Analysis of Shear Damage Evolution

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ABSTRACT: Shear damage development under constant stress is studied on the basis of the analytical solution of kinetic equations of damage accumulation. The existence of two threshold stresses is linked with the presence of three various stress intervals and respective regimes of damage growth. Utilization of the limit value of damage level as the failure criterion allows the scaling of the time-to-failure to be analyzed. The effect of initial damage on failure development is investigated.

KEY WORDS: damage accumulation, shear deformation, failure criterion, time-to-failure scaling.

INTRODUCTION

THE MULTIAXIAL LOADING of brittle and quasi-brittle materials (especially in the presence of the compressive component) usually results in a complicated scenario of the damage evolution and macroscopic crack nucleation and development. The latter is characterized by the branching, curvilinear crack propagation, which necessitates the introduction of mixed-mode fracture models (Liebowitz, 1968). The possibility for the description of such mutli-mode behavior in terms of continuum damage mechanics (CDM) is linked with the transition from the scalar damage parameter to one of the higher tensoral rank (see discussion and respective references on the models and history of topic in Chaboche, 1988; Chow and Wang, 1988; Murakami, 1988; Krajcinovic, 1989; Lemaitre and Chaboche, 1990; Lemaitre, 1992). The recent state of the introduction and utilization of various damage parameters, and their structure as well, is reflected, for instance, by Lubarda and Krajcinovic (1993); Lesne and Saanouni (1993) (see also references in these papers). Traditional CDM approaches to the multi-mode damage are formulated mainly as an extrapolation of the idea of effective stress, used by Murzewski (1957)—with damage being interpreted as the probability of microelements' fail-

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ure—and by Kachanov (1958) [some historical references on these concepts are given by Murzewski (1992)] for the uniaxial creep, to the multiaxial states. Another way, developed for the analysis of brittle fracture under compression, is utilization of a one-mode damage approach: description of the array of compression-induced tension cracks [see Deng and Nemat-Nasser (1994) and references there] in analysis of the shear faults formation. Such a procedure is applicable to isotropic homogeneous elastic bodies, as far as the crack's wings develop in them in a direction, minimizing shear—II-mode—component (Nemat-Nasser and Horii, 1982; Horii and Nemat-Nasser, 1985, 1986).

Still, in highly heterogeneous materials under compression both modes are to be accounted. The multiple experiments, especially in rock mechanics, vividly reflect the sufficient difference in the realization of various modes of fracture/damage (Hoek and Bieniawski, 1965; Kranz, 1983; V. G. Silberschmidt et al., 1992). It is naturally explained by the different micro- and mesomechanisms governing these processes under various conditions. The mechanisms' interaction results in the non-trivial damage development (for instance, cataclastic character of salt rock's failure when a complicated fracture surface with portions of orthogonal orientations is formed). Different modes of damage are linked with the evolution of the ensembles of various microdefects; nucleation, growth and coalescence of tensile microcracks (Naimark and Silberschmidt, 1991) can be referred to as damage of the mode I [in terms, traditional to the fracture mechanics (Liebowitz, 1968)], while the macroscopic result of the microshifts' development (closed microcracks, generated by the shear stresses, for instance, in the presence of a compressive stress component acting along the normal to their edges) will be called the shear damage, or the II-mode damage. General description of mixed-mode damage evolution should include evolution laws for both modes and also account for coupling effects. Still, at the initial stage of research, specific features of single-mode damage are also of interest, as far as respective model stress-state situation are obvious. Corresponding kinetic equations for both modes of damage are discussed in authors' recent papers (Silberschmidt, 1993b; Silberschmidt and Chaboche, 1994).

The fracture evolution in real brittle materials, linked with the spatio-temporal development of different damage modes, is characterized by the high level of stochasticity, which should be adequately reflected in the models. In such a case damage evolution kinetics is described in terms of stochastic differential equations, stationary solutions of which were obtained (Silberschmidt, 1993b; Silberschmidt and Chaboche, 1994). Unfortunately, high level of non-linearity of these equations excludes the possibility of the analytical solutions and necessitates the use of advanced numerical algorithms, which were recently developed (Kloeden and Platen, 1992; Milstein, 1995) and were adopted for damage accumulation kinetics (Silberschmidt, 1995, 1998). But this analysis should be preceded by the investigation of the specific temporal features of the deterministic behavior of the damage, i.e., in approximation of negligible level of effect of stochastic component(s).

