

Time-consistency of risk measures with GARCH volatilities and their estimation

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November 28, 2015

Abstract

In this paper we study time-consistent risk measures for returns that are given by a GARCH(1, 1) model. We present a construction of risk measures based on their static counterparts that overcomes the lack of time-consistency. We then study in detail our construction for the risk measures Value-at-Risk (VaR) and Average Value-at-Risk (AVaR). While in the VaR case we can derive an analytical formula for its time-consistent counterpart, in the AVaR case we derive lower and upper bounds to its time-consistent version. Furthermore, we incorporate techniques from Extreme Value Theory (EVT) to allow for a more tail-g geared statistical analysis of the corresponding risk measures. We conclude with an application of our results to a data set of stock prices.

2010 AMS subject classifications: 60G70, 91B30, 91G80, 91G70

2010 JEL classification: C02, C22, C58, G17, G32

Key words and phrases: dynamic risk measure, time-consistency, GARCH(1, 1), Extreme Value Theory, Value-at-Risk, Average Value-at-Risk, Expected Shortfall, Generalized Pareto distribution, aggregate returns.

1 Introduction

In the wake of the financial crisis risk management constitutes a constant active field that attracts both mathematical research and quantitative requirements for the practical implementation. Most financial institutions need to abide with the Basel II/III accords that prescribe certain risk management rules to be applied to internal risk control and that are under periodic regulatory supervision. Over the last two decades the key notion of risk management arose in the form of a risk measure referred to as Value-at-Risk (VaR). Simply put, VaR determines the risk capital of a financial institution as the quantile of a profit-and-loss distribution with respect to some prescribed (either by regulation or by internal rules) time horizon and confidence level. An axiomatic approach to the field of risk measures is given by Artzner et al. (1999) in which the notion of the *coherent* risk measure is introduced and where it has been realized that VaR does not always satisfy the property of coherence. Artzner et al. (1999) introduce a risk measure that amends the lack of coherence that is nowadays known as the Average-Value-at-Risk (AVaR). An extension to *convex* risk measures is given in Föllmer and Schied (2002), which integrates existing notions of risk into the mathematical framework of convex dual theory and, hence, allows for deep and powerful dual characterizations. In order to account for the dynamic stochastic evolution of profit-and-loss positions the static risk

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measurement has been extended to the class of *dynamic risk measures*, which treats the risk measure not only as a (nonlinear) expectation but as a stochastic process, see e.g. Detlefsen and Scandolo (2005) and Riedel (2004) for the extension to the dynamic setting by means of convex dual theory. It has been realized in this dynamic framework that most existing static risk measures do not transfer in a straightforward manner into processes without violating the required property of *time-consistency*. A time-consistent dynamic risk measure secures the consistent behavior of a risk measure that, if a portfolio is riskier than another portfolio at some future time, then this portfolio has been riskier than the other portfolio at any time before. The literature on time-consistency of risk measures is diverse and rich as different mathematical viewpoints can be adopted to prevent the consistency property. An incomplete chronicle of research done in the field of time-consistent risk measures includes Peng (2004), Riedel (2004), Detlefsen and Scandolo (2005), Weber (2006), Föllmer and Penner (2006), Roorda and Schumacher (2007), Penner (2007), Bion-Nadal (2009), and Bielecki et al. (2015). A major result from the research on time-consistency reveals that in the class of law-invariant risk measures there is only one risk measure that, upon transfer into a time-dynamic process setting, supports time-consistency, namely the entropic risk measure (cf. Föllmer and Knispel (2011)).

In parallel to the aforementioned theoretical work statistical models and methods have been developed to calibrate and integrate risk measures to real world data. As the industry standard VaR and its coherent counterpart AVaR are law-invariant risk measures, the main goal for an implementation of (A)VaR is to find a good estimate of the profit-and-loss distribution in the relevant region. In this field, the major class of estimation methods comprise the historical simulation method, methods based on Gaussian distribution assumptions and methods based on Extreme Value Theory (EVT). We refer to McNeil et al. (2005), in particular Chapter 2 and Chapter 7, for a detailed account and references to methods of profit-and-loss distribution estimation. More background on extreme value theory can be found in the monograph Embrechts et al. (1997). McNeil and Frey (2000) propose an implementation of VaR and AVaR that is based on an estimation of the log-returns distribution using a combination of a GARCH(1,1) model fit and an EVT approach for the residuals. Their method proceeds in a two-step scheme: first, the GARCH(1,1) model mimics the inherent stochastic volatility of financial time series, and the GARCH parameters are estimated by a pseudo maximum likelihood method. Second, they adopt a Peaks-over-Threshold (POT) approach to the residuals and only consider those residuals that exceed a critical value. The POT method justifies fitting a Generalized Pareto distribution (GPD) by means of a maximum likelihood method (e.g. Embrechts et al. (1997), Section 3.4 and Section 6.5) It is also in accord with the typically high confidence levels that are imposed on (A)VaR to zoom into the extreme branch of losses. Applying the POT method to the residuals rather than directly to the log-returns has the advantage that the fitting procedure to the extremes only needs to be applied once due to the white noise property of the residuals. Using these two steps, McNeil and Frey (2000) succeed to estimate (A)VaR by fitting a distribution that adequately accounts for the extremes in the tail and under mild conditions allows for closed form formulas for VaR and AVaR.

The goal of our paper is to incorporate dynamic time-consistency for VaR and AVaR. We investigate the extension of static risk measures to dynamic counterparts that satisfy time-consistency. A key property to succeed in this transfer is the dynamic programming principle, see Cheridito and Stadje (2009), Cheridito and Kupper (2011).

The two-step estimation scheme from McNeil and Frey (2000) using GARCH(1,1) and EVT allows us to derive a closed form expression for the dynamic time-consistent VaR that is easily implemented using the estimated GPD and the GARCH parameters. For AVaR however, such a closed form expression cannot be obtained and we derive closed form lower and upper approximations to AVaR. On top of being more conservative than their static counterparts, the dynamic time-consistent VaR

offers the benefit that the risk measurement of aggregated losses, which in e.g. McNeil and Frey (2000) have to be estimated by simulation methods, can now be estimated in a (semi-)closed way by simply aggregating the VaRs of the single positions at different future time points.

The paper is structured as follows. In Section 2 we present preliminaries on dynamic risk measures along with the dynamic programming principle characterization. Moreover, we introduce the GARCH(1,1) loss model, which establishes the model framework for the entire paper. In Section 3 we apply the new methodology from the previous section to derive a closed form expression for the time-consistent VaR and investigate its properties concerning the evolution over time and prove the linearization of aggregated losses. Section 4 is devoted to the study of AVaR. Since a closed form expression for time-consistent AVaR is not possible, as an alternative, we derive closed form expressions for pragmatic bounds to AVaR and study the properties as in the previous section. The proofs of the results of Sections 3 and 4 are postponed to the Appendix. In the last Section 5 we give a rehash on the part of extreme value theory that is relevant for our purpose, and apply our results to a data set of stock prices.

2 Conditional risk measures

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we consider a filtration $(\mathcal{F}_t)_{t=0}^T$ where $T \in \mathbb{N}$. We denote by $L^0(\mathcal{F}_t)$ with $t \in \{0, \dots, T\}$ the set of all \mathcal{F}_t -measurable random variables $X : \Omega \rightarrow \mathbb{R}$. In this paper, the space $L^0(\mathcal{F}_T)$ represents the space of all financial positions for which we need a risk assessment. Typically, we will be interested in losses, i.e. the negatives of log-returns of financial data.

Since conditional risk measures are random variables, all properties, equalities and inequalities below hold almost surely with respect to \mathbb{P} , and we assume this throughout without making extra mention of it.

Definition 2.1. For $t \in \{0, \dots, T\}$ a family of mappings $(\phi_t)_{t=0}^T$ with $\phi_t : L^0(\mathcal{F}_T) \rightarrow L^0(\mathcal{F}_t)$ is a dynamic monetary risk measure if it satisfies the following properties:

- (i) *Normalization:* $\phi_t(0) = 0$ for $t = 0, \dots, T$;
- (ii) *Monotonicity:* $\phi_t(X) \geq \phi_t(Y)$ for all $X, Y \in L^0(\mathcal{F}_T)$ such that $X \geq Y$, for $t = 0, \dots, T$;
- (iii) *Translation invariance:* $\phi_t(X + m) = \phi_t(X) + m$ for all $X \in L^0(\mathcal{F}_T)$ and $m \in L^0(\mathcal{F}_t)$, for $t = 0, \dots, T$.

If $L^0(\mathcal{F}_T)$ represents the space of all profit and loss variables, the above definition leads to the notion of a *dynamic monetary utility function*, see Definition 2.1 in Cheridito and Kupper (2011). If a dynamic monetary risk measure ϕ satisfies in addition to Definition 2.1 (i)-(iii)

- *Positive homogeneity:* $\phi_t(\lambda X) = \lambda \phi_t(X)$ for all $X \in L^0(\mathcal{F}_T)$ and $\lambda > 0$, for $t = 0, \dots, T$;
- *Subadditivity:* $\phi_t(X + Y) \leq \phi_t(X) + \phi_t(Y)$ for all $X, Y \in L^0(\mathcal{F}_T)$, for $t = 0, \dots, T$,

then we say that ϕ is a *coherent* (dynamic monetary) risk measure.

Definition 2.2. A dynamic monetary risk measure $\phi := (\phi_t)_{t=0}^T$ is time-consistent if

$$\phi_{t+1}(X) \geq \phi_{t+1}(Y) \quad \text{implies} \quad \phi_t(X) \geq \phi_t(Y),$$

for all $X, Y \in L^0(\mathcal{F}_T)$, for $t = 0, \dots, T - 1$.

The following useful characterization of time-consistency can be found in Cheridito and Kupper (2011).

Proposition 2.3. *A dynamic monetary risk measure $(\phi_t)_{t=0}^T$ is time-consistent if and only if it satisfies the Bellman principle*

$$\phi_t(X) = \phi_t(\phi_{t+1}(X)) \quad (2.1)$$

for all $X \in L^0(\mathcal{F}_T)$, and $t = 0, \dots, T-1$.

It has been noted in Cheridito and Stajde (2009) and Cheridito and Kupper (2011) that there is another way to construct time-consistent dynamic risk measures: let $(\rho_t)_{t=0}^{T-1}$ be an arbitrary dynamic monetary risk measure

$$\rho_t : L^0(\mathcal{F}_T) \rightarrow L^0(\mathcal{F}_t), \quad t = 0, \dots, T-1,$$

then the backward iteration

$$\phi_T(X) := X, \quad \phi_t(X) := \rho_t(\phi_{t+1}(X)), \quad t = 0, \dots, T-1, \quad (2.2)$$

defines a process $(\phi_t)_{t=0}^T$ which by definition is a time-consistent dynamic risk measure. The following property is a straightforward consequence of the construction of $(\phi_t)_{t=0}^T$.

Corollary 2.4. *For $X \in L^0(\mathcal{F}_T)$ we have for $t = 0, \dots, T-1$*

$$\phi_t(X) = \left(\rho_t \circ \rho_{t+1} \circ \dots \circ \rho_{T-1} \right)(X). \quad (2.3)$$

For a coherent risk measure ϕ , its subadditivity property implies that for any fixed $t \in \{0, \dots, T\}$ and $m \in \mathbb{N}$ such that $t+m \leq T$ and any $X_{t+k} \in L^0(\mathcal{F}_{t+k})$ for $k = 1, \dots, m$ we have

$$\phi_t\left(\sum_{k=1}^m X_{t+k}\right) \leq \sum_{k=1}^m \phi_t(X_{t+k}). \quad (2.4)$$

We construct our time-consistent dynamic risk measures by backwards iteration.

2.1 The GARCH(1,1) model for loss positions

Recall that we are interested in the risk assessment of losses. The focus of this paper is on a particular class of loss processes $(L_t)_{t=0}^T$: its dynamics is governed by a GARCH(1,1) process and typically represent (negative) log-returns. It holds that $(L_t)_{t=1}^T$ satisfies

$$\begin{aligned} L_t &= \sigma_t Z_t, \\ \sigma_t^2 &= a_0 + a_1 L_{t-1}^2 + b \sigma_{t-1}^2, \end{aligned} \quad (2.5)$$

where $a_0, a_1, b > 0$ are the model parameters, σ_0 and L_0 are \mathcal{F}_0 -measurable initial random variables, and $(Z_t)_{t=1}^T$ is a strict white noise process (independently identically distributed with zero mean and unit variance). Note also that by (2.5) σ_t is measurable with respect to \mathcal{F}_{t-1} for every $t = 1, \dots, T$.

