A one-sided symbol for Itô-Lévy processes

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Abstract

A one-sided symbol of a one-sided bounded Itô-Lévy process is introduced and possible ways for its computation as well as its relation to the infinitesimal generator of the process are provided. Further, an integral criterion for invariant distributions of the underlying process is derived which relies on the one-sided symbol. Several applications in settings of one-sided bounded as well as unbounded processes illustrate the advantages of the new criterion in comparison to previous methods.

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1 Introduction

Over the last two decades, the so-called symbol of a stochastic process has proven to be a useful tool in order to derive fine and global properties of the corresponding stochastic process (cf. [10], [19], [22] and Chapter 5 of [6]). Following ideas of Jacob [12] and Schilling [20], Schnurr has generalized this concept to homogeneous diffusions with jumps in the sense of Jacod and Shiryaev [13]. In its most general form the symbol of an $\mathbb{R}^d$-valued homogeneous diffusion with jumps $(X_t)_{t\geq 0}$ looks as follows. For $x, \xi \in \mathbb{R}^d$

$$p(x,\xi) := -\lim_{t\downarrow 0} \frac{\mathbb{E}^x e^{i(X_t^\sigma - x)'\xi} - 1}{t} \tag{1.1}$$

where X^σ is the process X stopped at time σ, the first exit time of a compact neighborhood of x, and x' denotes the transpose of the vector x. The stopping time σ is vital in order to derive the results on path properties for the most general classes of processes. Under mild

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regularity conditions the limit in (1.1) is always a continuous negative definite function in the sense of Schoenberg (cf. the monograph by Berg and Forst [4]). This means in particular that for each fixed x we obtain a Lévy-Khintchine exponent (cf. (2.1) below).

In [3] we derived an integral criterion for invariant measures of Itô-Lévy processes which is based on the symbol. In that article, we had to use the above formula without the stopping time as obviously distributional properties might be destroyed by stopping, while local path properties are stable under stopping. This resulted in the fact that for the method proposed in [3] the symbol has to satisfy a certain growth condition which corresponds to bounded semimartingale characteristics of the treated processes and hence to bounded coefficients of the SDEs in the background. Since this is a serious restriction, in this paper we aim to show an alternative to the classical symbol, which then can be used in cases without bounded characteristics/coefficients as long as the corresponding processes have a lower or upper bound (e.g. are restricted to be positive in each component). Hereby, we take up the classic idea of using Laplace transforms instead of Fourier transforms/ characteristic functions and define a one-sided symbol.

Thus, after setting the stage in Section 2, we will define the one-sided symbol in Section 3 which also includes several results on how to compute the one-sided symbol as well as on its relation to the generator of the stochastic process. Section 4 then is devoted to the derivation of an integral criterion for invariant measures of the underlying processes which relies on the one-sided symbol. Several applications are added to exhibit the usability of the derived criterion. In particular we also explore how the integral criterion may be applied even in cases of unbounded processes.

2 Preliminaries

Recall that a Lévy process $(L_t)_{t\geq 0}$ is a time- and space-homogeneous Markov process with càdlàg paths which starts a.s. in 0 and which is completely characterized by its *characteristic triplet* (ℓ, Q, N) or by its *Lévy-Khintchine exponent*, i.e.,

$$\phi_L(\xi) := -\log \mathbb{E}[\mathrm{e}^{iL_1'\xi}] = -i\ell'\xi + \frac{1}{2}\xi' Q\xi - \int_{\mathbb{R}^d} \left(\mathrm{e}^{i\xi' y} - 1 - i\xi' y \mathbb{1}_{\{\|y\|<1\}}(y)\right) N(\mathrm{d}y). \quad (2.1)$$

We refer to [18] for any further information on Lévy processes.

In this paper we will focus on the class of Itô-Lévy processes as defined below.

Definition 2.1. An *Itô-Lévy process* is a strong Markov process, which is a semimartingale with respect to every $\mathbb{P}^x$ having semimartingale characteristics of the form

$$\begin{aligned} B_t^{(j)}(\omega) &= \int_0^t \ell^{(j)}(X_s(\omega))\, \mathrm{d}s, \qquad j = 1, ..., d, \\ C_t^{jk}(\omega) &= \int_0^t Q^{jk}(X_s(\omega))\, \mathrm{d}s, \qquad j,k = 1, ..., d, \\ \nu(\omega; \mathrm{d}s, \mathrm{d}y) &= N(X_s(\omega), \mathrm{d}y)\, \mathrm{d}s, \end{aligned} \quad (2.2)$$

for every $x \in \mathbb{R}^d$ with respect to a fixed cut-off function χ. Here $\ell(x) = (\ell^{(1)}(x), ..., \ell^{(d)}(x))'$ is a vector in $\mathbb{R}^d$, $Q(x)$ is a positive semi-definite matrix and N is a Borel transition kernel such that $N(x, \{0\}) = 0$. We call ℓ, Q and $n := \int_{y\neq 0}(1 \wedge \|y\|^2)\ N(\cdot, \mathrm{d}y)$ the *differential characteristics* of the process.

These Itô-Lévy processes have been characterized in [7] as the set of solutions of very general SDEs of Skorokhod-type. In particular this class includes the set of solutions of Lévy driven SDEs which will play an important role in this paper. Recall that this class of processes is sometimes in the literature simply called *Itô processes* (cf. [3]) or *Lévy type processes* (cf. in particular [6]). Both terminologies have their advantages and drawbacks. Different classes are called Itô process in the literature while Lévy type implicates that these processes are 'close to Lévy' hiding the fact that entirely different techniques are to be used in order to deal with this general class of processes.

Continuity of the differential characteristics of the treated Itô-Lévy processes is always sufficient for our purposes. However, we have decided to use the concept of fine continuity in order to derive even more general results. Loosely speaking, the advantage of fine continuity is that one has to care about continuity only as far as the process can detect it. Intuitively speaking a measurable set is finely open, if the process starting in the set remains there for a positive amount of time. Let us recall the definition.

Definition 2.2. Let X be a Markov process with state space $(E, \mathcal{E})$.
(a) A set $A \subseteq E$ is called *finely open* if for each $x \in A$ there is a set $D \in \mathcal{E}^n$ such that $A^c \subseteq D$ and $\mathbb{P}^x(T_D > 0) = 1$. Here, $\mathcal{E}^n$ denotes as usual the nearly Borel sets (cf. [5, p. 60]) and T_D is the first exit time from D.

(b) Let $f : \mathbb{R}^d \to \mathbb{R}$ be a Borel-measurable function. Then f is called *finely continuous*, if it is continuous with respect to the topology defined above.

We will use fine continuity in a way that is governed by the subsequent result which was established in [5, Thm. II.4.8].

Proposition 2.3. *Let X be a Markov process and $f : \mathbb{R}^d \to \mathbb{R}$ be a Borel-measurable function. Then f is finely continuous if and only if the function*

$$t \mapsto f(X_t) = f \circ X_t \tag{2.3}$$

is right continuous at zero $\mathbb{P}^x$-a.s. for every $x \in \mathbb{R}^d$.

3 The Laplace symbol

3.1 Definition of the Laplace symbol

The Laplace symbol can be seen as a state-space dependent right hand side derivative at zero of the Laplace transform of a stochastic process. Since the Laplace transform characterizes the distribution of a random variable, the Laplace symbol in a certain way reflects the infinitesimal changes in the distribution over time.

Definition 3.1. Let $(X_t)_{t\geq 0}$ be a Markov process in $\mathbb{R}^d_+$. Define for every $x, \xi \in \mathbb{R}^d_+$ and $t \geq 0$ the quantity

$$h_\xi(x,t) := \frac{\mathbb{E}^x[\mathrm{e}^{-(X_t-x)'\xi}] - 1}{t}.$$

We call $\lambda : \mathbb{R}^d_+ \times \mathbb{R}^d_+ \to \mathbb{R}$ given by

$$\lambda(x,\xi) := -\lim_{t\downarrow 0} \frac{\mathbb{E}^x \mathrm{e}^{-(X_t-x)'\xi} - 1}{t} = -\lim_{t\downarrow 0} h_\xi(x,t) \tag{3.1}$$

the *Laplace symbol* of X with domain $D_\lambda \subseteq \mathbb{R}^d_+ \times \mathbb{R}^d_+$ whenever the limit exists for every $x, \xi \in D_\lambda$.