Note that traditional CDM-approaches usually utilize the last assumption. This paper is dedicated to the study of the II-mode (shear) damage evolution under the action of a constant external load and thus the study is devoted to creeping materials.

A MODEL

One of the principal questions in the CDM models is the formulation of the damage evolution (accumulation) law. Linked with its micromechanical background, damage develops as a result of (mainly) irreversible growth and coalescence of microdefects, the processes which occur even under the action of the non-changing external load (not to mention, for instance, thermal fluctuations). Thus, adequate description of the damage accumulation should account for not only stress-damage relations, but also for the temporal characteristics of such a process. The damage evolution equation is a part of general irreversible-thermodynamic formalism describing a macroscopic effect of microscopic processes. This concept was used, for instance, by Rice (1972, 1975) and Krajcinovic and Sumarac (1989) for description of the deformation behaviour of solids under quasi-static loading. It is worth mentioning that formulation of the evolution law naturally introduces a time axis into the model description that is very important in the analysis of the temporal effects of failure, reliability, etc. An elaboration of the general evolution law for the damage parameter—high-order tensor—is possible, for instance, in terms of the thermodynamical formalism (Naimark and Silberschmidt, 1991), but such an approach hides to a certain extent real mechanisms and left complicated problems of modes' interaction open. In order not to overcomplicate the analysis we will study the "pure" II-mode damage as the first approximation. In the following parts the analysis is performed in approximation of a physical point, thus the exact loading scheme—pure shear, compression, etc.—is of no interest. But at the level of spatial description—in terms of boundary-value problem, for instance—it becomes a dominant factor: stress distribution determines both the level of shear damage and its orientation, thus making the basement for general analysis of spatio-temporal effects of failure development.

Let analyze the shear damage development (characterized by a scalar parameter s), which is determined by an action of the maximal shear stress τ along the respective plane; its level and orientation depend on the stress state of the system. The introduced damage parameter has a deformation nature; it thus describes the part of general shear deformation linked with the macroscopic realization of the evolution of microshifts' array in contrast to traditional parameters of CDM which are (for the I-mode damage in our terms) either referred to the decrease of cross section or related to the part of the life expectancy. Such type of parameter is not linked with necessity for an introduction of—mainly artificial—norm; its limiting value is determined by material properties and a loading scheme.

The possibility to generate microshifts is linked with the necessity to overcome the resistance of the crack edge's interaction and of the friction, and while the former is a constant parameter, the latter depends on the level of the compressive force acting perpendicular to the shear plane (Nemat-Nasser and Horii, 1986; Krajcinovic, 1989; Bigel, 1992). The general form of the kinetic equation for the shear damage evolution then can be written as follows (Silberschmidt and Chaboche, 1994):

$$\frac{ds}{dt} = As^2 + Bs + D\langle \tau - \tau^* \rangle^n \tag{1}$$

where A, B and D are material parameters, τ^* is the threshold value, accounting both for friction and cohesion; $\langle\ \rangle$ denote the McCauley bracket; an exponent n allows to describe arbitrary extent of non-linearity (an introduction of it is proposed by J.-L. Chaboche, 1996). The system described by such an equation is characterized by the "trigger-like" kinetics (Silberschmidt and Chaboche, 1994): the process starts only after the overcoming the potential barrier, linked with the presence of resisting forces.

For a definite class of loading types (used in rock mechanics, for instance) τ^* can be considered to be a constant value for a given non-changing level of external load (rock pressure). In such a case Equation (1) can be solved analytically (Kamke, 1967). For the absence of initial damage ($s = 0$ for $t = 0$) the shear damage dependence on time is described by the following expression:

$$s = \begin{cases} \dfrac{2D(\Delta\tau)^n\left(1 - \exp(-t\widetilde{\Delta})\right)}{(B + \widetilde{\Delta})\exp(-t\widetilde{\Delta}) - (B - \widetilde{\Delta})}, & \Delta < 0 \\[2ex] \dfrac{B^2 t}{2A(2 - Bt)}, & \Delta = 0 \\[2ex] \dfrac{1}{2A}\left(\dfrac{\Delta \tan(t\widetilde{\Delta}/2) + B\widetilde{\Delta}}{\widetilde{\Delta} - B\tan(t\widetilde{\Delta}/2)} - B\right), & \Delta > 0 \end{cases} \tag{2}$$

