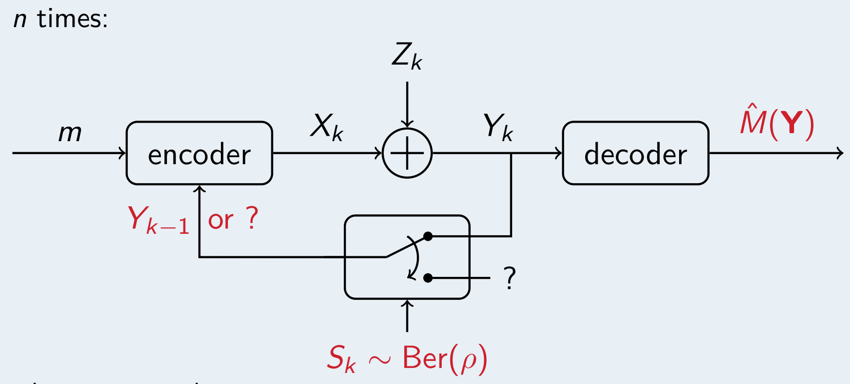
The Gaussian Channel with Intermittent Feedback

Lars Palzer
lars.palzer@tum.de

System Model

We wish to transmit a message $m \in \mathcal{M}$ over the time-discrete memoryless additive white Gaussian noise channel using the channel n times:



where at time-k

- X_k is the channel input,
- $Z_k \sim \mathcal{N}(0, \sigma^2)$ is i.i.d. Gaussian noise,
- Y_k is the channel output.

The transmitter receives *intermittent feedback* in the sense that Y_k is revealed to it strictly causally if, and only if, $S_k = 1$ with $\Pr(S_k = 1) = 1 - \Pr(S_k = 0) = \rho$ for all k. The receiver has no information about $\mathbf{S} = (S_1, \dots, S_n)$. We impose the average power constraint for some $P > 0$

$$\mathrm{E}\left[\sum_{k=1}^{n} X_k(m)^2\right] \le nP, \quad m \in \mathcal{M},$$

and assume that all messages are equiprobable.

Question

What is the asymptotically optimal error probability for communication via the AWGN channel with intermittent feedback?

Answer

If the probability of having feedback at any time is greater than one half, the error probability is doubly exponential in the block length for sufficiently small rates. Otherwise, it is exponential.

AWGN Channel with Feedback

No feedback (Shannon [1])

Exponential in the block length:

$$\mathrm{P}_e^{(n)} = e^{-n(E_{\mathrm{NoFB}} + o(1))}$$

Perfect feedback (e.g., Pinsker [2])

For noiseless feedback, the optimal error probability decreases essentially arbitrarily fast:

$$\mathrm{P}_e^{(n)} = \exp(-\exp \circ \cdots \circ \exp(n(E_{\mathrm{FB}} + o(1)))$$

Intermittent feedback with cognizant receiver [3]

Consider the system model above, but suppose that $\mathbf{S}$ is revealed to the receiver. For $\mathcal{M} = \{0, 1\}$,

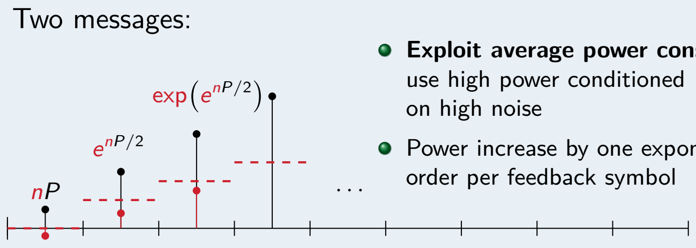
$$\mathrm{P}_e^{(n)} = \exp(-e^{n(-\log(1-\rho)+o(1))}).$$

Let $|\mathcal{M}| = e^{nR}$ and C be the capacity of the forward channel. Then,

$$\mathrm{P}_e^{(n)} = \exp(-e^{n(E_{\mathrm{IFB}} + o(1))}), \quad \text{for } R \in (0, \rho C),$$
$$\mathrm{P}_e^{(n)} = e^{-n(\tilde{E}_{\mathrm{IFB}} + o(1))}, \quad \text{for } R \in (\rho C, C).$$

Boosted Retransmission for Perfect Feedback

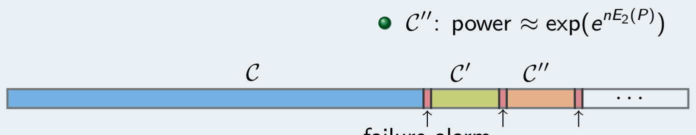
Two messages:



- **Exploit average power constraint:** use high power conditioned on high noise
- Power increase by one exponential order per feedback symbol

Positive Rates:

- $\mathcal{C}$: power $\approx nP$
- $\mathcal{C}'$: power $\approx e^{nE_1(P)}$
- $\mathcal{C}''$: power $\approx \exp(e^{nE_2(P)})$



References

[1] C. E. Shannon, "Probability of error for optimal codes in a Gaussian channel," *Bell Syst. Techn. J.*, vol. 38, pp. 611–656, 1959.