Remark 3.2. Obviously the Laplace symbol as in Definition 3.1 can also be defined for any Markov process which is componentwise bounded from below. Throughout this article we use the bound 0 to ease notation. Similarly, for Markov processes bounded from above, one may define a one-sided symbol $\lambda_- : \mathbb{R}^d_- \times \mathbb{R}^d_- \to \mathbb{R}$.

As a first example we consider subordinators.

Example 3.3. A subordinator is a Lévy process in $\mathbb{R}_+$ with only positive increments and it is well known (cf. [18]) that the Laplace transform of such a process $(X_t)_{t\geq 0}$ can be written as

$$\mathbb{E}^0\left[\mathrm{e}^{-\xi X_t}\right] = \mathrm{e}^{-t\lambda(\xi)}$$

where

$$\lambda(\xi) = \ell\xi - \int_{(0,\infty)} \left(\mathrm{e}^{-\xi y} - 1\right)\ N(\mathrm{d}y).$$

with $\ell \geq 0$ and a Lévy measure N on $\mathbb{R}_+$ such that $\int_{(0,\infty)}(1 \vee x)N(\mathrm{d}x) < \infty$. Using the fact that subordinators are homogeneous in space we obtain

$$-\frac{\mathbb{E}^x\left[\mathrm{e}^{-(X_t-x)\xi}\right] - 1}{t} = -\frac{\mathbb{E}^0\left[\mathrm{e}^{-X_t\xi}\right] - 1}{t} = -\frac{\mathrm{e}^{-t\lambda(\xi)} - 1}{t}$$

which converges to $\lambda(\xi)$ for $t \downarrow 0$. Therefore, in this case the Laplace symbol is constant in the first variable and coincides with the Laplace exponent.

Another class of processes for which the Laplace symbol is obviously relevant are Itô-Lévy processes conditioned to stay positive in some way. As an example we consider a Brownian motion absorbed at zero.

Example 3.4. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion in $\mathbb{R}$ and define $(X_t)_{t\geq 0}$ via $X^x_t = x + B_{t\wedge\tau}$, where $\tau = \inf\{t \geq 0, B_t = -x\}$. Then it follows by standard computations that

$$\lambda(x,\xi) = -\frac{d}{dt}\mathbb{E}^x[\mathrm{e}^{-(X_t-x)\xi}]|_{t=0} = -\frac{d}{dt}\mathbb{E}^0[\mathrm{e}^{-B_{t\wedge\tau}\xi}]|_{t=0} = -\frac{\xi^2}{2}, \quad x > 0, \xi \in \mathbb{R}_+,$$

while

$$\lambda(0,\xi) = -\frac{d}{dt}\mathbb{E}^0[\mathrm{e}^{-B_\tau\xi}]|_{t=0} = 0, \quad \xi \in \mathbb{R}_+.$$

Remark 3.5. Obviously, by comparing the symbol and the Laplace symbol defined above, one recognizes that at least formally we have

$$\text{`}\lambda(x,\xi) = p(x, i\xi)\text{'}.$$

Nevertheless, we chose not to define the Laplace symbol via an analytic extension of the symbol. The reason for this is simple. For a characteristic function to be analytic it is necessary that the corresponding distribution has moments of all orders (c.f. Lukacs [15, p. 198]). Since we are interested in an extension of the results in [3] to the case of unbounded characteristics/moments, this restriction obviously would have been to strong.
Still we will sometimes in this paper abuse the above notation, as it simplifies formulas a lot.

3.2 Computing the Laplace symbol

In order to establish the following main result of this section, we need a boundedness assumption which is very weak. In fact it is sufficient that for each differential characteristic d the function $d(\cdot)\exp(-\cdot)$ is bounded. Hence, polynomially bounded is sufficient in order to establish the following result and for all our applications.

Theorem 3.6. *Let $(X_t)_{t\geq 0}$ be an $\mathbb{R}^d_+$-valued Itô-Lévy process with polynomially bounded, finely continuous differential characteristics. For every $\xi \in \mathbb{R}^d$ the limit*

$$\lambda(x,\xi) := -\lim_{t\downarrow 0} \frac{\mathbb{E}^x \mathrm{e}^{-(X_t - x)'\xi} - 1}{t} = -\lim_{t\downarrow 0} h_\xi(x,t)$$

exists and the functions h_ξ are globally bounded in x (and t) for every $\xi \in \mathbb{R}^d_+$. As the limit we obtain

$$\lambda(x,\xi) = \ell(x)'\xi - \frac{1}{2}\xi' Q(x)\xi - \int_{\mathbb{R}^d\setminus\{0\}} \left(\mathrm{e}^{-y'\xi} - 1 + y'\xi \cdot \chi(y)\right) N(x, \mathrm{d}y). \tag{3.2}$$

Since the proof is very similar to the one of [3, Lemma 3.4] we only sketch it here.

Proof. We consider the one dimensional situation, since the multidimensional version works alike. Let $x, \xi \in \mathbb{R}_+$. First we use Itô's formula under the expectation and obtain

$$\frac{1}{t}\mathbb{E}^x\left(\mathrm{e}^{-(X_t-x)\xi} - 1\right) = \frac{1}{t}\mathbb{E}^x\left(\int_{(0,t]} -\xi \mathrm{e}^{-(X_{s-}-x)\xi}\,\mathrm{d}X_s\right) \tag{I}$$

$$+ \frac{1}{t}\mathbb{E}^x\left(\frac{1}{2}\int_{(0,t]} \xi^2 \mathrm{e}^{-(X_{s-}-x)\xi}\,\mathrm{d}[X,X]^c_s\right) \tag{II}$$

$$+ \frac{1}{t}\mathbb{E}^x\left(\mathrm{e}^{x\xi}\sum_{0<s\leq t}\left(\mathrm{e}^{-\xi X_s} - \mathrm{e}^{-\xi X_{s-}} + \xi \mathrm{e}^{-\xi X_{s-}}\Delta X_s\right)\right). \tag{III}$$

One has to deal with these terms one-by-one.

In order to calculate term (I) we use the canonical decomposition of a semimartingale (see [13, Thm. II.2.34])

$$X_t = X_0 + X_t^c + \int_{(0,t]} \chi(y)y \left(\mu^X(\cdot;\mathrm{d}s,\mathrm{d}y) - \nu(\cdot;\mathrm{d}s,\mathrm{d}y)\right) + \check{X}_t(\chi) + A_t(\chi). \tag{3.3}$$

where $A_t(\chi)$ is the finite variation part and $\check{X}_t = \sum_{s\le t}(\Delta X_s(1-\chi(\Delta X_s))$. It is easy to show that the integrals with respect to the martingale parts yield again martingales as they trivially are local martingales which can be shown to be bounded. To this end observe that

$$\begin{aligned}\left[\mathrm{e}^{-(X-x)\xi} \bullet X^c, \mathrm{e}^{-(X-x)\xi} \bullet X^c\right]_t &= \int_{(0,t]} (\mathrm{e}^{-(X_s-x)\xi})^2 \; \mathrm{d}[X^c,X^c]_s \\ &= \mathrm{e}^{2x\xi} \int_{(0,t]} \left(\mathrm{e}^{-2X_s\xi}Q(X_s)\right) \; \mathrm{d}s.\end{aligned}$$

The last term is uniformly bounded in ω and, therefore, finite for every $t \ge 0$. The integral with respect to the compensated sum of small jumps can be treated alike. Hence, the expected values of the two martingale parts are zero. The remaining jump part has to be put together with term (III).

