(1)}) + \tau_2 I(X_1^{(2)}; Y_3^{(2)} | X_2^{(2)}), \tau_1 I(X_1^{(1)}; Y_3^{(1)}) + \tau_2 I(X_1^{(2)} X_2^{(2)}; Y_3^{(2)}) \Big\} \\
R_{31} &\leq \min \Big\{ \tau_3 I(X_3^{(3)}; Y_2^{(3)}) + \tau_4 I(X_3^{(4)}; Y_1^{(4)} | X_2^{(4)}), \tau_3 I(X_3^{(3)}; Y_1^{(3)}) + \tau_4 I(X_3^{(4)} X_2^{(4)}; Y_1^{(4)}) \Big\}
\end{aligned}
$$

with $0 < \tau_l$ *and* $\tau_1 + \tau_2 + \tau_3 + \tau_4 \leq 1$*, for some joint probability distributions*

$$
\begin{aligned}
P(x_1^{(1)}, y_2^{(1)}, y_3^{(1)}) &= P(x_1^{(1)}) P(y_2^{(1)}, y_3^{(1)} | x_1^{(1)}) \\
P(x_1^{(2)}, x_2^{(2)}, y_3^{(2)}) &= P(x_2^{(2)}) P(x_1^{(2)} | x_2^{(2)}) P(y_3^{(2)} | x_1^{(2)}, x_2^{(2)}) \\
P(x_3^{(3)}, y_2^{(3)}, y_1^{(3)}) &= P(x_3^{(3)}) P(y_2^{(3)}, y_1^{(3)} | x_3^{(3)}) \\
P(x_3^{(4)}, x_2^{(4)}, y_1^{(4)}) &= P(x_2^{(4)}) P(x_3^{(4)} | x_2^{(4)}) P(y_1^{(4)} | x_2^{(4)}, x_3^{(4)})
\end{aligned}
$$

are achievable with a 4P-OWRC scheme and a decode-and-forward strategy.

Partial-Decode-and-Forward

Proposition 3.2.16 *All rate pairs of the half-duplex two-way relay channel that satisfy*

$$R_{13} \le \min\Big\{\tau_1 I(U_1^{(1)};Y_2^{(1)}) + \tau_1 I(X_1^{(1)};Y_3^{(1)}|U_1^{(1)}) + \tau_2 I(X_1^{(2)};Y_3^{(2)}|X_2^{(2)}),$$
$$\tau_1 I(X_1^{(1)};Y_3^{(1)}) + \tau_2 I(X_1^{(2)}X_2^{(2)};Y_3^{(2)})\Big\}$$
$$R_{31} \le \min\Big\{\tau_3 I(U_3^{(3)};Y_2^{(3)}) + \tau_3 I(X_3^{(3)};Y_1^{(3)}|U_3^{(3)}) + \tau_4 I(X_3^{(4)};Y_1^{(4)}|X_2^{(4)}),$$
$$\tau_3 I(X_3^{(3)};Y_1^{(3)}) + \tau_4 I(X_3^{(4)}X_2^{(4)};Y_1^{(4)})\Big\}$$

with $0 < \tau_l$ *and* $\tau_1 + \tau_2 + \tau_3 + \tau_4 \le 1$, *for some joint probability distributions*

$$\begin{aligned}
P(u_1^{(1)},x_1^{(1)},y_2^{(1)},y_3^{(1)}) &= P(u_1^{(1)})P(x_1^{(1)}|u_1^{(1)})P(y_2^{(1)},y_3^{(1)}|x_1^{(1)})\\
P(x_1^{(2)},x_2^{(2)},y_3^{(2)}) &= P(x_2^{(2)})P(x_1^{(2)}|x_2^{(2)})P(y_3^{(2)}|x_1^{(2)},x_2^{(2)})\\
P(u_3^{(3)},x_3^{(3)},y_2^{(3)},y_1^{(3)}) &= P(u_3^{(3)})P(x_3^{(3)}|u_3^{(3)})P(y_2^{(3)},y_1^{(3)}|x_3^{(3)})\\
P(x_3^{(4)},x_2^{(4)},y_1^{(4)}) &= P(x_2^{(4)})P(x_3^{(4)}|x_2^{(4)})P(y_1^{(4)}|x_2^{(4)},x_3^{(4)})
\end{aligned}$$

are achievable with a 4P-OWRC scheme and a partial-decode-and-forward strategy.

Compress-and-Forward

Proposition 3.2.17 *All rate pairs of the half-duplex two-way relay channel that satisfy*

$$\begin{aligned}
R_{13} &\le \tau_1 I(X_1^{(1)};\hat{Y}_2^{(1)}Y_3^{(1)}) + \tau_2 I(X_1^{(2)};Y_3^{(2)}|X_2^{(2)})\\
R_{31} &\le \tau_3 I(X_3^{(3)};\hat{Y}_2^{(3)}Y_1^{(3)}) + \tau_4 I(X_3^{(4)};Y_1^{(4)}|X_2^{(4)})
\end{aligned}$$

subject to

$$\begin{aligned}
\tau_1 I(\hat{Y}_2^{(1)};Y_2^{(1)}|Y_3^{(1)}) &\le \tau_2 I(X_2^{(2)};Y_3^{(2)})\\
\tau_3 I(\hat{Y}_2^{(3)};Y_2^{(3)}|Y_1^{(3)}) &\le \tau_4 I(X_2^{(4)};Y_1^{(4)}).
\end{aligned}$$

with $0 < \tau_l$ *and* $\tau_1 + \tau_2 + \tau_3 + \tau_4 \le 1$, *for some joint probability distributions*

$$\begin{aligned}
P(x_1^{(1)},y_2^{(1)},y_3^{(1)},\hat{y}_2^{(1)}) &= P(x_1^{(1)})P(y_2^{(1)},y_3^{(1)}|x_1^{(1)})P(\hat{y}_2^{(1)}|y_2^{(1)})\\
P(x_1^{(2)},x_2^{(2)},y_3^{(2)}) &= P(x_1^{(2)})P(x_2^{(2)})P(y_3^{(2)}|x_1^{(2)},x_2^{(2)})\\
P(x_3^{(3)},y_2^{(3)},y_1^{(3)},\hat{y}_2^{(3)}) &= P(x_3^{(3)})P(y_2^{(3)},y_1^{(3)}|x_3^{(3)})P(\hat{y}_2^{(3)}|y_2^{(3)})\\
P(x_3^{(4)},x_2^{(4)},y_1^{(4)}) &= P(x_2^{(4)})P(x_3^{(4)})P(y_1^{(4)}|x_2^{(4)},x_3^{(4)})
\end{aligned}$$

are achievable with a 4P-OWRC scheme and a compress-and-forward strategy.

Partial-Decode-Compress-and-Forward

Proposition 3.2.18 *All rate pairs of the half-duplex two-way relay channel that satisfy*

$$R_{13} \leq \min\left\{\tau_1 I(U_1^{(1)}; Y_2^{(1)}), \tau_1 I(U_1^{(1)}; Y_3^{(1)}) + \tau_2 I(V_2^{(2)}; Y_3^{(2)})\right\} + \tau_1 I(X_1^{(1)}; \hat{Y}_2^{(1)} Y_3^{(1)} | U_1^{(1)}) + \tau_2 I(X_1^{(2)}; Y_3^{(2)} | V_2^{(2)} X_2^{(2)})$$

$$R_{31} \leq \min\left\{\tau_3 I(U_3^{(3)}; Y_2^{(3)}), \tau_3 I(U_3^{(3)}; Y_1^{(3)}) + \tau_4 I(V_2^{(4)}; Y_1^{(4)})\right\} + \tau_3 I(X_3^{(3)}; \hat{Y}_2^{(3)} Y_1^{(3)} | U_3^{(3)}) + \tau_4 I(X_3^{(4)}; Y_1^{(4)} | V_2^{(4)} X_2^{(4)})$$

subject to

$$\tau_1 I(\hat{Y}_2^{(1)}; Y_2^{(1)} | U_1^{(1)} Y_3^{(1)}) \leq \tau_2 I(X_2^{(2)}; Y_3^{(2)} | V_2^{(2)})$$
$$\tau_3 I(\hat{Y}_2^{(3)}; Y_2^{(3)} | U_3^{(3)} Y_1^{(3)}) \leq \tau_4 I(X_2^{(4)}; Y_1^{(4)} | V_2^{(4)})$$

with $0 < \tau_l$ *and* $\tau_1 + \tau_2 + \tau_3 + \tau_4 \leq 1$, *for some joint probability distributions*

$$P(u_1^{(1)}, x_1^{(1)}, y_2^{(1)}, \hat{y}_2^{(1)}, y_3^{(1)}) = P(u_1^{(1)}) P(x_1^{(1)} | u_1^{(1)}) P(y_2^{(1)}, y_3^{(1)} | x_1^{(1)}) P(\hat{y}_2^{(1)} | y_2^{(1)}, u_1^{(1)})$$
$$P(v_2^{(2)}, x_2^{(2)}, x_1^{(2)}, y_3^{(2)}) = P(v_2^{(2)}) P(x_1^{(2)} | v_2^{(2)}) P(x_2^{(2)} | v_2^{(2)}) P(y_3^{(2)} | x_1^{(2)}, x_2^{(2)})$$
$$P(u_3^{(3)}, x_3^{(3)}, y_2^{(3)}, \hat{y}_2^{(3)}, y_1^{(3)}) = P(u_3^{(3)}) P(x_3^{(3)} | u_3^{(3)}) P(y_2^{(3)}, y_3^{(3)} | x_3^{(3)}) P(\hat{y}_2^{(3)} | y_2^{(3)}, u_3^{(3)})$$
$$P(v_2^{(4)}, x_2^{(4)}, x_3^{(4)}, y_1^{(4)}) = P(v_2^{(4)}) P(x_3^{(4)} | v_2^{(4)}) P(x_2^{(4)} | v_2^{(4)}) P(y_1^{(4)} | x_2^{(4)}, x_3^{(4)})$$

are achievable with a 4P-OWRC scheme and a partial-decode-compress-and-forward strategy.

3.2.5 6P Scheme

Until now 4 possible schemes have been studied. In order to make the analysis complete one could now try to verify all possible schemes and analyze each of them. If one restricts the scheme to n channel uses grouped by network state, containing at least one network state with active output and one network state with active input at node 2, while the ordering of individual phases is ignored, the number of possible schemes are

$$\begin{aligned}
&\text{2P}: && \binom{3}{1} \times \binom{3}{1} = 9 \\
&\text{3P}: && 2 \times \binom{3}{1} \times \binom{3}{2} = 18 \\
&\text{4P}: && 2 \times \binom{3}{1} \times \binom{3}{3} + \binom{3}{2} \times \binom{3}{2} = 15 \\
&\text{5P}: && 2 \times \binom{3}{2} \times \binom{3}{3} = 6 \\
&\text{6P}: && \binom{3}{3} \times \binom{3}{3} = 1.
\end{aligned} \tag{3.23}$$

In order to short-cut the effort of determining the schemes that make sense the most general approach is suggested here by a new scheme which uses all relevant network states. The scheme will be called 6P as it contains six basic transmission phases. Its network states are defined according to

$$\begin{aligned}
l = 1 &: s_1 = 1, s_2 = 0, s_3 = 0 \\
l = 2 &: s_1 = 0, s_2 = 0, s_3 = 1 \\
l = 3 &: s_1 = 1, s_2 = 0, s_3 = 1 \\
l = 4 &: s_1 = 0, s_2 = 1, s_3 = 0 \\
l = 5 &: s_1 = 0, s_2 = 1, s_3 = 1 \\
l = 6 &: s_1 = 1, s_2 = 1, s_3 = 0.
\end{aligned} \tag{3.24}$$

The outer bound on this scheme coincides with the problem outer bound. Achievable rates will be derived for DF and PDF relaying. The 6P scheme is depicted in Figure 3.14.

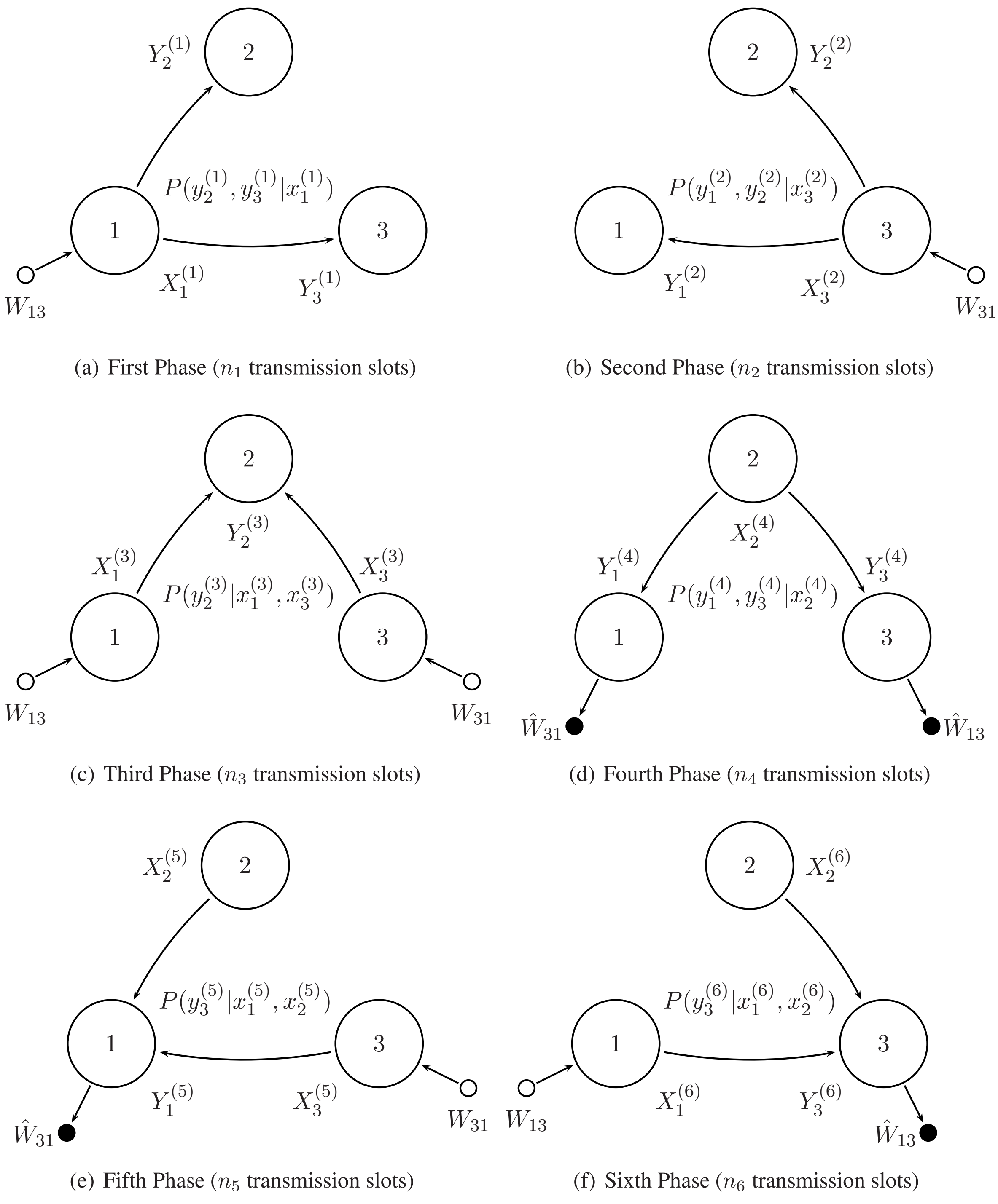


Figure 3.14: HD-TW Relay Channel: 6P Scheme

Decode-and-Forward For DF strategies it can be observed that the same basic code constructions show up in the different schemes studied until here. If one of the dialog nodes sends while node 2 and the other dialog node listen, node 2 fully decodes the message sent and the quiet dialog partner uses its output as side information for later decoding. If both dialog partners send to node 2 the channel is used as MAC. If node 2 sends to one of the dialog nodes the other node assists it or sends new information over the direct path, resulting in a MAC with correlated sources [5]. If node 2 sends to both dialog partners the channel is used as a *bi-directional broadcast channel* [3] [23] with a two-dimensional indexed codebook which has been shown to be optimal in such a situation. For the proof of the achievable rate pairs here first all three elementaries with active channel output at node 2 are used in order to "load" the relay with information. Subsequently, the three elementaries with active channel input at node 2 "unload" the relay. In order to contain all schemes studied before as special cases each of the first three phases is combined with each of the three last phases by distributing the codebook indices of each "load" phase at node 2 to all "unload" phases.

Proposition 3.2.19 *All rate pairs of the half-duplex two-way relay channel that satisfy*

$$
\begin{aligned}
R_{13} \leq \min \Big\{ & \tau_1 I(X_1^{(1)}; Y_2^{(1)}) + \tau_3 I(X_1^{(3)}; Y_2^{(3)} | X_3^{(3)}) + \tau_6 I(X_1^{(6)}; Y_3^{(6)} | X_2^{(6)}), \\
& \tau_1 I(X_1^{(1)}; Y_3^{(1)}) + \tau_4 I(X_2^{(4)}; Y_3^{(4)}) + \tau_6 I(X_1^{(6)} X_2^{(6)}; Y_3^{(6)}) \Big\} \\
R_{31} \leq \min \Big\{ & \tau_2 I(X_3^{(2)}; Y_2^{(2)}) + \tau_3 I(X_3^{(3)}; Y_2^{(3)} | X_1^{(3)}) + \tau_5 I(X_3^{(5)}; Y_1^{(5)} | X_2^{(5)}), \\
& \tau_2 I(X_3^{(2)}; Y_1^{(2)}) + \tau_4 I(X_2^{(4)}; Y_1^{(4)}) + \tau_5 I(X_2^{(5)} X_3^{(5)}; Y_1^{(5)}) \Big\} \\
R_{13} + R_{31} \leq & \tau_1 I(X_1^{(1)}; Y_2^{(1)}) + \tau_2 I(X_3^{(2)}; Y_2^{(2)}) + \tau_3 I(X_1^{(3)} X_3^{(3)}; Y_2^{(3)}) + \\
& \tau_5 I(X_3^{(5)}; Y_1^{(5)} | X_2^{(5)}) + \tau_6 I(X_1^{(6)}; Y_3^{(6)} | X_2^{(6)})
\end{aligned}
$$

with $0 < \tau_l$ *and* $\sum_{l=1}^{6} \tau_l \leq 1$*, for some joint probability distributions*

$$
\begin{aligned}
P(x_1^{(1)}, y_2^{(1)}, y_3^{(1)}) &= P(x_1^{(1)}) P(y_2^{(1)}, y_3^{(1)} | x_1^{(1)}) \\
P(x_3^{(2)}, y_2^{(2)}, y_1^{(2)}) &= P(x_3^{(2)}) P(y_1^{(2)}, y_2^{(2)} | x_3^{(2)}) \\
P(x_1^{(3)}, x_3^{(3)}, y_2^{(3)}) &= P(x_1^{(3)}) P(x_3^{(3)}) P(y_2^{(3)} | x_1^{(3)}, x_3^{(3)}) \\
P(x_2^{(4)}, y_1^{(4)}, y_3^{(4)}) &= P(x_2^{(4)}) P(y_1^{(4)}, y_3^{(4)} | x_2^{(4)}) \\
P(x_2^{(5)}, x_3^{(5)}, y_1^{(5)}) &= P(x_2^{(5)}) P(x_3^{(5)} | x_2^{(5)}) P(y_1^{(5)} | x_2^{(5)}, x_3^{(5)}) \\
P(x_1^{(6)}, x_2^{(6)}, y_3^{(6)}) &= P(x_2^{(6)}) P(x_1^{(6)} | x_2^{(6)}) P(y_3^{(6)} | x_1^{(6)}, x_2^{(6)})
\end{aligned}
$$

are achievable with a 6P scheme and a decode-and-forward strategy.

Proof see A6.1.

Generality At this point it would be desirable to know if the 6P-DF protocol can be used to determine the optimal DF scheme for fixed input distributions with respect to a rate objective.

Note that each of the individual six phases can asymptotically be turned off. Therefore, the 6P scheme contains as special cases all schemes which can be represented with the same ordering of phases and some phases having zero duration $\tau_l \to 0$. But can rate increase by a different ordering of the six phases? Also all achievability proofs assume a conservative ordering of networks states. Each state l that is used occurs in n_l subsequent channel uses. None of the protocols uses the possibility of an arbitrary permutation of network states on n channel uses. Although it might seem intuitively clear that such methods can not increase the rate of a DF scheme if the scheme itself is fixed a priori here some arguments are outlined in order to provide stronger evidence. First a decode-and-forward strategy is defined for the channel studied.

Definition 3.2.20 *Assume B transmission blocks each with n channel uses and a scheme using $l = 1, \dots, L \geq 2$ network states. $L_u \leq L - 1$ of the network states allow an active output and $L_d \leq L-1$ allow an active input at node 2. Each network state occurs $n_l < n$ times in each block b.*

A decode-and-forward strategy is used iff node 2 determines the full individual messages of nodes 1 and 3 from the output sequence $y_2^{n_l}$ each time after having observed n_l channel uses of the network state l with active output $Y_2^{(l)}$ and forwards the decoded messages in the subsequent $\sum_{L_d} n_l$ channel uses which allow an active channel input $X_2^{(l)}$.

Conjecture 3.2.21 *The codebook construction used for each individual phase in the proof of Proposition 3.2.19 is optimal with respect to the rate of a decode-and-forward strategy for the restricted half-duplex two-way relay channel.*

Arguments Follows from the comments on the 6P-DF protocol.

Conjecture 3.2.22 *For fixed input distributions the time allocation solution to Proposition 3.2.19, with respect to the maximization of any reasonable rate objective, provides the optimal decode-and-forward scheme for the restricted half-duplex two-way channel.*

Arguments Fix the input distributions and solve the time allocation problem of Proposition 3.2.19 with respect to a reasonable rate objective $f(R_{13}, R_{31})$. Here a reasonable objective is considered a function $f(R_{13}, R_{31})$ increasing in R_{13} and R_{31}. The optimal time allocation solution yields the achievable rates $R_{13}^\star$ and $R_{31}^\star$ and the objective value $f(R_{13}^\star, R_{31}^\star)$. The maximum amount of reliable information in bits D_{ij} transmitted between the nodes 1 and 3 after B blocks for a DF strategy with $L = 6$ network states is

$$D_{13} < R_{13}^\star \times B \times \sum_{l=1}^{L} n_l \qquad \text{and} \qquad D_{31} < R_{31}^\star \times B \times \sum_{l=1}^{L} n_l. \tag{3.25}$$

It is needed to show that changing the code in A6.1 or the scheme while using a DF strategy as defined above will not result in a higher objective, i.e., in higher rates. Consider different approaches in order to attack the conjecture:

- Change the duration of the used phases. With having already solved Proposition 3.2.19 for optimal time allocation with respect to the maximal $f(R_{13}, R_{31})$ this will not increase the objective.

- The scheme is left as in A6.1 but the codebook construction is improved. With Conjecture 3.2.21 being true this will not result in a higher rate for any of the two dialog messages.
- The scheme is changed while the codebook construction is the one of A6.1: The first three phases and the last three phases are interchanged arbitrarily among each other while all n channel uses stay grouped by network state. The proof on the achievable rates of such a scheme will result in rate expressions differing from Proposition 3.2.19 only in phase labels. Obviously, this does not improve any reasonable objective on the size of the message sets.
- The scheme is changed while the codebook construction is the one of A6.1: The first $\sum_{L_u} n_l$ channel uses are assigned to an arbitrary permutation of the network states L_u which allow active output at node 2. The last $\sum_{L_d} n_l$ channel uses are assigned to an arbitrary permutation of the network states L_d which allow active input at node 2. As the network states are known to all nodes they can order their input sequence by network state, decode from each subsequence and choose codewords (subsequences) for the following channel uses with active input. The output sequences are formed by the appropriate permutation of the subsequence entries. This equivalence to A6.1 does not allow a higher rate for any of the two dialog messages.
- The scheme is changed while codebook construction is as in A6.1: The six phases are interchanged such that the last three phases form the first three phases and the last three phases form the first three ones. The ordering inside the two groups of phases is arbitrary while all n channel uses stay grouped by network state. As a consequence the n_4 inputs of the first BC phase, the first $n_5 + n_6$ inputs at node 2 and the n_3 inputs of nodes 1 and 3 in the last MA phase are determined a priori with the scheme. The amount of information transmitted between the two nodes in B blocks can not exceed

$$D_{13} < R_{13}^{\star} \times \left(B \times \sum_{l=1}^{L} n_l - (n_3 + n_4) \right)$$
$$D_{31} < R_{31}^{\star} \times \left(B \times \sum_{l=1}^{L} n_l - (n_3 + n_4) \right). \tag{3.26}$$

- The scheme is arbitrary while the codebook construction is the one of A6.1: Node 2 decodes the first time after n_{dec} channel uses. Until the n_{dec}-th channel use the BC state has occurred n_{BC} times. As a consequence all the inputs at node 2 up to the n_{dec} channel use are determined a priori with the scheme. The amount of information transmitted between the two nodes in B blocks can not exceed

$$D_{13} < R_{13}^{\star} \times \left(B \times \sum_{l=1}^{L} n_l - n_{BC} \right)$$
$$D_{31} < R_{31}^{\star} \times \left(B \times \sum_{l=1}^{L} n_l - n_{BC} \right). \tag{3.27}$$

These arguments support the Conjecture 3.2.22.

Partial-Decode-and-Forward As mentioned in the introduction to this chapter partial-decode-and-forward strategies might be interesting especially for multi-antenna systems. Therefore, here additionally the PDF rates achievable with the six phase scheme are given.

Proposition 3.2.23 *All rate pairs of the half-duplex two-way relay channel that satisfy*

$$
\begin{aligned}
R_{13} \leq \min\Big\{&\tau_1 I(U_1^{(1)};Y_2^{(1)}) + \tau_1 I(X_1^{(1)};Y_3^{(1)}|U_1^{(1)}) + \tau_3 I(X_1^{(3)};Y_2^{(3)}|X_3^{(3)}) + \\
&\tau_6 I(X_1^{(6)};Y_3^{(6)}|X_2^{(6)}), \tau_1 I(X_1^{(1)};Y_3^{(1)}) + \tau_4 I(X_2^{(4)};Y_3^{(4)}) + \tau_6 I(X_1^{(6)}X_2^{(6)};Y_3^{(6)})\Big\} \\
R_{31} \leq \min\Big\{&\tau_2 I(U_3^{(2)};Y_2^{(2)}) + \tau_2 I(X_3^{(2)};Y_1^{(2)}|U_3^{(2)}) + \tau_3 I(X_3^{(3)};Y_2^{(3)}|X_1^{(3)}) + \\
&\tau_5 I(X_3^{(5)};Y_1^{(5)}|X_2^{(5)}), \tau_2 I(X_3^{(2)};Y_1^{(2)}) + \tau_4 I(X_2^{(4)};Y_1^{(4)}) + \tau_5 I(X_2^{(5)}X_3^{(5)};Y_1^{(5)})\Big\} \\
R_{13} + R_{31} \leq &\tau_1 I(U_1^{(1)};Y_2^{(1)}) + \tau_1 I(X_1^{(1)};Y_3^{(1)}|U_1^{(1)}) + \tau_2 I(U_3^{(2)};Y_2^{(2)}) + \tau_2 I(X_3^{(2)};Y_1^{(2)}|U_3^{(2)}) + \\
&\tau_3 I(X_1^{(3)}X_3^{(3)};Y_2^{(3)}) + \tau_5 I(X_3^{(5)};Y_1^{(5)}|X_2^{(5)}) + \tau_6 I(X_1^{(6)};Y_3^{(6)}|X_2^{(6)})
\end{aligned}
$$

with $0 < \tau_l$ *and* $\sum_{l=1}^{6} \tau_l \leq 1$, *for some joint probability distributions*

$$
\begin{aligned}
P(u_1^{(1)}, x_1^{(1)}, y_2^{(1)}, y_3^{(1)}) &= P(u_1^{(1)})P(x_1^{(1)}|u_1^{(1)})P(y_2^{(1)}, y_3^{(1)}|x_1^{(1)}) \\
P(u_3^{(2)}, x_3^{(2)}, y_2^{(2)}, y_1^{(2)}) &= P(u_3^{(2)})P(x_3^{(2)}|u_3^{(2)})P(y_1^{(2)}, y_2^{(2)}|x_3^{(2)}) \\
P(x_1^{(3)}, x_3^{(3)}, y_2^{(3)}) &= P(x_1^{(3)})P(x_3^{(3)})P(y_2^{(2)}|x_1^{(3)}, x_3^{(1)}) \\
P(x_2^{(4)}, y_1^{(4)}, y_3^{(4)}) &= P(x_2^{(4)})P(y_1^{(4)}, y_3^{(4)}|x_2^{(4)}) \\
P(x_2^{(5)}, x_3^{(5)}, y_1^{(5)}) &= P(x_2^{(5)})P(x_3^{(5)}|x_2^{(5)})P(y_1^{(5)}|x_2^{(5)}, x_3^{(5)}) \\
P(x_1^{(6)}, x_2^{(6)}, y_3^{(6)}) &= P(x_2^{(6)})P(x_1^{(6)}|x_2^{(6)})P(y_3^{(6)}|x_1^{(6)}, x_2^{(6)})
\end{aligned}
$$

are achievable with a 6P scheme and a partial-decode-and-forward strategy.

Proof extend A6.1 to superposition coding like from A4.1 to A4.2.

3.2.6 Wireline Communication

As in the one-way section the achievable rates for a wireline model are outlined.

Proposition 3.2.24 *All rate pairs of the half-duplex wireline two-way relay channel that satisfy*

$$
\begin{aligned}
R_{13} &\leq \min\Big\{\tau_1 b_{12}^{(1)} + \tau_3 b_{12}^{(3)} + \tau_6 b_{13}^{(6)}, \tau_1 b_{13}^{(1)} + \tau_4 b_{23}^{(4)} + \tau_6 (b_{13}^{(6)} + b_{23}^{(6)})\Big\} \\
R_{31} &\leq \min\Big\{\tau_2 b_{32}^{(2)} + \tau_3 b_{32}^{(3)} + \tau_5 b_{31}^{(5)}, \tau_2 b_{31}^{(2)} + \tau_4 b_{21}^{(4)} + \tau_5 (b_{21}^{(5)} + b_{31}^{(5)})\Big\}
\end{aligned}
$$

with $0 < \tau_l$ *and* $\sum_{l=1}^{6} \tau_l \leq 1$, *for some directed links of capacity* $b_{ij}^{(l)}$ *are achievable with a 6P scheme and a decode-and-forward strategy.*

Proposition 3.2.25 *All rate pairs of the half-duplex wireline two-way relay channel that satisfy*

$$R_{13} \leq \min\left\{\tau_1(b_{12}^{(1)} + b_{13}^{(1)}) + \tau_3 b_{12}^{(3)} + \tau_6 b_{13}^{(6)}, \tau_1 b_{13}^{(1)} + \tau_4 b_{23}^{(4)} + \tau_6(b_{13}^{(6)} + b_{23}^{(6)})\right\}$$

$$R_{31} \leq \min\left\{\tau_2(b_{32}^{(2)} + b_{31}^{(2)}) + \tau_3 b_{32}^{(3)} + \tau_5 b_{31}^{(5)}, \tau_2 b_{31}^{(2)} + \tau_4 b_{21}^{(4)} + \tau_5(b_{21}^{(5)} + b_{31}^{(5)})\right\}$$

with $0 < \tau_l$ *and* $\sum_{l=1}^{6} \tau_l \leq 1$*, for some directed links of capacity* $b_{ij}^{(l)}$ *are achievable with a 6P scheme and a partial-decode-and-forward strategy.*

Note that the sum-rate constraint is not active due to orthogonal channels in the MAC phase. The PDF coding method achieves the outer bound on this model. This implicates that PDF achieves the outer bound for the restricted half-duplex two-way relay channel with orthogonal channels. The requirement of all channels being orthogonal can be relaxed. Only the channels in the first three phases of the 6P scheme have to be orthogonal in order to achieve the outer bound.

3.3 Linear Problems

For linear rate and transmission cost objectives the rate expressions derived for different schemes have the property that optimal time allocation for fixed input distributions can be determined by solving linear programs of small size. It is shortly outlined how to formulate and solve some relevant problems. It is assumed that the channels have been specified by an conditional distribution P_c (or density p_c) and the input distributions are fixed to P_{in} (p_{in}).

Rate Maximization The first problem considered is the *Weighted Sum-Rate* maximization problem (WSRP). This problem has relevance if the communication rate through the channel should be maximized while the objective weights each user differently. Therefore, each user is assigned to a certain "priority" in the rate maximization. The objective is defined by

$$R_{\text{WSR}}(\lambda) = \lambda R_{13} + (1-\lambda) R_{31} \qquad \lambda \in [0,1]. \tag{3.28}$$

This objective can be used to determine a rate region by maximizing for λ from zero to one. A very similar problem with the same structure is the *Sum-Rate* maximization problem (SRP). The maximization of such an objective yields the maximum through-put via the communication channel. The objective is

$$R_{\text{SR}} = R_{13} + R_{31}.$$

The matrix $\boldsymbol{A}$ for both problems is determined by the used scheme and the expressions considered.

$$\boldsymbol{A} = \begin{bmatrix} {\boldsymbol{a}_{13,1}}^T \\ {\boldsymbol{a}_{13,2}}^T \\ {\boldsymbol{a}_{31,1}}^T \\ {\boldsymbol{a}_{31,2}}^T \\ {\boldsymbol{a}_s}^T \\ {\boldsymbol{a}_{q,1}}^T \\ {\boldsymbol{a}_{q,2}}^T \\ {\boldsymbol{a}_{q,3}}^T \\ \boldsymbol{a}_\tau^T \end{bmatrix} \tag{3.29}$$

where the constraints have the form

$$\begin{aligned} {\boldsymbol{a}_{13}}^T &= \begin{bmatrix} 1 & 0 & -I_{13}^{(1)}(P_c, P_{in}) & \dots & -I_{13}^{(L)}(P_c, P_{in}) \end{bmatrix} \\ {\boldsymbol{a}_{31}}^T &= \begin{bmatrix} 0 & 1 & -I_{31}^{(1)}(P_c, P_{in}) & \dots & -I_{31}^{(L)}(P_c, P_{in}) \end{bmatrix} \\ {\boldsymbol{a}_s}^T &= \begin{bmatrix} 1 & 1 & -I_s^{(1)}(P_c, P_{in}) & \dots & -I_s^{(L)}(P_c, P_{in}) \end{bmatrix} \\ {\boldsymbol{a}_q}^T &= \begin{bmatrix} 0 & 0 & \pm I_q^{(1)}(P_c, P_{in}) & \dots & \pm I_q^{(L)}(P_c, P_{in}) \end{bmatrix} \\ \boldsymbol{a}_\tau^T &= \begin{bmatrix} 0 & 0 & 1 & \dots & 1 \end{bmatrix} \\ \boldsymbol{b}^T &= \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix}. \end{aligned} \tag{3.30}$$

In order to optimize the communication for the highest possible symmetric rates the *MaxMin-Rate* maximization problem (MMP) has to be considered. The rate objective is defined by

$$R_{\text{MaxMin}} = \min\{R_{13}, R_{31}\}$$

and the constraints have a slightly different form

$$\begin{aligned}
\boldsymbol{a}_{13}^T &= \begin{bmatrix}1 & -I_{13}^{(1)}(P_c, P_{in}) & \dots & -I_{13}^{(L)}(P_c, P_{in})\end{bmatrix}\\
\boldsymbol{a}_{31}^T &= \begin{bmatrix}1 & -I_{31}^{(1)}(P_c, P_{in}) & \dots & -I_{31}^{(L)}(P_c, P_{in})\end{bmatrix}\\
\boldsymbol{a}_{s}^T &= \begin{bmatrix}2 & -I_{s}^{(1)}(P_c, P_{in}) & \dots & -I_{s}^{(L)}(P_c, P_{in})\end{bmatrix}\\
\boldsymbol{a}_{q}^T &= \begin{bmatrix}0 & \pm I_{q}^{(1)}(P_c, P_{in}) & \dots & \pm I_{q}^{(L)}(P_c, P_{in})\end{bmatrix}\\
\boldsymbol{a}_{\tau}^T &= \begin{bmatrix}0 & 1 & \dots & 1\end{bmatrix}\\
\boldsymbol{b}^T &= \begin{bmatrix}0 & \dots & 0 & 1\end{bmatrix}.
\end{aligned} \tag{3.31}$$

Note that depending on the expressions used some constraints might not be present and different mutual informations are zero. The sum-rate constraint is only present if a multiple-acess phase is used and the relay has to decode. The compression constraints $\boldsymbol{a}_q$ are only present for strategies with quantization at the relay. The vectors $\boldsymbol{c}$ for the different problems are

$$\begin{aligned}
\boldsymbol{c}(\lambda)_{\text{WSRP}}^T &= \begin{bmatrix}\lambda & (1-\lambda) & 0 & \dots & 0\end{bmatrix}\\
\boldsymbol{c}_{\text{SRP}}^T &= \begin{bmatrix}1 & 1 & 0 & \dots & 0\end{bmatrix}\\
\boldsymbol{c}_{\text{MMP}}^T &= \begin{bmatrix}1 & 0 & \dots & 0\end{bmatrix}.
\end{aligned} \tag{3.32}$$

With

$$\begin{aligned}
\boldsymbol{x}_{\text{WSRP/SRP}}^T &= \begin{bmatrix}R_{13} & R_{31} & \tau_1 & \dots & \tau_L\end{bmatrix}\\
\boldsymbol{x}_{\text{MMP}}^T &= \begin{bmatrix}R_{\text{MMP}} & \tau_1 & \dots & \tau_L\end{bmatrix}
\end{aligned} \tag{3.33}$$

the optimization problems have the form

$$\begin{aligned}
&\max \boldsymbol{c}^T\boldsymbol{x}\\
&s.t.\ \ \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}, \boldsymbol{0} \le \boldsymbol{x}, const(\boldsymbol{x})
\end{aligned} \tag{3.34}$$

where $const(\boldsymbol{x})$ denotes that additional constraints (optional) on $\boldsymbol{x}$ are fulfilled, e.g., discrete time slot lengths.

Minimizing Transmission Cost A second problem with linear structure is the *Transmission Cost* minimization problem (TCP). Each phase l is associated with a cost linear in activation time. The cost for each phase l and unit activation time is denoted $c_l > 0$. The objective is to minimize the cost for a certain rate requirement $\boldsymbol{R} = \begin{bmatrix}R_{13} & R_{31}\end{bmatrix}^T$ on the communication through the network. The constraints have the form

$$\begin{aligned}
\boldsymbol{a}_{13}^T &= \begin{bmatrix}-I_{13}^{(1)}(P_c, P_{in}) & \dots & -I_{13}^{(L)}(P_c, P_{in})\end{bmatrix}\\
\boldsymbol{a}_{31}^T &= \begin{bmatrix}-I_{31}^{(1)}(P_c, P_{in}) & \dots & -I_{31}^{(L)}(P_c, P_{in})\end{bmatrix}\\
\boldsymbol{a}_{s}^T &= \begin{bmatrix}-I_{s}^{(1)}(P_c, P_{in}) & \dots & -I_{s}^{(L)}(P_c, P_{in})\end{bmatrix}\\
\boldsymbol{a}_{q}^T &= \begin{bmatrix}\pm I_{q}^{(1)}(P_c, P_{in}) & \dots & \pm I_{q}^{(L)}(P_c, P_{in})\end{bmatrix}\\
\boldsymbol{a}_{\tau}^T &= \begin{bmatrix}1 & \dots & 1\end{bmatrix}\\
\boldsymbol{b}^T &= \begin{bmatrix}-R_{13} & -R_{31} & -(R_{13}+R_{31}) & 0 & \dots & 0 & 1\end{bmatrix}
\end{aligned} \tag{3.35}$$

and the cost vector is

$$\boldsymbol{c}^T = \begin{bmatrix} c_1 & \dots & c_L \end{bmatrix}. \tag{3.36}$$

With $\boldsymbol{x} = \begin{bmatrix} \tau_1 & \dots & \tau_L \end{bmatrix}^T$ the optimization problem has the form

$$\begin{aligned} &\min \boldsymbol{c}^T \boldsymbol{x} \\ &s.t. \quad \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}, \boldsymbol{0} \le \boldsymbol{x}, const(\boldsymbol{x}). \end{aligned} \tag{3.37}$$

For the wireline model the mutual informations have to be replaced by link capacities $b_{ij}^{(l)}$.

Complexity The problems above are identified as linear programs of small size. These can be solved at very low complexity. For example on the scalar Gaussian channels with 6P-OB or 6P-DF, as considered in the simulations, the parameters β, γ need to be determined before solving time allocation. If coherent signaling of two nodes is not possible ($\beta, \gamma = 0$), problems above are solved by one LP. For the wireline network with fixed link capacities also only one LP is needed. The challenge, especially for channels with multiple inputs and outputs or CF strategies, is to determine the right input distributions before solving for time allocation or to reformulate the problems in order to optimize the input distributions and time allocation jointly.

3.4 Gaussian Half-Duplex Two-Way Relay Channel

In this section the upper and lower bound expressions for channels with continuous Gaussian variables are derived. The same model assumptions like for the one-way channel are used. For the visualization of rate regions three scenarios in a line network will be considered where node 2 takes different positions

$$\begin{aligned}
&\text{"symmetric" :} && \boldsymbol{x}_2 = \begin{bmatrix}0.5 & 0\end{bmatrix}^T \\
&\text{"near" :} && \boldsymbol{x}_2 = \begin{bmatrix}0.25 & 0\end{bmatrix}^T \\
&\text{"close" :} && \boldsymbol{x}_2 = \begin{bmatrix}0.01 & 0\end{bmatrix}^T .
\end{aligned} \tag{3.38}$$

In the first scenario the channels of the dialog nodes to node 2 support symmetric rates. The other two configurations model modest and extremly asymmetric channels from the dialog nodes to the relay. Note that for the line/plane network model used here scalar channels have no direction, i.e., $h_{ij} = h_{ji}$. At some points throughout this section two asymptotic cases will be considered. The first one assumes infinite power available at one of the dialog nodes $P_{1/3}^{(l)} \to \infty$, the second infinite power available at node 2, $P_2^{(l)} \to \infty$. For these two asymptotic cases some strategies achieve the outer bounds of particular schemes.

3.4.1 2P-MA-BC Scheme

According to the channel model used the output sequences for all 2P-MA-BC strategies are

$$\begin{aligned}
\boldsymbol{Y}_2^{(1)} &= h_{12}\boldsymbol{X}_1^{(1)} + h_{32}\boldsymbol{X}_3^{(1)} + \boldsymbol{Z}_2^{(1)} \\
\boldsymbol{Y}_1^{(2)} &= h_{21}\boldsymbol{X}_2^{(2)} + \boldsymbol{Z}_1^{(2)} \\
\boldsymbol{Y}_3^{(2)} &= h_{23}\boldsymbol{X}_2^{(2)} + \boldsymbol{Z}_3^{(2)} .
\end{aligned} \tag{3.39}$$

Outer Bound For the outer bound of this scheme the input sequences are independent and given by

$$\begin{aligned}
\boldsymbol{X}_1^{(1)} &= \sqrt{P_1}\boldsymbol{f}_{11}^{(1)}(W_{13}) \\
\boldsymbol{X}_3^{(1)} &= \sqrt{P_3}\boldsymbol{f}_{31}^{(1)}(W_{13}) \\
\boldsymbol{X}_2^{(2)} &= \sqrt{P_2}\boldsymbol{f}_{21}^{(2)}(W_{13}, W_{31}),
\end{aligned} \tag{3.40}$$

resulting in the rate expressions

$$\begin{aligned}
R_{13} &\leq \min\left\{\tau_1 \log\left(1 + |h_{12}|^2 P_1^{(1)}\right), \tau_2 \log\left(1 + |h_{23}|^2 P_2^{(2)}\right)\right\} \\
R_{31} &\leq \min\left\{\tau_1 \log\left(1 + |h_{32}|^2 P_3^{(1)}\right), \tau_2 \log\left(1 + |h_{21}|^2 P_2^{(2)}\right)\right\}.
\end{aligned} \tag{3.41}$$

Note that for $P_1^{(1)} \to \infty$ the bound tends to

$$\begin{aligned}
R_{13} &\to \tau_2 \log\left(1 + |h_{23}|^2 P_2^{(2)}\right) \\
R_{31} &= \min\left\{\tau_1 \log\left(1 + |h_{32}|^2 P_3^{(1)}\right), \tau_2 \log\left(1 + |h_{21}|^2 P_2^{(2)}\right)\right\}
\end{aligned} \tag{3.42}$$

and for $P_2^{(2)} \to \infty$

$$\begin{aligned} R_{13} &\to \log\left(1+|h_{12}|^2 P_1^{(1)}\right) \\ R_{31} &\to \log\left(1+|h_{32}|^2 P_3^{(1)}\right). \end{aligned} \tag{3.43}$$

Decode-and-Forward The same input parametrization as for the outer bound yields the achievable rates for the DF strategy

$$\begin{aligned} R_{13} &\leq \min\left\{\tau_1 \log\left(1+|h_{12}|^2 P_1^{(1)}\right), \tau_2 \log\left(1+|h_{23}|^2 P_2^{(2)}\right)\right\} \\ R_{31} &\leq \min\left\{\tau_1 \log\left(1+|h_{32}|^2 P_3^{(1)}\right), \tau_2 \log\left(1+|h_{21}|^2 P_2^{(2)}\right)\right\} \\ R_{13}+R_{31} &\leq \tau_1 \log\left(1+|h_{12}|^2 P_1^{(1)}+|h_{32}|^2 P_3^{(1)}\right). \end{aligned} \tag{3.44}$$

The difference to the outer bound is the sum-rate constraint due to the MAC in the first transmission phase. For the case $P_1^{(1)}$ or $P_3^{(1)} \to \infty$ the sum-rate constraint is not active and the achievable rates and the scheme outer bound asymptotically coincide.

Compress-and-Forward (2LQ) For the CF strategy the input sequences are chosen to be

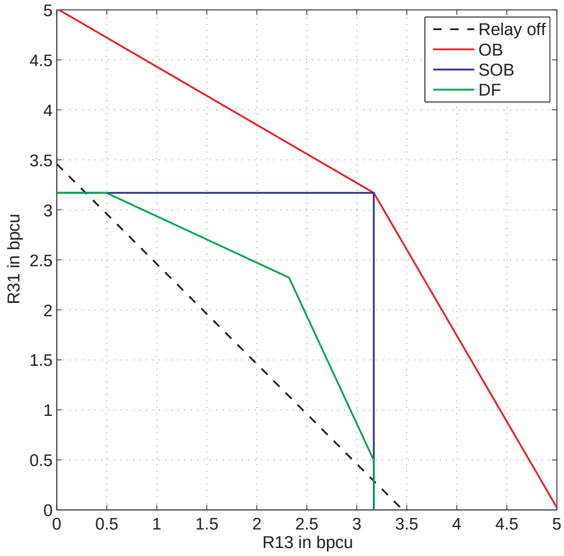
$$\begin{aligned} \boldsymbol{X}_1^{(1)} &= \sqrt{P_1^{(1)}}\boldsymbol{f}_{11}^{(1)}(W_{13}) \\ \boldsymbol{X}_3^{(1)} &= \sqrt{P_3^{(1)}}\boldsymbol{f}_{31}^{(1)}(W_{31}) \\ \boldsymbol{X}_2^{(2)} &= \underbrace{\sqrt{\beta P_2^{(2)}}\boldsymbol{f}_{21}^{(2)}(Q_{21})}_{\boldsymbol{U}_2^{(2)}}+\sqrt{(1-\beta)P_2^{(2)}}\boldsymbol{f}_{22}^{(2)}(Q_{22}) \qquad \beta \in [0,1] \end{aligned} \tag{3.45}$$

where the parameter β assigns the power of node 2 to the propagation of the "coarse" (Q_{21}) and the "refinement" (Q_{22}) quantization index. The quantized outputs are assumed to have the form

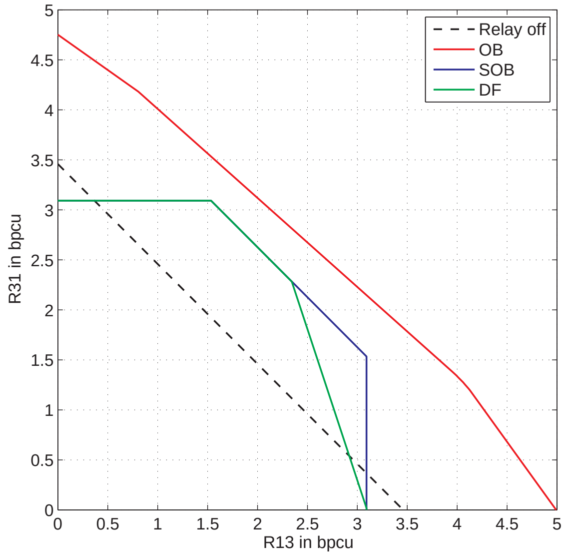
$$\begin{aligned} \hat{\boldsymbol{Y}}_{21}^{(1)} &= \boldsymbol{Y}_2^{(1)}+\hat{\boldsymbol{Z}}_{21}^{(1)}+\hat{\boldsymbol{Z}}_{22}^{(1)} \\ (\hat{\boldsymbol{Y}}_{21}^{(1)},\hat{\boldsymbol{Y}}_{22}^{(1)}) &= \boldsymbol{Y}_2^{(1)}+\hat{\boldsymbol{Z}}_{22}^{(1)} \end{aligned} \tag{3.46}$$

with independent $\hat{Z}_{21,k}^{(1)} \sim \mathcal{N}_{\mathbb{C}}(0,\hat{\sigma}_1^2)$ and $\hat{Z}_{22,k}^{(1)} \sim \mathcal{N}_{\mathbb{C}}(0,\hat{\sigma}_2^2)$. This yields the achievable rate expressions

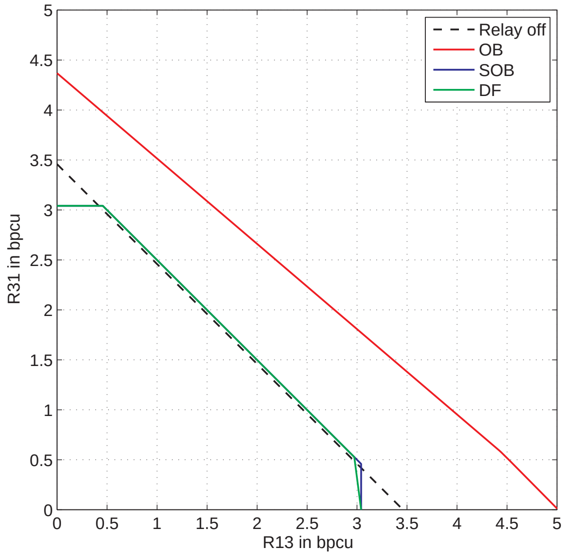
$$\begin{aligned} R_{13} &\leq \tau_1 \log\left(1+\frac{|h_{12}|^2 P_1^{(1)}}{1+\hat{\sigma}_1^2+\hat{\sigma}_2^2}\right) \\ R_{31} &\leq \tau_1 \log\left(1+\frac{|h_{32}|^2 P_3^{(1)}}{1+\hat{\sigma}_2^2}\right) \end{aligned} \tag{3.47}$$



(a) Symmetric



(b) Near



(c) Close

Figure 3.15: 2P-MA-BC with DF, Line Network with $P_i^{(l)} = 10$, $\alpha = 3$

subject to

$$
\begin{aligned}
\tau_1 \log\left(1+\frac{1+|h_{12}|^2 P_1^{(1)}}{\hat{\sigma}_1^2+\hat{\sigma}_2^2}\right) &\le \tau_2 \log\left(1+\frac{|h_{23}|^2 \beta P_2^{(2)}}{1+|h_{23}|^2 (1-\beta)P_2^{(2)}}\right)\\
\tau_1 \log\left(1+\frac{1+|h_{32}|^2 P_3^{(1)}}{\hat{\sigma}_1^2+\hat{\sigma}_2^2}\right) &\le \tau_2 \log\left(1+\frac{|h_{21}|^2 \beta P_2^{(2)}}{1+|h_{21}|^2 (1-\beta)P_2^{(2)}}\right)\\
\tau_1 \log\left(1+\frac{1+|h_{32}|^2 P_3^{(1)}}{\hat{\sigma}_2^2}\right) - \tau_1 \log\left(1+\frac{1+|h_{32}|^2 P_3^{(1)}}{\hat{\sigma}_1^2+\hat{\sigma}_2^2}\right) &\le \tau_2 \log\left(1+|h_{21}|^2 (1-\beta)P_2^{(2)}\right).
\end{aligned}
\tag{3.48}
$$

For the case $P_2^{(2)} \to \infty$ the achievable rates with CF and the scheme outer bound coincide.

