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K-RODE-TAYLOR SCHEMES OF ORDER 3 AND 4 FOR THE KANAI-TAJIMI EARTHQUAKE MODEL

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Abstract. Random ordinary differential equations (RODEs) represent an alternative way to formulate many stochastic ordinary differential equations (SODEs). RODEs allow us to rewrite SODEs in terms most scientists are familiar with, namely as a special type of ODEs. For the numerical solution of RODEs, different approaches have been proposed, such as averaged methods or K-RODE-Taylor schemes. K-RODE-Taylor schemes allow for higher-order discretisations at the prize of more complex, recursive formulations.

In this contribution we derive the K-RODE-Taylor schemes of order $K = 3$ and $K = 4$ for the specific RODE formulation of the Kanai-Tajimi earthquake model. Finally, we briefly discuss ways to compute approximations of the multiple integrals that appear in the corresponding formulas.

1. Introduction. Many interesting application scenarios in scientific computing nowadays involve random input. In the context of time-dependent problems, these problems have been treated in terms of stochastic ordinary differential equations (SODEs). An alternative (yet less prominent) way to describe these scenarios are random ordinary differential equations, which closely resemble regular ODEs (cf. [1]). The general form of a RODE initial value problem reads

$$\frac{dx}{dt} = F_\omega(t, x) := f(\omega(t), x), \quad x(t_0) = x_0, \quad (1.1)$$

for almost all $\omega \in \Omega$. Here, $\omega(t)$ represents a realisation in time of a stochastic process. The more common SODEs have the form

$$dX_t = a(X_t, t)dt + b(X_t, t)dW_t. \quad (1.2)$$

The functions $a(X_t, t)$ and $b(X_t, t)$ are commonly referred to as the drift and the diffusion terms respectively, and W_t is a usual Wiener process. The solution of (1.2) is given by

$$X_t = X_0 + \int_{t_0}^t a(X_s, s)ds + \int_{t_0}^t b(X_s, s)dW_s, \quad (1.3)$$

with initial value $X_{t_0} = X_0$. While the first integral in (1.3) is well defined, the second one requires us to integrate with respect to the Wiener process and has to be treated in a special way. This is the main downside of SODEs.

However, many SODEs can be formulated as RODEs and vice versa by means of the Doss-Sussmann / Imkeller-Schmalfuss (DSIS) correspondence [5, 10, 4]. Different numerical schemes have been developed for RODEs, most notably averaged methods and K-RODE-Taylor schemes [3, 8]. We focus on the higher-order K-RODE-Taylor schemes and apply them to a specific model, the Kanai-Tajimi (KT) earthquake model. The KT model approximates the stochastic motion of the earth's crust during earthquake excitations.

In this paper we derive in detail the K-RODE-Taylor schemes of orders $K = 3$ and $K = 4$ for the KT earthquake model explicitly. Since K-RODE-Taylor schemes are recursive, it is cumbersome to write down their explicit form for a given RODE. But once this is done one can simplify and optimize the computations of the individual terms, as we will see¹. The resulting schemes have been used for

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¹Alternatively, one could think of programming a K-RODE-Taylor scheme recursively for any K and a given RODE without explicitly calculating its final form, which lifts the effort of carrying out the calculations we performed, but it would not allow further optimizations or simplifications.

different publications concerning different implementation and performance aspects of the schemes, but their non-trivial derivation has not been described in detail. Furthermore, two ways to compute the resulting multiple time integrals over the Ornstein-Uhlenbeck (OU) process are briefly discussed. Thus, we have a reference covering all implementation aspects of the K-RODE-Taylor schemes of order 3 and 4.

The remainder of this paper is organized as follows: In Sec. 2 we briefly present the Kanai-Tajimi earthquake model and its RODE form, before reviewing the general form of the K-RODE-Taylor schemes in Sec. 3. We derive the K-RODE-Taylor scheme of orders $K = 3$ and $K = 4$ applied to the Kanai-Tajimi RODE model in Sec. 4 and 5, respectively. Section 6 describes two alternative ways to modify and approximate the multiple time integrals over the OU process, and we conclude this contribution in Sec. 7.

2. The Kanai-Tajimi Earthquake Model. We will apply the K-RODE-Taylor schemes of orders 3 and 4 to a variant of the Kanai-Tajimi earthquake model, which models earthquake-induced ground excitations properly for high frequency ranges [6, 7, 11].