[2] M. S. Pinsker, "The probability of error in block transmission in a memoryless Gaussian channel with feedback," *Prob. Peredachi Inf.*, vol. 4, no. 4, pp. 3–19, 1968.

[3] C. Bunte, A. Lapidoth and L. Palzer, "Coding for the Gaussian channel with intermittent feedback," *Proc. IEEE Int. Symp. Inf. Theory*, Honolulu, USA, Jun. 2014.

[4] L. Palzer, "Signaling over the Gaussian channel with intermittent feedback," *Proc. IEEE Int. Symp. Inf. Theory*, Hong Kong, Jun. 2015.

Direct Part Two Messages

Theorem 1

Let $\mathcal{M} = \{0, 1\}$. For any $P > 0$ and any $\rho \in (1/2, 1)$:

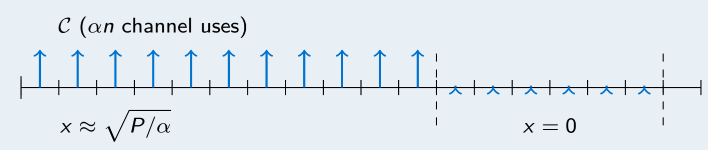
$$\liminf_{n\to\infty} \frac{1}{n} \log\left(-\log P_e^{(n)}\right) \ge \max_{\alpha \in (0,1)} \alpha D\left(\frac{1}{2} \middle\| \rho\left(1 - Q\left(\sqrt{\frac{P}{\alpha\sigma^2}}\right)\right)\right),$$

where $D(a\|b) = a \log\frac{a}{b} + (1-a)\log\frac{1-a}{1-b}$ and the maximization is subject to the constraint $Q\left(\sqrt{P/(\alpha\sigma^2)}\right) < \frac{\rho - 1/2}{\rho}$.

The optimal error probability for two messages decays double-exponentially in n. The second-order error exponent is upper bounded by $-\log(1-\rho)$ [3].

Sketch of the Scheme

Let $M = 0$. Use a binary repetition code $\mathcal{C}$ with hard decision decoding until time-αn, then save power until time-$(n-1)$.



- **If TX sees more than $\alpha n/2$ correct symbols, $\mathcal{C}$ cannot fail**
- Otherwise: boosted retransmission at time-n

Boosted Retransmission

For a single symbol,

$$\underbrace{\Pr(\&\ \substack{\text{feedback} \\ Y_k \text{ positive}})}_{:=p_s} = \rho\left(1 - Q\left(\sqrt{P/(\alpha\sigma^2)}\right)\right).$$

Taking α small enough to satisfy $p_s > 1/2$, we can use Sanov's theorem (large deviations on i.i.d. Bernoulli trials) to see that

$$\Pr(\text{retransmission}) = e^{-\alpha n D(1/2\|p_s) + o(n)},$$

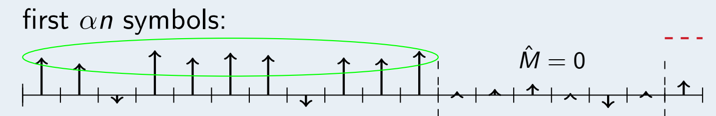
and the transmitter sends

$$x_n = e^{\frac{\alpha n}{2} D(1/2\|p_s) + o(n)},$$

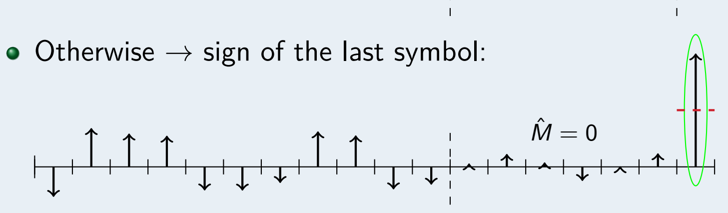
if less than $\alpha n/2$ positive symbols are fed back in the first phase and $x_n = 0$ otherwise.

What does the decoder do?

