Algebraic Multigrid Methods for mortar-based finite element discretizations in contact mechanics

Part1: Condensed formulation

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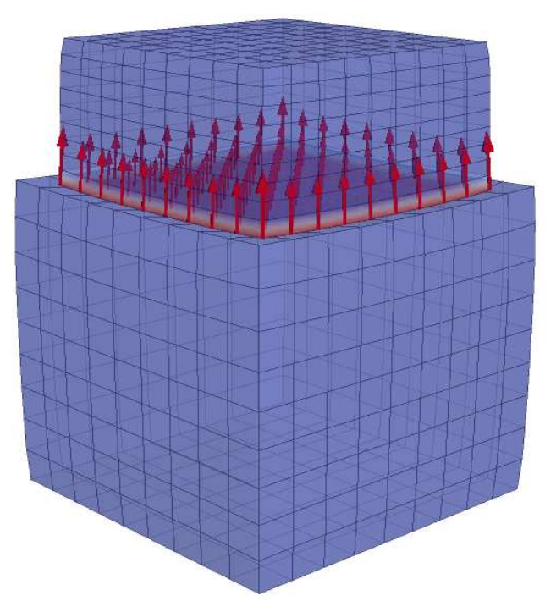
Motivation

Iterative linear solvers are crucial for solving large scale contact problems + Multigrid methods are known to be efficient solving strategies

Contact problems

Saddlepoint formulation

- Problem formulation based on mortar FE methods
- Initial boundary value problem of nonlinear elastodynamics
- KKT conditions for contact and Coulomb friction (optional)
- Direct Lagrange multiplier method


Two solid bodies example

$$\begin{pmatrix} \mathrm{K}_{\mathcal{N}_1\mathcal{N}_1} & \mathrm{K}_{\mathcal{N}_1\mathcal{M}} & 0 & 0 & 0 & 0\\ \mathrm{K}_{\mathcal{M}\mathcal{N}_1} & \mathrm{K}_{\mathcal{M}\mathcal{M}} & 0 & 0 & -a\mathrm{M}_{\mathcal{I}}^T & -a\mathrm{M}_{\mathcal{A}}^T\\ 0 & 0 & \mathrm{K}_{\mathcal{S}\mathcal{S}} & \mathrm{K}_{\mathcal{S}\mathcal{N}_2} & a\mathrm{D}_{\mathcal{I}}^T & a\mathrm{D}_{\mathcal{A}}^T\\ 0 & 0 & \mathrm{K}_{\mathcal{N}_2\mathcal{S}} & \mathrm{K}_{\mathcal{N}_2\mathcal{N}_2} & 0 & 0\\ 0 & 0 & 0 & 0 & \mathrm{I} & 0\\ 0 & \mathrm{N}_{\mathcal{M}} & \mathrm{N}_{\mathcal{S}} & 0 & 0 & 0\\ 0 & 0 & \mathrm{F}_{\mathcal{S}} & 0 & 0 & \mathrm{T}_{\mathcal{A}} \end{pmatrix} \begin{pmatrix} \Delta\mathbf{d}_{n+1,\mathcal{N}_1}\\ \Delta\mathbf{d}_{n+1,\mathcal{M}}\\ \Delta\mathbf{d}_{n+1,\mathcal{S}}\\ \Delta\mathbf{d}_{n+1,\mathcal{N}_2}\\ \Delta\lambda_{n+1,\mathcal{I}}\\ \Delta\lambda_{n+1,\mathcal{A}} \end{pmatrix} = -\begin{pmatrix} \mathbf{r}_{\mathcal{N}_1}\\ \mathbf{r}_{\mathcal{M}}\\ \mathbf{r}_{\mathcal{S}}\\ \mathbf{r}_{\mathcal{N}_2}\\ \mathbf{0}\\ \mathbf{g}_{\mathcal{A}}\\ \mathbf{0} \end{pmatrix}$$

- Structural equations (cartesian coordinates)
- Lagrange multipliers
- Contact constraints (normal-tangential formulation)

Condensed formulation

Dual (biorthogonal) basis functions
→ Condensation of Lagrange multipliers [1]

Matrix properties

- constant system size
- no saddlepoint structure enables usage of standard iterative solvers

