Dynamics of Stochastic Damage Evolution

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ABSTRACT: Characteristic features of damage accumulation under arbitrary stochastic conditions are studied in terms of continuum damage mechanics (CDM). A uniaxial tension case is chosen for a simplicity of discussion and clearness of results' interpretations. Modification for a kinetic equation of damage evolution for stochastic conditions is proposed. Numerical algorithms for three types of stochasticity—(a) additional noise (fluctuations in external load), (b) inner noise (as result of the non-uniform evolution of ensembles of microdefects) and (c) combination of previous two factors—are obtained. Introduction of a local failure criterion via a threshold damage concentration allows the time-to-fracture distributions and their change with the noise intensity to be analyzed.

KEY WORDS: damage accumulation, stochastic differential equations, failure criterion, noise, time-to-failure distribution.

INTRODUCTION

PROCESS OF DAMAGE accumulation in real brittle and quasi-brittle materials is characterised by the high level of randomness. Such stochastic behaviour is caused not only by the commonly present stochastic component in external load, but is mainly stipulated by the non-uniformity of material itself. The latter is linked with the spatial randomness of material's properties; various types of heterogeneity, texture, distribution of the second phase, presence of initial defects at various scale levels, etc. Action of external loading, no matter how uniform it can be, results in the complicated development of existing and/or in the generation of new defects. These defects are nucleated (or preferentially grow from their initial state), generally speaking, in the so-called "weak points"—regions in material, where its struc-

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ture results in the stress concentration. As far as such regions are randomly distributed, this process vividly exhibits stochastic dynamics. Such a scenario becomes even more complicated at the next stages of loading, when the inter-defect distance decreases—linked with defects' dimensions and/or density growth—and a respective interaction via stress field perturbations arises. Generation of discontinuities results in diminished load-bearing capacity of different regions of material and in a consequent load redistribution around such discontinuities, that, in its turn, activates damage accumulation in adjoining areas.

Thus, there are two—interacting in a common case—types of stochasticity in damage accumulation: spatial and temporal. The effect of spatial stochasticity was analysed by this author with co-authors elsewhere on the basis of lattice model and fractal approach (Silberschmidt, 1993; Silberschmidt and Chaboche, 1994a; Silberschmidt and Silberschmidt, 1994). Lattice approximation of stochastic media was performed by means of discretization of the region under study into elements—within which the macroscopic mechanic properties of the media were considered to be constant. Each unit cell was equivalent to a "physical point" with its own damage accumulation dynamics. The failure of such a cell results in a cascade of load redistributions on neighbouring elements within the radius of stress field's perturbation. Thus, the spatial material's randomness superimposed upon the determinate dynamics of damage accumulation generates the temporal stochasticity under the non-uniform failure of lattice elements. General aspects of such type of temporal randomness—though without introduction of damage parameters in the sense of CDM—were analysed by Hansen et al. (1990; see also respective references therein) for two classes of lattice models: (1) initially uniform medium with disorder caused by random (fracture) pattern formation, (2) pattern formation in disordered media.

This paper is dedicated to another type of temporal stochasticity. We want to study the macroscopic effect of both fluctuating external load and of stress fluctuations linked with processes of defects evolution on microscopic level. Both types are united by the necessity of transition from a traditional form of kinetic equation of damage accumulation to their stochastic analogues.

MODELS OF STOCHASTIC DAMAGE ACCUMULATION

Damage Parameter and Stochastic Kinetic Equations

Since the very beginning of CDM (Kachanov, 1958), researchers continue their attempts to introduce new or to modify old damage parameters and/or to relate them to some physical characteristics of failure process. Two main groups of approaches, based on linkage of damage parameter either to characteristics of imperfections (dimensions and/or density of defects, etc.) or to the extent of the depletion of structure's reliability, utilize different modeling techniques. The second group is usually related to the well-developed methods of fracture probability approaches (Eggwertz and Lind, 1985), but is more suitable for description of the structures' failure, as far as damage is treated as the general (universal) characteristic of the specimen, construction, etc., and thus, it is not considered in this paper. The first group of approaches regards the damage parameter as an additional inner variable, making possible the use of thermodynamic formalism for obtaining of constitutive equations (Chaboche, 1988; Lemaitre and Chaboche, 1991; Hansen and Schreyer, 1994; and references therein). Complicated cases of damage generation and development, caused by the material's heterogeneity, texture and/or non-uniform loading, necessitate the introduction of respective parameters of more complex structure (mainly by increasing its tensoral rank) (Krajcinovic, 1989; Lubarda and Krajcinovic, 1993; Lesne and Saanouni, 1993; see also references in these papers). Another opportunity is an introduction of various parameters for description of different damage modes (Silberschmidt and Chaboche, 1994b) with respective accumulation laws. This allows to distinguish and analyse the pure effect, though, of course, for the concrete narrow class of loading and leaving the question of modes' interaction for further research.

