Dynamics of Multibody Systems Containing Dependent Unilateral Constraints with Friction

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Abstract: In couplings of machines and mechanisms, backlash and friction phenomena are always occurring. Whether stick-slip phenomena take place depends on the structure of such couplings. These processes can be modeled as multibody systems with a time-varying topology. Making use of Lagrange multiplier methods with a mathematical formulation of the contact problem is very efficient for large systems with many constraints. The differential-algebraic equations of a system are transformed into a resolvable mathematical form by means of contact laws. In the following, rigid multibody systems with dependent constraints and planar friction will be considered. For the evaluation of such problems, an iterative algorithm is presented. This method is based on transformations of the kinematic secondary conditions in the form of inequalities to equations. In mathematical sense, these transformations are projections of the constraint forces on convex sets. Ultimately, we have a solvable nonlinear system of equations consisting of the differential equations of motion, the constraint equations and the projections of the constraint forces.

Key Words: Multibody systems, unilateral constraints, Coulomb friction, stick-slip phenomena, convex analysis, Lagrange multipliers

1. INTRODUCTION

The course of motion in multibody systems with three-dimensional unilateral constraints and dry friction is influenced by kinematic (geometric) restrictions. The state of every constraint is variable. During the phase of contact, a unilateral normal constraint is active while there is acting a passive constraint force. On the other hand, this constraint is passive during separation. Separation means that no constraint force acts in the normal and tangential direction.

The tangential constraint belonging to an active normal constraint can be active or passive if rigid body friction is considered in the contact area. Strictly speaking, the tangential constraint in the three-dimensional contact case consists of two simultaneously arising constraints, one for each direction of the tangential plane as long as the direction of friction is unknown. During the phase of stiction, a passive stiction force acts in the active tangential constraint. On the other hand, the constraint is passive during the phase of sliding. In this case, the sliding force is an active tangential force.

During a change in the structure of a constrained system, each activated constraint

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results in the loss of one degree of freedom, whereas each cancelled constraint increases the number of degrees of freedom by one. With these properties, the number of degrees of freedom is not constant during the evolution of the mechanical system. Therefore, the description of the system's dynamics is structure-varying. This is shown in the time-varying structure of the kinematic and kinetic basic equations, which are used for the mathematical description of multibody systems.

The numerical procedures for the solution of discrete or discretized contact problems can be divided into methods with Lagrange multipliers and methods with penalty functions. This division has its origin in the field of frictionless contact problems. Here, stable equilibrium displacement states can be found by minimizing the potential energy of a mechanical system, and the inequalities of unilateral constraints can be handled by the two mentioned methods.

Making use of Lagrange multipliers, the additional variables of the contact problem have the physical meaning of constraint forces. The inequality conditions are exactly satisfied by this method. When using the penalty method, a large value of the potential energy has the meaning of violating the constraint conditions, which will only be approximately satisfied.

When using Lagrange multipliers, two basically different procedures are applicable. In the first procedure, the solution of the contact problem is found by lining up the piecewise steady solutions at points of discontinuity (switch points) with appropriate transition conditions (contact laws). Usually the solution is not well-defined. At each switch point, the physically correct solution must be selected from a large number of possible states of the constraints. The combinatorial tests are computationally intense. One unilateral constraint with friction can have three different states: separation (no contact), contact without tangential relative velocity (stiction), and contact with tangential relative velocity (sliding). For a mechanical system with n unilateral constraints, the number of possible states is 3^n. We see that the number of combinatorial tests at each switch point is increasing rapidly with n. For mechanical systems with several constraints, the trial and error method is not feasible to determine the dynamics by checking the states of all constraints. In the second procedure, the contact problem is formulated as a resolvable mathematical problem, such as complementarity problems, quadratic programs, nonlinear programming problems, or variational inequalities.

In the literature, we find several algorithms for these mathematical problems often with proven convergence properties. For quasi-static contact problems, Klarbring (1986) gives a formulation of a parametric linear complementarity problem if the contact surface is constant during the loading history. In Kaneko (1978, 1980), we find an algorithm and a proof of the existence of a unique solution. Klarbring and Björkman (1988) have extended this algorithm for contact problems with varying contact surfaces. In Holmberg (1990) and Klarbring and Björkman (1988) an efficient algorithm for the solution of a quasi-static, linear elastic contact problem with planar friction is described. For this purpose, the so-called friction cone is linearized piecewise and a linear complementarity problem is solved.

Panagiotopoulos (1993) contains the mathematical and mechanical foundations for the formulation of hemivariational inequalities. The numerical treatment of such inequalities is shown in many not trivially solvable examples.

The unsteady dynamics of mechanical systems being composed of many rigid bodies and unilateral constraints with dry friction is the subject of Jean and Moreau (1992). The contact laws and differential equations of motion are formulated on the velocity level or impulse level, respectively. When multiple impacts with friction occur, the impact impulses act during finite time intervals, so that the dynamics are compatible with the actual states of all constraints. Because of the finite time intervals, the solution is found by means of time discretization and not by means of numerical integration.

Brach (1989, 1991) considers single collisions and formulates the impact equations on the impulse level by using Newton's law. Conditions for sticking or reversed sliding after the impact are derived from the kinetic energy by demanding an always dissipative impact law. This method is restricted for single impacts, because for multiple collisions such conditions cannot be derived from one scalar equation without additional assumptions. Haug, Wu, and Yang (1986) deal with multibody systems with friction contacts and collisions. The contact forces and active constraints are taken into consideration by Lagrange multipliers and secondary conditions, respectively. Keller (1986) classifies possible impact configurations and analytically solves the impact problem with friction for a single collision on the force level by using Poisson's and Coulomb's laws. In Smith (1991), an alternative contact law is used for single collisions. A more realistic estimate of the rebound is given, when the directions of the tangential relative velocity before and after the contact differ. For collisions of solid elastic spheres, for which a fairly detailed study of the mechanics of local deformation and sliding has been made, the proposed assumption appears to lead to an improvement in the prediction. Pereira and Nikravesh (1993) present a methodology in computational dynamics for the analysis of mechanical systems that undergo intermittent motion. A canonical form of the equations of motion is derived with a minimal set of coordinates; and these equations are used in a procedure for balancing the momenta of the system over the period of impact, calculating the jump in the body momenta, velocity discontinuities, and rebounds. The effect of dry friction is discussed, and a contact law is proposed. The application of this methodology is illustrated with the study of impacts of open loop and closed loop examples.

During the last eleven years, at the *Lehrstuhl B für Mechanik*, the theory for the treatment of multibody systems containing unilateral contacts was continuously advanced and applied to practical problems. At the beginning, Pfeiffer (1984) treats multiple frictionless impacts on the impulse level using Newton's law and assumes that impulses are transferred at each of the contacts. This replaces the unilateral character of the constraints in the normal direction by a bilateral formulation and leads to a set of linear equations for the relative velocities after the impact. Wapenhans (1989) and Braun (1989) investigate the dynamics of a landing aeroplane. They develop a nonlinear model to take into account the external landing impact and the impacts and stick-slip phenomena within the shock absorbers of the landing gear. The constraint forces are considered by means of constraint matrices in the equations of motion. Hajek (1990) investigates the reduction of blade vibration amplitudes in airborne gas turbines with special damper devices in which stick-slip phenomena are occurring. The simulation is based on combinatorial tests at the switch points, and each possible system state is described with different sets of minimal coordinates. A percussion drilling machine is examined in Glocker and Pfeiffer (1992). To avoid a description of each possible system state using different sets of minimal coordinates, the constrained motion is taken into account by algebraic relations. In Seyfferth (1993), Glocker and Pfeiffer (1994), as well as Seyfferth and Pfeiffer (1994), the derivation and solution of linear complementarity problems is given for dynamical two-dimensional contact problems with dependent unilateral constraints and impacts with friction. In addition, the practicability of complementarity methods is demonstrated by relevant practical applications, such as assembly processes with manipulators, turbine blade dampers and electropneumatic drilling machines. Glocker (1995) develops an impact model for two-dimensional contact situations with multiple impacts, which contains the main physical effects of a compliance element in the normal direction and a series of a compliance and a Coulomb friction elements in the tangential direction. The theory is applied to some basic examples that demonstrate the difference between Newton's and Poisson's hypotheses. In this work, we also find minimization principles that are equivalent to complementary conditions and theorems for the existence and uniqueness of the solution of linear complementarity problems.

Klarbring (1994) shows that the trial and error method yields no satisfactory results for three-dimensional contact problems with Coulomb friction. Even though a quasi-minimization principle is given for such problems by Kalker (1971) and Klarbring (1987), it is impracticable for numerical calculations because the objective function is no potential or complementary energy in a mechanical sense. Duvaut and Lions (1976) present basic mechanical energy principles by making simplifying assumptions. Based on these energy principles, Panagiotopoulos (1975) contains discrete minimization problems and a two-step iterative algorithm where a normal contact problem and a simplified friction problem are alternately solved until convergence occurs. Another method without using the quasi-minimization principle is seen for quasi-static, linear elastic contact problems in Alart and Curnier (1991), Klarbring (1992), as well as Simo and Laursen (1992). The theory of the so-called augmented Lagrange multiplier method is described in the following and applied to dynamical contact problems without impacts. The theory is verified by means of a basic example. In the appendix, the most important definitions and theorems of convex analysis are given, which are necessary for the derivation of the theory.

2. CONTACT KINEMATICS

A set of generalized coordinates

$$q(t) = \begin{pmatrix} q_1(t) \\ \vdots \\ q_f(t) \end{pmatrix} \in \mathbb{R}^f$$

is used for the mathematical description of the dynamics of a bilaterally constrained system of f degrees of freedom. To take into account additional unilateral constraints like contact or friction constraints, we have to derive some kinematic contact conditions. The possible motion of each body in a multibody system, which is compatible to the kinematic conditions, is restricted by conditions for normal distances, relative velocities, and relative accelerations in the potential contact points (active unilateral constraints).

We now consider a system with n_A contact points and introduce the four index sets

$$\begin{aligned} I_A &= \{1, 2, ..., n_A\} && \text{with } n_A \text{ elements} \\ I_C &= \{i \in I_A : g_{Ni} = 0\} && \text{with } n_C \text{ elements} \\ I_N &= \{i \in I_C : \dot{g}_{Ni} = 0\} && \text{with } n_N \text{ elements} \\ I_T &= \{i \in I_N : |\dot{\boldsymbol{g}}_{Ti}| = 0\} && \text{with } n_T \text{ elements,} \end{aligned} \tag{1}$$

which describe the kinematic state of each contact point. The set I_A consists of the n_A indices of all contact points. The elements of the set I_C are the n_C indices of the unilateral constraints with vanishing normal distance, but arbitrary relative velocity in the normal direction. In the index set I_N are the n_N indices of the potentially active normal constraints, which fulfill the necessary conditions for continuous contact (vanishing normal distance and no relative velocity in the normal direction). The n_T elements of the set I_T are the indices of the potentially active tangential constraints. The corresponding normal constraints are closed, and the relative velocities in the tangential direction are zero. The numbers of elements of the index sets I_C, I_N, and I_T are not constant because there are variable states of constraints due to separation and stick-slip phenomena.