Now we deal with term (II). Here we have $[X,X]_t^c = [X^c,X^c]_t = C_t = (Q(X_t) \bullet t)$. Hence

$$\frac{1}{2}\int_{(0,t]} \xi^2 \mathrm{e}^{-(X_{s-}-x)\xi} \; \mathrm{d}[X,X]_s^c = \frac{1}{2}\xi^2 \int_{(0,t]} \mathrm{e}^{-(X_{s-}-x)\xi} Q(X_s) \; \mathrm{d}s. \tag{3.4}$$

Since Q is finely continuous and since $w \mapsto \mathrm{e}^{-w\xi}Q(w)$ is bounded we obtain by dominated convergence

$$\lim_{t\downarrow 0} \frac{1}{2}\xi^2 \frac{1}{t}\mathbb{E}^x \int_{(0,t]} \mathrm{e}^{-(X_s-x)\xi}Q(X_s) \; \mathrm{d}s = \frac{1}{2}\xi^2 Q(x).$$

The remaining part of term (I) as well as the jump part work alike. Putting the terms together, we obtain in addition

$$\begin{aligned}\left|\frac{\mathbb{E}^x\mathrm{e}^{-(X_t-x)'\xi}-1}{t}\right| &= \Bigg| -\xi\frac{1}{t}\mathbb{E}^x \int_{(0,t]} \mathrm{e}^{-(X_{s-}-x)\xi}\ell(X_s) \; \mathrm{d}s + \frac{1}{2}\xi^2\frac{1}{t}\mathbb{E}^x \int_{(0,t]} \mathrm{e}^{-(X_{s-}-x)\xi}Q(X_s) \; \mathrm{d}s \\ &\qquad + \frac{1}{t}\mathbb{E}^x \int_{(0,t]} \mathrm{e}^{-(X_{s-}-x)\xi} \int_{\mathbb{R}\setminus\{0\}} \left(\mathrm{e}^{-y\xi} - 1 + y\xi\chi(y)\right) N(X_s,\mathrm{d}y) \; \mathrm{d}s \Bigg| \\ &\le |\xi|\,\frac{t}{t}\left\|\mathrm{e}^{-\xi\cdot}\ell(\cdot)\right\|_\infty + \xi^2 \frac{t}{2t}\left\|\mathrm{e}^{-\xi\cdot}Q(\cdot)\right\|_\infty \\ &\qquad + C_\xi \frac{t}{t}\left\|\mathrm{e}^{-\xi\cdot}\int_{y\neq 0}(1\wedge|y|^2)\; N(\cdot,\mathrm{d}y)\right\|_\infty,\end{aligned}$$

a bound which is uniform in t and x. □

Remark 3.7. As a consequence of the above result it is a simple task to derive the semimartingale characteristics of the process, if they have not been known a-priori. One just has to calculate the Laplace symbol and write it in the form (3.2), extract ℓ, Q and N and plug these quantities into equations (2.2).

In many cases Itô-Lévy processes are described by a Lévy driven SDE. If this is the case, one can directly determine the Laplace symbol from the SDE and the characteristic exponent of the driving Lévy process as shown in the following theorem. We will write X^x for the solution starting in x. Recall that it is always possible to enlarge the probability space and define a family of probability measures $(\mathbb{P}^x)_{x\in\mathbb{R}^d_+}$ on this enlargement in a way that $\mathbb{P}(X^x_t \in B) = \mathbb{P}^x(X_t \in B)$ (cf. [17, Section 5.6]).

Theorem 3.8. *Let $(L_t)_{t\geq 0}$ be an $\mathbb{R}^n$-valued Lévy process with characteristic exponent*

$$\phi(\xi) = -i\ell'\xi + \frac{1}{2}\xi'Q\xi - \int_{\mathbb{R}^n\setminus\{0\}} \left(e^{iy'\xi} - 1 - iy'\xi \cdot \mathbb{1}_{\{|y|<1\}}(y)\right) N(dy), \xi \in \mathbb{R}^n,$$

and consider the SDE

$$dX^x_t = \Phi(X^x_{t-})dL_t, \quad X^x_0 = x \in \mathbb{R}^d_+, \tag{3.5}$$

where $\Phi : \mathbb{R}^d_+ \to \mathbb{R}^{d\times n}_+$ is locally Lipschitz continuous and of linear growth. Then there exists a unique strong solution $(X^x_t)_{t\geq 0}$ of (3.5). Assume that $X^x_t \in \mathbb{R}^d_+$ holds a.s. for all $t \geq 0$, then this solution has Laplace symbol $\lambda : \mathbb{R}^d_+ \times \mathbb{R}^d_+ \to \mathbb{R}$ given by

$$\begin{aligned}&\lambda(x,\xi) = \phi(i\Phi(x)'\xi)\\ &= \ell'\Phi(x)'\xi - \frac{1}{2}(\Phi(x)'\xi)'Q(\Phi(x)'\xi) - \int_{\mathbb{R}^n\setminus\{0\}} \left(e^{-\Phi(x)'\xi y} - 1 + \Phi(x)'\xi y \cdot \mathbb{1}_{\{|y|<1\}}(y)\right) N(dy).\end{aligned}$$

Proof. By [13, Cond. IX.6.7] the given SDE has a unique solution. Since this solution is assumed to be positive (in all components) the Laplace symbol exists. The calculation of the Laplace symbol is very similar to the classical one which can be found in [21, Thm. 3.1]. Therefore, we only sketch the one-dimensional proof here.

Fix $x, \xi \in \mathbb{R}_+$. As in the proof of Theorem 3.6 we apply Itô's formula for jump semimartingales to the function $\exp(-(\cdot - x)\xi)$ and obtain

$$\begin{aligned}\frac{1}{t}\mathbb{E}^x\left(e^{-(X_t-x)\xi} - 1\right) = \frac{1}{t}\mathbb{E}^x\Bigg(&\int_{(0,t]} -\xi\, e^{-(X_{s-}-x)\xi}\, dX_s + \frac{1}{2}\int_{(0,t]} \xi^2\, e^{-(X_{s-}-x)\xi}\, d[X,X]^c_s\\ &+ e^{x\xi}\sum_{0<s\leq t}\left(e^{-X_s\xi} - e^{-X_{s-}\xi} + \xi e^{-X_{s-}\xi}\Delta X_s\right)\Bigg).\end{aligned} \tag{3.6}$$

For the first term we get

$$\begin{aligned}&\frac{1}{t}\mathbb{E}^x\int_{(0,t]}\left(-\xi\, e^{-(X_{s-}-x)\xi}\right) dX_s\\ &= \frac{1}{t}\mathbb{E}^x\int_{(0,t]}\left(-\xi\, e^{-(X_{s-}-x)\xi}\right) d\left(\int_{(0,s]}\Phi(X_{r-})\, dL_r\right)\\ &= \frac{1}{t}\mathbb{E}^x\int_{(0,t]}\left(-\xi\, e^{-(X_{s-}-x)\xi}\Phi(X_{s-})\right) d(\ell s) &(3.7)\\ &\quad + \frac{1}{t}\mathbb{E}^x\int_{(0,t]}\left(-\xi\, e^{-(X_{s-}-x)\xi}\Phi(X_{s-})\right) d\left(\sum_{0<r\leq s}\Delta L_r \mathbb{1}_{\{|\Delta L_r|\geq 1\}}\right) &(3.8)\end{aligned}$$

where we have used the Lévy-Itô decomposition. Since the integrands are bounded by our assumptions, the two martingale terms of the Lévy process yield martingales whose expected value is zero.

