A note on the Kesten–Goldie theorem

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Abstract

Consider the perpetuity equation $X \stackrel{\mathcal{D}}{=} AX + B$, where (A, B) and X on the right-hand side are independent. The Kesten–Goldie theorem states that $\mathbf{P}\{X > x\} \sim cx^{-\kappa}$ if $\mathbf{E}A^\kappa = 1, \mathbf{E}A^\kappa \log_+ A < \infty$ and $\mathbf{E}|B|^\kappa < \infty$. We assume that $\mathbf{E}|B|^\nu < \infty$ for some $\nu > \kappa$, and consider two cases (i) $\mathbf{E}A^\kappa = 1$ but $\mathbf{E}A^\kappa \log_+ A = \infty$; (ii) $\mathbf{E}A^\kappa < 1$ but $\mathbf{E}A^t = \infty$ for all $t > \kappa$. We show that under appropriate additional assumptions on A the asymptotic $\mathbf{P}\{X > x\} \sim cx^{-\kappa}\ell(x)$ holds, where ℓ is a nonconstant slowly varying function. We use Goldie's renewal theoretic approach.
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1 Introduction and results

Consider the perpetuity equation

$$X \stackrel{\mathcal{D}}{=} AX + B, \tag{1}$$

where (A, B) and X on the right-hand side are independent. To exclude degenerate cases as usual we assume that $\mathbf{P}\{Ax + B = x\} < 1$ for any $x \in \mathbb{R}$. We also assume that $A \geq 0$ and that $\log A$ conditioned on being nonzero is nonlattice.

The first results on existence of the solution and its tail behavior is due to Kesten [19], who proved (in d-dimension) that if $\mathbf{E}A^\kappa = 1$, $\mathbf{E}A^\kappa \log_+ A < \infty$, where $\log_+ x = \max\{\log x, 0\}$, $\mathbf{E}|B|^\kappa < \infty$ for some $\kappa > 0$ then the solution of (1) has Pareto-like tail, i.e.

$$\mathbf{P}\{X > x\} \sim c_+ x^{-\kappa} \text{ and } \mathbf{P}\{X < -x\} \sim c_- x^{-\kappa} \text{ as } x \to \infty, \tag{2}$$

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for some $c_+, c_- \geq 0, c_+ + c_- > 0$. (In the following any nonspecified limit relation is meant as $x \to \infty$.) Later Goldie [15] simplified the proof of the same result in the one-dimensional case (for more general equations) using renewal theoretic methods. Following Buraczewski, Damek and Mikosch [5] we refer to (2) as the Kesten–Goldie theorem. That is, under general conditions on A, if $\mathbf{P}\{A > 1\} > 0$ then the tail decreases polynomially. On the other hand, Goldie and Grüber [16] showed that the solution has at least exponential tail under the assumption $A \leq 1$ a.s. For further results in the thin-tailed case see Hitczenko and Wesołowski [18]. Returning to the heavy-tailed case Grey [17] showed that if $\mathbf{E}A^\kappa < 1$, $\mathbf{E}A^{\kappa+\epsilon} < \infty$ then the tail of X is regularly varying with parameter $-\kappa$ if and only if the tail of B is. That is, the regular variation of the solution X of (1) is either caused by A alone, or by B alone (under some weak condition on the other variable). Our intention in the present note is to explore more the role of A, i.e. to extend the Kesten–Goldie theorem. More precisely, we assume that $\mathbf{E}|B|^\nu < \infty$ for some $\nu > \kappa$, and we obtain sufficient conditions on A that imply $\mathbf{P}\{X > x\} \sim \ell(x)x^{-\kappa}$, where $\ell(\cdot)$ is some nonconstant slowly varying function.

The perpetuity equation (1) has a wide range of applications; we only mention the ARCH and GARCH models in financial time series analysis, see Embrechts, Klüppelberg and Mikosch [10, 8.4 Perpetuities and ARCH Processes]. For a complete account on the equation (1) refer to Buraczewski, Damek and Mikosch [5]. Equation (1) is also strongly related to exponential functional of Lévy processes, see Arista and Rivero [2] and Behme and Lindner [4] and the references therein.

The key idea in Goldie's proof is to introduce the new probability measure

$$\mathbf{P}_\kappa\{\log A \in C\} = \mathbf{E}[I(\log A \in C)A^\kappa], \tag{3}$$

where $I(\cdot)$ stands for the indicator function. Since $\mathbf{E}A^\kappa = 1$ this is indeed a probability measure. If F is the distribution function (df) of $\log A$ under $\mathbf{P}$ then under $\mathbf{P}_\kappa$

$$F_\kappa(x) = \mathbf{P}_\kappa\{\log A \leq x\} = \int_{-\infty}^{x} e^{\kappa y} F(\mathrm{d}y). \tag{4}$$

Under $\mathbf{P}_\kappa$ equation (1) can be rewritten as a renewal equation, where the renewal function corresponds to F_κ, the df of $\log A$. If $\mathbf{E}_\kappa \log A = \mathbf{E}A^\kappa \log A < \infty$ then a renewal theorem on the line implies the required tail asymptotics. Yet a smoothing transformation and a Tauberian argument is needed, since key renewal theorems apply only for direct Riemann integrable functions.