The normal distances and the relative velocities in the tangential direction are determined by means of relative kinematics. In general, the normal distances of all contact points

$$g_{Ni} = g_{Ni}(\boldsymbol{q}, t) \in \mathbb{R}^1 \;\; ; \;\; i = 1, ..., n_A \tag{2}$$

are functions of the generalized coordinate $\boldsymbol{q}$ and the time variable t. The normal distance g_{Ni} shows positive values for separation and negative values for penetration. Therefore, a changing sign of g_{Ni} indicates a transition from separation to contact. The relative velocities in the tangential direction

$$\dot{\boldsymbol{g}}_{Ti} = \dot{\boldsymbol{g}}_{Ti}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \in \mathbb{R}^2 \;\; ; \;\; i = 1, ..., n_A \tag{3}$$

are additionally dependent on the generalized velocities $\dot{\boldsymbol{q}}$. The relative velocities in the normal direction

$$\dot{g}_{Ni} = \dot{g}_{Ni}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = \frac{\partial g_{Ni}}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}} + \frac{\partial g_{Ni}}{\partial t} \tag{4}$$

are the first derivatives with respect to time of equation (2). The relative velocities in equations (4) and (3) can be written in the form

$$\dot{g}_{Ni} = \boldsymbol{w}_{Ni}^T \dot{\boldsymbol{q}} + \tilde{w}_{Ni} \quad ; \quad \dot{\boldsymbol{g}}_{Ti} = \boldsymbol{W}_{Ti}^T \dot{\boldsymbol{q}} + \tilde{\boldsymbol{w}}_{Ti} \tag{5}$$

with

$$\boldsymbol{w}_{Ni} = \left(\frac{\partial g_{Ni}}{\partial \boldsymbol{q}}\right)^T \quad ; \quad \boldsymbol{W}_{Ti} = \left(\frac{\partial \dot{\boldsymbol{g}}_{Ti}}{\partial \dot{\boldsymbol{q}}}\right)^T \quad ; \quad \tilde{w}_{Ni} = \frac{\partial g_{Ni}}{\partial t} \tag{6}$$

A negative value of the relative velocity $\dot{g}_{Ni}$ corresponds to an approaching process of the bodies. In the case of continual normal contact with $g_{Ni} = \dot{g}_{Ni} = 0$, we can use the relative velocity $\dot{g}_{Ti}$ to indicate a transition from sliding ($|\dot{g}_{Ti}| \neq 0$) to stiction or rolling ($|\dot{g}_{Ti}| = 0$). The relative accelerations of the contact points

$$\ddot{g}_{Ni} = \boldsymbol{w}_{Ni}^T \ddot{\boldsymbol{q}} + \overline{w}_{Ni} \quad ; \quad \ddot{\boldsymbol{g}}_{Ti} = \boldsymbol{W}_{Ti}^T \ddot{\boldsymbol{q}} + \overline{\boldsymbol{w}}_{Ti} \tag{7}$$

with

$$\overline{w}_{Ni} = \dot{\boldsymbol{w}}_{Ni}^T \dot{\boldsymbol{q}} + \dot{\tilde{w}}_{Ni} \quad ; \quad \overline{\boldsymbol{w}}_{Ti} = \dot{\boldsymbol{W}}_{Ti}^T \dot{\boldsymbol{q}} + \dot{\tilde{\boldsymbol{w}}}_{Ti} \tag{8}$$

are determined by differentiation of equation (5) with respect to time. Continual contact demands $g_{Ni} = \dot{g}_{Ni} = \ddot{g}_{Ni} = 0$, while separation is only possible if the relative acceleration $\ddot{g}_{Ni} > 0$. Transition from stiction to sliding occurs for a closed contact if the amount of the relative acceleration $|\ddot{\boldsymbol{g}}_{Ti}| > 0$.

3. DYNAMICS OF RIGID BODIES WITH SUPERIMPOSED UNILATERAL CONSTRAINTS

The kinetic basic equations (equations of motion) specify the connection between the forces acting on the system and the changes in the motion. Describing the motion of a structure-varying system starts from the differential equations of motion of the bilaterally constrained system.

$$\boldsymbol{M}(\boldsymbol{q}, t)\ddot{\boldsymbol{q}}(t) - \boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = \boldsymbol{0} \quad ; \quad \ddot{\boldsymbol{q}}(t) \in \mathbb{R}^f \tag{9}$$

with f degrees of freedom. The equations of motion can always be written in this form. The mass matrix $\boldsymbol{M}(\boldsymbol{q}, t) \in \mathbb{R}^{f,f}$ is symmetric and positive definite. The vector $\boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \in \mathbb{R}^f$ contains the gyroscopical accelerations together with the sum of all active forces and moments.

In a system with additional unilateral constraints, the number of degrees of freedom is variable. To avoid difficulties with many different sets of minimal coordinates, we take one set of generalized coordinates and consider the active unilateral constraints as secondary conditions. The occurring contact forces are taken into account in the equations of motion. Now we consider the unilateral constraints, which are closed contacts (elements of the index set I_N). The contact forces are included in equation (9) using the Lagrange multiplier method.

The constraint vectors $\boldsymbol{w}_{Ni}$ and the constraint matrices $\boldsymbol{W}_{Ti}$ in equation (6) are arranged as columns in the constraint matrices

$$\begin{aligned} \boldsymbol{W}_N &= [\ldots, \boldsymbol{w}_{Ni}, \ldots] \in \mathbb{R}^{f,n_N} \quad ; i \in I_N \\ \boldsymbol{W}_T &= [\ldots, \boldsymbol{W}_{Ti}, \ldots] \in \mathbb{R}^{f,2n_T} \quad ; i \in I_T \end{aligned} \tag{10}$$

for all active constraints. The constraint matrices are transformation matrices from the space of constraints to the configuration space. The transposed matrices are used for the transition from the configuration space to the space of constraints.

The contact forces have the amounts λ_{Ni} (normal forces) and the components λ_{Ti1} and λ_{Ti2} (tangential forces). These elements are combined in the vectors of constraint forces

$$\boldsymbol{\lambda}_N(t) = \begin{pmatrix} \vdots \\ \lambda_{Ni}(t) \\ \vdots \end{pmatrix} \in \mathbb{R}^{n_N} \quad ; \; i \in I_N$$

$$\boldsymbol{\lambda}_T(t) = \begin{pmatrix} \vdots \\ \boldsymbol{\lambda}_{Ti}(t) \\ \vdots \end{pmatrix} \in \mathbb{R}^{2n_T} \quad ; \; i \in I_T$$

with $\boldsymbol{\lambda}_{Ti}(t) = [\lambda_{Ti1}(t), \lambda_{Ti2}(t)]^T$. In general, the contact forces are time-varying quantities. By the constraint vectors and matrices in equation (6), the contact forces can be expressed in the configuration space. These forces are then added to equation (9),

$$\boldsymbol{M}\ddot{\boldsymbol{q}} - \boldsymbol{h} - \sum_{i \in I_N} (\boldsymbol{w}_{Ni}\lambda_{Ni} + \boldsymbol{W}_{Ti}\boldsymbol{\lambda}_{Ti}) = 0 \,. \tag{11}$$

The contact forces $\boldsymbol{\lambda}_{Ti}$, in equation (11) can be passive forces of sticking contacts or active forces of sliding contacts. We express the tangential forces of the $n_N - n_T$ sliding contacts by the corresponding normal forces using Coulomb's friction law by

$$\boldsymbol{\lambda}_{Ti} = -\mu_i \left(|\dot{\boldsymbol{g}}_{Ti}|\right) \frac{\dot{\boldsymbol{g}}_{Ti}}{|\dot{\boldsymbol{g}}_{Ti}|} \lambda_{Ni} \;\; ; i \in I_N \backslash I_T \tag{12}$$

where the coefficients $\mu_i \left(|\dot{\boldsymbol{g}}_{Ti}|\right)$ of sliding friction may depend on time. The negative sign relates to the opposite direction of relative velocity and friction force. The sliding forces of equation (12) in the configuration space are then

$$\boldsymbol{W}_{Ti}\boldsymbol{\lambda}_{Ti} = -\mu_i \left(|\dot{\boldsymbol{g}}_{Ti}|\right) \boldsymbol{W}_{Ti} \frac{\dot{\boldsymbol{g}}_{Ti}}{|\dot{\boldsymbol{g}}_{Ti}|} \lambda_{Ni} \;\; ; \;\; i \in I_N \backslash I_T \tag{13}$$

A substitution of these forces into equation (11) yields the equations of motion

$$\boldsymbol{M}(\boldsymbol{q}, t)\ddot{\boldsymbol{q}}(t) - \boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) - [\boldsymbol{W}_N + \boldsymbol{H}_R, \boldsymbol{W}_T] \begin{pmatrix} \boldsymbol{\lambda}_N(t) \\ \boldsymbol{\lambda}_T(t) \end{pmatrix} = \boldsymbol{0} \,, \tag{14}$$

with the additional contact forces as Lagrange multipliers. The matrices $\boldsymbol{W}_N$ and $\boldsymbol{W}_T$ are the constraint matrices of equation (10). The matrix $\boldsymbol{H}_R \in \mathbb{R}^{f, n_N}$ of the sliding contacts has the same dimension as the constraint matrix $\boldsymbol{W}_N$. For $n_T \leq n_N$, $\boldsymbol{H}_R$ consists of the $n_N - n_T$ columns

$$-\frac{\mu_i}{|\dot{\boldsymbol{g}}_{Ti}|} \boldsymbol{W}_{Ti} \, \dot{\boldsymbol{g}}_{Ti} \;\; ; \;\; i \in I_N \backslash I_T \,,$$

whereas the other n_T columns contain only zero-elements.

The relative accelerations of the active normal and tangential constraints in equation (7)

$$\begin{aligned} \ddot{g}_{Ni} &= \boldsymbol{w}_{Ni}^T \ddot{\boldsymbol{q}} + \overline{w}_{Ni} \ ; \ i \in I_N \\ \ddot{\boldsymbol{g}}_{Ti} &= \boldsymbol{W}_{Ti}^T \ddot{\boldsymbol{q}} + \overline{\boldsymbol{w}}_{Ti} \ ; \ i \in I_T \end{aligned} \tag{15}$$

can be combined by means of the constraint matrices (10) in the matrix notation. Together with equation (14), we get the system of equations

$$\begin{aligned} \boldsymbol{M}\ddot{\boldsymbol{q}} - \boldsymbol{h} - [\boldsymbol{W}_N + \boldsymbol{H}_R, \boldsymbol{W}_T] \begin{pmatrix} \boldsymbol{\lambda}_N \\ \boldsymbol{\lambda}_T \end{pmatrix} = \boldsymbol{0} \quad &\in \quad \mathbb{R}^f \\ \ddot{\boldsymbol{g}}_N = \boldsymbol{W}_N^T \ddot{\boldsymbol{q}} + \overline{\boldsymbol{w}}_N \quad &\in \quad \mathbb{R}^{n_N} \\ \ddot{\boldsymbol{g}}_T = \boldsymbol{W}_T^T \ddot{\boldsymbol{q}} + \overline{\boldsymbol{w}}_T \quad &\in \quad \mathbb{R}^{n_N} . \end{aligned} \tag{16}$$

The unknown quantities are the generalized accelerations $\ddot{\boldsymbol{q}} \in \mathbb{R}^f$, the contact forces in the normal direction $\boldsymbol{\lambda}_N \in \mathbb{R}^{n_N}$ and in the tangential direction $\boldsymbol{\lambda}_T \in \mathbb{R}^{2n_T}$, as well as the corresponding relative accelerations $\ddot{\boldsymbol{g}}_N \in \mathbb{R}^{n_N}$ and $\ddot{\boldsymbol{g}}_T \in \mathbb{R}^{2n_T}$. For the determination of the $f + 2(n_N + 2n_T)$ quantities, we now have up to $f + n_N + 2n_T$ equations. In the next section, the system of equations (16) will be completed by including the missing $n_N + 2n_T$ contact laws.

In general, the kinematic equations are dependent on each other if there is more than one contact point per rigid body. The situation is shown in linearly dependent columns of the constraint matrices $\boldsymbol{W}_N, \boldsymbol{W}_T$ in equation (10). Such constraints are designated as dependent constraints.

4. DETACHMENT AND STICK-SLIP TRANSITION

The conditions of transitions from contact to separation and sticking to sliding are formulated for coupled multibody systems. The main difficulty results from instantaneous changes in the contact forces at transitions from sliding to sticking or reversed sliding. Summarizing these considerations, we note that a vanishing tangential relative velocity does not necessarily lead to sticking, and each new active friction constraint generally produces an unsteady changing contact force. We look at systems with more than one contact point. If the contacts are coupled kinematically, the contact forces influence each other. Thus each new active friction constraint generally affects all of the other active contact constraints and produces jumps in the contact forces. Due to these jumps, induced transitions in the states of contacts may occur, which are either transitions to sliding or to take off. Therefore, we must answer the question of how many and which constraints change their state of contact, which are influenced by the new active tangential constraints. In this section, the conditions of transition are stated, which allows the evaluation of the transition problem avoiding the combinatorial problem of testing all possible contact state combinations for the solution without contradiction to the contact laws. These $n_N + 2n_T$

conditions are valid during continual contact and sticking, as well as for the transitions to sliding or separation.