First we deal with (3.8) containing the big jumps. Adding this integral to the third expression on the right-hand side of (3.6) we obtain

$$\begin{aligned}\frac{1}{t}\,\mathbb{E}^x \sum_{0<s\le t} \Big(\mathrm{e}^{-(X_{s-}-x)\xi} \left(\mathrm{e}^{-\Phi(X_{s-})\Delta L_s \xi} - 1 + \xi \Phi(X_{s-})\Delta L_s \mathbb{1}_{\{|\Delta X_s|<1\}} \right) \Big) \\ \xrightarrow{t\downarrow 0} \int_{\mathbb{R}\setminus\{0\}} \left(\mathrm{e}^{-\Phi(x)y\xi} - 1 + \xi\Phi(x)y\mathbb{1}_{\{|y|<1\}} \right) N(\mathrm{d}y)\end{aligned}$$

Here, as well as in the calculation of the other terms we make use of the simple fact that $\Phi(y)\cdot\exp(-y)$ is bounded. The calculation above uses some well known results about integration with respect to integer valued random measures, see [11, Section II.3], which allow us to integrate 'under the expectation' with respect to the compensating measure $\nu(\cdot\,;\mathrm{d}s,\mathrm{d}y)$ instead of the random measure itself. In the case of a Lévy process the compensator is of the form $\nu(\cdot\,;\mathrm{d}s,\mathrm{d}y) = N(\mathrm{d}y)\,\mathrm{d}s$, see [11, Example II.4.2].

We can treat the drift part (3.7) as well as the second expression on the right-hand side of (3.6) in a similar way. In the latter case we have to consider in addition

$$\begin{aligned}[X,X]^c_t &= \left(\left[\textstyle\int_{(0,\cdot]} \Phi(X_{r-})\mathrm{d}L_r,\ \int_{(0,\cdot]} \Phi(X_{r-})\mathrm{d}L_r \right]^c_t \right) = \int_{(0,t]} \Phi(X_{s-})^2\,\mathrm{d}[L,L]^c_s \\ &= \int_{(0,t]} \Phi(X_{s-})^2\,\mathrm{d}(Qs).\end{aligned}$$

In the end we obtain

$$\begin{aligned}\lambda(x,\xi) &= \ell(\Phi(x)\xi) - \frac{1}{2}(\Phi(x)\xi)Q(\Phi(x)\xi) \\ &\quad - \int_{\mathbb{R}\setminus\{0\}} \left(\mathrm{e}^{-(\Phi(x)\xi)y} - 1 + (\Phi(x)\xi)y\cdot\mathbb{1}_{\{|y|<1\}}(y) \right) N(\mathrm{d}y) \\ &= \phi(i\Phi(x)\xi).\end{aligned}$$

Note that in the multi-dimensional case the matrix $\Phi(x)$ has to be transposed. □

Remark 3.9. The condition of non-negativity of the solution of the given Lévy driven SDE as posed in the above theorem can be checked case by case. We are not aware of any general necessary and sufficient conditions for this. Sufficient conditions for non-negativity of the solution of a Lévy driven SDE can e.g. be found in [17, Thms. V.71 and V.72].

In the next special case, non-negativity of the solution of the SDE is easily checked. Therefore, we obtain the following corollary to Theorem 3.8.

Corollary 3.10. *Let $(L_t)_{t\ge 0}$ be a subordinator with Laplace exponent $\lambda_L(\xi), \xi \ge 0$ and consider the SDE*

$$\mathrm{d}X^x_t = \Phi(X^x_{t-})\mathrm{d}L_t, \quad X^x_0 = x \in \mathbb{R}^d_+, \tag{3.9}$$

where $\Phi : \mathbb{R}_+^d \to \mathbb{R}_+^{d\times 1}$ *is locally Lipschitz continuous and of linear growth. Then there exists a unique strong solution* $(X_t^x)_{t\geq 0}$, $X_t^x \in \mathbb{R}_+^d$, *of* (3.9) *and this solution has Laplace symbol* $\lambda : \mathbb{R}_+^d \times \mathbb{R}_+^d \to \mathbb{R}$ *given by*

$$\lambda(x,\xi) = \lambda_L(\Phi(x)\xi).$$

We end this section by establishing the connection of the Laplace symbol and the generator of an a.s. non-negative Itô-Lévy process. Therefore, we have to define the following subclass of $C_0^\infty((0,\infty)^d)$, the continuous, infinitely often differentiable functions on $(0,\infty)^d$, vanishing (componentwise) at infinity.

$$\mathcal{S} := \Bigg\{ f \in C_0^\infty((0,\infty)^d) : \int_{\mathbb{R}_+} \left| (-1)^k \frac{\partial^k f}{\partial x_i^k}(x)|_{x_i=k/t}(k/t)^{k+1} \right| \mathrm{d}t < \infty, \tag{3.10}$$
$$\text{for all } k=1,2,\dots, x_\ell \text{ fixed}, \ell \neq i = 1,\dots,d, \quad \text{and} \quad \text{for } x_\ell \text{ fixed}, \ell \neq i = 1,\dots,d,$$
$$\lim_{\substack{j\to\infty \\ k\to\infty}} \int_{\mathbb{R}_+} \left| (-1)^k \frac{\partial^k f}{\partial x_i^k}(x)|_{x_i=k/t}(k/t)^{k+1} - (-1)^j \frac{\partial^j f}{\partial x_i^j}(x)|_{x_i=j/t}(j/t)^{j+1} \right| \mathrm{d}t = 0 \Bigg\},$$

where x_i denotes the i'th component of the vector x.

Theorem 3.11. *Let* $(X_t)_{t\geq 0}$ *be an* $\mathbb{R}_+^d$*-valued Itô-Lévy process with polynomially bounded, finely continuous differential characteristics and with generator* $(\mathcal{A}, D(\mathcal{A}))$*. Then for all* $f \in D(\mathcal{A}) \cap \mathcal{S}$*, we have*

$$\mathcal{A}f(x) = -\int_{\mathbb{R}_+^d} \mathrm{e}^{-x'\xi} \lambda(x,\xi) \check{f}(\xi) \mathrm{d}\xi, \quad x \in \mathbb{R}_+^d, \tag{3.11}$$

where $\check{f}$ *denotes the inverse Laplace transform of* f*.*

Proof. First, observe that by a straightforward multivariate extension of [23, Thm. VII.17a], every $f \in \mathcal{S}$ admits an integrable inverse Laplace transform $\check{f}$. Hence, starting with the definition of the Laplace symbol we obtain

$$\begin{aligned}
\int_{\mathbb{R}_+^d} \mathrm{e}^{-x'\xi} \lambda(x,\xi) \check{f}(\xi) \mathrm{d}\xi &= -\int_{\mathbb{R}_+^d} \mathrm{e}^{-x'\xi} \lim_{t\to 0} \frac{1}{t} \left(\mathbb{E}^x[\mathrm{e}^{-(X_t-x)'\xi}] - 1 \right) \check{f}(\xi) \mathrm{d}\xi \\
&= -\int_{\mathbb{R}_+^d} \lim_{t\to 0} \frac{1}{t} \left(\mathbb{E}^x[\mathrm{e}^{-X_t'\xi}] - \mathrm{e}^{-x'\xi} \right) \check{f}(\xi) \mathrm{d}\xi \\
&= -\lim_{t\to 0} \frac{1}{t} \int_{\mathbb{R}_+^d} \left(\mathbb{E}^x[\mathrm{e}^{-X_t'\xi}] - \mathrm{e}^{-x'\xi} \right) \check{f}(\xi) \mathrm{d}\xi,
\end{aligned}$$

where in the last step we used Lebesgue's dominated convergence theorem, which is justified by Theorem 3.6. Further

$$\begin{aligned}
\int_{\mathbb{R}_+^d} \left(\mathbb{E}^x[\mathrm{e}^{-X_t'\xi}] - \mathrm{e}^{-x'\xi} \right) \check{f}(\xi) \mathrm{d}\xi &= \int_{\mathbb{R}_+^d} \left(\int_{\mathbb{R}_+^d} \mathrm{e}^{-z'\xi} \mathrm{d}P_{X_t^x}(z) - \mathrm{e}^{-x'\xi} \right) \check{f}(\xi) \mathrm{d}\xi \\
&= \int_{\mathbb{R}_+^d} f(z) \mathrm{d}P_{X_t^x}(z) - f(x) \\
&= \mathbb{E}^x[f(X_t)] - f(x)
\end{aligned}$$

by an application of Fubini's Theorem, such that altogether

$$\int_{\mathbb{R}_+^d} \mathrm{e}^{-x'\xi}\lambda(x,\xi)\check{f}(\xi)\mathrm{d}\xi = -\lim_{t\to 0}\frac{1}{t}\left(\mathbb{E}^x[f(X_t)] - f(x)\right) = -\mathcal{A}f(x).$$