Here $\Delta = 4AD(\Delta\tau)^n - B^2$; $\Delta\tau = \langle \tau - \tau^* \rangle$; $\widetilde{\Delta} = \sqrt{|\Delta|}$. Shear damage accumulation curves are shown in Figure 1 (here and below the results are presented for $n = 1$ and $A = 1$; $B = -2.5$; $D = 1$). It is obvious, that for the given material parameters, the increase in the shear stress level causes the change in the asymptotical behavior of the II-mode damage accumulation: for the relatively small level of the parameter $\Delta\tau$ a transition to a saturation regime is

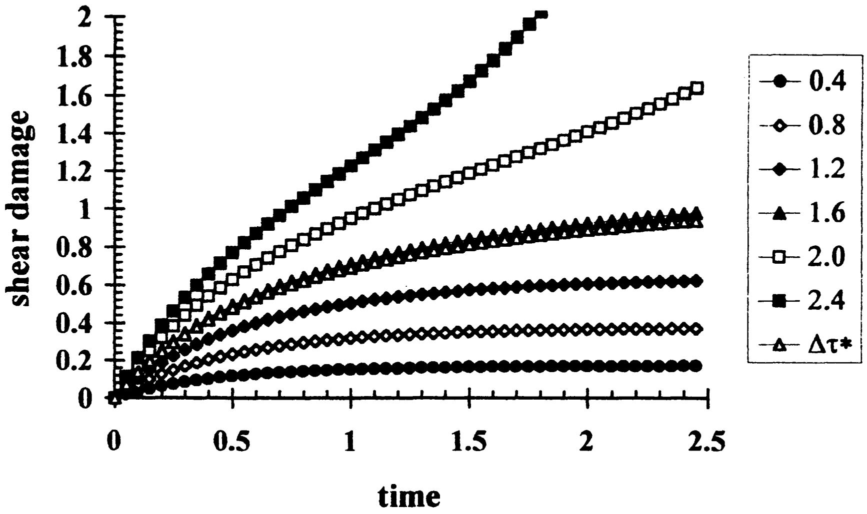

Figure 1. *Shear damage evolution for various levels of* $\Delta\tau$.

possible, while for sufficiently high $\Delta\tau$ the unrestricted growth of damage with time under the non-changing load is observed. The boundary between these two regimes is related to the transition value $\Delta\tau^* = (B^2/4AD)^{1/n}$ [note that it corresponds to Equation (2b)]. So, the interval of the shear stress' values can be divided into three sections: for $0 \leq \tau \leq \tau^*$ the shear damage initiation is impossible because of the resistance of friction and cohesion forces; for $\tau^* < \tau \leq \tau^* + \Delta\tau^*$ the value of shear damage is limited, the level of limit s° being determined by the τ value; for $\tau^* + \Delta\tau^* < \tau$ there is a transition to the infinite growth of damage. This last regime can be naturally associated with the damage localization and generation of macroscopic defect.

The expression for the value of $s^\circ = \lim_{t\to\infty} s$ has the following form for the first regime of shear damage accumulation:

$$s^\circ = \frac{2D(\Delta\tau)^n}{\tilde{\Delta} - B} \tag{3}$$

while for the second regime $s^\circ \to \infty$. The dependence of the saturation value of the II-mode damage on the acting shear stress is reflected in Figure 2. With increase in $\Delta\tau$ the nearly linear growth of the saturation value s° is accomplished by the singularity with $\Delta\tau \to \Delta\tau^*$

Account for the non-zero initial damage ($s = s_i$ for $t = 0$) transforms Equation (2) to the following form:

$$\tilde{s} = \begin{cases} \dfrac{\left(Bs_i - 2D(\Delta\tau)^n\right)\left[\exp\left(-t\tilde{\Delta}\right) - 1\right] + s_i\tilde{\Delta}\left[\exp\left(-t\tilde{\Delta}\right) + 1\right]}{(B + 2As_i)\left[\exp\left(-t\tilde{\Delta}\right) - 1\right] + \tilde{\Delta}\left[\exp\left(-t\tilde{\Delta}\right) + 1\right]}, & \Delta < 0 \\[2ex] \dfrac{4As_i + (B^2 + 2ABs_i)t}{2A[2 - (B + 2As_i)t]}, & \Delta = 0 \\[2ex] \dfrac{1}{2A}\dfrac{\Delta \tan\left(t\tilde{\Delta}/2\right) + (B + 2As_i)\left[\tilde{\Delta} + B\tan\left(t\tilde{\Delta}/2\right)\right] - B\tilde{\Delta}}{\tilde{\Delta} - (B + 2As_i)\tan\left(t\tilde{\Delta}/2\right)}, & \Delta > 0 \end{cases} \quad (4)$$