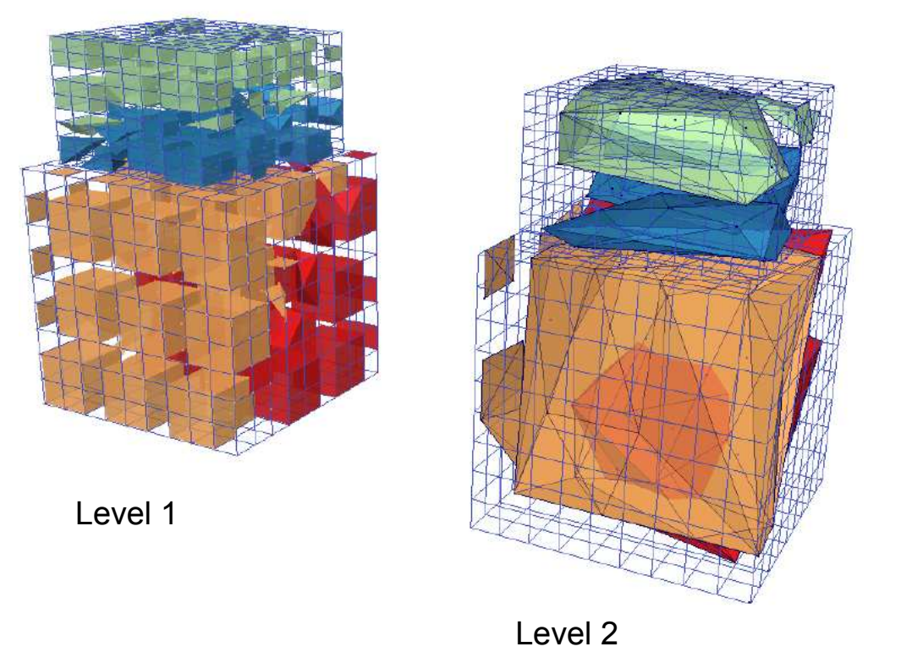
$$\begin{pmatrix} \mathrm{K}_{\mathcal{N}_1\mathcal{N}_1} & \mathrm{K}_{\mathcal{N}_1\mathcal{M}} & 0 & 0\\ \mathrm{K}_{\mathcal{M}\mathcal{N}_1}+\mathrm{P}_{\mathcal{A}}^T\mathrm{K}_{\mathcal{A}\mathcal{N}_1} & \mathrm{K}_{\mathcal{M}\mathcal{M}}+\mathrm{P}_{\mathcal{A}}^T\mathrm{K}_{\mathcal{A}\mathcal{M}} & \mathrm{K}_{\mathcal{M}\mathcal{S}}+\mathrm{P}_{\mathcal{A}}^T\mathrm{K}_{\mathcal{M}\mathcal{S}} & \mathrm{K}_{\mathcal{M}\mathcal{N}_2}+\mathrm{P}_{\mathcal{A}}^T\mathrm{K}_{\mathcal{A}\mathcal{N}_2}\\ \mathrm{K}_{\mathcal{I}\mathcal{N}_1} & \mathrm{K}_{\mathcal{I}\mathcal{M}} & \mathrm{K}_{\mathcal{I}\mathcal{S}} & \mathrm{K}_{\mathcal{I}\mathcal{N}_2}\\ 0 & \mathrm{N}_{\mathcal{M}} & \mathrm{N}_{\mathcal{S}} & 0\\ a\mathrm{T}_{\mathcal{A}}\mathrm{D}_{\mathcal{A}\mathcal{A}}^{-1}\mathrm{K}_{\mathcal{A}\mathcal{N}_1} & a\mathrm{T}_{\mathcal{A}}\mathrm{D}_{\mathcal{A}\mathcal{A}}^{-1}\mathrm{K}_{\mathcal{A}\mathcal{M}} & a\mathrm{T}_{\mathcal{A}}\mathrm{D}_{\mathcal{A}\mathcal{A}}^{-1}\mathrm{K}_{\mathcal{A}\mathcal{S}}-\mathrm{F}_{\mathcal{S}} & a\mathrm{T}_{\mathcal{A}}\mathrm{D}_{\mathcal{A}\mathcal{A}}^{-1}\mathrm{K}_{\mathcal{A}\mathcal{N}_2}\\ 0 & 0 & \mathrm{K}_{\mathcal{N}_2\mathcal{S}} & \mathrm{K}_{\mathcal{N}_2\mathcal{N}_2} \end{pmatrix} \begin{pmatrix} \Delta\mathbf{d}_{n+1,\mathcal{N}_1}\\ \Delta\mathbf{d}_{n+1,\mathcal{M}}\\ \Delta\mathbf{d}_{n+1,\mathcal{S}}\\ \Delta\mathbf{d}_{n+1,\mathcal{N}_2} \end{pmatrix} = -\begin{pmatrix} \mathbf{r}_{\mathcal{N}_1}+\mathrm{P}_{\mathcal{A}}^T\mathbf{r}_{\mathcal{A}}\\ \mathbf{r}_{\mathcal{I}}\\ \mathbf{g}_{\mathcal{A}}\\ a\mathrm{T}_{\mathcal{A}}\mathrm{D}_{\mathcal{A}\mathcal{A}}^{-1}\mathbf{r}_{\mathcal{A}}\\ \mathbf{r}_{\mathcal{N}_2}+\mathrm{P}_{\mathcal{A}}^T\mathbf{r}_{\mathcal{A}} \end{pmatrix}$$

