Bahadur–Kiefer Representations for Time Dependent Quantile Processes

Péter Kevei*
Center for Mathematical Sciences, Technische Universität München
Boltzmannstraße 3, 85748 Garching, Germany, and
MTA-SZTE Analysis and Stochastics Research Group
Bolyai Institute, Aradi vértanúk tere 1, 6720 Szeged, Hungary
e-mail: peter.kevei@tum.de

David M. Mason
Department of Applied Economics and Statistics, University of Delaware
213 Townsend Hall, Newark, DE 19716, USA
e-mail: davidm@udel.edu

Abstract

We define a time dependent empirical process based on n independent fractional Brownian motions and describe strong approximations to it by Gaussian processes. They lead to strong approximations and functional laws of the iterated logarithm for the quantile or inverse of this empirical process. They are obtained via time dependent Bahadur–Kiefer representations.

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1 Introduction

Swanson [13] using classical weak convergence theory proved that an appropriately scaled median of n independent Brownian motions converges weakly to a mean zero Gaussian process. More recently Kuelbs and Zinn [9], [10] have obtained central limit theorems for a time dependent quantile process based on n independent copies of a wide variety of random processes, which may be zero or perturbed to be not zero with probability 1 [w.p.1] at zero. These include certain self-similar processes of which fractional Brownian motion is a special case. Their approach is based on an extension of a result of Vervaat [16] on the weak convergence of inverse processes in combination with results from their deep study with Kurtz [Kurtz, Kuelbs and Zinn [8]] of central limit theorems for time dependent empirical processes.

We shall begin by defining a time dependent empirical process based on n independent fractional Brownian motions and describe a strong approximations to it recently obtained by Kevei and Mason [5]. We shall see that they lead to strong approximations and functional laws of the

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iterated logarithm for the quantile or inverse of these empirical processes and are obtained via time dependent Bahadur–Kiefer representations.

1.1 Swanson (2007) result

Our work is motivated by the following result of Swanson [13].

Let $\{B_j^{(1/2)}\}_{j\geq 1}$ be a sequence of i.i.d. standard Brownian motions and for each $n \geq 1$ and $t \geq 0$ let $M_n(t)$ denote the median of $B_1^{(1/2)}(t), \ldots, B_n^{(1/2)}(t)$. Swanson [13] using classical weak convergence theory proved that $\sqrt{n}M_n(t)$ converges weakly to a continuous centered Gaussian process X on $[0, \infty)$ with covariance function defined for $t_1, t_2 \in [0, \infty)$ by

$$E(X(t_1)X(t_2)) = \sqrt{t_1 t_2}\sin^{-1}\left(\frac{t_1 \wedge t_2}{\sqrt{t_1 t_2}}\right).$$

For a random particle motivation to look at such problems consult the Introduction in [13], where possible fractional Brownian motion generalizations are hinted at.

One of the aims of this paper is to place this result within the framework of what has been long known about the usual empirical and quantile processes.

1.2 Some classical quantile process lore

To put our study into a broader context, we recall here some classical quantile process lore. Let $X_1, X_2, \ldots,$ be i.i.d. F. For $\alpha \in (0,1)$ define the inverse or quantile function $Q(\alpha) = \inf\{x : F(x) \geq \alpha\}$ and the empirical quantile function $Q_n(\alpha) = \inf\{x : F_n(x) \geq \alpha\}$, where

$$F_n(x) = n^{-1}\sum_{j=1}^{n} 1\{X_j \leq x\}, \ x \in \mathbb{R},$$

is the empirical distribution function based on $X_1, \ldots, X_n$.