Partial-Decode-Compress-and-Forward (2LQ) The input sequences for the PDCF strategy can be assumed to have the form

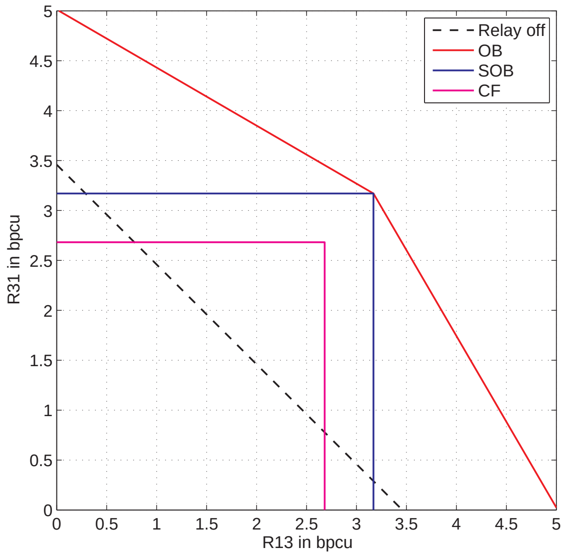
$$
\begin{aligned}
\boldsymbol{X}_1^{(1)} &= \underbrace{\sqrt{\beta P_1^{(1)}}\boldsymbol{f}_{11}^{(1)}(W_{13})}_{\boldsymbol{U}_1^{(1)}} + \sqrt{(1-\beta)P_1^{(1)}}\boldsymbol{f}_{12}^{(1)}(W_{13}) && \beta \in [0,1]\\
\boldsymbol{X}_3^{(1)} &= \underbrace{\sqrt{\gamma P_3^{(1)}}\boldsymbol{f}_{31}^{(1)}(W_{31})}_{\boldsymbol{U}_3^{(1)}} + \sqrt{(1-\gamma)P_3^{(1)}}\boldsymbol{f}_{32}^{(1)}(W_{31}) && \gamma \in [0,1]\\
\boldsymbol{X}_2^{(2)} &= \underbrace{\underbrace{\sqrt{\delta P_2^{(2)}}\boldsymbol{f}_{21}^{(2)}(W_{13},W_{31})}_{\boldsymbol{U}_2^{(2)}} + \sqrt{\zeta P_2^{(2)}}\boldsymbol{f}_{22}^{(2)}(Q_{21})}_{\boldsymbol{V}_2^{(2)}} + \sqrt{(1-\delta-\zeta)P_2^{(2)}}\boldsymbol{f}_{23}^{2)}(Q_{22}) && \delta,\zeta \in [0,1]
\end{aligned}
\tag{3.49}
$$

where $\delta+\zeta \le 1$. Here β and γ assign the powers of node 1 and 3 to the parts of their signals decoded at node 2. The parameters δ and ζ distribute the power at node 2 to the propagation of the decoded messages and the two quantization indices. The quantized outputs are assumed to have the form

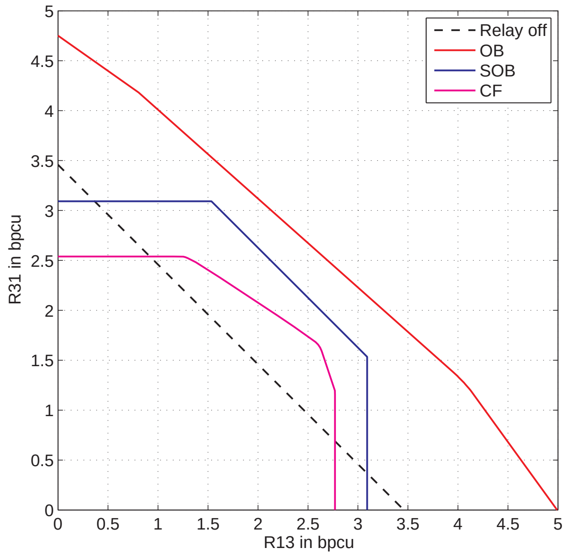
$$
\begin{aligned}
\hat{\boldsymbol{Y}}_{21}^{(1)} &= \boldsymbol{Y}_2^{(1)} + \hat{\boldsymbol{Z}}_{21}^{(1)} + \hat{\boldsymbol{Z}}_{22}^{(1)}\\
(\hat{\boldsymbol{Y}}_{21}^{(1)}, \hat{\boldsymbol{Y}}_{22}^{(1)}) &= \boldsymbol{Y}_2^{(1)} + \hat{\boldsymbol{Z}}_{22}^{(1)}
\end{aligned}
\tag{3.50}
$$

with independent $\hat{Z}_{21,k}^{(1)} \sim \mathcal{N}_{\mathbb{C}}(0,\hat{\sigma}_1^2)$ and $\hat{Z}_{22,k}^{(1)} \sim \mathcal{N}_{\mathbb{C}}(0,\hat{\sigma}_2^2)$. This yields the achievable rate expressions

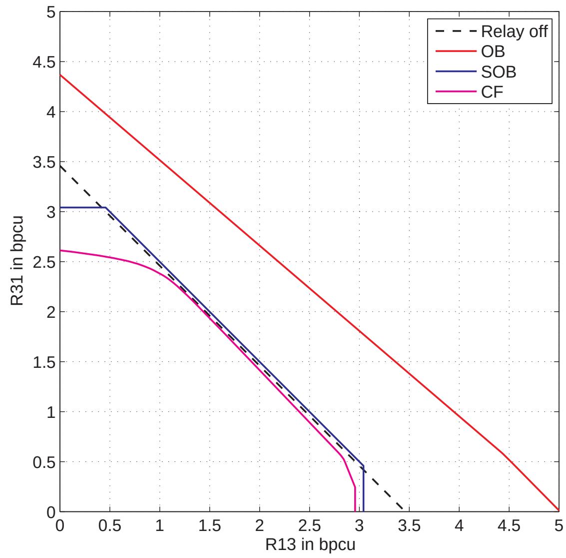
$$
\begin{aligned}
R_{13} \le \min\Bigg\{&\tau_1 \log\left(1+\frac{|h_{12}|^2 \beta P_1^{(1)}}{1+|h_{12}|^2 (1-\beta)P_1^{(1)} + |h_{32}|^2 (1-\gamma)P_3^{(1)}}\right),\\
&\tau_2 \log\left(1+\frac{|h_{23}|^2 \delta P_2^{(2)}}{1+|h_{23}|^2 (1-\delta)P_2^{(2)}}\right)\Bigg\} + \tau_1 \log\left(1+\frac{|h_{12}|^2 (1-\beta)P_1^{(1)}}{1+\hat{\sigma}_1^2+\hat{\sigma}_2^2}\right)
\end{aligned}
$$



(a) Symmetric



(b) Near



(c) Close

Figure 3.16: 2P-MA-BC with CF, Line Network with $P_i^{(l)} = 10$, $\alpha = 3$

$$R_{31} \leq \min\left\{\tau_1 \log\left(1 + \frac{|h_{32}|^2 \gamma P_3^{(1)}}{1 + |h_{12}|^2 (1-\beta) P_1^{(1)} + |h_{32}|^2 (1-\gamma) P_3^{(1)}}\right),\right.$$
$$\left.\tau_2 \log\left(1 + \frac{|h_{21}|^2 \delta P_2^{(2)}}{1 + |h_{21}|^2 (1-\delta) P_2^{(2)}}\right)\right\} + \tau_1 \log\left(1 + \frac{|h_{32}|^2 (1-\gamma) P_3^{(1)}}{1 + \hat{\sigma}_2^2}\right)$$

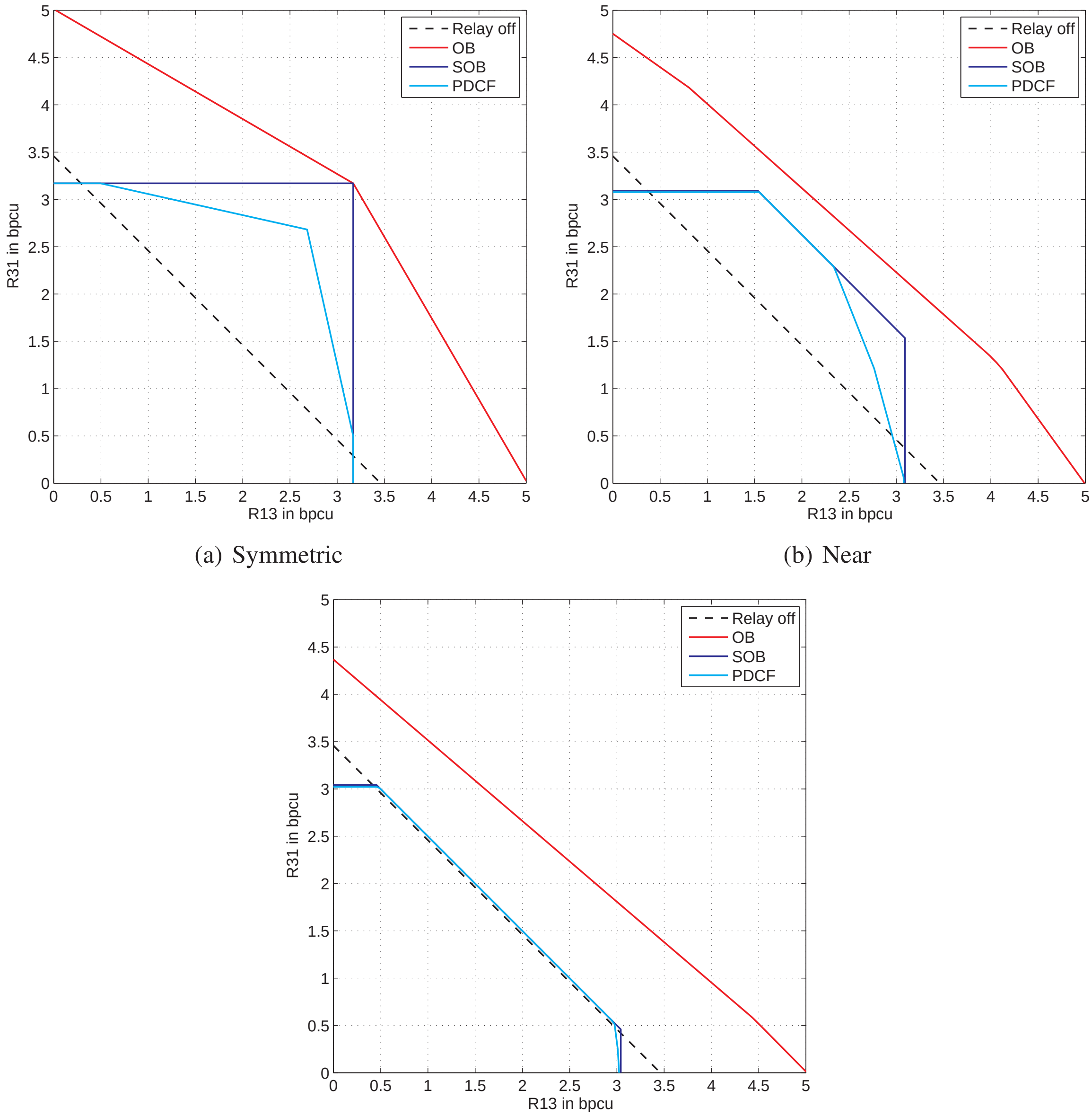
$$\begin{aligned} R_{13} + R_{31} \leq& \tau_1 \log\left(1 + \frac{|h_{12}|^2 \beta P_1^{(1)} + |h_{32}|^2 \gamma P_3^{(1)}}{1 + |h_{12}|^2 (1-\beta) P_1^{(1)} + |h_{32}|^2 (1-\gamma) P_3^{(1)}}\right) + \\ & \tau_1 \log\left(1 + \frac{|h_{12}|^2 (1-\beta) P_1^{(1)}}{1 + \hat{\sigma}_1^2 + \hat{\sigma}_2^2}\right) + \tau_1 \log\left(1 + \frac{|h_{32}|^2 (1-\gamma) P_3^{(1)}}{1 + \hat{\sigma}_2^2}\right) \end{aligned}$$

subject to

$$\begin{aligned} &\tau_1 \log\left(1 + \frac{1 + |h_{32}|^2 (1-\gamma) P_3^{(1)}}{\hat{\sigma}_1^2 + \hat{\sigma}_2^2}\right) \leq \tau_2 \log\left(1 + \frac{|h_{21}|^2 \zeta P_2^{(2)}}{1 + |h_{21}|^2 (1-\delta-\zeta) P_2^{(2)}}\right) \\ &\tau_1 \log\left(1 + \frac{1 + |h_{12}|^2 (1-\beta) P_1^{(1)}}{\hat{\sigma}_1^2 + \hat{\sigma}_2^2}\right) \leq \tau_2 \log\left(1 + \frac{|h_{23}|^2 \zeta P_2^{(2)}}{1 + |h_{23}|^2 (1-\delta-\zeta) P_2^{(2)}}\right) \\ &\tau_1 \log\left(1 + \frac{1 + |h_{32}|^2 (1-\gamma) P_3^{(1)}}{\hat{\sigma}_2^2}\right) - \tau_1 \log\left(1 + \frac{1 + |h_{32}|^2 (1-\gamma) P_3^{(1)}}{\hat{\sigma}_1^2 + \hat{\sigma}_2^2}\right) \\ &\leq \tau_2 \log\left(1 + |h_{21}|^2 (1-\delta-\zeta) P_2^{(2)}\right). \end{aligned} \tag{3.51}$$

The configuration $\beta = \gamma = \delta = 1$ would result in a DF strategy whereas $\beta = \gamma = \delta = 0$ determines a CF strategy. Note that with $\beta = 1, \gamma = 0$ it is also possible to operate with different strategies in both directions.

Simulations Figures 3.15-3.17 show the rate regions for the individual strategies. Figure 3.18 superimposes all plots. It show that for the chosen model and parameters the scheme outer bound meets the problem bound only at the sum-rate point for symmetric channels to node 2. However, in all situations the scheme outer bound (SOB) varies significantly from the problem outer bound (OB) and comes closer to the rate region without relay for asymmetric channels. The DF region approaches the scheme outer bound for the asymmetric case. The CF region shows the same shape as the SOB for the symmetric case. PDCF gives the convex hull of the points achievable with DF and CF.

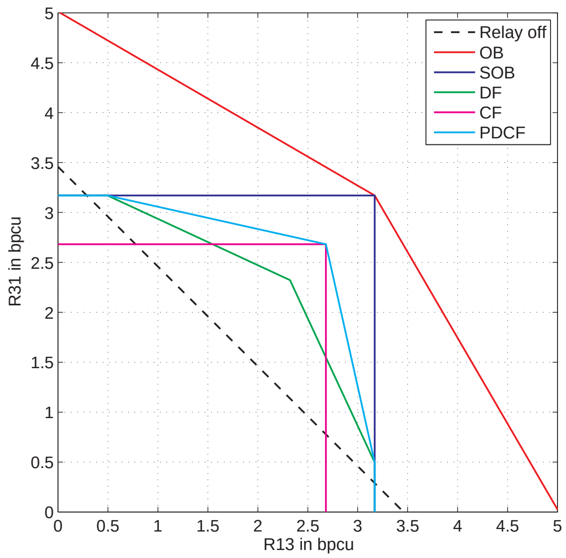


(a) Symmetric

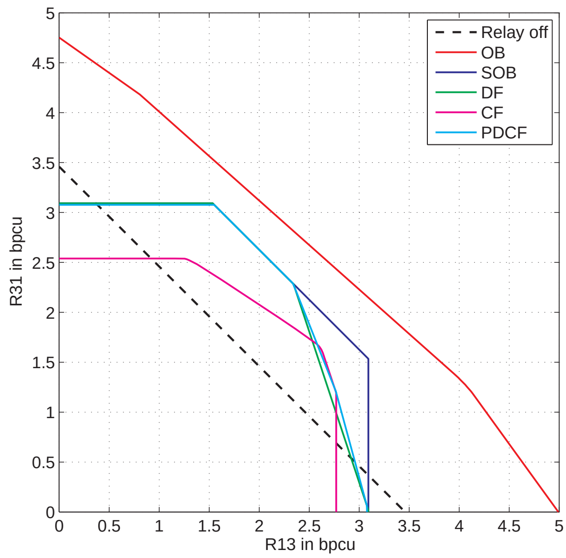
(b) Near

(c) Close

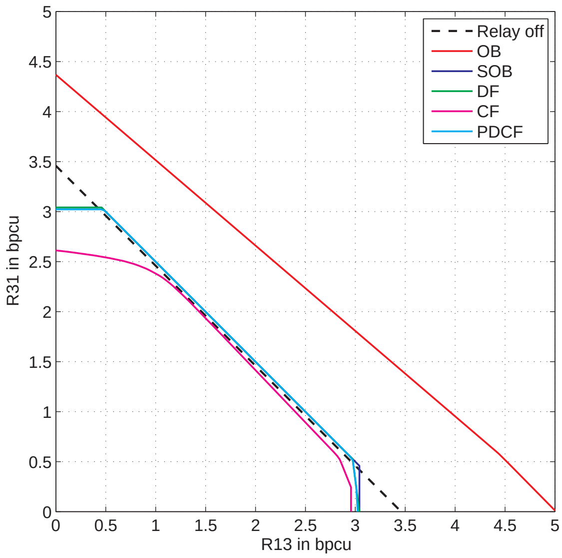
Figure 3.17: 2P-MA-BC with PDCF, Line Network with $P_i^{(l)} = 10$, $\alpha = 3$



(a) Symmetric



(b) Near



(c) Close

Figure 3.18: 2P-MA-BC, Line Network with $P_i^{(l)} = 10$, $\alpha = 3$

3.4.2 3P-BC Scheme

For all 3P-BC strategies the output sequences are given by

$$\begin{aligned}
\boldsymbol{Y}_2^{(1)} &= h_{12}\boldsymbol{X}_1^{(1)} + \boldsymbol{Z}_2^{(1)} \\
\boldsymbol{Y}_3^{(1)} &= h_{13}\boldsymbol{X}_1^{(1)} + \boldsymbol{Z}_3^{(1)} \\
\boldsymbol{Y}_1^{(2)} &= h_{31}\boldsymbol{X}_3^{(2)} + \boldsymbol{Z}_1^{(2)} \\
\boldsymbol{Y}_2^{(2)} &= h_{32}\boldsymbol{X}_3^{(2)} + \boldsymbol{Z}_2^{(2)} \\
\boldsymbol{Y}_1^{(3)} &= h_{21}\boldsymbol{X}_2^{(3)} + \boldsymbol{Z}_1^{(3)} \\
\boldsymbol{Y}_3^{(3)} &= h_{23}\boldsymbol{X}_2^{(3)} + \boldsymbol{Z}_3^{(3)}.
\end{aligned} \tag{3.52}$$

Outer Bound For the outer bound of this three phase scheme the input sequences are independent and given by

$$\begin{aligned}
\boldsymbol{X}_1^{(1)} &= \sqrt{P_1^{(1)}}\boldsymbol{f}_{11}^{(1)}(W_{13}) \\
\boldsymbol{X}_3^{(2)} &= \sqrt{P_3^{(2)}}\boldsymbol{f}_{31}^{(2)}(W_{31}) \\
\boldsymbol{X}_2^{(3)} &= \sqrt{P_2^{(3)}}\boldsymbol{f}_{21}^{(3)}(W_{13}, W_{31}),
\end{aligned} \tag{3.53}$$

resulting in the expressions

$$\begin{aligned}
R_{13} &\leq \min\left\{\tau_1 \log\left(1 + (|h_{12}|^2 + |h_{13}|^2)P_1^{(1)}\right), \tau_1 \log\left(1 + |h_{13}|^2 P_1^{(1)}\right) + \tau_3 \log\left(1 + |h_{23}|^2 P_2^{(3)}\right)\right\} \\
R_{31} &\leq \min\left\{\tau_2 \log\left(1 + (|h_{31}|^2 + |h_{32}|^2)P_3^{(2)}\right), \tau_2 \log\left(1 + |h_{32}|^2 P_3^{(2)}\right) + \tau_3 \log\left(1 + |h_{21}|^2 P_2^{(3)}\right)\right\}.
\end{aligned} \tag{3.54}$$

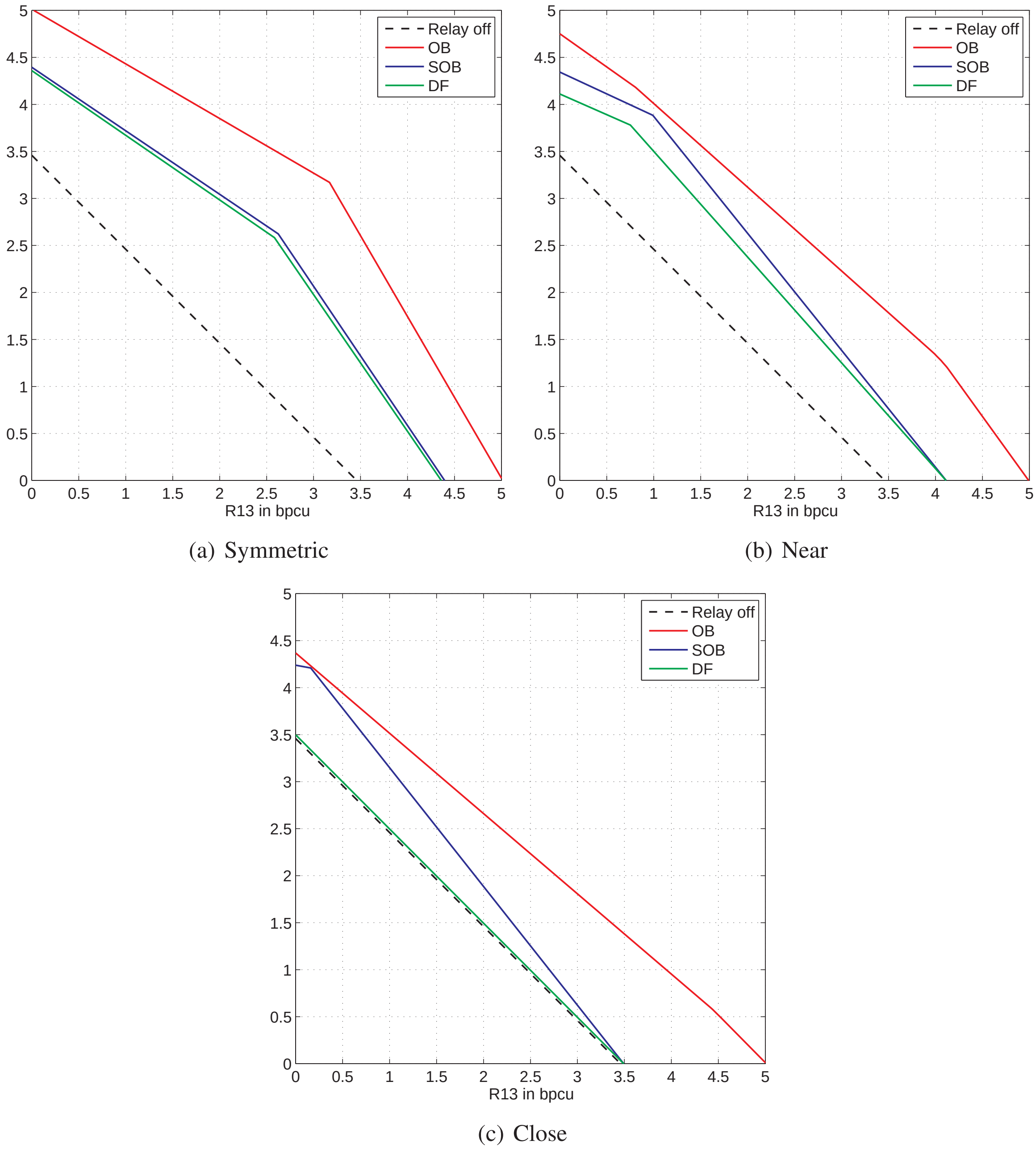
For the case $P_2^{(3)} \to \infty$ the bound tends to

$$\begin{aligned}
R_{13} &\to \tau_1 \log\left(1 + (|h_{12}|^2 + |h_{13}|^2)P_1^{(1)}\right) \\
R_{31} &\to \tau_2 \log\left(1 + (|h_{31}|^2 + |h_{32}|^2)P_3^{(2)}\right).
\end{aligned} \tag{3.55}$$

Decode-and-Forward The same input parameterization as for the outer bound yields the achievable rates with DF

$$\begin{aligned}
R_{13} &\leq \min\left\{\tau_1 \log\left(1 + |h_{12}|^2 P_1^{(1)}\right), \tau_1 \log\left(1 + |h_{13}|^2 P_1^{(1)}\right) + \tau_3 \log\left(1 + |h_{23}|^2 P_2^{(3)}\right)\right\} \\
R_{31} &\leq \min\left\{\tau_2 \log\left(1 + |h_{32}|^2 P_3^{(2)}\right), \tau_2 \log\left(1 + |h_{31}|^2 P_3^{(2)}\right) + \tau_3 \log\left(1 + |h_{21}|^2 P_2^{(3)}\right)\right\}.
\end{aligned} \tag{3.56}$$

Note that the expressions for PDF are not considered here as the strategy results in a DF strategy for scalar channels.



(a) Symmetric

(b) Near

(c) Close

Figure 3.19: 3P-BC with DF, Line Network with $P_i^{(l)} = 10$, $\alpha = 3$

Compress-and-Forward For the presented CF strategy the input sequence can be assumed to have the form

$$\begin{aligned}
\boldsymbol{X}_1^{(1)} &= \sqrt{P_1^{(1)}}\boldsymbol{f}_{11}^{(1)}(W_{13}) \\
\boldsymbol{X}_3^{(2)} &= \sqrt{P_3^{(2)}}\boldsymbol{f}_{31}^{(2)}(W_{31}) \\
\boldsymbol{X}_2^{(3)} &= \sqrt{\beta P_2^{(3)}}\boldsymbol{f}_{21}^{(3)}(Q_{23}) + \sqrt{(1-\beta)P_2^{(3)}}\boldsymbol{f}_{22}^{(3)}(Q_{21}) \qquad \beta \in [0,1].
\end{aligned} \tag{3.57}$$

For simpler notation here superposition coding instead of dirty-paper coding (DPC [4], giving the same rate expressions for scalar Gaussian broadcast channels [7]) is assumed. The parameter β controls the superposition power assignment at node 2. The quantized outputs at node 2 have the form

$$\begin{aligned}
\hat{\boldsymbol{Y}}_2^{(1)} &= \boldsymbol{Y}_2^{(1)} + \hat{\boldsymbol{Z}}_2^{(1)} \\
\hat{\boldsymbol{Y}}_2^{(2)} &= \boldsymbol{Y}_2^{(2)} + \hat{\boldsymbol{Z}}_2^{(2)}
\end{aligned} \tag{3.58}$$

with independent $\hat{Z}_{2,k}^{(1)} \sim \mathcal{N}_{\mathbb{C}}(0, \hat{\sigma}_1^2)$ and $\hat{Z}_{2,k}^{(2)} \sim \mathcal{N}_{\mathbb{C}}(0, \hat{\sigma}_2^2)$. The achievable rates are

$$\begin{aligned}
R_{13} &\le \tau_1 \log\left(1 + |h_{13}|^2 P_1^{(1)} + \frac{|h_{12}|^2 P_1^{(1)}}{1+\hat{\sigma}_1^2}\right) \\
R_{31} &\le \tau_2 \log\left(1 + |h_{31}|^2 P_3^{(2)} + \frac{|h_{32}|^2 P_3^{(2)}}{1+\hat{\sigma}_2^2}\right)
\end{aligned}$$

subject to

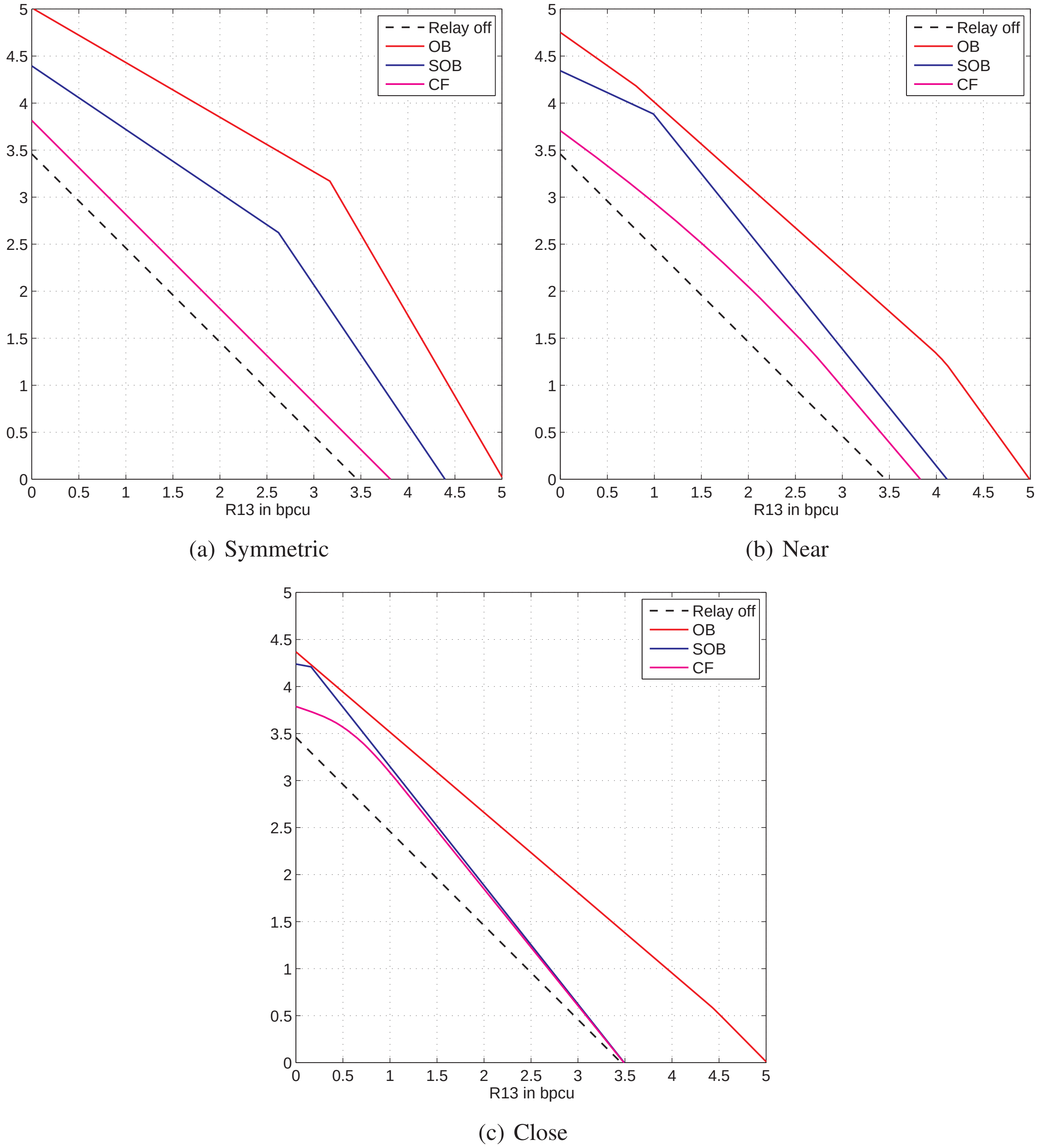
$$\begin{aligned}
\tau_1 \log\left(1 + \frac{1}{\hat{\sigma}_1^2}\left(1 + \frac{|h_{12}|^2 P_1^{(1)}}{1+|h_{13}|^2 P_1^{(1)}}\right)\right) &\le \tau_3 \log\left(1 + \frac{|h_{23}|^2 \beta P_2^{(3)}}{1+|h_{23}|^2 (1-\beta)P_2^{(3)}}\right) \\
\tau_2 \log\left(1 + \frac{1}{\hat{\sigma}_2^2}\left(1 + \frac{|h_{32}|^2 P_3^{(2)}}{1+|h_{31}|^2 P_3^{(2)}}\right)\right) &\le \tau_3 \log\left(1 + |h_{21}|^2 (1-\beta)P_2^{(3)}\right).
\end{aligned} \tag{3.59}$$

A second expression follows after interchanging the roles of nodes 1 and 3. For the asymptotic case $P_2^{(3)} \to \infty$ the rates with CF coincide with the scheme outer bound as $\hat{\sigma}_1^2, \hat{\sigma}_2^2 \to 0$.

Partial-Decode-Compress-and-Forward For this strategy the input sequences have the form

$$\begin{aligned}
\boldsymbol{X}_1^{(1)} &= \sqrt{\beta P_1^{(1)}}\boldsymbol{f}_{11}^{(1)}(W_{13}) + \sqrt{(1-\beta)P_1^{(1)}}\boldsymbol{f}_{12}^{(1)}(W_{13}) && \beta \in [0,1] \\
\boldsymbol{X}_3^{(2)} &= \sqrt{\gamma P_3^{(2)}}\boldsymbol{f}_{31}^{(2)}(W_{31}) + \sqrt{(1-\gamma)P_3^{(2)}}\boldsymbol{f}_{32}^{(2)}(W_{31}) && \gamma \in [0,1] \\
\boldsymbol{X}_2^{(3)} &= \sqrt{\delta P_2^{(3)}}\boldsymbol{f}_{21}^{(3)}(W_{13}, W_{31}) + \sqrt{\zeta P_2^{(3)}}\boldsymbol{f}_{22}^{(3)}(Q_{23}) + \sqrt{(1-\delta-\zeta)P_2^{(3)}}\boldsymbol{f}_{23}^{(3)}(Q_{21}) && \delta, \zeta \in [0,1].
\end{aligned} \tag{3.60}$$

where $\delta + \zeta \le 1$. Like for CF superposition coding instead of DPC is assumed. The parameters β and γ distribute powers at nodes 1 and 3 to the PDF and CF strategies. δ, ζ control the power



(a) Symmetric

(b) Near

(c) Close

Figure 3.20: 3P-BC with CF, Line Network with $P_i^{(l)} = 10$, $\alpha = 3$

assignment at node 2 to the propagation of the decoded messages and the quantization indices. The quantized outputs at node 2 are

$$\begin{aligned}\hat{\boldsymbol{Y}}_2^{(1)} &= \boldsymbol{Y}_2^{(1)} + \hat{\boldsymbol{Z}}_2^{(1)} \\ \hat{\boldsymbol{Y}}_2^{(2)} &= \boldsymbol{Y}_2^{(2)} + \hat{\boldsymbol{Z}}_2^{(2)}\end{aligned} \tag{3.61}$$

with independent $\hat{Z}_{2,k}^{(1)} \sim \mathcal{N}_{\mathbb{C}}(0, \hat{\sigma}_1^2)$ and $\hat{Z}_{2,k}^{(2)} \sim \mathcal{N}_{\mathbb{C}}(0, \hat{\sigma}_2^2)$. This results in the rates

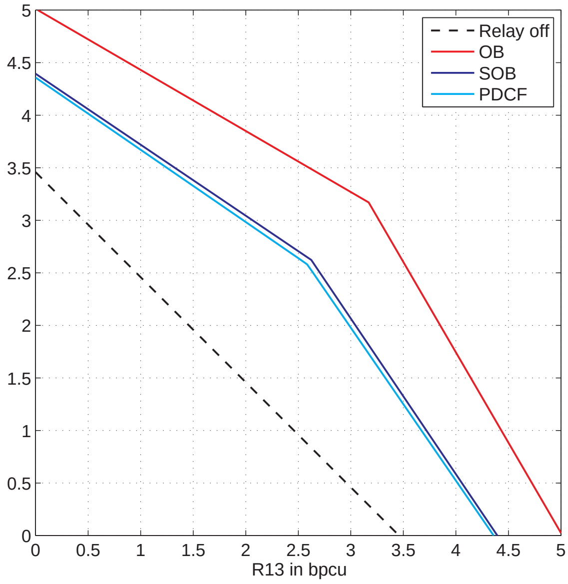
$$\begin{aligned}R_{13} \leq \min \Bigg\{ & \tau_1 \log\left(1 + \frac{|h_{13}|^2 \beta P_1^{(1)}}{1 + |h_{13}|^2 (1-\beta) P_1^{(1)}}\right) + \tau_3 \log\left(1 + \frac{|h_{23}|^2 \delta P_2^{(3)}}{1 + |h_{23}|^2 (1-\delta) P_2^{(3)}}\right), \\ & \tau_1 \log\left(1 + \frac{|h_{12}|^2 \beta P_1^{(1)}}{1 + |h_{12}|^2 (1-\beta) P_1^{(1)}}\right)\Bigg\} + \tau_1 \log\left(1 + \left(|h_{13}|^2 + \frac{|h_{12}|^2}{1+\hat{\sigma}_1^2}\right)(1-\beta) P_1^{(1)}\right) \\ R_{31} \leq \min \Bigg\{ & \tau_2 \log\left(1 + \frac{|h_{31}|^2 \gamma P_3^{(2)}}{1 + |h_{31}|^2 (1-\gamma) P_3^{(2)}}\right) + \tau_3 \log\left(1 + \frac{|h_{21}|^2 \delta P_2^{(3)}}{1 + |h_{21}|^2 (1-\delta) P_2^{(2)}}\right), \\ & \tau_2 \log\left(1 + \frac{|h_{32}|^2 \gamma P_3^{(2)}}{1 + |h_{32}|^2 (1-\gamma) P_3^{(2)}}\right)\Bigg\} + \tau_2 \log\left(1 + \left(|h_{31}|^2 + \frac{|h_{32}|^2}{1+\hat{\sigma}_2^2}\right)(1-\gamma) P_3^{(2)}\right)\end{aligned}$$

subject to

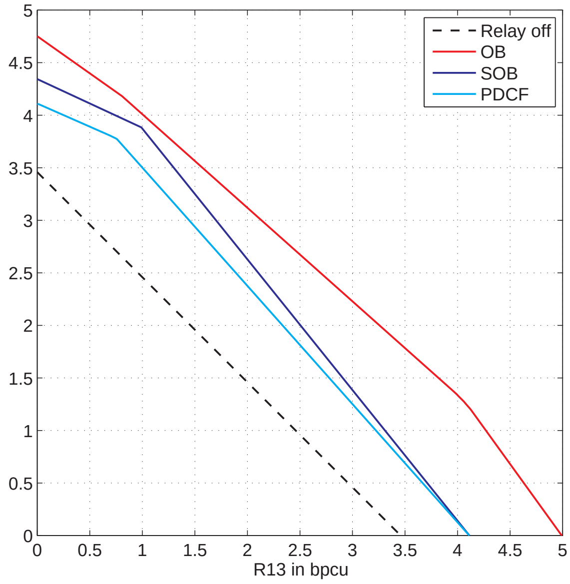
$$\begin{aligned}\tau_1 \log\left(1 + \frac{1}{\hat{\sigma}_1^2}\left(1 + \frac{|h_{12}|^2 (1-\beta) P_1^{(1)}}{1 + |h_{13}|^2 (1-\beta) P_1^{(1)}}\right)\right) &\leq \tau_3 \log\left(1 + \frac{|h_{23}|^2 \zeta P_2^{(3)}}{1 + |h_{23}|^2 (1-\zeta) P_2^{(3)}}\right) \\ \tau_2 \log\left(1 + \frac{1}{\hat{\sigma}_2^2}\left(1 + \frac{|h_{32}|^2 (1-\gamma) P_3^{(2)}}{1 + |h_{31}|^2 (1-\gamma) P_3^{(2)}}\right)\right) &\leq \tau_3 \log\left(1 + |h_{21}|^2 (1-\delta-\zeta) P_2^{(3)}\right).\end{aligned} \tag{3.62}$$

The configuration $\beta = \gamma = \delta = 1$ would result in a DF strategy whereas $\beta = \gamma = \delta = 0$ performs a CF strategy. Note that with $\beta = 1, \gamma = 0$ it is also possible to operate with different strategies in both directions. Further, there is the possibility to choose a superposition (with arbitrary weights) of CF and PDF strategies for both directions.

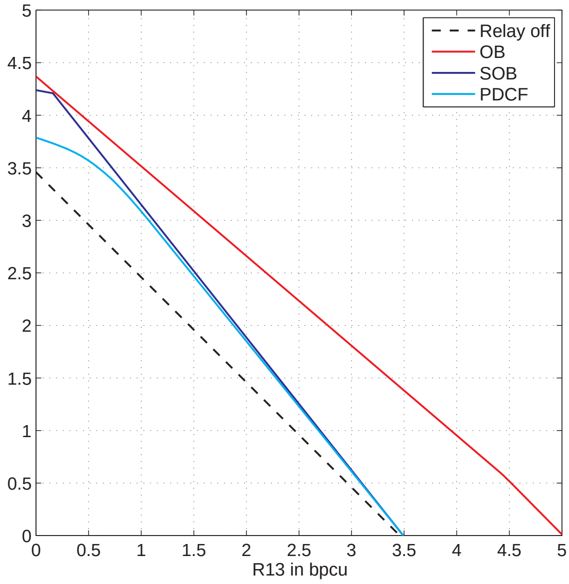
Simulations Figures 3.19-3.21 show the rate regions for the individual strategies. Figure 3.22 combines all plots. It shows that for the chosen model and parameters the scheme outer bound meets the general problem bound only at one point for very asymmetric channels. For symmetric channels the DF region is close to the scheme outer bound while for the asymmetric case it approaches the rate region without relay. CF outperforms DF for the "close" scenario. PDCF coincides with the region of the best strategy in each setting.



(a) Symmetric

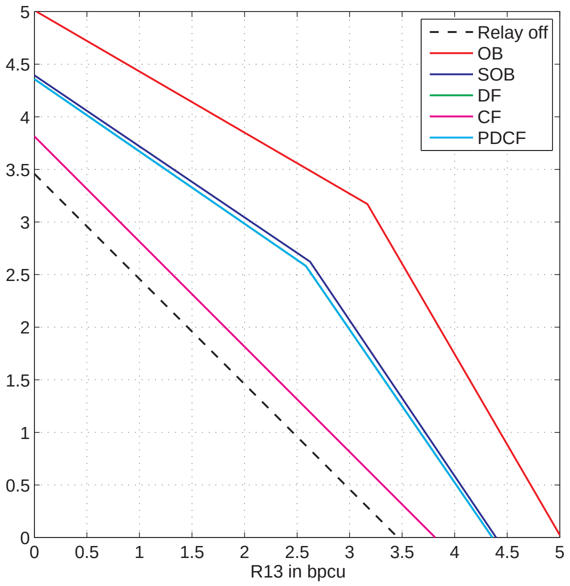


(b) Near

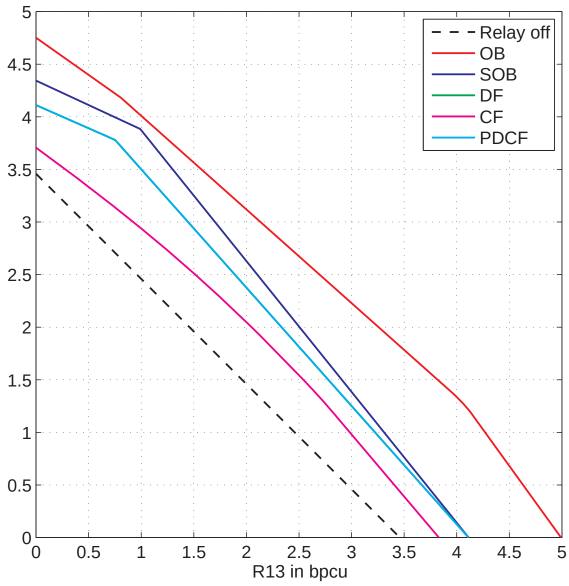


(c) Close

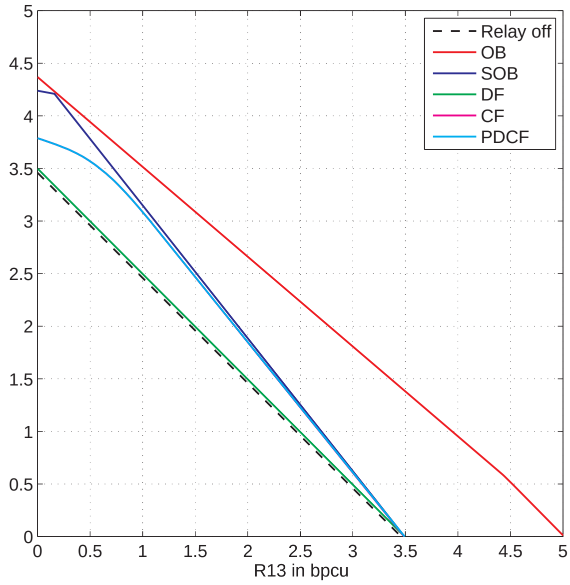
Figure 3.21: 3P-BC with PDCF, Line Network with $P_i^{(l)} = 10$, $\alpha = 3$



(a) Symmetric



(b) Near



(c) Close

Figure 3.22: 3P-BC, Line Network with $P_i^{(l)} = 10$, $\alpha = 3$

3.4.3 3P-MA Scheme

For all 3P-MA strategies the output sequences are given by

$$\begin{aligned}\boldsymbol{Y}_2^{(1)} &= h_{12}\boldsymbol{X}_1^{(1)} + h_{32}\boldsymbol{X}_3^{(1)} + \boldsymbol{Z}_2^{(1)}\\ \boldsymbol{Y}_3^{(2)} &= h_{13}\boldsymbol{X}_1^{(2)} + h_{23}\boldsymbol{X}_2^{(2)} + \boldsymbol{Z}_3^{(2)}\\ \boldsymbol{Y}_1^{(3)} &= h_{31}\boldsymbol{X}_3^{(3)} + h_{21}\boldsymbol{X}_2^{(3)} + \boldsymbol{Z}_1^{(3)}.\end{aligned} \tag{3.63}$$

Outer Bound For the outer bound of this three-phase scheme the input sequences are given by

$$\begin{aligned}\boldsymbol{X}_1^{(1)} &= \sqrt{P_1^{(1)}}\boldsymbol{f}_{11}^{(1)}(W_{13})\\ \boldsymbol{X}_3^{(1)} &= \sqrt{P_3^{(1)}}\boldsymbol{f}_{31}^{(1)}(W_{31})\\ \boldsymbol{X}_2^{(2)} &= \sqrt{P_2^{(2)}}\boldsymbol{f}_{21}^{(2)}(W_{13})\\ \boldsymbol{X}_1^{(2)} &= \underbrace{\sqrt{\beta P_1^{(2)}}\boldsymbol{f}_{21}^{(2)}(W_{13})}_{\sqrt{\beta P_1^{(2)}/P_2^{(2)}}\boldsymbol{X}_2^{(2)}} + \sqrt{(1-\beta)P_1^{(2)}}\boldsymbol{f}_{11}^{(2)}(W_{13}) \qquad \beta \in [0,1]\\ \boldsymbol{X}_2^{(3)} &= \sqrt{P_2^{(3)}}\boldsymbol{f}_{22}^{(3)}(W_{31})\\ \boldsymbol{X}_3^{(3)} &= \underbrace{\sqrt{\gamma P_3^{(3)}}\boldsymbol{f}_{22}^{(3)}(W_{31})}_{\sqrt{\gamma P_3^{(3)}/P_2^{(3)}}\boldsymbol{X}_2^{(3)}} + \sqrt{(1-\gamma)P_3^{(3)}}\boldsymbol{f}_{31}^{(3)}(W_{31}) \qquad \gamma \in [0,1]\end{aligned} \tag{3.64}$$

with the parameters β, γ determining the dependence between the inputs at nodes 1 or 3 and node 2. This gives the outer rate bound expressions

$$\begin{aligned}R_{13} \le \min\Big\{&\tau_1 \log\left(1 + |h_{12}|^2 P_1^{(1)}\right) + \tau_2 \log\left(1 + |h_{13}|^2 (1-\beta)P_1^{(2)}\right),\\ &\tau_2 \log\left(1 + |h_{13}|^2 P_1^{(2)} + |h_{23}|^2 P_2^{(2)} + 2\,|h_{13}h_{23}|\sqrt{\beta P_1^{(2)}P_2^{(2)}}\right)\Big\}\\ R_{31} \le \min\Big\{&\tau_1 \log\left(1 + |h_{32}|^2 P_3^{(1)}\right) + \tau_3 \log\left(1 + |h_{31}|^2 (1-\gamma)P_3^{(3)}\right),\\ &\tau_3 \log\left(1 + |h_{21}|^2 P_2^{(3)} + |h_{31}|^2 P_3^{(3)} + 2\,|h_{21}h_{31}|\sqrt{\gamma P_2^{(3)}P_3^{(3)}}\right)\Big\}.\end{aligned} \tag{3.65}$$

For the case $P_2^{(2)}, P_2^{(3)} \to \infty$ the bound tends to

$$\begin{aligned}R_{13} &\to \tau_1 \log\left(1 + |h_{12}|^2 P_1^{(1)}\right) + \tau_2 \log\left(1 + |h_{13}|^2 P_1^{(2)}\right)\\ R_{31} &\to \tau_1 \log\left(1 + |h_{32}|^2 P_3^{(1)}\right) + \tau_3 \log\left(1 + |h_{31}|^2 P_3^{(3)}\right).\end{aligned} \tag{3.66}$$

Decode-and-Forward With the same input parameterization as for the outer bound the achievable rates with a DF strategy are

$$
\begin{aligned}
R_{13} \leq \min\Big\{&\tau_1 \log\Big(1+|h_{12}|^2 P_1^{(1)}\Big) + \tau_2 \log\Big(1+|h_{13}|^2 (1-\beta) P_1^{(2)}\Big),\\
&\tau_2 \log\Big(1+|h_{13}|^2 P_1^{(2)} + |h_{23}|^2 P_2^{(2)} + 2\,|h_{13}h_{23}|\sqrt{\beta P_1^{(2)} P_2^{(2)}}\Big)\Big\}\\
R_{31} \leq \min\Big\{&\tau_1 \log\Big(1+|h_{32}|^2 P_3^{(1)}\Big) + \tau_3 \log\Big(1+|h_{31}|^2 (1-\gamma) P_3^{(3)}\Big),\\
&\tau_3 \log\Big(1+|h_{21}|^2 P_2^{(3)} + |h_{31}|^2 P_3^{(3)} + 2\,|h_{21}h_{31}|\sqrt{\gamma P_2^{(3)} P_3^{(3)}}\Big)\Big\}\\
R_{13}+R_{31} \leq &\tau_1 \log\Big(1+|h_{12}|^2 P_1^{(1)} + |h_{32}|^2 P_3^{(1)}\Big) + \tau_2 \log\Big(1+|h_{13}|^2 (1-\beta) P_1^{(2)}\Big) +\\
&\tau_3 \log\Big(1+|h_{31}|^2 (1-\gamma) P_3^{(3)}\Big).
\end{aligned} \tag{3.67}
$$

The parameters $\beta, \gamma \in [0;1]$ assign the powers at nodes 1 or 3 to the support of the input at node 2.

Compress-and-Forward For the CF strategy the input sequence can be assumed to have the form

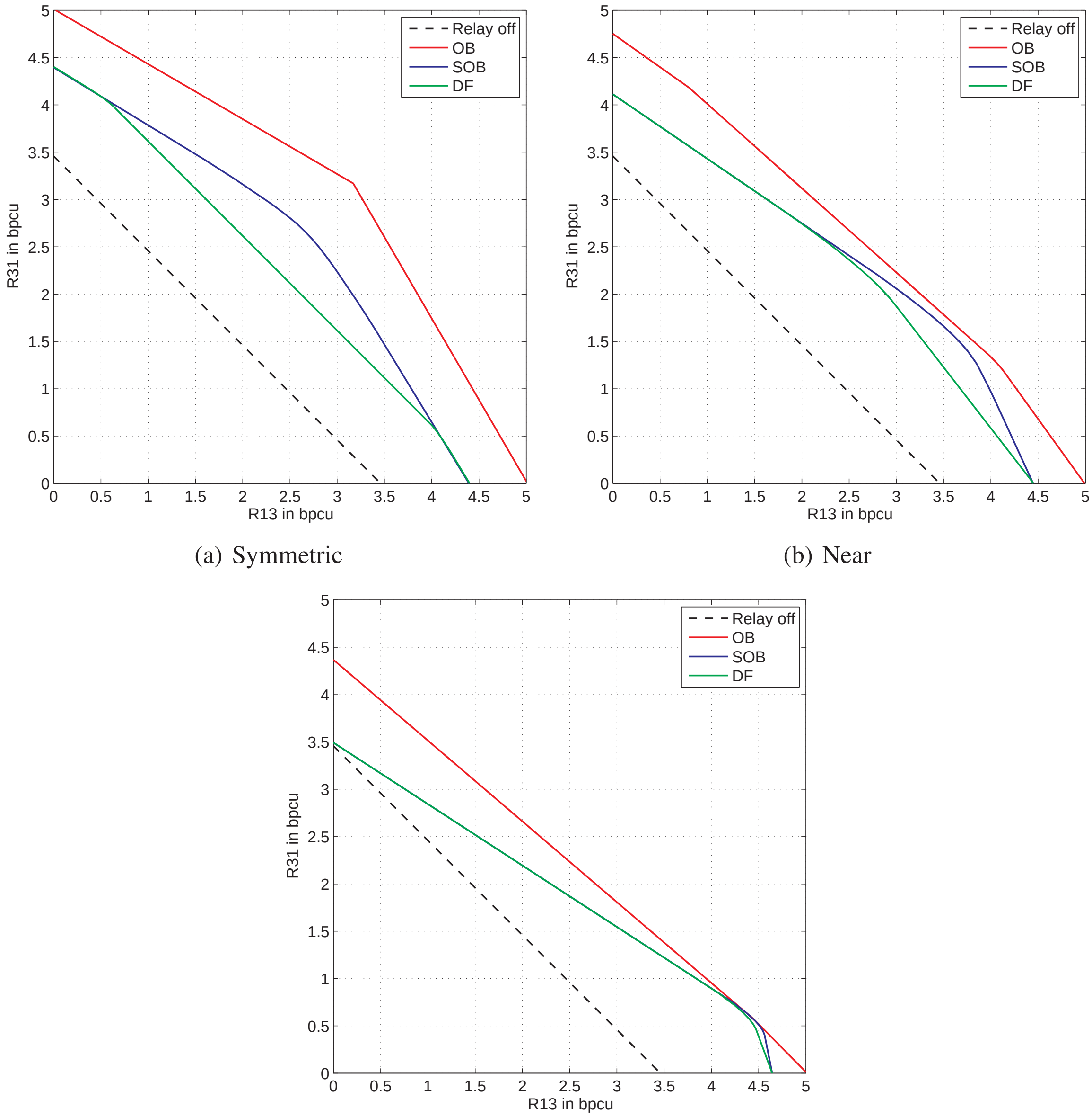
$$
\begin{aligned}
\boldsymbol{X}_1^{(1)} &= \sqrt{P_1^{(1)}}\boldsymbol{f}_{11}^{(1)}(W_{13})\\
\boldsymbol{X}_3^{(1)} &= \sqrt{P_3^{(1)}}\boldsymbol{f}_{31}^{(1)}(W_{31})\\
\boldsymbol{X}_2^{(2)} &= \sqrt{P_2^{(2)}}\boldsymbol{f}_{21}^{(2)}(Q_{23})\\
\boldsymbol{X}_1^{(2)} &= \sqrt{P_1^{(2)}}\boldsymbol{f}_{12}^{(2)}(W_{13})\\
\boldsymbol{X}_2^{(3)} &= \sqrt{P_2^{(3)}}\boldsymbol{f}_{22}^{(3)}(Q_{21})\\
\boldsymbol{X}_3^{(3)} &= \sqrt{P_3^{(3)}}\boldsymbol{f}_{32}^{(3)}(W_{31}).
\end{aligned} \tag{3.68}
$$

The quantized outputs at node 2 have the form

$$
\begin{aligned}
\hat{\boldsymbol{Y}}_{21}^{(1)} &= \boldsymbol{Y}_2^{(1)} + \hat{\boldsymbol{Z}}_1^{(1)}\\
\hat{\boldsymbol{Y}}_{22}^{(1)} &= \boldsymbol{Y}_2^{(1)} + \hat{\boldsymbol{Z}}_2^{(1)}
\end{aligned} \tag{3.69}
$$

with $\hat{Z}_{2,k}^{(1)} \sim \mathcal{N}_{\mathbb{C}}(0, \hat{\sigma}_1^2)$ and $\hat{Z}_{2,k}^{(2)} \sim \mathcal{N}_{\mathbb{C}}(0, \hat{\sigma}_2^2)$. This yields the achievable rate expressions

$$
\begin{aligned}
R_{13} &\leq \tau_1 \log\left(1+\frac{|h_{12}|^2 P_1^{(1)}}{1+\hat{\sigma}_1^2}\right) + \tau_2 \log\Big(1+|h_{13}|^2 P_1^{(2)}\Big)\\
R_{31} &\leq \tau_1 \log\left(1+\frac{|h_{32}|^2 P_3^{(1)}}{1+\hat{\sigma}_2^2}\right) + \tau_3 \log\Big(1+|h_{31}|^2 P_3^{(3)}\Big)
\end{aligned}
$$



(a) Symmetric (b) Near

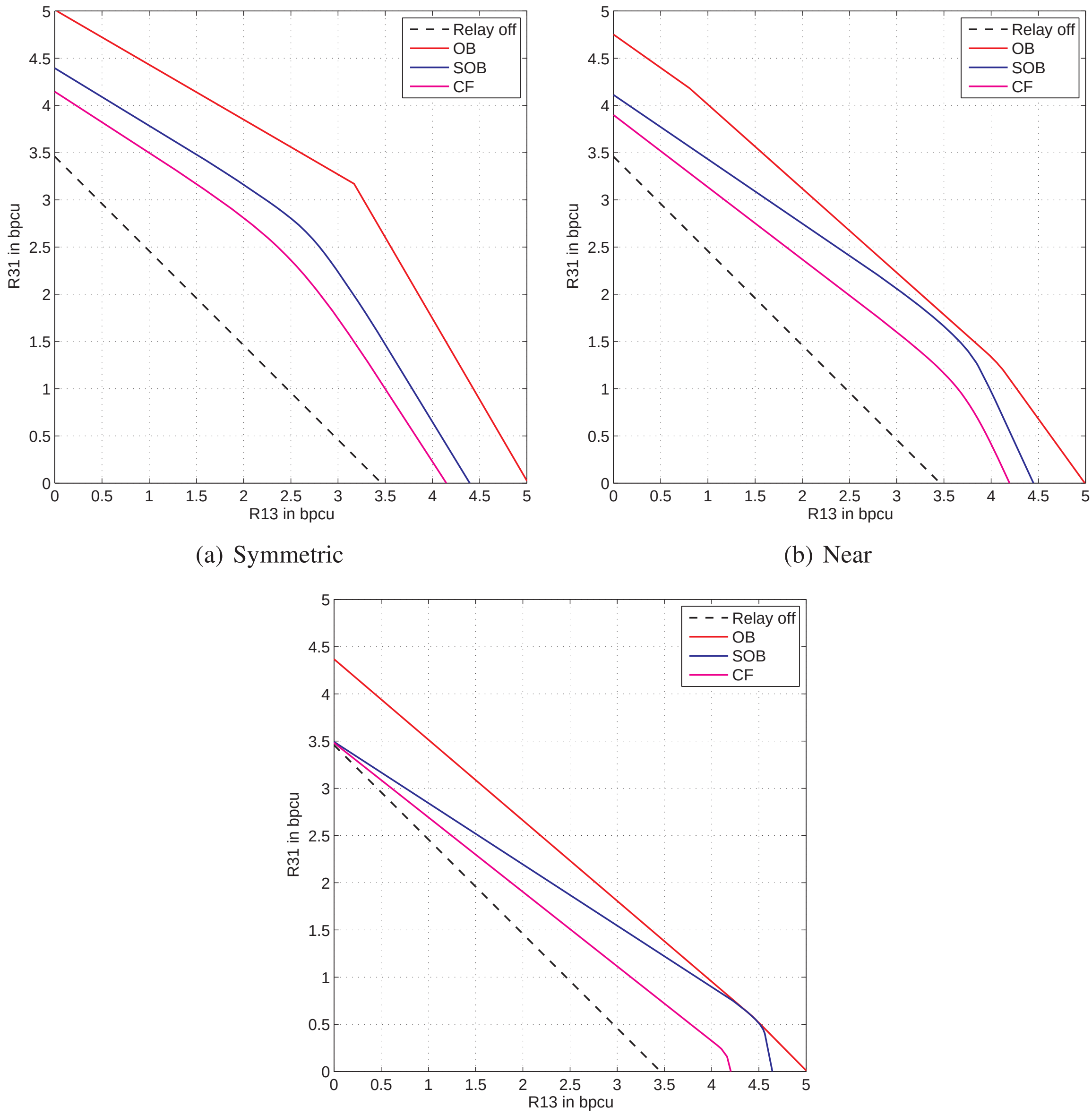
(c) Close

Figure 3.23: 3P-MA with DF, Line Network with $P_i^{(l)} = 10$, $\alpha = 3$

subject to

$$\begin{aligned}
\tau_1 \log\left(1+\frac{1+|h_{12}|^2 P_1^{(1)}}{\hat{\sigma}_1^2}\right) &\leq \tau_2 \log\left(1+\frac{|h_{23}|^2 P_2^{(2)}}{1+|h_{13}|^2 P_1^{(2)}}\right)\\
\tau_1 \log\left(1+\frac{1+|h_{32}|^2 P_3^{(1)}}{\hat{\sigma}_2^2}\right) &\leq \tau_3 \log\left(1+\frac{|h_{21}|^2 P_2^{(3)}}{1+|h_{31}|^2 P_3^{(3)}}\right)
\end{aligned} \tag{3.70}$$

which coincide asymptotically for the case $P_2^{(2)}, P_2^{(3)} \to \infty$ with the outer bound of the 3P-MA scheme as $\hat{\sigma}_1^2, \hat{\sigma}_2^2 \to 0$.