Traditional CDM approaches utilize, as a rule, deterministic form of damage accumulation equations, thus dealing with averaged parameters. Understanding of the sufficiently random character of real damage processes resulted in introduction of respective stochastic schemes of description. Woo and Li (1992) suggested the method of "stochastic dynamic probabilistic modeling" based on the presentation of the disturbance process in the system by small fluctuations (white Gaussian noise). The additional constitutive equation was presented in the form

$$dD_t = f(t,D_t)dt + G(t,D_t)dW_t \tag{1}$$

where D_t is randomized damage parameter, W_t is a Wiener process, a function f is the same as for deterministic equation and is determined from a dissipation function, G is obtained in an assumption that the intensity of fluctuations is directly proportional to the deterministic mean rate of damage evolution (Woo and Li, 1992). Still, the authors preferred to discuss the general mathematical aspects of stochastic processes' modeling [which were thoroughly analysed before by Gardiner (1983), Risken (1989)]; and did not demonstrate any solved problem, mentioning the applicability of approach to mainly ductile type of damage.

Diao (1995) tried to combine the traditional Kachanov's form of the kinetic equation for damage variable w with the stochastic relation for the growth rate of the defect of size a

$$\frac{da}{dt} = M(a) + \beta(a)f(t) \tag{2}$$

using also the weakest link theory of Weibull for the fracture probability $F(\sigma,t)$ of materials. Function $M(a)$ in Equation (2) is the migration growth rate of the defect, $f(t)$ is a white noise, $\beta(t)$ is a magnifying factor. The w-a relation is derived by means of introduction of direct linkage $w(\sigma, t) \equiv F(\sigma, t)$, though Kachanov's parameter is usually connected to the decrease of the bearing capacity of a cross section.

In our papers (Silberschmidt, 1993a; Silberschmidt and Chaboche, 1994b) two various modes of damage—I and II in terms of fracture mechanics—were introduced with respective constitutive equations. Introduction of the stochastic component transforms these kinetic equations to stochastic differential equations (SDE), stationary solutions of which were analysed (Silberschmidt and Chaboche, 1994b). The non-linearity of the right-hand parts of these relations excludes their analytical solutions, so, the recently developed methods of numerical solution of SDE (Kloeden and Platen, 1992) were used (Silberschmidt, 1995). This paper is dedicated to the analysis of stochastic damage evolution under uniform tension (I-mode damage), as far as the respective study for shear (II-mode) damage is discussed elsewhere (Silberschmidt and Silberschmidt, 1996).

Stochastic Differential Equation of Damage Accumulation and Its Analogues

A scalar damage parameter p can be used for a case under study—uniaxial tensile loading. More complicated stress-states can be analysed either with an introduction of the second-order damage tensor (Naimark and Silberschmidt, 1991) or utilizing a set of scalar damage parameters and relations for their interaction. A general form of the kinetic equation for a stochastic I-mode damage evolution is a nonlinear Langevin equation in a form:

$$\frac{dp}{dt} = f(p) + g(p)L(t) \tag{3}$$

where $f(p)$ is a right side of deterministic equation for I-mode damage (Silberschmidt, 1993b; Silberschmidt and Chaboche, 1994b), which is as follows:

$$f(p) = Ap^3 + Bp^2 + Cp - D\sigma \tag{4}$$

A, B, C, and D are the material's parameters, σ is a macroscopic tensile stress; $L(t)$ is a stochastic (Langevin) term with following properties:

(a) $\langle L(t) \rangle = 0$

(b) We consider stochastic term to change very rapidly, thus its autocorrelation function is a delta-function $\langle L(t)L(t') \rangle = \Gamma\delta(t - t')$, Γ is a constant

(c) $L(t)$ is a Gaussian (all odd moments vanish)

Experimental data (Galyarov, 1994) approve the possibility to use such an approximation for initial stage of failure process. Equation (3) is related to the following Fokker-Planck equation (Gardiner, 1983; Van Kampen, 1992) in approximation of σ = const

$$\frac{\partial P(p,t)}{\partial t} = -\frac{\partial}{\partial p}\left[f(p)+\frac{1}{2}\Gamma g(p)\frac{\partial g}{\partial p}\right]P+\Gamma\frac{\partial^2}{\partial p^2}[g(p)]^2 P \tag{5}$$

where $P(p, t)$ is the probability density of the solution of the original Equation (3). The corresponding Itô equation has the form (Risken, 1989; Kloeden and Platen, 1992)

$$dp(t) = f[p(t),t]dt + g[p(t),t]dW(t) \tag{6}$$

where $W(t)$ is a Wiener process. Integral form of Equation (6) is

$$p_t = p_0 + \int_0^t f(p_u)du + \int_0^t g(p_u)dW_u \tag{7}$$

As far as for the given type of non-linearity of $f(p)$ [Equation (4)] analytical solution of Equation (3) is impossible, the integral form of Itô process can be used for the elaboration of numerical schemes.

Numerical Schemes

The simplest time-discrete approximation is a stochastic generalization of Euler approximation, proposed by Maruyama (1955) [which is also called the Euler approximation (Kloeden and Platen, 1992)]:

$$p_{n+1} = p_n + f(p_n)\Delta_n + g(p_n)\Delta W_n \tag{8}$$

Here (and below) the time discretization $0 = t_0 < t_1 < \ldots < t_n < \ldots t_N = T$ of the interval $[0,T]$ with a constant time step $\delta = T/N$ is used; $\Delta_n = \delta$; $\Delta W_n = W_{\tau_{n+1}} - W_{\tau_n}$. The new random variables ΔW_n are normally distributed, that is, have the means $<\Delta W_n> = 0$ and variances $<(\Delta W_n)^2> = \Delta_n$ (Kloeden and Platen, 1992). Equation (8) can be utilized for slowly changing $f(s)$ and $g(s)$, but for more general cases a stochastic analogue of Taylor scheme with additional terms should be used (Milstein, 1995; Kloeden and Platen, 1992). This scheme is an expansion of the function around p_0 in terms of multiple stochastic integrals weighted by coefficients at this point:

$$p_t = p_0 + f(p_0)\int_{t_0}^{t} ds + g(p_0)\int_{t_0}^{t} dW_s$$

$$+ g(p_0)g'(p_0)\int_{t_0}^{t}\int_{t_0}^{s_2} dW_{s_1} dW_{s_2} + \ldots \tag{9}$$

The Euler-Maruyama Equation (8) then can be referred to as a strong Taylor approximation of order 0.5. Milstein (1995) proposed the approximation of order 1.0 by adding the next term to stochastic Taylor scheme:

$$p_{n+1} = p_n + f(p_n)\Delta_n + g(p_n)\Delta W_n + \frac{1}{2}g(p_n)g'(p_n)\left\{(\Delta W_n)^2 - \Delta_n\right\} \tag{10}$$

Note, that for a case $g(p)$ = const the Milstein scheme reduces to the Euler-Maruyama one. In our simulations we shall exploit the strong Taylor approximation of order 1.5, proposed by Platen and Wagner (1982):

$$p_{n+1} = p_n + f(p_n)\Delta_n + g(p_n)\Delta W_n + \frac{1}{2}g(p_n)g'(p_n)\left\{(\Delta W_n)^2 - \Delta_n\right\}$$

$$+ f'(p_n)g(p_n)\Delta Z_n + \frac{1}{2}\left[f(p_n)f'(p_n) + \frac{1}{2}g^2(p_n)f''(p_n)\right]\Delta_n^2$$

$$+ \left[f(p_n)g'(p_n) + \frac{1}{2}g^2(p_n)g''(p_n)\right]\{W_n\Delta_n - \Delta Z_n\}$$

$$+ \frac{1}{2}g(p_n)\left[g(p_n)g''(p_n) + (g'(p_n))^2\right]\left\{\frac{1}{3}(\Delta W_n)^2 - \Delta_n\right\}\Delta W_n \tag{11}$$