4.1. Contact Law for Normal Constraints

Each of the closed contact constraints $i \in I_N$ is characterized by a vanishing distance $g_{Ni} = 0$ and normal relative velocity $\dot{g}_{Ni} = 0$. Under the assumption of impenetrability ($g_{Ni} \geq 0$), only two situations may occur: contact is either maintained or a transition to separation takes place. In the first case, we know that the normal relative acceleration is vanishing and the normal contact forces must act with a compressive magnitude (due to the unilaterality of the normal constraint),

$$\ddot{g}_{Ni} = 0 \wedge \lambda_{Ni} \geq 0 \; ; \; i \in I_N. \tag{17}$$

The second case must describe the separation of the bodies. Separation is only achieved by nonnegative values of the normal relative acceleration and vanishing normal forces,

$$\ddot{g}_{Ni} \geq 0 \wedge \lambda_{Ni} = 0 \; ; \; i \in I_N. \tag{18}$$

With both cases in equations (17) and (18), we see that the normal contact law shows a complementary behavior. The product of the quantities λ_{Ni} and $\ddot{g}_{Ni}$ is always equal to zero,

$$\ddot{g}_{Ni}\lambda_{Ni} = 0 \; ; \; i \in I_N. \tag{19}$$

Thus the normal contact problem with all potential normal constraints is unambiguously determined by the n_N complementary conditions

$$\ddot{\boldsymbol{g}}_N \geq \boldsymbol{0} \; ; \; \boldsymbol{\lambda}_N \geq \boldsymbol{0} \; ; \; \ddot{\boldsymbol{g}}_N^T \boldsymbol{\lambda}_N = 0. \tag{20}$$

The variational inequality

$$-\ddot{\boldsymbol{g}}_N^T(\boldsymbol{\lambda}_N^* - \boldsymbol{\lambda}_N) \leq 0 \; ; \; \boldsymbol{\lambda}_N \in C_N \; ; \; \forall \boldsymbol{\lambda}_N^* \in C_N \tag{21}$$

is equivalent to the complementary conditions (20). Proof of this relation is given in Glocker (1995). The convex set

$$C_N = \{\boldsymbol{\lambda}_N^* \; : \; \boldsymbol{\lambda}_N^* \geq \boldsymbol{0}\} \tag{22}$$

contains all admissible contact forces λ_{Ni}^* in the normal direction. The meaning of the vector inequality in (22) is explained by theorem (47) in Appendix A. The definition (48) of convex sets is given in Appendix B.

4.2. Coulomb's Friction Law

The friction law of Coulomb states that the sliding friction force is proportional to the normal force of a contact. The amount of the static friction force is less than or equal to the maximum static friction force, which is also proportional to the normal force. For sliding friction, the friction force has the opposite direction of the relative velocity of the friction contact. For most material pairs of practical interest, the coefficients of static and

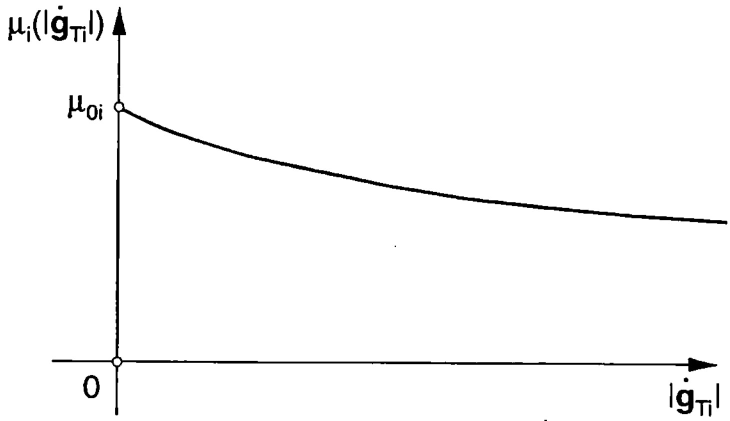

Figure 1. Friction characteristic.

sliding friction are different ($\mu_0 > \mu$). In general, the coefficient of sliding friction is a function of the relative velocity. In the following, we consider friction characteristics, as shown in Figure 1 for multiple contact problems with the property

$$\lim_{v_i \to 0} \mu_i(v_i) = \mu_{0i} \; ; \; v_i = |\dot{g}_{Ti}| \,. \tag{23}$$

With this property, Coulomb's friction law distinguishes between the two cases

$$\left.\begin{array}{lll} \text{stiction}: & |\boldsymbol{\lambda}_{Ti}| < \mu_{0i}\lambda_{Ni} \Rightarrow & |\dot{g}_{Ti}| = 0 \\ \text{sliding}: & |\boldsymbol{\lambda}_{Ti}| = \mu_i\lambda_{Ni} \Rightarrow & |\dot{g}_{Ti}| > 0 \end{array}\right\} i \in I_N. \tag{24}$$

For the frictional contact problem, we need a representation of the friction law (24) on the acceleration level to determine the tangential relative accelerations $\ddot{g}_T$ in equation (16). The friction forces of the sliding contacts are already taken into account in the first equation of (16) by $\boldsymbol{H}_R\boldsymbol{\lambda}_N$, so we have only to transform Coulomb's friction law on acceleration level for the n_T potential sticking contacts. This is possible because the tangential relative velocity and acceleration have the same direction for a transition from $|\dot{g}_{Ti}| = 0$ to $|\dot{g}_{Ti}| > 0$. Thus we get from (24) the cases

$$\left.\begin{array}{lll} \text{stiction}: & |\boldsymbol{\lambda}_{Ti}| < \mu_{0i}\lambda_{Ni} \Rightarrow & \ddot{\boldsymbol{g}}_{Ti} = \boldsymbol{0} \\ \text{sliding}: & |\boldsymbol{\lambda}_{Ti}| = \mu_{0i}\lambda_{Ni} \Rightarrow & \ddot{\boldsymbol{g}}_{Ti} = -\rho\boldsymbol{\lambda}_{Ti} \; ; \; \rho \geq 0 \end{array}\right\} i \in I_T \,. \tag{25}$$

In the first case, the state of stiction continues. In the second case, we have a transition from stiction to sliding, while the friction force comes up to its maximum value. The inequality

$$\ddot{\boldsymbol{g}}_{Ti}^T \boldsymbol{\lambda}_{Ti} \leq 0 \; ; \; i \in I_T \tag{26}$$

follows from equation (25) and describes the dissipative behavior of Coulomb friction. The variational inequality

$$-\ddot{\boldsymbol{g}}_{Ti}^T(\boldsymbol{\lambda}_{Ti}^* - \boldsymbol{\lambda}_{Ti}) \leq 0 \; ; \; \boldsymbol{\lambda}_{Ti} \in C_{Ti} \; ; \; \forall \boldsymbol{\lambda}_{Ti}^* \in C_{Ti} \; ; \; i \in I_T \tag{27}$$

is equivalent to equation (25). Proof of this relation is given in Glocker (1995). The convex set (see also definition [48] in Appendix B)

$$C_{Ti}(\lambda_{Ni}) = \{\boldsymbol{\lambda}_{Ti}^* \; : \; |\boldsymbol{\lambda}_{Ti}^*| \leq \mu_{0i}\lambda_{Ni}\} \tag{28}$$

contains all friction forces $\boldsymbol{\lambda}_{Ti}^*$, which fulfill Coulomb's friction law.

5. CONTACT PROBLEM IN MATHEMATICAL FORM

The mathematical system consisting of the equations of motion and the constraint equations in equation (16), as well as the variational inequalities (21) and (27)

$$\begin{gathered}
\boldsymbol{M}\ddot{\boldsymbol{q}} - [\boldsymbol{W}_N + \boldsymbol{H}_R, \boldsymbol{W}_T]\begin{pmatrix}\boldsymbol{\lambda}_N \\ \boldsymbol{\lambda}_T\end{pmatrix} - \boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = \boldsymbol{0} \\
\begin{pmatrix}\ddot{\boldsymbol{g}}_N \\ \ddot{\boldsymbol{g}}_T\end{pmatrix} = \begin{pmatrix}\boldsymbol{W}_N^T \\ \boldsymbol{W}_T^T\end{pmatrix}\ddot{\boldsymbol{q}} + \begin{pmatrix}\overline{\boldsymbol{w}}_N \\ \overline{\boldsymbol{w}}_T\end{pmatrix} \\
-\ddot{\boldsymbol{g}}_N^T(\boldsymbol{\lambda}_N^* - \boldsymbol{\lambda}_N) \leq 0 \; ; \; \boldsymbol{\lambda}_N \in C_N \; ; \; \forall \boldsymbol{\lambda}_N^* \in C_N \\
-\ddot{\boldsymbol{g}}_{Ti}^T(\boldsymbol{\lambda}_{Ti}^* - \boldsymbol{\lambda}_{Ti}) \leq 0 \; ; \; \boldsymbol{\lambda}_{Ti} \in C_{Ti}(\lambda_{Ni}) \; ; \; \forall \boldsymbol{\lambda}_{Ti}^* \in C_{Ti}(\lambda_{Ni}) \; ; \; i \in I_T,
\end{gathered} \tag{29}$$

with the convex sets (22) and (28) completely describes the contact problem. But the system (29) is not solvable in this form. Therefore, the variational inequalities are transformed into equalities. From this, we get a nonlinear system of equations that can be solved with iterative standard algorithms. This method is shown in Klarbring (1992).

5.1. Transformation for Normal Constraints

The variational inequality (21)

$$-\ddot{\boldsymbol{g}}_N^T(\boldsymbol{\lambda}_N^* - \boldsymbol{\lambda}_N) \leq 0 \; ; \; \boldsymbol{\lambda}_N \in C_N \; ; \; \forall \boldsymbol{\lambda}_N^* \in C_N$$

with the convex set of admissible normal contact forces $C_N = \{\boldsymbol{\lambda}_N^* \; : \; \boldsymbol{\lambda}_N^* \geq \boldsymbol{0}\}$ can be written in the form

$$[\boldsymbol{\lambda}_N - (\boldsymbol{\lambda}_N - r\ddot{\boldsymbol{g}}_N)]^T(\boldsymbol{\lambda}_N^* - \boldsymbol{\lambda}_N) \geq 0 \; ; \; \boldsymbol{\lambda}_N \in C_N \; ; \; \forall \boldsymbol{\lambda}_N^* \in C_N \tag{30}$$

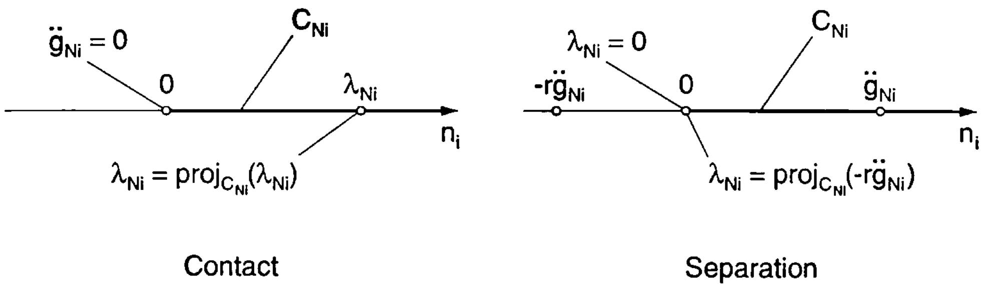

Figure 2. Projection for the normal force λ_{Ni} (normal direction n_i).

by premultiplication with an arbitrary factor

$$r > 0\ ;\ r \in \mathbb{R} \tag{31}$$

and by expansion with the identity $\boldsymbol{\lambda}_N - \boldsymbol{\lambda}_N = \mathbf{0}$. Corresponding to the definition of projection (59) in Appendix B, the variational inequality (30) is equivalent to the equation

$$\boldsymbol{\lambda}_N = \text{proj}_{C_N}(\boldsymbol{\lambda}_N - r\ddot{\boldsymbol{g}}_N). \tag{32}$$

By means of the definition of the convex set C_N (22), equation (32) can be simplified to

$$\boldsymbol{\lambda}_N = \text{proj}_{C_N}(\boldsymbol{\lambda}_N - r\ddot{\boldsymbol{g}}_N) = (\boldsymbol{\lambda}_N - r\ddot{\boldsymbol{g}}_N)^+ = \boldsymbol{\tau}_N^+, \tag{33}$$

where the abbreviation $\boldsymbol{\tau}_N = \boldsymbol{\lambda}_N - r\ddot{\boldsymbol{g}}_N$ has been used. We demonstrate the result of equation (33) by the two possible states of normal constraint i [see also equation (17) and (18)]. Together with the definition of positive vector parts (44) in Appendix A, we have the two cases

$$\begin{array}{llll}
\text{contact :} & \lambda_{Ni} > 0 \wedge \ddot{g}_{Ni} = 0 & \Rightarrow & \tau_{Ni} = \lambda_{Ni} - r\ddot{g}_{Ni} = \lambda_{Ni} \\
& \lambda_{Ni} > 0 & \Rightarrow & \lambda_{Ni}^+ = \lambda_{Ni} \\
& \lambda_{Ni} = \tau_{Ni} & \Rightarrow & \lambda_{Ni} = \tau_{Ni}^+ \\
\\
\text{separation :} & \lambda_{Ni} = 0 \wedge \ddot{g}_{Ni} > 0 & \Rightarrow & \tau_{Ni} = \lambda_{Ni} - r\ddot{g}_{Ni} = -r\ddot{g}_{Ni} \\
& -r\ddot{g}_{Ni} < 0 & \Rightarrow & \tau_{Ni}^+ = (-r\ddot{g}_{Ni})^+ = 0 \\
& \lambda_{Ni} = 0 & \Rightarrow & \lambda_{Ni} = \tau_{Ni}^+ = 0.
\end{array} \tag{34}$$

In a mathematical sense, the contact forces are projections of the vectorial difference between contact forces and weighted relative accelerations onto the convex set C_N. For both states of normal constraint i the projection is shown in Figure 2.