What we assume instead of the finiteness of the mean is that under $\mathbf{P}_\kappa$ the variable $\log A$ is in the domain of attraction of a stable law with index $\alpha \in (0,1)$, i.e. $\log A \in D(\alpha)$. Since

$$F_\kappa(-x) = \mathbf{P}_\kappa\{\log A \leq -x\} = \mathbf{E}I(\log A \leq -x)A^\kappa \leq e^{-\kappa x}, \tag{5}$$

under $\mathbf{P}_\kappa$ the variable $\log A \in D(\alpha)$ if and only if

$$1 - F_\kappa(x) = \overline{F}_\kappa(x) = \frac{\ell(x)}{x^\alpha}, \tag{6}$$

where ℓ is a slowly varying function. Let $U(x) = \sum_{n=0}^{\infty} F_{\kappa}^{*n}(x)$ denote the renewal measure of $\log A$ and put

$$m(x) = \int_0^x [F_\kappa(-u) + \overline{F}_\kappa(u)] \mathrm{d}u \sim \int_0^x \overline{F}_\kappa(u) \mathrm{d}u \sim \frac{\ell(x) x^{1-\alpha}}{1-\alpha}$$

for the truncated expectation; the first asymptotic follows from (5), the second from (6). To obtain the asymptotic behavior of the solution of the renewal equation we have to use a key renewal theorem for random variables with infinite mean. The two-sided infinite mean analog of the strong renewal theorem (SRT) is the convergence

$$\begin{aligned} \lim_{x\to\infty} m(x)[U(x+h) - U(x)] &= hC_\alpha, \\ \lim_{x\to\infty} m(x)[U(-x+h) - U(-x)] &= hC'_\alpha, \end{aligned} \tag{7}$$

where, in our case

$$C_\alpha = [\Gamma(\alpha)\Gamma(2-\alpha)]^{-1}, \ C'_\alpha = 0.$$

The first infinite mean SRT was shown by Garsia and Lamperti [14] in 1963 for nonnegative integer valued random variables, which was extended to the nonlattice case by Erickson [11, 12]. In both cases it was shown that for $\alpha \in (1/2, 1)$ assumption (6) implies the SRT, while for $\alpha \leq 1/2$ further assumptions are needed. For $\alpha \leq 1/2$ sufficient conditions for (7) were given by Chi [7], Doney [8], Vatutin and Topchii [22]. The necessary and sufficient condition for nonnegative random variables was given independently by Caravenna [6] and Doney [9]. They showed that if for a nonnegative random variable with df H (6) holds with $\alpha \leq 1/2$ then (7) holds if and only if

$$\begin{aligned} &\lim_{x\to\infty} x\overline{H}(x)[H(x+h) - H(x)] = 0, \text{ for any } h > 0, \text{ and} \\ &\lim_{\delta\to 0} \limsup_{x\to\infty} x\overline{H}(x) \int_1^{\delta x} \frac{1}{y\overline{H}(y)^2} H(x - \mathrm{d}y) = 0. \end{aligned} \tag{8}$$

Moreover, Doney conjectures that the same is true for random walks, i.e. $H \in D(\alpha)$ and the SRT holds if and only if (8) holds. This is actually shown for $\alpha > 1/3$. However, a closer inspection of Doney's proof shows that this is indeed the case for any $\alpha \in (0, 1)$ for exponentially negligible left-tail, which, by (5), is exactly our setup. For further results and history about the infinite mean SRT we refer to [6, 9] and the references therein. In what follows, we simply assume that (7) holds. In Lemma 1 below, which is a modification of Erickson's Theorem 2 [11], we prove the corresponding key renewal theorem. Since in the literature ([21, Lemma 3], [22, Theorem 4]) this lemma is stated incorrectly, we give a counterexample in the appendix. We use the notation $x_+ = \max\{x, 0\}$, $x_- = \max\{-x, 0\}$, $x \in \mathbb{R}$.

Theorem 1. *Assume that* $\mathbf{E}A^\kappa = 1$, $\mathbf{E}|B|^\nu < \infty$ *for some* $\nu > \kappa$, *(6) and (7) hold,* $\log A$ *conditioned on being nonzero is nonlattice, and that* $\mathbf{P}\{Ax + B = x\} < 1$ *for any* $x \in \mathbb{R}$. *Then for the tail of the solution of the perpetuity equation (1) we have*

$$\begin{aligned} \lim_{x\to\infty} m(\log x) x^\kappa \mathbf{P}\{X > x\} &= C_\alpha \frac{1}{\kappa} \mathbf{E}[(AX+B)_+^\kappa - (AX)_+^\kappa], \\ \lim_{x\to\infty} m(\log x) x^\kappa \mathbf{P}\{X \le -x\} &= C_\alpha \frac{1}{\kappa} \mathbf{E}[(AX+B)_-^\kappa - (AX)_-^\kappa]. \end{aligned} \tag{9}$$

Moreover, $\mathbf{E}[(AX+B)_+^\kappa - (AX)_+^\kappa] + \mathbf{E}[(AX+B)_-^\kappa - (AX)_-^\kappa] > 0$.

The conditions of the theorem are stated in terms of the measure-changed A. Following the remark by Korshunov after his Theorem 2 in [20] we give a sufficient condition to (6) in terms of the df F of $\log A$ under the original probability $\mathbf{P}$. Simple properties of regularly varying functions imply that if

$$e^{\kappa x} \overline{F}(x) = \frac{\alpha\, \ell(x)}{\kappa\, x^{\alpha+1}}$$

with a slowly varying function ℓ then (6) holds. However, the converse implication does not hold.