Here the indication $\tilde{s}$ is introduced in order to distinguish between this regime of damage accumulation and the initially non-damaged case. Calculations, based on Equation (4), have shown two different temporal asymptotes of the equation: (1) for $\tau \leq t^* + \Delta\tau^*$ the effect of initial damage level diminishes, i.e., $\tilde{s} \underset{t\to\infty}{\rightarrow} s^+$; the rate of such diminishment decreases with the increase in stress; (2) for $\tau > \tau^* + \Delta\tau$, in contrast, the difference $\tilde{s} - s$ after some period of nearly constant (or slightly enlarging) value sharply enters the zone of an abrupt growth. This zone of singularity begins earlier for greater levels of stress.

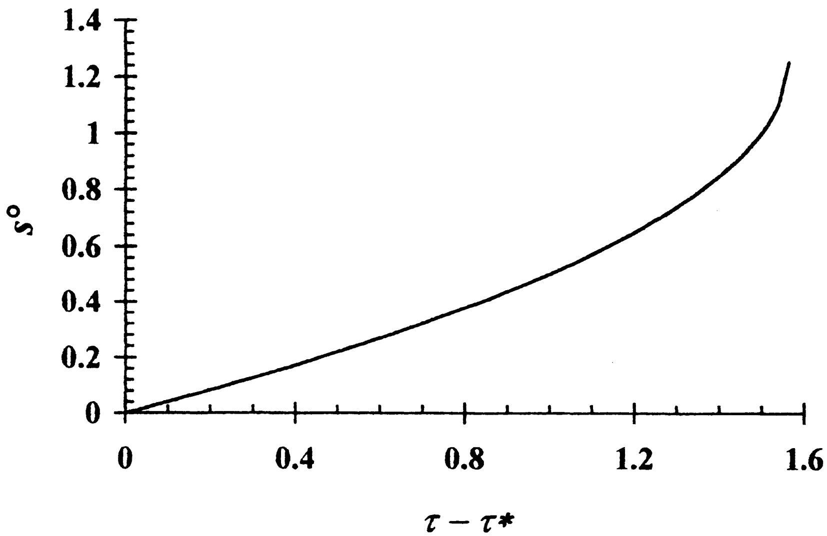

Figure 2. *Effect of stress on the limiting level of shear damage.*

DISCUSSION

The intervals of shear stress and respective regimes in II-mode damage accumulation reflect various types of material's response to the external load. The first threshold value—τ^*—corresponds to the minimal level of shear stress, which is necessary to initiate the II-mode damage development. The application of stresses, higher than this first threshold value but smaller than the second threshold—$\tau^* + \Delta\tau^*$—results in the finite response of the system to loading: no matter how long the load is applied, it cannot cause the overcoming of the saturation value $s^\circ(\Delta\tau)$. The transition over this second threshold sharply changes the temporal asymptotic behavior: from the saturation regime to the unrestricted increase in the level of damage even under the non-changing external load. In order to characterize temporal effects of the shear damage accumulation additional parameters should be introduced.

For the first type of behavior—restricted growth of s—the useful information is on the proximity of the system to the saturation regime. It is especially important for the cases of the dangerously high (pre-failure) levels of damage. So, lets introduce a characteristic time t_ε which corresponds to the moment when the damage enters the ε-zone of saturation—shear damage is equal to $s^\circ(1 - \varepsilon)$. Then, substituting Equation (3) into Equation (2a) and accounting for introduced determination for t_ε, one gets:

$$t_\varepsilon = \frac{1}{\tilde{\Delta}}\left(\ln \frac{B\left(B - \tilde{\Delta}\right) - 2AD(\Delta\tau)^n(2-\varepsilon)}{2AD(\Delta\tau)^n\varepsilon} + \ln \frac{B + \tilde{\Delta}}{B - \tilde{\Delta}}\right) \tag{5}$$

The respective dependence of t_ε on the level of shear stress is shown in Figure 3 for various values of ε. For the stresses far from the second threshold the introduced parameter changes insufficiently, much slower, than a respective level of saturation damage s° (compare Figure 3 with Figure 2). The infinite growth of the II-mode damage under the action of stresses, higher than the second threshold, excludes the possibility of introduction of the analogous characteristic for this type of material's behavior.