(a) Symmetric

(b) Near

(c) Close

Figure 3.24: 3P-MA with CF, Line Network with $P_i^{(l)} = 10$, $\alpha = 3$

Partial-Decode-Compress-and-Forward For this strategy the input sequences have the form

$$
\begin{aligned}
\boldsymbol{X}_1^{(1)} &= \underbrace{\sqrt{\beta P_1^{(1)}}\boldsymbol{f}_{11}^{(1)}(W_{13})}_{\boldsymbol{U}_1^{(1)}} + \sqrt{(1-\beta)P_1^{(1)}}\boldsymbol{f}_{12}^{(1)}(W_{13}) && \beta \in [0,1] \\
\boldsymbol{X}_3^{(1)} &= \underbrace{\sqrt{\gamma P_3^{(1)}}\boldsymbol{f}_{31}^{(1)}(W_{31})}_{\boldsymbol{U}_3^{(1)}} + \sqrt{(1-\gamma)P_3^{(1)}}\boldsymbol{f}_{32}^{(1)}(W_{31}) && \gamma \in [0,1] \\
\boldsymbol{X}_2^{(2)} &= \underbrace{\sqrt{\delta P_2^{(2)}}\boldsymbol{f}_{21}^{(2)}(W_{13})}_{\boldsymbol{U}_2^{(2)}} + \sqrt{(1-\delta)P_2^{(2)}}\boldsymbol{f}_{22}^{(2)}(Q_{23}) && \delta \in [0,1] \\
\boldsymbol{X}_1^{(2)} &= \underbrace{\sqrt{\zeta P_1^{(2)}}\boldsymbol{f}_{21}^{(2)}(W_{13})}_{\sqrt{\zeta P_1^{(2)}/\delta P_2^{(2)}}\boldsymbol{U}_2^{(2)}} + \sqrt{(1-\zeta)P_1^{(2)}}\boldsymbol{f}_{11}^{(2)}(W_{13}) && \zeta \in [0,1] \\
\boldsymbol{X}_2^{(3)} &= \underbrace{\sqrt{\theta P_2^{(3)}}\boldsymbol{f}_{21}^{(3)}(W_{31})}_{\boldsymbol{U}_2^{(3)}} + \sqrt{(1-\theta)P_2^{(3)}}\boldsymbol{f}_{22}^{(3)}(Q_{21}) && \theta \in [0,1] \\
\boldsymbol{X}_3^{(3)} &= \underbrace{\sqrt{\kappa P_3^{(3)}}\boldsymbol{f}_{21}^{(3)}(W_{31})}_{\sqrt{\kappa P_3^{(3)}/\theta P_2^{(3)}}\boldsymbol{U}_2^{(3)}} + \sqrt{(1-\kappa)P_3^{(3)}}\boldsymbol{f}_{31}^{(3)}(W_{31}) && \kappa \in [0,1].
\end{aligned}
\tag{3.71}
$$

The parameters β and γ divide the powers at nodes 1 and 3 to the DF and CF strategy, δ, θ determine the powers invested into the propagation of the decoded messages and quantization indices at node 2 and ζ, κ control the support of node 2 through nodes 1 and 3. The quantized outputs at node 2 have the form

$$
\begin{aligned}
\hat{\boldsymbol{Y}}_{21}^{(1)} &= \boldsymbol{Y}_2^{(1)} + \hat{\boldsymbol{Z}}_1^{(1)} \\
\hat{\boldsymbol{Y}}_{22}^{(1)} &= \boldsymbol{Y}_2^{(1)} + \hat{\boldsymbol{Z}}_2^{(1)}
\end{aligned}
\tag{3.72}
$$

with independent $\hat{Z}_{2,k}^{(1)} \sim \mathcal{N}_{\mathbb{C}}(0, \hat{\sigma}_1^2)$ and $\hat{Z}_{2,k}^{(2)} \sim \mathcal{N}_{\mathbb{C}}(0, \hat{\sigma}_2^2)$. This yields the achievable rates

$$
\begin{aligned}
R_{13} \leq \min\Bigg\{ &\tau_1 \log\left(1 + \frac{|h_{12}|^2 \beta P_1^{(1)}}{1 + |h_{12}|^2 (1-\beta)P_1^{(1)} + |h_{32}|^2 (1-\gamma)P_3^{(1)}}\right), \\
&\tau_2 \log\left(1 + \frac{|h_{13}|^2 \zeta P_1^{(2)} + |h_{23}|^2 \delta P_2^{(2)} + 2\,|h_{13}h_{23}|\sqrt{\delta\zeta P_1^{(2)} P_2^{(2)}}}{1 + |h_{13}|^2 (1-\zeta)P_1^{(2)} + |h_{23}|^2 (1-\delta)P_2^{(2)}}\right)\Bigg\} + \\
&\tau_1 \log\left(1 + \frac{|h_{12}|^2 (1-\beta)P_1^{(1)}}{1 + \hat{\sigma}_1^2}\right) + \tau_2 \log\left(1 + (1-\zeta)\,|h_{13}|^2 P_1^{(2)}\right)
\end{aligned}
$$

$$R_{31} \leq \min\Bigg\{\tau_1 \log\left(1 + \frac{|h_{32}|^2 \gamma P_3^{(1)}}{1 + |h_{32}|^2 (1-\gamma) P_3^{(1)} + |h_{12}|^2 (1-\beta) P_1^{(1)}}\right),$$
$$\tau_3 \log\left(1 + \frac{|h_{21}|^2 \theta P_2^{(3)} + |h_{31}|^2 \kappa P_3^{(3)} + 2\,|h_{21} h_{31}| \sqrt{\theta\kappa P_2^{(3)} P_3^{(3)}}}{1 + |h_{21}|^2 (1-\theta) P_2^{(3)} + |h_{31}|^2 (1-\kappa) P_3^{(3)}}\right)\Bigg\} +$$
$$\tau_1 \log\left(1 + \frac{|h_{32}|^2 (1-\gamma) P_3^{(1)}}{1 + \hat{\sigma}_2^2}\right) + \tau_3 \log\left(1 + (1-\kappa)\,|h_{31}|^2 P_3^{(3)}\right)$$

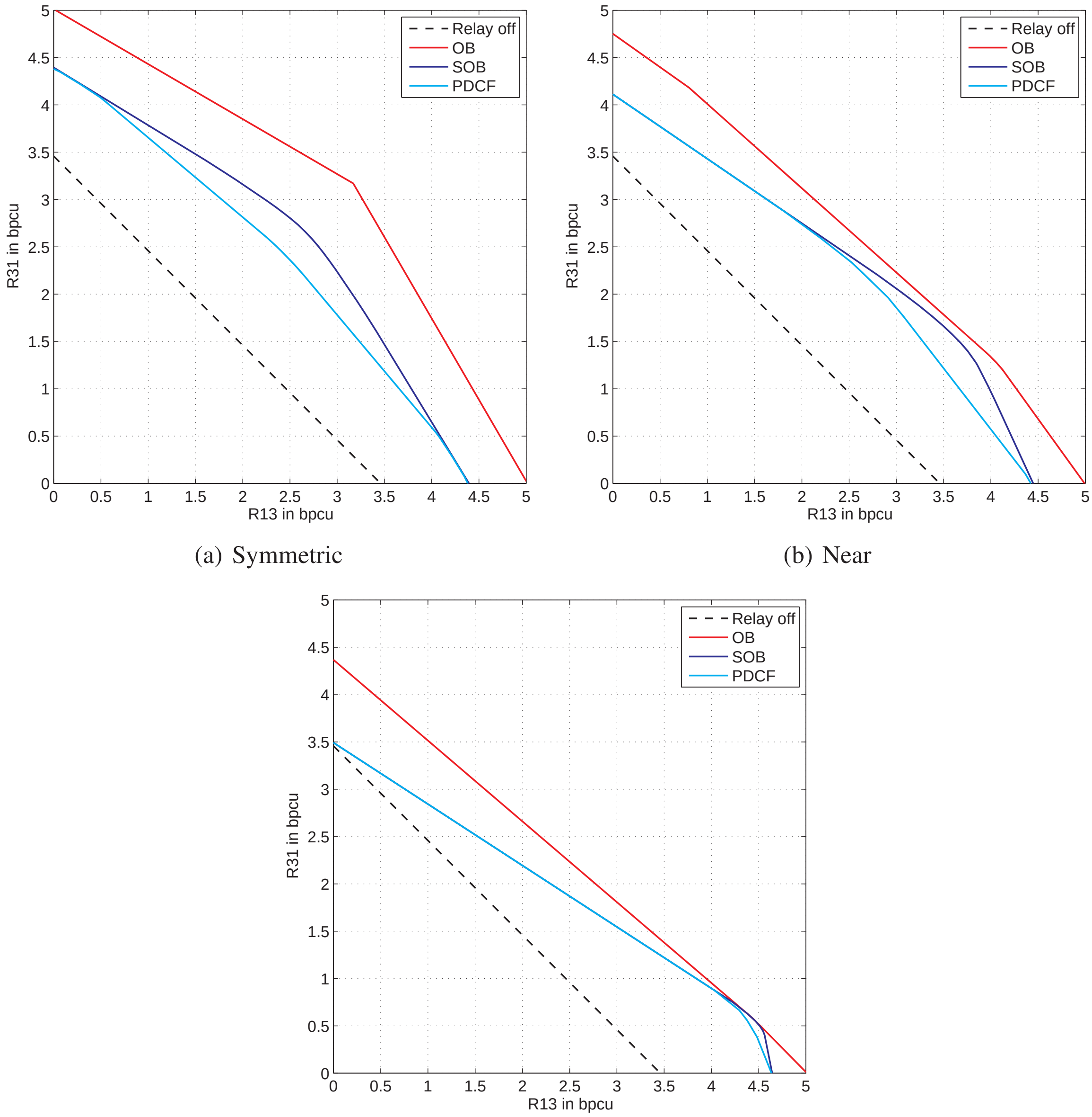
$$R_{13} + R_{31} \leq \tau_1 \log\left(1 + \frac{|h_{12}|^2 \beta P_1^{(1)} + |h_{32}|^2 \gamma P_3^{(1)}}{1 + |h_{12}|^2 (1-\beta) P_1^{(1)} + |h_{32}|^2 (1-\gamma) P_3^{(1)}}\right) +$$
$$\tau_1 \log\left(1 + \frac{|h_{12}|^2 (1-\beta) P_1^{(1)}}{1 + \hat{\sigma}_1^2}\right) + \tau_1 \log\left(1 + \frac{|h_{32}|^2 (1-\gamma) P_3^{(1)}}{1 + \hat{\sigma}_2^2}\right) +$$
$$\tau_2 \log\left(1 + |h_{13}|^2 (1-\zeta) P_1^{(2)}\right) + \tau_3 \log\left(1 + (1-\kappa)\,|h_{31}|^2 P_3^{(3)}\right)$$

subject to

$$\tau_1 \log\left(1 + \frac{1 + |h_{12}|^2 (1-\beta) P_1^{(1)}}{\hat{\sigma}_1^2}\right) \leq \tau_2 \log\left(1 + \frac{|h_{23}|^2 (1-\delta) P_2^{(2)}}{1 + |h_{13}|^2 (1-\zeta) P_1^{(2)}}\right)$$
$$\tau_1 \log\left(1 + \frac{1 + |h_{32}|^2 (1-\gamma) P_3^{(1)}}{\hat{\sigma}_2^2}\right) \leq \tau_3 \log\left(1 + \frac{|h_{21}|^2 (1-\theta) P_2^{(3)}}{1 + |h_{31}|^2 (1-\kappa) P_3^{(3)}}\right). \quad (3.73)$$

Like with the other PDCF schemes note the possibility to mix strategies individually for both directions.

Simulations Figures 3.23-3.25 show the rate regions for the individual strategies. Figure 3.26 combines all plots. For the chosen model and parameters simulations it reveals that the scheme outer bound meets the problem outer bound only at one point for very asymmetric channels. For symmetric channels the DF region is bounded away from the scheme outer bound by the MAC sum-rate constraint while it approaches this bound for asymmetric channels. The CF region shows the same shape as the scheme outer bound in the "symmetric" scenario. PDCF yields the convex hull of the rate points achievable with CF or DF.

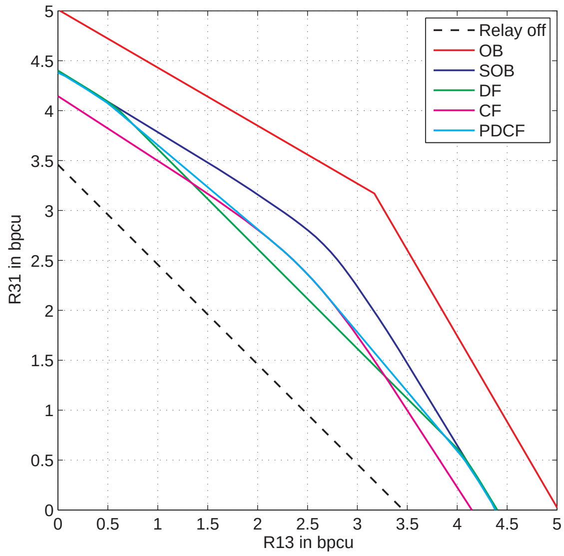


(a) Symmetric

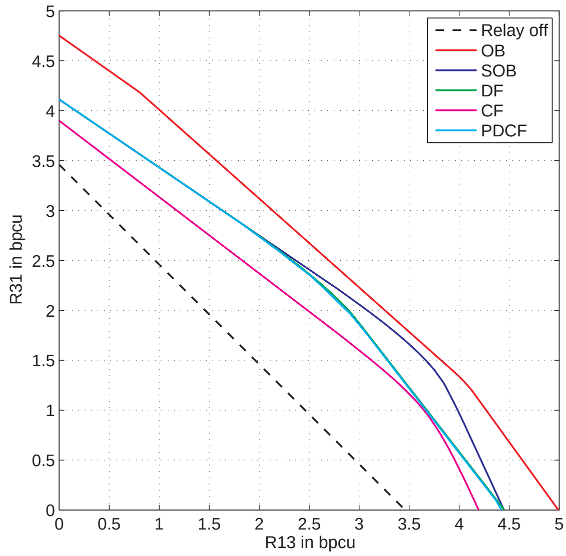
(b) Near

(c) Close

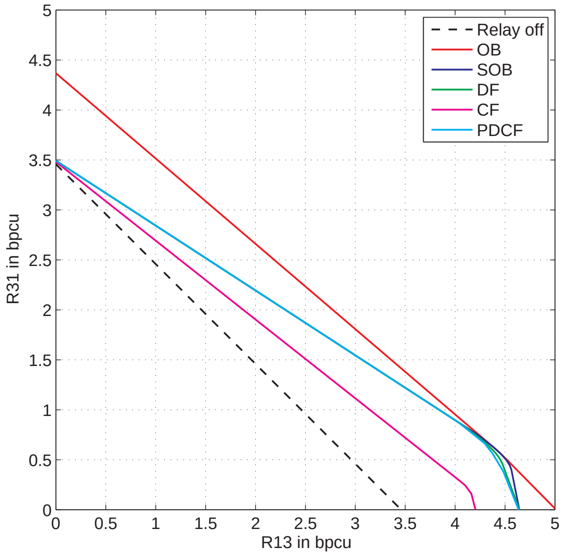
Figure 3.25: 3P-MA with PDCF, Line Network with $P_i^{(l)} = 10$, $\alpha = 3$



(a) Symmetric



(b) Near



(c) Close

Figure 3.26: 3P-MA, Line Network with $P_i^{(l)} = 10$, $\alpha = 3$

3.4.4 4P-OWRC Scheme

The output sequences of the four phase scheme are

$$\begin{aligned}
\boldsymbol{Y}_2^{(1)} &= h_{12}\boldsymbol{X}_1^{(1)} + \boldsymbol{Z}_2^{(1)} \\
\boldsymbol{Y}_3^{(1)} &= h_{13}\boldsymbol{X}_1^{(1)} + \boldsymbol{Z}_3^{(1)} \\
\boldsymbol{Y}_3^{(2)} &= h_{13}\boldsymbol{X}_1^{(2)} + h_{23}\boldsymbol{X}_2^{(2)} + \boldsymbol{Z}_3^{(2)} \\
\boldsymbol{Y}_2^{(3)} &= h_{32}\boldsymbol{X}_3^{(3)} + \boldsymbol{Z}_2^{(3)} \\
\boldsymbol{Y}_1^{(3)} &= h_{31}\boldsymbol{X}_3^{(3)} + \boldsymbol{Z}_1^{(3)} \\
\boldsymbol{Y}_1^{(4)} &= h_{31}\boldsymbol{X}_3^{(4)} + h_{21}\boldsymbol{X}_2^{(4)} + \boldsymbol{Z}_1^{(4)}.
\end{aligned} \tag{3.74}$$

Outer Bound For the outer bound expression the input sequences are of the form

$$\begin{aligned}
\boldsymbol{X}_1^{(1)} &= \sqrt{P_1^{(1)}}\boldsymbol{f}_{11}^{(1)}(W_{13}) \\
\boldsymbol{X}_1^{(2)} &= \underbrace{\sqrt{\beta P_1^{(2)}}\boldsymbol{f}_{21}^{(2)}(W_{13})}_{\sqrt{\beta P_1^{(2)}/P_2^{(2)}}\boldsymbol{X}_2^{(2)}} + \sqrt{(1-\beta)P_1^{(2)}}\boldsymbol{f}_{11}^{(2)}(W_{13}) && \beta \in [0;1] \\
\boldsymbol{X}_2^{(2)} &= \sqrt{P_2^{(2)}}\boldsymbol{f}_{21}^{(2)}(W_{13}) \\
\boldsymbol{X}_3^{(3)} &= \sqrt{P_3^{(3)}}\boldsymbol{f}_{31}^{(3)}(W_{31}) \\
\boldsymbol{X}_3^{(4)} &= \underbrace{\sqrt{\gamma P_3^{(4)}}\boldsymbol{f}_{21}^{(4)}(W_{31})}_{\sqrt{\gamma P_3^{(4)}/P_2^{(4)}}\boldsymbol{X}_2^{(4)}} + \sqrt{(1-\gamma)P_3^{(4)}}\boldsymbol{f}_{31}^{(4)}(W_{31}) && \gamma \in [0;1] \\
\boldsymbol{X}_2^{(4)} &= \sqrt{P_2^{(4)}}\boldsymbol{f}_{21}^{(4)}(W_{31})
\end{aligned} \tag{3.75}$$

with the parameters β, γ determining the dependence between the inputs at node 1 or 3 and node 2. The outer bound on the achievable rates is

$$\begin{aligned}
R_{13} \le \min\Big\{ &\tau_1 \log\left(1 + (|h_{12}|^2 + |h_{13}|^2)P_1^{(1)}\right) + \tau_2 \log\left(1 + |h_{13}|^2 (1-\beta)P_1^{(2)}\right), \\
&\tau_1 \log\left(1 + |h_{13}|^2 P_1^{(1)}\right) + \tau_2 \log\left(1 + |h_{13}|^2 P_1^{(2)} + |h_{23}|^2 P_2^{(2)} + 2\,|h_{13}h_{23}| \sqrt{\beta P_1^{(2)} P_2^{(2)}}\right)\Big\} \\
R_{31} \le \min\Big\{ &\tau_3 \log\left(1 + (|h_{32}|^2 + |h_{31}|^2)P_3^{(3)}\right) + \tau_4 \log\left(1 + |h_{31}|^2 (1-\gamma)P_3^{(4)}\right), \\
&\tau_3 \log\left(1 + |h_{31}|^2 P_3^{(3)}\right) + \tau_4 \log\left(1 + |h_{31}|^2 P_3^{(4)} + |h_{21}|^2 P_2^{(4)} + 2\,|h_{31}h_{21}| \sqrt{\gamma P_2^{(4)} P_3^{(4)}}\right)\Big\}.
\end{aligned} \tag{3.76}$$

Decode-and-Forward The input parameterization for the outer bound gives the achievable rates of the DF strategy

$$\begin{aligned}R_{13} \leq \min\Big\{&\tau_1 \log\left(1+|h_{12}|^2 P_1^{(1)}\right) + \tau_2 \log\left(1+|h_{13}|^2 (1-\beta)P_1^{(2)}\right),\\ &\tau_1 \log\left(1+|h_{13}|^2 P_1^{(1)}\right) + \tau_2 \log\left(1+|h_{13}|^2 P_1^{(2)} + |h_{23}|^2 P_2^{(2)} + 2|h_{13}h_{23}|\sqrt{\beta P_1^{(2)} P_2^{(2)}}\right)\Big\}\\ R_{31} \leq \min\Big\{&\tau_3 \log\left(1+|h_{32}|^2 P_3^{(3)}\right) + \tau_4 \log\left(1+|h_{31}|^2 (1-\gamma)P_3^{(4)}\right),\\ &\tau_3 \log\left(1+|h_{31}|^2 P_3^{(3)}\right) + \tau_4 \log\left(1+|h_{31}|^2 P_3^{(4)} + |h_{21}|^2 P_2^{(4)} + 2|h_{31}h_{21}|\sqrt{\gamma P_2^{(4)} P_3^{(4)}}\right)\Big\}.\end{aligned} \tag{3.77}$$

Compress-and-Forward For the CF strategy the inputs are

$$\begin{aligned}\boldsymbol{X}_1^{(1)} &= \sqrt{P_1^{(1)}} f_{11}^{(1)}(W_{13})\\ \boldsymbol{X}_1^{(2)} &= \sqrt{P_1^{(2)}} f_{11}^{(2)}(W_{13})\\ \boldsymbol{X}_2^{(2)} &= \sqrt{P_2^{(2)}} f_{21}^{(2)}(Q_{23})\\ \boldsymbol{X}_3^{(3)} &= \sqrt{P_3^{(3)}} f_{31}^{(3)}(W_{31})\\ \boldsymbol{X}_3^{(4)} &= \sqrt{P_3^{(4)}} f_{31}^{(4)}(W_{31})\\ \boldsymbol{X}_2^{(4)} &= \sqrt{P_2^{(4)}} f_{21}^{(4)}(Q_{21})\end{aligned} \tag{3.78}$$

with the quantized outputs at node 2

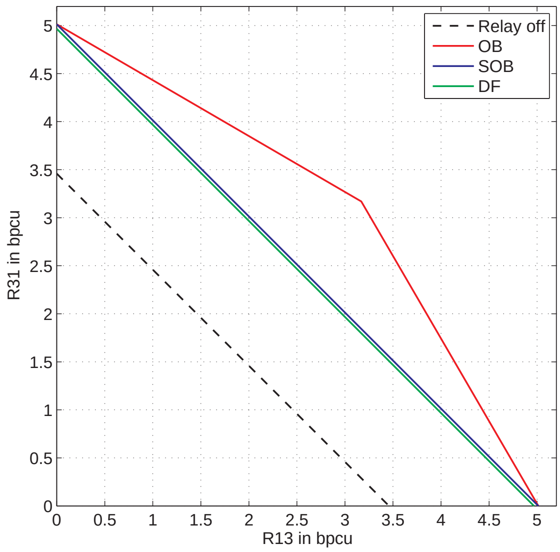
$$\begin{aligned}\hat{Y}_2^{(1)} &= Y_2^{(1)} + \hat{Z}_2^{(1)}\\ \hat{Y}_2^{(3)} &= Y_2^{(3)} + \hat{Z}_2^{(3)}.\end{aligned} \tag{3.79}$$

with independent $\hat{Z}_{2,k}^{(1)} \sim \mathcal{N}_{\mathbb{C}}(0, \hat{\sigma}_1^2)$ and $\hat{Z}_{2,k}^{(3)} \sim \mathcal{N}_{\mathbb{C}}(0, \hat{\sigma}_2^2)$. The achievable rates are

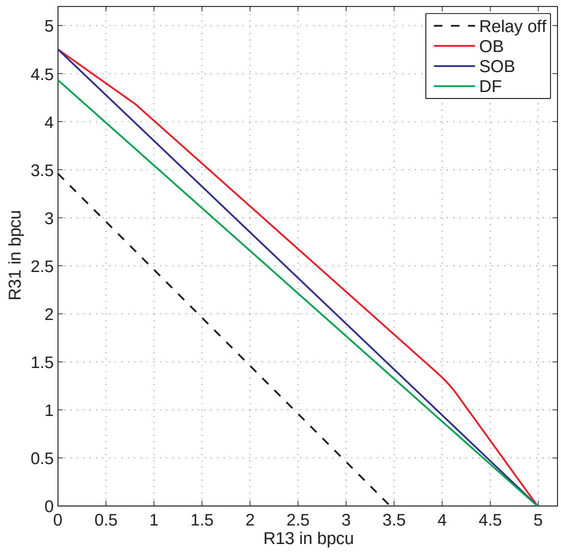
$$\begin{aligned}R_{13} &\leq \tau_1 \log\left(1+|h_{13}|^2 P_1^{(1)} + \frac{|h_{12}|^2 P_1^{(1)}}{1+\hat{\sigma}_1^2}\right) + \tau_2 \log\left(1+|h_{13}|^2 P_1^{(2)}\right)\\ R_{31} &\leq \tau_3 \log\left(1+|h_{31}|^2 P_3^{(3)} + \frac{|h_{32}|^2 P_3^{(3)}}{1+\hat{\sigma}_2^2}\right) + \tau_4 \log\left(1+|h_{31}|^2 P_3^{(4)}\right)\end{aligned} \tag{3.80}$$

subject to

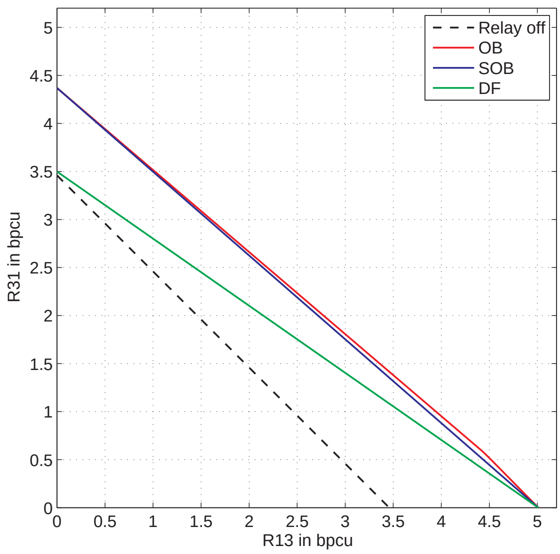
$$\begin{aligned}\tau_1 \log\left(1+\frac{1}{\hat{\sigma_1}^2}\left(1+\frac{|h_{12}|^2 P_1^{(1)}}{1+|h_{13}|^2 P_1^{(1)}}\right)\right) &\leq \tau_2 \log\left(1+\frac{|h_{23}|^2 P_2^{(2)}}{1+|h_{13}|^2 P_1^{(2)}}\right)\\ \tau_3 \log\left(1+\frac{1}{\hat{\sigma_2}^2}\left(1+\frac{|h_{32}|^2 P_3^{(3)}}{1+|h_{31}|^2 P_3^{(3)}}\right)\right) &\leq \tau_4 \log\left(1+\frac{|h_{21}|^2 P_2^{(4)}}{1+|h_{31}|^2 P_3^{(4)}}\right).\end{aligned} \tag{3.81}$$



(a) Symmetric

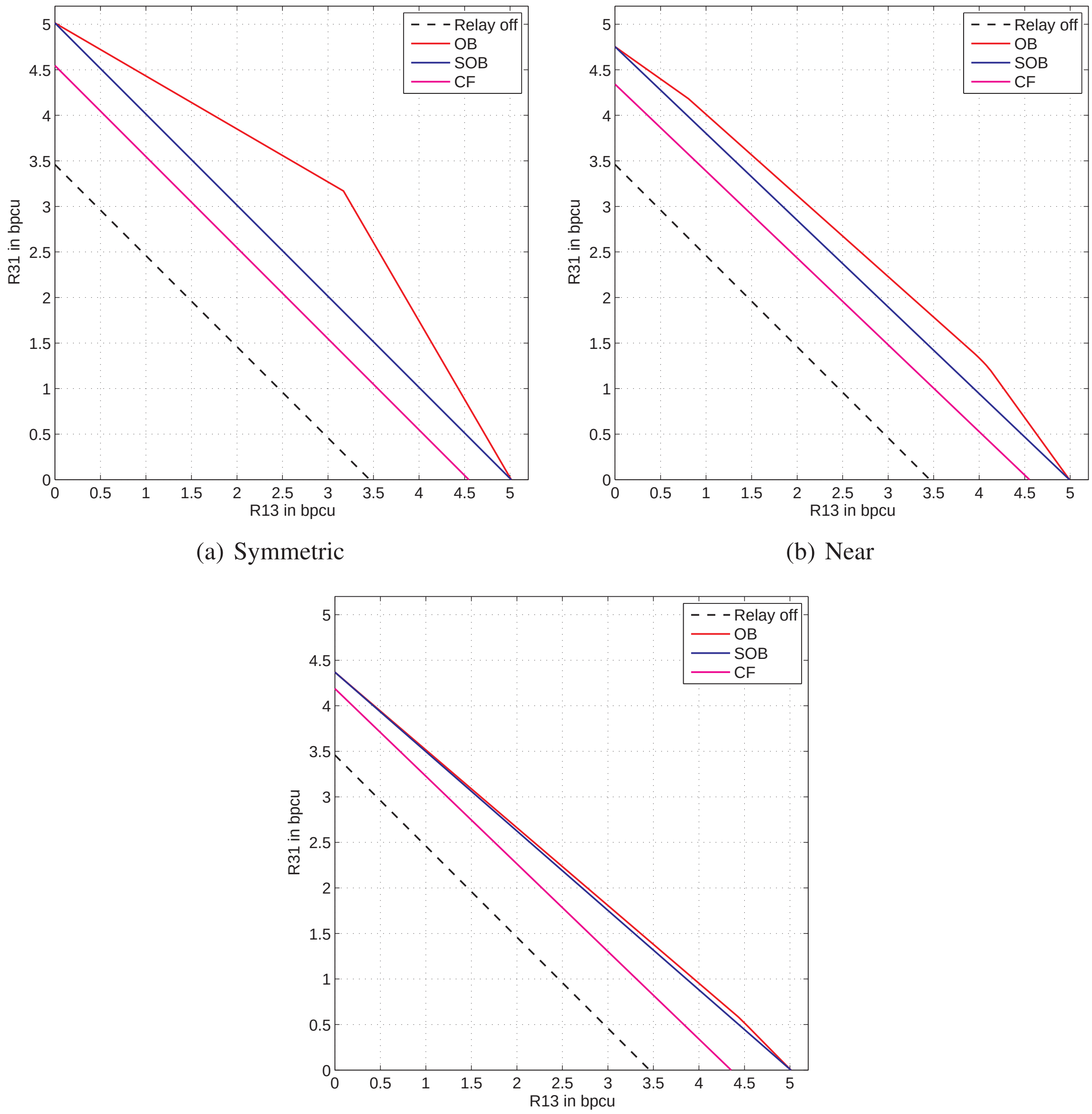


(b) Near



(c) Close

Figure 3.27: 4P-OWRC with DF, Line Network with $P_i^{(l)} = 10$, $\alpha = 3$



(a) Symmetric (b) Near

(c) Close

Figure 3.28: 4P-OWRC with CF, Line Network with $P_i^{(l)} = 10$, $\alpha = 3$

Partial-Decode-Compress-and-Forward For the PDCF strategy the input sequences are

$$
\begin{aligned}
\boldsymbol{X}_1^{(1)} &= \underbrace{\sqrt{\beta P_1^{(1)}}\boldsymbol{f}_{11}^{(1)}(W_{13})}_{\boldsymbol{U}_1^{(1)}} + \sqrt{(1-\beta)P_1^{(1)}}\boldsymbol{f}_{12}^{(1)}(W_{13}) && \beta \in [0;1] \\
\boldsymbol{X}_1^{(2)} &= \underbrace{\sqrt{\gamma P_1^{(2)}}\boldsymbol{f}_{21}^{(2)}(W_{13})}_{\sqrt{\gamma P_1^{(2)}/\delta P_2^{(2)}}\boldsymbol{V}_2^{(2)}} + \sqrt{(1-\gamma)P_1^{(2)}}\boldsymbol{f}_{13}^{(2)}(W_{13}) && \gamma \in [0;1] \\
\boldsymbol{X}_2^{(2)} &= \underbrace{\sqrt{\delta P_2^{(2)}}\boldsymbol{f}_{21}^{(2)}(W_{13})}_{\boldsymbol{V}_2^{(2)}} + \sqrt{(1-\delta)P_2^{(2)}}\boldsymbol{f}_{22}^{(2)}(Q_{23}) && \delta \in [0;1] \\
\boldsymbol{X}_3^{(3)} &= \underbrace{\sqrt{\zeta P_3^{(3)}}\boldsymbol{f}_{31}^{(3)}(W_{31})}_{\boldsymbol{U}_3^{(3)}} + \sqrt{(1-\zeta)P_3^{(3)}}\boldsymbol{f}_{32}^{(3)}(W_{31}) && \zeta \in [0;1] \\
\boldsymbol{X}_3^{(4)} &= \underbrace{\sqrt{\eta P_3^{(4)}}\boldsymbol{f}_{23}^{(4)}(W_{31})}_{\sqrt{\eta P_3^{(4)}/\theta P_2^{(4)}}\boldsymbol{V}_2^{(4)}} + \sqrt{(1-\eta)P_3^{(4)}}\boldsymbol{f}_{33}^{(4)}(W_{31}) && \eta \in [0;1] \\
\boldsymbol{X}_2^{(4)} &= \underbrace{\sqrt{\theta P_2^{(4)}}\boldsymbol{f}_{23}^{(4)}(W_{31})}_{\boldsymbol{V}_2^{(4)}} + \sqrt{(1-\theta)P_2^{(4)}}\boldsymbol{f}_{24}^{(4)}(Q_{21}) && \theta \in [0;1].
\end{aligned} \tag{3.82}
$$

The quantized outputs at node 2 are

$$
\begin{aligned}
\hat{\boldsymbol{Y}}_2^{(1)} &= \boldsymbol{Y}_2^{(1)} + \hat{\boldsymbol{Z}}_2^{(1)} \\
\hat{\boldsymbol{Y}}_2^{(3)} &= \boldsymbol{Y}_2^{(3)} + \hat{\boldsymbol{Z}}_2^{(3)}
\end{aligned} \tag{3.83}
$$

with independent $\hat{Z}_{2,k}^{(1)} \sim \mathcal{N}_{\mathbb{C}}(0, \hat{\sigma}_1^2)$ and $\hat{Z}_{2,k}^{(3)} \sim \mathcal{N}_{\mathbb{C}}(0, \hat{\sigma}_2^2)$. The achievable rates are

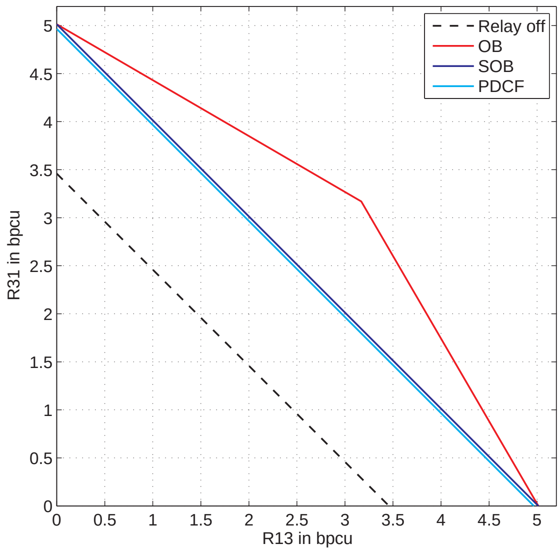
$$
\begin{aligned}
R_{13} \leq \min \Bigg\{ & \tau_1 \log\left(\frac{1+|h_{12}|^2 P_1^{(1)}}{1+|h_{12}|^2(1-\beta)P_1^{(1)}}\right), \tau_1 \log\left(\frac{1+|h_{13}|^2 P_1^{(1)}}{1+|h_{13}|^2(1-\beta)P_1^{(1)}}\right) + \\
& \tau_2 \log\left(\frac{1+|h_{13}|^2 P_1^{(2)} + |h_{23}|^2 P_2^{(2)} + 2|h_{13}h_{23}|\sqrt{\gamma\delta P_1^{(2)}P_2^{(2)}}}{1+|h_{13}|^2(1-\gamma)P_1^{(2)} + |h_{23}|^2(1-\delta)P_2^{(2)}}\right) \Bigg\} + \\
& \tau_1 \log\left(1+\left(|h_{13}|^2 + \frac{|h_{12}|^2}{1+\hat{\sigma}_1{}^2}\right)(1-\beta)P_1^{(1)}\right) + \tau_2 \log\left(1+|h_{13}|^2(1-\gamma)P_1^{(2)}\right)
\end{aligned}
$$

$$R_{31} \leq \min \left\{ \tau_3 \log \left(\frac{1 + |h_{32}|^2 P_3^{(3)}}{1 + |h_{32}|^2 (1-\zeta) P_3^{(3)}} \right), \tau_3 \log \left(\frac{1 + |h_{31}|^2 P_3^{(3)}}{1 + |h_{31}|^2 (1-\zeta) P_3^{(3)}} \right) + \right.$$
$$\left. \tau_4 \log \left(\frac{1 + |h_{31}|^2 P_3^{(4)} + |h_{21}|^2 P_2^{(4)} + 2 |h_{31} h_{21}| \sqrt{\eta \theta P_2^{(4)} P_3^{(4)}}}{1 + |h_{31}|^2 (1-\eta) P_3^{(4)} + |h_{21}|^2 (1-\theta) P_2^{(4)}} \right) \right\} +$$
$$\tau_3 \log \left(1 + \left(|h_{31}|^2 + \frac{|h_{32}|^2}{1 + \hat{\sigma_2}^2} \right) (1-\zeta) P_3^{(3)} \right) + \tau_4 \log \left(1 + |h_{31}|^2 (1-\eta) P_3^{(4)} \right)$$

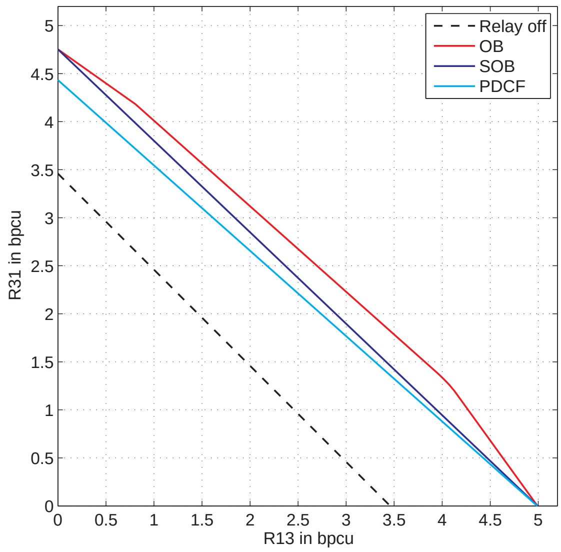
subject to

$$\tau_1 \log \left(1 + \frac{1}{\hat{\sigma}_1^2} \left(1 + \frac{|h_{12}|^2 (1-\beta) P_1^{(1)}}{1 + |h_{13}|^2 (1-\beta) P_1^{(1)}} \right) \right) \leq \tau_2 \log \left(1 + \frac{|h_{23}|^2 (1-\delta) P_2^{(2)}}{1 + |h_{13}|^2 (1-\gamma) P_1^{(2)}} \right)$$
$$\tau_3 \log \left(1 + \frac{1}{\hat{\sigma}_2^2} \left(1 + \frac{|h_{32}|^2 (1-\zeta) P_3^{(3)}}{1 + |h_{31}|^2 (1-\zeta) P_3^{(3)}} \right) \right) \leq \tau_4 \log \left(1 + \frac{|h_{21}|^2 (1-\theta) P_2^{(4)}}{1 + |h_{31}|^2 (1-\eta) P_3^{(4)}} \right). \quad (3.84)$$

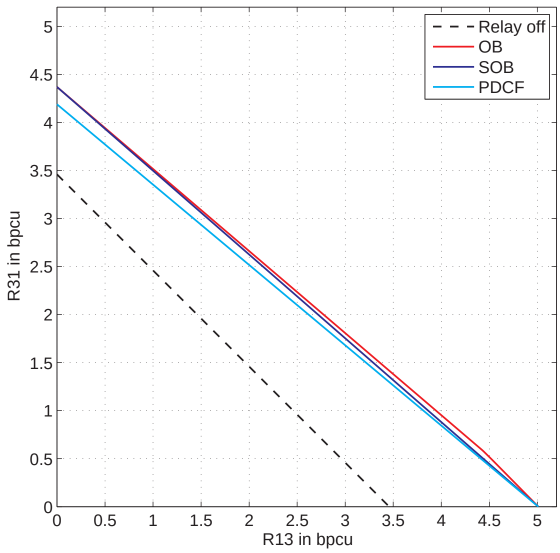
Simulations Figures 3.27-3.29 show the rate regions for the individual strategies. Figure 3.30 combines all plots. For the chosen model and parameters simulations in Figure 3.30 show that the scheme outer bound approaches the problem outer bound for asymmetric channels. For symmetric channels DF is close to the scheme outer bound. For asymmetric channels DF becomes optimal for the node closer to the relay. CF shows to be insensible to asymmetric channels from the dialog nodes to the relay as the rate regions change just slightly for the different scenarios. PDCF yields the convex hull of the rate points achievable with CF or DF.



(a) Symmetric

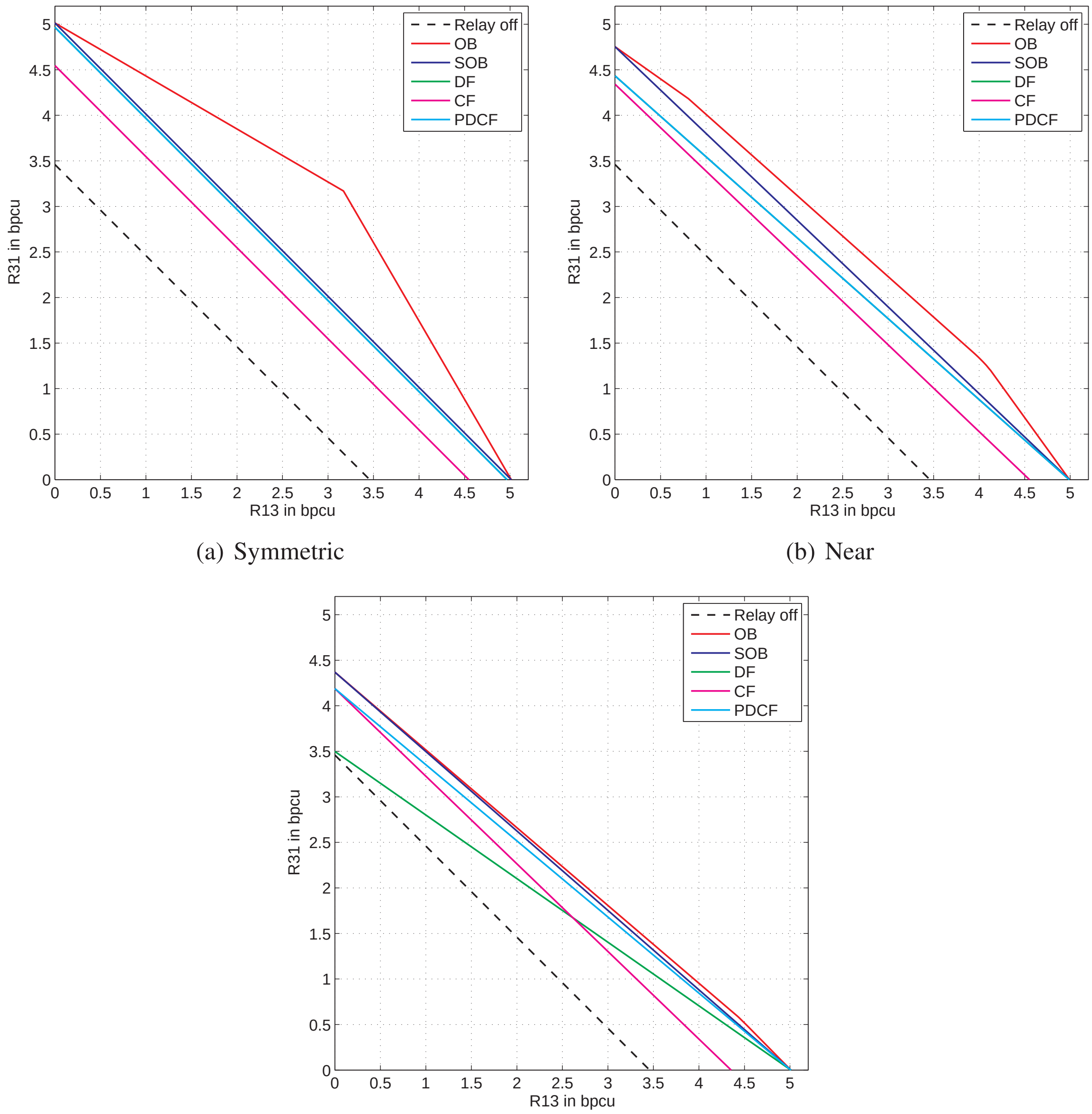


(b) Near



(c) Close

Figure 3.29: 4P-OWRC with PDCF, Line Network with $P_i^{(l)} = 10$, $\alpha = 3$



(a) Symmetric

(b) Near

(c) Close

Figure 3.30: 4P-OWRC, Line Network with $P_i^{(l)} = 10$, $\alpha = 3$

3.4.5 6P Scheme

The output sequences of the six phase scheme are

$$
\begin{aligned}
\boldsymbol{Y}_2^{(1)} &= h_{12}\boldsymbol{X}_1^{(1)} + \boldsymbol{Z}_2^{(1)} \\
\boldsymbol{Y}_3^{(1)} &= h_{13}\boldsymbol{X}_1^{(1)} + \boldsymbol{Z}_3^{(1)} \\
\boldsymbol{Y}_1^{(2)} &= h_{31}\boldsymbol{X}_3^{(2)} + \boldsymbol{Z}_1^{(2)} \\
\boldsymbol{Y}_2^{(2)} &= h_{32}\boldsymbol{X}_3^{(2)} + \boldsymbol{Z}_2^{(2)} \\
\boldsymbol{Y}_2^{(3)} &= h_{12}\boldsymbol{X}_1^{(3)} + h_{32}\boldsymbol{X}_3^{(3)} + \boldsymbol{Z}_2^{(3)} \\
\boldsymbol{Y}_1^{(4)} &= h_{21}\boldsymbol{X}_2^{(4)} + \boldsymbol{Z}_1^{(4)} \\
\boldsymbol{Y}_3^{(4)} &= h_{23}\boldsymbol{X}_2^{(4)} + \boldsymbol{Z}_3^{(4)} \\
\boldsymbol{Y}_1^{(5)} &= h_{21}\boldsymbol{X}_2^{(5)} + h_{31}\boldsymbol{X}_3^{(5)} + \boldsymbol{Z}_1^{(5)} \\
\boldsymbol{Y}_3^{(6)} &= h_{13}\boldsymbol{X}_1^{(6)} + h_{23}\boldsymbol{X}_2^{(6)} + \boldsymbol{Z}_3^{(6)}.
\end{aligned} \tag{3.85}
$$

Outer Bound For the outer bound the input sequences are of the form

$$
\begin{aligned}
\boldsymbol{X}_1^{(1)} &= \sqrt{P_1^{(1)}}\boldsymbol{f}_{11}^{(1)}(W_{13}) \\
\boldsymbol{X}_3^{(2)} &= \sqrt{P_3^{(2)}}\boldsymbol{f}_{31}^{(2)}(W_{31}) \\
\boldsymbol{X}_1^{(3)} &= \sqrt{P_1^{(3)}}\boldsymbol{f}_{11}^{(3)}(W_{13}) \\
\boldsymbol{X}_3^{(3)} &= \sqrt{P_3^{(3)}}\boldsymbol{f}_{31}^{(3)}(W_{31}) \\
\boldsymbol{X}_2^{(4)} &= \sqrt{P_2^{(4)}}\boldsymbol{f}_{21}^{(4)}(W_{13}, W_{31}) \\
\boldsymbol{X}_2^{(5)} &= \sqrt{P_2^{(5)}}\boldsymbol{f}_{21}^{(5)}(W_{31}) \\
\boldsymbol{X}_3^{(5)} &= \underbrace{\sqrt{\beta P_3^{(5)}}\boldsymbol{f}_{21}^{(5)}(W_{31})}_{\sqrt{\beta P_3^{(5)}/P_2^{(5)}}\boldsymbol{X}_2^{(5)}} + \sqrt{(1-\beta)P_3^{(5)}}\boldsymbol{f}_{31}^{(5)}(W_{31}) \qquad \beta \in [0;1] \\
\boldsymbol{X}_2^{(6)} &= \sqrt{P_2^{(6)}}\boldsymbol{f}_{21}^{(6)}(W_{13}) \\
\boldsymbol{X}_1^{(6)} &= \underbrace{\sqrt{\gamma P_1^{(6)}}\boldsymbol{f}_{21}^{(6)}(W_{13})}_{\sqrt{\gamma P_1^{(6)}/P_2^{(6)}}\boldsymbol{X}_2^{(6)}} + \sqrt{(1-\gamma)P_1^{(6)}}\boldsymbol{f}_{11}^{(6)}(W_{13}) \qquad \gamma \in [0;1]
\end{aligned} \tag{3.86}
$$

with the parameters β, γ controlling the dependence between the inputs at nodes 1 or 3 and node 2. The outer bound on achievable rates is

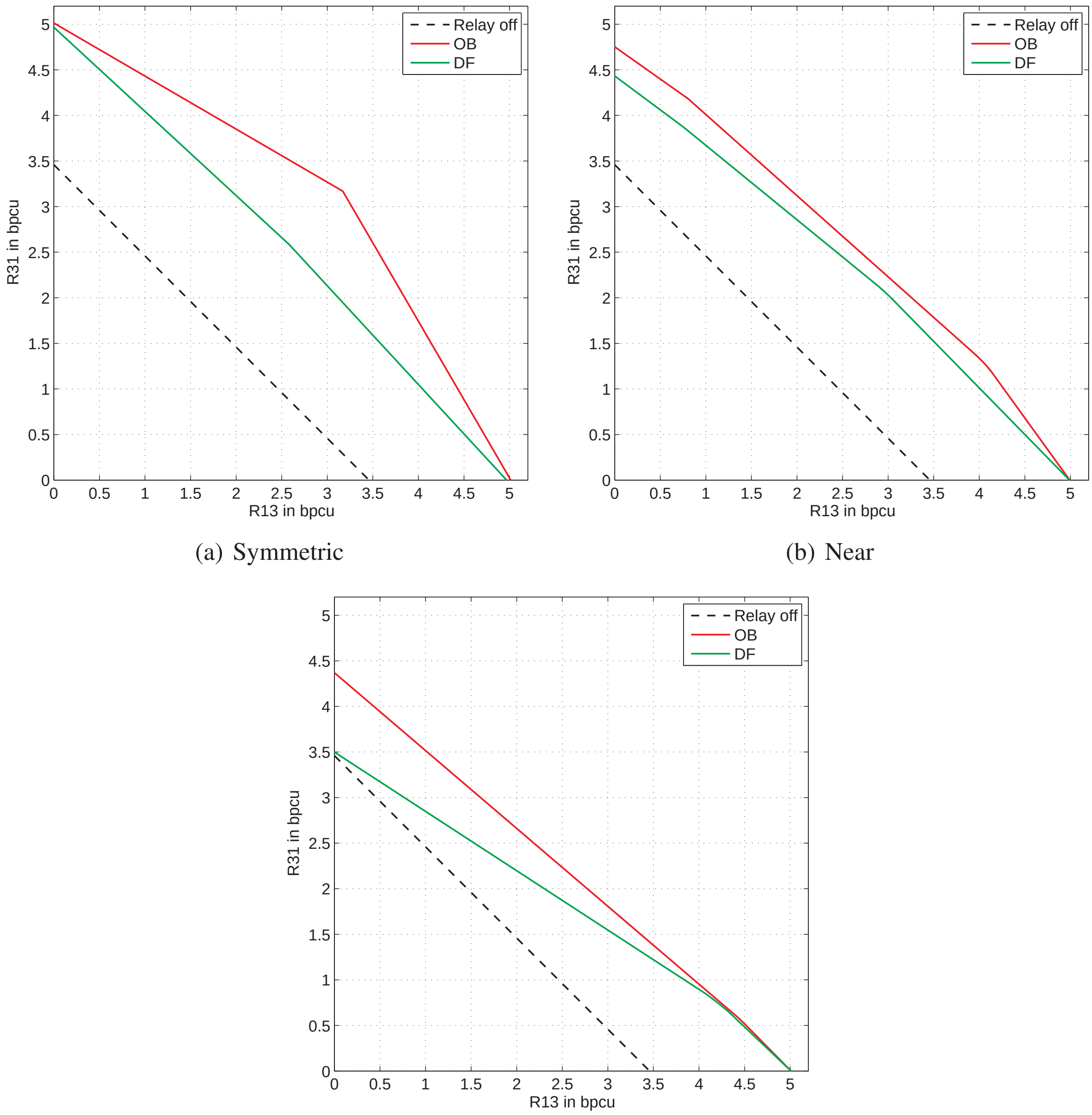
$$
\begin{aligned}
R_{13} \le \min\Big\{ &\tau_1 \log\left(1 + (|h_{12}|^2 + |h_{13}|^2)P_1^{(1)}\right) + \tau_3 \log\left(1 + |h_{12}|^2 P_1^{(3)}\right) + \\
&\tau_6 \log\left(1 + |h_{13}|^2 (1-\gamma)P_1^{(6)}\right), \tau_1 \log\left(1 + |h_{13}|^2 P_1^{(1)}\right) + \tau_4 \log\left(1 + |h_{23}|^2 P_2^{(4)}\right) + \\
&\tau_6 \log\left(1 + |h_{13}|^2 P_1^{(6)} + |h_{23}|^2 P_2^{(6)} + 2\,|h_{13}h_{23}|\sqrt{\gamma P_1^{(6)} P_2^{(6)}}\right)\Big\}
\end{aligned}
$$

$$R_{31} \leq \min\Big\{\tau_2 \log\left(1 + (|h_{32}|^2 + |h_{31}|^2)P_3^{(2)}\right) + \tau_3 \log\left(1 + |h_{32}|^2 P_3^{(3)}\right) + \\ \tau_5 \log\left(1 + |h_{31}|^2 (1-\beta)P_3^{(5)}\right), \tau_2 \log\left(1 + |h_{31}|^2 P_3^{(2)}\right) + \tau_4 \log\left(1 + |h_{21}|^2 P_2^{(4)}\right) + \\ \tau_5 \log\left(1 + |h_{31}|^2 P_3^{(5)} + |h_{21}|^2 P_2^{(5)} + 2\,|h_{31}h_{21}| \sqrt{\beta P_2^{(5)} P_3^{(5)}}\right)\Big\}. \tag{3.87}$$

Decode-and-Forward With the same input paramterization as the outer bound the DF achievable rates are

$$R_{13} \leq \min\Big\{\tau_1 \log\left(1 + |h_{12}|^2 P_1^{(1)}\right) + \tau_3 \log\left(1 + |h_{12}|^2 P_1^{(3)}\right) + \tau_6 \log\left(1 + |h_{13}|^2 (1-\gamma)P_1^{(6)}\right), \\ \tau_1 \log\left(1 + |h_{13}|^2 P_1^{(1)}\right) + \tau_4 \log\left(1 + |h_{23}|^2 P_2^{(4)}\right) + \\ \tau_6 \log\left(1 + |h_{13}|^2 P_1^{(6)} + |h_{23}|^2 P_2^{(6)} + 2\,|h_{13}h_{23}| \sqrt{\gamma P_1^{(6)} P_2^{(6)}}\right)\Big\} \\ R_{31} \leq \min\Big\{\tau_2 \log\left(1 + |h_{32}|^2 P_3^{(2)}\right) + \tau_3 \log\left(1 + |h_{32}|^2 P_3^{(3)}\right) + \tau_5 \log\left(1 + |h_{31}|^2 (1-\beta)P_3^{(5)}\right), \\ \tau_2 \log\left(1 + |h_{31}|^2 P_3^{(2)}\right) + \tau_4 \log\left(1 + |h_{21}|^2 P_2^{(4)}\right) + \\ \tau_5 \log\left(1 + |h_{31}|^2 P_3^{(5)} + |h_{21}|^2 P_2^{(5)} + 2\,|h_{31}h_{21}| \sqrt{\beta P_2^{(5)} P_3^{(5)}}\right)\Big\} \\ R_{13} + R_{31} \leq \tau_1 \log\left(1 + |h_{12}|^2 P_1^{(1)}\right) + \tau_2 \log\left(1 + |h_{32}|^2 P_3^{(2)}\right) + \tau_3 \log\left(1 + |h_{12}|^2 P_1^{(3)} + |h_{32}|^2 P_3^{(3)}\right) + \\ \tau_5 \log\left(1 + |h_{31}|^2 (1-\beta)P_3^{(5)}\right) + \tau_6 \log\left(1 + |h_{13}|^2 (1-\gamma)P_1^{(6)}\right). \tag{3.88}$$

Figure 3.31 shows the achievable rate regions for the three configurations considered.