The SODE form of the Kanai-Tajimi model for the ground acceleration $\ddot{u}_g(t)$ is

$$\ddot{u}_g = \ddot{x}_g + \xi_t = -2\zeta_g\omega_g\dot{x}_g - \omega_g^2x_g,$$

where x_g denotes the solution of a stochastic oscillator driven by zero-mean Gaussian white noise ξ_t

$$\ddot{x}_g + 2\zeta_g\omega_g\dot{x}_g + \omega_g^2x_g = -\xi_t, \quad x_g(0) = \dot{x}_g(0) = 0. \quad (2.1)$$

ζ_g and ω_g represent parameters to adapt the model to local geological site conditions. Using the definition of the OU process (cf. [2]) and the Doss-Sussmann/Imkeller-Schmalfuss correspondence (cf. [5, 10, 4]) one can formulate the SODE (2.1) as a first-order system of RODEs

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} -(z_2 + O_t) \\ -2\zeta_g\omega_g(z_2 + O_t) + \omega_g^2z_1 + O_t \end{pmatrix}, \quad (2.2)$$

with initial condition $z(0) = z_0 \in \mathbb{R}^2$. The exact update formula of the OU process is given by

$$O_{t+h} = O_t \cdot \mu_X + \sigma_X \cdot n_1 \quad (2.3)$$

for a timestep $h > 0$ and parameters $\mu_X = e^{-h/\tau}$ and $\sigma_X = (\frac{c\tau}{2}(1 - \mu_X^2))^{\frac{1}{2}}$. The constants c and τ are called *relaxation time* and *diffusion constant* respectively, and are problem dependent. (In the following we choose $\tau = c = 1$.) Also, $n_1 \sim \mathcal{N}(0, 1)$ is a normally distributed random variable.

3. Recursive Definition of the K-RODE-Taylor Schemes. The basic idea of the K-RODE-Taylor schemes is to derive higher-order schemes via Taylor expansions of the r.h.s. function f of Eq. (1.1), analogous to Runge-Kutta methods for deterministic ODEs. This is achieved by expanding f in the smooth variables X_t and ω , but not in time, where f is not smooth due to the dependence on the stochastic process. The resulting schemes are described in detailed in [8], and require a special notation, to which we now turn our attention.

Since we will use two-dimensional Taylor-like expansions of f (with respect to ω and x), we use multi-indices $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$. Their magnitude is represented by $|\alpha| := \alpha_1 + \alpha_2$ and can be generalised by a weight $\theta \in (0, 1]$ such that $|\alpha|_\theta := \theta\alpha_1 + \alpha_2$. In addition, for each $K \in \mathbb{R}_+$ with $K \geq |\alpha|_\theta$, we define $|\alpha|_\theta^K := K - |\alpha|_\theta$. Additionally, the factorial of the multi-index is given by $\alpha! := \alpha_1!\alpha_2!$ and we denote the partial derivatives w.r.t. the multi-index as $f_\alpha := (\partial_1)^{\alpha_1}(\partial_2)^{\alpha_2}f$. The order of the scheme is denoted $K \in \mathbb{R}_+$, and with it we can define the sets of multi-indices

$$\mathcal{A}_K := \{\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2 \quad : \quad |\alpha|_\theta = \theta\alpha_1 + \alpha_2 < K\}.$$

The explicit update formula of the K-RODE-Taylor scheme for the approximated solution $y_n^{K,h}$ then reads

$$y_{n+1}^{K,h} := y_n^{K,h} + \sum_{\mathcal{A}_K} N_\alpha^{(K)}(t_{n+1}, t_n, y_n^{K,h}), \quad (3.1)$$

where $h \in (0, 1]$ is the stepsize and

$$N_\alpha^{(K)}(\hat{t} + h, \hat{t}, \hat{y}) := \frac{1}{\alpha!} f_\alpha(\omega(\hat{t}), \hat{y}) \int_{\hat{t}}^{\hat{t}+h} (\Delta\omega_s)^{\alpha_1} \left(\Delta y_{\Delta s}^{(|\alpha|_{\hat{\theta}}^{(K)})}(\hat{t}, \hat{y}) \right)^{\alpha_2} ds, \quad (3.2a)$$

$$\Delta y_h^{(L)}(\hat{t}, \hat{y}) := \sum_{\mathcal{A}_L} N_\alpha^{(L)}(\hat{t} + h, \hat{t}, \hat{y}), \quad (3.2b)$$

with $\Delta\omega_s := \omega(s) - \omega(\hat{t})$, $\Delta s = s - \hat{t}$ for an arbitrary time $\hat{t} \in [t_0, T]$.