Due to different coordinate systems for structural equations and contact constraints
matrix A is not diagonally-dominant at slave DOF rows!

Algebraic Multigrid Methods

Basic idea

Reconstruct fine level solution from information of coarse representations of fine level problem



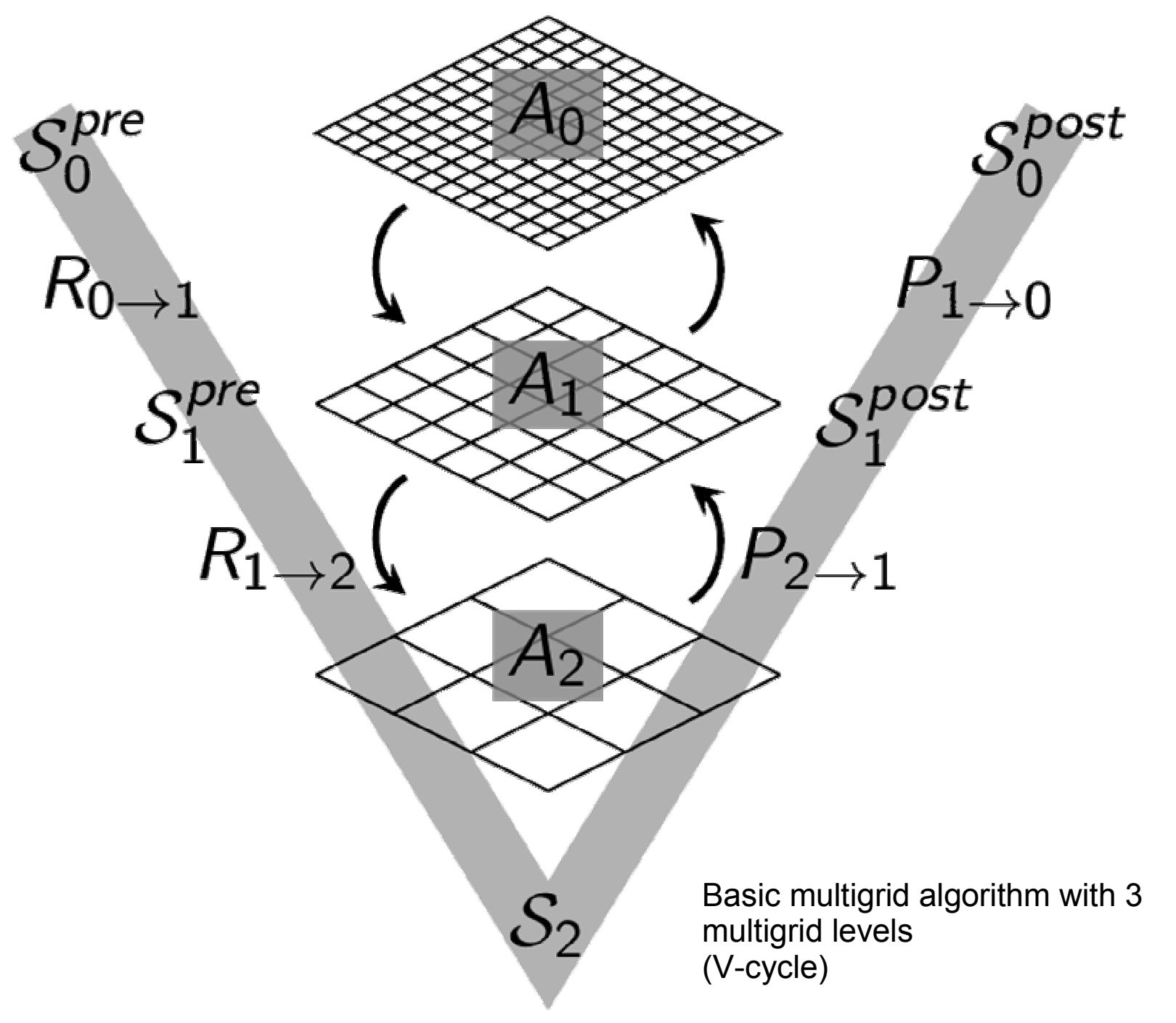
Exemplary aggregates for two solid bodies contact example on 4 processors