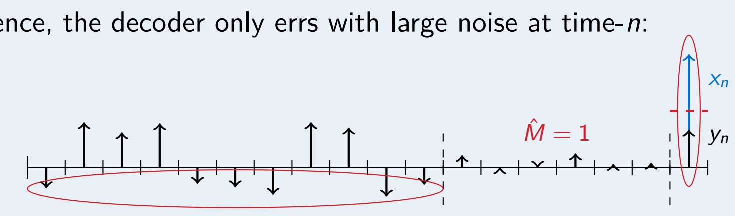
- If it detects no retransmission → hard decision decoding of the first αn symbols:



- Otherwise → sign of the last symbol:



Hence, the decoder only errs with large noise at time-n:



and thus

$$\Pr(\text{error}) \le Q\left(\frac{1}{2\sigma^2} e^{\frac{\alpha n}{2} D(1/2\|p_s) + o(n)}\right).$$

Direct Part Positive Rates

For sufficiently small rates, the optimal error probability is also doubly exponential in n.

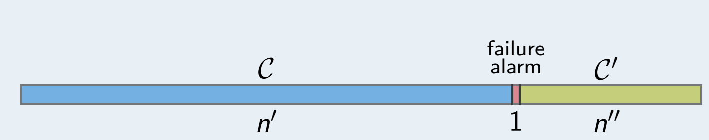
Theorem 2

Let $|\mathcal{M}| = e^{nR}$. For any $P > 0$ and $\rho \in (1/2, 1)$, there exists an $R_0 > 0$ such that for all $0 < R < R_0$:

$$\liminf_{n\to\infty} \frac{1}{n} \log\left(-\log P_e^{(n)}(R)\right) > 0.$$

Sketch of the scheme of block length $n = n' + 1 + n''$:

- Use q-ary non-feedback code $\mathcal{C}$ with minimum distance $\delta n'$
- Transmission of code word via pulse-amplitude modulation
- TX performs symbol-wise hard decisions on feedback
- Boosted failure alarm and retransmission with code $\mathcal{C}'$ if transmitter sees less than $(1 - \delta/2)n'$ correct symbols
- Varying q, R, n', we can choose any $\delta \in (0,1)$, hence $\Pr(\text{retransmission}) = e^{-O(n)}$.
- Failure alarm and $\mathcal{C}'$ use power growing exponentially in n → error probability is doubly exponential in n.



Converse Two Messages

Theorem 3

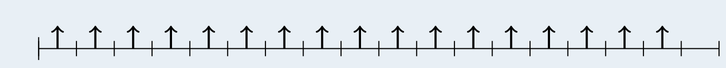
Let $\mathcal{M} = \{0, 1\}$. For any $\rho \in (0, 1/2)$ and any $P > 0$,

$$\limsup_{n\to\infty} -\frac{1}{n} \log P_e^{(n)} \le \frac{2\left(\sqrt{\rho P + \sigma^2/2} + \sqrt{(1-\rho)P}\right)^2}{\sigma^2}.$$

The optimal error probability decreases exponentially in n if feedback occurs with a probability smaller than one half. Note that for $\rho \nearrow 1/2$, the upper bound does not grow to infinity, although the first order error exponent is infinite for any $\rho > 1/2$ by Theorem 1.

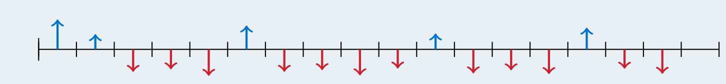
Heuristics for $\rho < 1/2$

Consider the binary repitition code:



What the transmitter sees (with high probability):

What the receiver sees (with exponentially decreasing probability):



Most of $\mathbf{Y}$ looks similar to the wrong code word but the transmitter sees none of the *bad symbols*

Intuition

Boosted retransmission is necessary in all cases with less than 50% feedback. For $\rho < 1/2$, the probability of this event tends to one and a boosted retransmission is not possible due to the power constraint.

Proof of Converse

1. Write error probability as

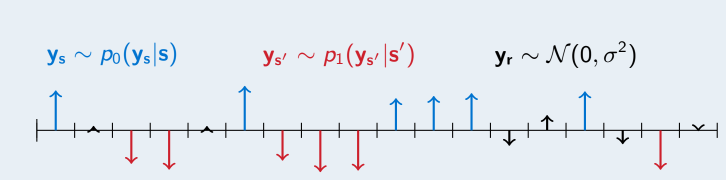
$$\Pr(\text{error}) = \frac{1}{2} \sum_{\mathbf{s} \in \{0,1\}^n} \Pr(\mathbf{S} = \mathbf{s}) \int_{\mathbf{y} \in \mathcal{D}_1} p_0(\mathbf{y}|\mathbf{s}) d\mathbf{y}$$
$$+ \frac{1}{2} \sum_{\mathbf{s}' \in \{0,1\}^n} \Pr(\mathbf{S}' = \mathbf{s}') \int_{\mathbf{y} \in \mathcal{D}_0} p_1(\mathbf{y}|\mathbf{s}') d\mathbf{y}$$