We denote by $F_Z : \mathbb{R} \rightarrow [0, 1]$ and $F_Z^{-1} : [0, 1] \rightarrow \mathbb{R}$ the distribution function and the left-continuous quantile function of each Z_t , respectively; i.e.,

$$F_Z(z) = \mathbb{P}(Z_t \leq z), \quad F_Z^{-1}(\alpha) = \inf\{x \in \mathbb{R} : F_Z(x) \geq \alpha\}, \quad \alpha \in (0, 1), \quad t = 0, \dots, T. \quad (2.6)$$

For properties of the quantile function F_Z^{-1} we refer to Resnick (1987), Section 0.2, or Embrechts et al. (1997), Proposition A1.6.

We assume that F_Z is strictly increasing, thus F_Z^{-1} is continuous, and that the right endpoint of Z_t is infinite; i.e.,

$$x_F = \inf\{x \in \mathbb{R} : F_Z(x) = 1\} = \infty.$$

If necessary we identify $F_Z^{-1}(1)$ with $x_F = \infty$. Since Z has infinite right endpoint, and α is close to 1, $F_Z^{-1}(\alpha)$ is as a rule positive. We shall also need the quantile function of Z^2 and note that for Z symmetric, $F_{Z^2}^{-1}(\alpha) = F_Z^{-1}(\frac{1}{2}(\alpha + 1))^2$. Note further that $\alpha \leq \frac{1}{2}(\alpha + 1)$ for $\alpha \in (0, 1)$, hence (for $F_Z^{-1}(\alpha) > 1$)

$$F_Z^{-1}(\alpha) \leq F_Z^{-1}(\alpha)^2 \leq F_Z^{-1}(\frac{1}{2}(\alpha + 1))^2 = F_{Z^2}^{-1}(\alpha). \quad (2.7)$$

We summarize the assumptions which we will assume throughout the paper.

Assumptions A: We assume that F_Z is strictly increasing with support \mathbb{R} and that $F_Z^{-1}(\alpha) > 0$. For simplicity, we also assume that Z is symmetric.

Since we often work with distribution tails, we note that F_Z^{-1} can also be represented as

$$F_Z^{-1}(\alpha) = \inf\{x \in \mathbb{R} : P(Z_t > x) \leq 1 - \alpha\}, \quad \alpha \in (0, 1), \quad t = 0, \dots, T. \quad (2.8)$$

3 Conditional time-consistent Value-at-Risk

In this section we study Value-at-Risk (VaR) in the framework of dynamic time-consistent risk measures. One typically considers $L \in L^0(\mathcal{F}_T)$ which represents a possibly large loss position, for which the probability of L exceeding a loss threshold $m > 0$ should be bounded by a small probability $1 - \alpha$, i.e. α is typically close to 1. The smallest loss threshold m which satisfies this bound is the VaR^α . Several versions of the (conditional) VaR definition can be found in the literature. In analogy to (2.8) we work throughout with the following, which caters best to the purpose of the treatments in this paper.

Definition 3.1. Given a loss position $L \in L^0(\mathcal{F}_T)$ the Value-at-Risk at level $\alpha \in (0, 1)$ at time $t \in \{0, \dots, T\}$ for L is defined by

$$\text{VaR}_t^\alpha(L) := \text{essinf}\{m \in L^0(\mathcal{F}_t) : \mathbb{P}(L \leq m \mid \mathcal{F}_t) \geq \alpha\}, \quad (3.1)$$

3.1 Time-consistent VaR for single day losses

We start this section with the following example, which is the core object of interest in McNeil and Frey (2000).

Example 3.2. For $t = 0, \dots, T - 1$ let L_{t+1} be given by (2.5). Then $\text{VaR}_t^\alpha(L_{t+1})$ is the *1-day-ahead-VaR*, which can be computed straightforwardly as

$$\text{VaR}_t^\alpha(L_{t+1}) = \text{essinf}\{m \in L^0(\mathcal{F}_t) : \mathbb{P}(\sigma_{t+1}Z_{t+1} \leq m \mid \mathcal{F}_t) \geq \alpha\}.$$

Since σ_{t+1} is \mathcal{F}_t -measurable, we also have that $\tilde{m} := m/\sigma_{t+1}$ is \mathcal{F}_t -measurable. Using the independence between Z_{t+1} and \mathcal{F}_t , and also (2.8), we can continue

$$\begin{aligned} \text{VaR}_t^\alpha(L_{t+1}) &= \sigma_{t+1} \text{essinf}\{\tilde{m} \in L_+^0(\mathcal{F}_t) : \mathbb{P}(Z_{t+1} \leq \tilde{m} \mid \mathcal{F}_t) \geq \alpha\} \\ &= \sigma_{t+1} \inf\{\tilde{m} \in \mathbb{R}_+ : \mathbb{P}(Z_{t+1} \leq \tilde{m}) \geq \alpha\} \end{aligned}$$

$$= \sigma_{t+1} F_Z^{-1}(\alpha). \quad (3.2)$$

Moreover, since $L_t \in \mathcal{F}_t$,

$$\text{VaR}_t^\alpha(L_t + L_{t+1}) = L_t + \text{VaR}_t^\alpha(L_{t+1}) = L_t + \sigma_{t+1} F_Z^{-1}(\alpha). \quad (3.3)$$

□

There are examples showing that Value-at-Risk from Definition 3.1 is not time-consistent (e.g. Cheridito and Stajda (2009) or Föllmer and Schied (2011, Example 11.13)). As the GARCH(1,1) model (2.5) is defined by an iteration, one could hope that for this specific model VaR^α is time-consistent. However, this is not true and we provide a counterexample, which makes use of Proposition 2.3.

Example 3.3. In order to see why in the framework of GARCH(1,1) losses VaR cannot be time-consistent, recall that according to Proposition 2.3 VaR^α is time-consistent if and only if it satisfies the dynamic programming principle

$$\text{VaR}_t^\alpha = \text{VaR}_t^\alpha \circ \text{VaR}_{t+1}^\alpha, \quad t = 0, \dots, T-1.$$

For $t \in \{0, \dots, T-2\}$, by (3.2), we have $\text{VaR}_{t+1}^\alpha(L_{t+2}) = \sigma_{t+2} F_Z^{-1}(\alpha)$ and, hence,

$$\text{VaR}_t^\alpha(\text{VaR}_{t+1}^\alpha(L_{t+2})) = \text{VaR}_t^\alpha(\sigma_{t+2} F_Z^{-1}(\alpha)).$$

We compute $\text{VaR}_t^\alpha(L_{t+2})$ and $\text{VaR}_t^\alpha(\sigma_{t+2} F_Z^{-1}(\alpha))$ for the GARCH(1,1) model:

$$\begin{aligned} \text{VaR}_t^\alpha(\sigma_{t+2} F_Z^{-1}(\alpha)) &= m^* = \text{essinf}\{m \in L^0(\mathcal{F}_t) : \mathbb{P}(\sigma_{t+2} F_Z^{-1}(\alpha) \leq m \mid \mathcal{F}_t) \geq \alpha\} \\ &= \text{essinf}\{m \in L^0(\mathcal{F}_t) : \mathbb{P}(\sqrt{a_0 + \sigma_{t+1}^2 (a_1 Z_{t+1}^2 + b)} F_Z^{-1}(\alpha) \leq m \mid \mathcal{F}_t) \geq \alpha\}. \end{aligned}$$

Since the function $\sqrt{(a_0 + \sigma_{t+1}^2 (a_1 Z_{t+1}^2 + b)) F_Z^{-1}(\alpha)^2}$ is strictly increasing in Z_{t+1}^2 and \mathcal{F}_t -measurable, we obtain

$$m^* = \sqrt{(a_0 + \sigma_{t+1}^2 (a_1 F_Z^{-1}(\alpha) + b)) F_Z^{-1}(\alpha)}. \quad (3.4)$$

Next we compute

$$\begin{aligned} \text{VaR}_t^\alpha(L_{t+2}) &= m^{**} = \text{essinf}\{m \in L^0(\mathcal{F}_t) : \mathbb{P}(\sigma_{t+2} Z_{t+2} \leq m \mid \mathcal{F}_t) \geq \alpha\} \\ &= \text{essinf}\{m \in L^0(\mathcal{F}_t) : \mathbb{P}(\sqrt{a_0 + \sigma_{t+1}^2 (a_1 Z_{t+1}^2 + b)} Z_{t+2} \leq m \mid \mathcal{F}_t) \geq \alpha\}. \end{aligned}$$

Now assume that $m^{**} = m^*$ for all $\alpha \in (0, 1)$. We denote by \mathbb{P}_t the conditional probability with respect to \mathcal{F}_t and calculate

$$\begin{aligned} \alpha &= \mathbb{P}_t(\sqrt{a_0 + \sigma_{t+1}^2 (a_1 Z_{t+1}^2 + b)} \leq \sqrt{a_0 + \sigma_{t+1}^2 (a_1 F_Z^{-1}(\alpha) + b)}) \\ &= \mathbb{P}_t(\sqrt{a_0 + \sigma_{t+1}^2 (a_1 Z_{t+1}^2 + b)} Z_{t+2} \leq \sqrt{a_0 + \sigma_{t+1}^2 (a_1 F_Z^{-1}(\alpha) + b)} F_Z^{-1}(\alpha)) \\ &= 2 \int_0^\infty \mathbb{P}_t(\sqrt{a_0 + \sigma_{t+1}^2 (a_1 Z_{t+1}^2 + b)} z \leq \sqrt{a_0 + \sigma_{t+1}^2 (a_1 F_Z^{-1}(\alpha) + b)} F_Z^{-1}(\alpha)) dF_Z(z) - 1 \\ &= 2 \int_0^{F_Z^{-1}(\alpha)} \mathbb{P}_t(\sqrt{a_0 + \sigma_{t+1}^2 (a_1 Z_{t+1}^2 + b)} z \leq \sqrt{a_0 + \sigma_{t+1}^2 (a_1 F_Z^{-1}(\alpha) + b)} F_Z^{-1}(\alpha)) dF_Z(z) \\ &\quad + 2 \int_{F_Z^{-1}(\alpha)}^\infty \mathbb{P}_t(\sqrt{a_0 + \sigma_{t+1}^2 (a_1 Z_{t+1}^2 + b)} z \leq \sqrt{a_0 + \sigma_{t+1}^2 (a_1 F_Z^{-1}(\alpha) + b)} F_Z^{-1}(\alpha)) dF_Z(z) - 1 \end{aligned}$$

Now note for the first integral that $F_Z^{-1}(\alpha)/z$ decreases in z to 0 and has minimum 1 over the integral range. This implies for the probability under the integral, that the left-hand random variable scaled by z decreases with z to α . Moreover, since the support of Z_{t+1} has infinite right endpoint, the second integral is positive. Hence, we estimate the right-hand side by

$$\geq 2\alpha^2 + 2a - 1,$$

where $a > 0$. However, for α close to 1 we have $\alpha + 1 < 2\alpha^2 + 2a$. \square

Cheridito and Stadje (2009) propose to amend the time-inconsistency of VaR using the backward iteration (2.2). This gives rise to the following definition.

Definition 3.4. *Given a loss position $L \in L^0(\mathcal{F}_T)$ and VaR^α from Definition 3.1. Then the time-consistent Value-at-Risk at level $\alpha \in (0, 1)$ for L is defined by*

$$\widetilde{\text{VaR}}_T^\alpha(L) := \text{VaR}_T^\alpha(L) = L, \quad \widetilde{\text{VaR}}_t^\alpha(L) := \text{VaR}_t^\alpha(\widetilde{\text{VaR}}_{t+1}^\alpha(L)), \quad t = 0, \dots, T-1. \quad (3.5)$$

In the notation of the construction from the recursion (2.2), this corresponds to $\rho_t := \text{VaR}_t^\alpha$ and $\phi_t := \widetilde{\text{VaR}}_t^\alpha$. As a consequence of the construction of $(\widetilde{\text{VaR}}_T^\alpha(X))_{t=0}^T$ we find for $L \in L^0(\mathcal{F}_T)$

$$\widetilde{\text{VaR}}_t^\alpha(L) = \left(\text{VaR}_t^\alpha \circ \text{VaR}_{t+1}^\alpha \circ \dots \circ \text{VaR}_{T-1}^\alpha \right)(L). \quad (3.6)$$

The choice of the GARCH(1,1) model (2.5) entails the convenient feature that the m -day ahead VaR assessment allows for a closed form solution. More precisely, we can derive an analytical solution for the time t risk assessment of the GARCH(1,1) loss at terminal time T as follows (as usual we set $\sum_{k=0}^{-1} a_n = 0$). The proof is given in Appendix A.