We define the empirical process

$$v_n(x) := \sqrt{n}\{F_n(x) - F(x)\}, \ x \in \mathbb{R},$$

and the quantile process

$$u_n(t) := \sqrt{n}\{Q_n(t) - Q(t)\}, \ t \in (0,1).$$

For a real-valued function Υ defined on a set S we shall use the notation

$$\|\Upsilon\|_S = \sup_{s\in S}|\Upsilon(s)|. \tag{1}$$

The empirical and quantile processes are closely connected to each other through the following Bahadur–Kiefer representation:

Let $X_1, X_2, \ldots,$ be i.i.d. F on $[0,1]$, where F is twice differentiable on $(0,1)$, $f(x) = F'(x)$, with

$$\inf_{x\in(0,1)} f(x) > 0 \text{ and } \sup_{x\in(0,1)}\left|F''(x)\right| < \infty.$$

We have (Kiefer [6]) the Bahadur–Kiefer representation

$$\limsup_{n\to\infty} \frac{n^{1/4}\,\|v_n(Q)+f(Q)u_n\|_{(0,1)}}{\sqrt[4]{\log\log n}\sqrt{\log n}} = \frac{1}{\sqrt[4]{2}}, \text{ a.s.} \tag{2}$$

The "Bahadur" is in reference to the original Bahadur [1] paper, where a less precise version of (2) was first established. The function $f(Q)$ is called the density quantile function. Deheuvels and Mason [4] developed a general approach to such theorems. For corresponding L^p versions of such results we refer to Csörgő and Shi [2].

Next using a strong approximation result of Komlós, Major and Tusnády [7] one has on the same probability space an i.i.d. F sequence $X_1, X_2, \ldots$, and a sequence of i.i.d. Brownian bridges $U_1, U_2, \ldots$, on $[0,1]$ such that

$$\left\| v_n\left(Q\right) - \frac{\sum_{j=1}^n U_j}{\sqrt{n}} \right\|_{(0,1)} = O\left(\frac{(\log n)^2}{\sqrt{n}}\right), \text{ a.s.} \tag{3}$$

Using (3) it is easy see that under the conditions for which the above the Bahadur–Kiefer representation (2) holds

$$\limsup_{n\to\infty} \frac{n^{1/4}\left\| \frac{\sum_{j=1}^n U_j}{\sqrt{n}} + f\left(Q\right)u_n \right\|_{(0,1)}}{\sqrt[4]{\log\log n}\sqrt{\log n}} = \frac{1}{\sqrt[4]{2}}, \text{ a.s.}$$

Deheuvels [3] has shown that this rate of strong approximation rate cannot be improved.

We shall develop analogues of these classical results for time dependent empirical and quantile processes based on independent copies of fractional Brownian motion. In particular, we shall extend the Swanson setup to fractional Brownian motion, which will put his result in a broader context.

2 A time dependent empirical process

In this section we recall some needed notation from [5]. Let $\left\{B^{(H)}\right\} \cup \left\{B_j^{(H)}\right\}_{j\geq 1}$ be a sequence of i.i.d. sample continuous fractional Brownian motions with Hurst index $0 < H < 1$ defined on $[0,\infty)$. Note that $B^{(H)}$ is a continuous mean zero Gaussian process on $[0,\infty)$ with covariance function defined for any $s,t \in [0,\infty)$

$$E\left(B^{(H)}(s)B^{(H)}(t)\right) = \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |s-t|^{2H}\right).$$

By the Lévy modulus of continuity theorem for sample continuous fractional Brownian motion $B^{(H)}$ with Hurst index $0 < H < 1$, (see Corollary 1.1 of [17]), we have for any $0 < T < \infty$, w.p. 1,

$$\sup_{0\leq s\leq t\leq T} \frac{\left|B^{(H)}\left(t\right) - B^{(H)}\left(s\right)\right|}{f_H(t-s)} =: L < \infty, \tag{4}$$

where for $u \geq 0$

$$f_H(u) = u^H\sqrt{1 \vee \log u^{-1}} \tag{5}$$

and $a \vee b = \max\{a, b\}$. We shall take versions of $\left\{B^{(H)}\right\} \cup \left\{B_j^{(H)}\right\}_{j\geq 1}$ such that (4) holds for all of their trajectories.