Starting from the first works on CDM, the problem of the macroscopic failure criterion, linked with the damage parameter, was intensively discussed (Krajcinovic, 1989; Lemaitre and Chaboche, 1990; Lemaitre, 1992; Kachanov, 1992). The initially unrealistic condition linked with the total loss of the bearing ability by the cross section (damage parameter being in this case the portion of the area occupied by the failed zones) was transformed to the criterion, based on the introduction of a norm—a limiting value—for the damage level (Kachanov, 1992; Lemaitre, 1992). Analogous ideas were used in micromechanical approaches [see reviews of Tvergaard (1995) and Pineau (1995) and references

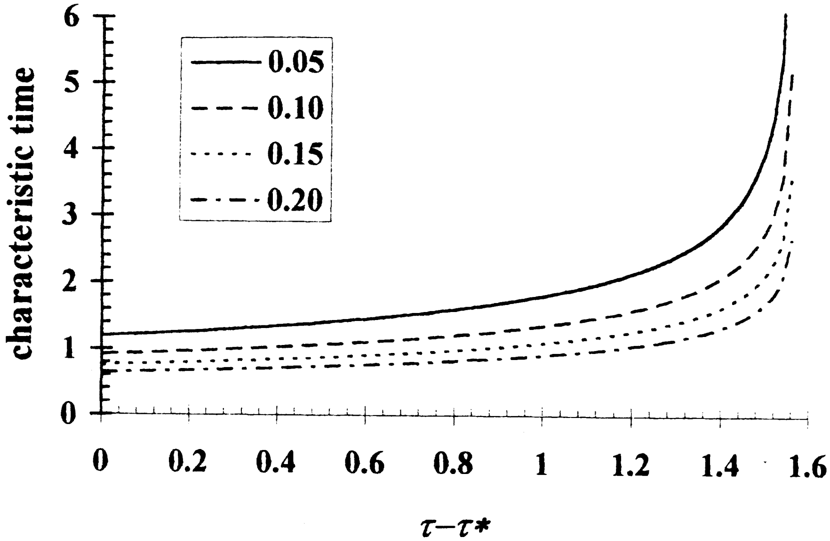

Figure 3. *Effect of stress on characteristic time of reaching the ε-zone of saturation regime for various levels of ε.*

there] linking the fracture occurrence with the critical value of volume fraction of cavities. In terms of such a description let introduce the critical value s^* for II-mode damage: the transition over this threshold causes an initiation of a macroscopic failure process (shear faults' formation, etc.). The value of s^* depends on the type of material and is its characteristic parameter. Then, with the obtained analytical linkage between the shear damage, stress and time, one can calculate the time-to-failure parameter $t°$:

$$t^\circ = \begin{cases} \dfrac{1}{\tilde{\Delta}}\left(\ln\left|\dfrac{B + 2As^* - \tilde{\Delta}}{B + 2As^* + \tilde{\Delta}}\right| + \ln\dfrac{B + \tilde{\Delta}}{B - \tilde{\Delta}}\right), & \Delta < 0 \\ \dfrac{4As^*}{B(B + 2As^*)}, & \Delta = 0 \\ \dfrac{2}{\tilde{\Delta}}\arctan\left(\dfrac{\tilde{\Delta}s^*}{2D(\Delta\tau)^n + Bs^*}\right), & \Delta > 0 \end{cases} \tag{6}$$

The next step is to study the time-to-fracture–stress relation. This dependence, obtained by the treatment of experimental data for traditional schemes of uni-axial loading, has a power-law form $t_f \propto \sigma^{-\alpha}$ with the scaling parameter α being approximately equal to 1. The results in our cases [described by Equation (6)] are presented in Figure 4 for $s^* = 0.1$. It is clearly seen that Equation (6) can be approximated by the power law $t^\circ \propto \Delta\tau^{-\beta}$. The scaling parameter β is only slightly higher than 1. Thus, exponent n, which was introduced in kinetic Equation (1), should have the value, differing not much from 1. Its exact value, of course, should be obtained by treatment of experimental data on damage accumulation under creep conditions. The deviation from the power-law behavior (from the straight line in double logarithmic coordinates) is linked with the existence of the limiting value $\Delta\tau^\circ$ for the cases when the damage threshold s^* is lower than the maximal possible [for given materials' parameters—coefficients of kinetic Equation (1)] saturation level of shear damage (determined as $\lim s|_{t\to\infty}^{\Delta\tau\to\Delta\tau^*}$). For shear stresses, less than