(a) Symmetric

(b) Near

(c) Close

Figure 3.31: 6P with DF , Line Network with $P_i^{(l)} = 10$, $\alpha = 3$

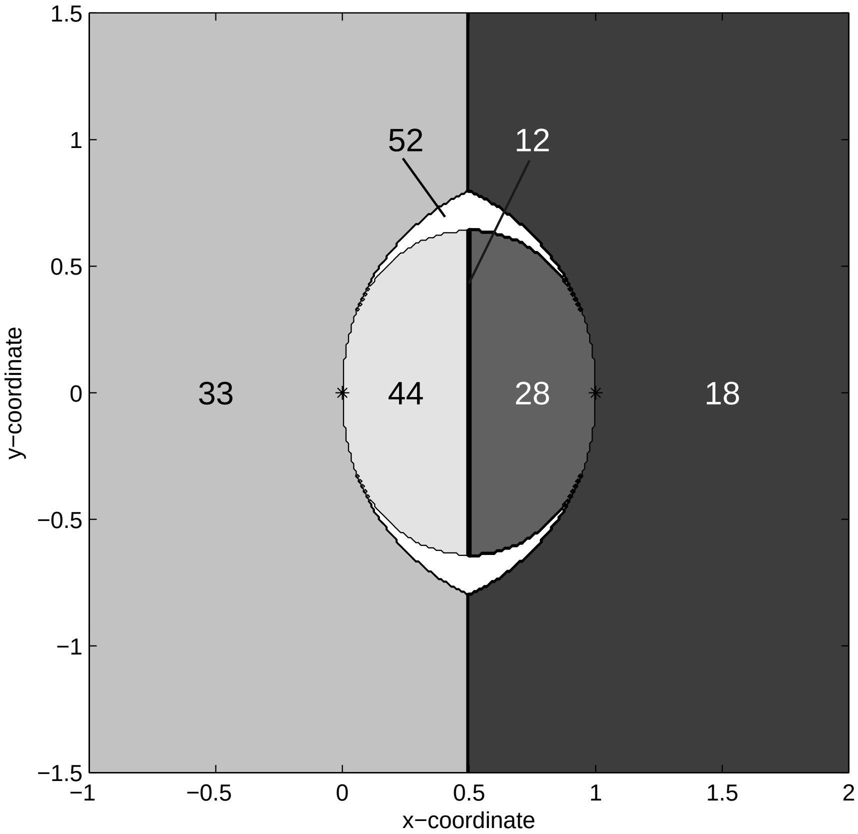
3.5 Towards Optimal Schemes

For fixed input distributions the problem outer bound and the 6P expressions can be used to identify potentially optimal schemes for any relaying strategy or strategies with decoding at the relay through time allocation. Here some simulation results are presented. For the problem bound and 6P DF expressions two linear rate objectives have been maximized in order to find the optimal time allocation to the six phases. On the plots the different schemes are depicted for the plane network. On the scheme plots the used schemes are numbered by the decimal version of the binary vector denoting the active phases. For example the schemes studied in detail in this work have the numbers

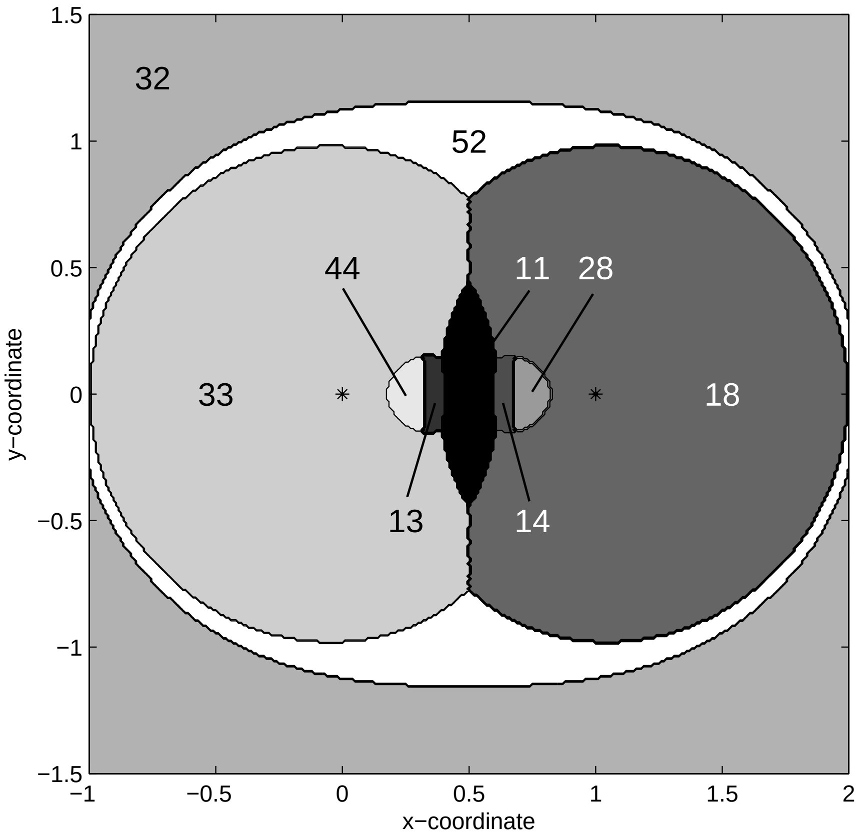
$$\begin{aligned}
\text{OWRC (1-3)} : 33 &= \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 1\end{bmatrix} \\
\text{OWRC (3-1)} : 18 &= \begin{bmatrix}0 & 1 & 0 & 0 & 1 & 0\end{bmatrix} \\
\text{2P-MA-BC} : 12 &= \begin{bmatrix}0 & 0 & 1 & 1 & 0 & 0\end{bmatrix} \\
\text{3P-BC} : 11 &= \begin{bmatrix}1 & 1 & 0 & 1 & 0 & 0\end{bmatrix} \\
\text{3P-MA} : 52 &= \begin{bmatrix}0 & 0 & 1 & 0 & 1 & 1\end{bmatrix} \\
\text{4P-OWRC} : 51 &= \begin{bmatrix}1 & 1 & 0 & 0 & 1 & 1\end{bmatrix} \\
\text{6P} : 63 &= \begin{bmatrix}1 & 1 & 1 & 1 & 1 & 1\end{bmatrix}.
\end{aligned} \tag{3.89}$$

A full legend of the schemes showing up in the following plots is found at the end of this section. Note that the binary vector only indicates which phase is active and not its individual time length. For all simulations the parameters are $P_i^{(l)} = 10$ and $\alpha = 3$.

Sum-Rate Maximization Figure 3.32 shows the resulting schemes for the problem outer bound and the 6P DF scheme after sampling over the input distributions (β, γ) and solving time allocation with respect to maximal sum-rate. It can be seen on the plot for the outer bound that the 2P-MA-BC (12) scheme shows up only in the position where the channels to node 2 support totally symmetric rates. A significant area is covered by one-way channels (18/33). The 3P-MA (52) scheme shows up in small areas. Between the two nodes the outer bound would suggest to use a scheme for sum-rate maximization which adds to the 2P-MA-BC a phase where relay and one dialog node send together (28/44). On the plot of the DF strategy in the outer area the relay is disconnected from the transmission process (32, here equivalent with 16). Again the two one-way schemes show up in a wide area around the nodes. In the middle the 3P-BC (11) scheme is used. The two asymmetric schemes (28/44) are used in small areas between the nodes. The schemes 13 and 14 add a phase to the 2P-MA-BC scheme where one of the dialog nodes sends alone. In Figure 3.33 the achievable sum-rate with 6P DF is compared to the outer bound and the rates of 2P-MA-BC DF. "Behind" each node the sum-rate bound is achieved with one-way channels. Between the nodes still improvements are possible, e.g. use 2P-MA-BC with CF. The comparison against the two phase scheme shows a gain up to 50 percent in the area between the nodes. On Figure 3.34 a comparison to the two-way channel without relay is depicted. Here a gain of up to 40 percent is observed

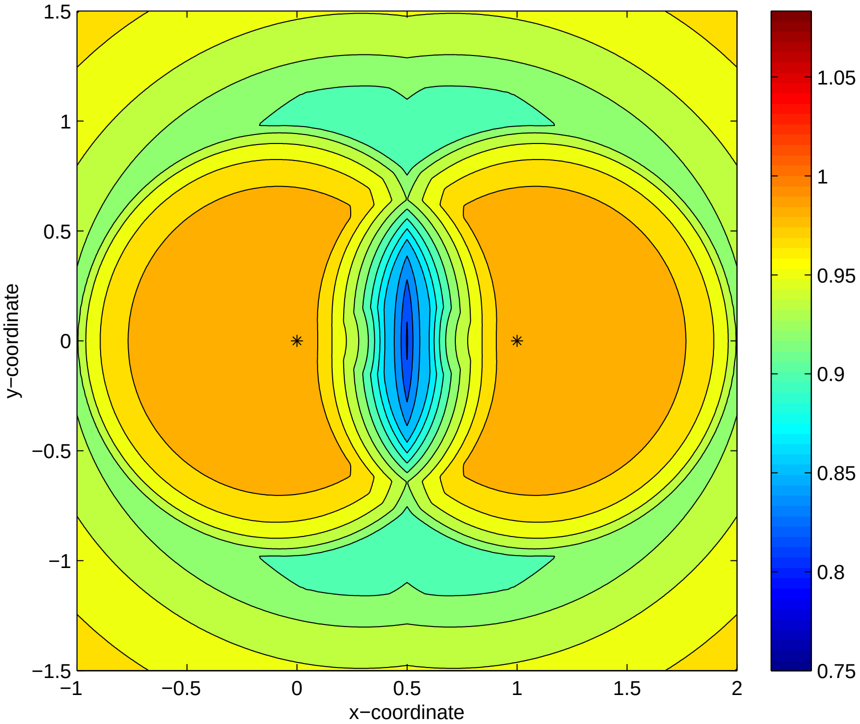


(a) Upper Bound

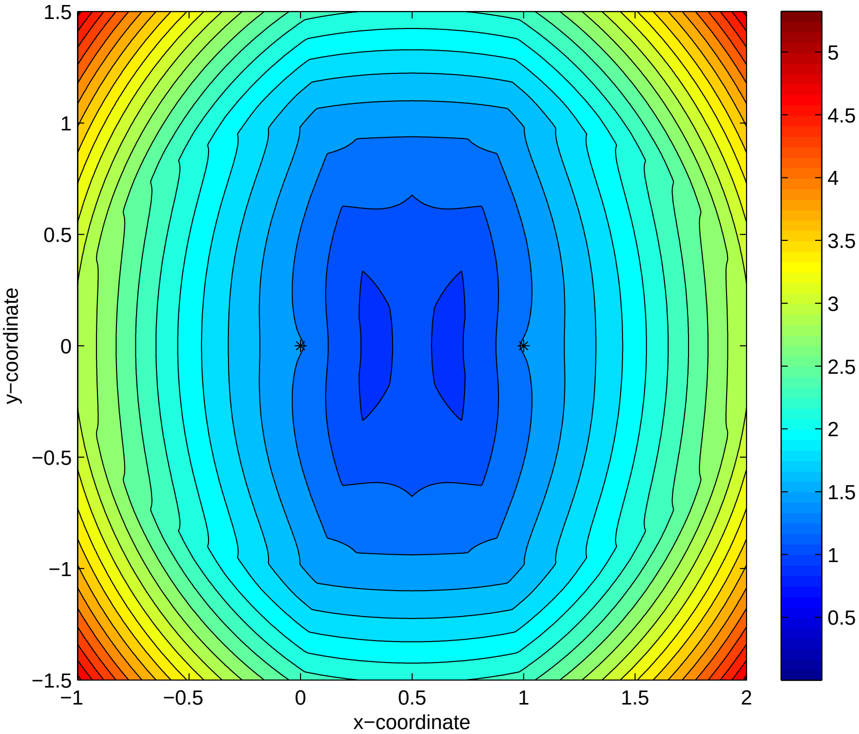


(b) 6P DF

Figure 3.32: Schemes, SR Maximization, Plane Network with $P_i^{(l)} = 10$, $\alpha = 3$

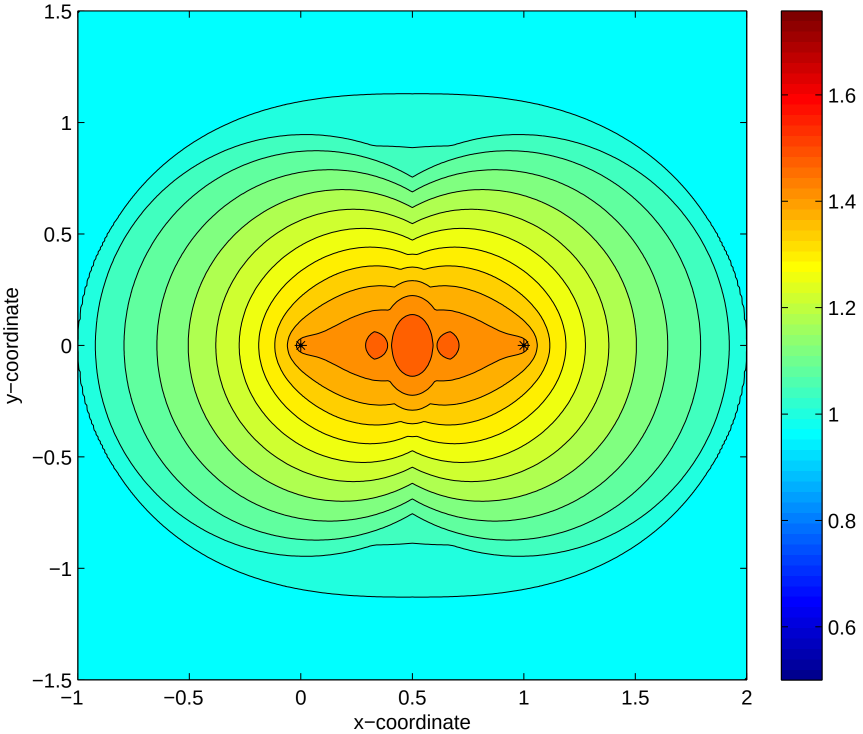


(a) 6P DF/UB



(b) 6P DF/2P-MA-BC DF

Figure 3.33: Comparison, SR Maximization, Plane Network with $P_i^{(l)} = 10$, $\alpha = 3$

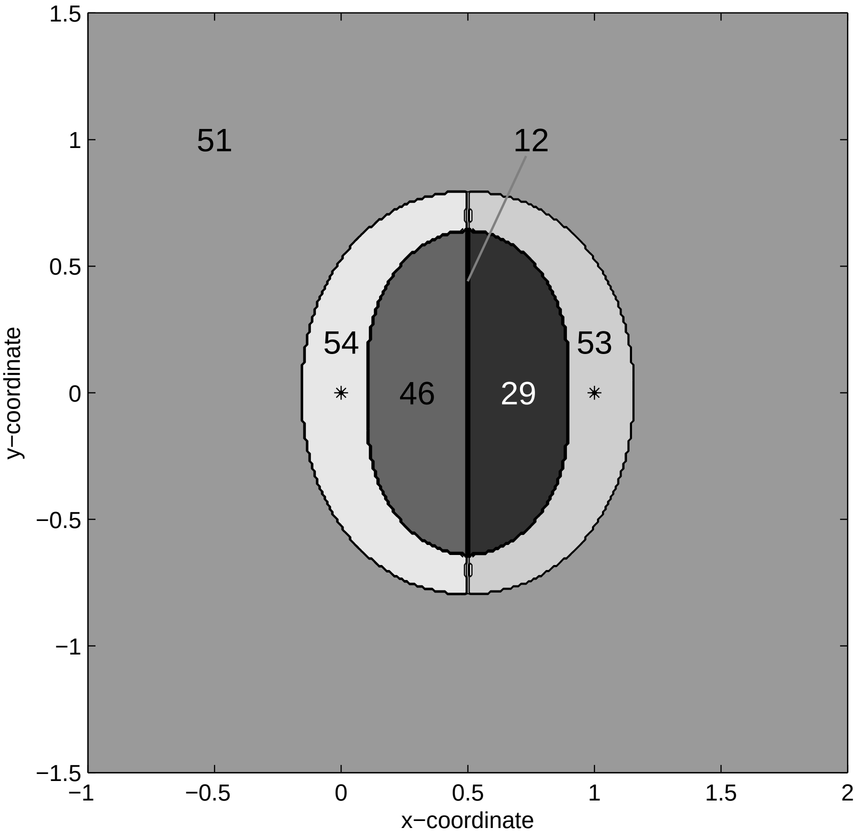


(a) 6P DF/TWC

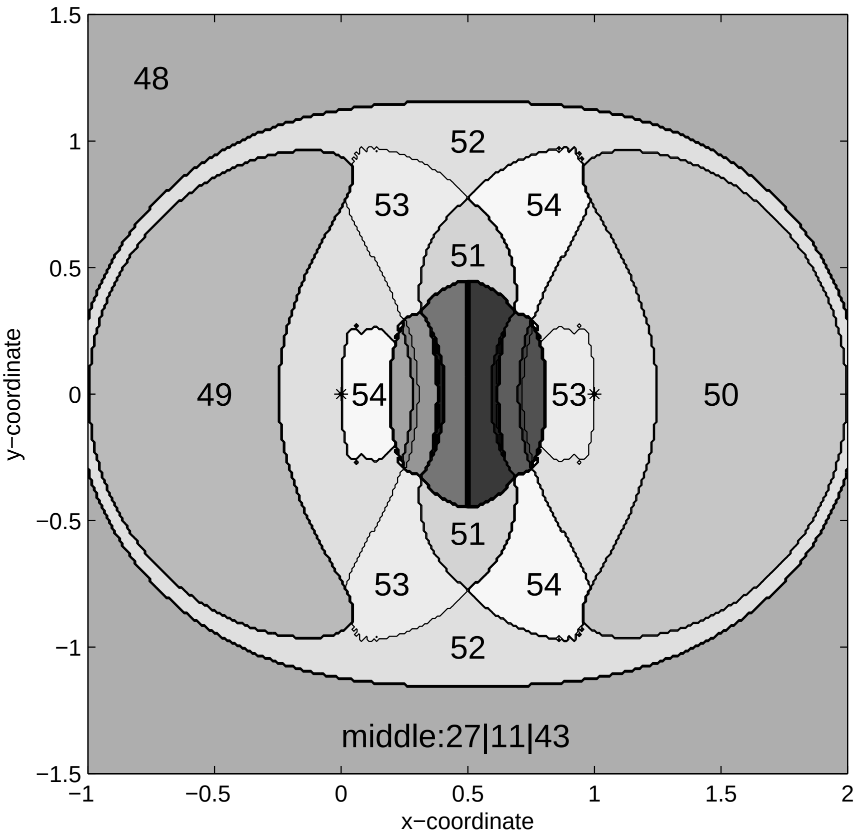
Figure 3.34: Comparison with TWC, SR Maximization, Plane Network with $P_i^{(l)} = 10$, $\alpha = 3$

Max-Min Maximization Figure 3.35 shows the resulting schemes after solving time allocation with respect to maximal maxmin-rate. The outer bound suggests to use the 4P-OWRC scheme in a very large area. In the middle the two phase scheme shows up again. The schemes 53 and 54 denote a scheme consisting of the 3P-MA scheme with an additional phase where one of the dialog nodes sends. 46 and 29 are a bit tricky with a 2P-MA-BC and two additional phases that can be interpreted as a OWRC with one phase in the reverse direction. In the DF plot many schemes show up. New are the schemes 49 and 50 (see scheme legend) "behind" each node. 48 denotes a two-way communication without relay. 27 and 43 are the 3P-BC scheme with an additional phase with relay and one dialog node active. The rate comparison (see Figure 3.36) shows that the 6P DF scheme operates at around 90 percent of the max-min upper bound in a large area. For the area between both nodes a significant distance can be observed. Comparing against the two-phase scheme shows a max-min gain of up to 50 percent for the area between and around the dialog nodes. Figure 3.37 shows that a significant max-min rate gain is possible in comparison to a simple two-way channel if the relay is located between both dialog nodes.

Insides The basic inside of such simulations is that there seems to be no ultimate scheme. By optimizing the time allocation to the different phases the 6P scheme degenerates into schemes with less than six phases as different channel configurations require different schemes. Each of the phases is active at some point and therefore useful for some particular situation. Only a 6P scheme has the potential to adapt all network configurations appropriately to the channels. It becomes also clear that the possible rate gain might be strongly dependent of the position of the relay. Note that

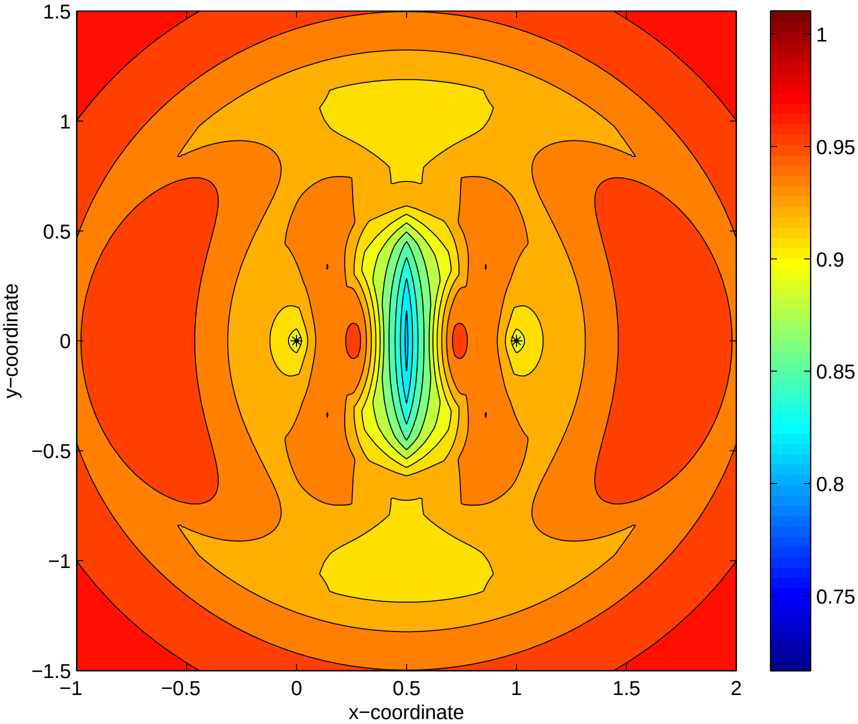


(a) Upper Bound

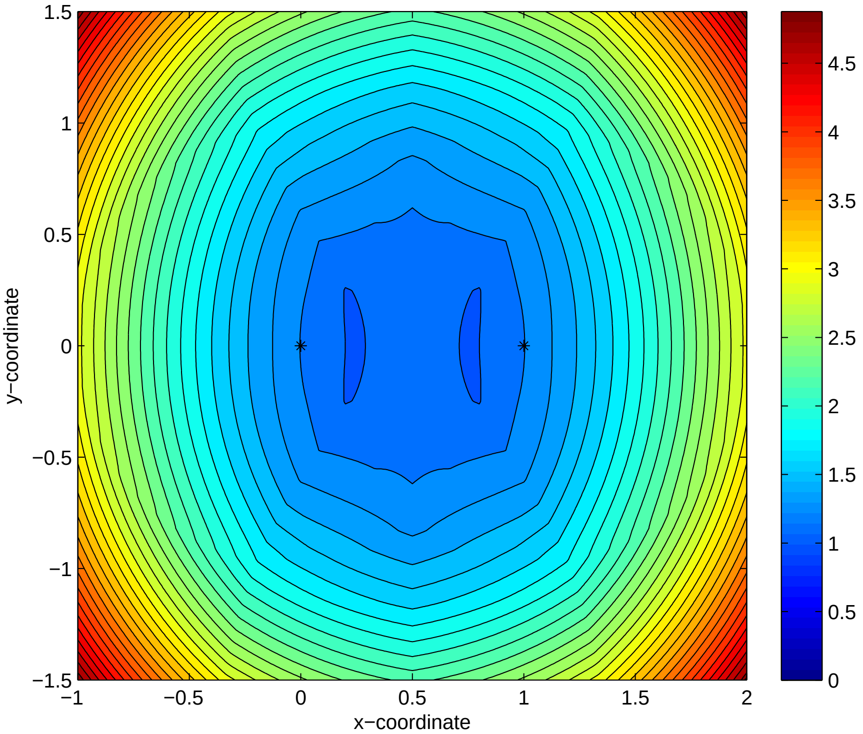


(b) 6P DF

Figure 3.35: Schemes, MM Maximization, Plane Network with $P_i^{(l)} = 10$, $\alpha = 3$

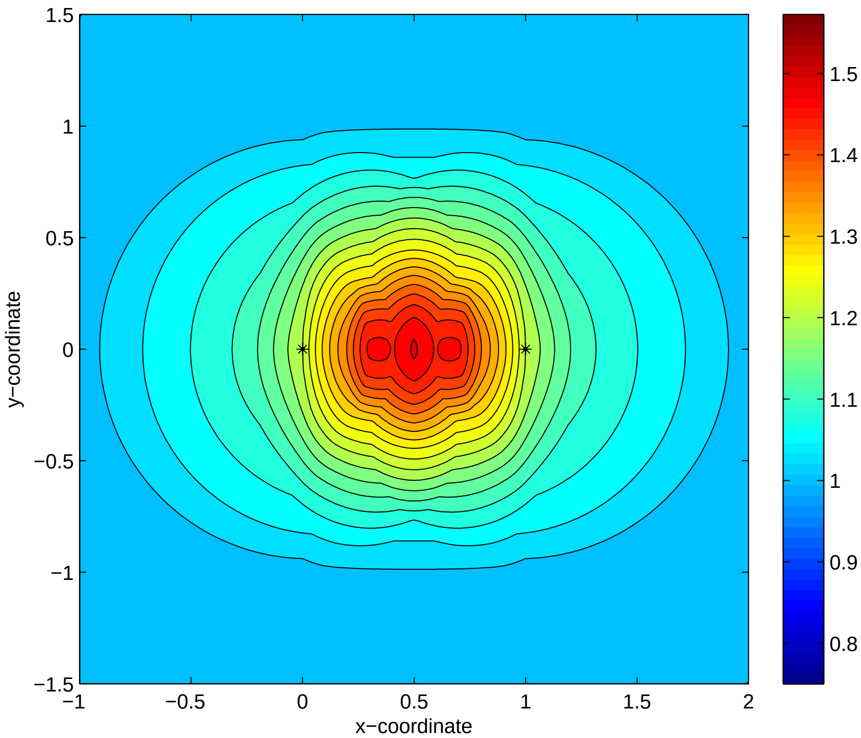


(a) 6P DF/UB



(b) 6P DF/2P-MA-BC DF

Figure 3.36: Comparison, MM Maximization, Plane Network with $P_i^{(l)} = 10$, $\alpha = 3$



(a) 6P DF/TWC

Figure 3.37: Comparison with TWC, MM Maximization, Plane Network with $P_i^{(l)} = 10$, $\alpha = 3$

the plots represent just an exemplary channel model. Especially with a different path-loss exponent the results might change.

Scheme Legend

$$\begin{aligned}
&11: \begin{bmatrix}1 & 1 & 0 & 1 & 0 & 0\end{bmatrix} && 12: \begin{bmatrix}0 & 0 & 1 & 1 & 0 & 0\end{bmatrix}\\
&13: \begin{bmatrix}1 & 0 & 1 & 1 & 0 & 0\end{bmatrix} && 14: \begin{bmatrix}0 & 1 & 1 & 1 & 0 & 0\end{bmatrix}\\
&18: \begin{bmatrix}0 & 1 & 0 & 0 & 1 & 0\end{bmatrix} && 27: \begin{bmatrix}1 & 1 & 0 & 1 & 1 & 0\end{bmatrix}\\
&28: \begin{bmatrix}0 & 0 & 1 & 1 & 1 & 0\end{bmatrix} && 29: \begin{bmatrix}1 & 0 & 1 & 1 & 1 & 0\end{bmatrix}\\
&32: \begin{bmatrix}0 & 0 & 0 & 0 & 0 & 1\end{bmatrix} && 33: \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 1\end{bmatrix}\\
&43: \begin{bmatrix}1 & 1 & 0 & 1 & 0 & 1\end{bmatrix} && 44: \begin{bmatrix}0 & 0 & 1 & 1 & 0 & 1\end{bmatrix}\\
&46: \begin{bmatrix}0 & 1 & 1 & 1 & 0 & 1\end{bmatrix} && 48: \begin{bmatrix}0 & 0 & 0 & 0 & 1 & 1\end{bmatrix}\\
&49: \begin{bmatrix}1 & 0 & 0 & 0 & 1 & 1\end{bmatrix} && 50: \begin{bmatrix}0 & 1 & 0 & 0 & 1 & 1\end{bmatrix}\\
&51: \begin{bmatrix}1 & 1 & 0 & 0 & 1 & 1\end{bmatrix} && 52: \begin{bmatrix}0 & 0 & 1 & 0 & 1 & 1\end{bmatrix}\\
&53: \begin{bmatrix}1 & 0 & 1 & 0 & 1 & 1\end{bmatrix} && 54: \begin{bmatrix}0 & 1 & 1 & 0 & 1 & 1\end{bmatrix}.
\end{aligned} \tag{3.90}$$

4. Conclusion

The wireline example outlined in the introduction of the second part of the thesis has raised the author's interest on the implications of the direct path in a wireless two-way relay channel. As a precise survey could not be found, an information theoretical approach has been carried out in order to understand the offered possibilities and obtain results that can be applied to SISO and MIMO wireless channels.

One-Way Relay Channel The first part of the work represents the attempt to understand the channel coding aspects on a half-duplex one-way model by applying the relaying methods of [6] (PDF, [18]). Therefore, self-contained proofs have been developed with extensive use of the methods and theorems presented in [18] and [7]. The reinterpretation of methods like binning, block-Markov coding and message-splitting has led to the strongly familiar technique of *reindex coding*. This disciplines the achievability proofs by making the exploitation of statistical dependencies in the network more transparent and straightforward. Surprisingly, the general results derived for the half-duplex model through coding proofs can not be found in literature. The expressions derived in [15] only hold for Gaussian channels. Simulations visualize the performance of different methods on the half-duplex model with scalar Gaussian channels and the benefits of time allocation.

Two-Way Relay Channel The insides and techniques of the first part have been used and adapted for the analysis of a restricted two-way relay channel with direct connection. A definition has been outlined and the Cut-set Theorem has been applied to deduce an outer bound on the achievable rates with fixed input distributions. Two schemes already proposed in literature have been revisited and slightly extended (2P-MA-BC with PDCF, 3P-BC with PDF/PDCF). A new three-phase scheme (3P-MA) has been suggested and studied in detail. Moreover, a four-phase scheme has been proposed. By separating the channel into two subsequent one-way channels the analysis of this scheme benefits directly from the half-duplex one-way results of the first part. The idea of using the problem outer bound to find a general scheme has led to a new scheme with six phases, here referred to as 6P. The achievable rates with full and partial decoding at the relay have been derived for this scheme. 6P DF is argued to contain all other possible fixed DF schemes as special cases. Simulations for scalar Gaussian channels visualize the performance of the different schemes and the relation of their individual performance bound to the more general problem bound. They show clearly that the prominent two-phase scheme can not attain the performance offered by a fully-connected network model. Especially for channels with asymmetric rates to the relay a two-phase approach will result in a significant performance loss. Simulations on a plane network model show the potential of 6P to outperform the two-way channel and the two-phase two-way relay channel with respect to communication rate. The results also indicate that an ultimate scheme might not exist and therefore justify the general approach which allows to use all possible network state configurations.

Extensions The thesis has focused on channel coding aspects. As time-sharing variables are a technical tool to extend derived expressions they have been omitted here. For the simulations carried out with scalar channel inputs and a per symbol power constraint this has no effect. They will be needed to exploit the potential of average power constraints and for vector-valued channel inputs. Care needs to be taken when determining their alphabet size as the rate regions might behave different than full-duplex regions. Cost problems might have deserved a more precise analysis and simulations. As frequently indicated a MIMO analysis will be highly interesting and reveal the full potential of methods with partial decoding at the relay.

Practical Implications As mentioned in the introduction the analysis makes severe assumptions that can not apply to real systems. However, for example the derived expressions for DF relaying might be used to approximate the time lengths of different phases or the bandwidth for the different channel parts for linear rate or cost objectives on SISO systems at low complexity, especially when coherent signaling is not possible. The random coding proofs do not shed light on how to build a good code with low complexity. But the underlying reindex method can serve as guideline for the codebook indexing structure, the index passing process through the network and the decoding order of practical code constructions.

Theoretical Implications The thesis has tried to study precisely the coding aspects on two small half-duplex network models. It is just natural that, at the end of such a survey, the question of the relation between full and half-duplex networks arises. For the one-way scenario the full-duplex coding methods are fairly understood for a long time. The application of these methods on a half-duplex model seem therefore more like a "reverse engineering" practice. The method how the cut-set bound is applied gives the impression that the achievable rates could also have been acquired formally from the full-duplex expressions of [6] or [18] without a coding proof. It should not be neglected that this would have been possible for example for CF. Note that for DF such a method yields lower rates. Here such an approach would have made it impossible to understand the underlying coding problem. Only this has allowed to study the two-way scenario in detail and to extend lower bounds to a scheme with six phases. A formal way from half-duplex expressions to full-duplex ones appears at the moment not obvious. This might indicate that half-duplex networks are a special case of full-duplex ones. It is interesting to observe that on the half-duplex model the problems in relation to the cut-set bound stay the same as in the full-duplex model but occur separated on orthogonal channel parts just connected through coding.

Appendix

In order to make the thesis easier to read, all achievability proofs have been moved to the appendix. A code is said to be reliable if $\Pr[\mathcal{E}]$ can be made arbitrary small with choosing n sufficiently large. $\mathcal{E}$ denotes the occurrence of an error at destination decoders. All rates that allow this or can be approached arbitrarily close with this property are considered to be *achievable*. Proofs of achievability follow the concept of defining encoders and decoders and to analyze the probability of error in dependence of the rate R and block length n. Here the proofs use properties of long random sequences and the performance of suboptimal decoders. The probability of error with a "good code" is not analyzed directly. Instead the symmetry of a random code construction is exploited in order to characterize the average probability of error over all codebooks in dependence of R and n. This allows to prove the existence of a reliable code under certain rate constraints.

A1. Letter-Typical Sequences

To make the work self-contained the definitions and theorems used are stated here without proofs. These and a comprehensive introduction can be found in [18] which is recommended as an overview on problems of information theory and the analysis of those using typicality. *Letter-typicality* is used in order to apply the Markov Lemma which is needed for the proofs with CF and PDCF. Definitions and applications of a weaker form called *entropy-typicality* can be found in [7].

Definition A1.1 (Letter-Typical Sequence)
Let $N(a|x^n)$ be the number of positions in the n-sequence x^n having the letter a. For $\epsilon \geq 0$ a sequence x^n is ϵ-letter typical with respect to $P_X(\cdot)$ if

$$\left|\frac{1}{n}N(a|x^n) - P_X(a)\right| \leq \epsilon P_X(a) \qquad \forall a \in \mathcal{X}\,.$$

Definition A1.2 (Letter-Typical Set)
The ϵ-letter typical set $T_\epsilon^n(P_X)$ with respect to $P_X(\cdot)$ is defined as

$$T_\epsilon^n(P_X) = \left\{x^n : \left|\frac{1}{n}N(a|x^n) - P_X(a)\right| \leq \epsilon P_X(a), \forall a \in \mathcal{X}\right\}.$$

Definition A1.3 (Jointly Letter-Typical Sequences)
Let $N(a, b|x^n, y^n)$ *be the number of occurences of* (a, b) *in the sequence* $(x_1, y_1), (x_2, y_2) \ldots (x_n, y_n)$. *For* $\epsilon \geq 0$ *the* ϵ*-letter typical set* $T_\epsilon^n(P_{XY})$ *with respect to* $P_{XY}(\cdot)$ *is defined as*

$$T_\epsilon^n(P_{XY}) = \left\{(x^n, y^n) : \left|\frac{1}{n}N(a, b|x^n, y^n) - P_{XY}(a, b)\right| \leq \epsilon P_{XY}(a, b), \forall (a, b) \in \mathcal{X} \times \mathcal{Y}\right\}.$$

Theorem A1.4 (Properties Letter-Typical Sequence)
Suppose $0 \leq \epsilon \leq \mu_X = \min_{a \in supp(P_X)} P_X(a)$, $x^n \in T_\epsilon^n(P_X)$ *and* X^n *is emitted by a discrete-memoryless source* $P_X(\cdot)$

$$\begin{aligned} 2^{-n(1+\epsilon)H(X)} &\leq P_X^n(x^n) \leq 2^{-n(1-\epsilon)H(X)} \\ (1 - \delta_\epsilon(n))2^{n(1-\epsilon)H(X)} &\leq |T_\epsilon^n(P_X)| \leq 2^{n(1+\epsilon)H(X)} \\ 1 - \delta_\epsilon(n) &\leq \Pr\left[X^n \in T_\epsilon^n(P_X)\right] \leq 1. \end{aligned}$$

where

$$\delta_\epsilon(n) = 2\,|\mathcal{X}|\, e^{-n\epsilon^2 \mu_X}.$$

Proof see [18, p. 272]

Theorem A1.5 (Properties Jointly Letter-Typical Sequences)
Suppose $0 \leq \epsilon_1 < \epsilon_2 \leq \mu_{XY} = \min_{(a,b) \in supp(P_{XY})} P_{XY}(a, b)$, $(x^n, y^n) \in T_{\epsilon_1}^n(P_{XY})$ *and* (X^n, Y^n) *emitted by a discrete-memoryless source* $P_{XY}(\cdot)$

$$\begin{aligned} 2^{-n(1+\epsilon_1)H(Y|X)} &\leq P_{Y|X}^n(y^n|x^n) \leq 2^{-n(1-\epsilon_1)H(Y|X)} \\ (1 - \delta_{\epsilon_1,\epsilon_2}(n))2^{n(1-\epsilon_2)H(Y|X)} &\leq \left|T_{\epsilon_2}^n(P_{XY}|x^n)\right| \leq 2^{n(1+\epsilon_2)H(Y|X)} \\ 1 - \delta_{\epsilon_1,\epsilon_2}(n) &\leq \Pr\left[Y^n \in T_{\epsilon_2}^n(P_{XY}|x^n)|X^n = x^n\right] \leq 1 \end{aligned}$$

where

$$\begin{aligned} T_\epsilon^n(P_{XY}|x^n) &= \{y^n : (x^n, y^n) \in T_\epsilon^n(P_{XY})\} \\ \delta_{\epsilon_1,\epsilon_2}(n) &= 2\,|\mathcal{X}|\,|\mathcal{Y}| \exp\left(-n\frac{(\epsilon_2 - \epsilon_1)^2}{1 + \epsilon_1}\mu_{XY}\right). \end{aligned}$$

Proof see [18, p. 280]

Theorem A1.6 (Properties Independent Letter-Typical Sequences)
Consider a joint distribution $P_{XY}(\cdot)$ *and suppose* $0 \le \epsilon_1 < \epsilon_2 \le \mu_{XY}$, $x^n \in T^n_{\epsilon_1}(P_X)$ *and* Y^n *is emitted by a discrete-memoryless source* $P_Y(\cdot)$

$$(1-\delta_{\epsilon_1,\epsilon_2}(n))2^{-n[I(X;Y)+2\epsilon_2 H(Y)]} \le \Pr\left[Y^n \in T^n_{\epsilon_2}(P_{XY}|x^n)\right] \le 2^{-n[I(X;Y)-2\epsilon_2 H(Y)]}.$$

where

$$\delta_{\epsilon_1,\epsilon_2}(n) = 2\,|\mathcal{X}|\,|\mathcal{Y}|\exp\left(-n\frac{(\epsilon_2-\epsilon_1)^2}{1+\epsilon_1}\mu_{XY}\right).$$

Proof see [18, p. 277]

Theorem A1.7
Suppose $0 \le \epsilon_1 < \epsilon_2 \le \mu_{UXY} = \min_{(a,b,c)\in supp(P_{UXY})} P_{UXY}(a,b,c)$, $(u^n, y^n) \in T^n_{\epsilon_1}(P_{UY})$ *and* X_i *is emitted by a discrete-memoryless source* $P_{X|U}(\cdot|u_i)$ *for* $i = 1 \ldots n$

$$\begin{aligned}(1-\delta_{\epsilon_1,\epsilon_2}(n))2^{-n[I(X;Y|U)+2\epsilon_2 H(X|U)]} &\le \Pr\left[X^n \in T^n_{\epsilon_2}(P_{UXY}|u^n, y^n)|U^n = u^n\right] \\ &\le 2^{-n[I(X;Y|U)-2\epsilon_2 H(X|U)]}\end{aligned}$$

where

$$\delta_{\epsilon_1,\epsilon_2}(n) = 2\,|\mathcal{U}|\,|\mathcal{X}|\,|\mathcal{Y}|\exp\left(-n\frac{(\epsilon_2-\epsilon_1)^2}{1+\epsilon_1}\mu_{UXY}\right).$$

Proof see [18, p. 338]

Definition A1.8 (Markov Chain)
For the discrete random variables X, Y *and* Z

$$X - Y - Z$$

form a Markov chain if

$$P(x,y,z) = \begin{cases} P(x,y)\,P(z|y) & \textit{if } P(y) > 0, \\ 0 & \textit{else.}\end{cases}$$

Theorem A1.9 (Markov Lemma)
Suppose

$$X - Y - Z$$

form a Markov chain, $0 \leq \epsilon_1 < \epsilon_2 \leq \mu_{XYZ} = \min_{(a,b,c)\in supp(P_{XYZ})} P_{XYZ}(a,b,c)$, $(x^n, y^n) \in T^n_{\epsilon_1}(P_{XY})$ *and* (X^n, Y^n, Z^n) *emitted by a discrete-memoryless source* $P_{XYZ}(\cdot)$

$$1 - \delta_{\epsilon_1,\epsilon_2}(n) \leq \Pr\left[Z^n \in T^n_{\epsilon_2}(P_{XYZ}|x^n, y^n)|Y^n = y^n\right]$$

where

$$\delta_{\epsilon_1,\epsilon_2}(n) = 2\,|\mathcal{X}|\,|\mathcal{Y}|\,|\mathcal{Z}|\exp\left(-n\frac{(\epsilon_2 - \epsilon_1)^2}{1+\epsilon_1}\mu_{XYZ}\right).$$

Proof see [18, p. 319]

A2. Proofs: Achievable Rates Half-Duplex Relay Channel

Comments and Assumptions Random encoding, jointly typical decoding and compression will be used to show which rates are achievable for the half-duplex one-way relay channel. For the following proofs it will be assumed that the transmission is performed with $n \geq 2$ channel uses and two phases $l = 1, 2$. Phase 1 features $n_1 \geq 1$ transmission slots, phase 2 supports $n_2 \geq 1$ transmission slots, with $n_1 + n_2 = n$. If n grows n_1 and n_2 are assumed to grow at the same rate, meaning that doubling n doubles n_l. For large n, $\frac{n_l}{n} \to \tau_l$ with $0 < \tau_l \leq 1$. The message $w \in [1; 2^{nR}]$ will be sent from node 1 to node 3. For all proofs $2^{nR} \in \mathbb{Z}_+$.

A2.1 Decode-and-Forward

Codebook $\mathbf{c_1}$ Generate $2^{n(R_1+R_2)}$ n_1-sequences $x_1^{n_1}(r, s)$, $r \in [1; 2^{nR_1}]$, $s \in [1; 2^{nR_2}]$ by choosing each element $x_{1,k}^{(1)}(r, s)$ independently according to $P_{X_1^{(1)}}(\cdot)$.

Codebook $\mathbf{c_2}$ Generate 2^{nR_1} n_2-sequences $x_2^{n_2}(r)$ by choosing each element $x_{2,k}^{(2)}(r)$ independently according to $P_{X_2^{(2)}}(\cdot)$.

Codebook $\mathbf{c_3}$ For each $x_2^{n_2}(r)$ generate 2^{nR_3} n_2-sequences $x_1^{n_2}(r, t)$, $t \in [1; 2^{nR_3}]$, by choosing each element $x_{1,k}^{(2)}(r, t)$ independently according to $P_{X_1^{(2)}|X_2^{(2)}}(\cdot|x_{2,k}^{(2)}(r))$.

Node 1 The message w is reindexed by (r, s, t). In the first phase node 1 transmits $x_1^{n_1}(r, s)$ within n_1 transmissions. In the second phase node 1 transmits $x_1^{n_2}(r, t)$ within n_2 transmissions.

Node 2 In the first phase $y_2^{n_1}$ is observed. After the first phase node 2 tries to find a pair $(\tilde{r}, \tilde{s})$ such that

$$(x_1^{n_1}(\tilde{r}, \tilde{s}), y_2^{n_1}) \in T_\epsilon^{n_1}(P_{X_1^{(1)}Y_2^{(1)}}). \tag{A1}$$

If there is none or more than one such pair $(\tilde{r}, \tilde{s})$, set $(\hat{r}(2), \hat{s}(2)) = (1, 1)$. Otherwise, the found index pair $(\tilde{r}, \tilde{s})$ is the estimate $(\hat{r}(2), \hat{s}(2))$ of node 2. In the second phase node 2 sends $x_2^{n_2}(\hat{r}(2))$ by the use of n_2 transmissions.

Node 3 In the first phase $y_3^{n_1}$ is observed. In the second phase $y_3^{n_2}$ is observed. After the second phase node 3 tries to find an index $\tilde{r}$ such that

$$(x_2^{n_2}(\tilde{r}), y_3^{n_2}) \in T_\epsilon^{n_2}(P_{X_2^{(2)}Y_3^{(2)}}). \tag{A2}$$

If there is none or more than one such index $\tilde{r}$, set $\hat{r}(3) = 1$. Otherwise, the found index $\tilde{r}$ is the estimate $\hat{r}(3)$ of node 3. Then node 3 tries to find an index $\tilde{t}$ such that

$$\big(x_1^{n_2}(\hat{r}(3), \tilde{t}), x_2^{n_2}(\hat{r}(3)), y_3^{n_2}\big) \in T_\epsilon^{n_2}(P_{X_1^{(2)}X_2^{(2)}Y_3^{(2)}}). \tag{A3}$$

If there is none or more than one such index $\tilde{t}$, set $\hat{t}(3) = 1$. Otherwise, the found index $\tilde{t}$ is the estimate $\hat{t}(3)$ of node 3. Finally, node 3 tries to find an index $\tilde{s}$ such that

$$(x_1^{n_1}(\hat{r}(3), \tilde{s}), y_3^{n_1}) \in T_\epsilon^{n_1}(P_{X_1^{(1)}Y_3^{(1)}}). \tag{A4}$$

If there is none or more than one such index $\tilde{s}$, set $\hat{s}(3) = 1$. Otherwise, the found index $\tilde{s}$ is the estimate $\hat{s}(3)$ of node 3. The output message $\hat{w}(3)$ of node 3 is found by reindexing $(\hat{r}(3), \hat{s}(3), \hat{t}(3))$.

Error Analysis The event

$$\mathcal{E} = \left\{\hat{W}(3) \neq W\right\} = \left\{\hat{R}(3) \neq R\right\} \cup \left\{\hat{S}(3) \neq S\right\} \cup \left\{\hat{T}(3) \neq T\right\} \tag{A5}$$

denotes the occurence of an error in the decode-and-forward communication scheme. The probability of this event is

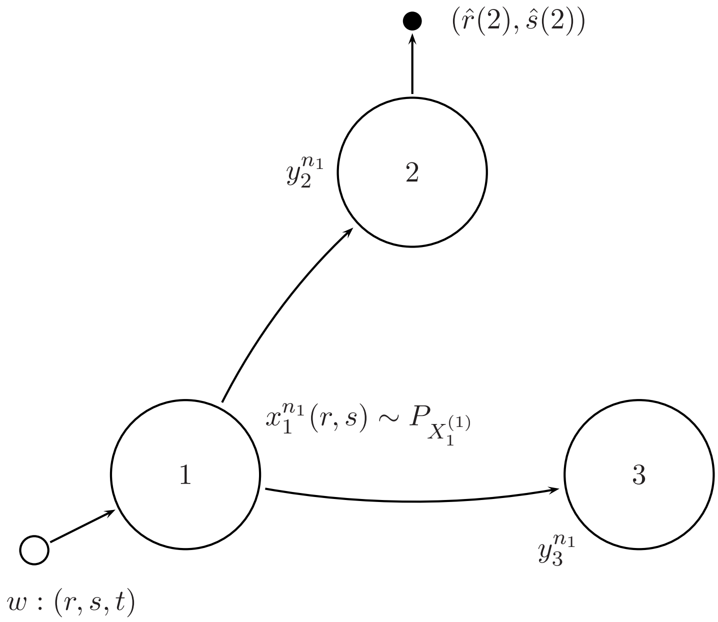
$$\begin{aligned}
&\Pr\left[\left\{\hat{R}(3) \neq R\right\} \cup \left\{\hat{S}(3) \neq S\right\} \cup \left\{\hat{T}(3) \neq T\right\}\right] \\
&\leq \Pr\left[\hat{R}(3) \neq R\right] + \Pr\left[\hat{T}(3) \neq T|\hat{R}(3) = R\right] + \Pr\left[\hat{S}(3) \neq S|\hat{R}(3) = R\right] \\
&\leq \Pr\left[\left\{\hat{R}(2) \neq R\right\} \cup \left\{\hat{R}(3) \neq \hat{R}(2)\right\}\right] + \Pr\left[\hat{T}(3) \neq T|\hat{R}(3) = R\right] + \Pr\left[\hat{S}(3) \neq S|\hat{R}(3) = R\right] \\
&\leq \Pr\left[\hat{R}(2) \neq R\right] + \Pr\left[\hat{R}(3) \neq \hat{R}(2)|\hat{R}(2) = R\right] + \Pr\left[\hat{T}(3) \neq T|\hat{R}(3) = R\right] \\
&+ \Pr\left[\hat{S}(3) \neq S|\hat{R}(3) = R\right] \\
&\leq \Pr\left[(\hat{R}(2), \hat{S}(2)) \neq (R, S)\right] + \Pr\left[\hat{R}(3) \neq \hat{R}(2)|\hat{R}(2) = R\right] + \Pr\left[\hat{T}(3) \neq T|\hat{R}(3) = R\right] \\
&+ \Pr\left[\hat{S}(3) \neq S|\hat{R}(3) = R\right].
\end{aligned} \tag{A6}$$

The probability of error can be upper bounded by the sum over the individual error probabilities of each decoder under the assumption that if the decoder needs side information this side information is error-free. Define the *conditional probabilities of error*

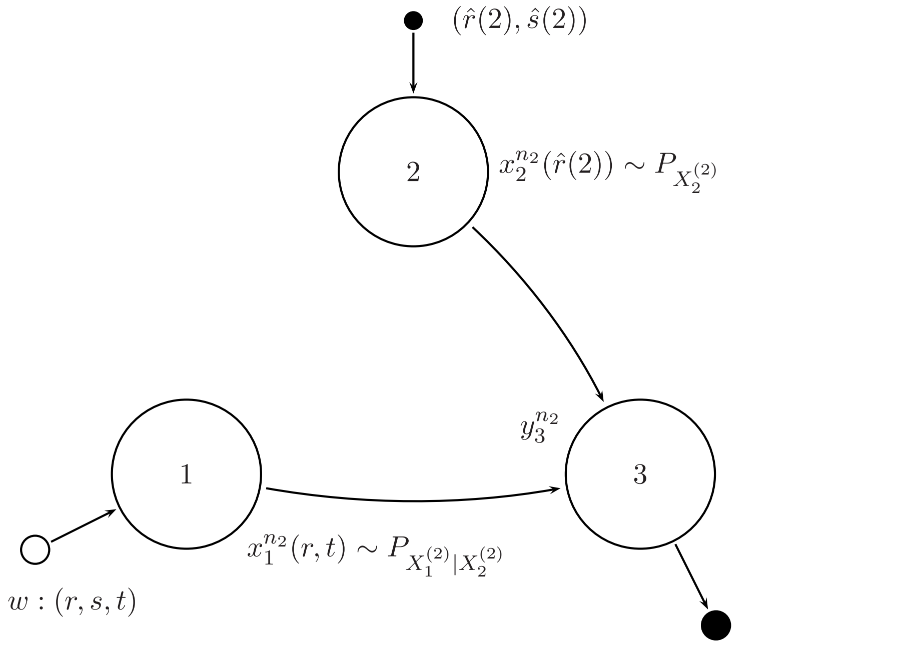
$$\begin{aligned}
\lambda_{2,(r,s)}(c_1) &= \Pr\left[(\hat{R}(2), \hat{S}(2)) \neq (r, s)|X_1^{n_1} = x_1^{n_1}(r, s)\right] = \sum_{\mathcal{Y}_2^{n_1}} P\left(y_2^{n_1}|x_1^{n_1}(r, s)\right) I\left(g_2(y_2^{n_1}) \neq (r, s)\right) \\
\lambda_{31,r}(c_2) &= \Pr\left[\hat{R}(3) \neq r|X_2^{n_2} = x_2^{n_2}(r)\right] = \sum_{\mathcal{Y}_3^{n_2}} P\left(y_3^{n_2}|x_2^{n_2}(\hat{r}(2))\right) I\left(g_{31}(y_3^{n_2}) \neq r\right) \\
\lambda_{32,(r,t)}(c_3) &= \Pr\left[\hat{T}(3) \neq t|X_1^{n_2} = x_1^{n_2}(r, t)\right] = \sum_{\mathcal{Y}_3^{n_2}} P\left(y_3^{n_2}|x_1^{n_2}(r, t)\right) I\left(g_{32}(y_3^{n_2}, r) \neq t\right) \\
\lambda_{33,(r,s)}(c_1) &= \Pr\left[\hat{S}(3) \neq s|X_1^{n_1} = x_1^{n_1}(r, s)\right] = \sum_{\mathcal{Y}_3^{n_1}} P\left(y_3^{n_1}|x_1^{n_1}(r, s)\right) I\left(g_{33}(y_3^{n_1}, r) \neq s\right)
\end{aligned} \tag{A7}$$

where $I(\cdot)$ is the indicator function and the functions

$$\begin{aligned}
g_2(y_2^{n_1}) &: \mathcal{Y}_2^{n_1} \to \mathcal{R} \times \mathcal{S} \\
g_{31}(y_3^{n_2}) &: \mathcal{Y}_3^{n_2} \to \mathcal{R} \\
g_{32}(y_3^{n_2}, \hat{r}(3)) &: \mathcal{Y}_3^{n_2} \times \mathcal{R} \to \mathcal{T} \\
g_{33}(y_3^{n_1}, \hat{r}(3)) &: \mathcal{Y}_3^{n_1} \times \mathcal{R} \to \mathcal{S}
\end{aligned} \tag{A8}$$



(a) First Phase (n_1 transmission slots)



(b) Second Phase (n_2 transmission slots)

Figure A1: Sketch Achievable Rate Proof, Decode-and-Forward are realized by jointly typical decoding as described in the transmission scheme. The *average probability of error* P_e over all codewords and codebooks is

$$\begin{aligned}P_e &= \sum_{\mathcal{W}} P(w) \sum_{\mathcal{C}_1} \sum_{\mathcal{C}_2} \sum_{\mathcal{C}_3} P(c_1) P(c_2) P(c_3|c_2) \Pr\left[\hat{W}(3) \neq W | W = w\right] \\ &\leq \sum_{\mathcal{R}} \sum_{\mathcal{S}} P(r) P(s) \sum_{\mathcal{C}_1} P(c_1) \left(\lambda_{2,(r,s)}(c_1) + \lambda_{33,(r,s)}(c_1)\right) + \sum_{\mathcal{R}} P(r) \sum_{\mathcal{C}_2} P(c_2) \lambda_{31,r}(c_2) \\ &\quad + \sum_{\mathcal{R}} \sum_{\mathcal{T}} P(r) P(t) \sum_{\mathcal{C}_2} P(c_2) \sum_{\mathcal{C}_3} P(c_3|c_2) \lambda_{32,(r,t)}(c_3). \end{aligned} \tag{A9}$$

Under the assumption that W and therefore R, S and T are uniformly distributed

$$\begin{aligned}P_e &\leq \frac{1}{2^{nR_1}} \frac{1}{2^{nR_2}} \sum_{\mathcal{R}} \sum_{\mathcal{S}} \sum_{\mathcal{C}_1} P(c_1) \left(\lambda_{2,(r,s)}(c_1) + \lambda_{33,(r,s)}(c_1)\right) + \frac{1}{2^{nR_1}} \sum_{\mathcal{R}} \sum_{\mathcal{C}_2} P(c_2) \lambda_{31,r}(c_2) \\ &\quad + \frac{1}{2^{nR_1}} \frac{1}{2^{nR_3}} \sum_{\mathcal{R}} \sum_{\mathcal{T}} \sum_{\mathcal{C}_3} P(c_3) \lambda_{32,(r,t)}(c_3). \end{aligned} \tag{A10}$$

Due to the symmetry of the code construction the conditional probabilities of error do not depend on the particular realizations of R, S and T. Therefore, it can be assumed that an index triple (r, s, t) has been sent by node 1 and $\hat{r}(2)$ by node 2. The upper bound on the average probability of the error event $\mathcal{E}$ reduces to

$$\begin{aligned}P_e &\leq \sum_{\mathcal{C}_1} P(c_1) \left(\lambda_{2,(r,s)}(c_1) + \lambda_{33,(r,s)}(c_1)\right) + \sum_{\mathcal{C}_2} P(c_2) \lambda_{31,\hat{r}(2)}(c_2) + \sum_{\mathcal{C}_3} P(c_3) \lambda_{32,(r,t)}(c_3) \\ &= \Pr\left[\mathcal{E}_2 \,|(R,S) = (r,s)\right] + \Pr\left[\mathcal{E}_{31} \,|\hat{R}(2) = \hat{r}(2)\right] + \Pr\left[\mathcal{E}_{32} \,|(\hat{R}(3), T) = (r,t)\right] \\ &\quad + \Pr\left[\mathcal{E}_{33} \,|(\hat{R}(3), S) = (r,s)\right] \end{aligned} \tag{A11}$$

where the events

$$\begin{aligned}\mathcal{E}_2 &= \left\{g_2(Y_2^{n_1}) \neq (R,S)\right\} \\ \mathcal{E}_{31} &= \left\{g_{31}(Y_3^{n_2}) \neq \hat{R}(2)\right\} \\ \mathcal{E}_{32} &= \left\{g_{32}(Y_3^{n_2}, \hat{R}(3)) \neq T\right\} \\ \mathcal{E}_{33} &= \left\{g_{33}(Y_3^{n_1}, \hat{R}(3)) \neq S\right\} \end{aligned} \tag{A12}$$

denote the occurrence of an error in the first, second, third and fourth decoding step. In the following it will be analyzed under which circumstances the upper bound on P_e can be made arbitrarily small. Note that if P_e can be made arbitrarily small one can argue [7, Section 7.7] that there must exist codes $\hat{c}_1$, $\hat{c}_2$ and $\hat{c}_3$ that can be modified to produce optimal codes $c_1^\star$, $c_2^\star$ and $c_3^\star$ which allow reliable communication over the half-duplex relay channel.