Algebraic Multigrid [3]

- Build multigrid hierarchy with an aggregation strategy using the fine level matrix information only
- Restriction and prolongation operators transfer information between different multigrid levels

Multigrid level smoothers

- Use smoothing effect of iterative methods (e.g. Jacobi, Gauss Seidel) to attack different components of the error on different multigrid levels
- Multigrid level matrices have to fulfill minimum prerequisites for convergence of the level smoothing algorithms
- Minimum requirement for convergence for most methods is at least a **diagonally-dominant** matrix



Basic multigrid algorithm with 3 multigrid levels (V-cycle)

Contact Algebraic Multigrid Method

Perumtation strategy

- fix mathematically non diagonally-dominant matrix rows by applying a column permutation strategy
- **Constrained permutation strategy:** permute only columns which belong to same mesh node/aggregate
- **Use contact information:** only permute columns that correspond to (possibly) problematic contact slave DOFs
- permute fine level matrix only Standard multigrid transfer operators preserve diagonal dominance of coarse level matrices

Constrained permutation strategy

Let $\mathcal{N}_{\mathcal{S}}$ be the set of slave DOF ids and $\mathcal{N}_{N_{\mathcal{S}}}$ the corresponding set of slave node ids with

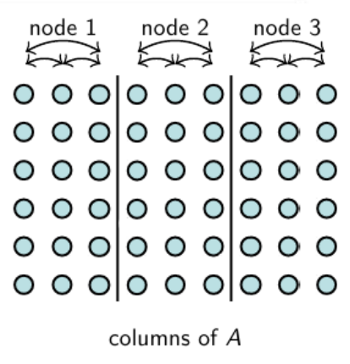
$$f:\mathcal{N}_{\mathcal{S}}\to\mathcal{N}_{N_{\mathcal{S}}}$$

a surjective mapping between the slave DOF ids and the corresponding slave node ids.
Find a permutation $p:\mathcal{N}_{\mathcal{S}}\to\mathcal{N}_{\mathcal{S}}, i\mapsto p(i)$ such that it is

$$\max_p \prod_i A_{i,p(i)}$$
$$\text{s.t. } f(p(i))-f(p(j))=0 \quad \forall i,j\in\mathcal{N}_{\mathcal{S}}.$$