□

4 Invariant Distributions

In [3] it was shown under certain boundedness conditions that a probability measure μ is invariant for a given rich Itô-Lévy process if and only if

$$\int_{\mathbb{R}^d} \mathrm{e}^{ix'\xi} p(x,\xi)\mu(\mathrm{d}x) = 0 \quad \text{for all } \ \xi \in \mathbb{R}^d, \tag{4.1}$$

where $p(x,\xi)$ is the (probabilistic) symbol of the Itô-Lévy process. Recall that an Itô-Lévy process is rich if and only if the domain of its generator contains the test functions $C_c^\infty(\mathbb{R}^d)$.
A similar criterion based on the Laplace symbol shall be established in the following. The main advantage of this new criterion is the fact, that we can drop the formerly needed boundedness conditions. Further, in the case of bounded Itô-Lévy processes richness can be a serious restriction as the bound directly influences the domain of the generator. As an example, recall that while e.g. Brownian motion is a rich Itô-Lévy process, the domain of the generator of the Brownian motion absorbed at zero as treated in Example 3.4 only contains functions with zero second derivative in zero.

Remark that proofs of necessity and sufficiency of (4.1) for μ to be invariant rely on completely different techniques, which is the reason why we will give two distinct theorems for the two directions in the following, starting with the necessity part.

Theorem 4.1. *Let $(X_t)_{t\geq 0}$ be an $\mathbb{R}_+^d$-valued Itô-Lévy process with polynomially bounded, finely continuous differential characteristics which admits an invariant distribution μ and whose Laplace symbol is given by $\lambda(x,\xi)$, $x,\xi \in \mathbb{R}_+^d$. Then*

$$\int_{\mathbb{R}_+^d} \mathrm{e}^{-x'\xi}\lambda(x,\xi)\mu(\mathrm{d}x) = 0 \quad \textit{for all } \ \xi \in \mathbb{R}_+^d. \tag{4.2}$$

Proof. Using Theorem 3.6 we obtain by Lebesgue's dominated convergence theorem

$$\begin{aligned}
\int_{\mathbb{R}_+^d} \mathrm{e}^{-x'\xi}\lambda(x,\xi)\mu(\mathrm{d}x) &= \int_{\mathbb{R}_+^d} \mathrm{e}^{-x'\xi} \lim_{t\to 0}\mathbb{E}^x\left[\frac{\mathrm{e}^{-(X_t-x)'\xi}-1}{t}\right]\mu(\mathrm{d}x) \\
&= \lim_{t\to 0}\frac{1}{t}\int_{\mathbb{R}_+^d} \mathrm{e}^{-x'\xi}\,\mathbb{E}^x[\mathrm{e}^{-(X_t-x)'\xi}-1]\mu(\mathrm{d}x), \\
&= \lim_{t\to 0}\frac{1}{t}\left(\int_{\mathbb{R}_+^d}\mathbb{E}^x\left[\mathrm{e}^{-X_t'\xi}\right]\mu(\mathrm{d}x) - \int_{\mathbb{R}_+^d}\mathrm{e}^{-x'\xi}\mu(\mathrm{d}x)\right) \\
&= 0,
\end{aligned}$$

as had to be shown. □

We continue with the sufficiency part.

Theorem 4.2. *Let $(X_t)_{t\geq 0}$ be an $\mathbb{R}_+^d$-valued Itô-Lévy process with polynomially bounded, finely continuous differential characteristics, with generator $(\mathcal{A}, D(\mathcal{A}))$ and with Laplace symbol $\lambda(x,\xi)$. Assume that the set of functions $f \in D(\mathcal{A}) \cap \mathcal{S}$ with $\mathcal{S}$ as in* (3.10) *contains a core. Further assume there exists a probability measure μ on $\mathbb{R}_+^d$ such that $\int_{\mathbb{R}_+^d} \mathrm{e}^{-x'\xi}|\lambda(x,\xi)|\mu(\mathrm{d}x) < \infty$ and $\int_{\mathbb{R}_+^d} \mathrm{e}^{-x'\xi}\lambda(x,\xi)\mu(\mathrm{d}x) = 0$ for all $x \in \mathbb{R}_+^d$. Then μ is invariant for $(X_t)_{t\geq 0}$.*

Proof. By Theorem 3.11 the generator $\mathcal{A}$ admits the representation (3.11) with integrable Laplace inverse $\check{f}$ for all $f \in D(\mathcal{A}) \cap \mathcal{S}$. Therefore, for these functions f we obtain using Fubini's theorem

$$\begin{aligned}\int_{\mathbb{R}_+^d} \mathcal{A}f(x)\mu(\mathrm{d}x) &= \int_{\mathbb{R}_+^d}\int_{\mathbb{R}_+} \mathrm{e}^{-x'\xi}\lambda(x,\xi)\check{f}(\xi)\mathrm{d}\xi f(x)\mu(\mathrm{d}x)\\ &= \int_{\mathbb{R}_+^d}\int_{\mathbb{R}_+} \mathrm{e}^{-x'\xi}\lambda(x,\xi)\mu(\mathrm{d}x)\check{f}(\xi)\mathrm{d}\xi\\ &= 0.\end{aligned}$$

Hence

$$\int_{\mathbb{R}_+^d} \mathcal{A}f(x)\mu(\mathrm{d}x) = 0, \quad \text{for all} \quad f \in D(\mathcal{A})$$

from which the assertion follows (see Remark 4.3). □

Remark 4.3. Observe that in the previous article [3] we made a distinction between invariant measures and so-called infinitesimal invariant measures, that is measures μ which fulfill $\int_{\mathbb{R}^d} \mathcal{A}f(x)\mu(\mathrm{d}x)$ but not necessarily

$$\int_{\mathbb{R}^d} T_t f(x)\mathrm{d}\mu(x) = \int_{\mathbb{R}^d} f(x)\mathrm{d}\mu(x), \quad \forall f \in C_0(\mathbb{R}^d), t \geq 0,$$

where $(T_t)_{t\geq 0}$ is the semigroup associated to the generator $(\mathcal{A}, D(\mathcal{A}))$ with $D(\mathcal{A})$ being a subset of $C_0(\mathbb{R}^d)$, the continuous, real-valued functions on $\mathbb{R}^d$ vanishing at infinity. This distinction can also be found in some of the literature concerning the subject as e.g. [1, 2]. Actually, in our setting such a distinction is not necessary as we are always a priori assuming the existence of a given Markov process and, therefore, are equipped with a given proper generator and semigroup. Hence e.g. the proof of [8, Prop. 9.2, b)⇔c)] can be applied without any further assumptions, implying that invariance and infinitesimal invariance are equivalent here (as well as in the setting of [3]).