First Step Choose $0 < \epsilon_1 < \epsilon \leq \mu_{X_1^{(1)}Y_2^{(1)}}$. The probability of the event $\mathcal{E}_2$ conditioned on the index pair $(R,S) = (r,s)$ being sent can be upper bounded by

$$\begin{aligned}
\Pr\left[\mathcal{E}_2 \,|(R,S)=(r,s)\right] &\leq \Pr\left[(X_1^{n_1}(r,s), Y_2^{n_1}) \notin T_{\epsilon_1}^{n_1}(P_{X_1^{(1)}Y_2^{(1)}})\right] \\
&\quad + \sum_{\substack{\tilde{r}=1,\tilde{s}=1 \\ (\tilde{r},\tilde{s})\neq(r,s)}}^{2^{nR_1},2^{nR_2}} \sum_{T_{\epsilon_1}^{n_1}(P_{Y_2^{(1)}})} P\left(y_2^{n_1}\right) \Pr\left[X_1^{n_1}(\tilde{r},\tilde{s}) \in T_{\epsilon}^{n_1}(P_{X_1^{(1)}Y_2^{(1)}}|y_2^{n_1})\right] \\
&\leq \delta_{\epsilon_1}(n_1) + \sum_{\substack{\tilde{r}=1,\tilde{s}=1 \\ (\tilde{r},\tilde{s})\neq(r,s)}}^{2^{nR_1},2^{nR_2}} \sum_{T_{\epsilon_1}^{n_1}(P_{Y_2^{(1)}})} P\left(y_2^{n_1}\right) 2^{-n_1(I(X_1^{(1)};Y_2^{(1)})-2\epsilon H(X_1^{(1)}))} \\
&\leq \delta_{\epsilon_1}(n_1) + \sum_{\substack{\tilde{r}=1,\tilde{s}=1 \\ (\tilde{r},\tilde{s})\neq(r,s)}}^{2^{nR_1},2^{nR_2}} 2^{-n_1(I(X_1^{(1)};Y_2^{(1)})-2\epsilon H(X_1^{(1)}))} \\
&\leq \delta_{\epsilon_1}(n_1) + 2^{n(R_1+R_2)} 2^{-n_1(I(X_1^{(1)};Y_2^{(1)})-2\epsilon H(X_1^{(1)}))} \\
&= \delta_{\epsilon_1}(n_1) + 2^{-n(\frac{n_1}{n} I(X_1^{(1)};Y_2^{(1)})-2\epsilon\frac{n_1}{n} H(X_1^{(1)})-R_1-R_2)}.
\end{aligned} \tag{A13}$$

where Theorems A1.4 and A1.6 have been used. This shows that one can drive $\Pr\left[\mathcal{E}_2 \,|(R,S)=(r,s)\right]$ to zero by choosing n large and satisfying

$$R_1 + R_2 < \tau_1 I(X_1^{(1)};Y_2^{(1)}) - 2\epsilon\tau_1 H(X_1^{(1)}). \tag{A14}$$

Second Step Choosing $0 < \epsilon_1 < \epsilon \leq \mu_{X_2^{(2)}Y_3^{(2)}}$ one can upper bound

$$\begin{aligned}
\Pr\left[\mathcal{E}_{31} \,|\hat{R}(2) = \hat{r}(2)\right] &\leq \Pr\left[(X_2^{n_2}(\hat{r}(2)), Y_3^{n_2}) \notin T_{\epsilon_1}^{n_2}(P_{X_2^{(2)}Y_3^{(2)}})\right] \\
&\quad + \sum_{\substack{\tilde{r}=1 \\ \tilde{r}\neq\hat{r}(2)}}^{2^{nR_1}} \sum_{T_{\epsilon_1}^{n_2}(P_{Y_3^{(2)}})} P\left(y_3^{n_2}\right) \Pr\left[X_2^{n_2}(\tilde{r}) \in T_{\epsilon}^{n_2}(P_{X_2^{(2)}Y_3^{(2)}}|y_3^{n_2})\right] \\
&\leq \delta_{\epsilon_1}(n_2) + 2^{-n(\frac{n_2}{n} I(X_2^{(2)};Y_3^{(2)})-2\epsilon\frac{n_2}{n} H(X_2^{(2)})-R_1)}
\end{aligned} \tag{A15}$$

by using Theorems A1.4 and A1.6. This shows that one can make $\Pr\left[\mathcal{E}_{31} \,|\hat{R}(2) = \hat{r}(2)\right]$ arbitrarily small by choosing n large and satisfying

$$R_1 < \tau_2 I(X_2^{(2)};Y_3^{(2)}) - 2\epsilon\tau_2 H(X_2^{(2)}). \tag{A16}$$

Third Step Choosing $0 < \epsilon_1 < \epsilon \leq \mu_{X_1^{(2)}X_2^{(2)}Y_3^{(2)}}$ one can upper bound

$$\begin{aligned}
&\Pr\left[\mathcal{E}_{32} \,|(\hat{R}(3),T) = (r,t)\right] \leq \Pr\left[(X_1^{n_2}(r,t), X_2^{n_2}(r), Y_3^{n_2}) \notin T_{\epsilon_1}^{n_2}(P_{X_1^{(2)}X_2^{(2)}Y_3^{(2)}})\right] \\
&+ \sum_{\substack{\tilde{t}=1 \\ \tilde{t}\neq t}}^{2^{nR_3}} \sum_{T_{\epsilon_1}^{n_2}(P_{X_2^{(2)}Y_3^{(2)}})} P\left(x_2^{n_2}(r), y_3^{n_2}\right) \Pr\left[X_1^{n_2}(r,\tilde{t}) \in T_{\epsilon}^{n_2}(P_{X_1^{(2)}X_2^{(2)}Y_3^{(2)}}|x_2^{n_2}(r), y_3^{n_2}) | X_2^{n_2}(r) = x_2^{n_2}(r)\right] \\
&\leq \delta_{\epsilon_1}(n_2) + 2^{-n(\frac{n_2}{n} I(X_1^{(2)};Y_3^{(2)}|X_2^{(2)})-2\epsilon\frac{n_2}{n} H(X_1^{(2)}|X_2^{(2)})-R_3)}.
\end{aligned} \tag{A17}$$

by using Theorem A1.4 and A1.7. This shows that one can make $\Pr\left[\mathcal{E}_{32}\,|(\hat{R}(3),T)=(r,t)\right]$ arbitrarily small by choosing n large and satisfying

$$R_3 < \tau_2 I(X_1^{(2)};Y_3^{(2)}|X_2^{(2)}) - 2\epsilon\tau_2 H(X_1^{(2)}|X_2^{(2)}). \tag{A18}$$

Fourth Step Choosing $0 < \epsilon_1 < \epsilon \le \mu_{X_1^{(1)}Y_3^{(1)}}$ one can upper bound

$$\begin{aligned}\Pr\left[\mathcal{E}_{33,(r,s)}\,|(\hat{R}(3),S)=(r,s)\right] &\le \Pr\left[(X_1^{n_1}(r,s),Y_3^{n_1})\notin T_{\epsilon_1}^{n_1}(P_{X_1^{(1)}Y_3^{(1)}})\right]\\ &\quad + \sum_{\substack{\tilde{s}=1\\ \tilde{s}\ne s}}^{2^{nR_2}} \sum_{T_{\epsilon_1}^{n_1}(P_{Y_3^{(1)}})} P\left(y_3^{n_1}\right)\Pr\left[X_1^{n_1}(r,\tilde{s})\in T_{\epsilon}^{n_1}(P_{X_1^{(1)}Y_3^{(1)}}|y_3^{n_1})\right]\\ &\le \delta_{\epsilon_1}(n_1) + 2^{-n(\frac{n_1}{n}I(X_1^{(1)};Y_3^{(1)})-2\epsilon\frac{n_1}{n}H(X_1^{(1)})-R_2)}.\end{aligned} \tag{A19}$$

by using Theorem A1.4 and A1.6. This shows that one can make $\Pr\left[\mathcal{E}_{33}\,|(\hat{R}(3),S)=(r,s)\right]$ arbitrarily small by choosing n large and satisfying

$$R_2 < \tau_1 I(X_1^{(1)};Y_3^{(1)}) - 2\epsilon\tau_1 H(X_1^{(1)}). \tag{A20}$$

Rates The error analysis reveals that for large n reliable communication requires

$$R_1 + R_2 < \tau_1 I(X_1^{(1)};Y_2^{(1)}) - 2\epsilon\tau_1 H(X_1^{(1)}) \tag{A21}$$

at node 2 and

$$\begin{aligned}R_1 &< \tau_2 I(X_2^{(2)};Y_3^{(2)}) - 2\epsilon\tau_2 H(X_2^{(2)})\\ R_3 &< \tau_2 I(X_1^{(2)};Y_3^{(2)}|X_2^{(2)}) - 2\epsilon\tau_2 H(X_1^{(2)}|X_2^{(2)})\\ R_2 &< \tau_1 I(X_1^{(1)};Y_3^{(1)}) - 2\epsilon\tau_1 H(X_1^{(1)})\end{aligned} \tag{A22}$$

at node 3. This gives

$$\begin{aligned}R = R_1+R_2+R_3 &< \tau_1 I(X_1^{(1)};Y_2^{(1)}) + \tau_2 I(X_1^{(2)};Y_3^{(2)}|X_2^{(2)}) - 2\epsilon\left(\tau_1 H(X_1^{(1)}) + \tau_2 H(X_1^{(2)}|X_2^{(2)})\right)\\ R = R_1+R_2+R_3 &< \tau_1 I(X_1^{(1)};Y_3^{(1)}) + \tau_2 I(X_2^{(2)};Y_3^{(2)}) + \tau_2 I(X_1^{(2)};Y_3^{(2)}|X_2^{(2)})\\ &\quad - 2\epsilon\left(\tau_1 H(X_1^{(1)}) + \tau_2 H(X_2^{(2)}) + \tau_2 H(X_1^{(2)}|X_2^{(2)})\right)\\ &= \tau_1 I(X_1^{(1)};Y_3^{(1)}) + \tau_2 I(X_1^{(2)}X_2^{(2)};Y_3^{(2)}) - 2\epsilon\left(\tau_1 H(X_1^{(1)}) + \tau_2 H(X_1^{(2)}X_2^{(2)})\right).\end{aligned} \tag{A23}$$

Choosing $\epsilon > 0$ but arbitrarily small establishes the proposition. ■

A2.2 Compress-and-Forward

Code Generate 2^{nR_1} n_1-sequences $x_1^{n_1}(r)$, $r \in [1; 2^{nR_1}]$, by choosing each element $x_{1,k}^{(1)}(r)$ independently according to $P_{X_1^{(1)}}(\cdot)$. Choose a "quantization channel" $P_{\hat{Y}_2^{(1)}|Y_2^{(1)}}(\cdot|\cdot)$ and calculate $P_{\hat{Y}_2^{(1)}}(\cdot)$ as the marginal distribution of $P_{\hat{Y}_2^{(1)}Y_2^{(1)}}(\cdot)$. Generate $2^{n(R_3+R_4)}$ sequences $\hat{y}_2^{n_1}(t,o)$, $t \in [1; 2^{nR_3}]$, $o \in [1; 2^{nR_4}]$ by choosing the elements of $\hat{y}_2^{n_1}(t,o)$ independently according to $P_{\hat{Y}_2^{(1)}}(\cdot)$. Generate 2^{nR_2} n_2-sequences $x_1^{n_2}(s)$, $s \in [1; 2^{nR_2}]$, by choosing each element $x_{1,k}^{(2)}(s)$ independently according to $P_{X_1^{(2)}}(\cdot)$. Generate 2^{nR_3} n_2-sequences $x_2^{n_2}(t)$, by choosing each element $x_{2,k}^{(2)}(t)$ independently according to $P_{X_2^{(2)}}(\cdot)$.

Node 1 The message w is reindexed by (r, s). In the first phase node 1 transmits $x_1^{n_1}(r)$ within n_1 transmission slots. In the second phase node 1 transmits $x_1^{n_2}(s)$ within n_2 transmission slots.

Node 2 In the first phase $y_2^{n_1}$ is observed. After the first phase node 2 tries to find a pair $(\tilde{t}, \tilde{o})$ such that

$$\left(\hat{y}_2^{n_1}(\tilde{t}, \tilde{o}), y_2^{n_1}\right) \in T_\epsilon^{n_1}(P_{\hat{Y}_2^{(1)}Y_2^{(1)}}). \tag{A24}$$

If there is none such pair $(\tilde{t}, \tilde{o})$ an error is declared. Otherwise, the found index pair $(\tilde{t}, \tilde{o})$ is the estimate $(\hat{t}(2), \hat{o}(2))$ of node 2 where the pair with the smallest linear index $2^{nR_3}(t-1)+o$ is selected if more than one pair was found. In the second phase node 2 transmits $x_2^{n_2}(\hat{t}(2))$ by using n_2 transmissions.

Node 3 In the first phase $y_3^{n_1}$ is observed. In the second phase $y_3^{n_2}$ is observed. After the second phase node 3 tries to find an index $\tilde{t}$ such that

$$\left(x_2^{n_2}(\tilde{t}), y_3^{n_2}\right) \in T_\epsilon^{n_2}(P_{X_2^{(2)}Y_3^{(2)}}). \tag{A25}$$

If there is none or more than one such index $\tilde{t}$, set $\hat{t}(3) = 1$. Otherwise, the found index $\tilde{t}$ is the estimate $\hat{t}(3)$ of node 3. Then node 3 tries to find an index $\tilde{s}$ such that

$$\left(x_1^{n_2}(\tilde{s}), x_2^{n_2}(\hat{t}(3)), y_3^{n_2}\right) \in T_\epsilon^{n_2}(P_{X_1^{(2)}X_2^{(2)}Y_3^{(2)}}). \tag{A26}$$

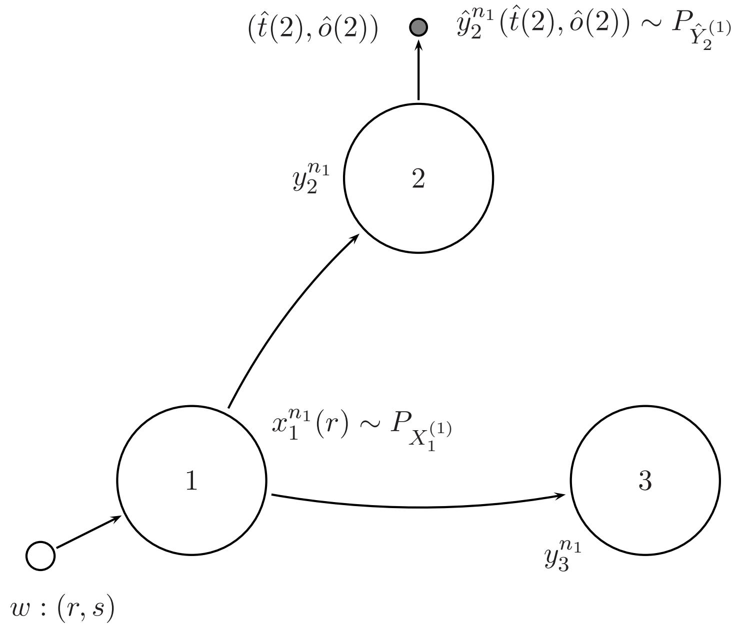
If there is none or more than one such index $\tilde{s}$, set $\hat{s}(3) = 1$. Otherwise, the found index $\tilde{s}$ is the estimate $\hat{s}(3)$ of node 3. Next, node 3 tries to find an index $\tilde{o}$, such that

$$\left(\hat{y}_2^{n_1}(\hat{t}(3), \tilde{o}), y_3^{n_1}\right) \in T_\epsilon^{n_1}(P_{\hat{Y}_2^{(1)}Y_3^{(1)}}). \tag{A27}$$

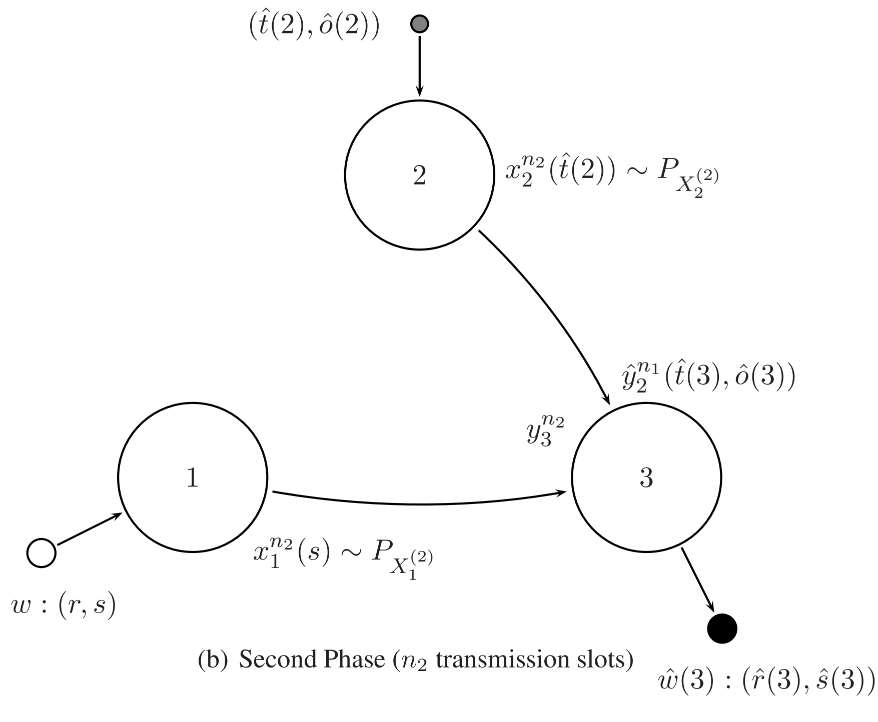
If there is none or more than one such index $\tilde{o}$, set $\hat{o}(3) = 1$. Otherwise, the found index $\tilde{o}$ is the estimate $\hat{o}(3)$ of node 3. Finally node 3 tries to find an index $\tilde{r}$ such that

$$\left(x_1^{n_1}(\tilde{r}), \hat{y}_2^{n_1}(\hat{t}(3), \hat{o}(3)), y_3^{n_1}\right) \in T_\epsilon^{n_1}(P_{X_1^{(1)}\hat{Y}_2^{(1)}Y_3^{(1)}}). \tag{A28}$$

If there is none or more than one such index $\tilde{r}$, set $\hat{r}(3) = 1$. Otherwise, the found index $\tilde{r}$ is the estimate $\hat{r}(3)$ of node 3. The output message $\hat{w}(3)$ of node 3 is found by reindexing $(\hat{r}(3), \hat{s}(3))$.



(a) First Phase (n_1 transmission slots)



(b) Second Phase (n_2 transmission slots)

Figure A2: Sketch Achievable Rate Proof, Compress-and-Forward

Error Analysis Using similar arguments as in the proof A2.1 the average probability of error for the compress-and-forward relaying scheme can be upper bounded by

$$
\begin{aligned}
P_e \leq \Pr\left[\mathcal{E}_q\right] &+ \Pr\left[\mathcal{E}_{31}\,|\hat{T}(2) = \hat{t}(2)\right] + \Pr\left[\mathcal{E}_{32}\,|(S, \hat{T}(3)) = (s, \hat{t}(2))\right] \\
&+ \Pr\left[\mathcal{E}_{33}\,|(\hat{T}(3), \hat{O}(2)) = (\hat{t}(2), \hat{o}(2))\right] + \Pr\left[\mathcal{E}_{34}\,|(R, \hat{T}(3), \hat{O}(3)) = (r, \hat{t}(2), \hat{o}(2))\right] \quad \text{(A29)}
\end{aligned}
$$

where the events

$$
\begin{aligned}
\mathcal{E}_{31} &= \left\{g_{31}(Y_3^{n_2}) \neq \hat{T}(2)\right\} \\
\mathcal{E}_{32} &= \left\{g_{32}(Y_3^{n_2}, \hat{T}(3)) \neq S\right\} \\
\mathcal{E}_{33} &= \left\{g_{33}(Y_3^{n_1}, \hat{T}(3)) \neq \hat{O}(2)\right\} \\
\mathcal{E}_{34} &= \left\{g_{34}(Y_3^{n_1}, \hat{T}(3), \hat{O}(3)) \neq R\right\} \qquad \text{(A30)}
\end{aligned}
$$

denote the occurrence of an error in the four decoding steps of node 3. The event $\mathcal{E}_q$ occures if node 2 declares an error. The functions

$$
\begin{aligned}
g_{31}(y_3^{n_2}) &: \mathcal{Y}_3^{n_2} \to \mathcal{T} \\
g_{32}(y_3^{n_2}, \hat{t}(3)) &: \mathcal{Y}_3^{n_2} \times \mathcal{T} \to \mathcal{S} \\
g_{33}(y_3^{n_1}, \hat{t}(3)) &: \mathcal{Y}_3^{n_1} \times \mathcal{T} \to \mathcal{O} \\
g_{33}(y_3^{n_1}, \hat{t}(3), \hat{o}(3)) &: \mathcal{Y}_3^{n_1} \times \mathcal{T} \times \mathcal{O} \to \mathcal{R} \qquad \text{(A31)}
\end{aligned}
$$

are realized by jointly typical decoding as described in the transmission scheme.

Quantization Choose $0 < \epsilon_1 < \epsilon \leq \mu_{Y_2^{(1)}\hat{Y}_2^{(1)}}$. The probability of the event $\mathcal{E}_q$ is

$$
\begin{aligned}
\Pr\left[\mathcal{E}_q\right] &\leq \Pr\left[Y_2^{n_1} \notin T_{\epsilon_1}^{n_1}(P_{Y_2^{(1)}})\right] \\
&\quad + \sum_{T_{\epsilon_1}^{n_1}(P_{Y_2^{(1)}})} P\left(y_2^{n_1}\right) \Pr\left[\bigcap_{\tilde{t}=1,\tilde{o}=1}^{2^{nR_3},2^{nR_4}} \left\{\hat{Y}_2^{n_1}(\tilde{t},\tilde{o}) \notin T_{\epsilon}^{n_1}(P_{Y_2^{(1)}\hat{Y}_2^{(1)}}|y_2^{n_1})\right\}\right] \\
&\leq \delta_{\epsilon_1}(n_1) + \sum_{T_{\epsilon_1}^{n_1}(P_{Y_2^{(1)}})} P\left(y_2^{n_1}\right)\left(1 - \Pr\left[\hat{Y}_2^{n_1} \in T_{\epsilon}^{n_1}(P_{Y_2^{(1)}\hat{Y}_2^{(1)}}|y_2^{n_1})\right]\right)^{2^{n(R_3+R_4)}} \\
&\leq \delta_{\epsilon_1}(n_1) + \left(1 - (1-\delta_{\epsilon_1,\epsilon}(n_1))2^{-n_1\left(I(\hat{Y}_2^{(1)};Y_2^{(1)}) + 2\epsilon H(\hat{Y}_2^{(1)})\right)}\right)^{2^{n(R_3+R_4)}} \\
&\leq \delta_{\epsilon_1}(n_1) + \exp\left(-(1-\delta_{\epsilon_1,\epsilon}(n_1))2^{n\left(R_3+R_4-\frac{n_1}{n}I(\hat{Y}_2^{(1)};Y_2^{(1)}) - 2\epsilon\frac{n_1}{n}H(\hat{Y}_2^{(1)})\right)}\right) \qquad \text{(A32)}
\end{aligned}
$$

where Theorem A1.4, A1.6 and the relation $(1-x)^m \leq \exp(-mx)$ were used. This shows that one can make $\Pr\left[\mathcal{E}_q\right]$ arbitrarily small by choosing n large and satisfying

$$
R_3 + R_4 > \tau_1 I(Y_2^{(1)}; \hat{Y}_2^{(1)}) + 2\epsilon\tau_1 H(\hat{Y}_2^{(1)}). \qquad \text{(A33)}
$$

First Step Choose $0 < \epsilon_1 < \epsilon \leq \mu_{X_2^{(2)}Y_3^{(2)}}$. Using Theorem A1.4 and A1.6 the probability of the event $\mathcal{E}_{31}$ conditioned on the index $\hat{T}(2) = \hat{t}(2)$ being sent by node 2 can be upper bounded by

$$
\begin{aligned}
\Pr\left[\mathcal{E}_{31}\,|\hat{T}(2)=\hat{t}(2)\right] &\leq \Pr\left[\left\{(X_2^{n_2}(\hat{t}(2)), Y_3^{n_2}) \notin T_{\epsilon_1}^{n_2}(P_{X_2^{(2)}Y_3^{(2)}})\right\}\right] \\
&\quad + \sum_{\substack{\tilde{t}=1\\ \tilde{t}\neq\hat{t}(2)}}^{2^{nR_3}} \sum_{T_{\epsilon_1}^{n_2}(P_{Y_3^{(2)}})} P\left(y_3^{n_2}\right) \Pr\left[\left\{X_2^{n_2}(\tilde{t}) \in T_{\epsilon}^{n_2}(P_{X_2^{(2)},Y_3^{(2)}}|y_3^{n_2})\right\}\right] \\
&\leq \delta_{\epsilon_1}(n_2) + \sum_{\substack{\tilde{t}=1\\ \tilde{t}\neq\hat{t}(2)}}^{2^{nR_3}} \sum_{T_{\epsilon_1}^{n_2}(P_{Y_3^{(2)}})} P\left(y_3^{n_2}\right) \Pr\left[\left\{X_2^{n_2}(\tilde{t}) \in T_{\epsilon}^{n_2}(P_{X_2^{(2)},Y_3^{(2)}}|y_3^{n_2})\right\}\right] \\
&\leq \delta_{\epsilon_1}(n_2) + 2^{-n(\frac{n_2}{n}I(X_2^{(2)};Y_3^{(2)})-2\epsilon\frac{n_2}{n}H(X_2^{(2)})-R_3)}. \qquad \text{(A34)}
\end{aligned}
$$

This shows that $\Pr\left[\mathcal{E}_{31}\,|\hat{T}(2)=\hat{t}(2)\right]$ gets arbitrarily small when choosing n large and satisfying

$$
R_3 < \tau_2 I(X_2^{(2)};Y_3^{(2)}) - 2\epsilon\tau_2 H(X_2^{(2)}). \qquad \text{(A35)}
$$

Second Step Choosing $0 < \epsilon_1 < \epsilon \leq \mu_{X_1^{(2)}X_2^{(2)}Y_3^{(2)}}$ one can upper bound

$$
\begin{aligned}
\Pr\left[\mathcal{E}_{32}\,|(S,\hat{T}(3))=(s,\hat{t}(2))\right] &\leq \Pr\left[(X_1^{n_2}, X_2^{n_2}, Y_3^{n_2}) \notin T_{\epsilon_1}^{n_2}(P_{X_1^{(2)}X_2^{(2)}Y_3^{(2)}})\right] \\
&\quad + \sum_{\substack{\tilde{s}=1\\ \tilde{s}\neq s}}^{2^{nR_2}} \sum_{T_{\epsilon_1}^{n_2}(P_{X_2^{(2)}Y_3^{(2)}})} P\left(x_2^{n_2}, y_3^{n_2}\right) \Pr\left[X_1^{n_2} \in T_{\epsilon}^{n_2}(P_{X_1^{(2)}X_2^{(2)}Y_3^{(2)}}|x_2^{n_2}, y_3^{n_2})\right] \\
&\leq \delta_{\epsilon_1}(n_2) + 2^{-n(\frac{n_2}{n}I(X_1^{(2)};X_2^{(2)}Y_3^{(2)})-2\epsilon\frac{n_2}{n}H(X_1^{(2)})-R_2)}. \qquad \text{(A36)}
\end{aligned}
$$

by using Theorem A1.4 and A1.6. This shows that $\Pr\left[\mathcal{E}_{32}\,|(S,\hat{T}(3))=(s,\hat{t}(2))\right]$ gets arbitrarily small when n is chosen large and, since $X_1^{(2)}, X_2^{(2)}$ are independent,

$$
\begin{aligned}
R_2 &< \tau_2 I(X_1^{(2)};X_2^{(2)}Y_3^{(2)}) - 2\epsilon\tau_2 H(X_1^{(2)}) \\
&= \tau_2 I(X_1^{(2)};Y_3^{(2)}|X_2^{(2)}) - 2\epsilon\tau_2 H(X_1^{(2)}) \qquad \text{(A37)}
\end{aligned}
$$

is satisfied.

Markov Lemma Note that

$$
(X_1^{(1)}, Y_3^{(1)}) - Y_2^{(1)} - \hat{Y}_2^{(1)} \qquad \text{(A38)}
$$

form a Markov chain. For the analysis of the last two steps it is now shown that

$$
\Pr\left[\left(X_1^{n_1}(r), Y_3^{n_1}, Y_2^{n_1}, \hat{Y}_2^{n_1}(\hat{t}(2), \hat{o}(2))\right) \notin T_{\epsilon_2}^{n_1}(P_{X_1^{(1)}Y_3^{(1)}Y_2^{(1)}\hat{Y}_2^{(1)}})\right] \qquad \text{(A39)}
$$

can be made arbitrarily small by choosing n large. Markov Lemma A1.9, Theorem A1.4 and choosing $0 < \epsilon_1 < \epsilon_2 < \epsilon \leq \mu_{X_1^{(1)}Y_3^{(1)}Y_2^{(1)}\hat{Y}_2^{(1)}}$ give

$$\begin{aligned}
&\Pr\left[\left(X_1^{n_1}(r), Y_3^{n_1}, Y_2^{n_1}, \hat{Y}_2^{n_1}(\hat{t}(2), \hat{o}(2))\right) \notin T_{\epsilon_2}^{n_1}(P_{X_1^{(1)}Y_3^{(1)}Y_2^{(1)}\hat{Y}_2^{(1)}})\right] \\
&\leq \Pr\left[(X_1^{n_1}(r), Y_3^{n_1}, Y_2^{n_1}) \notin T_{\epsilon_1}^{n_1}(P_{X_1^{(1)}Y_3^{(1)}Y_2^{(1)}})\right] \\
&+ \sum_{T_{\epsilon_1}^{n_1}(P_{X_1^{(1)}Y_3^{(1)}Y_2^{(1)}})} P\left(x_1^{n_1}, y_3^{n_1}, y_2^{n_1}\right) \Pr\left[\hat{Y}_2^{n_1} \notin T_{\epsilon_2}^{n_1}(P_{X_1^{(1)}Y_3^{(1)}Y_2^{(1)}\hat{Y}_2^{(1)}}|x_1^{n_1}, y_3^{n_1}, y_2^{n_1})|Y_2^{n_1} = y_2^{n_1}\right] \\
&\leq \delta_{\epsilon_1}(n_1) + \delta_{\epsilon_1,\epsilon_2}(n_1). \qquad \text{(A40)}
\end{aligned}$$

Third Step With the result above

$$\begin{aligned}
\Pr\left[\mathcal{E}_{33}\,|(\hat{T}(3), \hat{O}(2)) = (\hat{t}(2), \hat{o}(2))\right] &\leq \delta_{\epsilon_1}(n_1) + \delta_{\epsilon_1,\epsilon_2}(n_1) \\
&+ \sum_{\substack{\tilde{o}=1 \\ \tilde{o}\neq\hat{o}(2)}}^{2^{nR_4}} \sum_{T_{\epsilon_2}^{n_1}(P_{Y_3^{(1)}})} P\left(y_3^{n_1}\right) \Pr\left[\hat{Y}_2^{n_1} \in T_{\epsilon}^{n_1}(P_{Y_3^{(1)}\hat{Y}_2^{(1)}}|y_3^{n_1})\right] \\
&\leq \delta_{\epsilon_1}(n_1) + \delta_{\epsilon_1,\epsilon_2}(n_1) + 2^{-n(\frac{n_1}{n}I(\hat{Y}_2^{(1)};Y_3^{(1)}) - 2\epsilon\frac{n_1}{n}H(\hat{Y}_2^{(1)}) - R_4)}. \qquad \text{(A41)}
\end{aligned}$$

This shows that $\Pr\left[\mathcal{E}_{33}\,|(\hat{T}(3), \hat{O}(2)) = (\hat{t}(2), \hat{o}(2))\right]$ becomes arbitrarily small when n is large and

$$R_4 < \tau_1 I(\hat{Y}_2^{(1)}; Y_3^{(1)}) - 2\epsilon\tau_1 H(\hat{Y}_2^{(1)}) \qquad \text{(A42)}$$

is satisfied.

Fourth Step Again with the result of the Markov Lemma one upper bounds

$$\begin{aligned}
&\Pr\left[\mathcal{E}_{34}\,|(R, \hat{T}(3), \hat{O}(3)) = (r, \hat{t}(2), \hat{o}(2))\right] \\
&\leq \delta_{\epsilon_1}(n_1) + \delta_{\epsilon_1,\epsilon_2}(n_1) + \sum_{\substack{\tilde{r}=1 \\ \tilde{r}\neq r}}^{2^{nR_1}} \sum_{T_{\epsilon_2}^{n_1}(P_{\hat{Y}_2^{(1)}Y_3^{(1)}})} P\left(\hat{y}_2^{n_1}, y_3^{n_1}\right) \Pr\left[X_1^{n_1} \in T_{\epsilon}^{n_1}(P_{X_1^{(1)}\hat{Y}_2^{(1)}Y_3^{(1)}}|\hat{y}_2^{n_1}, y_3^{n_1})\right] \\
&\leq \delta_{\epsilon_1}(n_1) + \delta_{\epsilon_1,\epsilon_2}(n_1) + 2^{-n(\frac{n_1}{n}I(X_1^{(1)};\hat{Y}_2^{(1)}Y_3^{(1)}) - 2\epsilon\frac{n_1}{n}H(X_1^{(1)}) - R_1)}. \qquad \text{(A43)}
\end{aligned}$$

This shows that $\Pr\left[\mathcal{E}_{34}\,|(R, \hat{T}(3), \hat{O}(3)) = (r, \hat{t}(2), \hat{o}(2))\right]$ can be made arbitrarily small when n is large and

$$R_1 < \tau_1 I(X_1^{(1)}; \hat{Y}_2^{(1)}Y_3^{(1)}) - 2\epsilon\tau_1 H(X_1^{(1)}) \qquad \text{(A44)}$$

is satisfied.

Rates The error analysis reveals that for large n reliable communication requires

$$R_3 + R_4 > \tau_1 I(\hat{Y}_2^{(1)}; Y_2^{(1)}) + 2\epsilon\tau_1 H(\hat{Y}_2^{(1)}) \tag{A45}$$

at node 2 and

$$\begin{aligned}
R_3 &< \tau_2 I(X_2^{(2)}; Y_3^{(2)}) - 2\epsilon\tau_2 H(X_2^{(2)}) \\
R_2 &< \tau_2 I(X_1^{(2)}; Y_3^{(2)}|X_2^{(2)}) - 2\epsilon\tau_2 H(X_1^{(2)}) \\
R_4 &< \tau_1 I(\hat{Y}_2^{(1)}; Y_3^{(1)}) - 2\epsilon\tau_1 H(\hat{Y}_2^{(1)}) \\
R_1 &< \tau_1 I(X_1^{(1)}; \hat{Y}_2^{(1)} Y_3^{(1)}) - 2\epsilon\tau_1 H(X_1^{(1)}).
\end{aligned} \tag{A46}$$

at node 3. Consequently

$$R = R_1 + R_2 < \tau_1 I(X_1^{(1)}; \hat{Y}_2^{(1)} Y_3^{(1)}) + \tau_2 I(X_1^{(2)}; Y_3^{(2)}|X_2^{(2)}) - 2\epsilon \left(\tau_1 H(X_1^{(1)}) + \tau_2 H(X_1^{(2)})\right) \tag{A47}$$

subject to

$$\tau_1 I(Y_2^{(1)}; \hat{Y}_2^{(1)}) < \tau_1 I(\hat{Y}_2^{(1)}; Y_3^{(1)}) + \tau_2 I(X_2^{(2)}; Y_3^{(2)}) - 2\epsilon\tau_2 H(X_2^{(2)}) - 4\epsilon\tau_1 H(\hat{Y}_2^{(1)}). \tag{A48}$$

By the use of the Markov chain

$$Y_3^{(1)} - Y_2^{(1)} - \hat{Y}_2^{(1)} \tag{A49}$$

implying

$$I(\hat{Y}_2^{(1)}; Y_3^{(1)}|Y_2^{(1)}) = 0 \tag{A50}$$

the compression constraint can be reformulated

$$\begin{aligned}
\tau_1 I(\hat{Y}_2^{(1)}; Y_2^{(1)}) &< \tau_1 I(\hat{Y}_2^{(1)}; Y_3^{(1)}) + \tau_2 I(X_2^{(2)}; Y_3^{(2)}) - 2\epsilon\tau_2 H(X_2^{(2)}) - 4\epsilon\tau_1 H(\hat{Y}_2^{(1)}) \\
\tau_1 I(\hat{Y}_2^{(1)}; Y_2^{(1)}|Y_3^{(1)}) &< \tau_2 I(X_2^{(2)}; Y_3^{(2)}) - 2\epsilon\tau_2 H(X_2^{(2)}) - 4\epsilon\tau_1 H(\hat{Y}_2^{(1)}).
\end{aligned} \tag{A51}$$

Choosing $\epsilon > 0$ but arbitrarily small establishes the proposition. ■

A2.3 Partial-Decode-and-Forward

Code Generate $2^{n(R_1+R_2)}$ n_1-sequences $u_1^{n_1}(r,s)$, $r \in [1;2^{nR_1}]$, $s \in [1;2^{nR_2}]$, by choosing each element $u_{1,k}^{(1)}(r,s)$ independently according to $P_{U_1^{(1)}}(\cdot)$. For every $u_1^{n_1}(r,s)$ generate 2^{nR_3} n_1-sequences $x_1^{n_1}(r,s,t)$, $t \in [1;2^{nR_3}]$, by choosing each element $x_{1,k}^{(1)}(r,s,t)$ independently according to $P_{X_1^{(1)}|U_1^{(1)}}(\cdot|u_{1,k}^{(1)}(r,s))$. Generate 2^{nR_1} n_2-sequences $x_2^{n_2}(r)$ by choosing each element $x_{2,k}^{(2)}(r)$ independently according to $P_{X_2^{(2)}}(\cdot)$. For every $x_2^{n_2}(r)$ generate 2^{nR_4} n_2-sequences $x_1^{n_2}(r,o)$, $o \in [1;2^{nR_4}]$, by choosing each element $x_{1,k}^{(2)}(r,o)$ independently according to $P_{X_1^{(2)}|X_2^{(2)}}(\cdot|x_{2,k}^{(2)}(r))$.

Node 1 The message w is reindexed by (r,s,t,o). In the first phase node 1 transmits $x_1^{n_1}(r,s,t)$ within n_1 transmissions. In the second phase node 1 transmits $x_1^{n_2}(r,o)$ within n_2 transmissions.

Node 2 In the first phase $y_2^{n_1}$ is observed. After the first phase node 2 tries to find an index pair $(\tilde{r},\tilde{s})$ such that

$$(u_1^{n_1}(\tilde{r},\tilde{s}), y_2^{n_1}) \in T_\epsilon^{n_1}(P_{U_1^{(1)}Y_2^{(1)}}). \tag{A52}$$

If there is none or more than one such index pair $(\tilde{r},\tilde{s})$, set $(\hat{r}(2),\hat{s}(2)) = (1,1)$. Otherwise, the found pair $(\tilde{r},\tilde{s})$ is the estimate $(\hat{r}(2),\hat{s}(2))$ of node 2. In the second phase node 2 sends $x_2^{n_2}(\hat{r}(2))$ by using n_2 transmissions.

Node 3 In the first phase $y_3^{n_1}$ is observed. In the second phase $y_3^{n_2}$ is observed. After the second phase node 3 tries to find an index $\tilde{r}$ such that

$$(x_2^{n_2}(\tilde{r}), y_3^{n_2}) \in T_\epsilon^{n_2}(P_{X_2^{(2)}Y_3^{(2)}}). \tag{A53}$$

If there is none or more than one such index $\tilde{r}$, set $\hat{r}(3) = 1$. Otherwise, the found index $\tilde{r}$ is the estimate $\hat{r}(3)$ of node 3. Then node 3 tries to find an index $\tilde{o}$ such that

$$(x_1^{n_2}(\hat{r}(3),\tilde{o}), x_2^{n_2}(\hat{r}(3)), y_3^{n_2}) \in T_\epsilon^{n_2}(P_{X_1^{(2)}X_2^{(2)}Y_3^{(2)}}). \tag{A54}$$

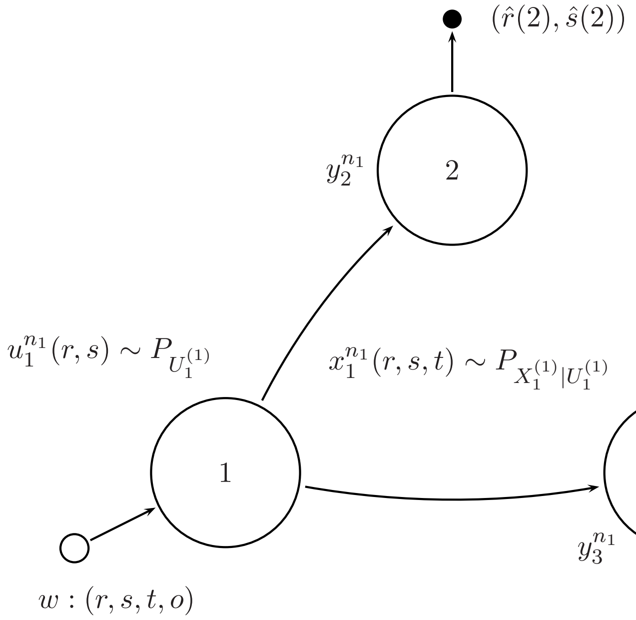
If there is none or more than one such index $\tilde{o}$, set $\hat{o}(3) = 1$. Otherwise, the found index $\tilde{o}$ is the estimate $\hat{o}(3)$ of node 3. Then node 3 tries to find an index $\tilde{s}$ such that

$$(u_1^{n_1}(\hat{r}(3),\tilde{s}), y_3^{n_1}) \in T_\epsilon^{n_1}(P_{U_1^{(1)}Y_3^{(1)}}). \tag{A55}$$

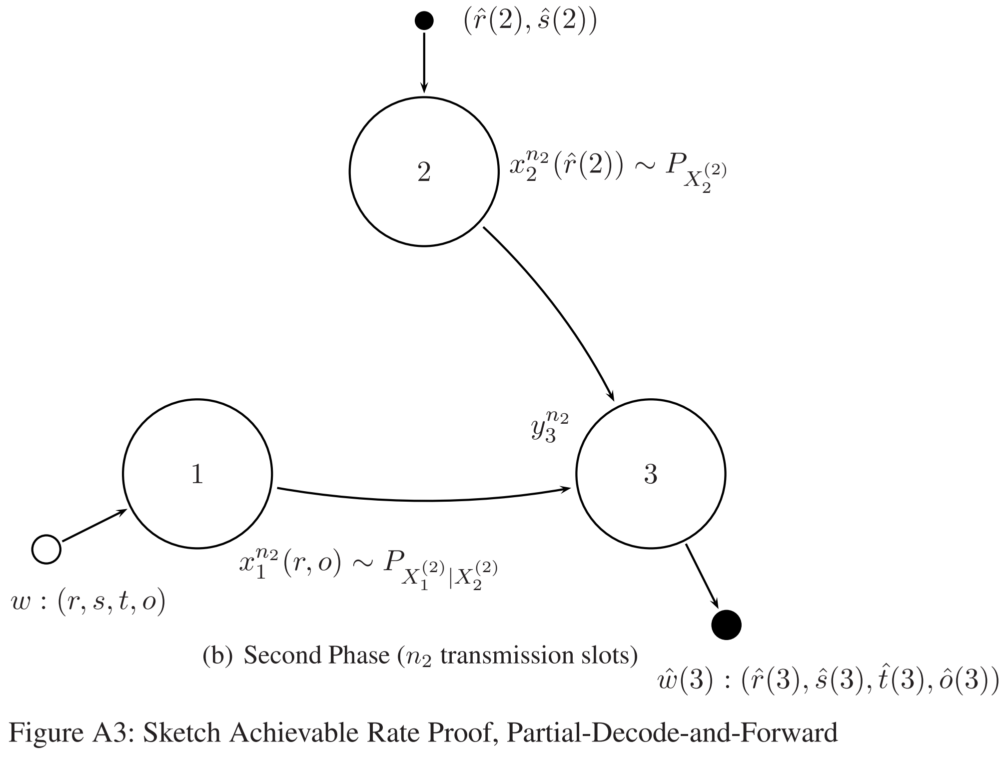
If there is none or more than one such index $\tilde{s}$, set $\hat{s}(3) = 1$. Otherwise, the found index $\tilde{s}$ is the estimate $\hat{s}(3)$ of node 3. Then node 3 tries to find an index $\tilde{t}$ such that

$$\left(x_1^{n_1}(\hat{r}(3),\hat{s}(3),\tilde{t}), u_1^{n_1}(\hat{r}(3),\hat{s}(3)), y_3^{n_1}\right) \in T_\epsilon^{n_1}(P_{U_1^{(1)}X_1^{(1)}Y_3^{(1)}}). \tag{A56}$$

If there is none or more than one such index $\tilde{t}$, set $\hat{t}(3) = 1$. Otherwise, the found index $\tilde{t}$ is the estimate $\hat{t}(3)$ of node 3. The output message $\hat{w}(3)$ of node 3 is found by reindexing $(\hat{r}(3),\hat{s}(3),\hat{t}(3),\hat{o}(3))$.



(a) First Phase (n_1 transmission slots)



(b) Second Phase (n_2 transmission slots)

Figure A3: Sketch Achievable Rate Proof, Partial-Decode-and-Forward

Rates An error analysis similar to the one in the proof A2.1 reveals that for large n reliable communication requires

$$R_1 + R_2 < \tau_1 I(U_1^{(1)}; Y_2^{(1)}) - 2\epsilon\tau_1 H(U_1^{(1)}) \tag{A57}$$

at node 2 and

$$\begin{aligned}
R_1 &< \tau_2 I(X_2^{(2)}; Y_3^{(2)}) - 2\epsilon\tau_2 H(X_2^{(2)}) \\
R_4 &< \tau_2 I(X_1^{(2)}; Y_3^{(2)}|X_2^{(2)}) - 2\epsilon\tau_2 H(X_1^{(2)}|X_2^{(2)}) \\
R_2 &< \tau_1 I(U_1^{(1)}; Y_3^{(1)}) - 2\epsilon\tau_1 H(U_1^{(1)}) \\
R_3 &< \tau_1 I(X_1^{(1)}; Y_3^{(1)}|U_1^{(1)}) - 2\epsilon\tau_1 H(X_1^{(1)}|U_1^{(1)})
\end{aligned} \tag{A58}$$

at node 3. Consequently, with $R = R_1 + R_2 + R_3 + R_4$

$$\begin{aligned}
R &< \tau_1 I(U_1^{(1)}; Y_2^{(1)}) + \tau_1 I(X_1^{(1)}; Y_3^{(1)}|U_1^{(1)}) + \tau_2 I(X_1^{(2)}; Y_3^{(2)}|X_2^{(2)}) \\
&\quad - 2\epsilon \left(\tau_1 H(U_1^{(1)}) + \tau_1 H(X_1^{(1)}|U_1^{(1)}) + \tau_2 H(X_1^{(2)}|X_2^{(2)})\right) \\
R &< \tau_1 I(U_1^{(1)}; Y_3^{(1)}) + \tau_1 I(X_1^{(1)}; Y_3^{(1)}|U_1^{(1)}) + \tau_2 I(X_2^{(2)}; Y_3^{(2)}) + \tau_2 I(X_1^{(2)}; Y_3^{(2)}|X_2^{(2)}) \\
&\quad - 2\epsilon \left(\tau_1 H(U_1^{(1)}) + \tau_1 H(X_1^{(1)}|U_1^{(1)}) + \tau_2 H(X_2^{(2)}) + \tau_2 H(X_1^{(2)}|X_2^{(2)})\right) \\
&= \tau_1 I(U_1^{(1)} X_1^{(1)}; Y_3^{(1)}) + \tau_2 I(X_1^{(2)} X_2^{(2)}; Y_3^{(2)}) - 2\epsilon \left(\tau_1 H(U_1^{(1)} X_1^{(1)}) + \tau_2 H(X_1^{(2)} X_2^{(2)})\right) \\
&= \tau_1 I(X_1^{(1)}; Y_3^{(1)}) + \tau_2 I(X_1^{(2)} X_2^{(2)}; Y_3^{(2)}) - 2\epsilon \left(\tau_1 H(U_1^{(1)} X_1^{(1)}) + \tau_2 H(X_1^{(2)} X_2^{(2)})\right).
\end{aligned} \tag{A59}$$

where the last step follows by using the fact that

$$U_1^{(1)} - X_1^{(1)} - Y_3^{(1)} \tag{A60}$$

form a Markov chain and therefore

$$I(U_1^{(1)} X_1^{(1)}; Y_3^{(1)}) = I(X_1^{(1)}; Y_3^{(1)}) + \underbrace{I(U_1^{(1)}; Y_3^{(1)}|X_1^{(1)})}_{=0}. \tag{A61}$$

Choosing $\epsilon > 0$ but arbitrarily small establishes the proposition. ■

A2.4 Partial-Decode-Compress-and-Forward

Code Generate $2^{n(R_1+R_2)}$ n_1-sequences $u_1^{n_1}(r,s)$, $r \in [1;2^{nR_1}]$, $s \in [1;2^{nR_2}]$, by choosing each element $u_{1,k}^{(i)}(r,s)$ independently according to $P_{U_1^{(1)}}(\cdot)$. For every $u_1^{n_1}(r,s)$ generate 2^{nR_3} n_1-sequences $x_1^{n_1}(r,s,t)$, $t \in [1;2^{nR_3}]$, by choosing each element $x_{1,k}^{(1)}(r,s,t)$ independently according to $P_{X_1^{(1)}|U_1^{(1)}}(\cdot|u_{1,k}^{(1)}(r,s))$. Choose a "quantization channel" $P(\hat{y}_2^{(1)}|y_2^{(1)},u_1^{(1)})$ and calculate $P(\hat{y}_2^{(1)}|u_1^{(1)})$ as the marginal distribution of $P(\hat{y}_2^{(1)},y_2^{(1)}|u_1^{(1)})$. For every $u_1^{n_1}(r,s)$ generate $2^{n(R_4+R_5)}$ n_1-sequences $\hat{y}_2^{n_1}(r,s,e,z)$, $e \in [1;2^{nR_4}]$, $z \in [1;2^{nR_5}]$, by choosing each element $\hat{y}_{2,k}^{(1)}(r,s,e,z)$ independently according to $P_{\hat{Y}_2^{(1)}|U_1^{(1)}}(\cdot|u_{1,k}^{(1)}(r,s))$. Generate 2^{nR_1} n_2-sequences $v_2^{n_2}(r)$ by choosing each element $v_{2,k}^{(2)}(r)$ independently according to $P_{V_2^{(2)}}(\cdot)$. For every $v_2^{n_2}(r)$ generate 2^{nR_4} n_2-sequences $x_2^{n_2}(r,e)$ by choosing each element $x_{2,k}^{(2)}(r,e)$ independently according to $P_{X_2^{(2)}|V_2^{(2)}}(\cdot|v_{2,k}^{(2)}(r))$. For every $v_2^{n_2}(r)$ generate 2^{nR_6} n_2-sequences $x_1^{n_2}(r,o)$, $o \in [1;2^{nR_6}]$, by choosing each element $x_{1,k}^{(2)}(r,o)$ independently according to $P_{X_1^{(2)}|V_2^{(2)}}(\cdot|v_{2,k}^{(2)}(r))$.

Node 1 The message w is reindexed by (r,s,t,o). In the first phase node 1 transmits $x_1^{n_1}(r,s,t)$ within n_1 transmissions. In the second phase node 1 transmits $x_1^{n_2}(r,o)$ within n_2 transmissions.

Node 2 In the first phase $y_2^{n_1}$ is observed. After the first phase node 2 tries to find a pair $(\tilde{r},\tilde{s})$ such that

$$(u_1^{n_1}(\tilde{r},\tilde{s}),y_2^{n_1}) \in T_\epsilon^{n_1}(P_{U_1^{(1)}Y_2^{(1)}}). \tag{A62}$$

If there is none or more than one such pair $(\tilde{r},\tilde{s})$, set $(\hat{r}(2),\hat{s}(2)) = (1,1)$. Otherwise, the found pair $(\tilde{r},\tilde{s})$ is the estimate $(\hat{r}(2),\hat{s}(2))$ of node 2. Then node 2 tries to find a pair $(\tilde{e},\tilde{z})$ such that

$$(\hat{y}_2^{n_1}(\hat{r}(2),\hat{s}(2),\tilde{e},\tilde{z}),u_1^{n_1}(\hat{r}(2),\hat{s}(2)),y_2^{n_1}) \in T_\epsilon^{n_1}(P_{\hat{Y}_2^{(1)}Y_2^{(1)}U_1^{(1)}}). \tag{A63}$$

If there is none such pair $(\tilde{e},\tilde{z})$ an error is declared. Otherwise, the found pair $(\tilde{e},\tilde{z})$ is the estimate $(\hat{e}(2),\hat{z}(2))$ of node 2 where the pair with the smallest linear index $2^{nR_4}(e-1)+z$ is selected if more than one pair was found. In the second phase node 2 sends $x_2^{n_2}(\hat{r}(2),\hat{e}(2))$ by using n_2 transmissions.