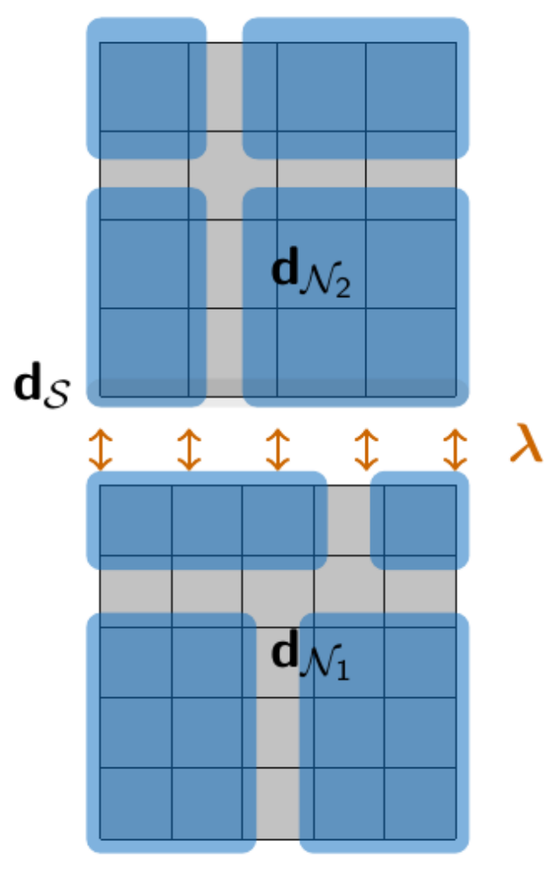
$A_{i,j}$ denotes the entry of matrix A in the i^{th} row and j^{th} column.

- no mix-up of node information through column permutations
- prerequisite for segregated aggregates



Aggregation strategy

- **Use contact information:** Build segregated aggregates which do not overlap between the distinct solid bodies
  → keep distinct solids separated on all multigrid levels
  → enables reuse of aggregates
- **Full multigrid:** no special handling of interface nodes in aggregation routine
  → consistent coarsening rate throughout whole domain



Visualization of segregated aggregates which do not overlap two distinct solids

Parallelization

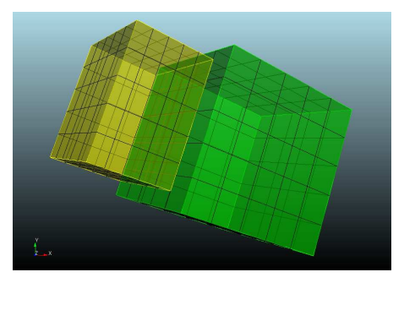
- **Uncoupled aggregates:** aggregates cannot overlap processor boundaries (simplifies implementation drastically)
- **Automatic rebalancing:** optimal choice of number of processors on coarse levels minimizes communication overhead

Test example

Two solid bodies contact

- Rotate problem configuration around y-axis and z-axis
- No change in physics through rotation

Problem setup and solver parameters



Simulation parameters

Material:	NeoHooke
ρ	$0.1\frac{kg}{m^3}$
E	10 GPa
ν	0.3
Timestep size:	0.02s
Timesteps:	25

- Iterative solver: GMRES
- Preconditioner: AMG (3 level)
- Coarse solver: direct (UMFPACK)
- Transfer operators: PG-AMG [4]
- Level smoother: 2 SGS (0.5)
- Min. aggregate size: 18 nodes

Expected behaviour: number of linear iterations independent of geometric configuration

		rotation around y-axis in [°]										
		0	10	20	30	40	45	50	60	70	80	90
rotation around z-axis in [°]	0	-	-	-	-	-	-	-	-	-	-	-
	10	-	-	-	-	-	-	-	-	-	-	-
	20	-	-	-	-	-	-	-	-	-	-	-
	30	-	-	-	-	-	-	-	-	-	-	-
	40	-	56.0	48				112.0	116.3	114.1	-	-
	45	-	39.4	38				68.5	117.2	114.2	-	-
	50	-	44.4	35.0	33.9	38.3	41.4	48.4	115.3	114.2	-	-
	60	-	-	36.5	34.4	32.8	36.5	37.1	89.4	114.2	-	-
	70	-	-	59.3	33.2	34.0	35.0	36.5	84.0	-	-	-
	80	-	-	-	59.7	43.8	48.5	53.4	-	-	-	-
	90	-	-	-	-	108.0	106.2	-	-	-	-	-

