Chair of Automatic Control

Technische Universität München

Stability-Preserving, Adaptive Model Reduction of DAEs by Krylov-Subspace Methods

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In this contribution, we exploit the specific structure of index-1 differential-algebraic equations (DAEs) in semi-explicit form and present two different methods for stability-preserving reduction. The first technique preserves strictly dissipativity of the underlying dynamics, the second takes advantage of $\mathcal{H}_2$-pseudo-optimal reduction and further allows for an adaptive selection of reduction parameters such as reduced order and Krylov shifts.

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Index-1 DAEs in semi-explicit form

Model reduction problem

Given a stable linear constant coefficient DAE

$$\begin{aligned} E\,\dot{x} &= A\,x + B\,u \\ y &= C\,x + D\,u \end{aligned} \qquad \begin{cases} x \in \mathbb{R}^N,\ u \in \mathbb{R}^m,\ y \in \mathbb{R}^p\ (p, m \ll N) \\ \det E = 0 \end{cases}$$

find a reduced order model

$$\begin{aligned} \overbrace{W^\top E V}^{E_r}\,\dot{x}_r &= \overbrace{W^\top A V}^{A_r}\,x_r + \overbrace{W^\top B}^{B_r}\,u \\ y_r &= \underbrace{C\,V}_{C_r}\,x_r + \underbrace{D}_{D_r}\,u \end{aligned} \qquad \begin{cases} x_r \in \mathbb{R}^n \\ n \ll N \end{cases}$$

that approximates the dynamics of the DAE while satisfying the algebraic constraints and preserving stability.

Index-1 semi-explicit DAEs (SE-DAE)

The special case of DAE considered takes the form

$$\begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_{22} \end{bmatrix} u$$
$$y = [C_{11}, C_{22}] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + Du \qquad (1)$$

Note that replacing $x_2 = -A_{22}^{-1}(A_{21}x_1 + B_{22}u)$ would yield the underlying ODE

$$\underbrace{E_{11}}_{E_1}\dot{x}_1 = \overbrace{(A_{11} - A_{12}A_{22}^{-1}A_{21})}^{A_1}\,x_1 + \overbrace{(B_{11} - A_{12}A_{22}^{-1}B_{22})}^{B_1}\,u$$
$$y = \underbrace{(C_{11} - C_{22}A_{22}^{-1}A_{21})}_{C_1}\,x_1 + \underbrace{\left(D \overbrace{- C_{22}A_{22}^{-1}B_{22}}^{D_{imp}}\right)}_{D_1}\,u \qquad (2)$$

Reduction by Krylov-subspace methods

Consider the input and output Krylov subspaces $\mathrm{Im}(V)$, $\mathrm{Im}(W)$ defined by following Sylvester equations

$$AV - EVS_V - BR = 0$$
$$A^\top W - E^\top W\,S_W{}^\top - C^\top L^\top = 0$$

and parametrized by the pairs (S_V, R) and (S_W, L).
Two-sided projection using Krylov-subspace methods for SE-DAEs as in (1) can be achieved by shifting the reduced matrices

$$\begin{aligned} E_r &= W^\top E V, & A_r &= W^\top A V + L\,D_{imp}\,R \\ B_r &= W^\top B + L\,D_{imp}, & C_r &= C, V + D_{imp}\,R \\ D_r &= D + D_{imp} \end{aligned} \qquad (3)$$

However, stability can be lost.

[Gugercin/Stykel/Wyatt '13]

Stability-preserving reduction

Selection of reduction strategy

SE-DAE

The selection of reduction strategy can be based on the structure of the DAE.

strictly proper — y / n

$D_{imp} = 0$

strictly dissipative — y / n

$E_1 = E_1^\top \succ 0,\ A_1 + A_1^\top \prec 0$

this can be achieved by

$T \cdot (1)$ with

$$T = \begin{bmatrix} E_{11}^\top P & -E_{11}^\top P A_{12} A_{22}^{-1} \\ 0 & I \end{bmatrix}$$

$E_{11}^\top P A_1 + A_1^\top P E_{11} \prec 0$

For strictly proper SE-DAEs, CUREd SPARK can be applied without changes.