The recursive structure of the scheme (3.1) is visible in Eqs. (3.2a)–(3.2b), since $\Delta y_{\Delta s}^{(|\alpha|_{\hat{\theta}}^{(K)})}$ is of order $|\alpha|_{\hat{\theta}}^{(K)} = K - |\alpha|_{\hat{\theta}} < K$. The value of the weight θ is chosen as the supremum of the Hölder coefficients of the sample paths of the underlying stochastic process (see [8] for details). The recursivity of the K-RODE-Taylor schemes makes the derivation and implementation of the scheme a nontrivial task. However, the particular form of our RODE (2.2) allows for an explicit form of the scheme for moderate K , which we derive in the next two sections.

4. The 3-RODE-Taylor Scheme. In this section, we formulate the K-RODE-Taylor scheme of order $K = 3$ for the Kanai-Tajimi earthquake model in the RODE form (2.2). Since the sample paths of the Ornstein-Uhlenbeck process are Hölder continuous of order $< 1/2$, we have $\theta = 1/2$, which results in the index set

$$\mathcal{A}_3 = \{(\alpha_1, \alpha_2) \in \mathbb{N}, \alpha_1 + 2\alpha_2 < 6\}. \quad (4.1)$$

So, $\mathcal{A}_3 = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (3, 0), (3, 1), (4, 0), (5, 0)\}$. The scheme has the form

$$y_{n+1}^{(3),h} = y_n^{(3),h} + \sum_{\mathcal{A}_3} N_\alpha^{(3)}(t_{n+1}, t_n, y_n^{(3),h}), \quad (4.2)$$

where the “step vectors” $N_\alpha^{(3)}$ are computed with $\Delta O_s := O_s - O_{t_n}$ and $\Delta s := s - t_n$ as

$$N_\alpha^{(3)}(t_{n+1}, t_n, y_n^{(3),h}) := \frac{1}{\alpha!} f_\alpha(O_{t_n}, y(t_n)) \int_{t_n}^{t_{n+1}} (\Delta O_s)^{\alpha_1} \left(\Delta y_{\Delta s}^{(|\alpha|_{1/2}^{(3)})}(t_n, y_n^{(3),h}) \right)^{\alpha_2} ds, \quad (4.3)$$

using

$$\Delta y_{\Delta s}^{(|\alpha|_{1/2}^{(L)})}(t_n, y_n^{(3),h}) := \sum_{|\alpha|_{1/2} < L} N_\alpha^{(L)}(t_n + \Delta s, t_n, y_n^{(3),h}). \quad (4.4)$$

Here, $L := K - |\alpha|_{\theta} < K$, i.e., we sum over a set smaller or equal to \mathcal{A}_K (which is equivalent to say that we use a method of lower order L).

Now we evaluate the sum in (4.2) for every pair (α_1, α_2) . However, since our function f is linear in both O_t and z , any derivative of order 2 or higher vanishes, as well as the pair $(1, 1)$ (since f is additive in z and O_t), so the only index pairs we need to evaluate are $\{(0, 0), (0, 1), (1, 0)\}$. Omitting the arguments of f and its partial derivatives for the sake of simplicity, we have:

- For $(0, 0)$,

$$N_{(0,0)}^{(3)}(t_{n+1}, t_n, y_n^{3,h}) = \frac{1}{0!} f_{(0,0)} \int_{t_n}^{t_{n+1}} (\Delta O_s)^0 \cdot \left(\Delta y_{\Delta s}^{(|\alpha|_{1/2}^{(3)})}(t_n, y_n^{3,h}) \right)^0 ds = hf.$$

- For $(1, 0)$,

$$N_{(1,0)}^{(3)}(t_{n+1}, t_n, y_n^{3,h}) = f_{(1,0)} \int_{t_n}^{t_{n+1}} \Delta O_s ds = \frac{\partial f}{\partial O_t} \int_{t_n}^{t_{n+1}} (O_s - O_{t_n}) ds.$$