not robust

average number of linear iterations over 25 timesteps

Multigrid without contact specific extensions

		rotation around y-axis in [°]										
		0	10	20	30	40	45	50	60	70	80	90
rotation around z-axis in [°]	0	32.4	32.3	32.1	31.7	31.7	31.6	31.7	31.7	31.8	32.0	31.9
	10	32.5	32.3	31.7	31.6	31.6	31.4	31.4	31.7	31.6	31.7	31.9
	20	32.4	32.1	31.6	31.6	31.4	31.4	31.2	31.2	31.5	31.8	32.1
	30	32.4	32.2	31.8	31.6	31.5	31.4	31.5	31.5	31.7	32.0	32.3
	40	32.3	29.9	27.3					28.5	30.8	31.9	32.8
	45	32.4	28.2	27.6					27.7	31.3	31.8	32.4
	50	34.6	29.7	28.9	31.2	27.5	27.4	27.8	28.9	30.0	31.2	31.2
	60	31.5	31.5	27.9	26.5	27.9	28.5	29.8	28.9	29.7	30.4	30.2
	70	30.1	30.0	30.1	29.4	27.9	32.5	32.5	32.3	29.5	29.4	29.8
	80	29.2	29.2	29.4	29.8	34.3	32.7	32.5	32.3	31.9	30.1	29.2
	90	29.0	29.2	30.0	30.9	34.5	37.9	32.0	32.0	31.7	31.8	30.9

robust

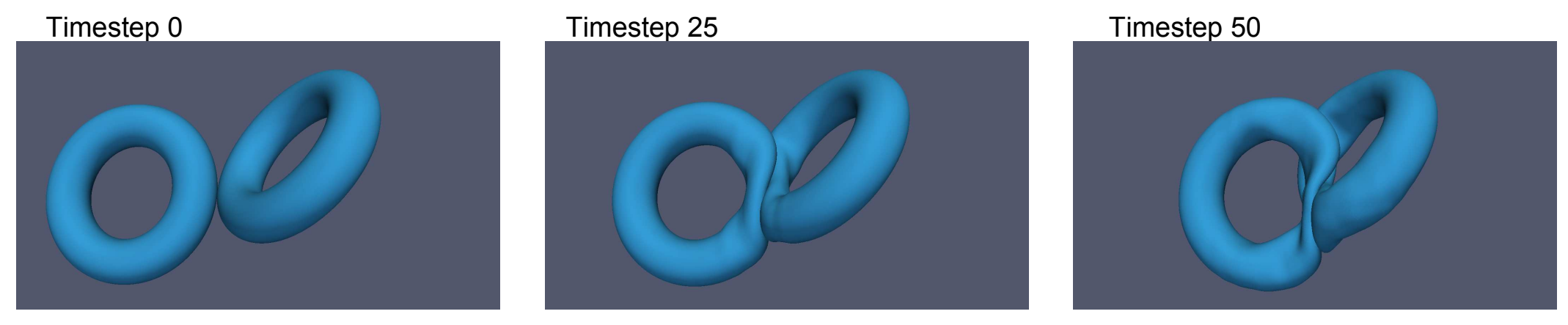
average number of linear iterations over 25 timesteps

Contact Algebraic Multigrid Method

Two tori impact example

Problem configuration

- #DOFs: 1050624, #Procs: 64, # timesteps: 50
- condensed contact formulation (Petrov Galerkin) [2]