Theorem 3.5. *Let $(L_t)_{t=0}^T$ be the loss process given by the GARCH(1,1) model (2.5). Then we have*

$$\widetilde{\text{VaR}}_t^\alpha(L_T) = F_Z^{-1}(\alpha) \sqrt{\mathcal{P}_t^T(a_1 F_Z^{-1}(\alpha) + b)}, \quad t = 0, \dots, T-1, \quad (3.7)$$

where $\mathcal{P}_t^T : \mathbb{R} \rightarrow \mathbb{R}$ is an \mathcal{F}_t -measurable mapping given by

$$\mathcal{P}_t^T(x) := a_0 \sum_{k=0}^{T-t-2} x^k + \sigma_{t+1}^2 x^{T-t-1}, \quad t = 0, \dots, T-1. \quad (3.8)$$

3.2 Time-consistent VaR for aggregated losses

We now come to the computation of the m -day-ahead VaR^α . So far, we have considered risk positions at a fixed day T that is ahead of time $t < T$ up to which information in the form of the filtration \mathcal{F}_t is available. The m -day-ahead VaR^α is a risk assessment of aggregated losses L_{t+k} from the time period $[t+1, t+m]$ for $t+m \leq T$. Next we show that $\widetilde{\text{VaR}}^\alpha$ linearizes across the aggregation of GARCH(1,1) losses.

Proposition 3.6. *Let $(L_t)_{t=0}^T$ be the loss process given by the GARCH(1,1) model (2.5), then we have for fixed $t \in \{1, \dots, T-1\}$ and $m \in \mathbb{N}$ such that $t+m \leq T$,*

$$\widetilde{\text{VaR}}_t^\alpha \left(\sum_{k=1}^m L_{t+k} \right) = F_Z^{-1}(\alpha) \sum_{k=1}^m \sqrt{\mathcal{P}_t^{t+k}(a_1 F_Z^{-1}(\alpha) + b)} = \sum_{k=1}^m \widetilde{\text{VaR}}_t^\alpha(L_{t+k}). \quad (3.9)$$

Proof. First note that for $m = 2$ we know from (3.3) that

$$\begin{aligned}\widetilde{\text{VaR}}_t^\alpha(L_{t+1} + L_{t+2}) &= \text{VaR}_t^\alpha(L_{t+1} + \text{VaR}_{t+1}^\alpha(L_{t+2})) \\ &= \text{VaR}_t^\alpha(\sigma_{t+1}Z_{t+1} + \sigma_{t+2}F_Z^{-1}(\alpha)),\end{aligned}$$

where the second line follows from (3.2). By (2.5) we have $\sigma_{t+2}^2 = a_0 + \sigma_{t+1}^2(a_1Z_{t+1}^2 + b)$ which transforms the last equation into

$$\widetilde{\text{VaR}}_t^\alpha(L_{t+1} + L_{t+2}) = \text{VaR}_t^\alpha(\sigma_{t+1}Z_{t+1} + F_Z^{-1}(\alpha)\sqrt{a_0 + \sigma_{t+1}^2(a_1Z_{t+1}^2 + b)}).$$

By the definition of VaR_t^α and the fact that $\sigma_{t+1}Z_{t+1} + F_Z^{-1}(\alpha)\sqrt{a_0 + \sigma_{t+1}^2(a_1Z_{t+1}^2 + b)}$ is a strictly increasing \mathcal{F}_t -measurable function of Z_{t+1}^2 , we find that

$$\widetilde{\text{VaR}}_t^\alpha(L_{t+1} + L_{t+2}) = \sigma_{t+1}F_Z^{-1}(\alpha) + F_Z^{-1}(\alpha)\sqrt{a_0 + \sigma_{t+1}^2(a_1F_Z^{-1}(\alpha)^2 + b)}, \quad (3.10)$$

which is equal to the sum $\widetilde{\text{VaR}}_t^\alpha(L_{t+1}) + \widetilde{\text{VaR}}_t^\alpha(L_{t+2})$ and also equal to the corresponding sum in the center. We proceed by induction and assume that (3.9) is true for $\sum_{k=1}^{m-1} L_{t+k}$. Since the sum is \mathcal{F}_{t+m} -measurable, we obtain by (3.6)

$$\begin{aligned}\widetilde{\text{VaR}}_t^\alpha\left(\sum_{k=1}^m L_{t+k}\right) &= \left(\text{VaR}_t^\alpha \circ \text{VaR}_{t+1}^\alpha \circ \cdots \circ \text{VaR}_{t+m-1}^\alpha\right)\left(\sum_{k=1}^{m-1} L_{t+k} + L_{t+m}\right) \\ &= \left(\text{VaR}_t^\alpha \circ \text{VaR}_{t+1}^\alpha \circ \cdots \circ \text{VaR}_{t+m-2}^\alpha\right)\left(\sum_{k=1}^{m-1} L_{t+k} + \text{VaR}_{t+m-1}^\alpha(L_{t+m})\right) \\ &= \text{VaR}_t^\alpha(L_{t+1} + \text{VaR}_{t+1}^\alpha(L_{t+2} + \cdots + \text{VaR}_{t+m-2}^\alpha(L_{t+m-1}) + \text{VaR}_{t+m-1}^\alpha(L_{t+m}))) \\ &= \text{VaR}_t^\alpha\left(L_{t+1} + \widetilde{\text{VaR}}_{t+1}^\alpha\left(\sum_{k=2}^m L_{t+k}\right)\right) = \text{VaR}_t^\alpha\left(L_{t+1} + \sum_{k=2}^m \widetilde{\text{VaR}}_{t+1}^\alpha(L_{t+k})\right)\end{aligned}$$

where the last identity follows by the induction hypothesis, which also implies

$$\begin{aligned}\sum_{k=2}^m \widetilde{\text{VaR}}_{t+1}^\alpha(L_{t+k}) &= F_Z^{-1}(\alpha) \sum_{k=2}^m \sqrt{\mathcal{P}_{t+1}^{t+k}(a_1F_Z^{-1}(\alpha) + b)} \\ &= F_Z^{-1}(\alpha) \sum_{k=2}^m \sqrt{a_0 \sum_{j=0}^{k-3} (a_1F_Z^{-1}(\alpha) + b)^j + \sigma_{t+2}^2(a_1F_Z^{-1}(\alpha) + b)^{k-2}}.\end{aligned}$$

We use $\sigma_{t+2}^2 = a_0 + \sigma_{t+1}^2(a_1Z_{t+1}^2 + b)$ from (2.5) and observe that

$$\begin{aligned}\sum_{k=2}^m \sqrt{a_0 \sum_{j=0}^{k-3} (a_1F_Z^{-1}(\alpha) + b)^j + (a_0 + \sigma_{t+1}^2(a_1Z_{t+1}^2 + b))(a_1F_Z^{-1}(\alpha) + b)^{k-2}} \\ = \sum_{k=2}^m \sqrt{a_0 \sum_{j=0}^{k-2} (a_1F_Z^{-1}(\alpha) + b)^j + (\sigma_{t+1}^2(a_1Z_{t+1}^2 + b))(a_1F_Z^{-1}(\alpha) + b)^{k-2}}\end{aligned}$$

is a strictly increasing function in Z_{t+1}^2 . Hence we can proceed by the same argument as in the pretext leading to (3.10) to achieve ultimately

$$\text{VaR}_t^\alpha\left(L_{t+1} + \sum_{k=2}^m \widetilde{\text{VaR}}_{t+1}^\alpha(L_{t+k})\right)$$

$$\begin{aligned}
&= \sigma_{t+1} F_Z^{-1}(\alpha) + F_Z^{-1}(\alpha) \sum_{k=2}^m \sqrt{a_0 \sum_{j=0}^{k-2} (a_1 F_{Z^2}^{-1}(\alpha) + b)^j + \sigma_{t+1}^2 (a_1 F_{Z^2}^{-1}(\alpha) + b)^{k-1}} \\
&= \sigma_{t+1} F_Z^{-1}(\alpha) + F_Z^{-1}(\alpha) \sum_{k=2}^m \sqrt{\mathcal{P}_t^{t+k} (a_1 F_{Z^2}^{-1}(\alpha) + b)} \\
&= F_Z^{-1}(\alpha) \sum_{k=1}^m \sqrt{\mathcal{P}_t^{t+k} (a_1 F_{Z^2}^{-1}(\alpha) + b)} \\
&= \sum_{k=1}^m \widetilde{\text{VaR}}_t^\alpha(L_{t+k}).
\end{aligned}$$

This finishes the proof. \square

4 Conditional time-consistent Average Value-at-Risk

This section is devoted to the study of time-consistent alternatives for the Average Value-at-Risk (AVaR). Due to coherence AVaR is commonly considered as a more reasonable rectification of VaR. For more details we refer to Föllmer and Schied (2011, Chapter 4). For $t \in \{0, \dots, T\}$ we define $L^1(\mathcal{F}_t)$ as the set of all \mathbb{P}_t -integrable losses, where \mathbb{P}_t denotes the conditional probability with respect to \mathcal{F}_t and \mathbb{E}_t the corresponding conditional expectation. The following definition relates AVaR to VaR.

Definition 4.1. *Given a loss position $L \in L^1(\mathcal{F}_T)$ the Average Value-at-Risk at level $\alpha \in (0, 1)$ at time $t \in \{0, \dots, T\}$ is given by*

$$\text{AVaR}_t^\alpha(L) = \frac{1}{1-\alpha} \int_\alpha^1 \text{VaR}_t^u(L) du. \quad (4.1)$$

with $\text{VaR}_t^\alpha(L)$ as in Definition 3.1.

Whereas VaR quantifies the risk associated to one particular level of risk, reflected in the choice of α , AVaR as an integrated VaR takes into account VaR at the entire bandwidth of risk levels between α and 1 and thus better reflects volume of extreme risks that VaR might neglect.

The following is the analog of a fact well-known for unconditional AVaR (e.g. Lemma 2.16 of McNeil et al. (2005)).

Remark 4.2. If the loss position $L \in L^1(\mathcal{F}_T)$ has a continuous distribution function, then

$$\text{AVaR}_t^\alpha(L) = \mathbb{E}_t[L \mid L > \text{VaR}_t^\alpha(L)], \quad t = 0, \dots, T. \quad (4.2)$$

Due to (4.2), AVaR is often also referred to as *conditional VaR* or *Expected Shortfall*. \square

Assumption B: Additionally to Assumptions A we require from now on also that Z has a continuous distribution function.

4.1 Time-consistent AVaR for single day losses

We focus again on the GARCH(1, 1) model from (2.5).

Example 4.3. Assume the setting as in Example 3.2. For $t = 0, \dots, T-1$ let L_{t+1} be given by (2.5). Then $\text{AVaR}_t^\alpha(L_{t+1})$ is the *1-day-ahead-AVaR*. If the innovations $(Z_t)_{t \geq 0}$ have a continuous

distribution function F_Z , then by linearity of the conditional expectation and \mathcal{F}_t -measurability of σ_{t+1} we get for $t = 0, \dots, T-1$,

$$\begin{aligned} \text{AVaR}_t^\alpha(L_{t+1}) &= \sigma_{t+1} \mathbb{E}_t[Z_{t+1} \mid Z_{t+1} > F_Z^{-1}(\alpha)] \\ &= \sigma_{t+1} \frac{1}{1-\alpha} \int_{F_Z^{-1}(\alpha)}^{\infty} y dF_Z(y) \\ &= \sigma_{t+1} \text{AVaR}^\alpha(Z). \end{aligned}$$

This calculation can also be found in McNeil and Frey (2000). □

In analogy to Section 3, a time-consistent version of AVaR is constructed as follows.

Definition 4.4. *Given a loss position $L \in L^1(\mathcal{F}_T)$ and the $\text{AVaR}_t^\alpha(L)$ as in Definition 4.1. Then the time-consistent Average Value-at-Risk at level $\alpha \in (0, 1)$ for L is defined by*

$$\widetilde{\text{AVaR}}_T^\alpha(L) := L, \quad \widetilde{\text{AVaR}}_t^\alpha(L) := \text{AVaR}_t^\alpha(\widetilde{\text{AVaR}}_{t+1}^\alpha(L)), \quad t = 0, \dots, T-1. \quad (4.3)$$

For the Average Value-at-Risk of the squared loss L_T^2 at time T we can derive an explicit formula similar to (3.7). Note that, though AVaR of L_T^2 allows for an interpretation as the conditional volatility at time T , our purpose of investigation is to employ AVaR of L_T^2 to derive pragmatic bounds to AVaR itself, see Section 4.2 below.

We start with a result analog to Theorem 3.5, and recall that $\sum_{k=0}^{-1} a_k = 0$. The proof is given in Appendix B.

Theorem 4.5. *Let $(L_t)_{t=0}^T$ be given by the GARCH(1,1) model (2.5). Then we have for the squared loss $L_T^2 \in L^1(\mathcal{F}_T)$ at terminal time T*

$$\widetilde{\text{AVaR}}_t^\alpha(L_T^2) = \frac{1}{1-\alpha} \int_\alpha^1 F_{Z^2}^{-1}(u) du \mathcal{P}_t^T \left(a_1 \frac{1}{1-\alpha} \int_\alpha^1 F_{Z^2}^{-1}(u) du + b \right), \quad t = 0, \dots, T-1, \quad (4.4)$$

where $\mathcal{P}_t^T : \mathbb{R} \rightarrow \mathbb{R}$ is an \mathcal{F}_t -measurable mapping given by

$$\mathcal{P}_t^T(x) = a_0 \sum_{k=0}^{T-t-2} x^k + \sigma_{t+1}^2 x^{T-t-1}, \quad t = 0, \dots, T-1.$$

It is also possible to derive expressions for m -day ahead $\widetilde{\text{AVaR}}^\alpha$. As usual we define $\prod_{j=1}^0 a_j = 1$.