Node 3 In the first phase $y_3^{n_1}$ is observed. In the second phase $y_3^{n_2}$ is observed. After the second phase node 3 tries to find an index $\tilde{r}$ such that

$$(v_2^{n_2}(\tilde{r}),y_3^{n_2}) \in T_\epsilon^{n_2}(P_{V_2^{(2)}Y_3^{(2)}}). \tag{A64}$$

If there is none or more than one such index $\tilde{r}$, set $\hat{r}(3) = 1$. Otherwise, the found index $\tilde{r}$ is the estimate $\hat{r}(3)$ of node 3. Then node 3 tries to find an index $\tilde{e}$ such that

$$(x_2^{n_2}(\hat{r}(3),\tilde{e}),v_2^{n_2}(\hat{r}(3)),y_3^{n_2}) \in T_\epsilon^{n_2}(P_{X_2^{(2)}V_2^{(2)}Y_3^{(2)}}). \tag{A65}$$

If there is none or more than one such index $\tilde{e}$, set $\hat{e}(3) = 1$. Otherwise, the found index $\tilde{e}$ is the estimate $\hat{e}(3)$ of node 3. Then node 3 tries to find an index $\tilde{o}$ such that

$$(x_1^{n_2}(\hat{r}(3),\tilde{o}),v_2^{n_2}(\hat{r}(3)),x_2^{n_2}(\hat{r}(3),\hat{e}(3)),y_3^{n_2}) \in T_\epsilon^{n_1}(P_{X_1^{(2)}V_2^{(2)}X_2^{(2)}Y_3^{(2)}}). \tag{A66}$$

If there is none or more than one such index $\tilde{o}$, set $\hat{o}(3) = 1$. Otherwise, the found index $\tilde{o}$ is the estimate $\hat{o}(3)$ of node 3. Then node 3 tries to find an index $\tilde{s}$ such that

$$(u_1^{n_1}(\hat{r}(3), \tilde{s}), y_3^{n_1}) \in T_\epsilon^{n_1}(P_{U_1^{(1)} Y_3^{(1)}}). \tag{A67}$$

If there is none or more than one such index $\tilde{s}$, set $\hat{s}(3) = 1$. Otherwise, the found index $\tilde{s}$ is the estimate $\hat{s}(3)$ of node 3. Then node 3 tries to find an index $\tilde{z}$, such that

$$(\hat{y}_2^{n_1}(\hat{r}(3), \hat{s}(3), \hat{e}(3), \tilde{z}), u_1^{n_1}(\hat{r}(3), \hat{s}(3)), y_3^{n_1}) \in T_\epsilon^{n_1}(P_{\hat{Y}_2^{(1)} Y_3^{(1)} U_1^{(1)}}). \tag{A68}$$

If there is none or more than one such index $\tilde{z}$, set $\hat{z}(3) = 1$. Otherwise, the found index $\tilde{z}$ is the estimate $\hat{z}(3)$ of node 3. Finally node 3 tries to find an index $\tilde{t}$, such that

$$\left(x_1^{n_1}(\hat{r}(3), \hat{s}(3), \tilde{t}), u_1^{n_1}(\hat{r}(3), \hat{s}(3)), \hat{y}_2^{n_1}(\hat{r}(3), \hat{s}(3), \hat{e}(3), \hat{z}(3)), y_3^{n_1}\right) \in T_\epsilon^{n_1}(P_{X_1^{(1)} \hat{Y}_2^{(1)} Y_3^{(1)} U_1^{(1)}}). \tag{A69}$$

If there is none or more than one such index $\tilde{t}$, set $\hat{t}(3) = 1$. Otherwise, the found index $\tilde{t}$ is the estimate $\hat{t}(3)$ of node 3. The output message $\hat{w}(3)$ of node 3 is found by reindexing $(\hat{r}(3), \hat{s}(3), \hat{t}(3), \hat{o}(3))$.

Rates An error analysis similar to the ones in the proofs A2.1 and A2.2 reveals that for large n reliable communication requires

$$\begin{aligned}
R_1 + R_2 &< \tau_1 I(U_1^{(1)}; Y_2^{(1)}) - 2\epsilon\tau_1 H(U_1^{(1)})\\
R_4 + R_5 &> \tau_1 I(\hat{Y}_2^{(1)}; Y_2^{(1)} | U_1^{(1)}) + 2\epsilon\tau_1 H(\hat{Y}_2^{(1)} | U_1^{(1)})
\end{aligned} \tag{A70}$$

at node 2 and

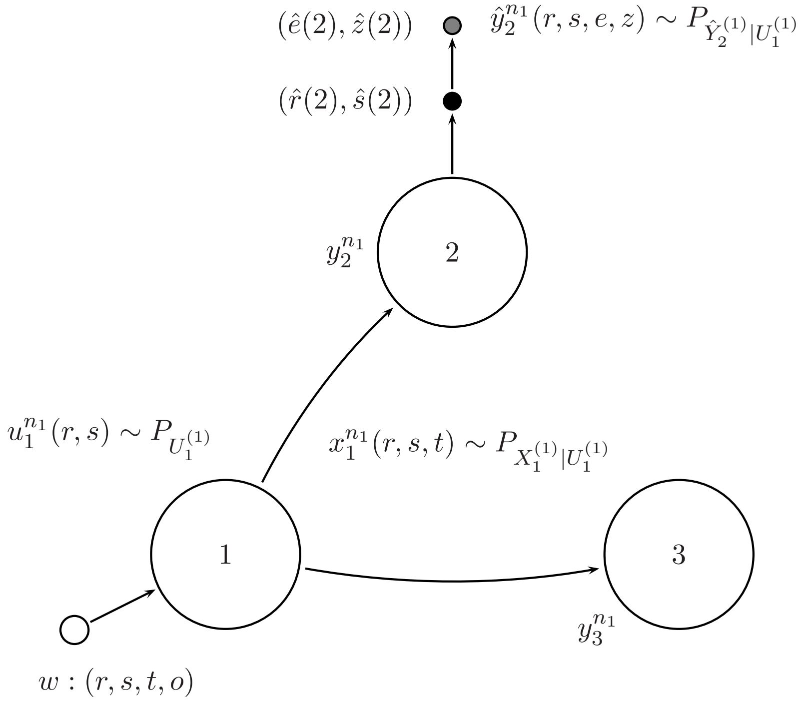
$$\begin{aligned}
R_1 &< \tau_2 I(V_2^{(2)}; Y_3^{(2)}) - 2\epsilon\tau_2 H(V_2^{(2)})\\
R_4 &< \tau_2 I(X_2^{(2)}; Y_3^{(2)} | V_2^{(2)}) - 2\epsilon\tau_2 H(X_2^{(2)} | V_2^{(2)})\\
R_6 &< \tau_2 I(X_1^{(2)}; X_2^{(2)} Y_3^{(2)} | V_2^{(2)}) - 2\epsilon\tau_2 H(X_1^{(2)} | V_2^{(2)})\\
&= \tau_2 I(X_1^{(2)}; Y_3^{(2)} | X_2^{(2)} V_2^{(2)}) - 2\epsilon\tau_2 H(X_1^{(2)} | V_2^{(2)})\\
R_2 &< \tau_1 I(U_1^{(1)}; Y_3^{(1)}) - 2\epsilon\tau_1 H(U_1^{(1)})\\
R_5 &< \tau_1 I(\hat{Y}_2^{(1)}; Y_3^{(1)} | U_1^{(1)}) - 2\tau_1\epsilon H(\hat{Y}_2^{(1)} | U_1^{(1)})\\
R_3 &< \tau_1 I(X_1^{(1)}; \hat{Y}_2^{(1)} Y_3^{(1)} | U_1^{(1)}) - 2\tau_1\epsilon H(X_1^{(1)} | U_1^{(1)})
\end{aligned} \tag{A71}$$

at node 3 where it has been used that

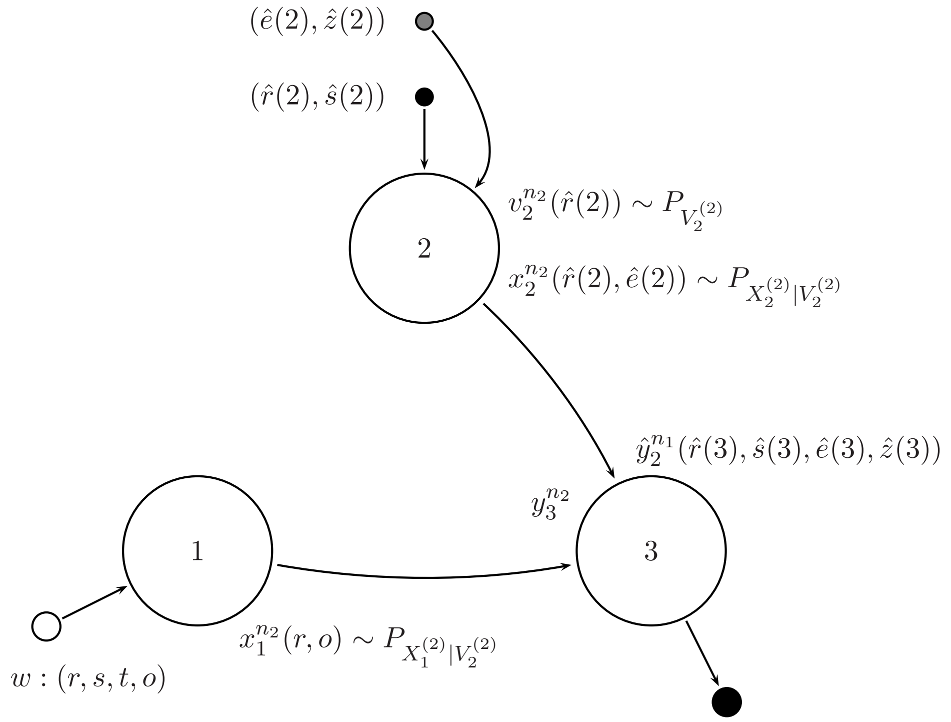
$$X_1^{(2)} - V_2^{(2)} - X_2^{(2)} \tag{A72}$$

form a Markov chain and therefore

$$I(X_1^{(2)}; X_2^{(2)} | V_2^{(2)}) = 0. \tag{A73}$$



(a) First Phase (n_1 transmission slots)



(b) Second Phase (n_2 transmission slots)

Figure A4: Sketch ARP, Partial-Decode-Compress-and-Forward

Consequently,

$$
\begin{aligned}
R &< \tau_1 I(U_1^{(1)}; Y_2^{(1)}) + \tau_1 I(X_1^{(1)}; \hat{Y}_2^{(1)} Y_3^{(1)} | U_1^{(1)}) + \tau_2 I(X_1^{(2)}; Y_3^{(2)} | V_2^{(2)} X_2^{(2)}) \\
&\quad - 2\epsilon \left(\tau_1 H(U_1^{(1)}) + \tau_1 H(X_1^{(1)} | U_1^{(1)}) + \tau_2 H(X_1^{(2)} | V_2^{(2)}) \right) \\
R &< \tau_1 I(U_1^{(1)}; Y_3^{(1)}) + \tau_1 I(X_1^{(1)}; \hat{Y}_2^{(1)} Y_3^{(1)} | U_1^{(1)}) + \tau_2 I(V_2^{(2)}; Y_3^{(2)}) + \tau_2 I(X_1^{(2)}; Y_3^{(2)} | V_2^{(2)} X_2^{(2)}) \\
&\quad - 2\epsilon \left(\tau_1 H(U_1^{(1)}) + \tau_1 H(X_1^{(1)} | U_1^{(1)}) + \tau_2 H(V_2^{(2)}) + \tau_2 H(X_1^{(2)} | V_2^{(2)}) \right)
\end{aligned} \tag{A74}
$$

subject to

$$
\begin{aligned}
\tau_1 I(\hat{Y}_2^{(1)}; Y_2^{(1)} | U_1^{(1)}) < \tau_1 I(\hat{Y}_2^{(1)}; Y_3^{(1)} | U_1^{(1)}) + \tau_2 I(X_2^{(2)}; Y_3^{(2)} | V_2^{(2)}) - 2\epsilon \tau_2 H(X_2^{(2)} | V_2^{(2)}) \\
- 4\epsilon \tau_1 H(\hat{Y}_2^{(1)} | U_1^{(1)}).
\end{aligned} \tag{A75}
$$

By the use of the Markov chain

$$
Y_3^{(1)} - (Y_2^{(1)}, U_1^{(1)}) - \hat{Y}_2^{(1)} \tag{A76}
$$

implying

$$
I(\hat{Y}_2^{(1)}; Y_3^{(1)} | U_1^{(1)} Y_2^{(1)}) = 0 \tag{A77}
$$

the constraint can be reformulated to

$$
\tau_1 I(\hat{Y}_2^{(1)}; Y_2^{(1)} | U_1^{(1)} Y_3^{(1)}) < \tau_2 I(X_2^{(2)}; Y_3^{(2)} | V_2^{(2)}) - 2\epsilon \tau_2 H(X_2^{(2)} | V_2^{(2)}) - 4\epsilon \tau_1 H(\hat{Y}_2^{(1)} | U_1^{(1)}). \tag{A78}
$$

Choosing $\epsilon > 0$ but arbitrarily small establishes the proposition. ■

A3. Proofs: Achievable Rates 2P-MA-BC Scheme

Comments and Assumptions Random encoding, jointly typical decoding and compression will be used to show which rates are achievable for the half-duplex two-way relay channel. For the following proofs it will be assumed that the transmission is performed with $n \geq 2$ channel uses and two phases $l = 1, 2$. Phase 1 features $n_1 \geq 1$ transmission slots, phase 2 supports $n_2 \geq 1$ transmission slots, with $n_1 + n_2 = n$. If n grows n_1 and n_2 are assumed to grow at the same rate. For large n, $\frac{n_l}{n} \to \tau_l > 0$. The message $w_{13} \in \{1, \ldots, 2^{nR_{13}}\}$ will be sent from node 1 to node 3 and the message $w_{31} \in \{1, \ldots, 2^{nR_{31}}\}$ will be sent from node 3 to node 1. For all proofs $2^{nR} \in \mathbb{Z}_+$.

A3.1 Decode-and-Forward

Code Generate 2^{nR_1} n_1-sequences $x_1^{n_1}(s)$, $s = 1, 2, \ldots, 2^{nR_1}$, by choosing each element $x_{1,k}^{(1)}(s)$ independently according to $P_{X_1^{(1)}}(\cdot)$. Generate 2^{nR_2} n_1-sequences $x_3^{n_1}(t)$, $t = 1, 2, \ldots, 2^{nR_2}$, by choosing each element $x_{3,k}^{(1)}(t)$ independently according to $P_{X_3^{(1)}}(\cdot)$. Generate $2^{n(R_1+R_2)}$ n_2-sequences $x_2^{n_2}(s,t)$ by choosing each element $x_{2,k}^{(2)}(s,t)$ independently according to $P_{X_2^{(2)}}(\cdot)$.

Node 1 (Input) The message w_{13} is reindexed by s. In the first phase node 1 transmits $x_1^{n_1}(s)$ with n_1 transmissions.

Node 3 (Input) The message w_{31} is reindexed by t. In the first phase node 3 transmits $x_3^{n_1}(t)$ with n_1 transmissions.

Node 2 In the first phase $y_2^{n_1}$ is observed. After the first phase node 2 tries to find a pair $(\tilde{s}, \tilde{t})$ such that

$$\left(x_1^{n_1}(\tilde{s}), x_3^{n_1}(\tilde{t}), y_2^{n_1}\right) \in T_\epsilon^{n_1}(P_{X_1^{(1)} X_3^{(1)} Y_2^{(1)}}). \tag{A79}$$

If there is none or more than one such pair $(\tilde{s}, \tilde{t})$, set $(\hat{s}(2), \hat{t}(2)) = (1, 1)$. Otherwise, $(\tilde{s}, \tilde{t})$ is the estimate $(\hat{s}(2), \hat{t}(2))$ of node 2. In the second phase node 2 sends $x_2^{n_2}(\hat{s}(2), \hat{t}(2))$ by the use of n_2 transmissions.

Node 1 (Output) In the second phase $y_1^{n_2}$ is observed. After the second phase node 1 tries to find an index $\tilde{t}$ such that

$$\left(x_2^{n_2}(s, \tilde{t}), y_3^{n_2}\right) \in T_\epsilon^{n_2}(P_{X_2^{(2)} Y_3^{(2)}}). \tag{A80}$$

If there is none or more than one such index $\tilde{t}$, set $\hat{t}(1) = 1$. Otherwise, $\tilde{t}$ is the estimate $\hat{t}(1)$ of node 1. The output message $\hat{w}_{31}(1)$ is found by reindexing $\hat{t}(1)$.

Node 3 (Output) In the second phase $y_3^{n_2}$ is observed. After the second phase node 3 tries to find an index $\tilde{s}$ such that

$$(x_2^{n_2}(\tilde{s}, t), y_3^{n_2}) \in T_\epsilon^{n_2}(P_{X_2^{(2)} Y_3^{(2)}}). \tag{A81}$$

If there is none or more than one such index $\tilde{s}$, set $\hat{s}(3) = 1$. Otherwise, $\tilde{s}$ is the estimate $\hat{s}(3)$ of node 3. The output message $\hat{w}_{13}(3)$ is found by reindexing $\hat{s}(3)$.

Rates An error analysis along the lines of A2.1 and A2.2 reveals that for large n, $\epsilon > 0$ but arbitrarily small, reliable communication requires

$$\begin{aligned}
R_1 &< \tau_1 I(X_1^{(1)}; X_3^{(1)} Y_2^{(1)}) \\
&= \tau_1 I(X_1^{(1)}; Y_2^{(1)} | X_3^{(1)}) \\
R_2 &< \tau_1 I(X_3^{(1)}; X_1^{(1)} Y_2^{(1)}) \\
&= \tau_1 I(X_3^{(1)}; Y_2^{(1)} | X_1^{(1)}) \\
R_1 + R_2 &< \tau_1 I(X_1^{(1)} X_3^{(1)}; Y_2^{(1)})
\end{aligned} \tag{A82}$$

at node 2,

$$R_2 < \tau_2 I(X_2^{(2)}; Y_1^{(2)}) \tag{A83}$$

at node 1 and

$$R_1 < \tau_2 I(X_2^{(2)}; Y_3^{(2)}) \tag{A84}$$

at node 3. This establishes the proposition. ■

A3.2 Compress-and-Forward with 2-Layer-Quantization

Code Generate 2^{nR_1} n_1-sequences $x_1^{n_1}(r)$, $r = 1, 2, \ldots, 2^{nR_1}$, by choosing each element $x_{1,k}^{(1)}(r)$ independently according to $P_{X_1^{(1)}}(\cdot)$. Generate 2^{nR_2} n_1-sequences $x_3^{n_1}(s)$, $s = 1, 2, \ldots, 2^{nR_2}$, by choosing each element $x_{3,k}^{(1)}(s)$ independently according to $P_{X_3^{(1)}}(\cdot)$. Choose a "coarse quantization channel" $P_{\hat{Y}_{21}^{(1)}|Y_2^{(1)}}(\cdot|\cdot)$ and calculate $P_{\hat{Y}_{21}^{(1)}}(\cdot)$ as the marginal distribution of $P_{\hat{Y}_{21}^{(1)}Y_2^{(1)}}(\cdot)$. Choose a "refinement quantization channel" $P_{\hat{Y}_{22}^{(1)}|Y_2^{(1)}\hat{Y}_{21}^{(1)}}(\cdot|\cdot)$ and calculate $P_{\hat{Y}_{22}^{(1)}|\hat{Y}_{21}^{(1)}}(\cdot|\cdot)$ as the marginal distribution of $P_{\hat{Y}_{22}^{(1)}Y_2^{(1)}|\hat{Y}_{21}^{(1)}}(\cdot)$. Generate $2^{n(R_3+R_4)}$ n_1-sequences $\hat{y}_{21}^{n_1}(t, o)$, $t = 1, 2, \ldots, 2^{nR_3}$, $o = 1, 2, \ldots, 2^{nR_4}$ by choosing each element $\hat{y}_{21,k}^{(1)}(t, o)$ independently according to $P_{\hat{Y}_{21}^{(1)}}(\cdot)$. For each $\hat{y}_{21}^{n_1}(t, o)$ generate $2^{n(R_5+R_6)}$ n_1-sequences $\hat{y}_{22}^{n_1}(t, o, e, z)$, $e = 1, 2, \ldots, 2^{nR_5}$, $z = 1, 2, \ldots, 2^{nR_6}$ by choosing each element $\hat{y}_{22,k}^{(1)}(t, o, e, z)$ independently according to $P_{\hat{Y}_{22}^{(1)}|\hat{Y}_{21}^{(1)}}(\cdot|\hat{y}_{21,k}^{(1)}(t, o))$. Generate 2^{nR_3} n_2-sequences $u_2^{n_2}(t)$ by choosing each element $u_{2,k}^{(2)}(t)$ independently according to $P_{U_2^{(2)}}(\cdot)$. For each $u_2^{n_2}(t)$ generate 2^{nR_5} n_2-sequences $x_2^{n_2}(t, e)$ by choosing each element $x_{2,k}^{(2)}(t, e)$ independently according to $P_{X_2^{(2)}|U_2^{(2)}}(\cdot|u_{2,k}^{(2)}(t))$.

Node 1 (Input) The message w_{13} is reindexed by r. In the first phase node 1 transmits $x_1^{n_1}(r)$ within n_1 transmissions.

Node 3 (Input) The message w_{31} is reindexed by s. In the first phase node 3 transmits $x_3^{n_1}(s)$ within n_1 transmissions.

Node 2 In the first phase $y_2^{n_1}$ is observed. After the first phase node 2 tries to find a pair $(\tilde{t}, \tilde{o})$ such that

$$\left(\hat{y}_{21}^{n_1}(\tilde{t}, \tilde{o}), y_2^{n_1}\right) \in T_\epsilon^{n_1}(P_{\hat{Y}_{21}^{(1)}Y_2^{(1)}}). \tag{A85}$$

If there is none such pair $(\tilde{t}, \tilde{o})$ node 2 declares an error. Otherwise, the found pair $(\tilde{t}, \tilde{o})$ is the estimate $(\hat{t}(2), \hat{o}(2))$ of node 2 where the pair with the smallest linear index $2^{nR_3}(o-1)+t$ is selected if more than one pair was found. Then node 2 tries to find a pair $(\tilde{e}, \tilde{z})$ such that

$$\left(\hat{y}_{22}^{n_1}(\hat{t}(2), \hat{o}(2), \tilde{e}, \tilde{z}), \hat{y}_{21}^{n_1}(\hat{t}(2), \hat{o}(2)), y_2^{n_1}\right) \in T_\epsilon^{n_1}(P_{\hat{Y}_{22}^{(1)}\hat{Y}_{21}^{(1)}Y_2^{(1)}}). \tag{A86}$$

If there is none such pair $(\tilde{e}, \tilde{z})$ an error is declared by node 2. Otherwise, the found pair $(\tilde{e}, \tilde{z})$ is the estimate $(\hat{e}(2), \hat{z}(2))$ of node 2 where the pair with the smallest linear index $2^{nR_5}(z-1)+e$ is selected if more than one pair was found. In the second phase node 2 sends $x_2^{n_2}(\hat{t}(2), \hat{e}(2))$ by the use of n_2 transmissions.

Node 1 (Output) In the second phase $y_1^{n_2}$ is observed. After the second phase node 1 tries to find an index $\tilde{t}$ such that

$$\left(u_2^{n_2}(\tilde{t}), y_1^{n_2}\right) \in T_\epsilon^{n_2}(P_{U_2^{(2)}Y_1^{(2)}}). \tag{A87}$$

If there is none or more than one such index $\tilde{t}$, set $\hat{t}(1) = 1$. Otherwise, the found index $\tilde{t}$ is the estimate $\hat{t}(1)$ of node 1. Then node 1 tries to find an index $\tilde{e}$ such that

$$\left(x_2^{n_2}(\hat{t}(1), \tilde{e}), u_2^{n_2}(\hat{t}(1)), y_1^{n_2}\right) \in T_\epsilon^{n_2}(P_{X_2^{(2)} U_2^{(2)} Y_1^{(2)}}). \tag{A88}$$

If there is none or more than one such index $\tilde{e}$, set $\hat{e}(1) = 1$. Otherwise, the found index $\tilde{e}$ is the estimate $\hat{e}(1)$ of node 1. Then node 1 tries to find an index $\tilde{o}$ such that

$$\left(\hat{y}_{21}^{n_1}(\hat{t}(1), \tilde{o}), x_1^{n_1}(r)\right) \in T_\epsilon^{n_1}(P_{\hat{Y}_{21}^{(1)} X_1^{(1)}}). \tag{A89}$$

If there is none or more than one such index $\tilde{o}$, set $\hat{o}(1) = 1$. Otherwise, the found index $\tilde{o}$ is the estimate $\hat{o}(1)$ of node 1. Then node 1 tries to find an index $\tilde{z}$ such that

$$\left(\hat{y}_{22}^{n_1}(\hat{t}(1), \hat{o}(1), \hat{e}(1), \tilde{z}), \hat{y}_{21}^{n_1}(\hat{t}(1), \hat{o}(1)), x_1^{n_1}(r)\right) \in T_\epsilon^{n_1}(P_{\hat{Y}_{22}^{(1)} \hat{Y}_{21}^{(1)} X_1^{(1)}}). \tag{A90}$$

If there is none or more than one such index $\tilde{z}$, set $\hat{z}(1) = 1$. Otherwise, the found index $\tilde{z}$ is the estimate $\hat{z}(1)$ of node 1. Finally node 1 tries to find an index $\tilde{s}$ such that

$$\left(x_3^{n_1}(\tilde{s}), x_1^{n_1}(r), \hat{y}_{21}^{n_1}(\hat{t}(1), \hat{o}(1)), \hat{y}_{22}^{n_1}(\hat{t}(1), \hat{o}(1), \hat{e}(1), \hat{z}(1))\right) \in T_\epsilon^{n_2}(P_{X_3^{(1)} X_1^{(1)} \hat{Y}_{21}^{(1)} \hat{Y}_{22}^{(1)}}). \tag{A91}$$

If there is none or more than one such index $\tilde{s}$, set $\hat{s}(1) = 1$. Otherwise, $\tilde{s}$ is the estimate $\hat{s}(1)$ of node 1. The output message $\hat{w}_{31}(1)$ of node 1 is found by reindexing $\hat{s}(1)$.

Node 3 (Output) In the second phase $y_3^{n_2}$ is observed. After the second phase node 3 tries to find an index $\tilde{t}$ such that

$$\left(u_2^{n_2}(\tilde{t}), y_3^{n_2}\right) \in T_\epsilon^{n_2}(P_{U_2^{(2)} Y_3^{(2)}}). \tag{A92}$$

If there is none or more than one such index $\tilde{t}$, set $\hat{t}(3) = 1$. Otherwise, the found index $\tilde{t}$ is the estimate $\hat{t}(3)$ of node 3. Then node 3 tries to find an index $\tilde{o}$ such that

$$\left(\hat{y}_{21}^{n_1}(\hat{t}(1), \tilde{o}), x_3^{n_1}(s)\right) \in T_\epsilon^{n_1}(P_{\hat{Y}_{21}^{(1)} X_3^{(1)}}). \tag{A93}$$

If there is none or more than one such index $\tilde{o}$, set $\hat{o}(1) = 1$. Otherwise, the found index $\tilde{o}$ is the estimate $\hat{o}(1)$ of node 1. Finally node 3 tries to find an index $\tilde{r}$ such that

$$\left(x_1^{n_1}(\tilde{r}), x_3^{n_1}(s), \hat{y}_{21}^{n_1}(\hat{t}(1), \hat{o}(1))\right) \in T_\epsilon^{n_1}(P_{X_1^{(1)} X_3^{(1)} \hat{Y}_{21}^{(1)}}). \tag{A94}$$

If there is none or more than one such index $\tilde{r}$, set $\hat{r}(3) = 1$. Otherwise, $\tilde{r}$ is the estimate $\hat{r}(3)$ of node 3. The output message $\hat{w}_{13}(3)$ of node 3 is found by reindexing $\hat{r}(3)$.

Rates An error analysis along the lines of A2.1 and A2.2 reveals that for large n, $\epsilon > 0$ but arbitrarily small reliable communication requires

$$\begin{aligned} R_3 + R_4 &> \tau_1 I(\hat{Y}_{21}^{(1)}; Y_2^{(1)}) \\ R_5 + R_6 &> \tau_1 I(\hat{Y}_{22}^{(1)}; Y_2^{(1)} | \hat{Y}_{21}^{(1)}) \end{aligned} \tag{A95}$$

at node 2,

$$\begin{aligned} R_3 &< \tau_2 I(U_2^{(2)}; Y_1^{(2)}) \\ R_5 &< \tau_2 I(X_2^{(2)}; Y_1^{(2)} | U_2^{(2)}) \\ R_4 &< \tau_1 I(X_1^{(1)}; \hat{Y}_{21}^{(1)}) \\ R_6 &< \tau_1 I(X_1^{(1)}; \hat{Y}_{22}^{(1)} | \hat{Y}_{21}^{(1)}) \\ R_2 &< \tau_1 I(X_3^{(1)}; \hat{Y}_{21}^{(1)} \hat{Y}_{22}^{(1)} | X_1^{(1)}) \end{aligned} \tag{A96}$$

at node 1 and

$$\begin{aligned} R_3 &< \tau_2 I(U_2^{(2)}; Y_3^{(2)}) \\ R_4 &< \tau_1 I(X_3^{(1)}; \hat{Y}_{21}^{(1)}) \\ R_1 &< \tau_1 I(X_1^{(1)}; \hat{Y}_{21}^{(1)} | X_3^{(1)}) \end{aligned} \tag{A97}$$

at node 3. Consequently,

$$\begin{aligned} R_{13} = R_1 &< \tau_1 I(X_1^{(1)}; \hat{Y}_{21}^{(1)} | X_3^{(1)}) \\ R_{31} = R_2 &< \tau_1 I(X_3^{(1)}; \hat{Y}_{21}^{(1)} \hat{Y}_{22}^{(1)} | X_1^{(1)}) \end{aligned} \tag{A98}$$

subject to

$$\begin{aligned} \tau_1 I(\hat{Y}_{21}^{(1)}; Y_2^{(1)}) &< \tau_1 I(X_1^{(1)}; \hat{Y}_{21}^{(1)}) + \tau_2 I(U_2^{(2)}; Y_1^{(2)}) \\ \tau_1 I(\hat{Y}_{21}^{(1)}; Y_2^{(1)}) &< \tau_1 I(X_3^{(1)}; \hat{Y}_{21}^{(1)}) + \tau_2 I(U_2^{(2)}; Y_3^{(2)}) \\ \tau_1 I(\hat{Y}_{22}^{(1)}; Y_2^{(1)} | \hat{Y}_{21}^{(1)}) &< \tau_1 I(X_1^{(1)}; \hat{Y}_{22}^{(1)} | \hat{Y}_{21}^{(1)}) + \tau_2 I(X_2^{(2)}; Y_1^{(2)} | U_2^{(2)}). \end{aligned} \tag{A99}$$

By the use of the Markov chains

$$\begin{aligned} X_1^{(1)} - Y_2^{(1)} - \hat{Y}_{21}^{(1)} \\ X_3^{(1)} - Y_2^{(1)} - \hat{Y}_{21}^{(1)} \\ X_1^{(1)} - (Y_2^{(1)}, \hat{Y}_{21}^{(1)}) - \hat{Y}_{22}^{(1)} \end{aligned} \tag{A100}$$

implying

$$\begin{aligned} I(\hat{Y}_{21}^{(1)}; X_1^{(1)} | Y_2^{(1)}) = 0 \\ I(\hat{Y}_{21}^{(1)}; X_3^{(1)} | Y_2^{(1)}) = 0 \\ I(\hat{Y}_{22}^{(1)}; X_1^{(1)} | Y_2^{(1)} \hat{Y}_{21}^{(1)}) = 0 \end{aligned} \tag{A101}$$

the compression constraints can be reformulated to

$$\begin{aligned} \tau_1 I(\hat{Y}_{21}^{(1)}; Y_2^{(1)} | X_1^{(1)}) &< \tau_2 I(U_2^{(2)}; Y_1^{(2)}) \\ \tau_1 I(\hat{Y}_{21}^{(1)}; Y_2^{(1)} | X_3^{(1)}) &< \tau_2 I(U_2^{(2)}; Y_3^{(2)}) \\ \tau_1 I(\hat{Y}_{22}^{(1)}; Y_2^{(1)} | X_1^{(1)} \hat{Y}_{21}^{(1)}) &< \tau_2 I(X_2^{(2)}; Y_1^{(2)} | U_2^{(2)}). \end{aligned} \tag{A102}$$

This establishes the proposition. ■

A second proposition follows by using the same arguments after having interchanged the role of nodes 1 and 3.

A3.3 Partial-Decode-Compress-and-Forward with 2-Layer-Quantization

Code Generate 2^{nR_1} n_1-sequences $u_1^{n_1}(a)$, $a = 1, 2, \ldots, 2^{nR_1}$, by choosing each element $u_{1,k}^{(1)}(a)$ independently according to $P_{U_1^{(1)}}(\cdot)$. For each $u_1^{n_1}(a)$ generate 2^{nR_2} n_1-sequences $x_1^{n_1}(a, b)$, $b = 1, 2, \ldots, 2^{nR_2}$, by choosing each element $x_{1,k}^{(1)}(a, b)$ independently according to $P_{X_1^{(1)}|U_1^{(1)}}(\cdot|u_{1,k}^{(1)}(a))$. Generate 2^{nR_3} n_1-sequences $u_3^{n_1}(c)$, $c = 1, 2, \ldots, 2^{nR_3}$, by choosing each element $u_{3,k}^{(1)}(c)$ independently according to $P_{U_3^{(1)}}(\cdot)$. For each $u_3^{n_1}(c)$ generate 2^{nR_4} n_1-sequences $x_3^{n_1}(c, d)$, $d = 1, 2, \ldots, 2^{nR_4}$, by choosing each element $x_{3,k}^{(1)}(c, d)$ independently according to $P_{X_3^{(1)}|U_3^{(1)}}(\cdot|u_{3,k}^{(1)}(c))$. Choose a "coarse quantization channel" $P_{\hat{Y}_{21}^{(1)}|Y_2^{(1)}U_1^{(1)}U_3^{(1)}}(\cdot|\cdot)$ and calculate $P_{\hat{Y}_{21}^{(1)}|U_1^{(1)}U_3^{(1)}}(\cdot)$ as the marginal distribution of $P_{\hat{Y}_{21}^{(1)}Y_2^{(1)}|U_1^{(1)}U_3^{(1)}}(\cdot)$. Choose a "refinement quantization channel" $P_{\hat{Y}_{22}^{(1)}|Y_2^{(1)}\hat{Y}_{21}^{(1)}U_1^{(1)}U_3^{(1)}}(\cdot|\cdot)$ and calculate $P_{\hat{Y}_{22}^{(1)}|\hat{Y}_{21}^{(1)}U_1^{(1)}U_3^{(1)}}(\cdot|\cdot)$ as the marginal distribution of $P_{\hat{Y}_{22}^{(1)}Y_2^{(1)}|\hat{Y}_{21}^{(1)}U_1^{(1)}U_3^{(1)}}(\cdot)$. For each $u_1^{n_1}(a)$ and $u_3^{n_1}(c)$ generate $2^{n(R_5+R_6)}$ n_1-sequences $\hat{y}_{21}^{n_1}(a, c, r, s)$, $r = 1, 2, \ldots, 2^{nR_5}$, $s = 1, 2, \ldots, 2^{nR_6}$, by choosing each element $\hat{y}_{21,k}^{(1)}(a, c, r, s)$ independently according to $P_{\hat{Y}_{21}^{(1)}|U_1^{(1)}U_3^{(1)}}(\cdot|u_{1,k}^{(1)}(a), u_{3,k}^{(1)}(c))$. For each $\hat{y}_{21}^{n_1}(a, c, r, s)$ generate $2^{n(R_7+R_8)}$ n_1-sequences $\hat{y}_{22}^{n_1}(a, c, r, s, t, o)$, $t = 1, 2, \ldots, 2^{nR_7}$, $o = 1, 2, \ldots, 2^{nR_8}$, by choosing each element $\hat{y}_{22,k}^{(1)}(a, c, r, s, t, o)$ independently according to $P_{\hat{Y}_{22}^{(1)}|\hat{Y}_{21}^{(1)}U_1^{(1)}U_3^{(1)}}(\cdot|\hat{y}_{21,k}^{(1)}(a, c, r, s), u_{1,k}^{(1)}(a), u_{3,k}^{(1)}(c))$. Generate $2^{n(R_1+R_3)}$ n_2-sequences $u_2^{n_2}(a, c)$, by choosing each element $u_{2,k}^{n_2}(a, c)$ independently according to $P_{U_2^{(2)}}(\cdot)$. For each $u_2^{n_2}(a, c)$ generate 2^{nR_5} n_2-sequences $v_2^{n_1}(a, c, r)$ by choosing each element $v_{2,k}^{(1)}(a, c, r)$ independently according to $P_{V_2^{(2)}|U_2^{(2)}}(\cdot|u_{2,k}^{(2)}(a, c))$. For each $v_2^{n_2}(a, c, r)$ generate 2^{nR_7} n_2-sequences $x_2^{n_1}(a, c, r, t)$ by choosing each element $x_{2,k}^{(1)}(a, c, r, t)$ independently according to $P_{X_2^{(2)}|V_2^{(2)}U_2^{(2)}}(\cdot|v_{2,k}^{(2)}(a, c, r), u_{2,k}^{(2)}(a, c))$.

Node 1 (Input) The message w_{13} is reindexed by (a, b). In the first phase node 1 transmits $x_1^{n_1}(a, b)$ within n_1 transmissions.

Node 3 (Input) The message w_{31} is reindexed by (c, d). In the first phase node 3 transmits $x_3^{n_1}(c, d)$ within n_1 transmissions.

Node 2 In the first phase $y_2^{n_1}$ is observed. After the first phase node 2 tries to find a pair $(\tilde{a}, \tilde{c})$ such that

$$(u_1^{n_1}(\tilde{a}), u_3^{n_1}(\tilde{c}), y_2^{n_1}) \in T_\epsilon^{n_1}(P_{U_1^{(1)}U_3^{(1)}Y_2^{(1)}}). \tag{A103}$$

If there is none or more than one such pair $(\tilde{a}, \tilde{c})$, set $(\hat{a}(2), \hat{c}(2)) = (1, 1)$. Otherwise, $(\tilde{a}, \tilde{c})$ is the estimate $(\hat{a}(2), \hat{c}(2))$ of node 2. Then node 2 tries to find a pair $(\tilde{r}, \tilde{s})$ such that

$$(\hat{y}_{21}^{n_1}(\hat{a}(2), \hat{c}(2), \tilde{r}, \tilde{s}), u_1^{n_1}(\hat{a}(2)), u_3^{n_1}(\hat{c}(2)), y_2^{n_1}) \in T_\epsilon^{n_1}(P_{\hat{Y}_{21}^{(1)}U_1^{(1)}U_3^{(1)}Y_2^{(1)}}). \tag{A104}$$

If there is none such pair $(\tilde{r}, \tilde{s})$ an error is declared. Otherwise, the found pair $(\tilde{r}, \tilde{s})$ is the estimate $(\hat{r}(2), \hat{s}(2))$ of node 2 where the pair with the smallest linear index $2^{nR_5}(s-1) + r$ is selected if more than one pair was found. Then node 2 tries to find a pair $(\tilde{t}, \tilde{o})$ such that

$$\begin{aligned}\big(\hat{y}_{22}^{n_1}(\hat{a}(2), \hat{c}(2), \hat{r}(2), \hat{s}(2), \tilde{t}, \tilde{o}), \hat{y}_{21}^{n_1}(\hat{a}(2), \hat{c}(2), \hat{r}(2), \hat{s}(2)), u_1^{n_1}(\hat{a}(2)), u_3^{n_1}(\hat{c}(2)), y_2^{n_1}\big)& \\ \in T_\epsilon^{n_1}(P_{\hat{Y}_{22}^{(1)}\hat{Y}_{21}^{(1)}U_1^{(1)}U_3^{(1)}Y_2^{(1)}}).&\end{aligned} \tag{A105}$$

If there is none such pair $(\tilde{t}, \tilde{o})$ declare an error. Otherwise, the found pair $(\tilde{t}, \tilde{o})$ is the estimate $(\hat{t}(2), \hat{o}(2))$ of node 2 where the pair with the smallest linear index $2^{nR_7}(o-1)+t$ is selected if more than one pair was found. In the second phase node 2 sends $x_2^{n_2}(\hat{a}(2), \hat{c}(2), \hat{r}(2), \hat{t}(2))$ by the use of n_2 transmissions.

Node 1 (Output) In the second phase $y_1^{n_2}$ is observed. After the second phase node 1 tries to find an index $\tilde{c}$ such that

$$(u_2^{n_2}(a, \tilde{c}), y_1^{n_2}) \in T_\epsilon^{n_2}(P_{U_2^{(2)} Y_1^{(2)}}). \tag{A106}$$

If there is none or more than one such index $\tilde{c}$, set $\hat{c}(1) = 1$. Otherwise, the found index $\tilde{c}$ is the estimate $\hat{c}(1)$ of node 1. Then node 1 tries to find an index $\tilde{r}$ such that

$$(v_2^{n_2}(a, \hat{c}(1), \tilde{r}), u_2^{n_2}(a, \hat{c}(1)), y_1^{n_2}) \in T_\epsilon^{n_2}(P_{V_2^{(2)} U_2^{(2)} Y_1^{(2)}}). \tag{A107}$$

If there is none or more than one such index $\tilde{r}$, set $\hat{r}(1) = 1$. Otherwise, the found index $\tilde{r}$ is the estimate $\hat{r}(1)$ of node 1. Then node 1 tries to find an index $\tilde{t}$ such that

$$\big(x_2^{n_2}(a, \hat{c}(1), \hat{r}(1), \tilde{t}), v_2^{n_2}(a, \hat{c}(1), \hat{r}(1)), u_2^{n_2}(a, \hat{c}(1)), y_1^{n_2}\big) \in T_\epsilon^{n_2}(P_{X_2^{(2)} V_2^{(2)} U_2^{(2)} Y_1^{(2)}}). \tag{A108}$$

If there is none or more than one such index $\tilde{t}$, set $\hat{t}(1) = 1$. Otherwise, the found index $\tilde{t}$ is the estimate $\hat{t}(1)$ of node 1. Then node 1 tries to find an index $\tilde{s}$ such that

$$(\hat{y}_{21}^{n_1}(a, \hat{c}(1), \hat{r}(1), \tilde{s}), u_1^{n_1}(a), u_3^{n_1}(\hat{c}(1)), x_1^{n_1}(a, b)) \in T_\epsilon^{n_1}(P_{\hat{Y}_{21}^{(1)} U_1^{(1)} U_3^{(1)} X_1^{(1)}}). \tag{A109}$$

If there is none or more than one such index $\tilde{s}$, set $\hat{s}(1) = 1$. Otherwise, the found index $\tilde{s}$ is the estimate $\hat{s}(1)$ of node 1. Then node 1 tries to find an index $\tilde{o}$ such that

$$\begin{aligned} &\big(\hat{y}_{22}^{n_1}(a, \hat{c}(1), \hat{r}(1), \hat{s}(1), \hat{t}(1), \tilde{o}), \hat{y}_{21}^{n_1}(a, \hat{c}(1), \hat{r}(1), \hat{s}(1)), u_1^{n_1}(a), u_3^{n_1}(\hat{c}(1)), x_1^{n_1}(a, b)\big) \\ &\qquad \in T_\epsilon^{n_1}(P_{\hat{Y}_{22}^{(1)} \hat{Y}_{21}^{(1)} U_1^{(1)} U_3^{(1)} X_1^{(1)}}). \end{aligned} \tag{A110}$$

If there is none or more than one such index $\tilde{o}$, set $\hat{o}(1) = 1$. Otherwise, the found index $\tilde{o}$ is the estimate $\hat{o}(1)$ of node 1. Finally node 1 tries to find an index $\tilde{d}$ such that

$$\begin{aligned} &\Big(x_3^{n_1}(\hat{c}(1), \tilde{d}), u_1^{n_1}(a), u_3^{n_1}(\hat{c}(1)), x_1^{n_1}(a, b), \hat{y}_{21}^{n_1}(a, \hat{c}(1), \hat{r}(1), \hat{s}(1)), \\ &\hat{y}_{22}^{n_1}(a, \hat{c}(1), \hat{r}(1), \hat{s}(1), \hat{t}(1), \hat{o}(1))\Big) \in T_\epsilon^{n_1}(P_{X_3^{(1)} U_1^{(1)} U_3^{(1)} X_1^{(1)} \hat{Y}_{21}^{(1)} \hat{Y}_{22}^{(1)}}). \end{aligned} \tag{A111}$$

If there is none or more than one such index $\tilde{o}$, set $\hat{o}(1) = 1$. Otherwise, the found index $\tilde{o}$ is the estimate $\hat{o}(1)$ of node 1. The output message $\hat{w}_{31}(1)$ of node 1 is found by reindexing $(\hat{c}(1), \hat{d}(1))$.

Node 3 (Output) In the second phase $y_3^{n_2}$ is observed. After the second phase node 3 tries to find an index $\tilde{a}$ such that

$$(u_2^{n_2}(\tilde{a}, c), y_3^{n_2}) \in T_\epsilon^{n_2}(P_{U_2^{(2)} Y_3^{(2)}}). \tag{A112}$$

If there is none or more than one such index $\tilde{a}$, set $\hat{a}(3) = 1$. Otherwise, $\tilde{a}$ is the estimate $\hat{a}(3)$ of node 3. Then node 3 tries to find an index $\tilde{r}$ such that

$$(v_2^{n_2}(\hat{a}(3), c, \tilde{r}), u_2^{n_2}(\hat{a}(3), c), y_3^{n_2}) \in T_\epsilon^{n_2}(P_{V_2^{(2)} U_2^{(2)} Y_3^{(2)}}). \tag{A113}$$

If there is none or more than one such index $\tilde{r}$, set $\hat{r}(3) = 1$. Otherwise, $\tilde{r}$ is the estimate $\hat{r}(3)$ of node 3. Then node 3 tries to find an index $\tilde{s}$ such that

$$(\hat{y}_{21}^{n_1}(\hat{a}(3), c, \hat{r}(3), \tilde{s}), u_1^{n_1}(\hat{a}(3)), u_3^{n_1}(c), x_3^{n_1}(c,d)) \in T_\epsilon^{n_1}(P_{\hat{Y}_{21}^{(1)}U_1^{(1)}U_3^{(1)}X_3^{(1)}}). \tag{A114}$$

If there is none or more than one such index $\tilde{s}$, set $\hat{s}(3) = 1$. Otherwise, $\tilde{s}$ is the estimate $\hat{s}(3)$ of node 3. Finally node 3 tries to find an index $\tilde{b}$ such that

$$\Big(x_1^{n_1}(\hat{a}(3), \tilde{b}), u_1^{n_1}(\hat{a}(3)), u_3^{n_1}(c), x_3^{n_1}(c,d), \hat{y}_{21}^{n_1}(\hat{a}(3), c, \hat{r}(3), \hat{s}(3))\Big) \tag{A115}$$

$$\in T_\epsilon^{n_1}(P_{X_3^{(1)}U_1^{(1)}U_3^{(1)}X_1^{(1)}\hat{Y}_{21}^{(1)}}). \tag{A116}$$

If there is none or more than one such index $\tilde{b}$, set $\hat{b}(3) = 1$. Otherwise, $\tilde{b}$ is the estimate $\hat{b}(3)$ of node 3. The output message $\hat{w}_{13}(3)$ of node 3 is found by reindexing $(\hat{a}(3), \hat{b}(3))$.

Rates An error analysis along the lines of A2.1 and A2.2 reveals that for large n, $\epsilon > 0$ but small, reliable communication requires

$$\begin{aligned}
R_1 &< \tau_1 I(U_1^{(1)}; Y_2^{(1)} | U_3^{(1)}) \\
R_3 &< \tau_1 I(U_3^{(1)}; Y_2^{(1)} | U_1^{(1)}) \\
R_1 + R_3 &< \tau_1 I(U_1^{(1)} U_3^{(1)}; Y_2^{(1)}) \\
R_5 + R_6 &> \tau_1 I(\hat{Y}_{21}^{(1)}; Y_2^{(1)} | U_1^{(1)} U_3^{(1)}) \\
R_7 + R_8 &> \tau_1 I(\hat{Y}_{22}^{(1)}; Y_2^{(1)} | \hat{Y}_{21}^{(1)} U_1^{(1)} U_3^{(1)})
\end{aligned} \tag{A117}$$

at node 2,

$$\begin{aligned}
R_3 &< \tau_2 I(U_2^{(2)}; Y_1^{(2)}) \\
R_5 &< \tau_2 I(V_2^{(2)}; Y_1^{(2)} | U_2^{(2)}) \\
R_7 &< \tau_2 I(X_2^{(2)}; Y_1^{(2)} | V_2^{(2)} U_2^{(2)}) \\
R_6 &< \tau_1 I(\hat{Y}_{21}^{(1)}; X_1^{(1)} | U_1^{(1)} U_3^{(1)}) \\
R_8 &< \tau_1 I(\hat{Y}_{22}^{(1)}; X_1^{(1)} | \hat{Y}_{21}^{(1)} U_1^{(1)} U_3^{(1)}) \\
R_4 &< \tau_1 I(X_3^{(1)}; \hat{Y}_{21}^{(1)} \hat{Y}_{22}^{(1)} | U_1^{(1)} U_3^{(1)} X_1^{(1)})
\end{aligned} \tag{A118}$$

at node 1 and

$$\begin{aligned}
R_1 &< \tau_2 I(U_2^{(2)}; Y_3^{(2)}) \\
R_5 &< \tau_2 I(V_2^{(2)}; Y_3^{(2)} | U_2^{(2)}) \\
R_6 &< \tau_1 I(\hat{Y}_{21}^{(1)}; X_3^{(1)} | U_1^{(1)} U_3^{(1)}) \\
R_2 &< \tau_1 I(X_1^{(1)}; \hat{Y}_{21}^{(1)} | U_1^{(1)} U_3^{(1)} X_3^{(1)})
\end{aligned} \tag{A119}$$

at node 3. Consequently,

$$\begin{aligned}
R_{13} &= R_1 + R_2 < \tau_1 I(U_1^{(1)}; Y_2^{(1)} | U_3^{(1)}) + \tau_1 I(X_1^{(1)}; \hat{Y}_{21}^{(1)} | U_1^{(1)} U_3^{(1)} X_3^{(1)}) \\
R_{13} &= R_1 + R_2 < \tau_2 I(U_2^{(2)}; Y_3^{(2)}) + \tau_1 I(X_1^{(1)}; \hat{Y}_{21}^{(1)} | U_1^{(1)} U_3^{(1)} X_3^{(1)}) \\
R_{31} &= R_3 + R_4 < \tau_1 I(U_3^{(1)}; Y_2^{(1)} | U_1^{(1)}) + \tau_1 I(X_3^{(1)}; \hat{Y}_{21}^{(1)} \hat{Y}_{22}^{(1)} | U_1^{(1)} U_3^{(1)} X_1^{(1)}) \\
R_{31} &= R_3 + R_4 < \tau_2 I(U_2^{(2)}; Y_1^{(2)}) + \tau_1 I(X_3^{(1)}; \hat{Y}_{21}^{(1)} \hat{Y}_{22}^{(1)} | U_1^{(1)} U_3^{(1)} X_1^{(1)}) \\
R_{13} + R_{31} &< \tau_1 I(U_1^{(1)} U_3^{(1)}; Y_2^{(1)}) + \tau_1 I(X_1^{(1)}; \hat{Y}_{21}^{(1)} | U_1^{(1)} U_3^{(1)} X_3^{(1)}) + \tau_1 I(X_3^{(1)}; \hat{Y}_{21}^{(1)} \hat{Y}_{22}^{(1)} | U_1^{(1)} U_3^{(1)} X_1^{(1)})
\end{aligned} \tag{A120}$$

subject to

$$\begin{aligned}
\tau_1 I(\hat{Y}_{21}^{(1)}; Y_2^{(1)} | U_1^{(1)} U_3^{(1)}) &< \tau_1 I(\hat{Y}_{21}^{(1)}; X_1^{(1)} | U_1^{(1)} U_3^{(1)}) + \tau_2 I(V_2^{(2)}; Y_1^{(2)} | U_2^{(2)}) \\
\tau_1 I(\hat{Y}_{21}^{(1)}; Y_2^{(1)} | U_1^{(1)} U_3^{(1)}) &< \tau_1 I(\hat{Y}_{21}^{(1)}; X_3^{(1)} | U_1^{(1)} U_3^{(1)}) + \tau_2 I(V_2^{(2)}; Y_3^{(2)} | U_2^{(2)}) \\
\tau_1 I(\hat{Y}_{22}^{(1)}; Y_2^{(1)} | \hat{Y}_{21}^{(1)} U_1^{(1)} U_3^{(1)}) &< \tau_1 I(\hat{Y}_{22}^{(1)}; X_1^{(1)} | \hat{Y}_{21}^{(1)} U_1^{(1)} U_3^{(1)}) + \tau_2 I(X_2^{(2)}; Y_1^{(2)} | V_2^{(2)} U_2^{(2)})
\end{aligned} \tag{A121}$$

By the use of the Markov chains

$$\begin{aligned}
X_1^{(1)} - (Y_2^{(1)}, U_1^{(1)}, U_3^{(1)}) - \hat{Y}_{21}^{(1)} \\
X_3^{(1)} - (Y_2^{(1)}, U_1^{(1)}, U_3^{(1)}) - \hat{Y}_{21}^{(1)} \\
X_1^{(1)} - (Y_2^{(1)}, \hat{Y}_{21}^{(1)}, U_1^{(1)}, U_3^{(1)}) - \hat{Y}_{22}^{(1)}
\end{aligned} \tag{A122}$$

the constraints on the compression rates can be reformulated to

$$\begin{aligned}
\tau_1 I(\hat{Y}_{21}^{(1)}; Y_2^{(1)} | U_1^{(1)} U_3^{(1)} X_1^{(1)}) &< \tau_2 I(V_2^{(2)}; Y_1^{(2)} | U_2^{(2)}) \\
\tau_1 I(\hat{Y}_{21}^{(1)}; Y_2^{(1)} | U_1^{(1)} U_3^{(1)} X_3^{(1)}) &< \tau_2 I(V_2^{(2)}; Y_3^{(2)} | U_2^{(2)}) \\
\tau_1 I(\hat{Y}_{22}^{(1)}; Y_2^{(1)} | \hat{Y}_{21}^{(1)} U_1^{(1)} U_3^{(1)} X_1^{(1)}) &< \tau_2 I(X_2^{(2)}; Y_1^{(2)} | V_2^{(2)} U_2^{(2)}).
\end{aligned} \tag{A123}$$

This establishes the proposition. ■

A4. Proofs: Achievable Rates 3P-BC Scheme

Comments and Assumptions Random encoding, jointly typical decoding and compression will be used to show which rates are achievable for the half-duplex two-way relay channel. For the following proofs it will be assumed that the transmission is performed with $n \geq 3$ channel uses and three phases $l = 1, 2, 3$. Phase 1 features $n_1 \geq 1$ transmission slots, phase 2 supports $n_2 \geq 1$ transmission slots and phase 3 supports $n_3 \geq 1$ transmission slots with $n_1 + n_2 + n_3 = n$. If n grows n_1, n_2 and n_3 are assumed to grow at the same rate. For large n, $\frac{n_l}{n} \to \tau_l > 0$. The message $w_{13} \in \{1, \ldots, 2^{nR_{13}}\}$ will be sent from node 1 to node 3 and the message $w_{31} \in \{1, \ldots, 2^{nR_{31}}\}$ will be sent from node 3 to node 1. For all proofs 2^{nR} denotes a positive integer.

A4.1 Decode-and-Forward

Code Generate $2^{n(R_1+R_2)}$ n_1-sequences $x_1^{n_1}(r, s)$, $r = 1, 2, \ldots, 2^{nR_1}$, $s = 1, 2, \ldots, 2^{nR_2}$, by choosing each element $x_{1,k}^{(1)}(r, s)$ independently according to $P_{X_1^{(1)}}(\cdot)$. Generate $2^{n(R_3+R_4)}$ n_2-sequences $x_3^{n_2}(t, o)$, $t = 1, 2, \ldots, 2^{nR_3}$, $o = 1, 2, \ldots, 2^{nR_4}$, by choosing each element $x_{3,k}^{(2)}(t, o)$ independently according to $P_{X_3^{(2)}}(\cdot)$. Generate $2^{n(R_1+R_3)}$ n_3-sequences $x_2^{n_3}(r, t)$, by choosing each element $x_{2,k}^{(3)}(r, t)$ independently according to $P_{X_2^{(3)}}(\cdot)$.