The asymptotic behavior of the solution X of (1) is closely related to the maximum $M = \max\{0, S_1, S_2, \ldots\}$ of the corresponding random walk $S_n = \log A_1 + \log A_2 + \ldots + \log A_n$, where $\log A_1, \log A_2, \ldots$ are iid $\log A$. The assumption $\mathbf{E}A^\kappa = 1$ implies that $\mathbf{E}\log A < 0$, so the random walk tends to $-\infty$, thus M is a.s. finite. Assuming (6) Korshunov [20] showed that for some constant $c > 0$

$$\lim_{x\to\infty} \mathbf{P}\{M > x\} e^{\kappa x} m(x) = c. \tag{10}$$

(In the following c, c', C are nonnegative constants, whose value may differ from line to line.) In specific cases this result is equivalent to our theorem. Let $(\xi_t)_{t\ge0}$ be a nonmonotone Lévy process, which tends to $-\infty$. Consider its exponential functional $J = \int_0^\infty e^{\xi_t} \mathrm{d}t$ and its supremum $\overline{\xi}_\infty = \sup_{t\ge0} \xi_t$. Arista and Rivero [2, Theorem 4] showed that $\mathbf{P}\{J > x\}$ is regularly varying with parameter $-\alpha$ if and only if $\mathbf{P}\{e^{\overline{\xi}_\infty} > x\}$ is regularly varying with the same parameter. Now, if (ξ_t) has finite jump activity and 0 drift then conditioning on its first jump time one has the perpetuity equation

$$J \overset{\mathcal{D}}{=} AJ + B,$$

with B being an exponential random variable, independent of A, which is the jump size. The equivalence of Theorem 1 and (10) readily follows. We use this argument at the end of the proof of Theorem 2.

Finally, we note that the tail behavior (9) with nontrivial slowly varying function was noted before by Rivero [21] for exponential functionals of Lévy processes. In Counterexample

1 [21] the following is shown. Let $(\sigma_t)_{t\geq 0}$ be a nonlattice subordinator, such that $\mathbf{E}e^{\kappa\sigma_1} < \infty$ and $m(x) = \mathbf{E}I(\sigma_1 > x)e^{\kappa\sigma_1}$ is regularly varying with index $-\alpha \in (-1/2, -1)$. Consider the Lévy process $(\xi_t)_{t\geq 0}$ obtained by killing σ at ζ, an independent exponential time with parameter $\log \mathbf{E}e^{\kappa\sigma_1}$. Then, in terms of the exponential functional $J = \int_0^\zeta e^{\xi_t} \mathrm{d}t$ Lemma 4 [21] states that $\lim_{x\to\infty} m(\log x)x^\kappa \mathbf{P}\{J > x\} = c$, for some $c > 0$. (Note that $\mathbb{P}_1\{T_0 > t\} = \mathbf{P}\{J > t\}$.)

Assume now that $\mathbf{E}A^\kappa = \theta < 1$ for some $\kappa > 0$, and $\mathbf{E}A^t = \infty$ for any $t > \kappa$. Consider the new probability measure

$$\mathbf{P}_\kappa\{\log A \in C\} = \theta^{-1}\mathbf{E}[I(\log A \in C)A^\kappa],$$

that is under the new measure $\log A$ has df

$$F_\kappa(x) = \theta^{-1} \int_{-\infty}^x e^{\kappa y} F(\mathrm{d}y).$$

Note that these are the same definitions as in (3) and (4), the only difference is that now $\theta < 1$. Therefore the same notation should not be confusing. The assumption $\mathbf{E}A^t = \infty$ for all $t > \kappa$ means that F_κ is heavy-tailed. Rewriting again (1) under the new measure $\mathbf{P}_\kappa$ leads now to a *defective* renewal equation for the tail of X. To analyze the asymptotic behavior of the resulting equation we use the techniques and results developed by Asmussen, Foss and Korshunov [3]. A slight modification of their setup is necessary, since our df F_κ is not concentrated on $[0, \infty)$.

For some $T \in (0, \infty]$ let $\Delta = (0, T]$. For a df H we put $H(x + \Delta) = H(x + T) - H(x)$. A df H on $\mathbb{R}$ is in the class $\mathcal{L}_\Delta$ if $H(x+t+\Delta)/H(x+\Delta) \to 1$ uniformly in $t \in [0, 1]$, and it belongs to the class of Δ-subexponential distributions, $H \in \mathcal{S}_\Delta$, if $H(x+\Delta) > 0$ for x large enough, $H \in \mathcal{L}_\Delta$, and $(H * H)(x + \Delta) \sim 2H(x + \Delta)$. If $H \in \mathcal{S}_\Delta$ for every $T > 0$ then it is called *locally subexponential*, $H \in \mathcal{S}_{loc}$. The definition of the class $\mathcal{S}_\Delta$ given by Asmussen, Foss and Korshunov [3] for distributions on $[0, \infty)$ and by Foss, Korshunov and Zachary [13, Section 4.7] for distributions on $\mathbb{R}$. In order to use a slight extension of Theorem 5 [3] we need the additional natural assumption $\sup_{y>x} F_\kappa(y + \Delta) = O(F_\kappa(x + \Delta))$ for x large enough. (This does not follow from the assumptions.) Properties of locally subexponential distributions, in particular its relation to infinitely divisible distributions were investigated by Watanabe and Yamamuro [23, 24].