Proposition 4.6. *Let $(L_t)_{t=0}^T$ be given by the GARCH(1,1) model (2.5). For $t > 0$ define $\mathcal{Q}_t^{t+1} = \sigma_{t+1}$ and for fixed $m \geq 2$*

$$\mathcal{Q}_t^{t+m}(z_1, z_2, \dots, z_{m-1}) := a_0 \sum_{k=0}^{m-2} \prod_{j=1}^k (a_1 z_j + b) + \sigma_{t+1}^2 \prod_{j=1}^{m-1} (a_1 z_k + b). \quad (4.5)$$

Then $\widetilde{\text{AVaR}}_t^\alpha(L_{t+m}) = \text{AVaR}^\alpha(Z) \sigma_{t+1}$ and for fixed $m \geq 2$

$$\widetilde{\text{AVaR}}_t^\alpha(L_{t+m}) = \frac{\text{AVaR}^\alpha(Z)}{(1-\alpha)^{m-1}} \underbrace{\int_{F_Z^{-1}(\alpha)}^{\infty} \dots \int_{F_Z^{-1}(\alpha)}^{\infty}}_{(m-1)\text{-times}} \sqrt{\mathcal{Q}_t^{t+m}(z_1, \dots, z_{m-1})} dF_{Z^2}(z_1) \dots dF_{Z^2}(z_{m-1}). \quad (4.6)$$

Proof. For $m = 1$ note that by (4.1) $\widetilde{\text{AVaR}}_t^\alpha(L_{t+1}) = \text{AVaR}_t^\alpha(L_{t+1}) = \sigma_{t+1} \frac{1}{1-\alpha} \int_{F_Z^{-1}(\alpha)}^\infty y d(y)$ as in Example 4.3. For simplicity we set $\bar{\kappa} = \frac{1}{1-\alpha} \int_{F_Z^{-1}(\alpha)}^\infty y d(y)$. For $\widetilde{\text{AVaR}}_t^\alpha(L_{t+2})$ we use this and then Lemma B.1 and compute for $m = 2$

$$\widetilde{\text{AVaR}}_t^\alpha(L_{t+2}) = \text{AVaR}_t^\alpha(\widetilde{\text{AVaR}}_{t+1}^\alpha(L_{t+2})) = \text{AVaR}_t^\alpha(\text{AVaR}_{t+1}^\alpha(L_{t+2})) \quad (4.7)$$

$$= \text{AVaR}_t^\alpha(\bar{\kappa} \sigma_{t+2}) \quad (4.8)$$

$$= \bar{\kappa} \text{AVaR}_t^\alpha(\sigma_{t+2}) \quad (4.9)$$

$$= \bar{\kappa} \mathbb{E}_t[\sigma_{t+2} \mid \sigma_{t+2} > \text{VaR}_{t+1}^\alpha(\sigma_{t+2})]$$

$$= \bar{\kappa} \mathbb{E}_t[\sigma_{t+2} \mid Z_{t+1}^2 > F_{Z^2}^{-1}(\alpha)]$$

$$= \frac{\bar{\kappa}}{1-\alpha} \mathbb{E}_t[\sqrt{a_0 + \sigma_{t+1}^2 (a_1 Z_{t+1}^2 + b)} \mathbb{1}_{\{Z_{t+1}^2 > F_{Z^2}^{-1}(\alpha)\}}]$$

$$= \frac{\bar{\kappa}}{1-\alpha} \int_{F_Z^{-1}(\alpha)}^\infty \sqrt{a_0 + \sigma_{t+1}^2 (a_1 z + b)} dF_{Z^2}(z),$$

since σ_{t+1} is \mathcal{F}_t -measurable. Assume that (4.6) holds for $L_{t+2}, \dots, L_{t+m-1}$. Then

$$\begin{aligned} \widetilde{\text{AVaR}}_t^\alpha(L_{t+m}) &= \text{AVaR}_t^\alpha(\widetilde{\text{AVaR}}_{t+1}^\alpha(L_{t+m})) \\ &= \text{AVaR}_t^\alpha\left(\frac{\bar{\kappa}}{(1-\alpha)^{m-2}} \underbrace{\int_{F_Z^{-1}(\alpha)}^\infty \cdots \int_{F_Z^{-1}(\alpha)}^\infty}_{(m-2)\text{-times}} \sqrt{\mathcal{Q}_{t+1}^{(t+1)+(m-1)}(z_1, \dots, z_{m-2})} dF_{Z^2}(z_1) \cdots dF_{Z^2}(z_{m-2})\right), \end{aligned}$$

where

$$\mathcal{Q}_{t+1}^{(t+1)+(m-1)}(z_1, z_2, \dots, z_{m-2}) := a_0 \sum_{k=0}^{m-3} \prod_{j=1}^k (a_1 z_j + b) + \sigma_{t+2}^2 \prod_{j=1}^{m-2} (a_1 z_k + b).$$

Setting $\sigma_{t+2}^2 = a_0 + \sigma_{t+1}^2 (a_1 Z_{t+1}^2 + b)$, then since σ_{t+1}^2 is \mathcal{F}_t -measurable, factorization of Z_{t+1}^2 gives another integral and another factor $1 - \alpha$ in the denominator. \square

4.2 Almost sure bounds for AVaR

Finding an analytical expression for $\widetilde{\text{AVaR}}_t^\alpha(L_T)$ for the (unsquared) GARCH(1,1) loss is not straightforward. It is however possible to derive closed form bounds for $\widetilde{\text{AVaR}}_t^\alpha(L_T)$.

4.2.1 AVaR-bounds for single day losses

We now derive a closed form upper bound to $\widetilde{\text{AVaR}}^\alpha$ which arises from an application of Jensen's inequality. For the proof of the following Proposition we refer to Appendix B.

Proposition 4.7. *Let L_T be the loss position at time $T > 0$ given by the GARCH(1, 1) model (2.5). Then*

$$\widetilde{\text{AVaR}}_t^\alpha(L_T) := \frac{1}{1-\alpha} \int_\alpha^1 F_Z^{-1}(y) dy \sqrt{\mathcal{P}_t^T\left(a_1 \frac{1}{1-\alpha} \int_\alpha^1 F_{Z^2}^{-1}(u) du + b\right)}, \quad t = 0, \dots, T-1, \quad (4.10)$$

where $\mathcal{P}_t^T(\cdot)$ is given by (3.8), satisfies

$$\widetilde{\text{AVaR}}_t^\alpha \leq \overline{\text{AVaR}}_t^\alpha \quad t = 0, \dots, T-1.$$

An easy alteration of the proof of the previous result yields a closed form lower bound $\underline{\text{AVaR}}^\alpha$ for $\widetilde{\text{AVaR}}^\alpha$.

Proposition 4.8. Let L_T be the loss position at time $T > 0$ given by the GARCH(1,1) model (2.5). Then

$$\underline{\text{AVaR}}_t^\alpha(L_T) := \frac{1}{1-\alpha} \int_\alpha^1 F_Z^{-1}(u) du \left(\frac{1}{1-\alpha} \int_\alpha^1 \sqrt{a_1 F_Z^{-1}(y) + b} dy \right)^{T-t-1} \sigma_{t+1}, \quad t = 0, \dots, T-1, \quad (4.11)$$

satisfies

$$\overline{\text{AVaR}}_t^\alpha(L_T) \geq \underline{\text{AVaR}}_t^\alpha(L_T) \quad t = 0, \dots, T-1.$$

4.2.2 AVaR-bounds for aggregated losses

Unfortunately, for AVaR there exists no result corresponding to Proposition 3.6, hence AVaR does not linearize across aggregation of GARCH losses. A key obstacle is that Lemma B.1 does not apply. However, due to the subadditivity of AVaR and the property (2.4), we can derive an upper bound for the aggregation of GARCH losses.

Proposition 4.9. Let $(L_t)_{t=0}^T$ be given by the GARCH(1,1) model (2.5). Then, for $t \in \{1, \dots, T-1\}$ and $m \in \mathbb{N}$ such that $t+m \leq T$, the m -day-ahead $\overline{\text{AVaR}}_t^\alpha$ of aggregated losses $\sum_{k=1}^m L_{t+k}$ is bounded by

$$\overline{\text{AVaR}}_t^\alpha \left(\sum_{k=1}^m L_{t+m+1} \right) \leq \sum_{k=1}^m \overline{\text{AVaR}}_t^\alpha(L_{t+m+1}). \quad (4.12)$$

Proof. By the subadditivity property (2.4) $\overline{\text{AVaR}}_t^\alpha \left(\sum_{k=1}^m L_{t+m+1} \right)$ satisfies

$$\overline{\text{AVaR}}_t^\alpha \left(\sum_{k=1}^m L_{t+m+1} \right) \leq \sum_{k=1}^m \overline{\text{AVaR}}_t^\alpha(L_{t+m+1}).$$

Now the assertion follows from an application of Proposition 4.7. □

Remark 4.10. Due to the lack of linearization across aggregation of GARCH losses, the aggregation of the m -ahead AVaR bounds from Proposition 4.8 do not produce a proper lower bound for $\overline{\text{AVaR}}_t^\alpha \left(\sum_{k=1}^m L_{t+m+1} \right)$. Whereas in case of the single m -day-ahead $\underline{\text{AVaR}}_t^\alpha(L_T)$ is a true lower bound to $\overline{\text{AVaR}}_t^\alpha(L_T)$, their aggregation $\sum_{k=1}^m \underline{\text{AVaR}}_t^\alpha(L_{t+m+1})$ is rather a lower bound to the upper bound $\sum_{k=1}^m \overline{\text{AVaR}}_t^\alpha(L_{t+m+1})$. It can happen that $\sum_{k=1}^m \underline{\text{AVaR}}_t^\alpha(L_{t+m+1})$ is either an upper bound or a lower bound for $\sum_{k=1}^m \overline{\text{AVaR}}_t^\alpha(L_{t+m+1})$. Nevertheless, we will use $\sum_{k=1}^m \underline{\text{AVaR}}_t^\alpha(L_{t+m+1})$ as a weak lower bound in our numerical experiments to get an orientation about how much the upper bound $\sum_{k=1}^m \overline{\text{AVaR}}_t^\alpha(L_{t+m+1})$ is tailing off.

5 Extreme value theory based quantile estimation

5.1 Generalized Pareto Distribution

Up to now we have not fixed the noise distribution, only assumed certain properties like infinite right endpoint or continuity of the distribution function. Throughout we worked with α close to 1 corresponding to the noise distribution function to be close to 1. Thus it is sufficient to specify the distribution function above some high threshold u . This is a typical assumption in extreme value theory, and we will apply the Peaks-over-Threshold method (as in McNeil and Frey (2000)). We first explain the setting in general.

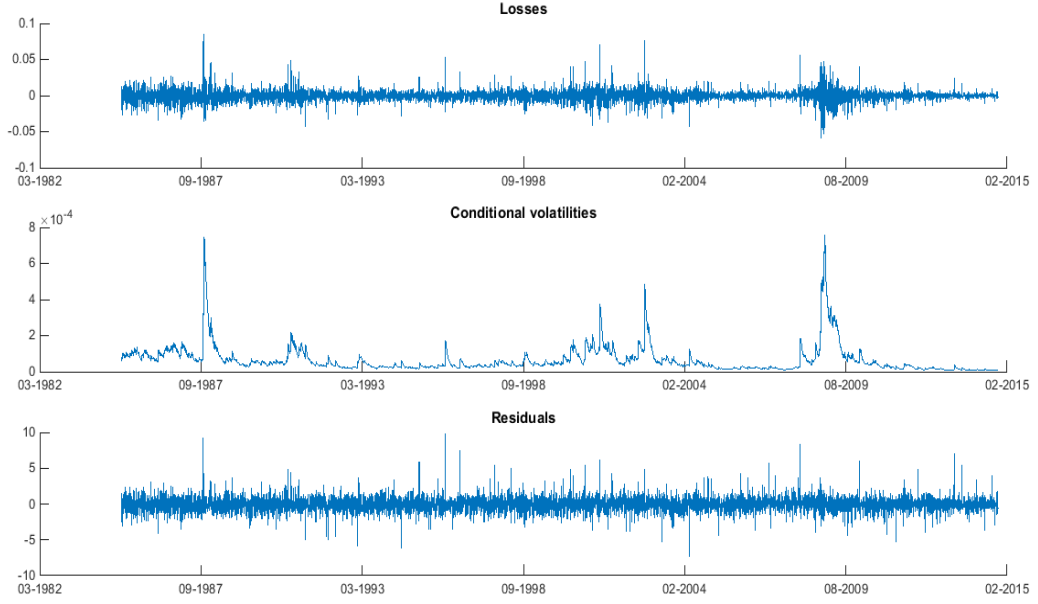


Figure 1: Motorola stock price analysis: loss data (top), conditional variances after fitting a GARCH(1,1) model (middle), and residuals of loss data (bottom).