Additional random variable ΔZ_n is necessary to represent the double stochastic integral, used in the stochastic Taylor approximation

$$\Delta Z_n = \int_{t_n}^{t_{n+1}}\int_{t_n}^{s_2} dW_{s_1} ds_2 \tag{12}$$

This variable has the following properties (Kloeden and Platen, 1992): (a) mean

$<\Delta Z_n> = 0$; (b) variance $<(\Delta Z_n)^2> = (1/3)(\Delta_n)^3$; (c) covariance $<\Delta W_n \, \Delta Z_n> = (1/2)(\Delta_n)^2$.

NUMERICAL SIMULATION AND DISCUSSION OF RESULTS

Even the case of a uniaxial load with a constant deterministic component presupposes various levels for description of stochastic damage accumulation. The first approximation is a simulation of stochastic action in terms of a fluctuating external load, using an assumption of an additive white noise for it, as far as this action does not depend on the level of damage. So, in this case, function $g(p)$ is equal to unity. The choice of the level for the constant Γ in Equations (3) and (5) should provide a correct value of the fluctuations' mean square for the stationary solution of kinetic equation.

For an additive white noise action the application of the strong Taylor 1.5 scheme (11) gives:

$$p_{n+1} = p_n + \Delta_n(Ap_n^3 + Bp_n^2 + Cp_n - D\sigma)\left\{1 + \frac{\Delta_n}{2}F_n\right\}$$

$$+ \frac{1}{2}\Delta_n^2 Q^2(3Ap_n + B) + Q\{\Delta W_n + F_n \Delta Z_n\} \tag{13}$$

Here $F_n = 3Ap_n^2 + 2Bp_n + C$, $Q = \sqrt{\Gamma}$. Four various statistical realizations (for different levels of noise intensity) are presented in Figure 1 together with the case of the pure deterministic loading (solid line). The values of the coefficients in the right-hand part of kinetic equation used in numerical simulations here and below are: $A = -1$; $B = 1.5$; $C = -0.6$; $D = -0.05$. The estimation of these coefficients for a given material can be obtained by the treatment of experimental data on creep.

Still, the evolution of defects' ensembles and respective dynamics of damage accumulation depends not only on the external fluctuating load. Even under the deterministic loading the evolution of defects at various scale levels is not uniform. It is linked with the complicated structure of real materials, stochastic distribution of defects and defects' nuclei. Generation of new microscopic discontinuities is a sufficiently more rapid process, if compared to a characteristic time of macroscopic damage evolution, and results in the load redistribution in their neighbourhood. As far as such events occur randomly, they can be described by a stochastic component which depends on the damage parameter p itself. Such self-action can be described in terms of inner (multiplicative) noise. As the first approximation an intensity of this noise can be considered as directly proportional to the achieved level of damage, i.e., $g(s) = s$. So, this stochastic action is stipulated only by inner sources; in the absence of damage its intensity equals zero. With damage, growing under load, more microscopic discontinuities