Theorem 2. *Assume that* $\mathbf{E}A^\kappa = \theta < 1$ *for some* $\kappa > 0$, $F_\kappa \in \mathcal{S}_{loc}$, $\sup_{y>x} F_\kappa(y + \Delta) = O(F_\kappa(x+\Delta))$ *for x large enough,* $\mathbf{E}|B|^\nu < \infty$ *for some* $\nu > \kappa$, $\log A$ *conditioned on A being nonzero is nonlattice, and* $\mathbf{P}\{Ax + B = x\} < 1$ *for any* $x \in \mathbb{R}$. *Then*

$$\begin{aligned}
\lim_{x\to\infty} g(\log x)^{-1}x^\kappa \mathbf{P}\{X > x\} &= \frac{1}{(1-\theta)^2\kappa}\mathbf{E}[(AX + B)_+^\kappa - (AX)_+^\kappa], \\
\lim_{x\to\infty} g(\log x)^{-1}x^\kappa \mathbf{P}\{X \leq -x\} &= \frac{1}{(1-\theta)^2\kappa}\mathbf{E}[(AX + B)_-^\kappa - (AX)_-^\kappa],
\end{aligned} \tag{11}$$

where $g(x) = F_\kappa((x, x+1])$. *Moreover,* $\mathbf{E}[(AX+B)_+^\kappa - (AX)_+^\kappa] + \mathbf{E}[(AX+B)_-^\kappa - (AX)_-^\kappa] > 0$.

The condition $F_\kappa \in \mathcal{S}_{loc}$ is much stronger than the corresponding regularly varying condition in Theorem 1. Typical examples satisfying this condition are the Pareto, lognormal and Weibull (with parameter less than 1) distributions, see [3, Section 4]. For example in the Pareto case, i.e. if for large enough x we have $\overline{F}_\kappa(x) = c\,x^{-\beta}$, for some $c > 0, \beta > 0$, then $g(x) \sim c\beta x^{-\beta-1}$, and so $\mathbf{P}\{X > x\} \sim c' x^{-\kappa} (\log x)^{-\beta-1}$. In the lognormal case, when $F_\kappa(x) = \Phi(\log x)$ for x large enough, with Φ being the standard normal df, (11) gives the asymptotic $\mathbf{P}\{X > x\} \sim c x^{-\kappa} e^{-(\log\log x)^2/2} / \log x$, $c > 0$. Finally, for Weibull tails $\overline{F}_\kappa(x) = e^{-x^\beta}$, $\beta \in (0,1)$, we obtain $\mathbf{P}\{X > x\} \sim c x^{-\kappa} (\log x)^{\beta-1} e^{-(\log x)^\beta}$, $c > 0$.

Theorem 2 and Theorem 4 [2] together immediately imply the following.

Corollary 1. *Let* $\log A_1, \log A_2, \ldots$ *be iid* $\log A$, *let* $S_n = \log A_1 + \log A_2 + \ldots + \log A_n$ *denote their partial sum, and* $M = \max\{0, S_1, S_2, \ldots\}$. *Assume that* $\mathbf{E}A^\kappa < 1$ *for some* $\kappa > 0$, $F_\kappa \in \mathcal{S}_{loc}$, $\sup_{y > x} F_\kappa(y + \Delta) = O(F_\kappa(x + \Delta))$ *for* x *large enough,* $\log A$ *conditioned on* A *being nonzero is nonlattice. Then*

$$\mathbf{P}\{M > x\} \sim c g(x) e^{\kappa x},$$

with some $c > 0$, *where* $g(x) = F_\kappa((x, x+1])$.

2 Proofs

2.1 Proof of Theorem 1

We follow closely Goldie's proof. It is enough to show the first limit in (9), since the second follows from this writing $-X$ and $-B$ in (1). Writing

$$\psi(x) = e^{\kappa x}(\mathbf{P}\{AX + B > e^x\} - \mathbf{P}\{AX > e^x\}), \; f(x) = e^{\kappa x}\mathbf{P}\{X > e^x\} \tag{12}$$

from (1) using that X and A are independent we obtain the equation

$$f(x) = \psi(x) + \mathbf{E} f(x - \log A) A^\kappa. \tag{13}$$

Using the definition (3) we see that $\mathbf{E}_\kappa g(\log A) = \mathbf{E}(g(\log A) A^\kappa)$ thus under the new measure equation (13) reads as

$$f(x) = \psi(x) + \mathbf{E}_\kappa f(x - \log A). \tag{14}$$

It is necessary to introduce the smoothing transformation, since the function ψ is not necessary directly Riemann integrable (dRi), which is needed to apply the key renewal theorem. For an integrable function g introduce its smoothed transform

$$\hat{g}(s) = \int_{-\infty}^{s} e^{-(s-x)} g(x) \mathrm{d}x. \tag{15}$$

Applying this transform to both sides of (14) we get the renewal equation

$$\hat{f}(s) = \hat{\psi}(s) + \mathbf{E}_\kappa \hat{f}(s - \log A). \tag{16}$$

Iterating (16) we obtain for any $n \geq 1$

$$\hat{f}(s) = \sum_{k=0}^{n-1} \int_{\mathbb{R}} \hat{\psi}(s-y) F_\kappa^{*k}(\mathrm{d}y) + \mathbf{E}_\kappa \hat{f}(s - S_n), \tag{17}$$

where $\log A_1, \log A_2, \ldots$ are iid $\log A$, independent of X, and $S_n = \log A_1 + \ldots + \log A_n$. Since $S_n \to -\infty$ $\mathbf{P}$-a.s.