- For $(0, 1)$,

$$\begin{aligned} N_{(0,1)}^{(3)}(t_{n+1}, t_n, y_n^{3,h}) &= f_{(0,1)} \int_{t_n}^{t_{n+1}} \Delta y_{\Delta s}^{(|\alpha|_{1/2}^{(3)})}(t_n, y_n^{3,h}) ds, \\ &= f_{(0,1)} \int_{t_n}^{t_{n+1}} \left(\sum_{\mathcal{A}_2} N_{\alpha}^{(2)}(t_n + \Delta s, t_n, y_n^{3,h}) \right) ds, \end{aligned}$$

where we evaluate $f_{(0,1)} = \partial_z f(z)$ as the Jacobian matrix of f (since $z \in \mathbb{R}^2$). From the last term $(0, 1)$, we obtain the sum over one order lower, \mathcal{A}_2 , which requires again the evaluation of the pairs $\{(0, 0), (1, 0), (0, 1)\}$, and so on, until the lowest order is achieved (namely, $K = 0$ where all contributions are zero due to empty sums in definition (4.4)). For the sake of clarity, let us write the last term explicitly (without writing the dependencies on t_n and y_n),

$$\begin{aligned} N_{(0,1)}^{(3)} &= f_{(0,1)} \int_{t_n}^{t_{n+1}} \left(\sum_{\mathcal{A}_2} N_{\alpha}^{(2)}(\Delta s) \right) ds \\ &= f_{(0,1)} \int_{t_n}^{t_{n+1}} \left(N_{(0,0)}^{(2)}(\Delta s) + N_{(1,0)}^{(2)}(\Delta s) + N_{(0,1)}^{(2)}(\Delta s) \right) ds \\ &= f_{(0,1)} \int_{t_n}^{t_{n+1}} \left(N_{(0,0)}^{(2)}(\Delta s) + N_{(1,0)}^{(2)}(\Delta s) + f_{(0,1)} \int_{t_n}^s \left(\sum_{\mathcal{A}_1} N_{\alpha}^{(1)}(\Delta v) \right) dv \right) ds \\ &= f_{(0,1)} \int_{t_n}^{t_{n+1}} \left(N_{(0,0)}^{(2)}(\Delta s) + N_{(1,0)}^{(2)}(\Delta s) + f_{(0,1)} \int_{t_n}^s \left(N_{(0,0)}^{(1)}(\Delta v) + N_{(1,0)}^{(1)}(\Delta v) \right) dv \right) ds \\ &= f_{(0,1)} \int_{t_n}^{t_{n+1}} \left(f \int_{t_n}^s + f_{(1,0)} \int_{t_n}^s \Delta O_v dv + f_{(0,1)} \int_{t_n}^s \left(f \int_{t_n}^v + f_{(1,0)} \int_{t_n}^v \Delta O_w dw \right) dv \right) ds \\ &= f_{(0,1)} f \frac{h^2}{2} + f_{(0,1)} \int_{t_n}^{t_{n+1}} \left(f_{(1,0)} \int_{t_n}^s \Delta O_v dv + f_{(0,1)} \int_{t_n}^s \left((v - t_n) f + f_{(1,0)} \int_{t_n}^v \Delta O_w dw \right) dv \right) ds \\ &= f_{(0,1)} f \frac{h^2}{2} + f_{(0,1)} f_{(1,0)} \int_{t_n}^{t_{n+1}} \int_{t_n}^s \Delta O_v dv ds + f_{(0,1)}^2 f \frac{h^3}{6} + f_{(0,1)}^2 f_{(1,0)} \int_{t_n}^{t_{n+1}} \int_{t_n}^s \int_{t_n}^v \Delta O_w dw dv ds. \end{aligned}$$

Note that inserting the different terms recursively requires to keep the dependency on Δs (which is indicated by stating only the first argument Δs of $N_{\alpha}^{(2)}$). This results in a rapidly increasing number of integrals with respect to s that have to be computed.