For any $t \in [0, \infty)$ and $x \in \mathbb{R}$ let $F(t, x) = P\left\{B^{(H)}(t) \leq x\right\}$. Note that

$$F(t, x) = \Phi\left(x/t^H\right), \tag{6}$$

where $\Phi(x) = P\{Z \leq x\}$, with Z being a standard normal random variable. For any $n \geq 1$ define the time dependent *empirical distribution function*

$$F_n(t, x) = n^{-1} \sum_{j=1}^{n} 1\left\{B_j^{(H)}(t) \leq x\right\}.$$

Applying Theorem 5 in [8] (also see their Remark 8) one can show for any choice of $0 < \gamma \leq 1 < T < \infty$ that the time dependent *empirical process* indexed by $(t, x) \in \mathcal{T}(\gamma)$,

$$v_n(t, x) = \sqrt{n}\left\{F_n(t, x) - F(t, x)\right\},$$

where

$$\mathcal{T}(\gamma) := [\gamma, T] \times \mathbb{R},$$

converges weakly to a uniformly continuous centered Gaussian process $G(t, x)$ indexed by $(t, x) \in \mathcal{T}(\gamma)$, whose trajectories are bounded, having covariance function

$$E(G(s, x) G(t, y)) = P\left\{B^{(H)}(s) \leq x, B^{(H)}(t) \leq y\right\} - P\left\{B^{(H)}(s) \leq x\right\} P\left\{B^{(H)}(t) \leq y\right\}. \tag{7}$$

Here we restrict ourselves in stating this weak convergence result to positive γ, since as pointed out in Section 8.1 of [8] the empirical process $v_n(t, x)$ indexed by $\mathcal{T}(0) := [0, T] \times \mathbb{R}$ does not converge weakly to a uniformly continuous centered Gaussian process indexed by $(t, x) \in \mathcal{T}(0)$, whose trajectories are bounded. In the sequel, $G(t, x)$ denotes a centered Gaussian process on $\mathcal{T}(0)$ with covariance (7) that is uniformly continuous on $\mathcal{T}(\gamma)$ with bounded trajectories for any $0 < \gamma \leq 1 < T < \infty$.

We shall also be using the following empirical process indexed by function notation. Let $X, X_1, X_2, \ldots,$ be i.i.d. random variables from a probability space $(\Omega, \mathcal{A}, P)$ to a measurable space $(S, \mathcal{S})$. Consider an empirical process indexed by a class $\mathcal{G}$ of bounded measurable real valued functions on $(S, \mathcal{S})$ defined by

$$\alpha_n(\varphi) := \sqrt{n}(P_n - P)\varphi = \frac{\sum_{i=1}^{n} \varphi(X_i) - nE\varphi(X)}{\sqrt{n}}, \quad \varphi \in \mathcal{G},$$

where

$$P_n(\varphi) = n^{-1} \sum_{i=1}^{n} \varphi(X_i) \text{ and } P(\varphi) = E\varphi(X).$$

Keeping this notation in mind, let $\mathcal{C}[0, T]$ be the class of continuous functions g on $[0, T]$ endowed with the topology of uniform convergence. Define the subclass of $\mathcal{C}[0, T]$

$$\mathcal{C}_\infty := \left\{g : \sup\left\{\frac{|g(s) - g(t)|}{f_H(|s - t|)},\ 0 \leq s, t \leq T\right\} < \infty\right\}.$$

Further, let $\mathcal{F}_{(\gamma,T)}$ be the class of functions of $g \in \mathcal{C}[0,T] \to \mathbb{R}$, indexed by $(t,x) \in \mathcal{T}(\gamma)$, of the form

$$h_{t,x}(g) = 1\{g(t) \le x, g \in \mathcal{C}_\infty\}.$$

Here we permit $\gamma = 0$. Since by (4) we can assume that each $B^{(H)}$, $B_j^{(H)}$, $j \ge 1$, is in $\mathcal{C}_\infty$, we see that for any $h_{t,x} \in \mathcal{F}_{(\gamma,T)}$,

$$\alpha_n(h_{t,x}) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(1\left\{B_i^{(H)}(t) \le x\right\} - P\left\{B^{(H)}(t) \le x\right\}\right) = v_n(t,x). \tag{8}$$

We shall be using the notation $\alpha_n(h_{t,x})$ and $v_n(t,x)$ interchangeably.