$$\mathbf{E}_\kappa \hat{f}(s - S_n) = e^{-s} \int_{-\infty}^{s} e^{(\kappa+1)y} \mathbf{P}\{X e^{S_n} > e^y\} \mathrm{d}y \to 0 \quad \text{as } n \to \infty,$$

therefore as $n \to \infty$ from (17) we have

$$\hat{f}(s) = \int_{\mathbb{R}} \hat{\psi}(s-y) U(\mathrm{d}y). \tag{18}$$

where $U(x) = \sum_{n=0}^{\infty} F_\kappa^{*n}(x)$ is the renewal measure of F_κ. The question is under what conditions of z the key renewal theorem

$$m(x) \int_{\mathbb{R}} z(x-y) U(\mathrm{d}y) \to C_\alpha \int_{\mathbb{R}} z(y) \mathrm{d}y \tag{19}$$

holds.

In the following lemma, which is a modification of Erickson's Theorem 2 [11], we give sufficient condition for z to (19) hold. We note that both in Lemma 3 [21] and in Theorem 4 of [22] the authors wrongly claim that (19) holds if z is dRi. A counterexample is given in the appendix.

Lemma 1. *Assume that z is dRi and $z(x) = O(x^{-1})$ as $x \to \infty$. Then (7) implies (19).*

Proof. We may and do assume that $z \geq 0$. Write

$$\begin{aligned}
& m(x) \int_{\mathbb{R}} z(x-y) U(\mathrm{d}y) \\
& = m(x) \left[\int_x^\infty z(x-y) U(\mathrm{d}y) + \int_0^x z(x-y) U(\mathrm{d}y) + \int_{-\infty}^0 z(x-y) U(\mathrm{d}y) \right] \\
& =: I_1(x) + I_2(x) + I_3(x).
\end{aligned}$$

We show that $I_1(x) \to C_\alpha \int_{-\infty}^0 z(y)\mathrm{d}y$ whenever z is dRi. Fix $h > 0$ and put $z_k(x) = I(x \in ((k-1)h, kh])$, $a_k = \inf\{z(x) : x \in ((k-1)h, kh]\}$, and $b_k = \sup\{z(x) : x \in ((k-1)h, kh]\}$, $k \in \mathbb{Z}$. Simply

$$m(x) \sum_{k=-\infty}^{0} a_k (U * z_k)(x) \leq I_1(x) \leq m(x) \sum_{k=-\infty}^{0} b_k (U * z_k)(x).$$

As $x \to \infty$ by (7) for any fix k

$$\begin{aligned} m(x)(U * z_k)(x) &= m(x)[U(x-kh+h) - U(x-kh)] \\ &= \frac{m(x)}{m(x-kh)} m(x-kh)[U(x-kh+h) - U(x-kh)] \to C_\alpha h, \end{aligned}$$

where the convergence $m(x)/m(x-kh) \to 1$ follows from the fact that m is regularly varying with index $1-\alpha$. Since $1-\alpha > 0$ and $k \leq 0$ this also gives us an integrable majorant uniformly in $k \leq 0$, i.e. for x large enough

$$\limsup_{k<0} m(x)(U * z_k)(x) \leq 2C_\alpha h.$$

Thus by Lebesgue's dominated convergence theorem

$$\begin{aligned} \lim_{x\to\infty} m(x) \sum_{k=-\infty}^{0} a_k (U * z_k)(x) &= C_\alpha \sum_{k=-\infty}^{0} a_k h, \\ \lim_{x\to\infty} m(x) \sum_{k=-\infty}^{0} b_k (U * z_k)(x) &= C_\alpha \sum_{k=-\infty}^{0} b_k h. \end{aligned}$$

Now the statement readily follows from the direct Riemann integrability of z.

The convergence $I_2(x) \to C_\alpha \int_0^\infty z(x)\mathrm{d}x$ is the statement of [11, Theorem 3]. From its proof we see that for any $\theta \in (0,1)$

$$m(x) \int_{\theta x}^{x} z(x-y) U(\mathrm{d}y) \to C_\alpha \int_0^\infty z(y)\mathrm{d}y$$

holds without the extra assumption $z(x) = O(1/x)$, which is only needed to show that

$$\lim_{\theta \downarrow 0} \limsup_{x\to\infty} m(x) \int_0^{\theta x} z(x-y) U(\mathrm{d}y) = 0.$$

Finally, we show that $I_3(x) \to 0$. Indeed, for any $\theta > 0$ the direct Riemann integrability of z combined with a Lebesgue's dominated convergence theorem implies

$$m(x) \int_{-\infty}^{-\theta x} z(x-y) U(\mathrm{d}y) \to 0,$$

while for small x we have with $K \geq \sup_{x>0} x z(x)$ that

$$\limsup_{x\to\infty} m(x) \int_{-\theta x}^{0} z(x-y) U(\mathrm{d}y) \leq K \limsup_{x\to\infty} \frac{U(-\theta x) m(x)}{x} = 0.$$

□

Recall the definition of ψ in (12). In what follows, we show that $\hat{\psi}$ satisfies the condition of Lemma 1.

Lemma 2. *Assume that* $\mathbf{E}A^\kappa \leq 1$, $\mathbf{E}|B|^\nu < \infty$ *for some* $\nu > \kappa$. *Then* $\hat{\psi}$ *is dRi and* $\hat{\psi}(s) = O(e^{-\delta s})$ *as* $s \to \infty$, *for some* $\delta > 0$.