Finally, the scheme (3.1) takes the following form:

$$\begin{aligned} y_{n+1}^{(3),h} &= y_n^{(3),h} + hf + f_{(1,0)} \int_{t_n}^{t_{n+1}} \Delta O_s ds + f_{(0,1)} f \frac{h^2}{2} + f_{(0,1)} f_{(1,0)} \int_{t_n}^{t_{n+1}} \int_{t_n}^s \Delta O_v dv ds \\ &\quad + f_{(0,1)}^2 f \frac{h^3}{6} + f_{(0,1)}^2 f_{(1,0)} \int_{t_n}^{t_{n+1}} \int_{t_n}^s \int_{t_n}^v \Delta O_w dw dv ds, \end{aligned} \quad (4.5)$$

again, with

$$f(O_t, z) = \begin{pmatrix} -(z_2 + O_t) \\ -2\zeta_g \omega_g (z_2 + O_t) + \omega_g^2 z_1 + O_t \end{pmatrix}, \quad (4.6)$$

$$f_{(1,0)}(O_t, z) = \begin{pmatrix} -1 \\ -2\zeta_g \omega_g + 1 \end{pmatrix}, \quad (4.7)$$

$$f_{(0,1)}(O_t, z) = \begin{pmatrix} 0 & -1 \\ \omega_g^2 & -2\zeta_g \omega_g \end{pmatrix}. \quad (4.8)$$

5. The 4-RODE-Taylor Scheme. The derivation of the 4-RODE-Taylor scheme for the KT model is similar to the one for $K = 3$. Additional terms with higher multiplicity in the integrals will appear due to the recursivity of $N_{(0,1)}^{(4)}$, but only due to that term.

For $K = 4$, we have

$$\mathcal{A}_4 = \{(\alpha_1, \alpha_2) \in \mathbb{N}, \alpha_1 + 2\alpha_2 < 8\}, \quad (5.1)$$

but the linearity of f limits this set again to $\{(0, 0), (0, 1), (1, 0)\}$ for evaluations. Thus, we have:

- For $(0, 0)$,

$$N_{(0,0)}^{(4)}(t_{n+1}, t_n, y_n^{4,h}) = \frac{1}{0!} f_{(0,0)} \int_{t_n}^{t_{n+1}} (\Delta O_s)^0 \cdot \left(\Delta y_{\Delta s}^{(|\alpha|_{1/2}^{(4)})}(t_n, y_n^{4,h}) \right)^0 ds = hf.$$

- For $(1, 0)$,

$$N_{(1,0)}^{(4)}(t_{n+1}, t_n, y_n^{4,h}) = f_{(1,0)} \int_{t_n}^{t_{n+1}} \Delta O_s ds = \frac{\partial f}{\partial O_t} \int_{t_n}^{t_{n+1}} (O_s - O_{t_n}) ds.$$

Calculating the remaining term $N_{(0,1)}^{(4)}$ results in

$$\begin{aligned} N_{(0,1)}^{(4)} &= f_{(0,1)} \int_{t_n}^{t_{n+1}} \left(\sum_{\mathcal{A}_3} N_{\alpha}^{(3)}(\Delta s) \right) ds \\ &= f_{(0,1)} \int_{t_n}^{t_{n+1}} \left(N_{(0,0)}^{(3)}(\Delta s) + N_{(1,0)}^{(3)}(\Delta s) + N_{(0,1)}^{(3)}(\Delta s) \right) ds \\ &= f_{(0,1)} \int_{t_n}^{t_{n+1}} \left(N_{(0,0)}^{(3)}(\Delta s) + N_{(1,0)}^{(3)}(\Delta s) + f_{(0,1)} \int_{t_n}^s \left(\sum_{\mathcal{A}_2} N_{\alpha}^{(2)}(\Delta v) \right) dv \right) ds \\ &= f_{(0,1)} f \frac{h^2}{2} + f_{(0,1)} f_{(1,0)} \int_{t_n}^{t_{n+1}} \int_{t_n}^s \Delta O_v dv ds \\ &\quad + f_{(0,1)} \int_{t_n}^{t_{n+1}} \left(f_{(0,1)} \int_{t_n}^s \left(N_{(0,0)}^{(2)}(\Delta v) + N_{(1,0)}^{(2)}(\Delta v) + N_{(0,1)}^{(2)}(\Delta v) \right) dv \right) ds \end{aligned}$$