Node 1 (Input) The message w_{13} is reindexed by (r, s). In the first phase node 1 transmits $x_1^{n_1}(r, s)$ within n_1 transmissions.

Node 3 (Input) The message w_{31} is reindexed by (t, o). In the second phase node 3 transmits $x_3^{n_2}(t, o)$ within n_2 transmissions.

Node 2 In the first phase $y_2^{n_1}$ is observed. In the second phase $y_2^{n_2}$ is observed. After the first phase node 2 tries to find a pair $(\tilde{r}, \tilde{s})$ such that

$$(x_1^{n_1}(\tilde{r}, \tilde{s}), y_2^{n_1}) \in T_\epsilon^{n_1}(P_{X_1^{(1)}Y_2^{(1)}}). \tag{A124}$$

If there is none or more than one such pair $(\tilde{r}, \tilde{s})$, set $(\hat{r}(2), \hat{s}(2)) = (1, 1)$. Otherwise, the found pair $(\tilde{r}, \tilde{s})$ is the estimate $(\hat{r}(2), \hat{s}(2))$ of node 2. After the second phase node 2 tries to find a pair $(\tilde{t}, \tilde{o})$ such that

$$\left(x_3^{n_2}(\tilde{t}, \tilde{o}), y_2^{n_2}\right) \in T_\epsilon^{n_2}(P_{X_3^{(2)}Y_2^{(2)}}). \tag{A125}$$

If there is none or more than one such pair $(\tilde{t}, \tilde{o})$, set $(\hat{t}(2), \hat{o}(2)) = (1, 1)$. Otherwise, the found pair $(\tilde{t}, \tilde{o})$ is the estimate $(\hat{t}(2), \hat{o}(2))$ of node 2. In the third phase node 2 sends $x_2^{n_3}(\hat{r}(2), \hat{t}(2))$.

Node 1 (Output) In the second phase $y_1^{n_2}$ is observed. In the third phase $y_1^{n_3}$ is observed. After the third phase node 1 tries to find an index $\tilde{t}$ such that

$$\left(x_2^{n_3}(r, \tilde{t}), y_1^{n_3}\right) \in T_\epsilon^{n_3}(P_{X_2^{(3)}Y_1^{(3)}}). \tag{A126}$$

If there is none or more than one such index $\tilde{t}$, set $\hat{t}(1) = 1$. Otherwise, $\tilde{t}$ is the estimate $\hat{t}(1)$ of node 1. Now node 1 tries to find an index $\tilde{o}$ such that

$$\left(x_3^{n_2}(\hat{t}(1), \tilde{o}), y_3^{n_2}\right) \in T_\epsilon^{n_2}(P_{X_3^{(2)}Y_3^{(2)}}). \tag{A127}$$

If there is none or more than one such index $\tilde{o}$, set $\hat{o}(1) = 1$. Otherwise, $\tilde{o}$ is the estimate $\hat{o}(1)$ of node 1.

Node 3 (Output) In the first phase $y_3^{n_1}$ is observed. In the third phase $y_3^{n_3}$ is observed. After the third phase node 3 tries to find an index $\tilde{r}$ such that

$$(x_2^{n_3}(\tilde{r}, t), y_3^{n_3}) \in T_\epsilon^{n_3}(P_{X_2^{(3)} Y_3^{(3)}}). \tag{A128}$$

If there is none or more than one such index $\tilde{r}$, set $\hat{r}(3)$. Otherwise, $\tilde{r}$ is the output $\hat{r}(3)$ of node 3. Now node 3 tries to find an index $\tilde{s}$ such that

$$(x_1^{n_1}(\hat{r}(3), \tilde{s}), y_3^{n_1}) \in T_\epsilon^{n_1}(P_{X_1^{(1)} Y_3^{(1)}}). \tag{A129}$$

If there is none or more than one such index $\tilde{s}$, set $\hat{s}(3) = 1$. Otherwise, $\tilde{s}$ is the estimate $\hat{s}(3)$ of node 3.

Rates An error analysis along the lines of A2.1 and A2.2 reveals that for large n, $\epsilon > 0$ but small, reliable communication requires

$$\begin{aligned} R_1 + R_2 &< \tau_1 I(X_1^{(1)}; Y_2^{(1)}) \\ R_3 + R_4 &< \tau_2 I(X_3^{(2)}; Y_2^{(2)}) \end{aligned} \tag{A130}$$

at node 2,

$$\begin{aligned} R_1 &< \tau_3 I(X_2^{(3)}; Y_3^{(3)}) \\ R_2 &< \tau_1 I(X_1^{(1)}; Y_3^{(1)}) \end{aligned} \tag{A131}$$

at node 3 and

$$\begin{aligned} R_3 &< \tau_3 I(X_2^{(3)}; Y_1^{(3)}) \\ R_4 &< \tau_2 I(X_3^{(2)}; Y_1^{(2)}) \end{aligned} \tag{A132}$$

at node 1. This establishes the proposition. ■

A4.2 Partial-Decode-and-Forward

Code Generate $2^{n(R_1+R_2)}$ n_1-sequences $u_1^{n_1}(a,b)$, $a=1,2,\ldots,2^{nR_1}$, $b=1,2,\ldots,2^{nR_2}$, by choosing each element $u_{1,k}^{(1)}(a,b)$ independently according to $P_{U_1^{(1)}}(\cdot)$. For each $u_1^{n_1}(a,b)$, generate 2^{nR_3} n_1-sequences $x_1^{n_1}(a,b,c)$, $c=1,2,\ldots,2^{nR_3}$, by choosing each element $x_{1,k}^{(1)}(a,b,c)$ independently according to $P_{X_1^{(1)}|U_1^{(1)}}(\cdot|u_{1,i}(a,b))$. Generate $2^{n(R_4+R_5)}$ n_2-sequences $u_3^{n_2}(r,s)$, $r=1,2,\ldots,2^{nR_4}$, $s=1,2,\ldots,2^{nR_5}$, by choosing each element $u_{3,k}^{(2)}(r,s)$ independently according to $P_{U_3^{(2)}}(\cdot)$. For each $u_3^{n_2}(r,s)$, generate 2^{nR_6} n_2-sequences $x_3^{n_2}(r,s,t)$, $t=1,2,\ldots,2^{nR_6}$, by choosing each element $x_{3,k}^{(2)}(r,s,t)$ independently according to $P_{X_3^{(2)}|U_3^{(2)}}(\cdot|u_{3,i}(r,s))$. Generate $2^{n(R_1+R_4)}$ n_3-sequences $x_2^{n_3}(a,r)$, $a=1,2,\ldots,2^{nR_1}$, $r=1,2,\ldots,2^{nR_4}$, by choosing each element $x_{2,k}^{(3)}(a,r)$ independently according to $P_{X_2^{(3)}}(\cdot)$.

Node 1 (Input) The message w_{13} is reindexed by (a,b,c). In the first phase node 1 transmits $x_1^{n_1}(a,b,c)$ within n_1 transmissions.

Node 3 (Input) The message w_{31} is reindexed by (r,s,t). In the second phase node 3 transmits $x_3^{n_2}(r,s,t)$ within n_2 transmissions.

Node 2 In the first phase $y_2^{n_1}$ is observed. After the first phase node 2 tries to find a pair $(\tilde{a},\tilde{b})$ such that

$$\left(u_1^{n_1}(\tilde{a},\tilde{b}),y_2^{n_1}\right)\in T_\epsilon^{n_1}(P_{U_1^{(1)}Y_2^{(1)}}). \tag{A133}$$

If there is none or more than one such tuple $(\tilde{a},\tilde{b})$, set $(\hat{a}(2),\hat{b}(2))=(1,1)$. Otherwise, the found pair $(\tilde{a},\tilde{b})$ is the estimate $(\hat{a}(2),\hat{b}(2))$ of node 2. In the second phase $y_2^{n_2}$ is observed. After the second phase node 2 tries to find a pair $(\tilde{r},\tilde{s})$ such that

$$(u_3^{n_2}(\tilde{r},\tilde{s}),y_2^{n_2})\in T_\epsilon^{n_2}(P_{U_3^{(2)}Y_2^{(2)}}). \tag{A134}$$

If there is none or more than one such pair $(\tilde{r},\tilde{s})$, set $(\hat{r}(2),\hat{s}(2))=(1,1)$. Otherwise, the found pair $(\tilde{r},\tilde{s})$ is the estimate $(\hat{r}(2),\hat{s}(2))$ of node 2. In the third phase node 2 sends $x_2^{n_3}(\hat{a}(2),\hat{r}(2))$.

Node 1 (Output) In the second phase $y_1^{n_2}$ is observed. In the third phase $y_1^{n_3}$ is observed. After the third phase node 1 tries to find an index $\tilde{r}$ such that

$$(x_2^{n_3}(a,\tilde{r}),y_1^{n_3})\in T_\epsilon^{n_3}(P_{X_2^{(3)},Y_1^{(3)}}). \tag{A135}$$

If there is none or more than one such index $\tilde{r}$, set $\hat{r}(1)=1$. Otherwise, the found index $\tilde{r}$ is the estimate $\hat{r}(1)$ of node 1. Then node 1 tries to find an index $\tilde{s}$ such that

$$(u_3^{n_2}(\hat{r}(1),\tilde{s}),y_1^{n_2})\in T_\epsilon^{n_2}(P_{U_3^{(2)}Y_1^{(2)}}). \tag{A136}$$

If there is none or more than one such index $\tilde{s}$, set $\hat{s}(1)=1$. Otherwise, the found index $\tilde{s}$ is the estimate $\hat{s}(1)$ of node 1. Finally node 1 tries to find an index $\tilde{t}$ such that

$$\left(x_3^{n_2}(\hat{r}(1),\hat{s}(1),\tilde{t}),u_3^{n_2}(\hat{r}(1),\hat{s}(1)),y_1^{n_2}\right)\in T_\epsilon^{n_2}(P_{X_3^{(2)}U_3^{(2)}Y_1^{(2)}}). \tag{A137}$$

If there is none or more than one such index $\tilde{t}$, set $\hat{t}(1)=1$. Otherwise, the found index $\tilde{t}$ is the estimate $\hat{t}(1)$ of node 1.

Node 3 (Output) In the first phase $y_3^{n_1}$ is observed. In the third phase $y_3^{n_3}$ is observed. After the third phase node 3 tries to find an index $\tilde{a}$ such that

$$(x_2^{n_3}(\tilde{a}, r), y_3^{n_3}) \in T_\epsilon^{n_3}(P_{X_2^{(3)}Y_3^{(3)}}). \tag{A138}$$

If there is none or more than one such index $\tilde{a}$, set $\hat{a}(1) = 1$. Otherwise, the found index $\tilde{a}$ is the estimate $\hat{a}(1)$ of node 3. Then node 3 tries to find an index $\tilde{b}$ such that

$$\left(u_1^{n_1}(\hat{a}(3), \tilde{b}), y_3^{n_1}\right) \in T_\epsilon^{n_1}(P_{U_1^{(1)}Y_3^{(1)}}). \tag{A139}$$

If there is none or more than one such index $\tilde{b}$, set $\hat{b}(3) = 1$. Otherwise, the found index $\tilde{b}$ is the estimate $\hat{b}(3)$ of node 3. Finally node 3 tries to find an index $\tilde{c}$ such that

$$\left(x_1^{n_1}(\hat{a}(3), \hat{b}(3), \tilde{c}), u_1^{n_1}(\hat{a}(3), \hat{b}(3)), y_3^{n_1}\right) \in T_\epsilon^{n_1}(P_{X_1^{(1)}U_1^{(1)}Y_3^{(1)}}). \tag{A140}$$

If there is none or more than one such index $\tilde{c}$, set $\hat{c}(3) = 1$. Otherwise, the found index $\tilde{c}$ is the estimate $\hat{c}(3)$ of node 3.

Rates An error analysis along the lines of A2.1 and A2.2 reveals that for large n, $\epsilon > 0$ but small, reliable communication requires

$$\begin{aligned} R_1 + R_2 &< \tau_1 I(U_1^{(1)}; Y_2^{(1)}) \\ R_4 + R_5 &< \tau_2 I(U_3^{(2)}; Y_2^{(2)}) \end{aligned} \tag{A141}$$

at node 2,

$$\begin{aligned} R_4 &< \tau_3 I(X_2^{(3)}; Y_1^{(3)}) \\ R_5 &< \tau_2 I(U_3^{(2)}; Y_1^{(2)}) \\ R_6 &< \tau_2 I(X_3^{(2)}; Y_1^{(2)}|U_3^{(2)}) \end{aligned} \tag{A142}$$

at node 1 and

$$\begin{aligned} R_1 &< \tau_3 I(X_2^{(3)}; Y_3^{(3)}) \\ R_2 &< \tau_1 I(U_1^{(1)}; Y_3^{(1)}) \\ R_3 &< \tau_1 I(X_1^{(1)}; Y_3^{(1)}|U_1^{(1)}) \end{aligned} \tag{A143}$$

at node 3. Therefore, with

$$\begin{aligned} U_1^{(1)} - X_1^{(1)} - Y_3^{(1)} \\ U_3^{(2)} - X_3^{(2)} - Y_1^{(2)} \end{aligned} \tag{A144}$$

forming Markov chains

$$\begin{aligned} R_{13} &< \tau_1 I(U_1^{(1)}; Y_2^{(1)}) + \tau_1(X_1^{(1)}; Y_3^{(1)}|U_1^{(1)}) \\ R_{13} &< \tau_1 I(X_1^{(1)}; Y_3^{(1)}) + \tau_3 I(X_2^{(3)}; Y_3^{(3)}) \\ R_{31} &< \tau_2 I(U_3^{(2)}; Y_2^{(2)}) + \tau_2(X_3^{(2)}; Y_1^{(2)}|U_3^{(2)}) \\ R_{31} &< \tau_3 I(X_2^{(3)}; Y_1^{(3)}) + \tau_2 I(X_3^{(2)}; Y_1^{(2)}). \end{aligned} \tag{A145}$$

This establishes the proposition. ■

A4.3 Compress-and-Forward

Code Generate $2^{nR_{13}}$ n_1-sequences $x_1^{n_1}(w_{13})$, $w_{13} = 1, 2, \ldots, 2^{nR_{13}}$, by choosing each element $x_{1,k}^{(1)}(w_{13})$ independently according to $P_{X_1^{(1)}}(\cdot)$. Generate $2^{nR_{31}}$ n_2-sequences $x_3^{n_2}(w_{31})$, $w_{31} = 1, 2, \ldots, 2^{nR_{31}}$, by choosing each element $x_{3,k}^{(2)}(w_{31})$ independently according to $P_{X_3^{(2)}}(\cdot)$. Choose a "quantization channel" $P_{\hat{Y}_2^{(1)}|Y_2^{(1)}}(\cdot|\cdot)$ and calculate $P_{\hat{Y}_2^{(1)}}(\cdot)$ as the marginal distribution of $P_{\hat{Y}_2^{(1)}Y_2^{(1)}}(\cdot)$. Generate $2^{n(R_1+R_2)}$ n_1-sequences $\hat{y}_2^{n_1}(r, s)$, $r = 1, 2, \ldots, 2^{nR_1}$, $s = 1, 2, \ldots, 2^{nR_2}$, by choosing each element $\hat{y}_{2,k}^{(1)}(r, s)$ independently according to $P_{\hat{Y}_2^{(1)}}(\cdot)$. Choose a "quantization channel" $P_{\hat{Y}_2^{(2)}|Y_2^{(2)}}(\cdot|\cdot)$ and calculate $P_{\hat{Y}_2^{(2)}}(\cdot)$ as the marginal distribution of $P_{\hat{Y}_2^{(2)}Y_2^{(2)}}(\cdot)$. Generate $2^{n(R_3+R_4)}$ n_2-sequences $\hat{y}_2^{n_2}(t, o)$, $t = 1, 2, \ldots, 2^{nR_3}$, $o = 1, 2, \ldots, 2^{nR_4}$, by choosing each element $\hat{y}_{2,k}^{(2)}(t, o)$ independently according to $P_{\hat{Y}_2^{(2)}}(\cdot)$. Generate $2^{n(R_3+R_5)}$ n_3-sequences $u_{21}^{n_3}(t, q_1)$, $q_1 = 1, 2, \ldots, 2^{nR_5}$, by choosing each element $u_{21,k}^{(3)}(t, q_1)$ independently according to $P_{U_{21}^{(3)}}(\cdot)$. Generate $2^{n(R_1+R_6)}$ n_3-sequences $u_{22}^{n_3}(r, q_2)$, $q_2 = 1, 2, \ldots, 2^{nR_6}$, by choosing each element $u_{22,k}^{(3)}(r, q_2)$ independently according to $P_{U_{22}^{(3)}}(\cdot)$.

Node 1 (Input) In the first phase node 1 transmits $x_1^{n_1}(w_{13})$ within n_1 transmissions.

Node 3 (Input) In the second phase node 3 transmits $x_3^{n_2}(w_{31})$ within n_2 transmissions.

Node 2 In the first phase $y_2^{n_1}$ is observed. After the first phase node 2 tries to find a pair $(\tilde{r}, \tilde{s})$ such that

$$(\hat{y}_2^{n_1}(\tilde{r}, \tilde{s}), y_2^{n_1}) \in T_\epsilon^{n_1}(P_{\hat{Y}_2^{(1)}Y_2^{(1)}}). \tag{A146}$$

If there is none such pair $(\tilde{r}, \tilde{s})$ an error is declared. Otherwise, the found pair $(\tilde{r}, \tilde{s})$ is the estimate $(\hat{r}(2), \hat{s}(2))$ of node 2, where the pair with the smallest linear index $2^{nR_1}(\tilde{s} - 1) + \tilde{r}$ is selected if more than one pair was found. In the second phase $y_2^{n_2}$ is observed. After the second phase node 2 tries to find a pair $(\tilde{t}, \tilde{o})$ such that

$$\big(\hat{y}_2^{n_2}(\tilde{t}, \tilde{o}), y_2^{n_2}\big) \in T_\epsilon^{n_2}(P_{\hat{Y}_2^{(2)}Y_2^{(2)}}). \tag{A147}$$

If there is none such pair $(\tilde{t}, \tilde{o})$ an error is declared. Otherwise, the found pair $(\tilde{t}, \tilde{o})$ is the estimate $(\hat{t}(2), \hat{o}(2))$ of node 2, where the pair with the smallest linear index $2^{nR_3}(\tilde{o} - 1) + \tilde{t}$ is selected if more than one pair was found. Then node 2 tries to find a pair $(\tilde{q}_1, \tilde{q}_2)$ such that

$$\big(u_{21}^{n_3}(\hat{t}(2), \tilde{q}_1), u_{22}^{n_3}(\hat{r}(2), \tilde{q}_2)\big) \in T_\epsilon^{n_3}(P_{U_{21}^{(3)}U_{22}^{(3)}}). \tag{A148}$$

If there is none such pair $(\tilde{q}_1, \tilde{q}_2)$ an error is declared. Otherwise, the found pair $(\tilde{q}_1, \tilde{q}_2)$ is the estimate $(\hat{q}_1(2), \hat{q}_2(2))$ of node 2, where the pair with the smallest linear index $2^{nR_5}(\tilde{q}_2 - 1) + \tilde{q}_1$ is selected if more than one pair was found. In the third phase node 2 sends

$$x_2^{n_3} = f^{n_3}(u_{21}^{n_3}(\hat{t}(2), \hat{q}_1(2)), u_{22}^{n_3}(\hat{r}(2), \hat{q}_2(2))) \tag{A149}$$

where $f(\cdot)$ is a function that maps symbols from $\mathcal{U}_{21} \times \mathcal{U}_{22}$ to $\mathcal{X}_2$.

Node 1 (Output) In the second phase $y_1^{n_2}$ is observed. In the third phase $y_1^{n_3}$ is observed. After the third phase node 1 tries to find a pair $(\tilde{t}, \tilde{q}_1)$ such that

$$\left(u_{21}^{n_3}(\tilde{t}, \tilde{q}_1), y_1^{n_3}\right) \in T_\epsilon^{n_3}(P_{U_{21}^{(3)} Y_1^{(3)}}). \tag{A150}$$

If there is none or more than one such pair $(\tilde{t}, \tilde{q}_1)$, set $(\hat{t}(1), \hat{q}_1(1)) = (1, 1)$. Otherwise, $(\tilde{t}, \tilde{q}_1)$ is the estimate $(\hat{t}(1), \hat{q}_1(1))$ of node 1. Then node 1 tries to find an index $\tilde{o}$ such that

$$\left(\hat{y}_2^{n_2}(\hat{t}(1), \tilde{o}), y_1^{n_2}\right) \in T_\epsilon^{n_2}(P_{\hat{Y}_2^{(2)} Y_1^{(2)}}). \tag{A151}$$

If there is none or more than one such index $\tilde{o}$, set $\hat{o}(1) = 1$. Otherwise, the found index $\tilde{o}$ is the estimate $\hat{o}(1)$ of node 1. Finally node 1 tries to find an index $\tilde{w}_{31}$ such that

$$\left(x_3^{n_2}(\tilde{w}_{31}), \hat{y}_2^{n_2}(\hat{t}(1), \hat{o}(1)), y_1^{n_2}\right) \in T_\epsilon^{n_2}(P_{X_3^{(2)} \hat{Y}_2^{(2)} Y_1^{(2)}}). \tag{A152}$$

If there is none or more than one such index $\tilde{w}_{31}$, set $\hat{w}_{31}(1) = 1$. Otherwise, $\tilde{w}_{31}$ is the output message $\hat{w}_{31}(1)$ of node 1.

Node 3 (Output) In the first phase $y_3^{n_1}$ is observed. In the third phase $y_3^{n_3}$ is observed. After the third phase node 3 tries to find a pair $(\tilde{r}, \tilde{q}_2)$ such that

$$(u_{22}^{n_3}(\tilde{r}, \tilde{q}_2), y_3^{n_3}) \in T_\epsilon^{n_3}(P_{U_{22}^{(3)} Y_3^{(3)}}). \tag{A153}$$

If there is none or more than one such $(\tilde{r}, \tilde{q}_2)$, set $(\hat{r}(3), \hat{q}_2(3)) = (1, 1)$. Otherwise, the found pair $(\tilde{r}, \tilde{q}_2)$ is the estimate $(\hat{r}(3), \hat{q}_2(3))$ of node 3. Then node 3 tries to find an index $\tilde{s}$ such that

$$(\hat{y}_2^{n_1}(\hat{r}(3), \tilde{s}), y_3^{n_1}) \in T_\epsilon^{n_1}(P_{\hat{Y}_2^{(1)} Y_3^{(1)}}) \tag{A154}$$

If there is none or more than one such index $\tilde{s}$, set $\hat{s}(3) = 1$. Otherwise, the found index $\tilde{s}$ is the estimate $\hat{s}(3)$ of node 3. Finally node 3 tries to find an index $\tilde{w}_{13}$ such that

$$(x_1^{n_1}(\tilde{w}_{13}), \hat{y}_2^{n_1}(\hat{r}(3), \hat{s}(3)), y_3^{n_1}) \in T_\epsilon^{n_1}(P_{X_1^{(1)} \hat{Y}_2^{(1)} Y_3^{(1)}}). \tag{A155}$$

If there is none or more than one such index $\tilde{w}_{13}$, set $\hat{w}_{13}(3) = 1$. Otherwise, the found index $\tilde{w}_{13}$ is the output message $\hat{w}_{13}(3)$ of node 3.

Rates An error analysis along the lines of A2.1 and A2.2 reveals that for large n, $\epsilon > 0$ but small, reliable communication requires

$$\begin{aligned}
R_1 + R_2 &> \tau_1 I(\hat{Y}_2^{(1)}; Y_2^{(1)}) \\
R_3 + R_4 &> \tau_2 I(\hat{Y}_2^{(2)}; Y_2^{(2)}) \\
R_5 + R_6 &> \tau_3 I(U_{21}^{(3)}; U_{22}^{(3)})
\end{aligned} \tag{A156}$$

at node 2,

$$\begin{aligned}
R_3 + R_5 &< \tau_3 I(U_{21}^{(3)}; Y_1^{(3)}) \\
R_4 &< \tau_2 I(\hat{Y}_2^{(2)}; Y_1^{(2)}) \\
R_{31} &< \tau_2 I(X_3^{(2)}; \hat{Y}_2^{(2)} Y_1^{(2)}).
\end{aligned} \tag{A157}$$

at node 1 and

$$\begin{aligned} R_1 + R_6 &< \tau_3 I(U_{22}^{(3)}; Y_3^{(3)}) \\ R_2 &< \tau_1 I(\hat{Y}_2^{(1)}; Y_3^{(1)}) \\ R_{13} &< \tau_1 I(X_1^{(1)}; \hat{Y}_2^{(1)} Y_3^{(1)}). \end{aligned} \tag{A158}$$

at node 3. Choosing

$$R_5 = \kappa \tau_3 I(U_{21}^{(3)}; U_{22}^{(3)}) \qquad \kappa \in [0; 1] \tag{A159}$$

gives

$$\begin{aligned} R_1 + R_2 &> \tau_1 I(\hat{Y}_2^{(1)}; Y_2^{(1)}) \\ R_3 + R_4 &> \tau_2 I(\hat{Y}_2^{(2)}; Y_2^{(2)}) \\ R_3 &< \tau_3 I(U_{21}^{(3)}; Y_1^{(3)}) - \kappa \tau_3 I(U_{21}^{(3)}; U_{22}^{(3)}) \\ R_4 &< \tau_2 I(\hat{Y}_2^{(2)}; Y_1^{(2)}) \\ R_1 &< \tau_3 I(U_{22}^{(3)}; Y_3^{(3)}) - (1-\kappa) \tau_3 I(U_{21}^{(3)}; U_{22}^{(3)}) \\ R_2 &< \tau_1 I(\hat{Y}_2^{(1)}; Y_3^{(1)}). \end{aligned} \tag{A160}$$

Using the fact that

$$\begin{aligned} &Y_3^{(1)} - Y_2^{(1)} - \hat{Y}_2^{(1)} \\ &Y_1^{(2)} - Y_2^{(2)} - \hat{Y}_2^{(2)} \end{aligned} \tag{A161}$$

form Markov chains the constraints can be reformulated to

$$\begin{aligned} \tau_1 I(\hat{Y}_2^{(1)}; Y_2^{(1)} | Y_3^{(1)}) &< \tau_3 I(U_{22}^{(3)}; Y_3^{(3)}) - (1-\kappa) \tau_3 I(U_{21}^{(3)}; U_{22}^{(3)}) \\ \tau_2 I(\hat{Y}_2^{(2)}; Y_2^{(2)} | Y_1^{(2)}) &< \tau_3 I(U_{21}^{(3)}; Y_1^{(3)}) - \kappa \tau_3 I(U_{21}^{(3)}; U_{22}^{(3)}). \end{aligned} \tag{A162}$$

Alternatively,

$$\begin{aligned} \tau_1 I(\hat{Y}_2^{(1)}; Y_2^{(1)} | Y_3^{(1)}) &< \tau_3 I(U_{22}^{(3)}; Y_3^{(3)}) \\ \tau_2 I(\hat{Y}_2^{(2)}; Y_2^{(2)} | Y_1^{(2)}) &< \tau_3 I(U_{21}^{(3)}; Y_1^{(3)}) \\ \tau_1 I(\hat{Y}_2^{(1)}; Y_2^{(1)} | Y_3^{(1)}) + \tau_2 I(\hat{Y}_2^{(2)}; Y_2^{(2)} | Y_1^{(2)}) &< \tau_3 I(U_{21}^{(3)}; Y_1^{(3)}) + \tau_3 I(U_{22}^{(3)}; Y_3^{(3)}) - \tau_3 I(U_{21}^{(3)}; U_{22}^{(3)}). \end{aligned} \tag{A163}$$

This establishes the proposition.■

A4.4 Partial-Decode-Compress-and-Forward

Code Generate $2^{n(R_1+R_2)}$ n_1-sequences $u_1^{n_1}(a,b)$, $a=1,2,\ldots,2^{nR_1}$, $b=1,2,\ldots,2^{nR_2}$, by choosing each element $u_{1,k}^{(1)}(a,b)$ independently according to $P_{U_1^{(1)}}(\cdot)$. For each $u_1^{n_1}(a,b)$, generate 2^{nR_3} n_1-sequences $x_1^{n_1}(a,b,c)$, $c=1,2,\ldots,2^{nR_3}$, by choosing each element $x_{1,k}^{(1)}(a,b,c)$ independently according to $P_{X_1^{(1)}|U_1^{(1)}}(\cdot|u_{1,k}^{(1)}(a,b))$. Choose a "quantization channel" $P_{\hat{Y}_2^{(1)}|Y_2^{(1)}U_1^{(1)}}(\cdot|\cdot)$ and calculate $P_{\hat{Y}_2^{(1)}|U_1^{(1)}}(\cdot)$ as the marginal distribution of $P_{\hat{Y}_2^{(1)}Y_2^{(1)}|U_1^{(1)}}(\cdot)$. For each $u_1^{n_1}(a,b)$, generate $2^{n(R_7+R_8)}$ n_1-sequences $\hat{y}_2^{n_1}(a,b,d,e)$, $d=1,2,\ldots,2^{nR_7}$, $e=1,2,\ldots,2^{nR_8}$, by choosing each element $\hat{y}_{2,k}^{(1)}(a,b,d,e)$ independently according to $P_{\hat{Y}_2^{(1)}|U_1^{(1)}}(\cdot|u_{1,i}(a,b))$. Generate $2^{n(R_4+R_5)}$ n_2-sequences $u_3^{n_2}(r,s)$, $r=1,2,\ldots,2^{nR_4}$, $s=1,2,\ldots,2^{nR_5}$, by choosing each element $u_{3,k}^{(2)}(r,s)$ independently according to $P_{U_3^{(2)}}(\cdot)$. For each $u_3^{n_2}(r,s)$, generate 2^{nR_6} n_2-sequences $x_3^{n_2}(r,s,t)$, $t=1,2,\ldots,2^{nR_6}$, by choosing each element $x_{3,k}^{(2)}(r,s,t)$ independently according to $P_{X_3^{(2)}|U_3^{(2)}}(\cdot|u_{3,k}^{(2)}(r,s))$. Choose a "quantization channel" $P_{\hat{Y}_2^{(2)}|Y_2^{(2)}U_3^{(2)}}(\cdot|\cdot)$ and calculate $P_{\hat{Y}_2^{(2)}|U_3^{(2)}}(\cdot)$ as the marginal distribution of $P_{\hat{Y}_2^{(2)}Y_2^{(2)}|U_3^{(2)}}(\cdot)$. For each $u_3^{n_2}(r,s)$, generate $2^{n(R_9+R_{10})}$ n_2-sequences $\hat{y}_2^{n_2}(r,s,o,z)$, $o=1,2,\ldots,2^{nR_9}$, $z=1,2,\ldots,2^{nR_{10}}$, by choosing each element $\hat{y}_{2,k}^{(2)}(r,s,o,z)$ independently according to $P_{\hat{Y}_2^{(2)}|U_3^{(2)}}(\cdot|u_{3,k}^{(2)}(r,s))$. Generate $2^{n(R_1+R_4)}$ n_3-sequences $u_2^{n_3}(a,r)$, $a=1,2,\ldots,2^{nR_1}$, $r=1,2,\ldots,2^{nR_4}$, by choosing each element $u_{2,k}^{(3)}(a,r)$ independently according to $P_{U_2^{(3)}}(\cdot)$. For each $u_2^{n_3}(a,r)$, generate $2^{n(R_9+R_{11})}$ n_3-sequences $v_{21}^{n_3}(a,r,o,q_1)$, $o=1,2,\ldots,2^{nR_9}$, $q_1=1,2,\ldots,2^{nR_{11}}$, by choosing each element $v_{21,k}^{(3)}(a,r,o,q_1)$ independently according to $P_{V_{21}^{(3)}|U_2^{(3)}}(\cdot|u_{2,k}^{(3)}(a,r))$. For each $u_2^{n_3}(a,r)$, generate $2^{n(R_7+R_{12})}$ n_3-sequences $v_{22}^{n_3}(a,r,d,q_2)$, $d=1,2,\ldots,2^{nR_7}$, $q_2=1,2,\ldots,2^{nR_{12}}$, by choosing each element $v_{22,k}^{(3)}(a,r,d,q_2)$ independently according to $P_{V_{22}^{(3)}|U_2^{(3)}}(\cdot|u_{2,k}^{(3)}(a,r))$.

Node 1 (Input) The message w_{13} is reindexed by (a,b,c). In the first phase node 1 transmits $x_1^{n_1}(a,b,c)$ within n_1 transmissions.

Node 3 (Input) The message w_{31} is reindexed by (r,s,t). In the second phase node 3 transmits $x_3^{n_2}(r,s,t)$ within n_2 transmissions.

Node 2 In the first phase $y_2^{n_1}$ is observed. After the first phase node 2 tries to find a pair $(\tilde{a},\tilde{b})$ such that

$$\left(u_1^{n_1}(\tilde{a},\tilde{b}),y_2^{n_1}\right)\in T_\epsilon^{n_1}(P_{U_1^{(1)}Y_2^{(1)}}). \tag{A164}$$

If there is none or more than one such pair $(\tilde{a},\tilde{b})$, set $(\hat{a}(2),\hat{b}(2))=(1,1)$. Otherwise, the found pair $(\tilde{a},\tilde{b})$ is the estimate $(\hat{a}(2),\hat{b}(2))$ of node 2. Then node 2 tries to find a pair $(\tilde{d},\tilde{e})$ such that

$$\left(\hat{y}_2^{n_1}(\hat{a}(2),\hat{b}(2),\tilde{d},\tilde{e}),u_1^{n_1}(\hat{a}(2),\hat{b}(2)),y_2^{n_1}\right)\in T_\epsilon^{n_1}(P_{\hat{Y}_2^{(1)}U_1^{(1)}Y_2^{(1)}}). \tag{A165}$$

If there is none such pair $(\tilde{d},\tilde{e})$ an error is declared. Otherwise, the found pair $(\tilde{d},\tilde{e})$ is the estimate $(\hat{d}(2),\hat{e}(2))$ of node 2 where the pair with the smallest linear index $2^{nR_7}(e-1)+d$ is selected if more than one pair was found. In the second phase $y_2^{n_2}$ is observed. After the second phase node 2 tries to find a pair $(\tilde{r}, \tilde{s})$ such that

$$(u_3^{n_2}(\tilde{r}, \tilde{s}), y_2^{n_2}) \in T_\epsilon^{n_2}(P_{U_3^{(2)}Y_2^{(2)}}). \tag{A166}$$

If there is none or more than one such pair $(\tilde{r}, \tilde{s})$, set $(\hat{r}(2), \hat{s}(2)) = (1, 1)$. Otherwise, the found pair $(\tilde{r}, \tilde{s})$ is the estimate $(\hat{r}(2), \hat{s}(2))$ of node 2. Then node 2 tries to find a pair $(\tilde{o}, \tilde{z})$ such that

$$(\hat{y}_2^{n_2}(\hat{r}(2), \hat{s}(2), \tilde{o}, \tilde{z}), u_3^{n_2}(\hat{r}(2), \hat{s}(2)), y_2^{n_2}) \in T_\epsilon^{n_2}(P_{\hat{Y}_2^{(2)}U_3^{(2)}Y_2^{(2)}}). \tag{A167}$$

If there is none such pair $(\tilde{o}, \tilde{z})$ an error is declared. Otherwise, the found pair $(\tilde{o}, \tilde{z})$ is the estimate $(\hat{o}(2), \hat{z}(2))$ of node 2 where the pair with the smallest linear index $2^{nR_9}(z-1)+o$ is selected if more than one pair was found. Then node 2 tries to find a pair $(\tilde{q}_1, \tilde{q}_2)$ such that

$$\Big(v_{21}^{n_3}(\hat{a}(2), \hat{r}(2), \hat{o}(2), \tilde{q}_1), v_{22}^{n_3}(\hat{a}(2), \hat{r}(2), \hat{d}(2), \tilde{q}_2), u_2^{n_3}(\hat{a}(2), \hat{r}(2))\Big) \in T_\epsilon^{n_3}(P_{V_{21}^{(3)}V_{22}^{(3)}U_2^{(3)}}). \tag{A168}$$

If there is none such pair $(\tilde{q}_1, \tilde{q}_2)$ an error is declared. Otherwise, the found pair $(\tilde{q}_1, \tilde{q}_2)$ is the estimate $(\hat{q}_1(2), \hat{q}_2(2))$ of node 2, where the pair with the smallest linear index $2^{nR_{11}}(\tilde{q}_2-1)+\tilde{q}_1$ is selected if more than one pair was found. In the third phase node 2 sends

$$x_2^{n_3} = f^{n_3}(v_{21}^{n_3}(\hat{a}(2), \hat{r}(2), \hat{o}(2), \hat{q}_1(2)), v_{22}^{n_3}(\hat{a}(2), \hat{r}(2), \hat{d}(2), \hat{q}_2(2)), u_2^{n_3}(\hat{a}(2), \hat{r}(2))) \tag{A169}$$

where $f(\cdot)$ is a function that maps symbols from $\mathcal{V}_{21} \times \mathcal{V}_{22} \times \mathcal{U}_2$ to $\mathcal{X}_2$.

Node 1 (Output) In the second phase $y_1^{n_2}$ is observed. In the third phase $y_1^{n_3}$ is observed. After the third phase node 1 tries to find an index $\tilde{r}$ such that

$$(u_2^{n_3}(a, \tilde{r}), y_1^{n_3}) \in T_\epsilon^{n_3}(P_{U_2^{(3)}Y_1^{(3)}}). \tag{A170}$$

If there is none or more than one such index $\tilde{r}$, set $\hat{r}(1) = 1$. Otherwise, the found index $\tilde{r}$ is the estimate $\hat{r}(1)$ of node 1. Then node 1 tries to find a pair $(\tilde{o}, \tilde{q}_1)$ such that

$$(v_{21}^{n_3}(a, \hat{r}(1), \tilde{o}, \tilde{q}_1), u_2^{n_3}(a, \hat{r}(1)), y_1^{n_3}) \in T_\epsilon^{n_3}(P_{V_{21}^{(3)}U_2^{(3)}Y_1^{(3)}}). \tag{A171}$$

If there is none or more than one such pair $(\tilde{o}, \tilde{q}_1)$, set $(\hat{o}(1), \hat{q}_1(1)) = (1, 1)$. Otherwise, the found pair $(\tilde{o}, \tilde{q}_1)$ is the estimate $(\hat{o}(1), \hat{q}_1(1))$ of node 1. Then node 1 tries to find an index $\tilde{s}$ such that

$$(u_3^{n_2}(\hat{r}(1), \tilde{s}), y_1^{n_2}) \in T_\epsilon^{n_2}(P_{U_3^{(2)}Y_1^{(2)}}). \tag{A172}$$

If there is none or more than one such index $\tilde{z}$, set $\hat{z}(1) = 1$. Otherwise, the found index $\tilde{z}$ is the estimate $\hat{z}(1)$ of node 1. Then node 1 tries to find an index $\tilde{z}$ such that

$$(\hat{y}_2^{n_2}(\hat{r}(1), \hat{s}(1), \hat{o}(1), \tilde{z}), u_3^{n_2}(\hat{r}(1), \hat{s}(1)), y_1^{n_2}) \in T_\epsilon^{n_2}(P_{\hat{Y}_2^{(2)}U_3^{(2)}Y_1^{(2)}}). \tag{A173}$$

If there is none or more than one such index $\tilde{s}$, set $\hat{s}(1) = 1$. Otherwise, the index $\tilde{s}$ is the estimate $\hat{s}(1)$ of node 1. Finally node 1 tries to find an index $\tilde{t}$ such that

$$\big(x_3^{n_2}(\hat{r}(1), \hat{s}(1), \tilde{t}), u_3^{n_2}(\hat{r}(1), \hat{s}(1)), \hat{y}_2^{n_2}(\hat{r}(1), \hat{s}(1), \hat{o}(1), \hat{z}(1)), y_1^{n_2}\big) \in T_\epsilon^{n_2}(P_{X_3^{(2)}U_3^{(2)}\hat{Y}_2^{(2)}Y_1^{(2)}}). \tag{A174}$$

If there is none or more than one such index $\tilde{t}$, set $\hat{t}(1) = 1$. Otherwise, the found index $\tilde{t}$ is the estimate $\hat{t}(1)$ of node 1.

Node 3 (Output) In the first phase $y_3^{n_1}$ is observed. In the third phase $y_3^{n_3}$ is observed. After the third phase node 3 tries to find an index $\tilde{a}$ such that

$$(u_2^{n_3}(\tilde{a}, r), y_3^{n_3}) \in T_\epsilon^{n_3}(P_{U_2^{(3)} Y_3^{(3)}}). \tag{A175}$$

If there is none or more than one such index $\tilde{a}$, set $\hat{a}(3) = 1$. Otherwise, the found index $\tilde{a}$ is the estimate $\hat{a}(3)$ of node 3. Then node 3 tries to find pair $(\tilde{d}, \tilde{q}_2)$ such that

$$\left(v_{22}^{n_3}(\hat{a}(3), r, \tilde{d}, \tilde{q}_2), u_2^{n_3}(\hat{a}(3), r), y_3^{n_3}\right) \in T_\epsilon^{n_3}(P_{V_{22}^{(3)} U_2^{(3)} Y_3^{(3)}}). \tag{A176}$$

If there is none or more than one such $(\tilde{d}, \tilde{q}_2)$, set $(\hat{d}(3), \hat{q}_2(3)) = (1, 1)$. Otherwise, the found pair $(\tilde{d}, \tilde{q}_2)$ is the estimate $(\hat{d}(3), \hat{q}_2(3))$ of node 3. Then node 3 tries to find an index $\tilde{b}$ such that

$$\left(u_1^{n_1}(\hat{a}(3), \tilde{b}), y_3^{n_1}\right) \in T_\epsilon^{n_1}(P_{U_1^{(1)} Y_3^{(1)}}). \tag{A177}$$

If there is none or more than one such index $\tilde{b}$, set $\hat{b}(3) = 1$. Otherwise, the found index $\tilde{b}$ is the output $\hat{b}(3)$ of node 3. Then node 3 tries to find an index $\tilde{e}$ such that

$$\left(\hat{y}_1^{n_1}(\hat{a}(3), \hat{b}(3), \hat{d}(3), \tilde{e}), u_1^{n_1}(\hat{a}(3), \hat{b}(3)), y_3^{n_1}\right) \in T_\epsilon^{n_1}(P_{\hat{Y}_2^{(1)} U_1^{(1)} Y_3^{(1)}}). \tag{A178}$$

If there is none or more than one such index $\tilde{e}$, set $\hat{e}(3) = 1$. Otherwise, the found index $\tilde{e}$ is the estimate $\hat{e}(3)$ of node 3. Finally node 3 tries to find an index $\tilde{c}$ such that

$$\left(x_1^{n_1}(\hat{a}(3), \hat{b}(3), \tilde{c}), u_1^{n_1}(\hat{a}(3), \hat{b}(3)), \hat{y}_2^{n_1}(\hat{a}(3), \hat{b}(3), \hat{d}(3), \hat{e}(3)), y_3^{n_1}\right) \in T_\epsilon^{n_1}(P_{X_1^{(1)} U_1^{(1)} \hat{Y}_2^{(1)} Y_3^{(1)}}). \tag{A179}$$

If there is none or more than one such index $\tilde{c}$, set $\hat{c}(3) = 1$. Otherwise, the found index $\tilde{c}$ is the estimate $\hat{c}(3)$ of node 3.

Rates An error analysis along the lines of A2.1 and A2.2 reveals that for large n, $\epsilon > 0$ but small, reliable communication requires

$$\begin{aligned} R_1 + R_2 &< \tau_1 I(U_1^{(1)}; Y_2^{(1)}) \\ R_4 + R_5 &< \tau_2 I(U_3^{(2)}; Y_2^{(2)}) \\ R_7 + R_8 &> \tau_1 I(\hat{Y}_2^{(1)}; Y_2^{(1)} | U_1^{(1)}) \\ R_9 + R_{10} &> \tau_2 I(\hat{Y}_2^{(2)}; Y_2^{(2)} | U_3^{(2)}) \\ R_{11} + R_{12} &> \tau_3 I(V_{21}^{(3)}; V_{22}^{(3)} | U_2^{(3)}) \end{aligned} \tag{A180}$$

at node 2,

$$\begin{aligned} R_4 &< \tau_3 I(U_2^{(3)}; Y_1^{(3)}) \\ R_9 + R_{11} &< \tau_3 I(V_{21}^{(3)}; Y_1^{(3)} | U_2^{(3)}) \\ R_5 &< \tau_2 I(U_3^{(2)}; Y_1^{(2)}) \\ R_{10} &< \tau_2 I(\hat{Y}_2^{(2)}; Y_1^{(2)} | U_3^{(2)}) \\ R_6 &< \tau_2 I(X_3^{(2)}; \hat{Y}_2^{(2)} Y_1^{(2)} | U_3^{(2)}) \end{aligned} \tag{A181}$$

at node 1,

$$
\begin{aligned}
R_1 &< \tau_3 I(U_2^{(3)}; Y_3^{(3)}) \\
R_7 + R_{12} &< \tau_3 I(V_{22}^{(3)}; Y_3^{(3)} | U_2^{(3)}) \\
R_2 &< \tau_1 I(U_1^{(1)}; Y_3^{(1)}) \\
R_8 &< \tau_1 I(\hat{Y}_2^{(1)}; Y_3^{(1)} | U_1^{(1)}) \\
R_3 &< \tau_1 I(X_1^{(1)}; \hat{Y}_2^{(1)} Y_3^{(1)} | U_1^{(1)})
\end{aligned} \tag{A182}
$$

at node 3. Therefore,

$$
\begin{aligned}
R_{13} &< \tau_1 I(U_1^{(1)}; Y_3^{(1)}) + \tau_1 I(X_1^{(1)}; \hat{Y}_2^{(1)} Y_3^{(1)} | U_1^{(1)}) + \tau_3 I(U_2^{(3)}; Y_3^{(3)}) \\
R_{13} &< \tau_1 I(U_1^{(1)}; Y_2^{(1)}) + \tau_1 I(X_1^{(1)}; \hat{Y}_2^{(1)} Y_3^{(1)} | U_1^{(1)}) \\
R_{31} &< \tau_2 I(U_3^{(2)}; Y_1^{(2)}) + \tau_2 I(X_3^{(2)}; \hat{Y}_2^{(2)} Y_1^{(2)} | U_3^{(2)}) + \tau_3 I(U_2^{(3)}; Y_1^{(3)}) \\
R_{31} &< \tau_2 I(U_3^{(2)}; Y_2^{(2)}) + \tau_2 I(X_3^{(2)}; \hat{Y}_2^{(2)} Y_1^{(2)} | U_3^{(2)})
\end{aligned} \tag{A183}
$$

subject to

$$
\begin{aligned}
\tau_1 I(\hat{Y}_2^{(1)}; Y_2^{(1)} | Y_3^{(1)} U_1^{(1)}) &< \tau_3 I(V_{22}^{(3)}; Y_3^{(3)} | U_2^{(3)}) - (1-\alpha)\tau_3 I(V_{21}^{(3)}; V_{22}^{(3)} | U_2^{(3)}) \\
\tau_2 I(\hat{Y}_2^{(2)}; Y_2^{(2)} | Y_1^{(2)} U_3^{(2)}) &< \tau_3 I(V_{21}^{(3)}; Y_1^{(3)} | U_2^{(3)}) - \alpha\tau_3 I(V_{21}^{(3)}; V_{22}^{(3)} | U_2^{(3)})
\end{aligned} \tag{A184}
$$

where

$$
R_{11} = \alpha\tau_3 I(V_{21}^{(3)}; V_{22}^{(3)} | U_2^{(3)}) \tag{A185}
$$

and the fact

$$
\begin{aligned}
&Y_3^{(1)} - (Y_2^{(1)}, U_1^{(1)}) - \hat{Y}_2^{(1)} \\
&Y_1^{(2)} - (Y_2^{(2)}, U_3^{(2)}) - \hat{Y}_2^{(2)}
\end{aligned} \tag{A186}
$$

was used. This establishes the proposition. ■

A5. Proofs: Achievable Rates 3P-MA Scheme

Comments and Assumptions Random encoding, jointly typical decoding and compression will be used to show which rates are achievable for the half-duplex two-way relay channel. For the following proofs it will be assumed that the transmission is performed with $n \geq 3$ channel uses and three phases $l = 1, 2, 3$. Phase 1 features $n_1 \geq 1$ transmission slots, phase 2 supports $n_2 \geq 1$ transmission slots and phase 3 supports $n_3 \geq 1$ transmission slots with $n_1 + n_2 + n_3 = n$. If n grows n_1, n_2 and n_3 are assumed to grow at the same rate. For large n, $\frac{n_l}{n} \to \tau_l > 0$. The message $w_{13} \in \{1, \ldots, 2^{nR_{13}}\}$ will be sent from node 1 to node 3 and the message $w_{31} \in \{1, \ldots, 2^{nR_{31}}\}$ will be sent from node 3 to node 1. For all proofs 2^{nR} denotes a positive integer.

A5.1 Decode-and-Forward

Code Generate 2^{nR_1} n_1-sequences $x_1^{n_1}(r)$, $r = 1, 2, \ldots, 2^{nR_1}$, choosing each element $x_{1,k}^{(1)}(r)$ independently according to $P_{X_1^{(1)}}(\cdot)$. Generate 2^{nR_1} n_2-sequences $x_2^{n_2}(r)$ choosing each element $x_{2,k}^{(2)}(r)$ independently according to $P_{X_2^{(2)}}(\cdot)$. For each $x_2^{n_2}(r)$ generate 2^{nR_2} n_2-sequences $x_1^{n_2}(r, s)$, $s = 1, 2, \ldots, 2^{nR_2}$, choosing each element $x_{1,k}^{(2)}(r, s)$ independently according to $P_{X_1^{(2)}|X_2^{(2)}}(\cdot)$. Generate 2^{nR_3} n_1-sequences $x_3^{n_1}(t)$, $t = 1, 2, \ldots, 2^{nR_3}$, choosing each element $x_{3,k}^{(1)}(t)$ independently according to $P_{X_3^{(1)}}(\cdot)$. Generate 2^{nR_3} n_3-sequences $x_2^{n_3}(t)$ choosing each element $x_{2,k}^{(3)}(t)$ independently according to $P_{X_2^{(3)}}(\cdot)$. Generate $2^{n(R_3+R_4)}$ n_3-sequences $x_3^{n_3}(t, o)$, $o = 1, 2, \ldots, 2^{nR_4}$ choosing each element $x_{3,k}^{(3)}(t, o)$ independently according to $P_{X_3^{(3)}|X_2^{(3)}}(\cdot)$.

Node 1 (Input) The message w_{13} is reindexed by (r, s). In the first phase node 1 transmits $x_1^{n_1}(r)$ within n_1 transmissions. In the second phase node 1 transmits $x_1^{n_2}(r, s)$ within n_2 transmissions.

Node 3 (Input) The message w_{31} is reindexed by (t, o). In the first phase node 3 transmits $x_3^{n_1}(t)$ within n_1 transmissions. In the third phase node 3 transmits $x_3^{n_3}(t, o)$ within n_3 transmissions.

Node 2 In the first phase $y_2^{n_1}$ is observed. After the first phase node 2 tries to find a pair $(\tilde{r}, \tilde{t})$ such that

$$\left(x_1^{n_1}(\tilde{r}), x_3^{n_1}(\tilde{t}), y_2^{n_1}\right) \in T_\epsilon^{n_1}(P_{X_1^{(1)}X_3^{(1)}Y_2^{(1)}}). \tag{A187}$$

If there is none or more than one such pair $(\tilde{r}, \tilde{t})$, set $(\hat{r}(2), \hat{t}(2)) = (1, 1)$. Otherwise, the found pair $(\tilde{r}, \tilde{t})$ is the estimate $(\hat{r}(2), \hat{t}(2))$ of node 2. In the second phase node 2 sends $x_2^{n_2}(\hat{r}(2))$. In the third phase node 2 sends $x_2^{n_3}(\hat{t}(2))$.

Node 1 (Output) In the third phase $y_1^{n_3}$ is observed. After the third phase node 1 tries to find an index $\tilde{t}$ such that

$$\left(x_2^{n_3}(\tilde{t}), y_1^{n_3}\right) \in T_\epsilon^{n_3}(P_{X_2^{(3)}Y_1^{(3)}}). \tag{A188}$$

If there is none or more than one such index $\tilde{t}$, set $\hat{t}(1) = 1$. Otherwise, the found index $\tilde{t}$ is the estimate $\hat{t}(1)$ of node 1. Then node 1 tries to find an index $\tilde{o}$ such that

$$\left(x_3^{n_3}(\hat{t}(1), \tilde{o}), y_1^{n_3}\right) \in T_\epsilon^{n_3}(P_{X_3^{(3)}Y_1^{(3)}}). \tag{A189}$$

If there is none or more than one such index $\tilde{o}$, set $\hat{o}(1) = 1$. Otherwise, the found index $\tilde{o}$ is the estimate $\hat{o}(1)$ of node 1.

Node 3 (Output) In the second phase $y_3^{n_2}$ is observed. After the second phase node 3 tries to find an index $\tilde{r}$ such that

$$(x_2^{n_2}(\tilde{r}), y_3^{n_2}) \in T_\epsilon^{n_2}(P_{X_2^{(2)}Y_3^{(2)}}). \tag{A190}$$

If there is none or more than one such index $\tilde{r}$, set $\hat{r}(3) = 1$. Otherwise, the found index $\tilde{r}$ is the estimate $\hat{r}(3)$ of node 3. Then node 3 tries to find an index $\tilde{s}$ such that

$$(x_1^{n_2}(\hat{r}(3), \tilde{s}), y_3^{n_2}) \in T_\epsilon^{n_2}(P_{X_1^{(2)}Y_3^{(2)}}). \tag{A191}$$

If there is none or more than one such index $\tilde{s}$, set $\hat{s}(3) = 1$. Otherwise, the found index $\tilde{s}$ is the estimate $\hat{s}(3)$ of node 3.

Rates An error analysis along the lines of A2.1 and A2.2 reveals that for large n, $\epsilon > 0$ but small, reliable communication requires

$$\begin{aligned} R_1 &< \tau_1 I(X_1^{(1)}; Y_2^{(1)}|X_3^{(1)}) \\ R_3 &< \tau_1 I(X_3^{(1)}; Y_2^{(1)}|X_1^{(1)}) \\ R_1 + R_3 &< \tau_1 I(X_1^{(1)} X_3^{(1)}; Y_2^{(1)}) \end{aligned} \tag{A192}$$

at node 2,

$$\begin{aligned} R_3 &< \tau_3 I(X_2^{(3)}; Y_1^{(3)}) \\ R_4 &< \tau_3 I(X_3^{(3)}; Y_1^{(3)}|X_2^{(3)}) \end{aligned} \tag{A193}$$

at node 1 and

$$\begin{aligned} R_1 &< \tau_2 I(X_2^{(2)}; Y_3^{(2)}) \\ R_2 &< \tau_2 I(X_1^{(2)}; Y_3^{(2)}|X_2^{(2)}) \end{aligned} \tag{A194}$$

at node 3. Consequently,

$$\begin{aligned} R_{13} &< \tau_1 I(X_1^{(1)}; Y_2^{(1)}|X_3^{(1)}) + \tau_2 I(X_1^{(2)}; Y_3^{(2)}|X_2^{(2)}) \\ R_{13} &< \tau_2 I(X_1^{(2)}, X_2^{(2)}; Y_3^{(2)}) \\ R_{31} &< \tau_1 I(X_3^{(1)}; Y_2^{(1)}|X_1^{(1)}) + \tau_3 I(X_3^{(3)}; Y_1^{(3)}|X_2^{(3)}) \\ R_{31} &< \tau_3 I(X_2^{(3)} X_3^{(3)}; Y_1^{(3)}) \end{aligned} \tag{A195}$$

subject to

$$R_{13} + R_{31} < \tau_1 I(X_1^{(1)}, X_3^{(1)}; Y_2^{(1)}) + \tau_2 I(X_1^{(2)}; Y_3^{(2)}|X_2^{(2)}) + \tau_3 I(X_3^{(3)}; Y_1^{(3)}|X_2^{(3)}). \tag{A196}$$

This establishes the proposition. ■

A5.2 Compress-and-Forward

Code Generate 2^{nR_1} n_1-sequences $x_1^{n_1}(a)$, $a = 1, 2, \dots, 2^{nR_1}$, by choosing each element $x_{1,k}^{(1)}(a)$ independently according to $P_{X_1^{(1)}}(\cdot)$. Generate 2^{nR_2} n_2-sequences $x_1^{n_2}(b)$, $b = 1, 2, \dots, 2^{nR_2}$, by choosing each element $x_{1,k}^{(2)}(b)$ independently according to $P_{X_1^{(2)}}(\cdot)$. Generate 2^{nR_3} n_1-sequences $x_3^{n_1}(c)$, $c = 1, 2, \dots, 2^{nR_3}$, by choosing each element $x_{3,k}^{(1)}(c)$ independently according to $P_{X_3^{(1)}}(\cdot)$. Generate 2^{nR_4} n_3-sequences $x_3^{n_3}(o)$, $o = 1, 2, \dots, 2^{nR_4}$, by choosing each element $x_{3,k}^{(3)}(o)$ independently according to $P_{X_3^{(3)}}(\cdot)$. Choose a "quantization channel" $P_{\hat{Y}_{21}^{(1)}|Y_2^{(1)}}(\cdot|\cdot)$ and calculate $P_{\hat{Y}_{21}^{(1)}}(\cdot)$ as the marginal distribution of $P_{\hat{Y}_{21}^{(1)}Y_2^{(1)}}(\cdot)$. Generate $2^{n(R_5+R_6)}$ n_1-sequences $\hat{y}_{21}^{n_1}(r,s)$, $r = 1, 2, \dots, 2^{nR_5}$, $s = 1, 2, \dots, 2^{nR_6}$, by choosing each element $\hat{y}_{21,k}^{(1)}(r,s)$ independently according to $P_{\hat{Y}_{21}^{(1)}}(\cdot)$. Choose a "quantization channel" $P_{\hat{Y}_{22}^{(1)}|Y_2^{(1)}}(\cdot|\cdot)$ and calculate $P_{\hat{Y}_{22}^{(1)}}(\cdot)$ as the marginal distribution of $P_{\hat{Y}_{22}^{(1)}Y_2^{(1)}}(\cdot)$. Generate $2^{n(R_7+R_8)}$ n_1-sequences $\hat{y}_{22}^{n_1}(t,z)$, $t = 1, 2, \dots, 2^{nR_7}$, $z = 1, 2, \dots, 2^{nR_8}$, by choosing each element $\hat{y}_{22,k}^{(1)}(t,z)$ independently according to $P_{\hat{Y}_{22}^{(1)}}(\cdot)$. Generate 2^{nR_5} n_2-sequences $x_2^{n_2}(r)$ by choosing each element $x_{2,k}^{(2)}(r)$ independently according to $P_{X_2^{(2)}}(\cdot)$. Generate 2^{nR_7} n_3-sequences $x_2^{n_3}(t)$ by choosing each element $x_{2,k}^{(3)}(t)$ independently according to $P_{X_2^{(3)}}(\cdot)$.