5.2. Transformation for Tangential Constraints

The variational inequality (27)

$$-\ddot{\boldsymbol{g}}_{Ti}^T(\boldsymbol{\lambda}_{Ti}^* - \boldsymbol{\lambda}_{Ti}) \leq 0\ ;\ \boldsymbol{\lambda}_{Ti} \in C_{Ti}(\lambda_{Ni})\ ;\ \forall \boldsymbol{\lambda}_{Ti}^* \in C_{Ti}(\lambda_{Ni})\ ;\ \ i \in I_T$$

with the convex set of admissible tangential contact forces $C_{Ti}(\lambda_{Ni}) = \{\boldsymbol{\lambda}_{Ti}^* \,:\, |\boldsymbol{\lambda}_{Ti}^*| \leq \mu_{0i}\lambda_{Ni}\}$ can be written in the form

$$[\boldsymbol{\lambda}_{Ti} - (\boldsymbol{\lambda}_{Ti} - r\ddot{\boldsymbol{g}}_{Ti})]^T(\boldsymbol{\lambda}_{Ti}^* - \boldsymbol{\lambda}_{Ti}) \geq 0 \;;\; \boldsymbol{\lambda}_{Ti} \in C_{Ti}(\lambda_{Ni}) \;;\; \forall \boldsymbol{\lambda}_{Ti}^* \in C_{Ti}(\lambda_{Ni}) \quad (35)$$

by premultiplication with (31) and by expansion with the identity $\boldsymbol{\lambda}_{Ti} - \boldsymbol{\lambda}_{Ti} = \mathbf{0}$. Corresponding to the definition of projection (59) in Appendix B, the variational inequality (35) is equivalent to the equation

$$\boldsymbol{\lambda}_{Ti} = \text{proj}_{C_{Ti}(\lambda_{Ni})}(\boldsymbol{\lambda}_{Ti} - r\ddot{\boldsymbol{g}}_{Ti}) = \text{proj}_{C_{Ti}(\tau_{Ni}^+)}(\boldsymbol{\tau}_{Ti}), \quad (36)$$

where r > 0 and the abbreviations $\tau_{Ni}^+ = \lambda_{Ni}$ and $\boldsymbol{\tau}_{Ti} = \boldsymbol{\lambda}_{Ti} - r\ddot{\boldsymbol{g}}_{Ti}$ have been used. Equation (36) can be written in a simpler form,

$$\boldsymbol{\lambda}_{Ti} = \text{proj}_{C_{Ti}(\tau_{Ni}^+)}(\boldsymbol{\tau}_{Ti}) = \begin{cases} \boldsymbol{\tau}_{Ti} & \text{if } |\boldsymbol{\tau}_{Ti}| < \mu_{0i}\tau_{Ni}^+ \\ \mu_{0i}\tau_{Ni}^+ \dfrac{\boldsymbol{\tau}_{Ti}}{|\boldsymbol{\tau}_{Ti}|} & \text{if } |\boldsymbol{\tau}_{Ti}| \geq \mu_{0i}\tau_{Ni}^+ \end{cases} \quad (37)$$

with the convex set $C_{Ti}(\tau_{Ni}^+) = \{\boldsymbol{\tau}_{Ti}^* : |\boldsymbol{\tau}_{Ti}^*| \leq \mu_{0i}\tau_{Ni}^+\}$. We demonstrate the result by the two different states of tangential constraint i [see also equation (25)]. The two cases are

$$\begin{array}{lll} \text{stiction :} & |\boldsymbol{\lambda}_{Ti}| < \mu_{0i}\lambda_{Ni} \wedge \ddot{\boldsymbol{g}}_{Ti} = \mathbf{0} & \Rightarrow \; \boldsymbol{\tau}_{Ti} = \boldsymbol{\lambda}_{Ti} - r\ddot{\boldsymbol{g}}_{Ti} = \boldsymbol{\lambda}_{Ti} \\ & |\boldsymbol{\lambda}_{Ti}| < \mu_{0i}\lambda_{Ni} & \Rightarrow \; |\boldsymbol{\tau}_{Ti}| < \mu_{0i}\tau_{Ni}^+ \\ \text{sliding :} & |\boldsymbol{\lambda}_{Ti}| = \mu_{0i}\lambda_{Ni} \wedge \ddot{\boldsymbol{g}}_{Ti} = -\rho_i\boldsymbol{\lambda}_{Ti} & \Rightarrow \; \boldsymbol{\tau}_{Ti} = \boldsymbol{\lambda}_{Ti} - r\ddot{\boldsymbol{g}}_{Ti} \\ & & \qquad\;\; = \underbrace{(1 + r\rho_i)}_{\geq 1}\boldsymbol{\lambda}_{Ti} \\ & \boldsymbol{\tau}_{Ti} = (1 + r\rho_i)\boldsymbol{\lambda}_{Ti} & \Rightarrow \; |\boldsymbol{\tau}_{Ti}| \geq |\boldsymbol{\lambda}_{Ti}| \wedge \boldsymbol{\lambda}_{Ti} \,||\, \boldsymbol{\tau}_{Ti} \\ & |\boldsymbol{\tau}_{Ti}| \geq |\boldsymbol{\lambda}_{Ti}| & \Rightarrow \; |\boldsymbol{\tau}_{Ti}| \geq \mu_{0i}\tau_{Ni}^+ \\ & \boldsymbol{\lambda}_{Ti} \,||\, \boldsymbol{\tau}_{Ti} & \Rightarrow \; \boldsymbol{\lambda}_{Ti} = |\boldsymbol{\lambda}_{Ti}| \dfrac{\boldsymbol{\lambda}_{Ti}}{|\boldsymbol{\lambda}_{Ti}|} \\ & & \qquad\;\; = \mu_{0i}\tau_{Ni}^+ \dfrac{\boldsymbol{\lambda}_{Ti}}{|\boldsymbol{\lambda}_{Ti}|} \,. \end{array} \quad (38)$$

With these two cases, we get the interdependence of $\boldsymbol{\lambda}_{Ti}$ and $\boldsymbol{\tau}_{Ti}$:

$$\begin{array}{lll} \text{stiction :} & |\boldsymbol{\tau}_{Ti}| < \mu_{0i}\tau_{Ni}^+ & \Rightarrow \; \boldsymbol{\lambda}_{Ti} = \boldsymbol{\tau}_{Ti} \\ \text{sliding :} & |\boldsymbol{\tau}_{Ti}| \geq \mu_{0i}\tau_{Ni}^+ & \Rightarrow \; \boldsymbol{\lambda}_{Ti} = \mu_{0i}\tau_{Ni}^+ \dfrac{\boldsymbol{\tau}_{Ti}}{|\boldsymbol{\tau}_{Ti}|}. \end{array} \quad (39)$$

In a mathematical sense, the contact force is a projection of the vectorial difference between contact force and weighted relative acceleration onto the convex set C_{Ti}. For both states of tangential constraint i, the projection is shown in Figure 3.

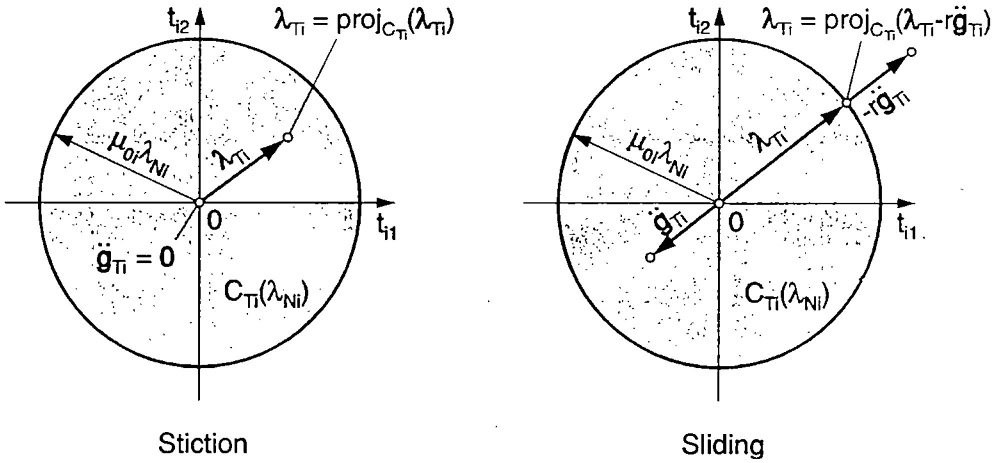

Figure 3. Projection for the tangential force $\boldsymbol{\lambda}_{Ti}$ (tangential directions t_{i1} and t_{i2}).

6. SOLUTION OF THE CONTACT PROBLEM

By means of the projected constraint forces in equations (33) and (37), we have the non-linear system of equations

$$\boldsymbol{M}\ddot{\boldsymbol{q}} - [\boldsymbol{W}_N + \boldsymbol{H}_R, \boldsymbol{W}_T]\begin{pmatrix}\boldsymbol{\lambda}_N\\ \boldsymbol{\lambda}_T\end{pmatrix} - \boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = \boldsymbol{0}$$

$$\begin{pmatrix}\ddot{\boldsymbol{g}}_N\\ \ddot{\boldsymbol{g}}_T\end{pmatrix} = \begin{pmatrix}\boldsymbol{W}_N^T\\ \boldsymbol{W}_T^T\end{pmatrix}\ddot{q} + \begin{pmatrix}\overline{\boldsymbol{w}}_N\\ \overline{\boldsymbol{w}}_T\end{pmatrix}$$

$$\boldsymbol{\lambda}_N = \text{proj}_{C_N}(\boldsymbol{\lambda}_N - r\ddot{\boldsymbol{g}}_N) = (\boldsymbol{\lambda}_N - r\ddot{\boldsymbol{g}}_N)^+ = \boldsymbol{\tau}_N^+ \tag{40}$$

$$\boldsymbol{\lambda}_{Ti} = \text{proj}_{C_{Ti}(\lambda_{Ni})}(\boldsymbol{\lambda}_{Ti} - r\ddot{\boldsymbol{g}}_{Ti}) = \begin{cases}\boldsymbol{\tau}_{Ti} & \text{if } |\boldsymbol{\tau}_{Ti}| < \mu_{0i}\tau_{Ni}^+\\ \mu_{0i}\tau_{Ni}^+\dfrac{\boldsymbol{\tau}_{Ti}}{|\boldsymbol{\tau}_{Ti}|} & \text{if } |\boldsymbol{\tau}_{Ti}| \geq \mu_{0i}\tau_{Ni}^+\end{cases}$$

The convex sets of the admissible constraint forces in the system of equations (40) are

$$C_N = \{\boldsymbol{\lambda}_N^* \;:\; \boldsymbol{\lambda}_N^* \geq \boldsymbol{0}\}$$

$$C_{Ti}(\lambda_{Ni}) = \{\boldsymbol{\lambda}_{Ti}^* : |\boldsymbol{\lambda}_{Ti}^*| \leq \mu_{0i}\lambda_{Ni}\}.$$

The projections in equation (40) show that the constraint forces $\boldsymbol{\lambda}_N$ and $\boldsymbol{\lambda}_T$ are functionals of $\boldsymbol{\lambda}_N, \ddot{\boldsymbol{g}}_N$ or $\boldsymbol{\lambda}_N, \boldsymbol{\lambda}_T, \ddot{\boldsymbol{g}}_N, \ddot{\boldsymbol{g}}_T$, respectively. These functionals can be written in the form

$$\begin{aligned} \boldsymbol{\lambda}_N &= \boldsymbol{\Pi}_N(\boldsymbol{\lambda}_N, \ddot{\boldsymbol{q}}) \\ \boldsymbol{\lambda}_T &= \boldsymbol{\Pi}_T(\boldsymbol{\lambda}_N, \boldsymbol{\lambda}_T, \ddot{\boldsymbol{q}}) \end{aligned} \tag{41}$$

if the constraint equations of (40) are considered by eliminating the relative accelerations in the functionals. Now we are able to replace the constraint forces in the equation of motion in (14) by the functionals (41) and we get an implicit equation for the generalized accelerations,

$$\boldsymbol{M}\ddot{\boldsymbol{q}} - [\boldsymbol{W}_N + \boldsymbol{H}_R]\boldsymbol{\Pi}_N(\boldsymbol{\lambda}_N, \ddot{\boldsymbol{q}}) - \boldsymbol{W}_T\boldsymbol{\Pi}_T(\boldsymbol{\lambda}_N, \boldsymbol{\lambda}_T, \ddot{\boldsymbol{q}}) - \boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = \boldsymbol{0}. \tag{42}$$

The nonlinear system of equations (41) and (42) are resolvable by means of an iterative Newton algorithm.