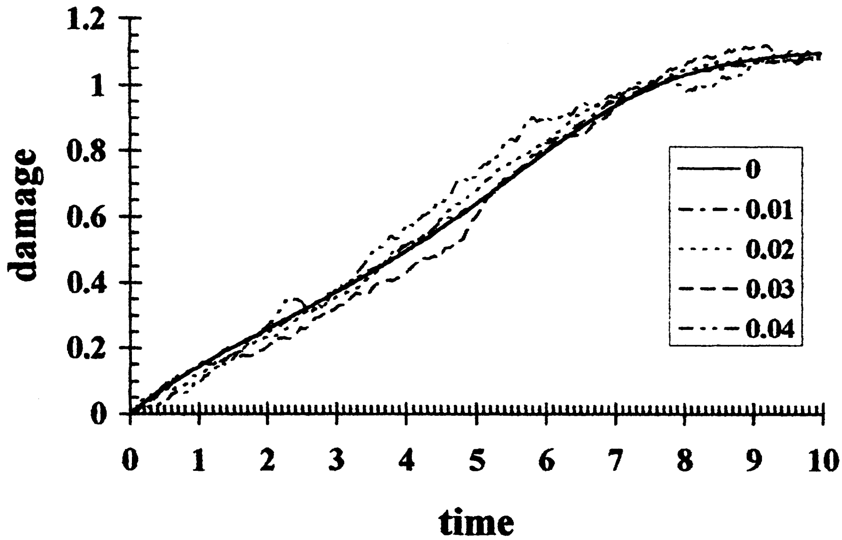

***Figure 1.** Effect of additive noise of various intensity on damage accumulation (stress σ = 4).*

are being generated, decreasing the inter-defect distance and causing multiple load redistributions in their neighbourhood. These load redistributions mean—on the macroscopic level—the increase of noise intensity, as far as the latter is the result of the superposition of these microevents. The strong Taylor 1.5 scheme for the inner noise has the following form:

$$p_{n+1} = p_n + \Delta_n(Ap_n^3 + Bp_n^2 + Cp_n - D\sigma)\left\{1 + \frac{\Delta_n}{2}F_n\right\}$$

$$+ \frac{1}{2}\Delta_n^2 Q^2(3Ap_n + B)p_n - \frac{Q^3}{8\sqrt{p_n}}(\Delta W_n \Delta_n - \Delta Z_n)$$

$$+ Q\sqrt{p_n}\,\{\Delta W_n + F_n \Delta Z_n\} + \frac{Q^2}{4}[(\Delta W_n)^2 - \Delta_n] \qquad (14)$$

The character of the inner noise effect (for the same statistical realization), numerically calculated with the use of Equation (14), are shown in Figure 2 for the same level of load. Results for a coupled action of deterministic load component of various levels and multiplicative noise of the same intensity are presented in Figure 3. The principal difference of the inner noise action from the one of an additive

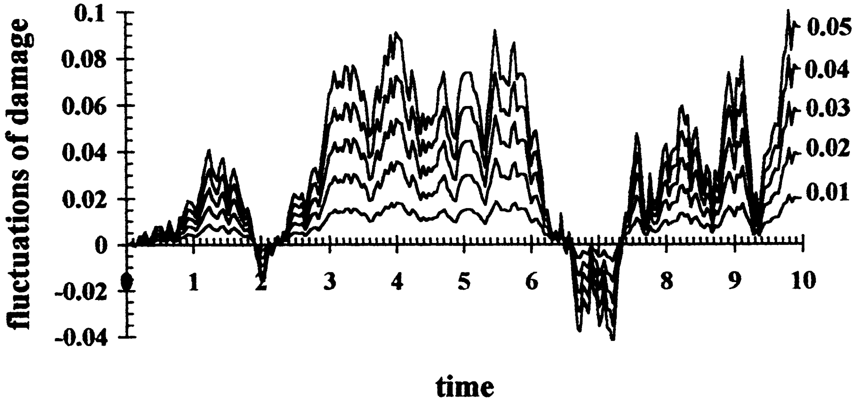

Figure 2. *Pure effect of inner noise of various intensity (σ = 4).*

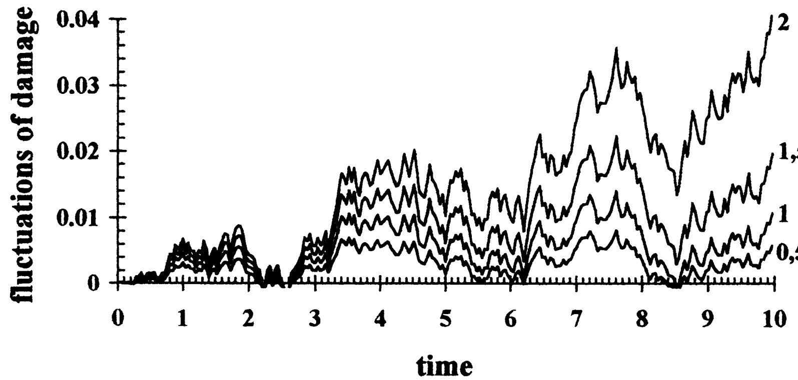

Figure 3. *Effect of stress of various levels on damage accumulation under inner noise action (Q = 0.02).*