4.1 Some applications

Example 4.4. The Cox-Ingersoll-Ross (CIR) process is defined as the solution of the SDE

$$\mathrm{d}X_t^x = a(b - X_t^x)\mathrm{d}t + \sigma\sqrt{X_t^x}\mathrm{d}B_t, \quad t \geq 0, \quad X_0^x = x > 0,$$

for some constants $a, b, \sigma > 0$ and a real-valued standard Brownian motion $(B_t)_{t\geq 0}$. This process is a.s. non-negative (given $2ab \geq \sigma^2$ it even is a.s. positive) and using Theorem 3.8 we hence obtain the Laplace symbol of $(X_t^x)_{t\geq 0}$ as

$$\lambda(x,\xi) = a(b-x)\xi - \frac{1}{2}\sigma^2 x\xi^2 = ab\xi - (a\xi + \frac{1}{2}\sigma^2\xi^2)x, \quad x, \xi \geq 0.$$

Hence by Theorem 4.1 any stationary distribution μ of the CIR process has to fulfil

$$\begin{aligned} 0 &= \int_{\mathbb{R}_+^d} \mathrm{e}^{-x'\xi} \lambda(x,\xi)\mu(\mathrm{d}x) \\ &= ab\xi \int_{\mathbb{R}_+^d} \mathrm{e}^{-x'\xi}\mu(\mathrm{d}x) - (a\xi + \frac{1}{2}\sigma^2\xi^2) \int_{\mathbb{R}_+^d} \mathrm{e}^{-x'\xi} x\mu(\mathrm{d}x) \\ &= ab\xi\psi_\mu(\xi) + (a\xi + \frac{1}{2}\sigma^2\xi^2)\psi_\mu'(\xi). \end{aligned}$$

This first order linear ODE is uniquely solved by

$$\psi(\xi) = c(2a + \xi\sigma^2)^{-\frac{2ab}{\sigma^2}}$$

for some constant c. Since $\lim_{\xi\to 0}\psi(\xi) = 1$ we observe that $c = (2a)^{\frac{2ab}{\sigma^2}}$ such that

$$\psi(\xi) = \left(\frac{2a}{2a + \xi\sigma^2}\right)^{\frac{2ab}{\sigma^2}}$$

which is the Laplace transform of a Gamma distribution, the well-known unique stationary distribution of the CIR process.

Example 4.5. The stochastic Verhulst model is a stochastic population growth model where the population size $(X_t)_{t\geq 0}$ solves

$$\mathrm{d}X_t^x = X_t^x(a - X_t^x)\mathrm{d}t + \sigma X_t^x \mathrm{d}B_t, \quad X_0^x = x \geq 0, \tag{4.3}$$

with $a, \sigma > 0$ and a real-valued standard Brownian motion $(B_t)_{t\geq 0}$. This SDE has an a.s. non-negative unique solution (cf. [16, Exercise 5.15]) and hence we obtain its Laplace symbol by Theorem 3.8 as

$$\lambda(x,\xi) = x(a-x)\xi - \frac{1}{2}\sigma^2 x^2\xi^2, \quad x, \xi \geq 0.$$

Hence if the stochastic Verhulst model had an invariant law, its Laplace transform ψ had to fulfil

$$0 = -a\xi\psi'(\xi) - (\xi + \frac{1}{2}\sigma^2\xi^2)\psi''(\xi). \tag{4.4}$$

This second order linear ODE is for $2a \neq \sigma^2$ uniquely solved by

$$\psi(\xi) = \frac{c_1(2 + \sigma^2\xi)^{1-\frac{2a}{\sigma^2}}}{\sigma^2 - 2a} + c_2$$

for constants c_1, c_2. From $\lim_{\xi\to 0}\psi(\xi) = 1$ we further derive c_2 and obtain that

$$\psi(\xi) = 1 + c_1 \frac{(2+\sigma^2\xi)^{1-\frac{2a}{\sigma^2}} - 2^{1-\frac{2a}{\sigma^2}}}{\sigma^2 - 2a}.$$

The case $c_1 = 0$ corresponds to the Dirac measure in 0 and obviously $X_t^0 = 0$ is a stationary solution of (4.3) with starting value 0.
From Bernsteins theorem we further observe that if ψ is a Laplace transform of a probability measure, then

$$\psi'(\xi) = c_1 \frac{(2+\sigma^2\xi)^{-\frac{2a}{\sigma^2}}}{\sigma^2} \leq 0,$$

such that $c_1 \leq 0$. This then implies automatically $(-1)^n\psi^{(n)}(\xi) \geq 0$ for all $n \in \mathbb{N}\backslash\{0\}$. So for ψ to define the Laplace transform of a non-trivial probability measure it is necessary and sufficient that $c_1 < 0$ and $\psi(\xi) \geq 0$ for all $\xi > 0$. Letting ξ tend to infinity in the above formula in the case $\sigma^2 > 2a$ this yields to a contradiction. In the case $\sigma^2 < 2a$ $\psi(\xi) \geq 0$ is equivalent to

$$(2+\sigma^2\xi)^{1-\frac{2a}{\sigma^2}} - 2^{1-\frac{2a}{\sigma^2}} \geq \frac{\sigma^2 - 2a}{c_1} > 0$$

which leads to a contradiction for $\xi \to 0$.
In the case $2a = \sigma^2$ the unique solution to (4.4) is given by

$$\psi(\xi) = c_1 \frac{\log(a\xi + 1)}{a} + c_2$$

for some constants c_1, c_2 and a similar reasoning as above again leads either to a Dirac measure in 0 or to a contradiction.
Hence in the stochastic Verhulst model no (non-trivial) invariant probability measure exists.

Example 4.6. Consider a continuous-state branching process with immigration (CBI), i.e. a Markov process on $\mathbb{R}_+$ whose Laplace symbol is given by

$$\lambda(x,\xi) = F(\xi) + xG(\xi),$$

where F and G are of Lévy-Khintchine form, that is

$$F(\xi) = a_F\xi + \int_{\mathbb{R}_+} (1 - \mathrm{e}^{-u\xi})\nu_F(\mathrm{d}u), \quad \text{and}$$
$$G(\xi) = a_G\xi - \sigma_G^2\xi^2 + \int_{\mathbb{R}_+} (1 - \mathrm{e}^{-u\xi} - u\xi\mathbb{1}_{(0,1]}(u))\nu_G(\mathrm{d}u),$$

where $a_F, \sigma_G^2 \geq 0$, $a_G \in \mathbb{R}$ and ν_F, ν_G are Lévy measures (see e.g. [14] for details).
Then for any probability measure μ with Laplace transform ψ_μ we have

$$\int_{\mathbb{R}_+} \mathrm{e}^{-x\xi}\lambda(x,\xi)\mu(\mathrm{d}x) = F(\xi)\psi_\mu(\xi) + G(\xi)\psi'_\mu(\xi), \quad \xi \geq 0,$$

and this equals 0 if and only if

$$\psi_\mu(\xi) = \exp\left(\int_{(0,\xi)} \frac{F(u)}{G(u)} \mathrm{d}u\right)$$

which is the form of the Laplace transform of the invariant measure of a CBI process as shown in [14, Thm. 2.6].

4.2 Symmetric processes

Stationarity of a Markov process is kept when applying a measurable function on the values of the process. We can use this fact to apply the above results also in case of processes which are not one-sided bounded as will be demonstrated in the following proposition and example.