Proof. The direct Riemann integrability holds without the extra moment assumption; this is shown in [15, Lemma 9.2].

Let us choose $\varepsilon > 0$ so small that

$$\kappa + \frac{3\kappa\varepsilon}{1-\varepsilon} < \nu, \text{ if } \kappa \geq 1, \text{ and } \kappa + \varepsilon \leq \min\{1, \nu\}, \text{ for } \kappa < 1. \tag{20}$$

Using the inequality

$$|\mathbf{P}\{AX + B > y\} - \mathbf{P}\{AX > y\}| \leq \mathbf{P}\{AX + B > y \geq AX\} + \mathbf{P}\{AX > y \geq AX + B\},$$

we obtain

$$|\hat{\psi}(s)| \leq \int_{-\infty}^{s} e^{-(s-x)} e^{\kappa x} [\mathbf{P}\{AX + B > e^x \geq AX\} + \mathbf{P}\{AX > e^x \geq AX + B\}] \mathrm{d}x. \tag{21}$$

Changing variables and using Fubini's theorem we have for the first term

$$\begin{aligned}
& e^{-s} \int_{-\infty}^{s} e^{(\kappa+1)x} \mathbf{P}\{AX + B > e^x \geq AX\} \mathrm{d}x \\
& = e^{-s} \int_0^{e^s} y^\kappa \mathbf{P}\{AX + B > y \geq AX\} \mathrm{d}y \\
& \leq e^{-\varepsilon s} \int_0^{e^s} e^{-(1-\varepsilon)s} y^\kappa \mathbf{P}\{AX + B > y \geq AX\} \mathrm{d}y \\
& \leq e^{-\varepsilon s} \int_0^{e^s} y^{\kappa - 1 + \varepsilon} \mathbf{P}\{AX + B > y \geq AX\} \mathrm{d}y \\
& \leq e^{-\varepsilon s} \int_0^{\infty} y^{\kappa - 1 + \varepsilon} \mathbf{E} I(AX + B > y \geq AX) \mathrm{d}y \\
& \leq e^{-\varepsilon s} (\kappa + \varepsilon)^{-1} \mathbf{E} I(B \geq 0)((AX + B)_+^{\kappa+\varepsilon} - (AX)_+^{\kappa+\varepsilon}).
\end{aligned}$$

The same calculation for the second term in (21) implies

$$|\hat{\psi}(s)| \leq e^{-\varepsilon s} (\kappa + \varepsilon)^{-1} \mathbf{E} ||AX + B|^{\kappa+\varepsilon} - |AX|^{\kappa+\varepsilon}|.$$

We show that the expectation on the right-hand side is finite. Indeed, for $a, b \in \mathbb{R}$ we have $||a + b|^\gamma - |a|^\gamma| \leq |b|^\gamma$ for $\gamma \leq 1$ and $||a + b|^\gamma - |a|^\gamma| \leq 2\gamma |b| (|a|^{\gamma-1} + |b|^{\gamma-1})$ for $\gamma > 1$. From Theorem 1.4 by Alsmeyer, Iksanov and Rösler [1] we know that $\mathbf{E}|X|^\gamma < \infty$ for any

$\gamma < \kappa$. Assume that $\kappa \geq 1$ and let $p = \kappa + 2\kappa\varepsilon/(1-\varepsilon)$, $1/q = 1 - 1/p$. By Hölder's inequality and by the choice of ε in (20)

$$\begin{aligned} \mathbf{E}||AX+B|^{\kappa+\varepsilon} - |AX|^{\kappa+\varepsilon}| &\leq 2(\kappa+\varepsilon)\left[\mathbf{E}|B||AX|^{\kappa+\varepsilon-1} + \mathbf{E}|B|^{\kappa+\varepsilon}\right] \\ &\leq 2(\kappa+\varepsilon)\left[\mathbf{E}|X|^{\kappa+\varepsilon-1}(\mathbf{E}|B|^p)^{1/p}(\mathbf{E}A^{q(\kappa+\varepsilon-1)})^{1/q} + \mathbf{E}|B|^{\kappa+\varepsilon}\right] < \infty, \end{aligned}$$

which proves the statement for $\kappa \geq 1$. For $\kappa < 1$ we choose ε such that $\kappa + \varepsilon \leq 1$, so

$$\mathbf{E}||AX+B|^{\kappa+\varepsilon} - |AX|^{\kappa+\varepsilon}| \leq \mathbf{E}|B|^{\kappa+\varepsilon} < \infty.$$

□

Note that for Lemma 1 we only need that $\hat{\psi}(s) = O(s^{-1})$. Obvious modification of the proof shows that this holds if instead of assuming $\mathbf{E}|B|^\nu < \infty$ for some $\nu > \kappa$ we assume that

$$\mathbf{E}|B|^\kappa(\log_+|B|)^{\max\{1,\kappa\}} < \infty, \ \mathbf{E}|B|^\kappa \log_+ A < \infty, \text{ and } \mathbf{E}|B|A^{\kappa-1}\log_+ A < \infty.$$

Clearly, the latter condition is weaker for independent A and B.