Let $\mathbb{G}_{(\gamma,T)}$ denote the mean zero Gaussian process indexed by $\mathcal{F}_{(\gamma,T)}$, having covariance function defined for $h_{s,x}, h_{t,y} \in \mathcal{F}_{(\gamma,T)}$

$$E\left(\mathbb{G}_{(\gamma,T)}(h_{s,x})\,\mathbb{G}_{(\gamma,T)}(h_{t,y})\right) = P\left\{B^{(H)}(s) \le x, B^{(H)}(t) \le y, B^{(H)} \in \mathcal{C}_\infty\right\}$$

$$-P\left\{B^{(H)}(s) \le x, B^{(H)} \in \mathcal{C}_\infty\right\} P\left\{B^{(H)}(t) \le y, B^{(H)} \in \mathcal{C}_\infty\right\},$$

which since $P\left\{B^{(H)} \in \mathcal{C}_\infty\right\} = 1$,

$$= E\left(G(s,x)\,G(t,y)\right),$$

i.e. $\mathbb{G}_{(\gamma,T)}(h_{t,x})$ defines a probabilistically equivalent version of the Gaussian process $G(t,x)$ for $(t,x) \in \mathcal{T}(\gamma)$. We shall say that a process $\widetilde{\mathcal{Y}}$ is a *probabilistically equivalent version* of $\mathcal{Y}$ if $\widetilde{\mathcal{Y}} \stackrel{\mathrm{D}}{=} \mathcal{Y}$.

2.1 The Kevei and Mason (2016) strong approximation results for α_n

For future reference we record here two strong approximations for α_n that were recently established by Kevei and Mason [5]. In the results that follow

$$\nu_0 = 2 + \frac{2}{H} \quad \text{and} \quad H_0 = 1 + H. \tag{9}$$

The main results in [5] are the following two strong approximation theorems.

Theorem 1. *([5]) For any $1 \ge \gamma > 0$, for all $1/(2\tau_1(0)) < \alpha < 1/\tau_1(0)$ and $\xi > 1$ there exist a $\rho(\alpha,\xi) > 0$, a sequence of i.i.d. $B_1^{(H)}, B_2^{(H)}, \ldots$, and a sequence of independent copies $\mathbb{G}_{(\gamma,T)}^{(1)}, \mathbb{G}_{(\gamma,T)}^{(2)}, \ldots$, of $\mathbb{G}_{(\gamma,T)}$ sitting on the same probability space such that*

$$\max_{1 \le m \le n} \left\| \sqrt{m}\alpha_m - \sum_{i=1}^{m} \mathbb{G}_{(\gamma,T)}^{(i)} \right\|_{\mathcal{F}_{(\gamma,T)}} = O\left(n^{1/2-\tau(\alpha)} (\log n)^{\tau_2}\right), \ \text{a.s.}, \tag{10}$$

where $\tau(\alpha) = (\alpha\tau_1(0) - 1/2)/(1+\alpha) > 0$, $\tau_1(0) = 1/(2+5\nu_0)$, $\tau_2 = (19H+25)/(24H+20)$ and ν_0 is defined in (9).

For any $\kappa > 0$ let

$$\mathcal{G}(\kappa) = \{t^{\kappa} h_{t,x} : (t,x) \in [0,T] \times \mathbb{R}\}.$$

For $g \in \mathcal{G}(\kappa)$, with some abuse of notation, we shall write

$$\mathbb{G}_{(0,T)}(g) = t^{\kappa} \mathbb{G}_{(0,T)}(h_{t,x}).$$

Also, in analogy with (1), in the following theorem,

$$\left\| \sqrt{m}\alpha_m - \sum_{i=1}^{m} \mathbb{G}_{(0,T)}^{(i)} \right\|_{\mathcal{G}(\kappa)} := \sup \left\{ \left| t^{\kappa} \alpha_n(h_{t,x}) - t^{\kappa} \sum_{i=1}^{m} \mathbb{G}_{(0,T)}^{(i)}(h_{t,x}) \right| : (t,x) \in [0,T] \times \mathbb{R} \right\}.$$