The *Generalized Pareto distribution (GPD)* is given by

$$G_{\xi,\beta}(x) = \begin{cases} 1 - \left(1 + \frac{\xi}{\beta}x\right)^{-1/\xi}, & \xi \neq 0, \\ 1 - \exp\left(-\frac{x}{\beta}\right), & \xi = 0, \end{cases} \quad (5.1)$$

where $\beta > 0$ and $\xi \in \mathbb{R}$. If $\xi > 0$ (5.1) is defined for $x \geq 0$ and if $\xi < 0$ (5.1) is defined on $x \in [0, -\beta/\xi]$, see e.g. Section 3.4 in Embrechts et al. (1997). Assume that we fix some high threshold $u > 0$. Given a random variable X with distribution function F and right endpoint x_F , its associated *excess distribution function* is defined as

$$F_u(y) = \mathbb{P}(X - u \leq y \mid X > u) = \frac{F(y+u) - F(u)}{1 - F(u)}, \quad 0 \leq y < x_F - u. \quad (5.2)$$

The strength of the GPD is compressed in a result by Pickands (1975) and Balkema and de Haan (1974) which classifies the GPD as the limit distribution of a large class of excess distributions. More precisely, under mild conditions there exists a measurable non-negative parameter $\beta = \beta(u)$ such that

$$\lim_{u \rightarrow x_F} \sup_{0 \leq x \leq x_F - u} |F_u(x) - G_{\xi,\beta(u)}| = 0$$

holds, see Theorem 3.4.13 in Embrechts et al. (1997) for a rigorous statement of this result. The density of (5.1) is given by

$$g_{\xi,\beta}(x) = \begin{cases} \frac{1}{\beta} \left(1 + \frac{\xi}{\beta}x\right)^{-1/\xi-1}, & \xi \neq 0, \\ \frac{1}{\beta} \exp\left(-\frac{x}{\beta}\right), & \xi = 0. \end{cases} \quad (5.3)$$

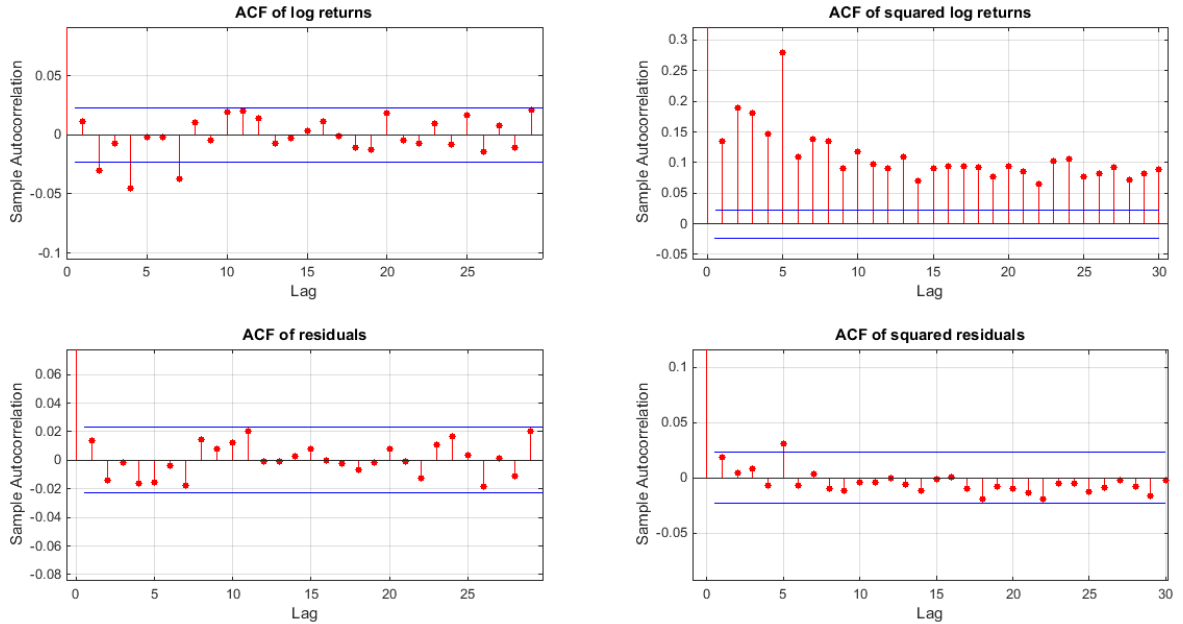


Figure 2: Motorola stock price analysis: sample autocorrelation functions for loss data (top) and residuals after fitting a GARCH(1,1) model (bottom).

Under the assumption that Z has the distribution function F_Z , which for some high enough threshold $u > 0$ satisfies $F_u(x) = G_{\xi, \beta}(x)$ for $0 \leq x \leq x_F - u$ and for some $\xi \in \mathbb{R}$ and $\beta > 0$, we find for $\alpha \geq F(u)$ (for $\xi = 0$ we interpret this quantile as the quantile of the corresponding exponential distribution)

$$F_Z^{-1}(\alpha) = u + \frac{\beta}{\xi} \left(\left(\frac{1 - \alpha}{1 - F(u)} \right)^{-\xi} - 1 \right), \quad (5.4)$$

$$\frac{1}{1 - \alpha} \int_{\alpha}^1 F_Z^{-1}(y) dy = \frac{F_Z^{-1}(\alpha)}{1 - \xi} + \frac{\beta - \xi u}{1 - \xi}. \quad (5.5)$$

By (2.7) we obtain

$$F_{Z^2}^{-1}(\alpha) = F_Z^{-1} \left(\frac{1}{2}(\alpha + 1) \right)^2 = \left(u + \frac{\beta}{\xi} \left(\left(\frac{\frac{1}{2}(1 - \alpha)}{1 - F(u)} \right)^{-\xi} - 1 \right) \right)^2.$$

Unfortunately, there is no explicit expression for $\frac{1}{\alpha - 1} \int_{\alpha}^1 F_{Z^2}^{-1}(y) dy$.

5.2 Statistical model fitting

In this section we apply the theory and formulas derived previously to a data set. We choose the historical daily closing prices of the Motorola stock from 1st March 1985 until 15th October 2014 as this data set provides several canonical features of financial time series. We transform prices into negative log-returns; i.e., into losses, and fit the GARCH(1,1) parameters using Quasi Maximum Likelihood Estimation (QMLE) (e.g. Franq and Zakoian (2010), Chapter 7). The parameter estimates can be found in Table 1, and the outcome is depicted in Figure 1.

We see in the middle plot of Figure 1 major clustering of volatility in October 1987 (Black Monday), in a pronounced period between 2000 until 2002 (Dot-com bubble and wake of 9/11 attacks) and in a longer lasting period following the financial crisis between 2008 until 2010.

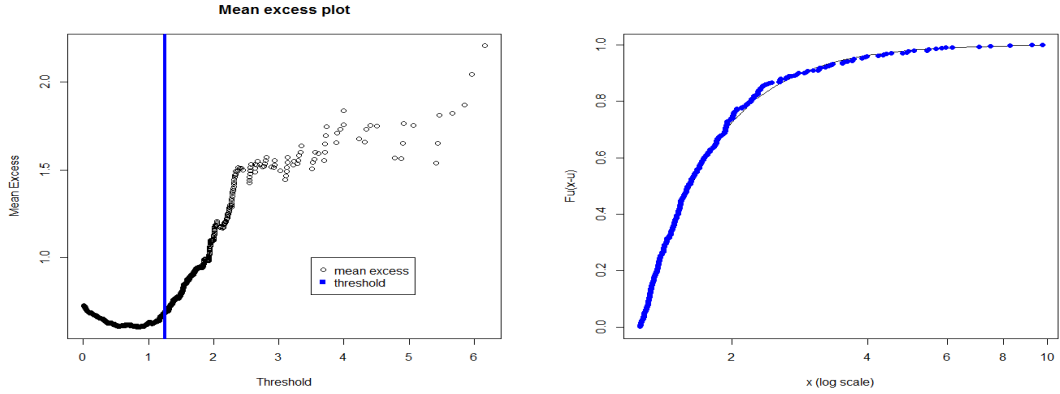


Figure 3: Fit of the Generalized Pareto Distribution. Left: mean excess plot of the positive residuals, and QQ-plot of the threshold exceeding residuals against the fitted GPD. Right: excess distribution $F_u(x-u)$ from the fitted GPD model (solid line) against the empirical estimates of excess probabilities (dotted points).

Parameter	Value	Standard error
\hat{a}_0	2e-07	1.09e-07
\hat{a}_1	0.0451	0.0014
\hat{b}	0.9531	0.0013

Table 1: Estimated GARCH(1,1) parameters by QMLE.

In a next step we examine the sample autocorrelation functions of the loss data and the residuals after fitting a GARCH(1,1) model. In Figure 2 the bottom plots depict the acf of the residuals and the squared residuals and is supportive for our assumption of i.i.d. GARCH residuals Z_t . This is also reflected in several runs of the Ljung-Box for various lags for the residuals. The residuals also pass the augmented Dickey-Fuller and the KPSS stationarity tests.

As explained in Section 5.1 we fit a GPD to the upper tail of the residuals. We first have to choose a high enough threshold value u and we choose it as the approximate 92% quantile of the residuals. This is supported by studying the mean excess plot of the nonnegative residuals in Figure 3: the 92% quantile of the residuals (solid blue line) yields a threshold which sufficiently marks the beginning of the linear behaviour of the mean excess plot. Since the empirical mean excesses are increasing, we

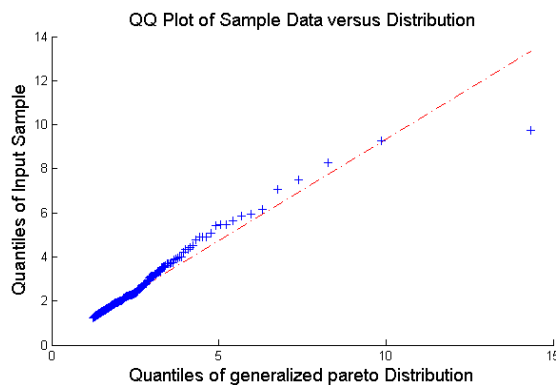


Figure 4: QQ-plot of the threshold exceeding residuals against the fitted GPD.

may assume that the shape parameter ξ is positive. This is confirmed by the parameter estimates for ξ and β . The Maximum Likelihood Estimators are $\widehat{\xi} = 0.3376$ with a 95% confidence interval $[0.2272, 0.4481]$ and $\widehat{\beta} = 0.4609$ with a 95% confidence interval $[0.4023, 0.5280]$.

In Figure 3, the right hand plot depicts the GPD fit of the excess distribution $F_u(x - u) = \mathbb{P}(X \leq x | X > u)$ superimposed on empirical estimates of excess probabilities. Note how well the GPD model fits to the empirical estimates of the excess probabilities.

A QQ-plot of the empirical quantiles against the fitted quantiles is depicted in Figure 4. Note again the good correspondence of the fitted GPD with the empirical estimates.

5.3 Fitting time-consistent risk measures to data

We now compute the corresponding time-consistent risk measures from Sections 3 and 4.

5.3.1 Time consistent VaR estimation

In a first step, for a single loss position L_t we compute the m -day-ahead time-consistent VaR given by Proposition 3.5 for different levels of α ; i.e., we fix t and consider $\widehat{\text{VaR}}_t^\alpha(L_{t+m})$ for various $m \in \mathbb{N}$. In Figure 5 we plot $\widehat{\text{VaR}}_t^\alpha(L_{t+m})$ for $m = 1, \dots, 10$.

Once the single time consistent risk measures $\widehat{\text{VaR}}_t^\alpha(L_{t+m})$ are computed, we simultaneously get the risk measure of the aggregated losses over m days from Proposition 3.6 by aggregation; i.e.,

$$\widehat{\text{VaR}}_t^\alpha\left(\sum_{j=1}^m L_{t+j}\right) = \sum_{j=1}^m \widehat{\text{VaR}}_t^\alpha(L_{t+j}).$$

In Figure 6 we plot $\widehat{\text{VaR}}_t^\alpha(\sum_{j=1}^m L_{t+j})$ for $m = 1, \dots, 10$. Table 2 shows the values of $\widehat{\text{VaR}}^\alpha$ for single losses L_{t+m} and aggregated losses $\sum_{j=1}^m L_{t+j}$ for various levels of α and $m = 1, \dots, 10$.