6.1. Solution of the System of Nonlinear Equations

The generalized accelerations $\ddot{\boldsymbol{q}}$ are determined with Newton's method. In a first step, the constraint forces are kept constant during the solution of equation (42). After each determination of the accelerations, the constraint forces are updated according to the functionals (41). The two following steps are repeated with $l = l + 1$ until convergence occurs:

1. Determination of $\ddot{\boldsymbol{q}}^l$ according to the nonlinear equation

   $$\boldsymbol{M}\ddot{\boldsymbol{q}}^l - [\boldsymbol{W}_N + \boldsymbol{H}_R]\boldsymbol{\Pi}_N(\boldsymbol{\lambda}_N^l, \ddot{\boldsymbol{q}}^l) - \boldsymbol{W}_T\boldsymbol{\Pi}_T(\boldsymbol{\lambda}_N^l, \boldsymbol{\lambda}_T^l, \ddot{\boldsymbol{q}}^l) - \boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = \boldsymbol{0}$$

   $\boldsymbol{\lambda}_N^l$ and $\boldsymbol{\lambda}_T^l$ remain constant.

2. Update of the constraint forces according to the functionals

   $$\boldsymbol{\lambda}_N^{l+1} = \boldsymbol{\Pi}_N(\boldsymbol{\lambda}_N^l, \ddot{\boldsymbol{q}}^l)$$

   $$\boldsymbol{\lambda}_T^{l+1} = \boldsymbol{\Pi}_T(\boldsymbol{\lambda}_N^l, \boldsymbol{\lambda}_T^l, \ddot{\boldsymbol{q}}^l).$$

In the following application of the algorithm, we use the subroutine NEWT, which is described in Press, Teukolsky, Vetterling, and Flannery (1992). This is a globally convergent method for the solution of nonlinear systems of equations.

6.2. Numerical Integration

When the generalized accelerations and constraint forces are known, we get the generalized velocities and positions by means of numerical integration. For this purpose, the equations of motion will be extended to the double dimension of the state space. The state equation has the form

$$\begin{pmatrix} \dot{\boldsymbol{q}} \\ \ddot{\boldsymbol{q}} \end{pmatrix} = \begin{pmatrix} \dot{\boldsymbol{q}} \\ \boldsymbol{b}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{\lambda}_N, \boldsymbol{\lambda}_T, t) \end{pmatrix} \in \mathbb{R}^{2f} \tag{43}$$

with the abbreviation

$$\boldsymbol{b}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{\lambda}_N, \boldsymbol{\lambda}_T, t) = \boldsymbol{M}^{-1}\boldsymbol{h} + \boldsymbol{M}^{-1}[\boldsymbol{W}_N + \boldsymbol{H}_R, \boldsymbol{W}_T] \begin{pmatrix} \boldsymbol{\lambda}_N \\ \boldsymbol{\lambda}_T \end{pmatrix}.$$

In the following application of the algorithm, we use the method of Runge-Kutta-Fehlberg with automatical control of the integration interval for the numerical integration of state equation (43).

7. APPLICATION

We consider an oscillator with one mass and three contact points on an oblique plane. The rigid body is excited by a periodically rotating unbalanced mass. The angle of inclination and the speed of the unbalanced mass are adjusted in the way that stick-slip phenomena occur during the motion of the body. The experimental set-up is shown in Figure 4. By means of two light-emitting diodes, which are fixed on the top of the body, we observe and photograph the motion in the plane and the rotation around the vertical axis of the body.

7.1. Mechanical Model

The body without constraints has three translational degrees of freedom—$x(t)$, $y(t)$, $z(t)$—and three rotating degrees of freedom—$\alpha(t)$, $\beta(t)$, $\gamma(t)$. The translational degrees of freedom represent the positions of the center of mass, which moves due to the rotating unbalanced mass. All six degrees of freedom are combined in the vector of generalized coordinates

$$\boldsymbol{q}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ \alpha(t) \\ \beta(t) \\ \gamma(t) \end{pmatrix} = \begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \\ q_4(t) \\ q_5(t) \\ q_6(t) \end{pmatrix} \in \mathbb{R}^6.$$

Because of the selected test conditions (parameters in Appendix C) no take off occurs. All contacts are permanently closed and no impacts occur during the simulation. So the results for the coordinate $z(t)$ in the vertical direction will not be considered in the following. The twistings $\alpha(t)$ and $\beta(t)$ around the longitudinal and lateral axis, respectively, are always zero.

Due to the rotating unbalance mass, the rotating time-varying force vector

$$\boldsymbol{F}(t) = \begin{pmatrix} F_x(t) \\ F_y(t) \\ F_z(t) \end{pmatrix} = \begin{pmatrix} F_0 \cos\Omega t \\ 0 \\ F_0 \sin\Omega t \end{pmatrix}$$

acts as an external force on the body. The magnitude of $\boldsymbol{F}(t)$ is the centrifugal force $F_0 = m_E r_E \Omega^2$. On the center of mass S_1 the two components of the gravity force

$$F_N = mg\,\cos\delta$$

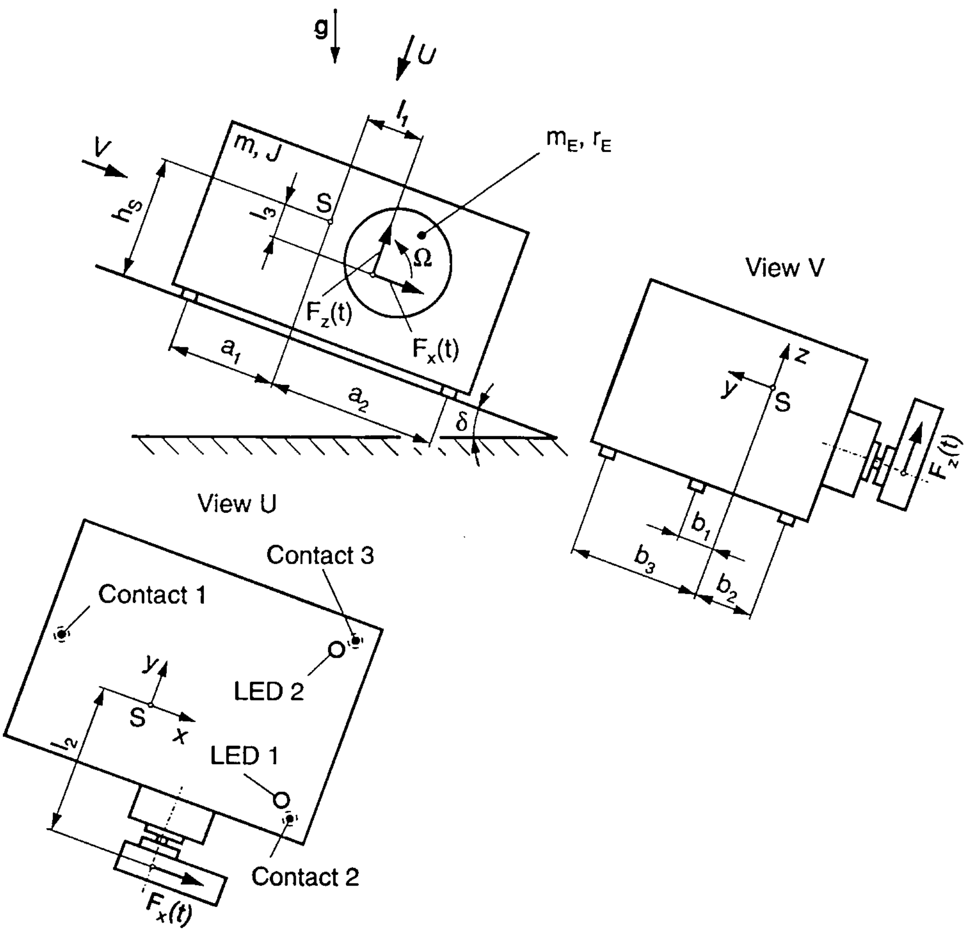

Figure 4. Oscillator on oblique plane.

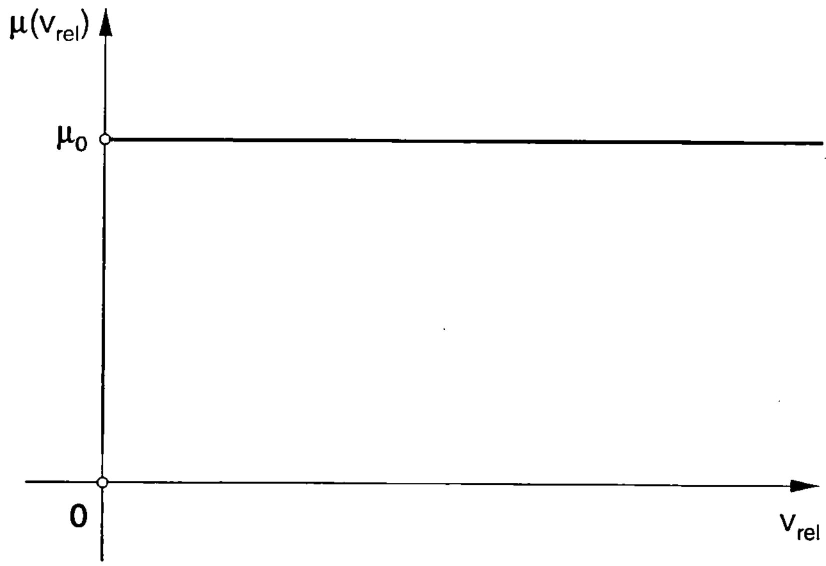

Figure 5. Friction characteristic.

$$F_H = mg \sin\delta$$

are acting. For the consideration of friction, we use a friction characteristic as shown in Figure 5. The coefficient of sliding friction μ is independent on the relative velocity and has the same value as the coefficient of static friction μ_0. The values of the model parameters and the initial conditions for the numerical integration are given in Appendix C.

7.2. Contact Kinematics

In principle, separation of any closed contact is possible. Then the angular positions $\alpha(t)$ and $\beta(t)$ are very small compared with the position $\gamma(t)$. Therefore, the sine of an angle can be replaced by the angle itself, and the cosine of an angle has approximately the value 1. If separation is occurring, the translational position is also very small. Moreover, the terms of second order

$$\left.\begin{array}{r} q_i q_j \approx 0 \\ q_i \dot{q}_j \approx 0 \\ \dot{q}_i \dot{q}_j \approx 0 \end{array}\right\} \forall i, j = 3, 4, 5$$

are negligible. With these simplifications, we determine the contact kinematics. Because of the rotating unbalanced mass the distances a_1, a_2, and l_1 periodically change with the amplitude Δa. The distances h_S and l_3 change with the amplitude Δh. These distances are shown in Figure 4. All changes happen with the angular frequency Ω.