2. Define sets of
   - Strongly typical feedback patterns $\mathcal{G} = \{\mathbf{s} : \text{approx. } (\rho \pm \delta)n \text{ feedback symbols}\}$, $\delta > 0$
   - Energy limited output sequences $\mathcal{T}_y = \{\mathbf{y} : \|\mathbf{y}\|^2 < n\alpha^2\}$, $\alpha > 0$
   - Output sequences that yield energy limited channel inputs $\mathcal{T}_m(\mathbf{s}) = \{\mathbf{y} : \|\mathbf{x}_m\|^2 < n\beta^2\}$, $\beta > 0$
3. Note that

$$\Pr(\text{error}) \ge \frac{1}{2} \sum_{\mathbf{s} \in \mathcal{G}} \Pr(\mathbf{S} = \mathbf{s}) \int_{\mathbf{y} \in \mathcal{D}_1 \cap \mathcal{T}_y \cap \mathcal{T}_0(\mathbf{s})} p_0(\mathbf{y}|\mathbf{s}) d\mathbf{y}$$
$$+ \frac{1}{2} \sum_{\mathbf{s}' \in \mathcal{G}} \Pr(\mathbf{S}' = \mathbf{s}') \int_{\mathbf{y} \in \mathcal{D}_0 \cap \mathcal{T}_y \cap \mathcal{T}_1(\mathbf{s}')} p_1(\mathbf{y}|\mathbf{s}') d\mathbf{y}$$

4. Lower bound output density conditioned on $\mathbf{s}$ and $\mathbf{s}'$:

$$p_0(\mathbf{y}|\mathbf{s}) \ge q(\mathbf{y}|\mathbf{s}, \mathbf{s}') e^{-n\frac{(\alpha+\beta)^2}{2\sigma^2}},$$
$$p_1(\mathbf{y}|\mathbf{s}') \ge q(\mathbf{y}|\mathbf{s}, \mathbf{s}') e^{-n\frac{(\alpha+\beta)^2}{2\sigma^2}}.$$



Under $q(\mathbf{y}|\mathbf{s}, \mathbf{s}')$

- $\approx \rho n$ symbols $\sim p_0(.|\mathbf{s})$
- $\approx \rho n$ symbols $\sim p_1(.|\mathbf{s}')$
- $\approx (1 - 2\rho)n$ symbols noise
- $\mathbf{s}, \mathbf{s}'$ have no symbol in common (this fails for $\rho > 1/2$)

5. Combine terms for $m = 0$ and $m = 1$:

$$\int_{\mathbf{y} \in \mathcal{D}_1 \cap \mathcal{T}_y \cap \mathcal{T}_0(\mathbf{s}) \cap \mathcal{T}_1(\mathbf{s}')} q(\mathbf{y}|\mathbf{s}, \mathbf{s}') d\mathbf{y} + \int_{\mathbf{y} \in \mathcal{D}_0 \cap \mathcal{T}_y \cap \mathcal{T}_0(\mathbf{s}) \cap \mathcal{T}_1(\mathbf{s}')} q(\mathbf{y}|\mathbf{s}, \mathbf{s}') d\mathbf{y}$$
$$= \int_{\mathbf{y} \in \mathcal{T}_y \cap \mathcal{T}_0(\mathbf{s}) \cap \mathcal{T}_1(\mathbf{s}')} q(\mathbf{y}|\mathbf{s}, \mathbf{s}') d\mathbf{y}$$

6. Define new joint probability measure Q and use Markov's inequality with suitable choices for α and β:

$$\Pr(\text{error})$$
$$\ge \frac{1}{2} e^{-n\frac{(\alpha+\beta)^2}{2\sigma^2}} \mathrm{P}(\mathbf{S} \in \mathcal{F}) \mathrm{Q}(\mathbf{Y} \in \mathcal{T}_y \cap \mathcal{T}_0(\mathbf{S}) \cap \mathcal{T}_1(\mathbf{S}'))$$
$$\ge \frac{1}{2} \varepsilon\, e^{-n\frac{4\left(\sqrt{(\rho P + (1+2\delta)\sigma^2/2} + \sqrt{(1-\rho)P}\right)^2}{2\sigma^2(1-\varepsilon)^2}}$$

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