Node 1 (Input) The message w_{13} is reindexed by a and b. In the first phase node 1 transmits $x_1^{n_1}(a)$ within n_1 transmissions. In the second phase node 1 transmits $x_1^{n_2}(b)$ within n_2 transmissions.

Node 3 (Input) The message w_{31} is reindexed by c and o. In the first phase node 3 transmits $x_3^{n_1}(c)$ within n_1 transmissions. In the third phase node 3 transmits $x_3^{n_3}(o)$ within n_3 transmissions.

Node 2 In the first phase $y_2^{n_1}$ is observed. After the first phase node 2 tries to find a pair $(\tilde{r}, \tilde{s})$ such that

$$(\hat{y}_{21}^{n_1}(\tilde{r}, \tilde{s}), y_2^{n_1}) \in T_\epsilon^{n_1}(P_{\hat{Y}_{21}^{(1)}Y_2^{(1)}}). \tag{A197}$$

If there is none such pair $(\tilde{r}, \tilde{s})$ an error is declared. Otherwise, the found pair $(\tilde{r}, \tilde{s})$ is the estimate $(\hat{r}(2), \hat{s}(2))$ of node 2 where the pair with the smallest linear index $2^{nR_5}(s-1)+r$ is selected if more than one pair was found. Then node 2 tries to find a a pair $(\tilde{t}, \tilde{z})$ such that

$$\left(\hat{y}_{22}^{n_1}(\tilde{t}, \tilde{z}), y_2^{n_1}\right) \in T_\epsilon^{n_1}(P_{\hat{Y}_{22}^{(1)}Y_2^{(1)}}). \tag{A198}$$

If there is none such $(\tilde{t}, \tilde{z})$ an error is declared. Otherwise, $(\tilde{t}, \tilde{z})$ is the estimate $(\hat{t}(2), \hat{z}(2))$ of node 2 where the pair with the smallest linear index $2^{nR_7}(z-1)+t$ is selected if more than one pair was found. In the second phase node 2 sends $x_2^{n_2}(\hat{r}(2))$. In the third phase node 2 sends $x_2^{n_3}(\hat{t}(2))$.

Node 1 (Output) In the third phase $y_1^{n_3}$ is observed. After the third phase node 1 tries to find an index $\tilde{t}$ such that

$$\left(x_2^{n_3}(\tilde{t}), y_1^{n_3}\right) \in T_\epsilon^{n_3}(P_{X_2^{(3)}Y_1^{(3)}}). \tag{A199}$$

If there is none or more than one such index $\tilde{t}$, set $\hat{t}(1) = 1$. Otherwise, the found index $\tilde{t}$ is the estimate $\hat{t}(1)$ of node 1. Then node 1 tries to find an index $\tilde{o}$ such that

$$\left(x_3^{n_3}(\tilde{o}), x_2^{n_3}(\hat{t}(1)), y_1^{n_3}\right) \in T_\epsilon^{n_3}(P_{X_3^{(3)} X_2^{(3)} Y_1^{(3)}}). \tag{A200}$$

If there is none or more than one such index $\tilde{o}$, set $\hat{o}(1) = 1$. Otherwise, the found index $\tilde{o}$ is the estimate $\hat{o}(1)$ of node 1. Then node 1 tries to find an index $\tilde{z}$ such that

$$\left(\hat{y}_{22}^{n_1}(\hat{t}(1), \tilde{z}), x_1^{n_1}(a)\right) \in T_\epsilon^{n_1}(P_{\hat{Y}_{22}^{(1)} X_1^{(1)}}). \tag{A201}$$

If there is none or more than one such index $\tilde{z}$, set $\hat{z}(1) = 1$. Otherwise, the found index $\tilde{z}$ is the estimate $\hat{z}(1)$ of node 1. Finally node 1 tries to find an index $\tilde{c}$ such that

$$\left(x_3^{n_1}(\tilde{c}), x_1^{n_1}(a), \hat{y}_{22}^{n_1}(\hat{t}(1), \hat{z}(1))\right) \in T_\epsilon^{n_1}(P_{X_3^{(1)} X_1^{(1)} \hat{Y}_{22}^{(1)}}). \tag{A202}$$

If there is none or more than one such index $\tilde{c}$, set $\hat{c}(1) = 1$. Otherwise, the found index $\tilde{c}$ is the estimate $\hat{c}(1)$ of node 1.

Node 3 (Output) In the second phase $y_3^{n_2}$ is observed. After the second phase node 3 tries to find an index $\tilde{r}$ such that

$$(x_2^{n_2}(\tilde{r}), y_3^{n_2}) \in T_\epsilon^{n_2}(P_{X_2^{(2)} Y_3^{(2)}}). \tag{A203}$$

If there is none or more than one such index $\tilde{r}$, set $\hat{r}(3) = 1$. Otherwise, the found index $\tilde{r}$ is the estimate $\hat{r}(3)$ of node 3. Then node 3 tries to find an index $\tilde{b}$ such that

$$\left(x_1^{n_2}(\tilde{b}), x_2^{n_2}(\hat{r}(3)), y_3^{n_2}\right) \in T_\epsilon^{n_2}(P_{X_1^{(2)} X_2^{(2)} Y_3^{(2)}}). \tag{A204}$$

If there is none or more than one such index $\tilde{b}$, set $\hat{b}(3) = 1$. Otherwise, the found index $\tilde{b}$ is the output $\hat{b}(3)$ of node 3. Then node 3 tries to find an index $\tilde{s}$ such that

$$(\hat{y}_{21}^{n_1}(\hat{r}(3), \tilde{s}), x_3^{n_1}(c)) \in T_\epsilon^{n_1}(P_{\hat{Y}_{21}^{(1)} X_3^{(1)}}). \tag{A205}$$

If there is none or more than one such index $\tilde{s}$, set $\hat{s}(3) = 1$. Otherwise, the found index $\tilde{s}$ is the estimate $\hat{s}(3)$ of node 3. Finally node 3 tries to find an index $\tilde{a}$ such that

$$(x_1^{n_1}(\tilde{a}), x_3^{n_1}(c), \hat{y}_{21}^{n_1}(\hat{r}(3), \hat{s}(3))) \in T_\epsilon^{n_1}(P_{X_1^{(1)} X_3^{(1)} \hat{Y}_{21}^{(1)}}). \tag{A206}$$

If there is none or more than one such index $\tilde{a}$, set $\hat{a}(1) = 1$. Otherwise, the found index $\tilde{a}$ is the output $\hat{a}(1)$ of node 1.

Rates An error analysis along the lines of A2.1 and A2.2 reveals that for large n, $\epsilon > 0$ but small, reliable communication requires

$$\begin{aligned} R_5 + R_6 &> \tau_1 I(\hat{Y}_{21}^{(1)}; Y_2^{(1)}) \\ R_7 + R_8 &> \tau_1 I(\hat{Y}_{22}^{(1)}; Y_2^{(1)}) \end{aligned} \tag{A207}$$

at node 2,

$$\begin{aligned} R_7 &< \tau_3 I(X_2^{(3)}; Y_1^{(3)}) \\ R_4 &< \tau_3 I(X_3^{(3)}; Y_1^{(3)}|X_2^{(3)}) \\ R_8 &< \tau_1 I(\hat{Y}_{22}^{(1)}; X_1^{(1)}) \\ R_3 &< \tau_1 I(X_3^{(1)}; \hat{Y}_{22}^{(1)}|X_1^{(1)}) \end{aligned} \tag{A208}$$

at node 1 and

$$\begin{aligned} R_5 &< \tau_2 I(X_2^{(2)}; Y_3^{(2)}) \\ R_2 &< \tau_2 I(X_1^{(2)}; Y_3^{(2)}|X_2^{(2)}) \\ R_6 &< \tau_1 I(\hat{Y}_{21}^{(1)}; X_3^{(1)}) \\ R_1 &< \tau_1 I(X_1^{(1)}; \hat{Y}_{21}^{(1)}|X_3^{(1)}) \end{aligned} \tag{A209}$$

at node 3. Consequently,

$$\begin{aligned} R_{13} = R_1 + R_2 &< \tau_1 I(X_1^{(1)}; \hat{Y}_{21}^{(1)}|X_3^{(1)}) + \tau_2 I(X_1^{(2)}; Y_3^{(2)}|X_2^{(2)}) \\ R_{31} = R_3 + R_4 &< \tau_1 I(X_3^{(1)}; \hat{Y}_{22}^{(1)}|X_1^{(1)}) + \tau_3 I(X_3^{(3)}; Y_1^{(3)}|X_2^{(3)}) \end{aligned} \tag{A210}$$

subject to

$$\begin{aligned} \tau_1 I(\hat{Y}_{21}^{(1)}; Y_2^{(1)}) &< \tau_1 I(\hat{Y}_{21}^{(1)}; X_3^{(1)}) + \tau_2 I(X_2^{(2)}; Y_3^{(2)}) \\ \tau_1 I(\hat{Y}_{22}^{(1)}; Y_2^{(1)}) &< \tau_1 I(\hat{Y}_{22}^{(1)}; X_1^{(1)}) + \tau_3 I(X_2^{(3)}; Y_1^{(3)}). \end{aligned} \tag{A211}$$

By the use of the Markov chains

$$\begin{aligned} X_3^{(1)} - Y_2^{(1)} - \hat{Y}_{21}^{(1)} \\ X_1^{(1)} - Y_2^{(1)} - \hat{Y}_{22}^{(1)} \end{aligned} \tag{A212}$$

implying

$$\begin{aligned} I(\hat{Y}_{21}^{(1)}; X_3^{(1)}|Y_2^{(1)}) = 0 \\ I(\hat{Y}_{22}^{(1)}; X_1^{(1)}|Y_2^{(1)}) = 0 \end{aligned} \tag{A213}$$

the constraints can be reformulated to

$$\begin{aligned} \tau_1 I(\hat{Y}_{21}^{(1)}; Y_2^{(1)}|X_3^{(1)}) &< \tau_2 I(X_2^{(2)}; Y_3^{(2)}) \\ \tau_1 I(\hat{Y}_{22}^{(1)}; Y_2^{(1)}|X_1^{(1)}) &< \tau_3 I(X_2^{(3)}; Y_1^{(3)}). \end{aligned} \tag{A214}$$

This establishes the proposition. ■

A5.3 Partial-Decode-Compress-and-Forward

Code Generate 2^{nR_1} n_1-sequences $u_1^{n_1}(a)$, $a = 1, 2, \ldots, 2^{nR_1}$, by choosing each element $u_{1,k}^{(1)}(a)$ independently according to $P_{U_1^{(1)}}(\cdot)$. For each $u_1^{n_1}(a)$ generate 2^{nR_2} n_1-sequences $x_1^{n_1}(a,b)$, $b = 1, 2, \ldots, 2^{nR_2}$, by choosing each element $x_{1,k}^{(1)}(a,b)$ independently according to $P_{X_1^{(1)}|U_1^{(1)}}(\cdot|u_{1,k}^{(1)}(a))$. Generate 2^{nR_4} n_1-sequences $u_3^{n_1}(r)$, $r = 1, 2, \ldots, 2^{nR_4}$, by choosing each element $u_{3,k}^{(1)}(r)$ independently according to $P_{U_3^{(1)}}(\cdot)$. For each $u_3^{n_1}(r)$ generate 2^{nR_5} n_1-sequences $x_3^{n_1}(r,s)$, $s = 1, 2, \ldots, 2^{nR_5}$, by choosing each element $x_{3,k}^{(1)}(r,s)$ independently according to $P_{X_3^{(1)}|U_3^{(1)}}(\cdot|u_{3,k}^{(1)}(r))$. Choose a "quantization channel" $P_{\hat{Y}_{21}^{(1)}|Y_2^{(1)}U_1^{(1)}U_3^{(1)}}(\cdot|\cdot)$ and calculate $P_{\hat{Y}_{21}^{(1)}|U_1^{(1)}U_3^{(1)}}(\cdot|\cdot)$ as the marginal distribution of $P_{\hat{Y}_{21}^{(1)}Y_2^{(1)}|U_1^{(1)}U_3^{(1)}}(\cdot|\cdot)$. For each pair $u_1^{n_1}(a)$, $u_3^{n_1}(r)$, generate $2^{n(R_7+R_8)}$ n_1-sequences $\hat{y}_{21}^{n_1}(a,r,o,e)$, $o = 1, 2, \ldots, 2^{nR_7}$, $e = 1, 2, \ldots, 2^{nR_8}$, by choosing each element $\hat{y}_{21,k}^{(1)}(a,r,o,e)$ independently according to $P_{\hat{Y}_{21}^{(1)}|U_1^{(1)}U_3^{(1)}}(\cdot)$. Choose a "quantization channel" $P_{\hat{Y}_{22}^{(1)}|Y_2^{(1)}U_1^{(1)}U_3^{(1)}}(\cdot|\cdot)$ and calculate $P_{\hat{Y}_{22}^{(1)}|U_1^{(1)}U_3^{(1)}}(\cdot|\cdot)$ as the marginal distribution of $P_{\hat{Y}_{22}^{(1)}Y_2^{(1)}|U_1^{(1)}U_3^{(1)}}(\cdot|\cdot)$. For each pair $u_1^{n_1}(a)$, $u_3^{n_1}(r)$, generate $2^{n(R_9+R_{10})}$ n_1-sequences $\hat{y}_{22}^{n_1}(a,r,q,z)$, $q = 1, 2, \ldots, 2^{nR_9}$, $z = 1, 2, \ldots, 2^{nR_{10}}$, by choosing each element $\hat{y}_{22,k}^{(1)}(a,r,q,z)$ independently according to $P_{\hat{Y}_{22}^{(1)}|U_1^{(1)}U_3^{(1)}}(\cdot)$. Generate 2^{nR_1} n_2-sequences $u_2^{n_2}(a)$ by choosing each element $u_{2,k}^{(2)}(a)$ independently according to $P_{U_2^{(2)}}(\cdot)$. For each $u_2^{n_2}(a)$ generate 2^{nR_7} n_2-sequences $x_2^{n_2}(a,o)$ by choosing each element $x_{2,k}^{(2)}(a,o)$ independently according to $P_{X_2^{(2)}|U_2^{(2)}}(\cdot|u_{2,k}^{(2)}(a))$. For each $u_2^{n_2}(a)$ generate 2^{nR_3} n_2-sequences $x_1^{n_2}(a,c)$ by choosing each element $x_{1,k}^{(2)}(a,c)$ independently according to $P_{X_1^{(2)}|U_2^{(2)}}(\cdot|u_{2,k}^{(2)}(a))$. Generate 2^{nR_4} n_3-sequences $u_2^{n_3}(r)$ by choosing each element $u_{2,k}^{(3)}(r)$ independently according to $P_{U_2^{(3)}}(\cdot)$. For each $u_2^{n_3}(r)$ generate 2^{nR_9} n_3-sequences $x_2^{n_3}(r,q)$ by choosing each element $x_{2,k}^{(3)}(r,q)$ independently according to $P_{X_2^{(3)}|U_2^{(3)}}(\cdot|u_{2,k}^{(3)}(r))$. For each $u_2^{n_3}(r)$ generate 2^{nR_6} n_3-sequences $x_3^{n_3}(r,t)$ by choosing each element $x_{3,k}^{(3)}(r,t)$ independently according to $P_{X_3^{(3)}|U_2^{(3)}}(\cdot|u_{2,k}^{(3)}(r))$.

Node 1 (Input) The message w_{13} is reindexed by (a,b,c). In the first phase node 1 transmits $x_1^{n_1}(a,b)$ within n_1 transmissions. In the second phase node 1 transmits $x_2^{n_2}(a,c)$ within n_2 transmissions.

Node 3 (Input) The message w_{31} is reindexed by (r,s,t). In the first phase node 3 transmits $x_3^{n_1}(r,s)$ within n_1 transmissions. In the third phase node 3 transmits $x_3^{n_3}(r,t)$ within n_3 transmissions.

Node 2 In the first phase $y_2^{n_1}$ is observed. After the first phase node 2 tries to find a pair $(\tilde{a},\tilde{r})$ such that

$$(u_1^{n_1}(\tilde{a}), u_3^{n_1}(\tilde{r}), y_2^{n_1}) \in T_\epsilon^{n_1}(P_{U_1^{(1)}U_3^{(1)}Y_2^{(1)}}). \tag{A215}$$

If there is none or more than one such pair $(\tilde{a}, \tilde{r})$, set $(\hat{a}(2), \hat{r}(2)) = (1, 1)$. Otherwise, the found pair $(\tilde{a}, \tilde{r})$ is the estimate $(\hat{a}(2), \hat{r}(2))$ of node 2. Then node 2 tries to find a pair $(\tilde{o}, \tilde{e})$ such that

$$(\hat{y}_{21}^{n_1}(\hat{a}(2), \hat{r}(2), \tilde{o}, \tilde{e}), u_1^{n_1}(\hat{a}(2)), u_3^{n_1}(\hat{r}(2)), y_2^{n_1}) \in T_\epsilon^{n_1}(P_{\hat{Y}_{21}^{(1)} U_1^{(1)} U_3^{(1)} Y_2^{(1)}}). \tag{A216}$$

If there is none such pair $(\tilde{o}, \tilde{e})$ an error is declared. Otherwise, the found pair $(\tilde{o}, \tilde{e})$ is the estimate $(\hat{o}(2), \hat{e}(2))$ of node 2 where the pair with the smallest linear index $2^{nR_7}(e-1)+o$ is selected if more than one pair was found. Then node 2 tries to find a a pair $(\tilde{q}, \tilde{z})$ such that

$$(\hat{y}_{22}^{n_1}(\hat{a}(2), \hat{r}(2), \tilde{q}, \tilde{z}), u_1^{n_1}(\hat{a}(2)), u_3^{n_1}(\hat{r}(2)), y_2^{n_1}) \in T_\epsilon^{n_1}(P_{\hat{Y}_{22}^{(1)} U_1^{(1)} U_3^{(1)} Y_2^{(1)}}). \tag{A217}$$

If there is none such pair $(\tilde{q}, \tilde{z})$ an error is declared. Otherwise, the found pair $(\tilde{q}, \tilde{z})$ is the estimate $(\hat{q}(2), \hat{z}(2))$ of node 2 where the pair with the smallest linear index $2^{nR_9}(z-1)+q$ is selected if more than one pair was found. In the second phase node 2 sends $x_2^{n_2}(\hat{a}(2), \hat{o}(2))$. In the third phase node 2 sends $x_2^{n_3}(\hat{r}(2), \hat{q}(2))$.

Node 1 (Output) In the third phase $y_1^{n_3}$ is observed. After the third phase node 1 tries to find an index $\tilde{r}$ such that

$$(u_2^{n_3}(\tilde{r}), y_1^{n_3}) \in T_\epsilon^{n_3}(P_{U_2^{(3)} Y_1^{(3)}}). \tag{A218}$$

If there is none or more than one such index $\tilde{r}$, set $\hat{r}(1) = 1$. Otherwise, the found index $\tilde{r}$ is the estimate $\hat{r}(1)$ of node 1. Then node 1 tries to find an index $\tilde{q}$ such that

$$(x_2^{n_3}(\hat{r}(1), \tilde{q}), u_2^{n_3}(\hat{r}(1)), y_1^{n_3}) \in T_\epsilon^{n_3}(P_{X_2^{(3)} U_2^{(3)} Y_1^{(3)}}). \tag{A219}$$

If there is none or more than one such index $\tilde{q}$, set $\hat{q}(1) = 1$. Otherwise, the found index $\tilde{q}$ is the estimate $\hat{q}(1)$ of node 1. Then node 1 tries to find an index $\tilde{t}$ such that

$$\big(x_3^{n_3}(\hat{r}(1), \tilde{t}), x_2^{n_3}(\hat{r}(1), \hat{q}(1)), u_2^{n_3}(\hat{r}(1)), y_1^{n_3}\big) \in T_\epsilon^{n_3}(P_{X_3^{(3)} X_2^{(3)} U_2^{(3)} Y_1^{(3)}}). \tag{A220}$$

If there is none or more than one such $\tilde{t}$, set $\hat{t}(1) = 1$. Otherwise, the found index $\tilde{t}$ is the output $\hat{t}(1)$ of node 1. Then node 1 tries to find an index $\tilde{z}$ such that

$$(\hat{y}_{22}^{n_1}(a, \hat{r}(1), \hat{q}(1), \tilde{z}), u_1^{n_1}(a), u_3^{n_1}(\hat{r}(1)), x_1^{n_1}(a, b)) \in T_\epsilon^{n_1}(P_{\hat{Y}_{22}^{(1)} U_1^{(1)} U_3^{(1)} X_1^{(1)}}). \tag{A221}$$

If there is none or more than one such index $\tilde{z}$, set $\hat{z}(1)$. Otherwise, the found index $\tilde{z}$ is the estimate $\hat{z}(1)$ of node 1. Finally node 1 tries to find an index $\tilde{s}$ such that

$$\begin{aligned}(x_3^{n_1}(\hat{r}(1), \tilde{s}), x_1^{n_1}(a, b), u_1^{n_1}(a), u_3^{n_1}(\hat{r}(1)), \hat{y}_{22}^{n_1}(a, \hat{r}(1), \hat{q}(1), \hat{z}(1)))&\\ \in T_\epsilon^{n_1}(P_{X_3^{(1)} X_1^{(1)} U_1^{(1)} U_3^{(1)} \hat{Y}_{22}^{(1)}}).&\end{aligned} \tag{A222}$$

If there is none or more than one such index $\tilde{s}$, set $\hat{s}(1) = 1$. Otherwise, the found index $\tilde{s}$ is the estimate $\hat{s}(1)$ of node 1.

Node 3 (Output) In the second phase $y_3^{n_2}$ is observed. After the second phase node 3 tries to find an index $\tilde{a}$ such that

$$(u_2^{n_2}(\tilde{a}), y_3^{n_2}) \in T_\epsilon^{n_2}(P_{U_2^{(2)}Y_3^{(2)}}). \tag{A223}$$

If there is none or more than one such index $\tilde{a}$, set $\hat{a}(3) = 1$. Otherwise, the found index $\tilde{a}$ is the estimate $\hat{a}(3)$ of node 3. Then node 3 tries to find an index $\tilde{o}$ such that

$$(x_2^{n_2}(\hat{a}(3), \tilde{o}), u_2^{n_2}(\hat{a}(3)), y_3^{n_2}) \in T_\epsilon^{n_2}(P_{X_2^{(2)}U_2^{(2)}Y_3^{(2)}}). \tag{A224}$$

If there is none or more than one such index $\tilde{o}$, set $\hat{o}(3) = 1$. Otherwise, the found index $\tilde{o}$ is the estimate $\hat{o}(3)$ of node 3. Then node 3 tries to find an index $\tilde{c}$ such that

$$(x_1^{n_2}(\hat{a}(3), \tilde{c}), x_2^{n_2}(\hat{a}(3), \hat{o}(3)), u_2^{n_2}(\hat{a}(3)), y_3^{n_2}) \in T_\epsilon^{n_2}(P_{X_1^{(2)}X_2^{(2)}U_2^{(2)}Y_3^{(2)}}). \tag{A225}$$

If there is none or more than one such index $\tilde{c}$, set $\hat{c}(3) = 1$. Otherwise, the found index $\tilde{c}$ is the estimate $\hat{c}(3)$ of node 3. Then node 3 tries to find an index $\tilde{e}$ such that

$$(\hat{y}_{21}^{n_1}(\hat{a}(3), r, \hat{o}(3), \tilde{e}), x_3^{n_1}(r, s), u_1^{n_1}(\hat{a}(3)), u_3^{n_1}(r)) \in T_\epsilon^{n_1}(P_{\hat{Y}_{22}^{(1)}X_3^{(1)}U_1^{(1)}U_3^{(1)}}). \tag{A226}$$

If there is none or more than one such index $\tilde{e}$, set $\hat{e}(3) = 1$. Otherwise, the found index $\tilde{e}$ is the estimate $\hat{e}(3)$ of node 3. Finally node 3 tries to find an index $\tilde{b}$ such that

$$\begin{aligned}\Big(x_1^{n_1}(\hat{a}(3), \tilde{b}), x_3^{n_1}(r, s), u_1^{n_1}(\hat{a}(3)), u_3^{n_1}(r), \hat{y}_{21}^{n_1}(\hat{a}(3), r, \hat{o}(3), \hat{e}(3))\Big)& \\ \in T_\epsilon^{n_1}(P_{X_1^{(1)}X_3^{(1)}U_1^{(1)}U_3^{(1)}\hat{Y}_{21}^{(1)}}).&\end{aligned} \tag{A227}$$

If there is none or more than one such index $\tilde{b}$, set $\hat{b}(3) = 1$. Otherwise, the found index $\tilde{b}$ is the estimate $\hat{b}(3)$ of node 3.

Rates An error analysis along the lines of A2.1 and A2.2 reveals that for large n, $\epsilon > 0$ but small, reliable communication requires

$$\begin{aligned}
R_1 &< \tau_1 I(U_1^{(1)}; Y_2^{(1)}|U_3^{(1)}) \\
R_4 &< \tau_1 I(U_3^{(1)}; Y_2^{(1)}|U_1^{(1)}) \\
R_1 + R_4 &< \tau_1 I(U_1^{(1)}, U_3^{(1)}; Y_2^{(1)}) \\
R_7 + R_8 &> \tau_1 I(\hat{Y}_{21}^{(1)}; Y_2^{(1)}|U_1^{(1)}U_3^{(1)}) \\
R_9 + R_{10} &> \tau_1 I(\hat{Y}_{22}^{(1)}; Y_2^{(1)}|U_1^{(1)}U_3^{(1)})
\end{aligned} \tag{A228}$$

at node 2,

$$\begin{aligned}
R_4 &< \tau_3 I(U_2^{(3)}; Y_1^{(3)}) \\
R_9 &< \tau_3 I(X_2^{(3)}; Y_1^{(3)}|U_2^{(3)}) \\
R_6 &< \tau_3 I(X_3^{(3)}; Y_1^{(3)}|X_2^{(3)}U_2^{(3)}) \\
R_{10} &< \tau_1 I(\hat{Y}_{22}^{(1)}; X_1^{(1)}|U_1^{(1)}U_3^{(1)}) \\
R_5 &< \tau_1 I(X_3^{(1)}; \hat{Y}_{22}^{(1)}|U_1^{(1)}U_3^{(1)}X_1^{(1)})
\end{aligned} \tag{A229}$$

at node 1 and

$$\begin{aligned}
R_1 &< \tau_2 I(U_2^{(2)}; Y_3^{(2)}) \\
R_7 &< \tau_2 I(X_2^{(2)}; Y_3^{(2)} | U_2^{(2)}) \\
R_3 &< \tau_2 I(X_1^{(2)}; Y_3^{(2)} | X_2^{(2)} U_2^{(2)}) \\
R_8 &< \tau_1 I(\hat{Y}_{21}^{(1)}; X_3^{(1)} | U_1^{(1)} U_3^{(1)}) \\
R_2 &< \tau_1 I(X_1^{(1)}; \hat{Y}_{21}^{(1)} | U_1^{(1)} U_3^{(1)} X_3^{(1)}).
\end{aligned} \tag{A230}$$

at node 3. Consequently,

$$\begin{aligned}
R_{13} &< \tau_1 I(U_1^{(1)}; Y_2^{(1)} | U_3^{(1)}) + \tau_1 I(X_1^{(1)}; \hat{Y}_{21}^{(1)} | U_1^{(1)} U_3^{(1)} X_3^{(1)}) + \tau_2 I(X_1^{(2)}; Y_3^{(2)} | X_2^{(2)} U_2^{(2)}) \\
R_{13} &< \tau_1 I(X_1^{(1)}; \hat{Y}_{21}^{(1)} | U_1^{(1)} U_3^{(1)} X_3^{(1)}) + \tau_2 I(U_2^{(2)}; Y_3^{(2)}) + \tau_2 I(X_1^{(2)}; Y_3^{(2)} | X_2^{(2)} U_2^{(2)}) \\
R_{31} &< \tau_1 I(U_3^{(1)}; Y_2^{(1)} | U_1^{(1)}) + \tau_1 I(X_3^{(1)}; \hat{Y}_{22}^{(1)} | U_1^{(1)} U_3^{(1)} X_1^{(1)}) + \tau_3 I(X_3^{(3)}; Y_1^{(3)} | X_2^{(3)} U_2^{(3)}) \\
R_{31} &< \tau_1 I(X_3^{(1)}; \hat{Y}_{22}^{(1)} | U_1^{(1)} U_3^{(1)} X_1^{(1)}) + \tau_3 I(U_2^{(3)}; Y_1^{(3)}) + \tau_3 I(X_3^{(3)}; Y_1^{(3)} | X_2^{(3)} U_2^{(3)}) \\
R_{13} + R_{31} &< \tau_1 I(U_1^{(1)} U_3^{(1)}; Y_2^{(1)}) + \tau_1 I(X_1^{(1)}; \hat{Y}_{21}^{(1)} | U_1^{(1)} U_3^{(1)} X_3^{(1)}) + \tau_1 I(X_3^{(1)}; \hat{Y}_{22}^{(1)} | U_1^{(1)} U_3^{(1)} X_1^{(1)}) + \\
&\quad \tau_2 I(X_1^{(2)}; Y_3^{(2)} | X_2^{(2)} U_2^{(2)}) + \tau_3 I(X_3^{(3)}; Y_1^{(3)} | X_2^{(3)} U_2^{(3)})
\end{aligned} \tag{A231}$$

subject to

$$\begin{aligned}
\tau_1 I(\hat{Y}_{21}^{(1)}; Y_2^{(1)} | U_1^{(1)}, U_3^{(1)}) &< \tau_1 I(\hat{Y}_{21}^{(1)}; X_3^{(1)} | U_1^{(1)} U_3^{(1)}) + \tau_2 I(X_2^{(2)}; Y_3^{(2)} | U_2^{(2)}) \\
\tau_1 I(\hat{Y}_{22}^{(1)}; Y_2^{(1)} | U_1^{(1)}, U_3^{(1)}) &< \tau_1 I(\hat{Y}_{22}^{(1)}; X_1^{(1)} | U_1^{(1)} U_3^{(1)}) + \tau_3 I(X_2^{(3)}; Y_1^{(3)} | U_2^{(3)}).
\end{aligned} \tag{A232}$$

By the use of the Markov chains

$$\begin{aligned}
X_1^{(1)} - (Y_2^{(1)}, U_1^{(1)}, U_3^{(1)}) - \hat{Y}_{22}^{(1)} \\
X_3^{(1)} - (Y_2^{(1)}, U_1^{(1)}, U_3^{(1)}) - \hat{Y}_{21}^{(1)}
\end{aligned} \tag{A233}$$

implying

$$\begin{aligned}
I(\hat{Y}_{21}^{(1)}; X_3^{(1)} | Y_2^{(1)} U_1^{(1)} U_3^{(1)}) = 0 \\
I(\hat{Y}_{22}^{(1)}; X_1^{(1)} | Y_2^{(1)} U_1^{(1)} U_3^{(1)}) = 0
\end{aligned} \tag{A234}$$

the constraints can be reformulated to

$$\begin{aligned}
\tau_1 I(\hat{Y}_{21}^{(1)}; Y_2^{(1)} | U_1^{(1)} U_3^{(1)} X_3^{(1)}) &< \tau_2 I(X_2^{(2)}; Y_3^{(2)} | U_2^{(2)}) \\
\tau_1 I(\hat{Y}_{22}^{(1)}; Y_2^{(1)} | U_1^{(1)} U_3^{(1)} X_1^{(1)}) &< \tau_3 I(X_2^{(3)}; Y_1^{(3)} | U_2^{(3)}).
\end{aligned} \tag{A235}$$

This establishes the proposition. ■

A6. Proofs: Achievable Rates 6P Scheme

Comments and Assumptions Random encoding and jointly typical decoding will be used to show which rates are achievable for the half-duplex two-way relay channel. For the following proofs it will be assumed that the transmission is performed with $n \geq 6$ channel uses and six phases $l = 1, \ldots, L = 6$. Phase l features $n_l \geq 1$ transmission slots with $\sum_{l=1}^{L} n_l = n$. If n grows each n_l is assumed to grow at the same rate. For large n, $\frac{n_l}{n} \to \tau_l > 0$. The message $w_{13} \in \{1, \ldots, 2^{nR_{13}}\}$ will be sent from node 1 to node 3 and the message $w_{31} \in \{1, \ldots, 2^{nR_{31}}\}$ will be sent from node 3 to node 1. For all proofs 2^{nR} denotes a positive integer.

A6.1 Decode-and-Forward

Code Generate $2^{n(R_1+R_2+R_3)}$ n_1-sequences $x_1^{n_1}(a,b,c)$, $a = 1,2,\ldots,2^{nR_1}$, $b = 1,2,\ldots,2^{nR_2}$, $c = 1,2,\ldots,2^{nR_3}$, by choosing each element $x_{1,k}^{(1)}(a,b,c)$ independently according to $P_{X_1^{(1)}}(\cdot)$. Generate $2^{n(R_7+R_8+R_9)}$ n_2-sequences $x_3^{n_2}(r,s,t)$, $r = 1,2,\ldots,2^{nR_7}$, $s = 1,2,\ldots,2^{nR_8}$, $t = 1,2,\ldots,2^{nR_9}$, by choosing each element $x_{3,k}^{(2)}(r,s,t)$ independently according to $P_{X_3^{(2)}}(\cdot)$. Generate $2^{n(R_4+R_5)}$ n_3-sequences $x_1^{n_3}(d,e)$, $d = 1,2,\ldots,2^{nR_4}$, $e = 1,2,\ldots,2^{nR_5}$, by choosing each element $x_{1,k}^{(3)}(d,e)$ independently according to $P_{X_1^{(3)}}(\cdot)$. Generate $2^{n(R_{10}+R_{11})}$ n_3-sequences $x_3^{n_3}(o,q)$, $o = 1,2,\ldots,2^{nR_{10}}$, $q = 1,2,\ldots,2^{nR_{11}}$, by choosing each element $x_{3,k}^{(3)}(o,q)$ independently according to $P_{X_3^{(3)}}(\cdot)$. Generate $2^{n(R_1+R_4+R_7+R_{10})}$ n_4-sequences $x_2^{n_4}(a,d,r,o)$, by choosing each element $x_{2,k}^{(4)}(a,d,r,o)$ independently according to $P_{X_2^{(4)}}(\cdot)$. Generate $2^{n(R_8+R_{11})}$ n_5-sequences $x_2^{n_5}(s,q)$, by choosing each element $x_{2,k}^{(5)}(s,q)$ independently according to $P_{X_2^{(5)}}(\cdot)$. For each $x_2^{n_5}(s,q)$ generate $2^{nR_{12}}$ n_5-sequences $x_3^{n_5}(s,q,z)$, $z = 1,2,\ldots,2^{nR_{12}}$, choosing each element $x_{3,k}^{(5)}(s,q,z)$ independently according to $P_{X_3^{(5)}|X_2^{(5)}}(\cdot|x_{2,k}^{(5)}(s,q))$. Generate $2^{n(R_2+R_5)}$ n_6-sequences $x_2^{n_6}(b,e)$, by choosing each element $x_{2,k}^{(6)}(b,e)$ independently according to $P_{X_2^{(6)}}(\cdot)$. For each $x_2^{n_6}(b,e)$ generate 2^{nR_6} n_6-sequences $x_1^{n_6}(b,e,f)$, $f = 1,2,\ldots,2^{nR_6}$, choosing each element $x_{1,k}^{(6)}(b,e,f)$ independently according to $P_{X_1^{(6)}|X_2^{(6)}}(\cdot|x_{2,k}^{(6)}(b,e))$.

Node 1 (Input) The message w_{13} is reindexed by (a,b,c,d,e,f). In the first phase node 1 transmits $x_1^{n_1}(a,b,c)$ within n_1 transmissions. In the third phase node 1 transmits $x_1^{n_3}(d,e)$ within n_3 transmissions. In the sixth phase node 1 transmits $x_1^{n_6}(b,e,f)$ within n_6 transmissions.

Node 3 (Input) The message w_{31} is reindexed by (r,s,t,o,q,z). In the second phase node 3 transmits $x_3^{n_2}(r,s,t)$ within n_2 transmissions. In the third phase node 3 transmits $x_3^{n_3}(o,q)$ within n_3 transmissions. In the fifth phase node 3 transmits $x_3^{n_5}(s,q,z)$ within n_5 transmissions.

Node 2 In the first phase $y_2^{n_1}$ is observed. In the second phase $y_2^{n_2}$ is observed. In the third phase $y_2^{n_3}$ is observed. After the first phase node 2 tries to find a triple $(\tilde{a},\tilde{b},\tilde{c})$ such that

$$\left(x_1^{n_1}(\tilde{a},\tilde{b},\tilde{c}), y_2^{n_1}\right) \in T_\epsilon^{n_1}(P_{X_1^{(1)}Y_2^{(1)}}). \tag{A236}$$

If there is none or more than one such triple $(\tilde{a},\tilde{b},\tilde{c})$, set $(\hat{a}(2),\hat{b}(2),\hat{c}(2)) = (1,1,1)$. Otherwise, the found triple $(\tilde{a},\tilde{b},\tilde{c})$ is the estimate $(\hat{a}(2),\hat{b}(2),\hat{c}(2))$ of node 2. After the second phase node 2 tries to find a triple $(\tilde{r}, \tilde{s}, \tilde{t})$ such that

$$\left(x_3^{n_2}(\tilde{r}, \tilde{s}, \tilde{t}), y_2^{n_2}\right) \in T_\epsilon^{n_2}(P_{X_3^{(2)} Y_2^{(2)}}). \tag{A237}$$

If there is none or more than one such triple $(\tilde{r}, \tilde{s}, \tilde{t})$, set $(\hat{r}(2), \hat{s}(2), \hat{t}(2)) = (1, 1, 1)$. Otherwise, the found triple $(\tilde{r}, \tilde{s}, \tilde{t})$ is the estimate $(\hat{r}(2), \hat{s}(2), \hat{t}(2))$ of node 2. After the third phase node 2 tries to find a quadruple $(\tilde{d}, \tilde{e}, \tilde{o}, \tilde{q})$ such that

$$\left(x_1^{n_3}(\tilde{d}, \tilde{e}), x_3^{n_3}(\tilde{o}, \tilde{q}), y_2^{n_3}\right) \in T_\epsilon^{n_3}(P_{X_1^{(3)} X_3^{(3)} Y_2^{(3)}}). \tag{A238}$$

If there is none or more than one such quadruple $(\tilde{d}, \tilde{e}, \tilde{o}, \tilde{q})$, set $(\hat{d}(2), \hat{e}(2), \hat{o}(2), \hat{q}(2)) = (1, 1, 1, 1)$. Otherwise, the found quadruple $(\tilde{d}, \tilde{e}, \tilde{o}, \tilde{q})$ is the estimate $(\hat{d}(2), \hat{e}(2), \hat{o}(2), \hat{q}(2))$ of node 2.

In the fourth phase node 2 sends $x_2^{n_4}(\hat{a}(2), \hat{d}(2), \hat{r}(2), \hat{o}(2))$. In the fifth phase node 2 sends $x_2^{n_5}(\hat{s}(2), \hat{q}(2))$. In the sixth phase node 2 sends $x_2^{n_6}(\hat{b}(2), \hat{e}(2))$.

Node 1 (Output) In the second phase $y_1^{n_2}$ is observed. In the fourth phase $y_1^{n_4}$ is observed. In the fifth phase $y_1^{n_5}$ is observed. After the fourth phase node 1 tries to find a pair $(\tilde{r}, \tilde{o})$ such that

$$(x_2^{n_4}(a, d, \tilde{r}, \tilde{o}), y_1^{n_4}) \in T_\epsilon^{n_4}(P_{X_2^{(4)} Y_1^{(4)}}). \tag{A239}$$

If there is none or more than one such pair $(\tilde{r}, \tilde{o})$, set $(\hat{r}(1), \hat{o}(1))$. Otherwise, $(\tilde{r}, \tilde{o})$ is the estimate $(\hat{r}(1), \hat{o}(1))$ of node 1. Now node 1 tries to find a pair $(\tilde{s}, \tilde{q})$ such that

$$(x_2^{n_5}(\tilde{s}, \tilde{q}), y_1^{n_5}) \in T_\epsilon^{n_5}(P_{X_2^{(5)} Y_1^{(5)}}). \tag{A240}$$

If there is none or more than one such pair $(\tilde{s}, \tilde{q})$, set $(\hat{s}(1), \hat{q}(1)) = (1, 1)$. Otherwise, $(\tilde{s}, \tilde{q})$ is the estimate $(\hat{s}(1), \hat{q}(1))$ of node 1. Then node 1 tries to find an index $\tilde{z}$ such that

$$(x_3^{n_5}(\hat{s}(1), \hat{q}(1), \tilde{z}), x_2^{n_5}(\hat{s}(1), \hat{q}(1)), y_1^{n_5}) \in T_\epsilon^{n_5}(P_{X_3^{(5)} X_2^{(5)} Y_1^{(5)}}). \tag{A241}$$

If there is none or more than one such index $\tilde{z}$, set $\hat{z}(1) = 1$. Otherwise, $\tilde{z}$ is the estimate $\hat{z}(1)$ of node 1. Finally node 1 tries to find an index $\tilde{t}$ such that

$$\left(x_3^{n_2}(\hat{r}(1), \hat{s}(1), \tilde{t}), y_1^{n_2}\right) \in T_\epsilon^{n_2}(P_{X_3^{(2)} Y_1^{(2)}}). \tag{A242}$$

If there is none or more than one such index $\tilde{t}$, set $\hat{t}(1) = 1$. Otherwise, $\tilde{t}$ is the estimate $\hat{t}(1)$ of node 1. The message $\hat{w}_{31}(1)$ is found by reindexing $(\hat{r}(1), \hat{s}(1), \hat{t}(1), \hat{o}(1), \hat{q}(1), \hat{z}(1))$.

Node 3 (Output) In the first phase $y_3^{n_1}$ is observed. In the fourth phase $y_3^{n_4}$ is observed. In the sixth phase $y_3^{n_5}$ is observed. After the fourth phase node 3 tries to find a pair $(\tilde{a}, \tilde{d})$ such that

$$\left(x_2^{n_4}(\tilde{a}, \tilde{d}, r, o), y_3^{n_4}\right) \in T_\epsilon^{n_4}(P_{X_2^{(4)} Y_3^{(4)}}). \tag{A243}$$

If there is none or more than one such pair $(\tilde{a}, \tilde{d})$, set $(\hat{a}(3), \hat{d}(3))$. Otherwise, $(\tilde{a}, \tilde{d})$ is the estimate $(\hat{a}(3), \hat{d}(3))$ of node 3. Now node 3 tries to find a pair $(\tilde{b}, \tilde{e})$ such that

$$\left(x_2^{n_6}(\tilde{b}, \tilde{e}), y_3^{n_6}\right) \in T_\epsilon^{n_6}(P_{X_2^{(6)} Y_3^{(6)}}). \tag{A244}$$

If there is none or more than one such pair $(\tilde{b}, \tilde{e})$, set $(\hat{b}(3), \hat{e}(3)) = (1, 1)$. Otherwise, $(\tilde{b}, \tilde{e})$ is the estimate $(\hat{b}(3), \hat{e}(3))$ of node 3. Then node 3 tries to find an index $\tilde{f}$ such that

$$\left(x_1^{n_6}(\hat{b}(3), \hat{e}(3), \tilde{f}), x_2^{n_6}(\hat{b}(3), \hat{e}(3)), y_3^{n_6}\right) \in T_\epsilon^{n_6}(P_{X_1^{(6)} X_2^{(6)} Y_3^{(6)}}). \tag{A245}$$

If there is none or more than one such index $\tilde{f}$, set $\hat{f}(3) = 1$. Otherwise, $\tilde{f}$ is the estimate $\hat{f}(3)$ of node 3. Finally node 3 tries to find an index $\tilde{c}$ such that

$$\left(x_1^{n_1}(\hat{a}(3), \hat{b}(3), \tilde{c}), y_3^{n_1}\right) \in T_\epsilon^{n_1}(P_{X_1^{(1)} Y_3^{(1)}}). \tag{A246}$$

If there is none or more than one such index $\tilde{c}$, set $\hat{c}(3) = 1$. Otherwise, $\tilde{c}$ is the estimate $\hat{c}(3)$ of node 3. The message $\hat{w}_{13}(3)$ is found by reindexing $(\hat{a}(3), \hat{b}(3), \hat{c}(3), \hat{d}(3), \hat{e}(3), \hat{f}(3))$.

Rates An error analysis along the lines of A2.1 reveals that for large n, $\epsilon > 0$ but small, reliable communication requires

$$\begin{aligned} R_1 + R_2 + R_3 &< \tau_1 I(X_1^{(1)}; Y_2^{(1)}) \\ R_7 + R_8 + R_9 &< \tau_2 I(X_3^{(2)}; Y_2^{(2)}) \\ R_4 + R_5 &< \tau_3 I(X_1^{(3)}; Y_2^{(3)} | X_3^{(3)}) \\ R_{10} + R_{11} &< \tau_3 I(X_3^{(3)}; Y_2^{(3)} | X_1^{(3)}) \\ R_4 + R_5 + R_{10} + R_{11} &< \tau_3 I(X_1^{(3)} X_3^{(3)}; Y_2^{(3)}) \end{aligned} \tag{A247}$$

at node 2,

$$\begin{aligned} R_7 + R_{10} &< \tau_4 I(X_2^{(4)}; Y_1^{(4)}) \\ R_8 + R_{11} &< \tau_5 I(X_2^{(5)}; Y_1^{(5)}) \\ R_{12} &< \tau_5 I(X_3^{(5)}; Y_1^{(5)} | X_2^{(5)}) \\ R_9 &< \tau_2 I(X_3^{(2)}; Y_1^{(2)}) \end{aligned} \tag{A248}$$

at node 1 and

$$\begin{aligned} R_1 + R_4 &< \tau_4 I(X_2^{(4)}; Y_3^{(4)}) \\ R_2 + R_5 &< \tau_6 I(X_2^{(6)}; Y_3^{(6)}) \\ R_6 &< \tau_6 I(X_1^{(6)}; Y_3^{(6)} | X_2^{(6)}) \\ R_3 &< \tau_1 I(X_1^{(1)}; Y_3^{(1)}) \end{aligned} \tag{A249}$$

at node 3. Consequently,

$$\begin{aligned} R_{13} &< \tau_1 I(X_1^{(1)}; Y_2^{(1)}) + \tau_3 I(X_1^{(3)}; Y_2^{(3)} | X_3^{(3)}) + \tau_6 I(X_1^{(6)}; Y_3^{(6)} | X_2^{(6)}) \\ R_{13} &< \tau_1 I(X_1^{(1)}; Y_3^{(1)}) + \tau_4 I(X_2^{(4)}; Y_3^{(4)}) + \tau_6 I(X_2^{(6)}; Y_3^{(6)}) + \tau_6 I(X_1^{(6)}; Y_3^{(6)} | X_2^{(6)}) \\ R_{31} &< \tau_2 I(X_3^{(2)}; Y_2^{(2)}) + \tau_3 I(X_3^{(3)}; Y_2^{(3)} | X_1^{(3)}) + \tau_5 I(X_3^{(5)}; Y_1^{(5)} | X_2^{(5)}) \\ R_{31} &< \tau_2 I(X_3^{(2)}; Y_1^{(2)}) + \tau_4 I(X_2^{(4)}; Y_1^{(4)}) + \tau_5 I(X_2^{(5)}; Y_1^{(5)}) + \tau_5 I(X_3^{(5)}; Y_1^{(5)} | X_2^{(5)}) \end{aligned} \tag{A250}$$

subject to

$$R_{13} + R_{31} < \tau_1 I(X_1^{(1)}; Y_2^{(1)}) + \tau_2 I(X_3^{(2)}; Y_2^{(2)}) + \tau_3 I(X_1^{(3)} X_3^{(3)}; Y_2^{(3)}) + \\ + \tau_5 I(X_3^{(5)}; Y_1^{(5)} | X_2^{(5)}) + \tau_6 I(X_1^{(6)}; Y_3^{(6)} | X_2^{(6)}). \tag{A251}$$

This establishes the proposition. ■

A7. Algorithms

Algorithm 1 HD-OWRC, Time Allocation for Rate Maximization

Approximate communication scenario by one of the propositions or the upper bound
Specify all channels by conditional distributions P_c (or densities p_c)
Specify all relevant fix input distributions P_k (p_k) with $k = 1, \ldots, K$

$R_o^\star = 0$
for all $k = 1$ to K **do**
  Calculate all mutual informations $I(P_c, P_k)$
  specify $\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}$ (see 2.2)
  Solve: $\max \boldsymbol{c}^T \boldsymbol{x}$ s.t. $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{0} \leq \boldsymbol{x}, const(\boldsymbol{x}) \rightarrow \boldsymbol{x}_k$
  $R_{o,k} = \boldsymbol{c}^T \boldsymbol{x}_k$
  if $R_{o,k} > R_o^\star$ **then**
    $R_o^\star = R_o$
    $\boldsymbol{x}^\star = \boldsymbol{x}_k$
    $P^\star = P_k$
  end if
end for

$R_o^\star \rightarrow$ achievable rate
$P^\star \rightarrow$ optimal input distribution
$\boldsymbol{x}^\star \rightarrow$ optimal time allocation $\boldsymbol{\tau}^\star$

Algorithm 2 HD-OWRC, Time Allocation for Transmission Cost Minimization with Rate Request

Approximate communication scenario by one of the propositions or the upper bound
Specify all channels by conditional distributions P_c (or densities p_c)
Specify all relevant fix input distributions P_k (p_k) with $k = 1, \ldots, K$
Associate costs $c_{l,k}(\tau = 1, P_c, P_k)$ with each phase l of unit time

Rate request: R
$TC^\star = \infty$
flag=false
for all $k = 1$ to K **do**
  Calculate all mutual informations $I(P_c, P_k)$
  specify $\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}$ (see 2.2)
  Solve: $\min \boldsymbol{c}^T \boldsymbol{x}$ s.t. $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{0} \leq \boldsymbol{x}, const(\boldsymbol{x}) \rightarrow \boldsymbol{x}_k$
  if LP has solution **then**
    flag=true
    $TC_k = \boldsymbol{c}^T \boldsymbol{x}_k$
    if $TC_k < TC^\star$ **then**
      $TC^\star = TC$
      $\boldsymbol{x}^\star = \boldsymbol{x}_k$
      $P^\star = P_k$
    end if
  end if
end for

if flag **then**
  $TC^\star \rightarrow$ transmission cost
  $P^\star \rightarrow$ optimal input distribution
  $\boldsymbol{x}^\star \rightarrow$ optimal time allocation $\boldsymbol{\tau}^\star$
else
  R is not achievable
end if

Algorithm 3 HD-TWRC, Time Allocation for Weighted Sum-Rate Maximization

Approximate communication scenario by one of the propositions or scheme upper bounds
Specify all channels by conditional distributions P_c (or densities p_c)
Specify all relevant fix input distributions P_k (p_k) with $k = 1, \ldots, K$

Choose a weight: α
$R_o^\star = 0$
for all $k = 1$ to K **do**
  Calculate all mutual informations $I(P_c, P_k)$
  Specify $\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}(\alpha)$ (see 3.3)
  Solve: $\max \boldsymbol{c}^T \boldsymbol{x}$ s.t. $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{0} \leq \boldsymbol{x}, const(\boldsymbol{x}) \rightarrow \boldsymbol{x}_k$
  $R_{o,k} = \boldsymbol{c}^T \boldsymbol{x}_k$
  if $R_{o,k} > R_o^\star$ **then**
    $R_o^\star = R_o$
    $\boldsymbol{x}^\star = \boldsymbol{x}_k$
    $P^\star = P_k$
  end if
end for

$R_o^\star \rightarrow$ achievable weighted sum-rate
$P^\star \rightarrow$ optimal input distributions
$\boldsymbol{x}^\star \rightarrow R_{13}^\star$, $R_{31}^\star$ and optimal time allocation $\boldsymbol{\tau}^\star$

Algorithm 4 HD-TWRC, Time Allocation for Sum-Rate/MaxMin-Rate Maximization

Approximate communication scenario by one of the propositions or scheme upper bounds
Specify all channels by conditional distributions P_c (or densities p_c)
Specify all relevant input distributions P_k (p_k) with $k = 1, \ldots, K$

$R_o^\star = 0$
for all $k = 1$ to K **do**
  Calculate all mutual informations $I(P_c, P_k)$
  Specify $\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}$ (see 3.3)
  Solve: $\max \boldsymbol{c}^T \boldsymbol{x}$ s.t. $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{0} \leq \boldsymbol{x}, const(\boldsymbol{x}) \rightarrow \boldsymbol{x}_k$
  $R_{o,k} = \boldsymbol{c}^T \boldsymbol{x}_k$
  if $R_{o,k} > R_o^\star$ **then**
    $R_o^\star = R_o$
    $\boldsymbol{x}^\star = \boldsymbol{x}_k$
    $P^\star = P_k$
  end if
end for

$R_o^\star \rightarrow$ achievable sum-rate/maxmin-rate
$P^\star \rightarrow$ optimal input distributions
$\boldsymbol{x}^\star \rightarrow$ Rates R_{13}, R_{31} / R_{MMP} and time allocation solution $\boldsymbol{\tau}^\star$

Algorithm 5 HD-TWRC, Time Allocation for Transmission Cost Minimization with Rate Request

Approximate communication scenario by one of the propositions or scheme upper bounds
Specify all channels by conditional distributions P_c (or densities p_c)
Specify all relevant fix input distributions P_k (p_k) with $k = 1, \dots, K$
Associate costs $c_{l,k}(\tau = 1, P_c, P_k)$ with each phase l of unit time duration

Rate request: $\boldsymbol{R} = [R_{13} \quad R_{31}]^T$
$TC^\star = \infty$
flag=false
for all $k = 1$ to K **do**
  Calculate all mutual informations $I(P_c, P_k)$ from proposition or upper bound
  Specify $\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}$ (see 3.3)
  Solve: $\min \boldsymbol{c}^T \boldsymbol{x}$ s.t. $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{0} \leq \boldsymbol{x}, const(\boldsymbol{x}) \rightarrow \boldsymbol{x}_k$
  if LP has solution **then**
    flag=true
    $TC_k = \boldsymbol{c}^T \boldsymbol{x}_k$
    if $TC_k < TC^\star$ **then**
      $TC^\star = TC$
      $\boldsymbol{x}^\star = \boldsymbol{x}_k$
      $P^\star = P_k$
    end if
  end if
end for

if flag **then**
  $TC^\star \rightarrow$ transmission cost
  $P^\star \rightarrow$ optimal input distribution
  $\boldsymbol{x}^\star \rightarrow$ optimal time allocation $\boldsymbol{\tau}^\star$
else
  R is not achievable
end if

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