$$\Delta\tau^\circ = \left(-\frac{s^*}{D}(As^* + B)\right)^{1/n} \quad (7)$$

the transition to the macroscopic failure is impossible—it is stable damage accumulation regime, while respective parameter $t^\circ \xrightarrow[\Delta\tau\to\Delta\tau^\circ_+]{} \infty$. Analysis of the time-to-failure–stress dependence for the various levels of the damage threshold (Figure 5) proves its universal character: all respective graphs are straight lines in

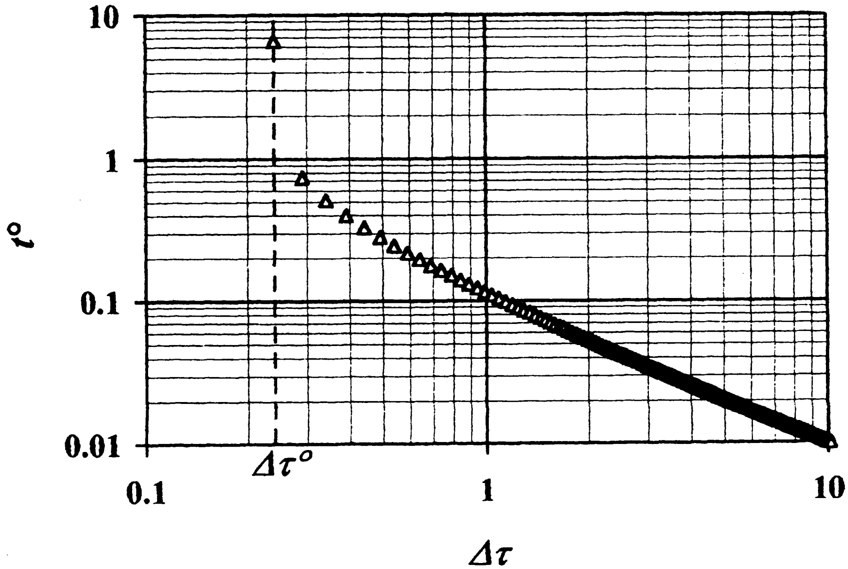

Figure 4. *Time-to-failure–stress dependence for* s* = *0.1.*

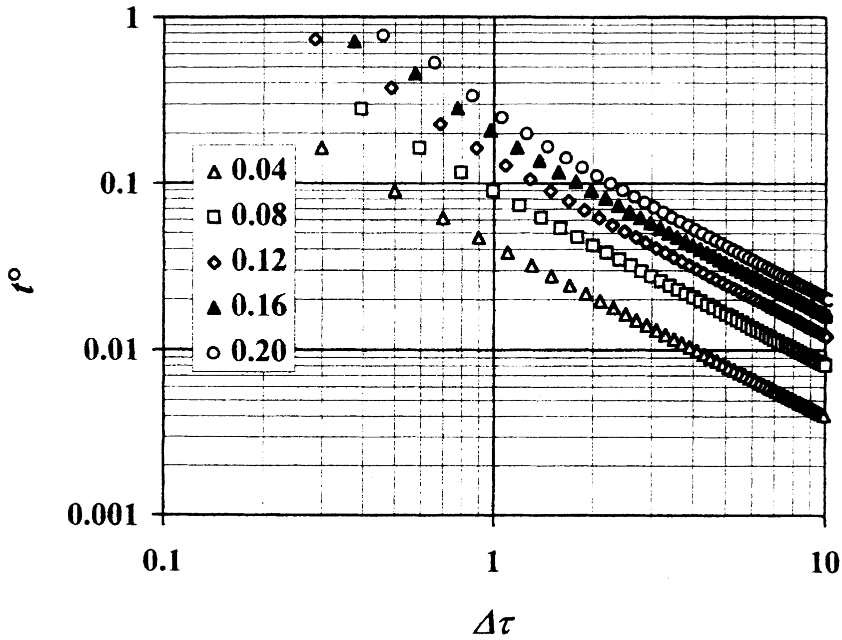

Figure 5. *Effect of the level of threshold value* s* *on time-to-failure–stress dependence.*