Proposition 4.7. *Let L, Z be independent Lévy processes in $\mathbb{R}$ and assume that Z is symmetric. Consider the unique solution of the SDE*

$$\mathrm{d}X_t = \Phi(X_{t-})\mathrm{d}L_t + \mathrm{d}Z_t, \quad t \geq 0. \tag{4.5}$$

where Φ is Lipschitz continuous, polynomially bounded and odd. Let μ be invariant for X, then for all $\xi > 0$

$$\begin{aligned}
&2\sigma_L^2\xi^2 \int_{\mathbb{R}} \mathrm{e}^{-\xi x^2} x^2 \Phi^2(x)\mu(\mathrm{d}x) + 2\sigma_Z^2\xi^2 \int_{\mathbb{R}} \mathrm{e}^{-\xi x^2} x^2 \mu(\mathrm{d}x) \\
&= 2\ell_L \xi \int_{\mathbb{R}} \mathrm{e}^{-\xi x^2} x\Phi(x)\mu(\mathrm{d}x) + \sigma_L^2 \xi \int_{\mathbb{R}} \mathrm{e}^{-\xi x^2} \Phi^2(x)\mu(\mathrm{d}x) + \sigma_Z^2 \xi \int_{\mathbb{R}} \mathrm{e}^{-\xi x^2} \mu(\mathrm{d}x) \\
&\quad + \int_{\mathbb{R}} \mathrm{e}^{-\xi x^2} \int_{\mathbb{R}} (1 - \mathrm{e}^{-u^2\Phi^2(x)\xi}) N_L(\mathrm{d}u)\mu(\mathrm{d}x) + \int_{\mathbb{R}} (1 - \mathrm{e}^{-u^2\xi}) N_Z(\mathrm{d}u) \int_{\mathbb{R}} \mathrm{e}^{-\xi x^2} \mu(\mathrm{d}x).
\end{aligned}$$

In particular if Z is symmetric strictly α-stable, $0 < \alpha < 2$, with characteristic exponent $\phi_Z(\xi) = |\xi|^\alpha$ we obtain

$$\begin{aligned}
2\sigma_L^2\xi^2 \int_{\mathbb{R}} \mathrm{e}^{-\xi x^2} x^2 \Phi^2(x)\mu(\mathrm{d}x) &= 2\ell_L \xi \int_{\mathbb{R}} \mathrm{e}^{-\xi x^2} x\Phi(x)\mu(\mathrm{d}x) + \sigma_L^2 \xi \int_{\mathbb{R}} \mathrm{e}^{-\xi x^2} \Phi^2(x)\mu(\mathrm{d}x) \\
&\quad + \int_{\mathbb{R}} \mathrm{e}^{-\xi x^2} \int_{\mathbb{R}} (1 - \mathrm{e}^{-u^2\Phi^2(x)\xi}) N_L(\mathrm{d}u)\mu(\mathrm{d}x) + \frac{1}{\alpha}\Gamma(1-\alpha/2)\xi^{\alpha/2} \int_{\mathbb{R}} \mathrm{e}^{-\xi x^2} \mu(\mathrm{d}x).
\end{aligned}$$

Proof. Since Z is symmetric, we also have that X given by

$$X_t = X_0 + \int_{(0,t]} \Phi(X_{t-})\mathrm{d}L_t + Z_t$$

is symmetric, whenever the starting distribution of X_0 is symmetric.

Consider the process $Y_t := X_t^2$, then by Itô's formula

$$\begin{aligned}
dY_t = \mathrm{d}X_t^2 &= 2X_{t-}dX_t + d[X,X]_t - 2X_{t-}\Delta X_t \\
&= 2X_{t-}(\Phi(X_{t-})\mathrm{d}L_t + \mathrm{d}Z_t) + (\Phi(X_{t-}))^2 d[L,L]_t + \mathrm{d}[Z,Z]_t \\
&\quad - 2X_{t-}(\Phi(X_{t-})\Delta L_t + \Delta Z_t) \\
&=: 2X_{t-}(\Phi(X_{t-})\mathrm{d}L_t^c + \mathrm{d}Z_t^c) + (\Phi(X_{t-}))^2 d[L,L]_t + \mathrm{d}[Z,Z]_t \\
&= 2\mathrm{sgn}(X_{t-})\sqrt{Y_{t-}}(\Phi(\mathrm{sgn}(X_{t-})\sqrt{Y_{t-}})\mathrm{d}L_t^c + \mathrm{d}Z_t^c) \\
&\quad + (\Phi(\mathrm{sgn}(X_{t-})\sqrt{Y_{t-}}))^2 d[L,L]_t + \mathrm{d}[Z,Z]_t \\
&= 2\sqrt{Y_{t-}}\Phi(\sqrt{Y_{t-}})\mathrm{d}L_t^c + 2\mathrm{sgn}(X_{t-})\sqrt{Y_{t-}}\mathrm{d}Z_t^c \\
&\quad + (\Phi(\sqrt{Y_{t-}}))^2 d[L,L]_t + \mathrm{d}[Z,Z]_t
\end{aligned}$$

where $Z_t^c = Z_t - \sum_{0<s\le t} \Delta Z_s$ is just the Brownian part of Z, since Z is symmetric. Further

$$[L,L]_t = \sigma_L^2 t + \sum_{0<s\le t} \Delta L_s^2 \text{ has Laplace exponent } \lambda_{[L,L]}(\xi) = \sigma_L^2\xi + \int_{\mathbb{R}} (1-\mathrm{e}^{-u^2\xi}) N_L(\mathrm{d}u),$$

and the Laplace exponent of $[Z,Z]_t$ looks similar. Hence by Theorem 3.8, the Laplace symbol of Y is given by

$$\begin{aligned}
\lambda_Y(y,\xi) &= \phi_{L^c}(i2\sqrt{y}\Phi(\sqrt{y})\xi) + \phi_{Z^c}(i2\mathrm{sgn}(x)\sqrt{y}\xi) + \lambda_{[L,L]}(\Phi^2(\sqrt{y})\xi) + \lambda_{[Z,Z]}(\xi) \\
&= 2\ell_L\sqrt{y}\Phi(\sqrt{y})\xi - 4\frac{\sigma_L^2}{2} y\Phi^2(\sqrt{y})\xi^2 - 4\frac{\sigma_Z^2}{2} y\xi^2 \\
&\quad + \sigma_L^2\Phi^2(\sqrt{y})\xi + \int_{\mathbb{R}} (1-\mathrm{e}^{-u^2\Phi^2(\sqrt{y})\xi}) N_L(\mathrm{d}u) + \sigma_Z^2\xi + \int_{\mathbb{R}} (1-\mathrm{e}^{-u^2\xi}) N_Z(\mathrm{d}u),
\end{aligned}$$

where ϕ_{L^c} and ϕ_{Z^c} denote the characteristic exponents of L^c and Z^c, respectively.
If X has an invariant (symmetric) distribution μ, then Y has the invariant distribution $\tilde{\mu}$ with $\tilde{\mu} = T(\mu)$ and $T : x \mapsto x^2$. Hence by Theorem 4.1 above

$$\begin{aligned}
0 &= \int_{(0,\infty)} \mathrm{e}^{-\xi y} \lambda_Y(y,\xi)\tilde{\mu}(\mathrm{d}y) \\
&= 2\ell_L\xi \int_{(0,\infty)} \mathrm{e}^{-\xi y}\sqrt{y}\Phi(\sqrt{y})\tilde{\mu}(\mathrm{d}y) - 2\sigma_L^2\xi^2 \int_{(0,\infty)} \mathrm{e}^{-\xi y} y\Phi^2(\sqrt{y})\tilde{\mu}(\mathrm{d}y) \\
&\quad - 2\sigma_Z^2\xi^2 \int_{(0,\infty)} \mathrm{e}^{-\xi y} y\tilde{\mu}(\mathrm{d}y) + \sigma_L^2\xi \int_{(0,\infty)} \mathrm{e}^{-\xi y}\Phi^2(\sqrt{y})\tilde{\mu}(\mathrm{d}y) + \sigma_Z^2\xi \int_{(0,\infty)} \mathrm{e}^{-\xi y}\tilde{\mu}(\mathrm{d}y) \\
&\quad + \int_{(0,\infty)} \mathrm{e}^{-\xi y} \int_{\mathbb{R}} (1-\mathrm{e}^{-u^2\Phi^2(\sqrt{y})\xi}) N_L(\mathrm{d}u)\tilde{\mu}(\mathrm{d}y) + \int_{\mathbb{R}} (1-\mathrm{e}^{-u^2\xi}) N_Z(\mathrm{d}u) \int_{(0,\infty)} \mathrm{e}^{-\xi y}\tilde{\mu}(\mathrm{d}y) \\
&= 2\ell_L\xi \int_{\mathbb{R}} \mathrm{e}^{-\xi x^2} x\Phi(x)\mu(\mathrm{d}x) - 2\sigma_L^2\xi^2 \int_{\mathbb{R}} \mathrm{e}^{-\xi x^2} x^2\Phi^2(x)\mu(\mathrm{d}x) \\
&\quad - 2\sigma_Z^2\xi^2 \int_{\mathbb{R}} \mathrm{e}^{-\xi x^2} x^2\mu(\mathrm{d}x) + \sigma_L^2\xi \int_{\mathbb{R}} \mathrm{e}^{-\xi x^2}\Phi^2(x)\mu(\mathrm{d}x) + \sigma_Z^2\xi \int_{\mathbb{R}} \mathrm{e}^{-\xi x^2}\mu(\mathrm{d}x) \\
&\quad + \int_{\mathbb{R}} \mathrm{e}^{-\xi x^2} \int_{\mathbb{R}} (1-\mathrm{e}^{-u^2\Phi^2(x)\xi}) N_L(\mathrm{d}u)\mu(\mathrm{d}x) + \int_{\mathbb{R}} (1-\mathrm{e}^{-u^2\xi}) N_Z(\mathrm{d}u) \int_{\mathbb{R}} \mathrm{e}^{-\xi x^2}\mu(\mathrm{d}x)
\end{aligned}$$