5.3.2 Time consistent AVaR estimation

In a second step, we compute the approximate upper and lower AVaR bounds for single loss position L_t and we compute the m -day-ahead for different levels of α ; i.e., we fix t and consider $\underline{\text{AVaR}}_t^\alpha(L_{t+m})$ and $\overline{\text{AVaR}}_t^\alpha(L_{t+m})$ for various $m \in \mathbb{N}$. The risk measure of the aggregated losses over m days we obtain from Propositions 4.7.

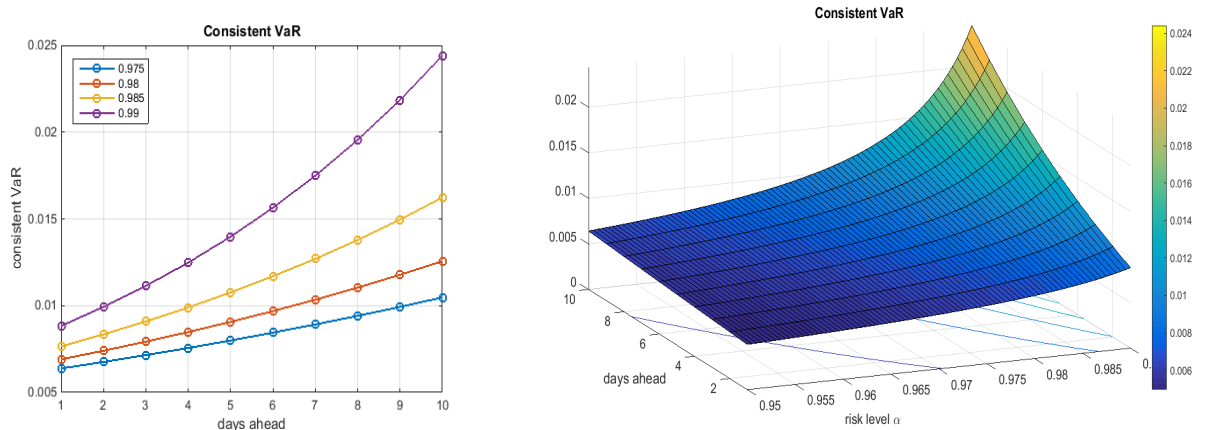


Figure 5: Time-consistent VaR estimation for single loss positions m days ahead.

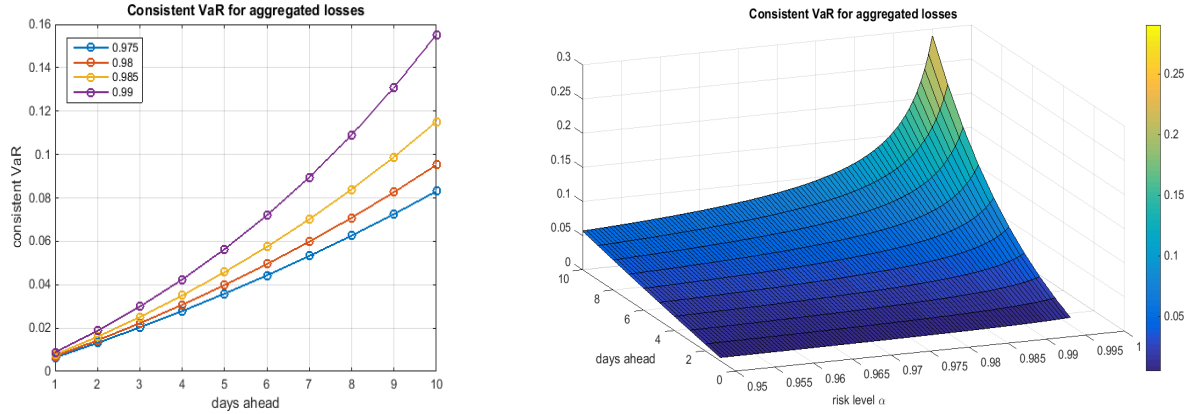


Figure 6: Time-consistent VaR estimation for aggregated loss positions m days ahead.

α/m	single loss				aggr. loss			
	97.5%	98%	98.5%	99%	97.5%	98%	98.5%	99%
1	0.0064	0.0069	0.0077	0.0088	0.0064	0.0069	0.0077	0.0088
2	0.0068	0.0074	0.0083	0.0099	0.0132	0.0143	0.0160	0.0187
3	0.0072	0.0079	0.0091	0.0111	0.0204	0.0222	0.0251	0.0297
4	0.0076	0.0085	0.0099	0.0125	0.0280	0.0307	0.0350	0.0422
5	0.0080	0.0091	0.0108	0.0140	0.0360	0.0398	0.0458	0.0562
6	0.0084	0.0097	0.0117	0.0156	0.0444	0.0495	0.0575	0.0718
7	0.0089	0.0103	0.0127	0.0175	0.0533	0.0598	0.0702	0.0893
8	0.0094	0.0110	0.0138	0.0196	0.0627	0.0708	0.0840	0.1089
9	0.0099	0.0118	0.0150	0.0218	0.0726	0.0826	0.0990	0.1307
10	0.0105	0.0126	0.0162	0.0244	0.0831	0.0952	0.1152	0.1551

Table 2: Values for $\widehat{\text{VaR}}^\alpha$ for single losses L_{t+m} and aggregated losses $\sum_{j=1}^m L_{t+j}$.

α/m	lower bound				upper bound			
	97.5%	98%	98.5%	99%	97.5%	98%	98.5%	99%
1	0.0098	0.0106	0.0118	0.0135	0.0098	0.0106	0.0118	0.0135
2	0.0115	0.0127	0.0145	0.0177	0.0131	0.0146	0.0168	0.0206
3	0.0134	0.0152	0.0180	0.0231	0.0173	0.0199	0.0239	0.0313
4	0.0157	0.0182	0.0222	0.0302	0.0230	0.0271	0.0340	0.0475
5	0.0183	0.0217	0.0275	0.0394	0.0304	0.0370	0.0483	0.0721
6	0.0214	0.0260	0.0340	0.0515	0.0402	0.0504	0.0686	0.1093
7	0.0250	0.0311	0.0420	0.0672	0.0532	0.0687	0.0974	0.1658
8	0.0291	0.0371	0.0520	0.0878	0.0704	0.0936	0.1384	0.2515
9	0.0340	0.0444	0.0643	0.1146	0.0932	0.1276	0.1966	0.3813
10	0.0398	0.0531	0.0795	0.1497	0.1232	0.1738	0.2792	0.5783

Table 3: Values for $\underline{\text{AVaR}}^\alpha$ and $\overline{\text{AVaR}}^\alpha$ for single losses L_{t+m} with $m = 1, \dots, 10$.

5.4 Conclusions

Obviously, the interpretation for the dynamic time-consistent (A)VaR differs considerably to that of the static (A)VaR: the dynamic (A)VaR evolves via the composition of the static (A)VaR over time.

α/m	aggr. $\underline{\text{AVaR}}^\alpha$				aggr. $\overline{\text{AVaR}}^\alpha$			
	97.5%	98%	98.5%	99%	97.5%	98%	98.5%	99%
1	0.0098	0.0106	0.0118	0.0135	0.0098	0.0106	0.0118	0.0135
2	0.0213	0.0233	0.0263	0.0312	0.0229	0.0252	0.0286	0.0341
3	0.0347	0.0385	0.0443	0.0543	0.0402	0.0451	0.0525	0.0654
4	0.0504	0.0567	0.0665	0.0845	0.0632	0.0722	0.0865	0.1129
5	0.0687	0.0784	0.0940	0.1239	0.0936	0.1092	0.1348	0.1850
6	0.0901	0.1044	0.1280	0.1754	0.1338	0.1596	0.2034	0.2943
7	0.1151	0.1355	0.1700	0.2426	0.1870	0.2283	0.3008	0.4601
8	0.1442	0.1726	0.2220	0.3304	0.2574	0.3219	0.4392	0.7116
9	0.1782	0.2170	0.2863	0.4450	0.3506	0.4495	0.6358	1.0929
10	0.2180	0.2701	0.3658	0.5947	0.4738	0.6233	0.9150	1.6712

Table 4: $\underline{\text{AVaR}}^\alpha$ and $\overline{\text{AVaR}}^\alpha$ for aggregated losses $\sum_{j=1}^m L_{t+j}$.

This results in a much more conservative risk measurement as the risky positions that are due far in the future not only enter the risk assessment through their own dynamics at the future maturity but rather enter through their risk assessment along any time point up to maturity. This has the intended effect that risky effects which arise until maturity are cushioned at any time. As one would expect, the higher safety margins α are required the more dramatic is the increase of safety capital when more days-ahead risk management is envisioned.

Table 2 contrasts single and aggregated time-consistent VaR values for different α and maturities m . It shows convincingly, how much higher capital reserves are needed to guarantee uniform safety at the same level over the whole time to maturity. Already at a level of $\alpha = 0.975$ the time-consistent aggregate loss VaR more than doubles from maturity 1 to 2 and multiplies by a factor of more than 12 to maturity 10. There is a high price to pay to safeguard against all uncertainties, which may lie in the far future.

For a comparison recall a standard industry method to estimate a 10-day VaR based on the central limit theorem, or normality of future losses (e.g. McNeil et al. (2010), Section 2.3.4). Recall that the loss from time t over the next m periods can be written as the sum over the negative returns during this period. If returns are iid with mean zero and variance σ^2 (or even normally distributed), then this sum is again (approximately) normally distributed with mean zero and variance $m\sigma^2$. This motivates the estimation of the sum of losses over m days by the estimation of the 1-day VaR and multiply it by \sqrt{m} .

Let us compare the values for $\overline{\text{VaR}}_t^\alpha$ from Table 2 with this industry standard. We find for $\alpha = 0.975$ a 10-day VaR of $0.0064\sqrt{10} = 0.0202$ (which we have to compare with the time-consistent $\overline{\text{VaR}}_t^\alpha(\sum_{j=1}^{10} L_{t+j}) = 0.0830$, which is more than 4 times as large), and for $\alpha = 0.99$ a 10-day VaR of $0.0088\sqrt{10} = 0.0278$ (which we have to compare with the time-consistent $\overline{\text{VaR}}_t^\alpha(\sum_{j=1}^{10} L_{t+j}) = 0.1553$, which is more than 5 times as large). One reason for this huge difference is the well-known fact that GARCH losses do not scale with \sqrt{m} , but scaling depends strongly on the parameters; cf. Franq and Zakoian (2010), Chapter 4. However, this alone does not explain the huge difference between the simple industry standard and the time-consistent VaR for the aggregated losses.

Due to their construction the composed VaR and AVaR for aggregated future losses produce much more conservative reserve requirements than the standard VaR and AVaR for the same level of α . As an implication the standard reserving requirement of excessively high levels of α like 99% or 99.9%

covering 100- or 1000-year events may be put to a test taking into consideration reduced levels of α , e.g. in the bandwidth 90% – 97.5%. The reduction of such extremely high levels would also be very reasonable from a statistical point of view as lower level quantiles give rise to much more reliable estimators.