Theorem 2. *([5]) For any $\kappa > 0$, for all $1/(2\tau_1') < \alpha < 1/\tau_1'$, and $\xi > 1$ there exist a $\rho'(\alpha, \xi) > 0$, a sequence of i.i.d. $B_1^{(H)}, B_2^{(H)}, \ldots,$ and a sequence of independent copies $\mathbb{G}_{(0,T)}^{(1)}, \mathbb{G}_{(0,T)}^{(2)}, \ldots,$ of $\mathbb{G}_{(0,T)}$ sitting on the same probability space such that*

$$\max_{1 \le m \le n} \left\| \sqrt{m}\alpha_m - \sum_{i=1}^{m} \mathbb{G}_{(0,T)}^{(i)} \right\|_{\mathcal{G}(\kappa)} = O\left(n^{1/2 - \tau'(\alpha)} (\log n)^{\tau_2} \right), \text{ a.s.,} \tag{11}$$

where $\tau'(\alpha) = (\alpha\tau_1' - 1/2)/(1+\alpha) > 0$ and $\tau_1' = \tau_1'(\kappa) = \kappa/(5H_0 + \kappa(2 + 5\nu_0))$.

Notice that (10) and (11) trivially imply that for some $1/2 > \xi > 0$

$$\max_{1 \le m \le n} \left\| \sqrt{m}\alpha_m - \sum_{i=1}^{m} \mathbb{G}_{(\gamma,T)}^{(i)} \right\|_{\mathcal{F}_{(\gamma,T)}} = O\left(n^{-\xi}\right), \quad \text{a.s.,}$$

and

$$\max_{1 \le m \le n} \left\| \sqrt{m}\alpha_m - \sum_{i=1}^{m} \mathbb{G}_{(0,T)}^{(i)} \right\|_{\mathcal{G}(\kappa)} = O\left(n^{-\xi}\right), \quad \text{a.s.}$$

2.2 Applications to LIL

Kevei and Mason [5] point out that the following compact law of the iterated logarithm (LIL) for α_n follows from their Theorem 1, namely

$$\left\{ \frac{\alpha_n(h_{t,x})}{\sqrt{2\log\log n}} : h_{t,x} \in \mathcal{F}_{(\gamma,T)} \right\} = \left\{ \frac{v_n(t,x)}{\sqrt{2\log\log n}} : (t,x) \in \mathcal{T}(\gamma) \right\} \tag{12}$$

is, w.p. 1, relatively compact in $\ell_\infty\left(\mathcal{F}_{(\gamma,T)}\right)$ (the space of bounded functions Υ on $\mathcal{F}_{(\gamma,T)}$ equipped with supremum norm $\|\Upsilon\|_{\mathcal{F}_{(\gamma,T)}} = \sup_{\varphi \in \mathcal{F}_{(\gamma,T)}} |\Upsilon(\varphi)|$) and its limit set is the unit ball of the reproducing kernel Hilbert space determined by the covariance function $E(\mathbb{G}_{(\gamma,T)}(h_{s,x})\mathbb{G}_{(\gamma,T)}(h_{t,y}))$ $= E(G(s,x)G(t,y))$. In particular we get that

$$\limsup_{n\to\infty} \frac{\|\alpha_n\|_{\mathcal{F}_{(\gamma,T)}}}{\sqrt{2\log\log n}} = \limsup_{n\to\infty} \sup_{(t,x)\in\mathcal{T}(\gamma)} \left| \frac{v_n(t,x)}{\sqrt{2\log\log n}} \right| = \sigma(\gamma,T), \text{ a.s.}$$

where

$$\sigma^2(\gamma,T) = \sup\left\{ E\left(\mathbb{G}_{(\gamma,T)}^2(h_{t,x})\right) : h_{t,x} \in \mathcal{F}_{(\gamma,T)} \right\} = \frac{1}{4}.$$