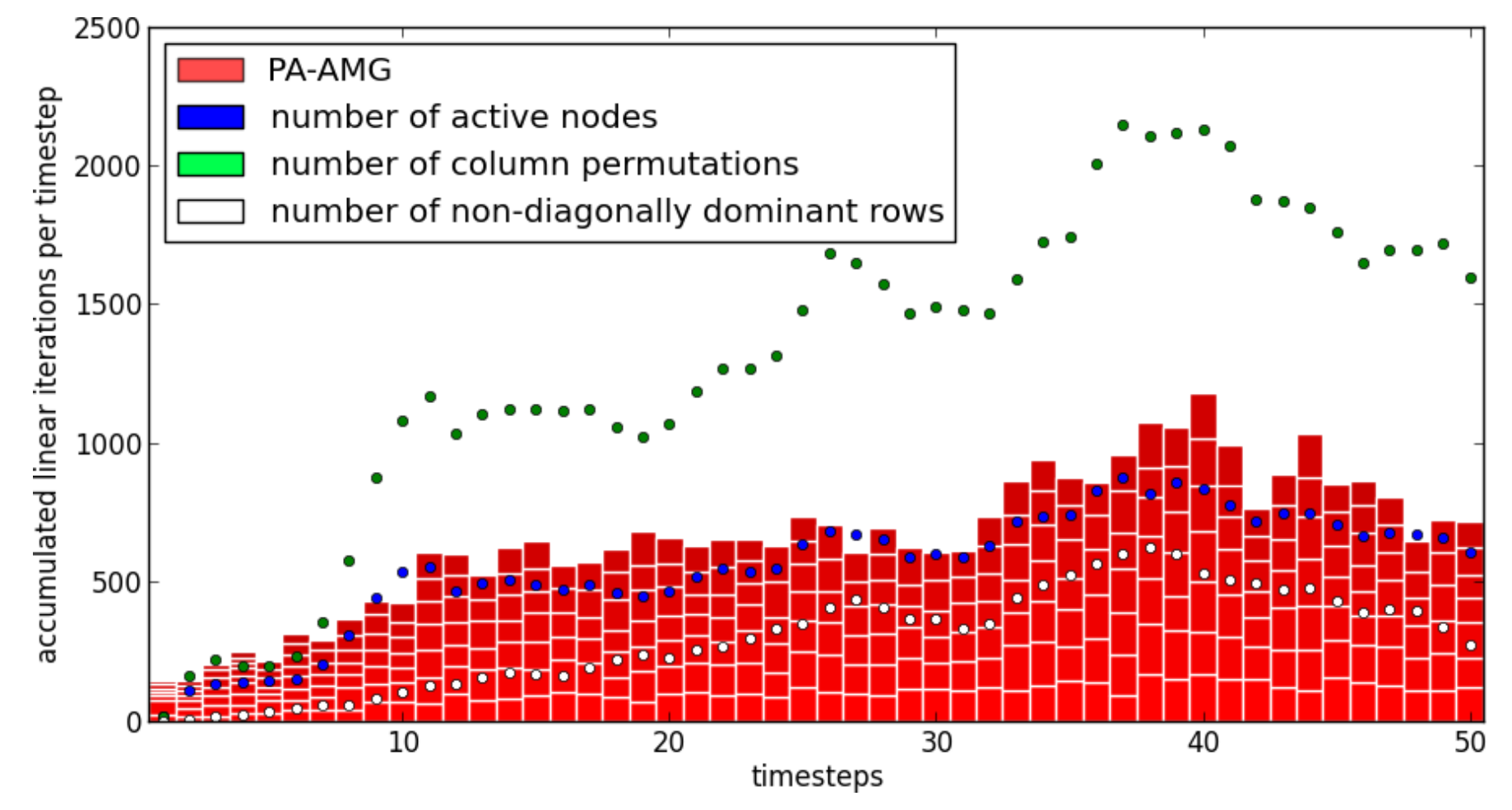


Solver parameters

- Iterative solver: GMRES
- Preconditioner: AMG (4 level)
- Coarse solver: direct (UMFPACK)
- Transfer operators: PA-AMG
- Level smoother: 2 SGS (0.7)
- Min. aggregate size: 27 nodes

Findings

- number of non diagonally-dominant rows corresponds to number of active contact nodes
- number of linear iterations depends on number of non diagonally-dominant rows



Conclusions

- Full Multigrid method for contact problems in condensed formulation.
- Robust and flexible preconditioner for large scale problems.
- Fully parallelized algorithm with optimal rebalancing of transfer operators.

References

[1] Wohlmuth, B.I., "A mortar finite element method using dual spaces for the Lagrange multiplier", SIAM Journal on Numerical Analysis, 38, 989-1012, (2000).
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[3] Vanek, P., Mandel, J. and Brezina, M., "Algebraic Multigrid by Smoothed Aggregation for Second and Fourth Order Elliptic problems", Computing, 56, 179-196, (1996).
[4] Sala, M., Tuminaro, R.S., "A new Petrov-Galerkin Smoothed Aggregation Preconditioner for nonsymmetric Linear Systems.", SIAM Journal on Scientific Computing, 31(1), 143-166, (2008).
[5] Wiesner, T.A., Tuminaro, R.S., Wall, W.A. and Gee, M.W., "Multigrid Transfers for Nonsymmetric Systems Based on Schur Complements and Galerkin Projections", Numerical Linear Algebra with Applicaötions, in press, (2013).