$$\begin{aligned}
&= f_{(0,1)} f \frac{h^2}{2} + f_{(0,1)} f_{(1,0)} \int_{t_n}^{t_{n+1}} \int_{t_n}^s \Delta O_v dv ds \\
&\quad + f_{(0,1)} f_{(0,1)} f \int_{t_n}^{t_{n+1}} \int_{t_n}^s \int_{t_n}^v dw dv ds + f_{(0,1)} f_{(0,1)} f_{(1,0)} \int_{t_n}^{t_{n+1}} \int_{t_n}^s \int_{t_n}^v \Delta O_w dw dv ds \\
&\quad + f_{(0,1)} \int_{t_n}^{t_{n+1}} \left(f_{(0,1)} \int_{t_n}^s \left(f_{(0,1)} \int_{t_n}^v \left(\sum_{\mathcal{A}_1} N_\alpha^{(1)}(\Delta w) \right) dw \right) dv \right) ds \\
&= f_{(0,1)} f \frac{h^2}{2} + f_{(0,1)} f_{(1,0)} \int_{t_n}^{t_{n+1}} \int_{t_n}^s \Delta O_v dv ds + f_{(0,1)}^2 f \frac{h^3}{6} + f_{(0,1)}^2 f_{(1,0)} \int_{t_n}^{t_{n+1}} \int_{t_n}^s \int_{t_n}^v \Delta O_w dw dv ds \\
&\quad + f_{(0,1)} \int_{t_n}^{t_{n+1}} \left(f_{(0,1)} \int_{t_n}^s \left(f_{(0,1)} \int_{t_n}^v \left(f \int_{t_n}^w dx + f_{1,0} \int_{t_n}^w \Delta O_x dx \right) dw \right) dv \right) ds \\
&= f_{(0,1)} f \frac{h^2}{2} + f_{(0,1)} f_{(1,0)} \int_{t_n}^{t_{n+1}} \int_{t_n}^s \Delta O_v dv ds + f_{(0,1)}^2 f \frac{h^3}{6} + f_{(0,1)}^2 f_{(1,0)} \int_{t_n}^{t_{n+1}} \int_{t_n}^s \int_{t_n}^v \Delta O_w dw dv ds \\
&\quad + f_{(0,1)}^3 f \frac{h^4}{24} + f_{(0,1)}^3 f_{(1,0)} \int_{t_n}^{t_{n+1}} \int_{t_n}^s \int_{t_n}^v \int_{t_n}^w \Delta O_x dx dw dv ds .
\end{aligned}$$

Therefore, the overall 4-RODE-Taylor scheme for the KT model has the form

$$\begin{aligned}
y_{n+1}^{(4),h} &= y_n^{(4),h} + hf + f_{(1,0)} \int_{t_n}^{t_{n+1}} \Delta O_s ds + f_{(0,1)} f \frac{h^2}{2} + f_{(0,1)} f_{(1,0)} \int_{t_n}^{t_{n+1}} \int_{t_n}^s \Delta O_v dv ds \\
&\quad + f_{(0,1)}^2 f \frac{h^3}{6} + f_{(0,1)}^2 f_{(1,0)} \int_{t_n}^{t_{n+1}} \int_{t_n}^s \int_{t_n}^v \Delta O_w dw dv ds \\
&\quad + f_{(0,1)}^3 f \frac{h^4}{24} + f_{(0,1)}^3 f_{(1,0)} \int_{t_n}^{t_{n+1}} \int_{t_n}^s \int_{t_n}^v \int_{t_n}^w \Delta O_x dx dw dv ds ,
\end{aligned} \tag{5.2}$$

with f , $f_{(1,0)}$, and $f_{(0,1)}$ defined via (4.6)–(4.8).

6. Integration of the OU process terms. The final step is to calculate the (multiple) integrals of the OU process in (4.5) and (5.2). We restrict our discussion to the case of order $K = 3$ which is complex enough to observe higher multiplicity of the integrals but moderate enough to keep the notation compact. We first follow a direct approach to compute the multiple integrals by reformulating the problem using properties of the underlying OU process. Then, a more general approach is briefly discussed which allows to transform the relevant multiple integrals into one-dimensional counterparts.