The normal distances between the potential contact points are

$$\boldsymbol{g}_N = \begin{pmatrix} g_{N1} \\ g_{N2} \\ g_{N3} \end{pmatrix} \in \mathbb{R}^3$$

with the components

$$\begin{aligned} g_{N1} &= q_3 + q_5\left[(a_1 + \Delta a \cos\Omega t)\ \cos q_6 + b_1 \sin q_6\right] \\ &\quad + q_4\left[-(a_1 + \Delta a \cos\Omega t)\sin q_6 + b_1 \cos q_6\right] \end{aligned}$$

$$\begin{aligned} g_{N2} &= q_3 + q_5\left[(-a_2 + \Delta a \cos\Omega t)\ \cos q_6 - b_2 \sin q_6\right] \\ &\quad + q_4\left[(a_2 - \Delta a \cos\Omega t)\sin q_6 - b_2 \cos q_6\right] \end{aligned}$$

$$\begin{aligned} g_{N3} &= q_3 + q_5\left[(-a_2 + \Delta a \cos\Omega t)\ \cos q_6 + b_3 \sin q_6\right] \\ &\quad + q_4\left[(a_2 - \Delta a \cos\Omega t)\sin q_6 + b_3 \cos q_6\right] . \end{aligned}$$

The relative velocities in the tangential plane of the contact points are

$$\dot{\boldsymbol{g}}_T = \begin{pmatrix} \dot q_1 - \dot q_6 b_1 \cos q_6 + \dot q_6\,(a_1 + \Delta a \cos\Omega t)\ \sin q_6 - \dot q_5\,(h_s + \Delta h \sin\Omega t) \\ \dot q_2 - \dot q_6 b_1 \sin q_6 - \dot q_6\,(a_1 + \Delta a \cos\Omega t)\ \cos q_6 + \dot q_4\,(h_s + \Delta h \sin\Omega t) \\ \dot q_1 + \dot q_6 b_2 \cos q_6 - \dot q_6\,(a_2 - \Delta a \cos\Omega t)\ \sin q_6 - \dot q_5\,(h_s + \Delta h \sin\Omega t) \\ \dot q_2 + \dot q_6 b_2 \sin q_6 + \dot q_6\,(a_2 - \Delta a \cos\Omega t)\ \cos q_6 + \dot q_4\,(h_s + \Delta h \sin\Omega t) \\ \dot q_1 - \dot q_6 b_3 \cos q_6 - \dot q_6\,(a_2 - \Delta a \cos\Omega t)\ \sin q_6 - \dot q_5\,(h_s + \Delta h \sin\Omega t) \\ \dot q_2 - \dot q_6 b_3 \sin q_6 + \dot q_6\,(a_2 - \Delta a \cos\Omega t)\ \cos q_6 + \dot q_4\,(h_s + \Delta h \sin\Omega t) \end{pmatrix} \in \mathbb{R}^6.$$

The normal distance will be twice and the tangential relative velocity once differentiated with respect to time. In the deviations, we find the constraint matrix of the normal constraints

$$\boldsymbol{W}_N = [\boldsymbol{w}_{N1}, \boldsymbol{w}_{N2}, \boldsymbol{w}_{N3}] \in \mathbb{R}^{6,3}$$

with the constraint vectors

$$\boldsymbol{w}_{N1} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -(a_1 + \Delta a \cos\Omega t)\ \sin q_6 + b_1 \cos q_6 \\ (a_1 + \Delta a \cos\Omega t)\ \cos q_6 + b_1 \sin q_6 \\ -(a_1 + \Delta a \cos\Omega t)(q_5 \cos q_6 + q_4 \sin q_6) + b_1(q_4 \cos q_6 - q_5 \cos q_6) \end{pmatrix} \in \mathbb{R}^6$$

$$\boldsymbol{w}_{N2} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ (a_2 - \Delta a \cos\Omega t)\ \sin q_6 - b_2 \cos q_6 \\ (-a_2 + \Delta a \cos\Omega t)\ \cos q_6 - b_2 \sin q_6 \\ (a_2 - \Delta a \cos\Omega t)\,(q_5 \sin q_6 + q_4 \cos q_6) - b_2\,(q_4 \sin q_6 + q_5 \cos q_6) \end{pmatrix} \in \mathbb{R}^6$$

$$\boldsymbol{w}_{N3} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ (a_2 - \Delta a \cos\Omega t)\ \sin q_6 + b_3 \cos q_6 \\ (-a_2 + \Delta a \cos\Omega t)\ \cos q_6 + b_3 \sin q_6 \\ (a_2 - \Delta a \cos\Omega t)\,(q_5 \sin q_6 + q_4 \cos q_6) + b_3\,(q_5 \cos q_6 - q_4 \sin q_6) \end{pmatrix} \in \mathbb{R}^6$$

and the constraint matrix of the tangential constraints

$$\boldsymbol{W}_T = [\boldsymbol{w}_{T11}, \boldsymbol{w}_{T12}, \boldsymbol{w}_{T21}, \boldsymbol{w}_{T22}, \boldsymbol{w}_{T31}, \boldsymbol{w}_{T32}] \in \mathbb{R}^{6,6}$$

with the constraint vectors

$$\boldsymbol{w}_{T11} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -(h_S + \Delta h \sin\Omega t) \\ -b_1 \cos q_6 + (a_1 + \Delta a \cos\Omega t)\ \sin q_6 \end{pmatrix} \in \mathbb{R}^6$$

$$\boldsymbol{w}_{T12} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ h_S + \Delta h \sin\Omega t \\ 0 \\ -b_1 \sin q_6 - (a_1 + \Delta a \cos\Omega t)\ \cos q_6 \end{pmatrix} \in \mathbb{R}^6$$

$$\boldsymbol{w}_{T21} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -(h_S + \Delta h \sin\Omega t) \\ b_2 \cos q_6 - (a_2 - \Delta a \cos\Omega t)\ \sin q_6 \end{pmatrix} \in \mathbb{R}^6$$

$$\boldsymbol{w}_{T22} = \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ h_S + \Delta h \sin\Omega t \\ 0 \\ b_2 \sin q_6 + (a_2 - \Delta a \cos\Omega t) \cos q_6 \end{pmatrix} \in \mathbb{R}^6$$

$$\boldsymbol{w}_{T31} = \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ 0 \\ -(h_S + \Delta h \sin\Omega t) \\ -b_3 \cos q_6 - (a_2 - \Delta a \cos\Omega t) \sin q_6 \end{pmatrix} \in \mathbb{R}^6$$

$$\boldsymbol{w}_{T32} = \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \\ 0 \\ h_S + \Delta h \sin\Omega t \\ \mathbf{0} \\ -b_3 \sin q_6 + (a_2 - \Delta a \cos\Omega t) \cos q_6 \end{pmatrix} \in \mathbb{R}^6 .$$

The remaining terms of relative kinematics are contained in the vectors

$$\tilde{\boldsymbol{w}}_N = \begin{pmatrix} \Delta a\Omega (q_4 \sin q_6 - q_5 \cos q_6) \sin\Omega t \\ \Delta a\Omega (q_4 \sin q_6 - q_5 \cos q_6) \sin\Omega t \\ \Delta a\Omega (q_4 \sin q_6 - q_5 \cos q_6) \sin\Omega t \end{pmatrix} \in \mathbb{R}^3$$

$$\tilde{\boldsymbol{w}}_T = \mathbf{0} \in \mathbb{R}^6$$

$$\overline{\boldsymbol{w}}_N = [\overline{w}_{N1}, \overline{w}_{N2}, \overline{w}_{N3}]^T \in \mathbb{R}^3$$

$$\begin{aligned} \overline{w}_{N1} = {} & -2\dot{q}_4\dot{q}_6 (a_1 + \Delta a \cos\Omega t) \cos q_6 - 2\dot{q}_5\dot{q}_6 (a_1 + \Delta a \cos\Omega t) \sin q_6 \\ & + q_4\dot{q}_6^2 (a_1 + \Delta a \cos\Omega t) \sin q_6 - q_5\dot{q}_6^2 (a_1 + \Delta a \cos\Omega t) \cos q_6 \\ & + 2b_1\dot{q}_6 (\dot{q}_5 \cos q_6 - \dot{q}_4 \sin q_6) - b_1\dot{q}_6^2 (q_4 \cos q_6 + q_5 \sin q_6) \\ & + \Omega^2 (q_4 \sin q_6 - q_5 \cos q_6) \Delta a \cos\Omega t \\ \overline{w}_{N2} = {} & 2\dot{q}_4\dot{q}_6 (a_2 - \Delta a \cos\Omega t) \cos q_6 + 2\dot{q}_5\dot{q}_6 (a_2 - \Delta a \cos\Omega t) \sin q_6 \\ & - q_4\dot{q}_6^2 (a_2 - \Delta a \cos\Omega t) \sin q_6 + q_5\dot{q}_6^2 (a_2 - \Delta a \cos\Omega t) \cos q_6 \end{aligned}$$

$$
\begin{aligned}
& +2b_2\dot{q}_6\left(\dot{q}_4\,\mathrm{sin}q_6-\dot{q}_5\,\mathrm{cos}q_6\right)+b_2\dot{q}_6^2\left(q_4\,\mathrm{cos}q_6+q_5\,\mathrm{sin}q_6\right)\\
& +\Omega^2\left(q_4\,\mathrm{sin}q_6-q_5\,\mathrm{cos}q_6\right)\Delta a\,\mathrm{cos}\Omega t\\
\overline{w}_{N3} \;=\; & 2\dot{q}_4\dot{q}_6\left(a_2-\Delta a\,\mathrm{cos}\Omega t\right)\,\mathrm{cos}q_6+2\dot{q}_5\dot{q}_6\left(a_2-\Delta a\,\mathrm{cos}\Omega t\right)\,\mathrm{sin}q_6\\
& -q_4\dot{q}_6^2\left(a_2-\Delta a\,\mathrm{cos}\Omega t\right)\,\mathrm{sin}q_6+q_5\dot{q}_6^2\left(a_2-\Delta a\,\mathrm{cos}\Omega t\right)\,\mathrm{cos}q_6\\
& +2b_3\dot{q}_6\left(\dot{q}_5\,\mathrm{cos}q_6-\dot{q}_4\,\mathrm{sin}q_6\right)-b_3\dot{q}_6^2\left(q_4\,\mathrm{cos}q_6+q_5\,\mathrm{sin}q_6\right)\\
& +\Omega^2\left(q_4\,\mathrm{sin}q_6-q_5\,\mathrm{cos}q_6\right)\Delta a\,\mathrm{cos}\Omega t\\
\overline{\boldsymbol{w}}_T \;=\; & \left[\overline{w}_{T1},\overline{w}_{T2},\overline{w}_{T3},\overline{w}_{T4},\overline{w}_{T5},\overline{w}_{T6}\right]^T\in\mathbb{R}^6
\end{aligned}
$$

$$
\begin{aligned}
\overline{w}_{T1} \;=\; & \dot{q}_6^2b_1\,\mathrm{sin}q_6+\dot{q}_6^2\left(a_1+\Delta a\,\mathrm{cos}\Omega t\right)\,\mathrm{cos}q_6\\
& -\dot{q}_6\Omega\Delta a\,\mathrm{sin}\Omega t\,\mathrm{sin}q_6-\dot{q}_5\Omega\Delta h\,\mathrm{cos}\Omega t
\end{aligned}
$$

$$
\begin{aligned}
\overline{w}_{T2} \;=\; & -\dot{q}_6^2b_1\,\mathrm{cos}q_6+\dot{q}_6^2\left(a_1+\Delta a\,\mathrm{cos}\Omega t\right)\,\mathrm{sin}q_6\\
& +\dot{q}_6\Omega\Delta a\,\mathrm{sin}\Omega t\,\mathrm{cos}q_6+\dot{q}_4\Omega\Delta h\,\mathrm{cos}\Omega t
\end{aligned}
$$

$$
\begin{aligned}
\overline{w}_{T3} \;=\; & -\dot{q}_6^2b_2\,\mathrm{sin}q_6-\dot{q}_6^2\left(a_2-\Delta a\,\mathrm{cos}\Omega t\right)\,\mathrm{cos}q_6\\
& -\dot{q}_6\Omega\Delta a\,\mathrm{sin}\Omega t\,\mathrm{sin}q_6-\dot{q}_5\Omega\Delta h\,\mathrm{cos}\Omega t
\end{aligned}
$$

$$
\begin{aligned}
\overline{w}_{T4} \;=\; & \dot{q}_6^2b_2\,\mathrm{cos}q_6-\dot{q}_6^2\left(a_2-\Delta a\,\mathrm{cos}\Omega t\right)\,\mathrm{sin}q_6\\
& +\dot{q}_6\Omega\Delta a\,\mathrm{sin}\Omega t\,\mathrm{cos}q_6+\dot{q}_4\Omega\Delta h\,\mathrm{cos}\Omega t
\end{aligned}
$$

$$
\begin{aligned}
\overline{w}_{T5} \;=\; & \dot{q}_6^2b_3\,\mathrm{sin}q_6-\dot{q}_6^2\left(a_2-\Delta a\,\mathrm{cos}\Omega t\right)\,\mathrm{cos}q_6\\
& -\dot{q}_6\Omega\Delta a\,\mathrm{sin}\Omega t\,\mathrm{sin}q_6-\dot{q}_5\Omega\Delta h\,\mathrm{cos}\Omega t
\end{aligned}
$$

$$
\begin{aligned}
\overline{w}_{T6} \;=\; & -\dot{q}_6^2b_3\,\mathrm{cos}q_6-\dot{q}_6^2\left(a_2-\Delta a\,\mathrm{cos}\Omega t\right)\,\mathrm{sin}q_6\\
& +\dot{q}_6\Omega\Delta a\,\mathrm{sin}\Omega t\,\mathrm{cos}q_6+\dot{q}_4\Omega\Delta h\,\mathrm{cos}\Omega t.
\end{aligned}
$$

7.3. Equations of Motion

The differential equations of motion for the system with four degrees of freedom has the form

$$\boldsymbol{M}\ddot{\boldsymbol{q}}-\boldsymbol{h}=\boldsymbol{0}\,.$$

The mass matrix is

$$
\boldsymbol{M}=\begin{pmatrix}
m & 0 & 0 & 0 & 0 & 0\\
0 & m & 0 & 0 & 0 & 0\\
0 & 0 & m & 0 & 0 & 0\\
0 & 0 & 0 & A\cos^2 q_6+B\sin^2 q_6 & (A-B)\,\mathrm{cos}q_6\,\mathrm{sin}q_6 & Cq_5\\
0 & 0 & 0 & (A-B)\,\mathrm{cos}q_6\,\mathrm{sin}q_6 & A\sin^2 q_6+B\cos^2 q_6 & 0\\
0 & 0 & 0 & Cq_5 & 0 & C
\end{pmatrix}\in\mathbb{R}^{6,6},
$$

which has block-diagonal form. A, B, and C are the diagonal elements of the inertia tensor relating to the center of mass. The mass moments of inertia are approximately constant because of the small displacements of the center of mass due to the rotating unbalanced mass.