noise is that averaged curves do not coincide with the graphs for a deterministic action (absence of fluctuations). This is known as the noise-induced shift (Van Kampen, 1992) and is vividly seen also in our case (Figure 4 presents damage accumulation curves, averaged for 10,000 statistical realizations). In order to more carefully analyse such a behaviour, both the deviation from the deterministic curve (with zero level of fluctuations) (Figure 5) and a standard deviation for distributions of 10,000 solutions (Figure 6) were calculated. The noise of relatively low intensity results in a nearly constant (after some initial period) level of damage fluctuations, while the increase in intensity causes a linear and even more rapid growth of fluctuations in time.

In order to complete the description, the numerical algorithm for a coupled action of both external and internal noise is proposed. In such a case $g(p) = 1 + Rp$ and stochastic Taylor scheme gives

$$
\begin{aligned}
p_{n+1} = {} & p_n + \Delta_n (Ap_n^3 + Bp_n^2 + Cp_n - D\sigma)\left\{1 + \frac{\Delta_n}{2} F_n \right. \\
& \left. + \frac{QR}{2\sqrt{1+Rp_n}} (\Delta W_n \Delta_n - \Delta Z_n)\right\} + Q\sqrt{1+Rp_n}\{\Delta W_n + F_n \Delta Z_n\} \\
& + \frac{1}{2}\Delta_n^2 Q^2 (3Ap_n + B)(1 + Rp_n) - \frac{Q^3 R^2}{8\sqrt{1+Rp_n}} (\Delta W_n \Delta_n - \Delta Z_n) \\
& \times \frac{Q^2 R}{4} \left[(\Delta W_n)^2 - \Delta_n\right]
\end{aligned}
\tag{15}
$$

The results of calculations for this case are not presented in this paper. It is caused by the difficulty of interpretation linked with the multi-parameter character of phenomena (effects of stress, noise intensity and ratio of noises contribution—coefficient R—are to be analysed).

Time-to-Fracture Distribution

Damage accumulation, caused even by a constant load of the sub-critical magnitude, can result in generation of macroscopic failures and their propagation till the total rupture of specimen (if only the saturation—quasi-equilibrium—level of damage with relatively low concentration of defects is not reached). Thus, the local failure criterion can be introduced not in the traditional form, as a threshold value for component(s) and invariant(s) of stress tensor or for any combinations of

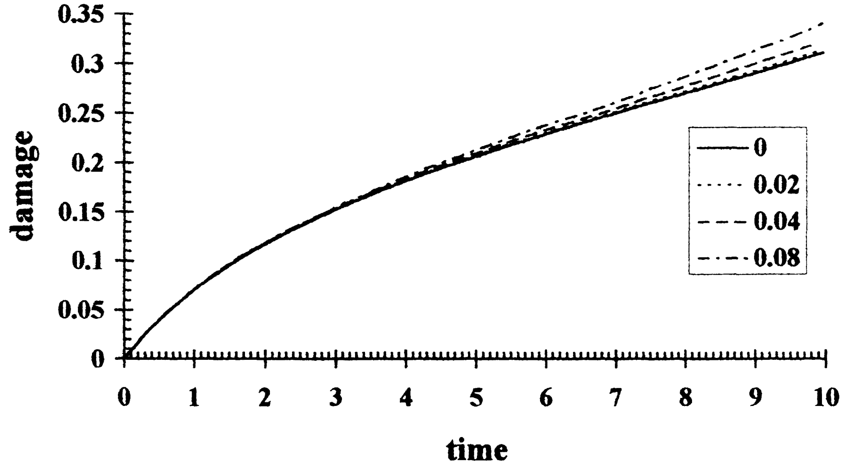

Figure 4. Averaged curves of damage accumulation under inner noise of various intensity ($\sigma = 2$).