6.1. Direct Approach for Integrating OU terms. From the theory of stochastic processes [2], we know that the OU process O_t is the solution of the following stochastic ordinary differential equation (known as the *Langevin* equation):

$$dO_t = -\frac{1}{\tau} O_t h + c^{1/2} dW_t. \tag{6.1}$$

The solution of the OU process at a time $t + h$ for some $h > 0$ is then given by Eq. (2.3). Furthermore, the integral of the OU process (which we will denote \tilde{O}_t) can also be written down explicitly. It is defined through the relation $\tilde{O}_{t+h} = \tilde{O}_t + O_t + h$, and its explicit expression is given by

$$\tilde{O}_{t+\Delta t} = \tilde{O}_t + O_t \tau (1 - \mu) + \left(\sigma_Y^2 - \frac{\kappa_{XY}^2}{\sigma_X^2} \right)^{1/2} n_2 + \frac{\kappa_{XY}}{\sigma_X} n_1, \tag{6.2}$$

where $\mu := e^{-h/\tau}$, $\sigma_Y^2 := c\tau^3[\Delta t/\tau - 2(1-\mu) + (1/2)(1-\mu^2)]$ and $\kappa_{XY} := (c\tau^2/2)(1-\mu)^2$. With the use of Eq. (6.2) it is possible to evaluate the first integrals in our numerical scheme (4.5). For instance

$$\begin{aligned} \int_{t_n}^{t_{n+1}} \Delta O_s ds &= \int_{t_n}^{t_{n+1}} (O_s - O_{t_n}) ds = \Delta \tilde{O}_{t_{n+1}} - hO_{t_n} = \tilde{O}_{t_{n+1}} - \tilde{O}_{t_n} - hO_{t_n}, \\ &= O_{t_n}[\tau(1-\mu) - h] + \left(\sigma_Y^2 - \frac{\kappa_{XY}^2}{\sigma_X^2} \right)^{1/2} n_2 + \frac{\kappa_{XY}}{\sigma_X} n_1. \end{aligned}$$

Since the integral is given in terms of the OU process itself, it can be integrated again, but with the additional difficulty that μ depends on Δt . Thus, the expressions for the second and third integrals are complicated. Therefore, if one does not have a closed expression for the integrals of the stochastic process involved in the differential equation, one can always calculate the integrals numerically, taking care that the order of precision of the integration method is at least as high as that of the K-RODE Taylor scheme.

The double integral reduces to

$$\int_{t_n}^{t_{n+1}} \int_{t_n}^s \Delta O_v dv ds = \int_{t_n}^{t_{n+1}} \left(\tilde{O}_s - \tilde{O}_{t_n} - (s-t_n)O_{t_s} \right) ds = \int_{t_n}^{t_{n+1}} \Delta \tilde{O}_s ds - \frac{h^2}{2} O_{t_n}.$$

Finally, for the triple integral, we have

$$\int_{t_n}^{t_{n+1}} \int_{t_n}^s \int_{t_n}^v \Delta O_w dw dv ds = \int_{t_n}^{t_{n+1}} \int_{t_n}^s \Delta \tilde{O}_v dv ds - \frac{h^3}{6} O_{t_n}.$$

Thus, with the help of the explicit expressions for O_t and \tilde{O}_t we have simplified our scheme (4.5) from calculating three integrals of ΔO_t (one simple, one double, and one triple) to only two integrals of $\Delta \tilde{O}_t$ (one simple and one double). Let us write it once again:

$$\begin{aligned} y_{n+1}^{(3),h} &= y_n^{(3),h} + hf + f_{(1,0)} \left(\Delta \tilde{O}_{t_{n+1}} - hO_{t_n} \right) + f_{(0,1)} f \frac{h^2}{2} + f_{(0,1)} f_{(1,0)} \left(\int_{t_n}^{t_{n+1}} \Delta \tilde{O}_s ds - \frac{h^2}{2} O_{t_n} \right) \\ &+ \frac{h^3}{6} f_{(0,1)}^2 f + f_{(0,1)}^2 f_{(1,0)} \left(\int_{t_n}^{t_{n+1}} \int_{t_n}^s \Delta \tilde{O}_v dv ds - \frac{h^3}{6} O_{t_n} \right), \end{aligned} \quad (6.3)$$