A Proofs of Section 3

Proof of Theorem 3.5 We proceed by backward induction. Firstly, by (3.2), at $T - 1$ we have the 1-day-ahead-VaR

$$\widehat{\text{VaR}}_{T-1}^\alpha(L_T) = F_Z^{-1}(\alpha) \sigma_T$$

which agrees with (3.7) for $t = T - 1$. Assume that (3.7) holds for all $s = t, \dots, T - 1$. We have

$$\begin{aligned} \widehat{\text{VaR}}_{t-1}^\alpha(L_T) &= \text{VaR}_{t-1}^\alpha(\widehat{\text{VaR}}_t^\alpha(L_T)) \\ &= \text{VaR}_{t-1}^\alpha(F_Z^{-1}(\alpha) \sqrt{\mathcal{P}_t^T(a_1 F_{Z^2}^{-1}(\alpha) + b)}) \\ &= \text{essinf}\{m \in L^0(\mathcal{F}_{t-1}) : \mathbb{P}(F_Z^{-1}(\alpha) \sqrt{\mathcal{P}_t^T(a_1 F_{Z^2}^{-1}(\alpha) + b)} \leq m \mid \mathcal{F}_{t-1}) \geq \alpha\}. \end{aligned}$$

We denote by \mathbb{P}_{t-1} the conditional probability with respect to \mathcal{F}_{t-1} . Note that

$$\begin{aligned} \mathbb{P}_{t-1}(F_Z^{-1}(\alpha) \sqrt{\mathcal{P}_t^T(a_1 F_{Z^2}^{-1}(\alpha) + b)} \leq m) &= \mathbb{P}_{t-1}(F_Z^{-1}(\alpha)^2 \mathcal{P}_t^T(a_1 F_{Z^2}^{-1}(\alpha) + b) \leq m^2) \\ &= \mathbb{P}\left(\sigma_{t+1}^2 (a_1 F_{Z^2}^{-1}(\alpha) + b)^{T-t-1} \leq \left(\frac{m}{F_Z^{-1}(\alpha)}\right)^2 - a_0 \sum_{k=0}^{T-t-2} (a_1 F_{Z^2}^{-1}(\alpha) + b)^k \mid \mathcal{F}_{t-1}\right). \end{aligned}$$

Using the definition of the GARCH volatility (2.5) for σ_{t+1}^2 this can be continued by

$$\begin{aligned} \mathbb{P}_{t-1}(F_Z^{-1}(\alpha) \sqrt{\mathcal{P}_t^T(a_1 F_{Z^2}^{-1}(\alpha) + b)} \leq m) \\ = \mathbb{P}_{t-1}\left(a_1 \sigma_t^2 Z_t^2 \leq \frac{1}{(a_1 F_{Z^2}^{-1}(\alpha) + b)^{T-t-1}} \left(\left(\frac{m}{F_Z^{-1}(\alpha)}\right)^2 - a_0 \sum_{k=0}^{T-t-2} (a_1 F_{Z^2}^{-1}(\alpha) + b)^k \right) - a_0 - b \sigma_t^2 \right) \end{aligned}$$

Since σ_t is \mathcal{F}_{t-1} -measurable and Z_t is independent of \mathcal{F}_{t-1} we conclude that

$$\begin{aligned} \widehat{\text{VaR}}_{t-1}^\alpha(L_T) &= F_Z^{-1}(\alpha) \sqrt{\left(a_0 + (a_1 F_{Z^2}^{-1}(\alpha) + b) \sigma_t^2 \right) (a_1 F_{Z^2}^{-1}(\alpha) + b)^{T-t-1} + a_0 \sum_{k=0}^{T-t-2} (a_1 F_{Z^2}^{-1}(\alpha) + b)^k} \\ &= F_Z^{-1}(\alpha) \sqrt{a_0 \sum_{k=0}^{T-t-1} (a_1 F_{Z^2}^{-1}(\alpha) + b)^k + \sigma_t^2 (a_1 F_{Z^2}^{-1}(\alpha) + b)^{T-t}} \\ &= F_Z^{-1}(\alpha) \sqrt{\mathcal{P}_{t-1}^T(a_1 F_{Z^2}^{-1}(\alpha) + b)}. \end{aligned}$$

This finishes the proof. □

B Proofs of Section 4

We need the following lemma.

Lemma B.1. For $t = 0, \dots, T-2$ assume that $f_t: \mathbb{R} \rightarrow \mathbb{R}$ is a \mathcal{F}_t -measurable, and strictly increasing mapping. Then we have

$$\{\omega \in \Omega : f_t(Z_{t+1}^2) > \text{VaR}_t^\alpha(f_t(Z_{t+1}^2))\} = \{\omega \in \Omega : Z_{t+1}^2 > F_{Z^2}^{-1}(\alpha)\}.$$

In particular,

$$\{\omega \in \Omega : \sigma_{t+2} > \text{VaR}_t^\alpha(\sigma_{t+2})\} = \{\omega \in \Omega : Z_{t+1}^2 > F_{Z^2}^{-1}(\alpha)\}.$$

Proof. Due to the assumptions on f_t it is invertible. According to the definition of VaR_t^α we have

$$\begin{aligned} \text{VaR}_t^\alpha(f_t(Z_{t+1}^2)) &= \text{essinf}\{m \in L^0(\mathcal{F}_t) : \mathbb{P}(f_t(Z_{t+1}^2) \leq m \mid \mathcal{F}_t) \geq \alpha\} \\ &= \text{essinf}\{m \in L^0(\mathcal{F}_t) : \mathbb{P}(Z_{t+1}^2 \leq f_t^{-1}(m)) \geq \alpha\} \\ &= \text{essinf}\{m \in L^0(\mathcal{F}_t) : F_{Z^2}(f_t^{-1}(m)) \geq \alpha\} \\ &= \text{essinf}\{m \in L^0(\mathcal{F}_t) : m \geq f_t(F_{Z^2}^{-1}(\alpha))\} \\ &= f_t(F_{Z^2}^{-1}(\alpha)) \end{aligned}$$

where the third line follows from the independence between Z_{t+1} and \mathcal{F}_t . Thus

$$f_t(Z_{t+1}^2) > \text{VaR}_t^\alpha(f_t(Z_{t+1}^2)) = f_t(F_{Z^2}^{-1}(\alpha))$$

holds if and only if $Z_{t+1}^2 > F_{Z^2}^{-1}(\alpha)$. For the second part, note that by (3.4),

$$\text{VaR}_t^\alpha(\sigma_{t+2}) = \sqrt{a_0 + \sigma_{t+1}^2(a_1 F_{Z^2}^{-1}(\alpha) + b)}.$$

From the definition of the GARCH(1,1) model (2.5) we conclude

$$\sigma_{t+2} = \sqrt{a_0 + \sigma_{t+1}^2(a_1 Z_{t+1}^2 + b)} = f_t(Z_{t+1}^2) > \text{VaR}_t^\alpha(f_t(Z_{t+1}^2)) = \sqrt{a_0 + \sigma_{t+1}^2(a_1 F_{Z^2}^{-1}(\alpha) + b)} = \text{VaR}_t^\alpha(\sigma_{t+2})$$

if and only if $Z_{t+1}^2 > F_{Z^2}^{-1}(\alpha)$. \square

Proof of Theorem 4.5 We apply again backward induction. From (4.1) and Example 4.3 we have

$$\widetilde{\text{AVaR}}_{T-1}^\alpha(L_T^2) = \sigma_T^2 \frac{1}{1-\alpha} \int_\alpha^1 F_{Z^2}^{-1}(u) du,$$

which agrees with (4.4) for $t = T-1$. For simplicity we write $\bar{\kappa}_2 = \frac{1}{1-\alpha} \int_\alpha^1 F_{Z^2}^{-1}(u) du$. Now assume that (4.4) holds for all $s = t, \dots, T-1$. Then it remains to prove (4.4) for $t-1$. We have by (4.1) and (2.5)

$$\widetilde{\text{AVaR}}_{t-1}^\alpha(L_T^2) = \text{AVaR}_{t-1}^\alpha(\widetilde{\text{AVaR}}_t^\alpha(L_T^2)) = \text{AVaR}_{t-1}^\alpha(\bar{\kappa}_2 \mathcal{P}_t^T(a_1 \bar{\kappa}_2 + b)).$$

We denote $G_t := \mathcal{P}_t^T(a_1 \bar{\kappa}_2 + b)$, which is a measurable function of σ_{t+1} , and take the constant out of the expectation, which yields

$$\widetilde{\text{AVaR}}_{t-1}^\alpha(L_T^2) = \bar{\kappa}_2 \mathbb{E}_{t-1}[G_t \mid G_t > \text{VaR}_{t-1}^\alpha(G_t)].$$

Now note that by Definition 3.1

$$\text{VaR}_{t-1}^\alpha(G_t) = \text{essinf}\{m \in L^0(\mathcal{F}_{t-1}) : \mathbb{P}(G_t \leq m \mid \mathcal{F}_{t-1}) > \alpha\}.$$

We denote by \mathbb{P}_{t-1} the conditional probability with respect to \mathcal{F}_{t-1} . We compute further, using the definition of the GARCH volatility (2.5) for σ_{t+1}^2

$$\begin{aligned}\mathbb{P}_{t-1}(G_t \leq m) &= \mathbb{P}_{t-1}\left(a_0 \sum_{k=0}^{T-t-2} (a_1 \bar{\kappa}_2 + b)^k + \sigma_{t+1}^2 (a_1 \bar{\kappa}_2 + b)^{T-t-1} \leq m\right) \\ &= \mathbb{P}_{t-1}\left(\sigma_{t+1} \leq \frac{m - a_0 \sum_{k=0}^{T-t-2} (a_1 \bar{\kappa}_2 + b)^k}{(a_1 \bar{\kappa}_2 + b)^{T-t-1}}\right) \\ &= \mathbb{P}\left(Z_t^2 \leq \left(\frac{m - a_0 \sum_{k=0}^{T-t-2} (a_1 \bar{\kappa}_2 + b)^k}{(a_1 \bar{\kappa}_2 + b)^{T-t-1}} - a_0 - b\sigma_t\right) \frac{1}{a_1 \sigma_t^2}\right),\end{aligned}$$

where in the last line we have used that σ_t is \mathcal{F}_{t-1} -measurable and the independence of Z_t and \mathcal{F}_{t-1} . We can thus conclude that

$$\text{VaR}_{t-1}^\alpha(G_t) = \left(a_0 + \sigma_t^2 (a_1 F_{Z^2}^{-1}(\alpha) + b)\right) (a_1 \bar{\kappa}_2 + b)^{T-t-1} + a_0 \sum_{k=0}^{T-t-2} (a_1 \bar{\kappa}_2 + b)^k.$$

From Lemma B.1 we know that $\{G_t > \text{VaR}_{t-1}^\alpha(G_t)\} = \{Z_t^2 > F_{Z^2}^{-1}(\alpha)\}$. Hence, it follows from the independence of Z_t and \mathcal{F}_{t-1} that

$$\mathbb{E}_{t-1}[G_t \mid G_t > \text{VaR}_{t-1}^\alpha(G_t)] = \mathbb{E}_{t-1}[G_t \mid Z_t^2 > F_{Z^2}^{-1}(\alpha)] = \frac{1}{1-\alpha} \mathbb{E}_{t-1}[G_t \mathbb{1}_{\{Z_t^2 > F_{Z^2}^{-1}(\alpha)\}}]. \quad (\text{B.1})$$

Moreover, we calculate

$$\begin{aligned}\mathbb{E}_{t-1}[G_t \mathbb{1}_{\{Z_t^2 > F_{Z^2}^{-1}(\alpha)\}}] &= (1-\alpha) a_0 \sum_{k=0}^{T-t-2} (a_1 \bar{\kappa}_2 + b)^k + (a_1 \bar{\kappa}_2 + b)^{T-t-1} \int_{F_{Z^2}^{-1}(\alpha)}^{\infty} (a_0 + \sigma_t^2 (a_1 u + b)) dF_{Z^2}(u) \\ &= (1-\alpha) a_0 \sum_{k=0}^{T-t-2} (a_1 \bar{\kappa}_2 + b)^k + (a_1 \bar{\kappa}_2 + b)^{T-t-1} \int_{\alpha}^1 (a_0 + \sigma_t^2 (a_1 F_{Z^2}^{-1}(u) + b)) du \\ &= (1-\alpha) a_0 \sum_{k=0}^{T-t-1} (a_1 \bar{\kappa}_2 + b)^k + (a_1 \bar{\kappa}_2 + b)^{T-t-1} (a_1 \bar{\kappa}_2 + (1-\alpha)b) \sigma_t^2,\end{aligned}$$

which in combination with (B.1) yields

$$\mathbb{E}_{t-1}[G_t \mid G_t > \text{VaR}_{t-1}^\alpha(G_t)] = a_0 \sum_{k=0}^{T-t-1} (a_1 \bar{\kappa}_2 + b)^k + (a_1 \bar{\kappa}_2 + b)^{T-t} \sigma_t^2 = \mathcal{P}_{t-1}^T(a_1 \bar{\kappa}_2 + b).$$

This finally amounts to

$$\begin{aligned}\widetilde{\text{AVaR}}_{t-1}^\alpha(L_T^2) &= \frac{1}{1-\alpha} \int_{\alpha}^1 F_Z^{-1}(u) du \mathbb{E}_{t-1}[G_t \mid G_t > \text{VaR}_{t-1}^\alpha(G_t)] \\ &= \frac{1}{1-\alpha} \int_{\alpha}^1 F_Z^{-1}(u) du \mathcal{P}_{t-1}^T(a_1 \bar{\kappa}_2 + b),\end{aligned}$$

which proves the assertion. \square

Proof of Proposition 4.7 A careful proof tracking reveals its similarity to the proof of Theorem 4.5. For simplicity we set $\bar{\kappa} = \frac{1}{1-\alpha} \int_{\alpha}^1 F_Z^{-1}(y) dy$ and $\bar{\kappa}_2 = \frac{1}{1-\alpha} \int_{\alpha}^1 F_{Z^2}^{-1}(y) dy$.