For systems with implicit feedthrough term, CUREd SPARK is applied on a SE-DAE realization of the strictly proper part.

Proposed reduction procedures

1) Stability-preserving reduction for strictly proper, strictly dissipative SE-DAEs by orthogonal projection

Note: Strictly dissipativity implies asymptotic stability

$$E_1 = E_1^\top \succ 0,\ A_1 + A_1^\top \prec 0 \implies \Lambda(A_1, E_1) \subset \mathbb{C}^-$$

and is preserved by orthogonal projection

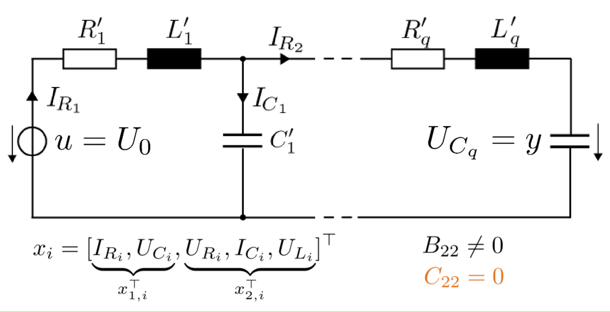
$$V^\top E_1 V = \left(V^\top E_1 V\right)^\top \succ 0,\quad V^\top \left(A_1 + A_1^\top\right) V \prec 0$$

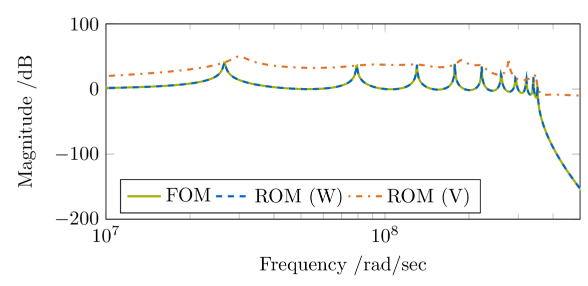
Question: How can we conduct an orthogonal projection of the underlying dynamics (2) by projecting the implicit DAE in (1)?

[Silveira/Kamon/Elfadel/White '99, Panzer '14]

Problem: The reduction strategy in (3) generally fails to reduce the underlying ODE (2) for orthogonal projections!

Solution: If $B_{22} = 0$ compute V as an input, if $C_{22} = 0$ compute W as an output Krylov subspace





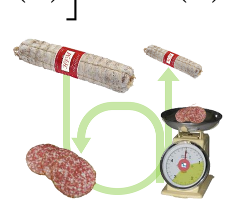
2) Stability preservation with adaptive choice of reduced order and Krylov parameters (CUREd SPARK)

Cumulative reduction (CURE) based on factorization

$$G_e(s) = \underbrace{\left[\begin{array}{c|c} E, A & B_\perp \\ \hline C & 0 \end{array}\right]}_{G_\perp(s)} \cdot \underbrace{\left[\begin{array}{c|c} E_r, A_r & B_r \\ \hline R & I \end{array}\right]}_{\widetilde{G}_r(s)}$$

allows for an adaptive choice of reduced order n.

$$\begin{aligned} G(s) &= G_r(s) + \boxed{G_\perp(s)} \cdot \widetilde{G}_r(s) \\ &= G_r(s) + \boxed{G_{r,2}(s) + G_{\perp,2}(s)\cdot\widetilde{G}_{r,2}(s)} \cdot \widetilde{G}_r(s) \\ &\ \ \vdots \\ &= G_r^\Sigma(s) + G_{\perp,k}(s) \cdot \widetilde{G}_r^\Sigma(s) \end{aligned}$$



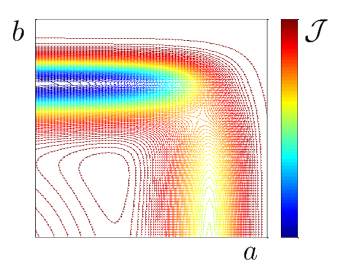
[Panzer/Jaensch/Wolf/Lohmann '13, Wolf/Panzer/Lohmann '13]