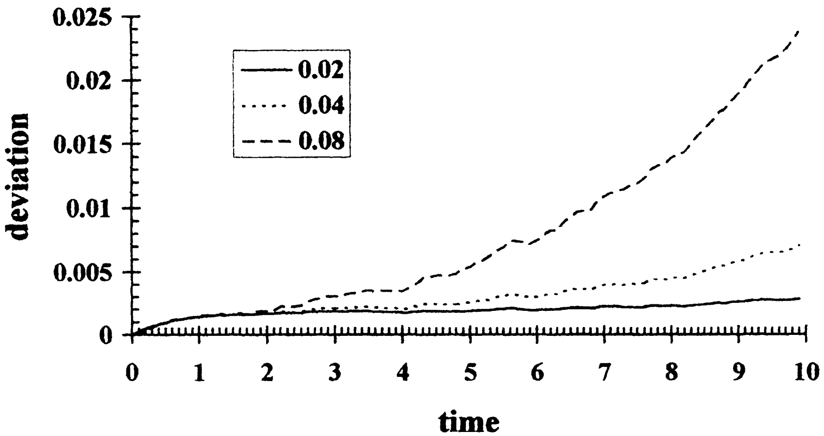

Figure 5. Deviation of averaged curves of damage accumulation under inner noise of various intensity from the deterministic case ($\sigma = 2$).

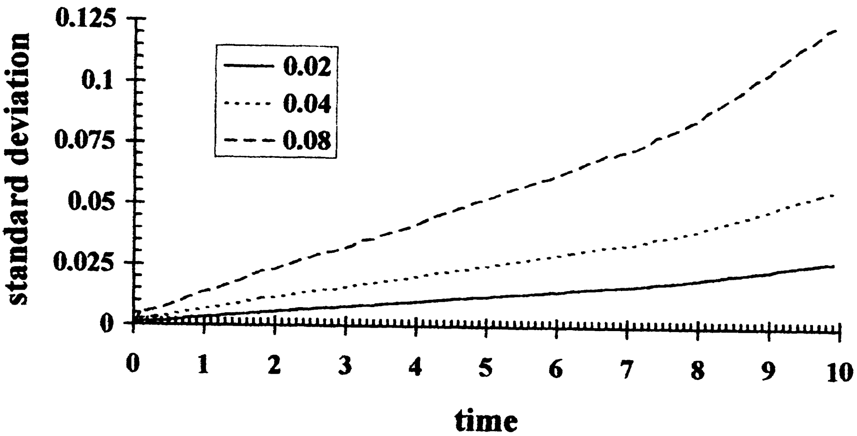

Figure 6. *Change of standard deviation with time for action of inner noise of various intensity (10,000 realizations, $\sigma = 2$).*

these parameters, but in terms of a limiting value for the damage level. Such an approach is supported by the experimental evidence of the invariant character of the local prefracture concentration of defects in material for a vast interval of loading conditions (Betekhtin et al., 1989; Silberschmidt, 1993b). The same idea is used, for instance, by Pineau (1995; see also references therein), where the overcoming of critical volume fraction of cavities results in an acceleration of damage accumulation. So, the local failure criterion is introduced in the following form:

$$p \geq p^* \tag{16}$$

It is noted, that overcoming the threshold value p^* results in a sharp acceleration of damage growth and rapid (with respect to a characteristic time of quasi-stable damage accumulation) failure of specimen/structure. The combination of the local failure criterion (16) with numerical computation of damage dynamics naturally introduces a time-to-fracture parameter in consideration. Action of stochastic factors, either external or internal, causes the randomness of this parameter, when a set of statistical realizations is analysed. This situation is an analogue for an experimental treatment of twin specimens under given conditions. Thus, a description of the system's evolution in terms of "physical point" transforms to a study of effective characteristics of failure.

Figure 7 presents the effect of load fluctuations (in approximation of an additive noise) of different intensity on time to fracture (dashed straight line corresponds to the absence of the stochastic component—only deterministic load of the same magnitude) for a set of 10,000 simulations. It is vividly seen, that an increase in