as claimed.
If Z is symmetric, strictly α-stable, $0<\alpha<2$, with characteristic exponent $\phi_Z(\xi)=|\xi|^\alpha$ we have $N_Z(\mathrm{d}u)=|u|^{-1-\alpha}\mathrm{d}u$ and thus

$$\begin{aligned}\int_{\mathbb{R}}(1-\mathrm{e}^{-u^2\xi})N_Z(\mathrm{d}u) &= \int_{\mathbb{R}}(1-\mathrm{e}^{-u^2\xi})|u|^{-1-\alpha}\mathrm{d}u \\ &= \frac{1}{2}\int_{(0,\infty)}(1-\mathrm{e}^{-v\xi})v^{-1-\alpha/2}dv \\ &= \frac{1}{2}\left(\frac{\alpha}{2}\right)^{-1}\Gamma(1-\alpha/2)\xi^{\alpha/2}.\end{aligned}$$

□

Example 4.8. Consider a symmetric generalized Ornstein-Uhlenbeck (GOU) process solving (4.5) with $\Phi(x)=x$. Then its invariant measure fulfills by the above theorem for all $\xi>0$

$$\begin{aligned}&2\sigma_L^2\xi^2\int_{\mathbb{R}}\mathrm{e}^{-\xi x^2}x^4\mu(\mathrm{d}x)+2\sigma_Z^2\xi^2\int_{\mathbb{R}}\mathrm{e}^{-\xi x^2}x^2\mu(\mathrm{d}x)\\ &=2\ell_L\xi\int_{\mathbb{R}}\mathrm{e}^{-\xi x^2}x^2\mu(\mathrm{d}x)+\sigma_L^2\xi\int_{\mathbb{R}}\mathrm{e}^{-\xi x^2}x^2\mu(\mathrm{d}x)+\sigma_Z^2\xi\int_{\mathbb{R}}\mathrm{e}^{-\xi x^2}\mu(\mathrm{d}x)\\ &\quad+\int_{\mathbb{R}}\mathrm{e}^{-\xi x^2}\int_{\mathbb{R}}(1-\mathrm{e}^{-u^2x^2\xi})N_L(\mathrm{d}u)\mu(\mathrm{d}x)+\int_{\mathbb{R}}(1-\mathrm{e}^{-u^2\xi})N_Z(\mathrm{d}u)\int_{\mathbb{R}}\mathrm{e}^{-\xi x^2}\mu(\mathrm{d}x).\end{aligned}$$

Denoting the Laplace transform $\int_{\mathbb{R}}\mathrm{e}^{-\xi x^2}\mu(\mathrm{d}x)$ by $\lambda_2(\xi)$ this further reduces to

$$\begin{aligned}&2\sigma_L^2\xi^2\lambda_2''(\xi)-2\sigma_Z^2\xi^2\lambda_2'(\xi)\\ &=-2\ell_L\xi\lambda_2'(\xi)-\sigma_L^2\xi\lambda_2'(\xi)+\sigma_Z^2\xi\lambda_2(\xi)\\ &\quad+\int_{\mathbb{R}}\mathrm{e}^{-\xi x^2}\int_{\mathbb{R}}(1-\mathrm{e}^{-u^2x^2\xi})N_L(\mathrm{d}u)\mu(\mathrm{d}x)+\int_{\mathbb{R}}(1-\mathrm{e}^{-u^2\xi})N_Z(\mathrm{d}u)\lambda_2(\xi).\end{aligned}$$

In particular, for the well-studied special case of Z being a Brownian motion and $L_t=\ell_L t$ being deterministic we deduce

$$2(\ell_L-\sigma_Z^2\xi)\lambda_2'(\xi)=\sigma_Z^2\lambda_2(\xi),$$

which yields

$$\lambda_2(\xi)=\left(\frac{\ell_L}{\ell_L-\sigma_Z^2\xi}\right)^{1/2}=\left(1-\frac{\sigma_Z^2}{\ell_L}\xi\right)^{-1/2}$$

from which we see that for $\ell_L<0$, the squared process admits a Gamma$(\frac{1}{2},\frac{-\ell_L}{\sigma_Z^2})$ invariant distribution from which we recover that the GOU process has a normal invariant law.

4.3 Processes on $\mathbb{R}^d$

Using a similar idea as in the last subsection, we observe that the Laplace symbol can even be used in order to analyze the stationary distribution of processes defined on the

whole space $\mathbb{R}^d$. To this end one has to trnasform the given process by a C^2-bijection $f : \mathbb{R}^d \to (0,\infty)^d$ which grows at most polynomially.
More precisely, let X be an Itô-Lévy process with state space $\mathbb{R}$ (to keep the notation simple). Denote the finely continuous and polynomially bounded differential characteristics of X by ℓ, Q and $n := \int_{y\neq 0}(1 \wedge \|y\|^2)\ N(\cdot, \mathrm{d}y)$. As we are in the one-dimensional case, we could use the C^2-bijection

$$f(x) := \exp(x) \cdot \mathbb{1}_{(-\infty,0]}(x) + \left(\frac{1}{2}x^2 + x + 1\right) \cdot \mathbb{1}_{(0,\infty)}(x)$$

It is a simple consequence of Itô's formula that we obtain for the transformed process $\widetilde{X} := f(X)$ with state space $\mathbb{R}_+\backslash\{0\}$ the differential characteristics

$$\begin{aligned}
\widetilde{\ell}(X_{t-}) &= f'(X_{t-})\ell(X_{t-}) + \frac{1}{2}f''(X_{t-})Q(X_{t-}) \\
&\quad + \int (\widetilde{\chi}(f(X_{t-}+y) - f(X_{t-})) - f'(X_{t-})\chi(y))\ N(X_{t-}, \mathrm{d}y) \\
\widetilde{Q}(X_{t-}) &= f'(X_{t-})Q(X_{t-})f'(X_{t-}) \\
\widetilde{N}(X_{t-}, A) &= \int_A (f(X_{t-}+y) - f(X_{t-}))\ N(X_{t-}, \mathrm{d}y) \text{ for } A \in \mathcal{B}(\mathbb{R}\backslash\{0\}).
\end{aligned} \tag{4.6}$$

Here, χ and $\widetilde{\chi}$ denote the truncation functions on $\mathbb{R}$ with respect to which the characteristics of X, respectively $\widetilde{X}$, are defined.

Substituting X_{t-} by $f^{-1}(\widetilde{X}_{t-})$ in (4.6) we obtain that $\widetilde{X}$ is again an Itô-Lévy process. If the differential characteristics of this process are again polynomially bounded, we can use our methods on this new process. Its Laplace symbol $\lambda(\widetilde{x}, \xi)$ is derived by plugging (4.6) into (3.2). The process $\widetilde{X}$ can then be analyzed by the method above. If we find a stationary distribution $\widetilde{\mu}$, we can transform it by f^{-1} to derive the stationary distribution of the original process X.

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