The vector of all external forces is

$$\boldsymbol{h} = \begin{pmatrix} mg\,\sin\delta + F_0\,(\cos q_6\,\cos\Omega t + q_5\,\sin\Omega t) \\ \\ F_0\,(\sin q_6\,\cos\Omega t - q_4\,\sin\Omega t) \\ \\ -mg\,\cos\delta + F_0\,(q_4\,\sin q_6\,\cos\Omega t - q_5\,\cos q_6\,\cos\Omega t + \sin\Omega t) \\ \\ F_0\,(l_1 - \Delta a\,\cos\Omega t)\,\sin q_6\,\sin\Omega t + F_0 l_2\,(q_5\,\cos\Omega t - \cos q_6\,\sin\Omega t) \\ +F_0\,(l_3 + \Delta h\,\sin\Omega t)\,\sin q_6\,\cos\Omega t \\ +\dot{q}_5\dot{q}_6\left[(A-B)\,(1 - 2\cos^2 q_6) - C\right] + 2\dot{q}_4\dot{q}_6(A-B)\,\cos q_6\,\sin q_6 \\ \\ -F_0\,(l_1 - \Delta a\,\cos\Omega t)\,\cos q_6\,\sin\Omega t - F_0 l_2\,\sin q_6\,\sin\Omega t \\ -F_0\,(l_3 + \Delta h\,\sin\Omega t)\,\cos q_6\,\cos\Omega t \\ +\dot{q}_4\dot{q}_6\left[(A-B)\,(1 - 2\cos^2 q_6) + C\right] + 2\dot{q}_5\dot{q}_6(B-A)\,\cos q_6\,\sin q_6 \\ \\ F_0 l_2\,\cos\Omega t \end{pmatrix} \in \mathbb{R}^6$$

with the abbreviation $F_0 = m_E r_E \Omega^2$. For the simplification of the matrix $\boldsymbol{M}$ and the vector $\boldsymbol{h}$, we used the assumptions in Subsection 7.2.

7.4. Results and Comparison with Measurement

The above theory has been applied to the stick-slip process of the oscillating mass shown in Figure 4. Some numerical results for the model parameters and initial conditions of Appendix C are shown in Figures 6, 7, and 8. Along the oblique plane, we have the x-direction; the lateral direction in the plane is called y-direction. The angle γ describes the position of the mass around the vertical z-axis. In Figure 6, the time courses of the positions are shown in a small time interval of 1 s. All positions of the mass simultaneously remain constant during time intervals that are periodically occurring phases of stiction. By means of the photographed traces of both light-emitting diodes under stroboscopic exposure, the position of the mass has been reconstructed for discrete times. With regard to the moving center of mass due to the rotating unbalanced mass, these measured points are also shown in Figure 7. The correspondence between simulation and measurement is good. The diagrams of Figures 8 show the computed accelerations $\ddot{x}, \ddot{y}, \ddot{\gamma}$ as a function of their velocities $\dot{x}, \dot{y}, \dot{\gamma}$. The unsteady changes in the accelerations arise due to stick-slip transitions.

8. CONCLUSIONS

Stick-slip phenomena in rigid multibody systems with many frictional contacts imply some fundamental problems, especially when the contacts are coupled and three-dimensional.

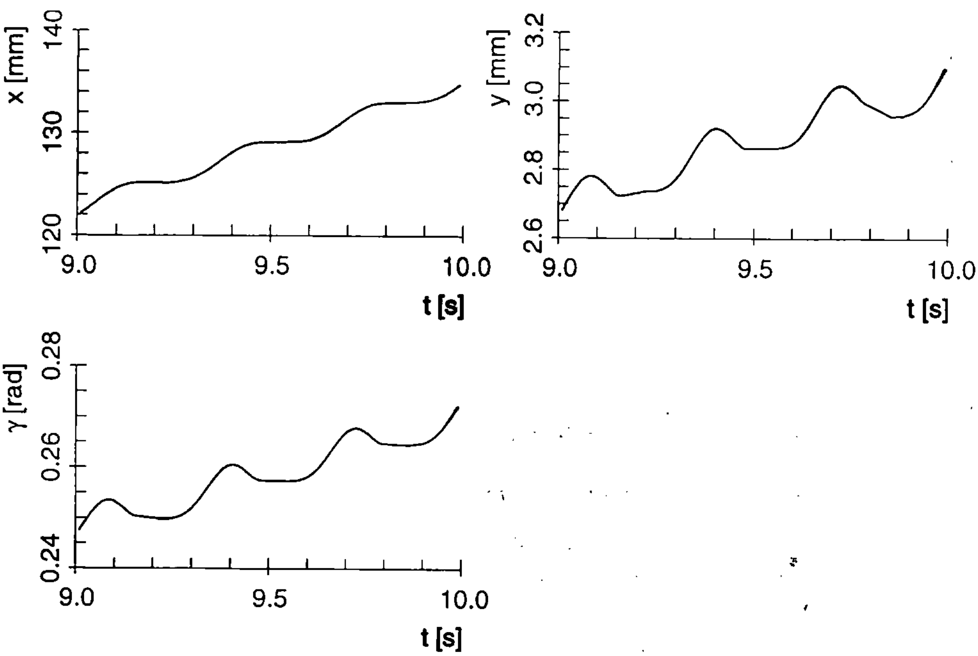

Figure 6. Positions of the oscillating mass.

The behavior of one contact can influence the state of all others. This situation leads to a compatibility problem with respect to constraint dynamics. Considering all possibilities of motion in one contact and combining them with all possibilities in all the other contacts leads to a huge number of formal combinations of possible constraints of which only one is physically meaningful. For three-dimensional problems, this selection cannot be evaluated by a linear complementarity problem, as the friction direction of an active tangential constraint in the first moment after the transition from stiction to sliding is unknown. This situation leads to a nonlinear complementarity problem, which is solvable by means of several algorithms. If the contacts are additionally dependent, no formulation of the contact problem as a nonlinear complementarity problem is available up to now. The projection method described in this paper is a way to deal with such contact problems. The constraint forces are considered by means of constraint matrices in the equations of motion and have the mathematical meaning of Lagrange multipliers. By transforming the kinematic secondary conditions, we get equations for determination of the constraint forces. These are projections of the vectorial differences between contact forces and weighted relative accelerations onto the convex sets of the admissible contact forces. The method has been successfully applied to the example given in this paper.

APPENDIX

A. VECTOR ALGEBRA

Here we find the most important definitions and theorems for calculating with vectors.

- Definition of the positive vector part:
  $\boldsymbol{x} = \{x_i\}$ is a vector of $\mathbb{R}^n$. *The positive part* of vector $\boldsymbol{x}$ is

$$\boldsymbol{x}^+ = \{x_i\}^+ = \frac{1}{2}\{|x_i| + x_i\} \quad \Rightarrow \quad x_i^+ = \begin{cases} 0 & \text{if } x_i \leq 0 \\ x_i & \text{if } x_i > 0 \ . \end{cases} \tag{44}$$

- Definition of the negative vector part:
  The *negative part* of vector $\boldsymbol{x}$ is

$$\boldsymbol{x}^- = \{x_i\}^- = \frac{1}{2}\{|x_i| - x_i\} \quad \Rightarrow \quad x_i^- = \begin{cases} -x_i & \text{if } x_i < 0 \\ 0 & \text{if } x_i \geq 0 \ . \end{cases} \tag{45}$$

- Theorem for the connection of both parts:
  The difference of both parts of vector $\boldsymbol{x}$ according to

$$\boldsymbol{x} = \boldsymbol{x}^+ - \boldsymbol{x}^- \tag{46}$$

  is the vector $\boldsymbol{x}$ itself.

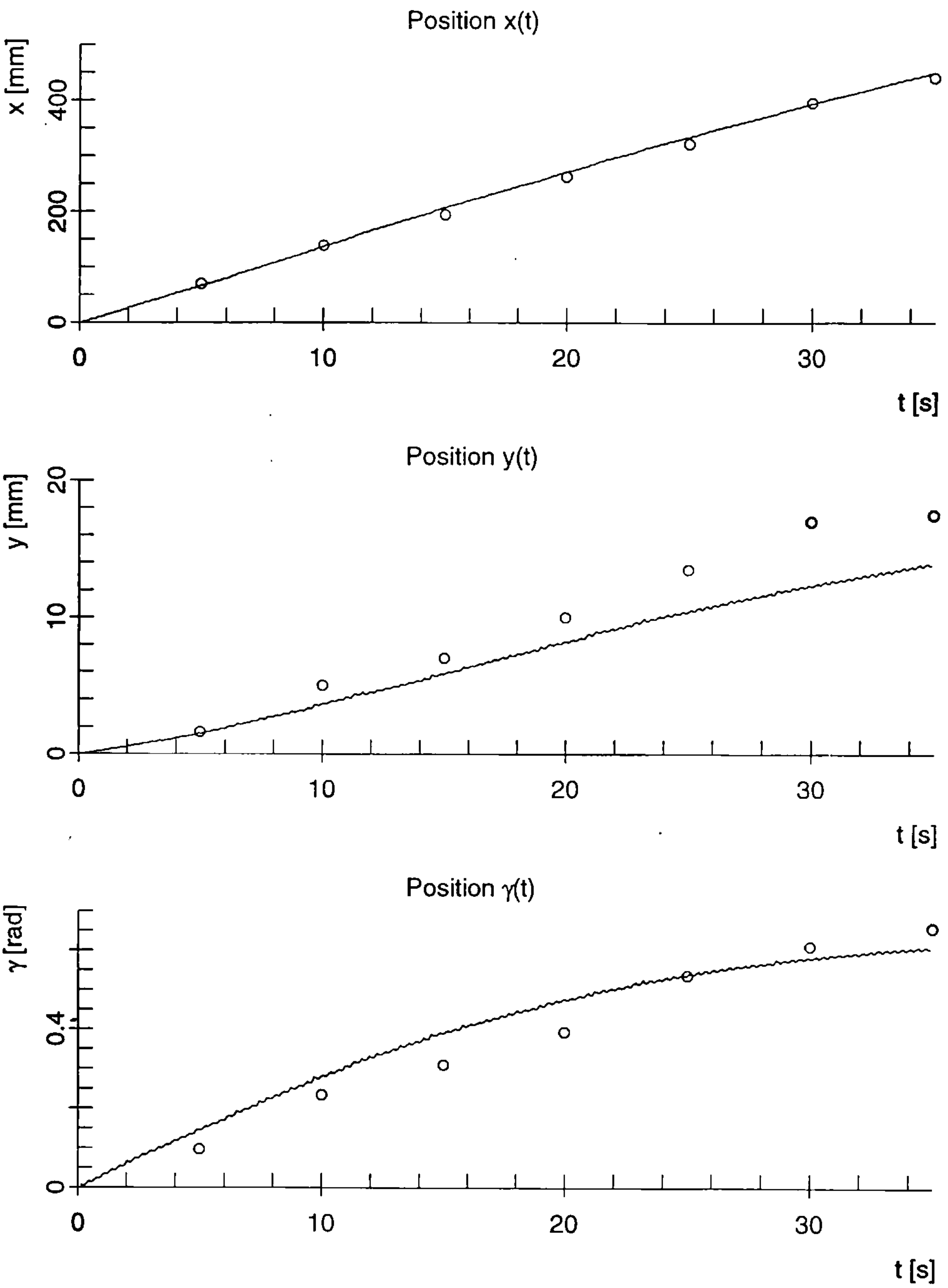

Figure 7. Comparison of theory (—) and measurement (o).