Following [8], we can further simplify the double integral using integration by parts, namely

$$\begin{aligned} \int_{t_n}^{t_{n+1}} \int_{t_n}^s \Delta \tilde{O}_v dv ds &= \int_{t_n}^{t_{n+1}} ds \int_{t_n}^{t_{n+1}} \Delta \tilde{O}_s ds - \int_{t_n}^{t_{n+1}} \int_{t_n}^s \Delta \tilde{O}_s dv ds, \\ &= h \int_{t_n}^{t_{n+1}} \Delta \tilde{O}_s ds - \int_{t_n}^{t_{n+1}} \Delta \tilde{O}_s (s-t_n) ds = h \int_0^h \left(1 - \frac{u}{h} \right) \Delta \tilde{O}_{t_n+u} du, \\ &\approx h\delta \sum_{k=1}^m \left(1 - \frac{k\delta}{h} \right) \Delta \tilde{O}_{t_n+k\delta} = h\delta \underbrace{\sum_{k=1}^m \Delta \tilde{O}_{t_n+k\delta}}_{=:I_1} - \delta^2 \underbrace{\sum_{k=1}^m k \cdot \Delta \tilde{O}_{t_n+k\delta}}_{=:I_2}, \\ &= hI_1 - I_2, \end{aligned}$$

with $\delta = h/m$ and m sufficiently large so that the desired order is obtained. Note that we used a simple Riemann sum to approximate the integrals. A higher order method—for instance, Simpson's rule—would not attain its order, since the integrands do not have the required smoothness. Thus, using a higher order method would not bring any advantages.

Now we have all the ingredients to directly implement the numerical scheme, given in its final form as

$$\begin{aligned} y_{n+1}^{(3),h} &= y_n^{(3),h} + hf + f_{(1,0)} \left(\Delta \tilde{O}_{t_{n+1}} - hO_{t_n} \right) + f_{(0,1)} f \frac{h^2}{2} + f_{(0,1)} f_{(1,0)} \left(I_1 - \frac{h^2}{2} O_{t_n} \right) \\ &+ \frac{h^3}{6} f_{(0,1)}^2 f + f_{(0,1)}^2 f_{(1,0)} \left(hI_1 - I_2 - \frac{h^3}{6} O_{t_n} \right). \end{aligned} \quad (6.4)$$

6.2. Reduction of Multiple Integrals: General Approach. In this section we briefly describe a general approach to transform the multiple integrals into a one-dimensional one that is then going to be approximated by a quadrature rule.

As described in [5] and [9], multiple integrals can be simplified in the following way:

$$\int_{t_n}^{t_{n+1}} \int_{t_n}^{x_d} \int_{t_n}^{x_{d-1}} \cdots \int_{t_n}^{x_1} f(z) dz dx_1 \cdots dx_d = \int_{t_n}^{t_{n+1}} \frac{1}{d!} (t_{n+1} - z)^d f(z) dz. \quad (6.5)$$

Using this method for the multiple time integrals over the difference of the OU process (i.e. ΔO_z etc. as integrand f), we can directly apply low-order numerical quadrature rules such as Riemann sums²,

$$\int_{t_n}^{t_{n+1}} \frac{1}{d!} (t_{n+1} - z)^d \Delta O_z dz \approx \delta \sum_{j=1}^M \frac{1}{d!} (t_{n+1} - z_j)^d \Delta O_{z_j} = \delta \sum_{j=1}^M \frac{1}{d!} (h - j\delta)^d (O_{t_n + j\delta} - O_{t_n}), \quad (6.6)$$

where $h = t_{n+1} - t_n$, $\delta = h/M$, and $z_j := t_n + j\delta$. Note, again, that due to a lack of smoothness in time, higher-order quadrature rules would not attain their order and thus present no advantages. Furthermore, M has to be chosen such that the quadrature approximation error is of the same order as the K-RODE-Taylor scheme itself. The general approach consists then in reducing the multiple integrals to one-dimensional integrals that can be approximated numerically with low-order quadrature rules.

7. Conclusion. We have shown how to apply high-order K-RODE-Taylor schemes to approximate a given RODE, namely the Kanai-Tajimi earthquake model. Since the schemes are recursive it is difficult to write them down explicitly, even for simple RODEs. We described the necessary steps to do this for $K = 3$ and $K = 4$, and we made use of our knowledge of the OU process to simplify several multiple integrals. These can be rewritten as one-dimensional integrals which are easy to compute. This can justify the extra effort of deducing closed expressions for the recursive schemes, especially if performance is an issue.

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²The trapezoidal sums have also been tested and they result in the same amount of computational load and equal approximation error.

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