We return to the proof of the main result. From Lemmas 1 and 2 we obtain that for the solution of (18)

$$\lim_{s\to\infty} m(s)\hat{f}(s) = C_\alpha \int_{\mathbb{R}} \psi(y)\mathrm{d}y =: c, \tag{22}$$

where we also used the simple fact that $\int_{\mathbb{R}} \psi(y)\mathrm{d}y = \int_{\mathbb{R}} \hat{\psi}(s)\mathrm{d}s$. From (22) the statement follows again the same way as in [15, Lemma 9.3]. Indeed, since $m(x)$ is regularly varying $m(\log x)$ is slowly varying, thus from (22) we obtain for any $0 < a < 1 < b < \infty$

$$m(\log x)\frac{1}{x}\int_{ax}^{bx} y^\kappa \mathbf{P}\{X > y\}\mathrm{d}y \to (b-a)c.$$

With $a = 1 < b$ and $a < 1 = b$ it follows that

$$\begin{aligned} c\frac{b-1}{b^{\kappa+1}-1}(\kappa+1) &\leq \liminf_{x\to\infty} x^\kappa m(\log x)\mathbf{P}\{X > x\} \\ &\leq \limsup_{x\to\infty} x^\kappa m(\log x)\mathbf{P}\{X > x\} \\ &\leq c\frac{1-a}{1-a^{\kappa+1}}(\kappa+1). \end{aligned}$$

As $a \uparrow 1, b \downarrow 1$ the convergence follows. The form of the constant follows by evaluating the integral $\int_{\mathbb{R}} \psi(x)\mathrm{d}x$, which goes along the same lines as in the proof of Lemma 2.

Finally, the positivity of the limit follows exactly the same way as in [15]. Goldie shows [15, p.157] that for some positive constants $c, C > 0$

$$\mathbf{P}\{|X| > x\} \geq c\,\mathbf{P}\left\{\sup\left\{\log A_1 + \ldots + \log A_k : k \geq 1\right\} > C + \log x\right\}. \tag{23}$$

Now the positivity follows from (10).

2.2 Proof of Theorem 2

Again it is enough to prove the result for the right tail. Following the same steps as in the previous proof, we end up with the defective renewal equation

$$f(x) = \psi(x) + \theta \mathbf{E}_\kappa f(x - \log A).$$

Applying the smoothing transform in (15) we obtain

$$\hat{f}(s) = \hat{\psi}(s) + \theta \mathbf{E}_\kappa \hat{f}(s - \log A).$$

The same way as we showed that (16) implies (18) we obtain that

$$\hat{f}(s) = \int_{\mathbb{R}} \hat{\psi}(s-y) U(\mathrm{d}y),$$

where U is the defective renewal measure $U(x) = \sum_{n=0}^\infty (\theta F_\kappa)^{*n}(x)$. Since $\theta < 1$ we have $U(\mathbb{R}) = (1-\theta)^{-1} < \infty$. A modification of Theorem 5 [3] gives the following. Recall that $g(x) = \theta[F_\kappa(x+1) - F_\kappa(x)]$.

Lemma 3. *Assume that F_κ satisfies the condition of Theorem 2, z is dRi, and $z(x) = o(g(x))$. Then*

$$\int_{\mathbb{R}} z(x-y) U(\mathrm{d}y) \sim g(x) \frac{\int_{\mathbb{R}} z(y) \mathrm{d}y}{(1-\theta)^2}.$$

Proof. Assume that z is nonnegative. We again cut the integral

$$\int_{\mathbb{R}} z(x-y) U(\mathrm{d}y) = I_1(x) + I_2(x) + I_3(x),$$

where I_1, I_2 and I_3 are the integrals on (x,∞), $(0,x]$ and on $(-\infty,0]$, respectively.

The asymptotics $I_1(x) \sim g(x) \int_{-\infty}^0 z(y)\mathrm{d}y/(1-\theta)^2$ follows along the same lines as in the proof of Lemma 1. Theorem 5(i) [3] gives $I_2(x) \sim g(x) \int_0^\infty z(y)\mathrm{d}y/(1-\theta)^2$. (In the appendix we explain why the results for Δ-subexponential distributions on $[0,\infty)$ remain true in our case.) Finally, for I_3 we have

$$I_3(x) \le U(-\infty, 0) \sup_{y \ge x} z(y) = o(g(x)),$$

where we use that $\sup_{y\ge x} F_\kappa(y+\Delta) = O(F_\kappa(x+\Delta))$. □

From Lemma 2 we have $\hat{\psi}(x) = O(e^{-\delta x})$ for some $\delta > 0$. Since F_κ is subexponential $\hat{\psi}(x) = o(g(x))$. That is, the condition of Lemma 3 holds, and we obtain the asymptotic

$$\hat{f}(s) \sim g(s) \frac{\int_{\mathbb{R}} z(y)\mathrm{d}y}{(1-\theta)^2}, \quad s \to \infty.$$

Since $g(x)$ is subexponential, $g(\log x)$ is slowly varying, and the proof can be finished exactly the same way as in Theorem 1.