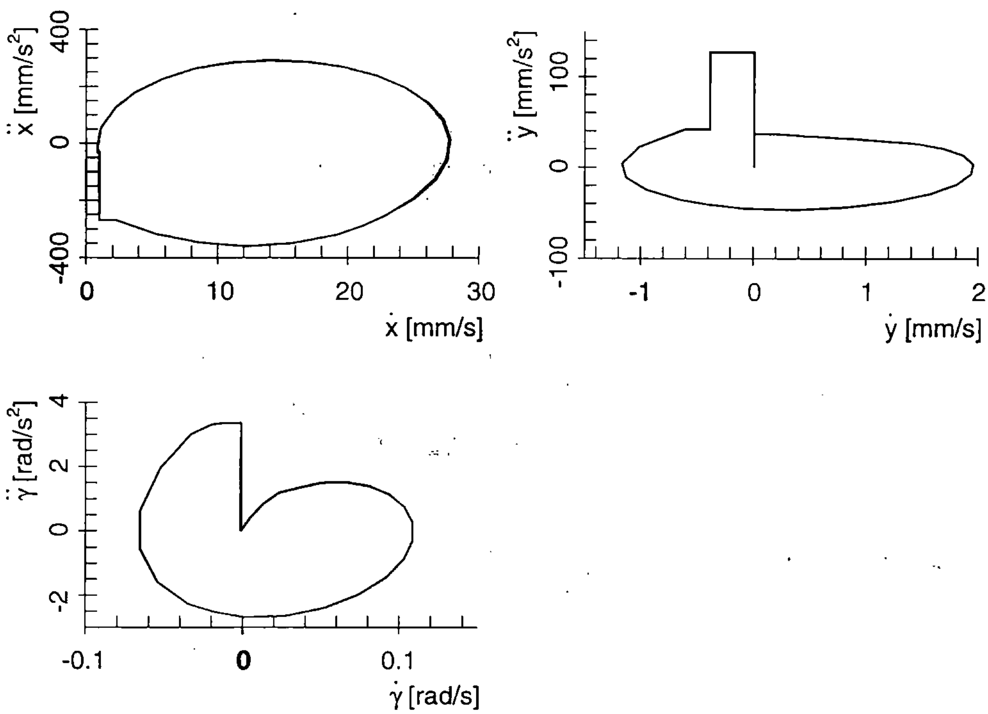

Figure 8. Dynamical behavior of the oscillating mass.

- Theorem for vector inequalities:
  $\boldsymbol{x} = \{x_i\}$ and $\boldsymbol{y} = \{y_i\}$ are two vectors of $\mathbb{R}^n$. The inequalities between $\boldsymbol{x}$ and $\boldsymbol{y}$ have the meaning

$$\left.\begin{array}{ll} \boldsymbol{x} \geq \boldsymbol{y} & \Rightarrow x_i \geq y_i \\ \boldsymbol{x} \leq \boldsymbol{y} & \Rightarrow x_i \leq y_i \end{array}\right\} \forall_i = 1, ..., n \tag{47}$$

B. CONVEX ANALYSIS

Here we find important definitions, theorems, and calculation rules for convex sets and functions (see also Rockafellar [1972]).

B.1. Convex Sets

- Definition of the convex set:
  A set $C \subset \mathbb{R}^n$ is called *convex* if

$$(1-\lambda)\boldsymbol{x} + \lambda\boldsymbol{y} \in C \;\; ; \;\; \forall \boldsymbol{x} \in C \;\; ; \;\; \forall \boldsymbol{y} \in C \tag{48}$$

  for an arbitrary λ with $0 < \lambda < 1$.

- Definition of the normal cone:
  Let $\boldsymbol{x} \in C$, where C is a convex set. The set of all vectors $\boldsymbol{y}$ which are perpendicular to vector $\boldsymbol{x}$

$$N_C(\boldsymbol{x}) = \left\{\boldsymbol{y} : \boldsymbol{y}^T(\boldsymbol{x}^* - \boldsymbol{x}) \leq 0 \;\; ; \;\; \boldsymbol{x} \in C \;\; ; \;\; \forall \boldsymbol{x}^* \in C\right\} \tag{49}$$

  is called the *normal cone* of C in $\boldsymbol{x}$. The equivalent relation

$$\boldsymbol{y} \in N_C(\boldsymbol{x}) \;\; \Leftrightarrow \;\; \boldsymbol{y}^T(\boldsymbol{x}^* - \boldsymbol{x}) \leq 0 \;\; ; \;\; \boldsymbol{x} \in C \;\; ; \;\; \forall \boldsymbol{x}^* \in C \tag{50}$$

  has the same value.

B.2. Convex Functions

- Definition of the epigraph:
  Let $f(\boldsymbol{x})$ be a real-valued function on the domain $S \subset \mathbb{R}^n$. The set

$$\text{epi } f = \{(\boldsymbol{x}, \mu) \; : \; \boldsymbol{x} \in S \; ; \; \mu \geq f(\boldsymbol{x}) \; ; \; \mu \in \mathbb{R}\} \tag{51}$$

  is called the *epigraph* of f.

- Definition of the convex function:
  The function $f(\boldsymbol{x})$ is called a *convex function* on $S \subset \mathbb{R}^n$ if epi $f \subset \mathbb{R}^{n+1}$ is a convex set.

- Definition of the indicator function:
  Let $C \subset \mathbb{R}^n$ be a convex set. The function

$$\Psi_C(\boldsymbol{x}) = \begin{cases} 0 & \text{if } \boldsymbol{x} \in C \\ +\infty & \text{if } \boldsymbol{x} \notin C \ . \end{cases} \tag{52}$$

  is called the *indicator function* of the set C. $\Psi_C(\boldsymbol{x})$ is a convex function.

B.3. Subgradient and Subdifferential

- Definition of the subgradient:
  Let $f(\boldsymbol{x}^*)$ be a convex function. A vector $\boldsymbol{y}$ is called the *subgradient* of f in $\boldsymbol{x}$ if

$$f(\boldsymbol{x}^*) \geq f(\boldsymbol{x}) + \boldsymbol{y}^T(\boldsymbol{x}^* - \boldsymbol{x}) \ ; \ \forall \boldsymbol{x}^*. \tag{53}$$

- Definition of the subdifferential:
  Let $f(\boldsymbol{x}^*)$ be convex function. The set of all subgradients $\boldsymbol{y}$ of f in $\boldsymbol{x}$

$$\partial f(\boldsymbol{x}) = \left\{ \boldsymbol{y} : f(\boldsymbol{x}^*) \geq f(\boldsymbol{x}) + \boldsymbol{y}^T(\boldsymbol{x}^* - \boldsymbol{x}) \ ; \ \forall \boldsymbol{x}^* \right\} \tag{54}$$

  is called the *subdifferential* of f in $\boldsymbol{x}$. The equivalent relation

$$\boldsymbol{y} \in \partial f(\boldsymbol{x}) \quad \Leftrightarrow f(\boldsymbol{x}^*) \geq f(\boldsymbol{x}) + \boldsymbol{y}^T(\boldsymbol{x}^* - \boldsymbol{x}) \ ; \ \forall \boldsymbol{x}^* \tag{55}$$

  has the same value.

- Subdifferential of the indicator function:
  With the definition of the subdifferential (54), the subdifferential of the indicator function (52) is

$$\partial \Psi_C(\boldsymbol{x}) = \left\{ \boldsymbol{y} : 0 \geq \boldsymbol{y}^T(\boldsymbol{x}^* - \boldsymbol{x}) \ ; \ \boldsymbol{x} \in C \ ; \ \forall \boldsymbol{x}^* \in C \right\} . \tag{56}$$

  The comparison with the definition of the normal cone (49) shows the equality

$$\partial \Psi_C(\boldsymbol{x}) = N_C(\boldsymbol{x}). \tag{57}$$

B.4. Minimization of Convex Functions

- Definition of the minimization problem:
  A convex function $f_0(\boldsymbol{x})$ over the convex set C has to be minimized.

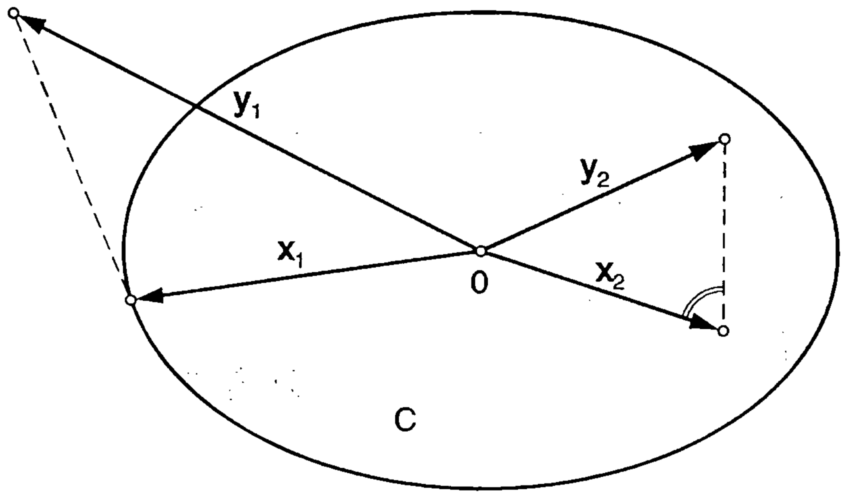

Figure 9. Projection onto convex sets.

- Definition of the objective function:
  The convex function

$$f(\boldsymbol{x}) = f_0(\boldsymbol{x}) + \Psi_C(\boldsymbol{x}) \ ; \ \boldsymbol{x} \in \mathbb{R}^n \tag{58}$$

  is called the *objective function* of the convex minimization problem. The infimum $\inf_{\boldsymbol{x}} f(\boldsymbol{x})$ is called the *optimal value* and the points $\overline{\boldsymbol{x}}$ at which f reaches its infimum are called the *optimal solutions* of the minimization problem.

- Definition of the projection:
  Let $\boldsymbol{y} \in \mathbb{R}^n$ be an arbitrary point and $C \subset \mathbb{R}^n$ a convex set. The projection of $\boldsymbol{y}$ onto C is that point $\boldsymbol{x} \in C$ which has the smallest distance to $\boldsymbol{y}$. With definition (58) the objective function of the minimization problem is

$$f(\boldsymbol{x}) = \frac{1}{2} |\boldsymbol{x} - \boldsymbol{y}|^2 + \Psi_C(\boldsymbol{x}).$$

  A necessary and sufficient condition for an infimum of $f(\boldsymbol{x})$ is

$$\begin{aligned} \mathbf{0} \in \partial f(\boldsymbol{x}) \quad &\Rightarrow \quad \mathbf{0} \in (\boldsymbol{x} - \boldsymbol{y}) + N_C(\boldsymbol{x}) \\ &\Rightarrow \quad \boldsymbol{y} - \boldsymbol{x} \in N_C(\boldsymbol{x}). \end{aligned}$$

  With definition (50) the equivalent relation for the projection is

$$(\boldsymbol{y} - \boldsymbol{x})^T(\boldsymbol{x}^* - \boldsymbol{x}) \leq 0 \ ; \ \boldsymbol{x} \in C \ ; \ \forall \boldsymbol{x}^* \in C \quad \Leftrightarrow \quad \boldsymbol{x} = \mathrm{proj}_C(\boldsymbol{y}). \tag{59}$$

  Figure 9 shows the projections of two different points $\boldsymbol{y}$.

C. DATA SET FOR THE EXAMPLE

C.1. Model Parameters

mass	m	=	663.0 g
moment of inertia	A	=	$0.85 \cdot 10^{-3} kgm^2$
moment of inertia	B	=	$1.6 \cdot 10^{-3} kgm^2$
moment of inertia	C	=	$1.75 \cdot 10^{-3} kgm^2$
acceleration due to gravity	g	=	$9.81\, m/s^2$
coefficient of sliding friction	μ	=	0.098
coefficient of static friction	μ_0	=	0.098
distance	a_1	=	42.4 mm
distance	a_2	=	49.6 mm
amplitude of distance change	Δa	=	3.5 mm
distance	b_1	=	1.0 mm
distance	b_2	=	25.0 mm
distance	b_3	=	31.0 mm
distance	h_S	=	37.4 mm
amplitude of distance change	Δh	=	4.5 mm
distance	l_1	=	10.4 mm
distance	l_2	=	43.0 mm
distance	l_3	=	16.5 mm
angle of inclination	δ	=	0.115 rad
unbalanced mass	m_E	=	55.0 g
eccentricity	r_E	=	11.0 mm
angular frequency of unbalanced mass	Ω	=	19.6 rad/s

C.2. Initial Conditions

The numerical integration is started with the initial values of the generalized coordinates and velocities

$$\boldsymbol{q}_0 = \boldsymbol{q}(t=0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \dot{\boldsymbol{q}}_0 = \dot{\boldsymbol{q}}(t=0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

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