$\mathcal{H}_2$-pseudo-optimal rational Krylov (PORK) for SE-DAEs

Algorithm 1 SE-DAE PORK
Input: (E, A, B, C, D), (S_V, R)
Output: $\mathcal{H}_2$-pseudo-optimal reduced system matrices
1: $V \leftarrow AV - EVS_V - BR = 0$
2: $P_r^{-1} = \mathrm{lyap}(-S_V^\top, R^\top R)$
3: $B_r = -P_r R^\top$, $A_r = S_V + B_r R$, $E_r = I$
4: $C_r = CV + D_{imp}R$, $D_r = D + D_{imp}$

Optimal choice of (S_V, R) with SPARK

$$\min \|G - G_r\|_{\mathcal{H}_2} \Rightarrow \max \|G_r\|_{\mathcal{H}_2}$$
$$\mathcal{J}(a, b) = -\|G_r\|_{\mathcal{H}_2}^2 = -C_r P_r C_r^\top \qquad (4)$$



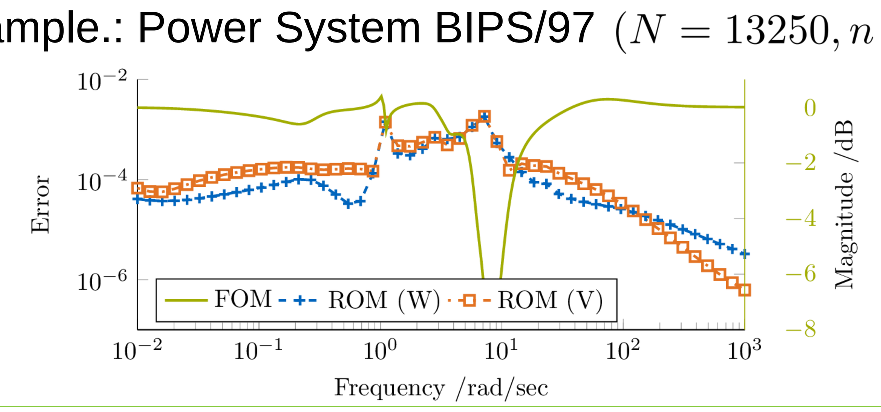
2*) CUREd SPARK for SE-DAEs with implicit feedthrough $D_{imp} \neq 0$

Problem: The implicit feedthrough term makes the cost function (4) meaningless

Solution: Compute a realization for the strictly proper part of (1)

$$G = G^{sp} + D_{imp}, \quad G^{sp} \begin{cases} E\dot{x} &= Ax + \begin{bmatrix} B_{11} - A_{12}A_{22}^{-1}B_{22} \\ 0 \end{bmatrix} u \\ y &= Cx \end{cases}$$

Example.: Power System BIPS/97 $(N = 13250, n = 50)$



Conclusions

SE-DAEs, arising frequently in electrical systems and power networks, can now be reduced without loss of stability by Krylov subspace methods. By the extension of CUREd SPARK to this class of systems, it is possible to adaptively choose reduced order and Krylov parameters.
This procedure currently works only for SISO systems.

[Gugercin/Stykel/Wyatt '13] Model reduction of descriptor systems by interpolatory projection methods
[Panzer/Jaensch/Wolf/Lohmann '13] A greedy rational Krylov method for H2-pseudooptimal model order reduction with preservation of stability
[Panzer '14] Model Order Reduction by Krylov Subspace Methods with Global Error Bounds and Automatic Choice of Parameters
[Silveira/Kamon/Elfadel/White '99] A coordinate-transformed Arnoldi algorithm for generating guaranteed stable reduced-order models of RLC circuits
[Wolf/Panzer/Lohmann '13] H2 pseudo-optimality in model order reduction by Krylov subspace methods

COMING SOON
The reduction was conducted with **sssMOR**, a sparse state space and model reduction toolbox for MATLAB.
Expected release: **Nov 2015**

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