It remains to show the positivity of the sum of the constants. By inequality (23) it is enough to show that $\mathbf{P}\{e^M > x\} \sim c x^{-\kappa} g(\log x)$, with some $c > 0$, where $M = \sup_{n \geq 1}(\log A_1 + \ldots + \log A_n)$ is the maximum of the random walk, is the same as after Theorem 1 and in Corollary 1. Let us choose B to be an standard exponential random variable, independent of A. Since $B > 0$ the constant of the right-tail in (11) is positive. Consider the Lévy process S_{N_t}, where N_t is a standard Poisson process, and $S_n = \log A_1 + \ldots + \log A_n$. Since $S_{N_t} \to -\infty$ the exponential functional $J = \int_0^\infty e^{S_{N_t}} \mathrm{d}t$ is finite, and conditioning on the first jump we see that it satisfies the perpetuity equation

$$J \stackrel{\mathcal{D}}{=} AJ + B.$$

We have just proved that the solution of the latter equation has regularly varying tail with index $-\kappa$. Theorem 4 by Arista and Rivero [2] implies that

$$\frac{\mathbf{P}\{J > x\}}{\mathbf{P}\{e^M > x\}} \to c' > 0,$$

where the constant c' can be determined. The proof is ready. This also implies Corollary 1.

3 Appendix

3.1 A counterexample

Here we give a counterexample to [21, Lemma 3] and [22, Theorem 4], which shows that alone from the direct Riemann integrability of z the key renewal theorem (19) does not follow.

Let $a_n = n^{-\beta}$, with some $\beta > 1$, and let $d_n \uparrow \infty$ a sequence of integers. Define

$$z(x) = \begin{cases} 0, & x = d_n \pm 1/2, \\ a_n, & x = d_n, \\ \text{linear}, & x \in (d_n - 1/2, d_n) \cup (d_n, d_n + 1/2), \\ 0, & \text{elsewhere}. \end{cases}$$

Since $\sum_{n=1}^\infty a_n < \infty$ the function z is directly Riemann integrable.

For $\alpha \in (0,1)$ consider the measure $V(\mathrm{d}y) = y^{\alpha-1}\mathrm{d}y$. Then

$$\int_0^x z(x-y) y^{\alpha-1} \mathrm{d}y = \int_0^x z(y)(x-y)^{\alpha-1} \mathrm{d}y = x^{\alpha-1} \int_0^x z(y)(1-y/x)^{\alpha-1} \mathrm{d}y.$$

It is clear that by Lebesgue's dominated convergence theorem for any $\nu \in (0,1)$

$$\int_0^{\nu x} z(y)(1-y/x)^{\alpha-1} \mathrm{d}y \to \int_0^\infty z(y) \mathrm{d}y.$$

On the other hand for $x = d_n$

$$\int_{d_n-1/2}^{d_n} z(y)(1-y/d_n)^{\alpha-1}\mathrm{d}y = \int_{d_n-1/2}^{d_n} a_n 2[y-(d_n-1/2)]\left(\frac{d_n-y}{d_n}\right)^{\alpha-1}\mathrm{d}y$$
$$= a_n d_n^{1-\alpha}2^{-\alpha}\int_0^1 u(1-u)^{\alpha-1}\mathrm{d}u$$

Choosing $d_n = n^2$ and β such that $2\alpha+\beta<2$ we see that the latter integral goes to infinity, so the asymptotic (19) does not hold.

This example also shows that a growth condition on z, something like Erickson's $z(x) = O(1/x)$, is needed.

3.2 Local subexponentiality

We claim that Theorem 5 in [3] remains true in our setup. Additionally to the local subexponential property, we assume that $\sup_{y\geq x} H(y+\Delta) = O(H(x+\Delta))$. The main technical tool in [3] is the equivalence in Proposition 2. In our setup it has the following form.

Lemma 4. *Assume that* $H \in \mathcal{L}_\Delta$*, and* $\sup_{y\geq x} H(y+\Delta) = O(H(x+\Delta))$*. Let* X, Y *be iid* H*. The following are equivalent:*

(i) $H \in \mathcal{S}_\Delta$*;*

(ii) there is a function h *such that* $h(x)\to\infty$*,* $h(x) < x/2$*,* $H(x-y+\Delta) \sim H(x+\Delta)$ *uniformly in* $|y| \leq h(x)$*, and*

$$\mathbf{P}\{X+Y\in x+\Delta, X>h(x), Y>h(x)\} = o(H(x+\Delta)).$$

Proof. Write

$$\mathbf{P}\{X+Y\in x+\Delta\} = \mathbf{P}\{X+Y\in x+\Delta, X\leq h(x)\} + \mathbf{P}\{X+Y\in x+\Delta, Y\leq h(x)\}$$
$$+\mathbf{P}\{X+Y\in x+\Delta, X>h(x), Y>h(x)\}.$$

Since

$$\int_{-h(x)}^{h(x)} \mathbf{P}\{Y\in x-y+\Delta\}H(\mathrm{d}y) \sim H(x+\Delta),$$

and

$$\int_{-\infty}^{-h(x)} \mathbf{P}\{Y\in x-y+\Delta\}H(\mathrm{d}y) \leq H(-h(x))\sup_{y\geq x} H(y+\Delta) = o(H(x+\Delta)),$$

for the first two terms

$$\mathbf{P}\{X+Y\in x+\Delta, X\leq h(x)\} \sim H(x+\Delta),$$

and the result follows. □

Similarly, assuming the extra growth condition all the results in [3] hold true with the obvious modification of the proof.

Alternatively, we could use Theorem 1.1 by Watanabe and Yamamuro [23], from which the extension of Theorem 5 in [3] follows. In Theorem 1.1 [23] finite exponential moment for the left-tail is assumed, which is satisfied in our setup by (5). Also note that Theorem 1.1 is a much stronger result what we need: it gives an equivalence of certain tails, and we only need implication (ii) $\Rightarrow$ (iii).

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