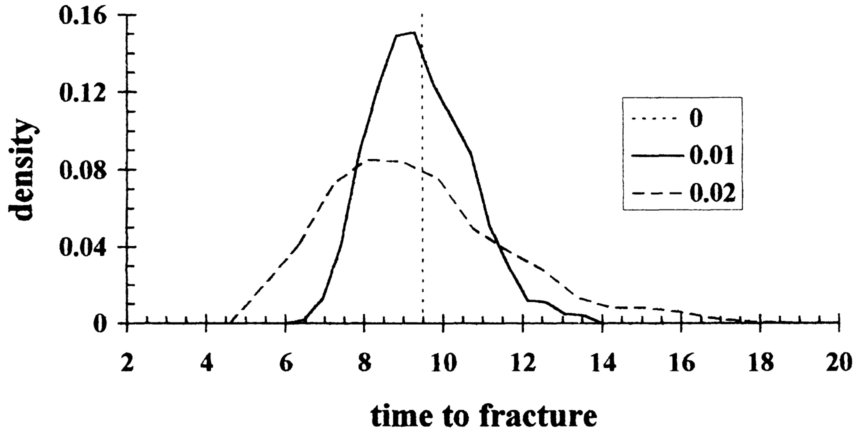

Figure 7. *Effect of additive noise of various intensity on time-to-fracture distribution* (p* = 0.3 *and* $\sigma = 2$).

noise intensity results in a sufficient widening of distributions, with possible local failure occurring at times, up to 50% less than predicted for a deterministic case. Such a decrease in life expectancy cannot be, of course, neglected in reliability estimations for real constructions. Inner noise, as can be expected from the previous results, causes not only the expansion of the intervals of possible time-to-fracture, but also a shift of the distribution's median to the left—the region of diminished reliability (Figure 8).

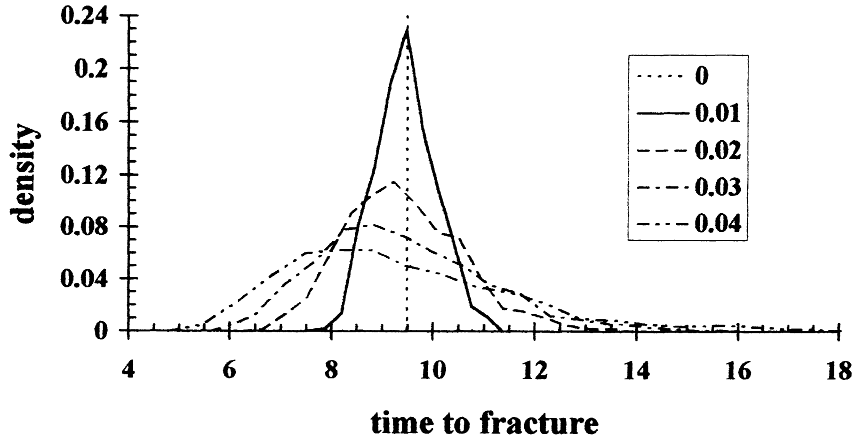

Figure 8. *Effect of inner noise of various intensity on time-to-fracture distribution* (p* = 0.3 *and* $\sigma = 2$).

CONCLUSIONS

Direct introduction of the stochasticity in CDM-model (though at this stage for a relatively simple case of uniaxial tension) provides additional possibilities for analysis, not only of the damage accumulation itself, but also for more understanding of the reliability problem. Even the absence of a fluctuating component in an external load (more correctly, its negligibility) does not result in pure deterministic case of damage accumulation and failure evolution in real brittle and quasi-brittle solids. The non-uniformly distributed microscopic defects and/or their nuclei randomly develop even under macroscopically uniform force field. The consequence of such a behaviour is well-known from the experimental practice: temporal fracture characteristics demonstrate a sufficient scatter for twin-specimens under the same load level. The proposed model description provides quantitative tools for simulation and analysis of this phenomenon, and thus can serve as a base for reliability estimations. Numerical results distinctly show that a noise action of relatively small intensity can cause nearly 100% scatter of effective failure characteristic—time-to-fracture. The more important factor is the noise-induced (in the case of inner and combined noises) shift in the general, i.e., averaged, response of material to external load; and this trend, unfortunately, leads to the decrease of reliability. Therefore, it should be adequately accounted for in estimates. The effect of noise-intensity level on the extent of the reliability decrease can be a base for an assessment of the necessity of accomplishment of traditional computations (in deterministic approximation) by stochastic ones. As noted in the introduction, this type of randomness is only a part of the whole process; which was reduced to the physical-point analysis in order to investigate the characteristic features of stochastic dynamics of damage in detail. The unified approach, accounting also for the effect of spatial randomness (linked with material properties' non-uniformity), can be the next approximation in simulation of the general picture of the damage/failure process.

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