At $t = T - 1$ we have $\widetilde{\text{AVaR}}_{T-1}^\alpha(L_T) = \bar{\kappa} \sigma_T$ which coincides with $\text{AVaR}_{T-1}^\alpha(L_T)$. Since by Definition and (4.8) and (4.9),

$$\widetilde{\text{AVaR}}_{T-2}^\alpha(L_T) = \text{AVaR}_{T-2}^\alpha(\widetilde{\text{AVaR}}_{T-1}^\alpha(L_T)) = \bar{\kappa} \text{AVaR}_{T-2}^\alpha(\sigma_T),$$

we obtain

$$\begin{aligned}\text{AVaR}_{T-2}^\alpha(\sigma_T) &= \mathbb{E}_{T-2}\left[\sigma_T \mid \sigma_T > \text{VaR}_{T-2}^\alpha(\sigma_T)\right] \\ &= \mathbb{E}_{T-2}\left[\sigma_T \mid Z_{T-1}^2 > F_{Z^2}^{-1}(\alpha)\right].\end{aligned}$$

by Lemma B.1. An application of Jensen's inequality yields

$$\mathbb{E}_{T-2}\left[\sigma_T \mid Z_{T-1}^2 > F_{Z^2}^{-1}(\alpha)\right] \leq \left(\mathbb{E}_{T-2}\left[\sigma_T^2 \mid Z_{T-1}^2 > F_{Z^2}^{-1}(\alpha)\right]\right)^{1/2}.$$

We obtain further

$$\begin{aligned}\mathbb{E}_{T-2}\left[\sigma_T^2 \mid Z_{T-1}^2 > F_{Z^2}^{-1}(\alpha)\right] &= \mathbb{E}_{T-2}\left[a_0 + \sigma_{T-1}^2(a_1 Z_{T-1}^2 + b) \mid Z_{T-1}^2 > F_{Z^2}^{-1}(\alpha)\right] \\ &= \frac{1}{1-\alpha} \int_{F_{Z^2}^{-1}(\alpha)}^\infty \left(a_0 + \sigma_{T-1}^2(a_1 y + b)\right) dF_{Z^2}(y) \\ &= a_0 + \sigma_{T-1}^2 \left(b + a_1 \frac{1}{1-\alpha} \int_{F_{Z^2}^{-1}(\alpha)}^\infty y dF_{Z^2}(y)\right) \\ &= a_0 + \sigma_{T-1}^2 \left(b + a_1 \bar{\kappa}_2\right),\end{aligned}$$

which amounts to

$$\widehat{\text{AVaR}}_{T-2}^\alpha(L_T) \leq \bar{\kappa} \sqrt{a_0 + \sigma_{T-1}^2(a_1 \bar{\kappa}_2 + b)} = \bar{\kappa} \sqrt{\mathcal{P}_{T-2}^T(a_1 \bar{\kappa} + b)} = \overline{\text{AVaR}}_{T-2}^\alpha(L_T).$$

This proves for $t = T - 2$ that $\overline{\text{AVaR}}_{T-2}^\alpha(L_T)$ is an upper bound for $\widehat{\text{AVaR}}_{T-2}^\alpha(L_T)$.

Now assume that $\overline{\text{AVaR}}_s^\alpha(L_T) \geq \widehat{\text{AVaR}}_s^\alpha(L_T)$ holds true for $s = T - 1, \dots, t + 1$. We show next that also

$$\overline{\text{AVaR}}_t^\alpha(L_T) \geq \widehat{\text{AVaR}}_t^\alpha(L_T).$$

To this end notice that

$$\widehat{\text{AVaR}}_t^\alpha(L_T) = \text{AVaR}_t^\alpha(\widehat{\text{AVaR}}_{t+1}^\alpha(L_T)) \leq \text{AVaR}_t^\alpha(\overline{\text{AVaR}}_{t+1}^\alpha(L_T)). \quad (\text{B.2})$$

Moreover, we have by the induction assumption

$$\begin{aligned}\text{AVaR}_t^\alpha(\overline{\text{AVaR}}_{t+1}^\alpha(L_T)) &= \text{AVaR}_t^\alpha\left(\bar{\kappa}_2 \sqrt{\mathcal{P}_{t+1}^T\left(\frac{a_1}{1-\alpha} \int_\alpha^1 F_{Z^2}^{-1}(u) du + b\right)}\right) \\ &= \bar{\kappa}_2 \mathbb{E}_t\left[\sqrt{\mathcal{P}_{t+1}^T(a_1 \bar{\kappa}_2 + b)} \mid \sqrt{\mathcal{P}_{t+1}^T(a_1 \bar{\kappa}_2 + b)} > \text{VaR}_t^\alpha\left(\sqrt{\mathcal{P}_{t+1}^T(a_1 \bar{\kappa}_2 + b)}\right)\right].\end{aligned}$$

By a similar calculation as in the proof of Theorem 4.5 and Lemma B.1, we can see that the above expression simplifies to

$$\begin{aligned}\text{AVaR}_t^\alpha(\overline{\text{AVaR}}_{t+1}^\alpha(L_T)) &= \bar{\kappa}_2 \mathbb{E}_t\left[\sqrt{\mathcal{P}_{t+1}^T(a_1 \bar{\kappa}_2 + b)} \mid Z_{t+1}^2 > F_{Z^2}^{-1}(\alpha)\right] \\ &\leq \frac{1}{1-\alpha} \int_\alpha^1 F_Z^{-1}(y) dy \left(\mathbb{E}_t\left[\mathcal{P}_{t+1}^T(a_1 \bar{\kappa}_2 + b) \mid Z_{t+1}^2 > F_{Z^2}^{-1}(\alpha)\right]\right)^{1/2},\end{aligned}$$

where the last line follows from Jensen's inequality. Note that

$$\mathbb{E}_t\left[\mathcal{P}_{t+1}^T(a_1 \bar{\kappa}_2 + b) \mid Z_{t+1}^2 > F_{Z^2}^{-1}(\alpha)\right]$$

$$\begin{aligned}
&= \frac{1}{1-\alpha} \int_{F_{Z^2}^{-1}(\alpha)}^{\infty} \left(a_0 \sum_{k=0}^{T-t-3} (a_1 \bar{\kappa}_2 + b)^k + (a_0 + \sigma_{t+1}^2 (a_1 y + b)) (a_1 \bar{\kappa}_2 + b)^{T-t-2} \right) dF_{Z^2}(y) \\
&= a_0 \sum_{k=0}^{T-t-2} (a_1 \bar{\kappa}_2 + b)^k + \frac{1}{1-\alpha} (a_1 \bar{\kappa}_2 + b)^{T-t-2} \sigma_{t+1}^2 \int_{\alpha}^1 (a_1 F_{Z^2}^{-1}(y) + b) dy \\
&= a_0 \sum_{k=0}^{T-t-2} (a_1 \bar{\kappa}_2 + b)^k + \sigma_{t+1}^2 (a_1 \bar{\kappa}_2 + b)^{T-t-1} \\
&= \mathcal{P}_t^T(a_1 + b),
\end{aligned}$$

which implies

$$\text{AVaR}_t^\alpha(\overline{\text{AVaR}}_{t+1}^\alpha(L_T)) \leq \frac{1}{1-\alpha} \int_{\alpha}^1 F_Z^{-1}(y) dy \sqrt{\mathcal{P}_t^T(a_1 \bar{\kappa}_2 + b)} = \overline{\text{AVaR}}_t^\alpha(L_T).$$

Finally it follows from (B.2) that $\overline{\text{AVaR}}_t^\alpha(L_T) \leq \overline{\text{AVaR}}_t^\alpha(L_T)$. \square

Proof of Proposition 4.8 At $t = T - 1$, $\underline{\text{AVaR}}_{T-1}^\alpha(L_T)$ coincides with $\overline{\text{AVaR}}_{T-1}^\alpha(L_T)$. At $t = T - 2$ we obtain

$$\overline{\text{AVaR}}_{T-2}^\alpha(L_T) = \text{AVaR}_{T-2}^\alpha(\overline{\text{AVaR}}_{T-1}^\alpha(L_T)) = \frac{1}{1-\alpha} \int_{\alpha}^1 F_Z^{-1}(y) dy \text{AVaR}_{T-2}^\alpha(\sigma_T).$$

For simplicity we write $\bar{\kappa} = \frac{1}{1-\alpha} \int_{\alpha}^1 F_Z^{-1}(y) dy$. By continuity of the distribution function inherited from Z ,

$$\text{AVaR}_{T-2}^\alpha(\sigma_T) = \mathbb{E}_{T-2}[\sigma_T \mid \sigma_T > \text{VaR}_{T-2}^\alpha(\sigma_T)],$$

which by Lemma B.1 rewrites as

$$\text{AVaR}_{T-2}^\alpha(\sigma_T) = \mathbb{E}_{T-2}[\sigma_T \mid Z_{T-1}^2 > F_{Z^2}^{-1}(\alpha)].$$

Using the \mathcal{F}_{T-2} -measurability of σ_{T-1} and $a_0 > 0$, we calculate further,

$$\begin{aligned}
\mathbb{E}_{T-2}[\sigma_T \mid Z_{T-1}^2 > F_{Z^2}^{-1}(\alpha)] &= \mathbb{E}_{T-2}[\sqrt{a_0 + \sigma_{T-1}^2 (a_1 Z_{T-1}^2 + b)} \mid Z_{T-1}^2 > F_{Z^2}^{-1}(\alpha)] \\
&\geq \sigma_{T-1} \mathbb{E}_{T-2}[\sqrt{a_1 Z_{T-1}^2 + b} \mid Z_{T-1}^2 > F_{Z^2}^{-1}(\alpha)] \\
&= \sigma_{T-1} \frac{1}{1-\alpha} \int_{F_{Z^2}^{-1}(\alpha)}^{\infty} \sqrt{a_1 y + b} dF_{Z^2}(y).
\end{aligned} \tag{B.3}$$

Hence, it follows that

$$\overline{\text{AVaR}}_{T-2}^\alpha(L_T) \geq \bar{\kappa} \frac{1}{1-\alpha} \int_{\alpha}^1 \sqrt{a_1 F_{Z^2}^{-1}(y) + b} dy \sigma_{T-1} = \underline{\text{AVaR}}_{T-2}^\alpha(L_T).$$

This proves for $t = T - 2$ that $\underline{\text{AVaR}}_{T-2}^\alpha(L_T)$ as in (4.11) is a lower bound for $\overline{\text{AVaR}}_{T-2}^\alpha(L_T)$.

Now assume that $\underline{\text{AVaR}}_s^\alpha(L_T) \leq \overline{\text{AVaR}}_s^\alpha(L_T)$ holds true for $s = T - 1, \dots, t + 1$. We show next that also

$$\underline{\text{AVaR}}_t^\alpha(L_T) \leq \overline{\text{AVaR}}_t^\alpha(L_T).$$

To this end notice that

$$\overline{\text{AVaR}}_t^\alpha(L_T) = \text{AVaR}_t^\alpha(\overline{\text{AVaR}}_{t+1}^\alpha(L_T)) \geq \text{AVaR}_t^\alpha(\underline{\text{AVaR}}_{t+1}^\alpha(L_T)). \tag{B.4}$$

Moreover, we have by the induction assumption

$$\begin{aligned}
\text{AVaR}_t^\alpha(\underline{\text{AVaR}}_{t+1}^\alpha(L_T)) &= \text{AVaR}_t^\alpha\left(\bar{\kappa} \sigma_{T-1} \frac{1}{1-\alpha} \int_\alpha^1 \sqrt{a_1 F_{Z^2}^{-1}(y) + b} dy\right) \\
&= \bar{\kappa} \left(\frac{1}{1-\alpha} \int_{F_{Z^2}^{-1}(\alpha)}^\infty \sqrt{a_1 y + b} dF_{Z^2}(y) \right)^{T-t-2} \mathbb{E}_t[\sigma_{t+2} \mid \sigma_{t+2} > \text{VaR}_t^\alpha(\sigma_{t+2})] \\
&= \bar{\kappa} \left(\frac{1}{1-\alpha} \int_{F_{Z^2}^{-1}(\alpha)}^\infty \sqrt{a_1 y + b} dF_{Z^2}(y) \right)^{T-t-2} \mathbb{E}_t[\sigma_{t+2} \mid Z_{t+1}^2 > F_{Z^2}^{-1}(\alpha)],
\end{aligned}$$

where the last equality follows from Lemma B.1. Then by the same calculation which lead to (B.3),

$$\mathbb{E}_t[\sigma_{t+2} \mid Z_{t+1}^2 > F_{Z^2}^{-1}(\alpha)] = \sigma_{t+1} \int_\alpha^1 \sqrt{a_1 F_{Z^2}^{-1}(y) + b} dy,$$

this yields together with (B.4),

$$\widetilde{\text{AVaR}}_t^\alpha(L_T) \geq \bar{\kappa} \left(\frac{1}{1-\alpha} \int_\alpha^1 \sqrt{a_1 F_{Z^2}^{-1}(y) + b} dy \right)^{T-t-1} \sigma_{t+1} = \underline{\text{AVaR}}_t^\alpha(L_T).$$

□

Acknowledgements

We are grateful to one of the Reviewers and Marcin Pitera, who pointed out some errors in a previous version of this paper.

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