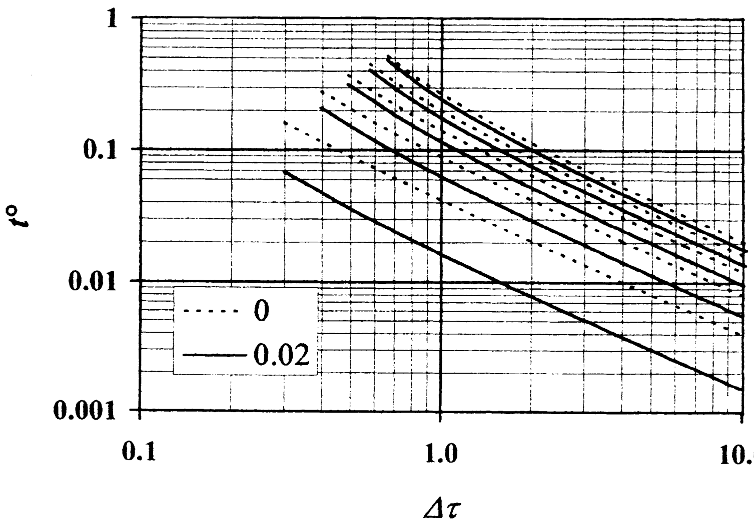

Figure 6. *Effect of initial damage* s_i *on time-to-failure. Values of* s* *for each pair of graphs—for initial non-damaged (dashed lines) and damaged (solid lines) cases—correspond to respective ones in Figure 5.*

double logarithmic coordinates and have the same slope. Additional corroboration of such a universality is the analysis of the case with initial non-zero damage. Figure 6 presents two sets of time-to-fracture–stress graphs: for conditions of initially non-damaged state and for cases with initial damage. It is obvious that presence of defects at the beginning of damage accumulation causes a drastic reduction of life-expectancy of specimen—up to two times. But the type of universality is still the same: all the graphs in both sets have the same slope (scaling parameter). Note that the analogous scaling character was obtained for the stochastic brittle failure under tensile loading for both the I-mode damage accumulation (Silberschmidt and Silberschmidt, 1994) and crack propagation (with account for crack-damage interaction) (Silberschmidt and Yakubovich, 1993; Silberschmidt, 1993a, 1995). Thus, these considerations can be used in the various applications for reliability analysis, etc.

CONCLUSION

The evaluation of a "pure" effect of shear damage is investigated in order to exclude the necessity to account for effects of interaction of various damage modes. The introduced damage parameter represents the macroscopic result (in terms of the respective portion of deformation) of the initiation and development of microscopic shifts. The proposed model for the II-mode damage accumulation demonstrates the differences in material's response to loading for three intervals of shear stresses. The first threshold value for stress divides the zones of the absence of damage (an external action is not sufficient for overcoming the friction and cohesion) and of its initiation/growth. The second threshold marks the transition (under the increase in the stress level) from stable damage accumulation (with saturation character for the long-time action) to an unstable one, characterized by the sharp change in temporal asymptote (singularity). The possibility of the unrestricted growth of II-mode damage under the non-changing stress on finite time intervals can be referred to as a macroscopic failure initiation.

The analytical solution of the kinetic equation for the shear damage accumulation provides a basis for the estimation of the temporal effects in stress-damage relation. The characteristic time for reaching the stationary regime—an approach to the saturation level of damage for respective stress intervals—is introduced. The scaling character of the time-to-failiure–stress dependence is shown to be the universal feature of the macroscopic fracture caused by the damage accumulation; the traditional CDM failure criterion limiting the damage level is used for such an analysis. It is proved that for the given material properties (including the threshold value of damage) there is a definite interval of load which cannot cause the transition to macroscopic failure, no matter how long the load is applied.

The non-zero initial damage can sufficiently influence the damage accumulation kinetics especially in a case of relatively high stresses (for small levels of external action, in contrast, this effect diminishes with time). Respective time-to-failure decreases sharply, but the scaling character remains the same.

The obtained analytical results for the deterministic conditions (neglecting the fluctuations linked with loading and/or with the damage evolution) can be utilized as a base for the study of the stochasticity effect on damage accumulation [using the stochastic differential equation for description of damage kinetics (Silberschmidt and Chaboche, 1994)].

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