Furthermore, they derive from their Theorem 2 the following compact LIL, for all $0<\kappa<\infty$,

$$\left\{\frac{t^{\kappa}\alpha_n\left(h_{t,x}\right)}{\sqrt{2\log\log n}}:h_{t,x}\in\mathcal{F}_{(0,T)}\right\}=\left\{\frac{t^{\kappa}v_n\left(t,x\right)}{\sqrt{2\log\log n}}:(t,x)\in[0,T]\times\mathbb{R}\right\} \tag{13}$$

is, w.p. 1, relatively compact in $\ell_\infty\left(\mathcal{G}\left(\kappa\right)\right)$ and its limit set is the unit ball of the reproducing kernel Hilbert space determined by the covariance function $E\left(s^{\kappa}t^{\kappa}\mathbb{G}_{(\gamma,T)}\left(h_{s,x}\right)\mathbb{G}_{(\gamma,T)}\left(h_{t,y}\right)\right)=E\left(s^{\kappa}t^{\kappa}G\left(s,x\right)G\left(t,y\right)\right)$. This implies that

$$\limsup_{n\to\infty}\frac{\|\alpha_n\|_{\mathcal{G}(\kappa)}}{\sqrt{2\log\log n}}=\limsup_{n\to\infty}\sup_{(t,x)\in[0,T]\times\mathbb{R}}\left|\frac{t^{\kappa}v_n\left(t,x\right)}{\sqrt{2\log\log n}}\right|=\sigma_\kappa\left(T\right),\text{ a.s.,} \tag{14}$$

where

$$\sigma_\kappa^2\left(T\right)=\sup\left\{E\left(\mathbb{G}_{(0,T)}^2\left(t^{\kappa}h_{t,x}\right)\right):t^{\kappa}h_{t,x}\in\mathcal{G}\left(\kappa\right)\right\}=\frac{T^{2\kappa}}{4}. \tag{15}$$

3 Bahadur–Kiefer representations and strong approximations for time dependent quantile processes

3.1 A time dependent quantile process

For each $t\in(0,\infty)$ and $\alpha\in(0,1)$ define the time dependent *inverse* or *quantile function*

$$\tau_\alpha(t)=\inf\left\{x:F\left(t,x\right)\geq\alpha\right\},$$

and the time dependent *empirical inverse* or *empirical quantile function*

$$\tau_\alpha^n(t)=\inf\left\{x:F_n\left(t,x\right)\geq\alpha\right\}, \tag{16}$$

and the corresponding time dependent *quantile process*

$$u_n(t,\alpha):=\sqrt{n}\left(\tau_\alpha^n(t)-\tau_\alpha(t)\right).$$

Notice that by (6), for each fixed $t>0$, $F\left(t,x\right)$ has density

$$f\left(t,x\right)=\frac{1}{t^H\sqrt{2\pi}}\exp\left(-\frac{x^2}{2t^{2H}}\right),\ -\infty<x<\infty.$$

Further, for each $t\in(0,\infty)$ and $\alpha\in(0,1)$, $\tau_\alpha\left(t\right)$ is uniquely defined by

$$\tau_\alpha\left(t\right)=t^Hz_\alpha,\ \text{where } P\left\{Z\leq z_\alpha\right\}=\alpha, \tag{17}$$

which says that $f\left(t,\tau_\alpha\left(t\right)\right)=\frac{1}{t^H\sqrt{2\pi}}\exp\left(-\frac{z_\alpha^2}{2}\right)$.

3.2 Our results for time dependent quantile processes

We shall prove the following uniform time dependent Bahadur–Kiefer representations for the quantile process $u_n\left(t,\alpha\right)$. We shall see that one easily infers from them LIL and strong approximations for such processes.

Introduce the condition on a sequence of constants $0<\gamma_n\leq 1$

$$\infty>-\frac{\log\gamma_n}{\log n}\to\eta,\quad\text{as } n\to\infty. \tag{18}$$

Theorem 3. *Whenever $0 < \gamma = \gamma_n \leq 1$ satisfies (18) for some $0 \leq \eta < 1/(2H)$, then for any $0 < \rho < 1/2$ and $T > 1$*

$$\begin{aligned} &\sup_{(t,\alpha)\in[\gamma_n,T]\times[\rho,1-\rho]} |v_n(t,\tau_\alpha(t)) + f(t,\tau_\alpha(t))u_n(t,\alpha)| \\ &= O\left(n^{-1/4}\gamma_n^{-H/2}(\log\log n)^{1/4}(\log n)^{1/2}\right), \ \ a.s. \end{aligned} \tag{19}$$

Remark 1. It is noteworthy here to point out that when $\gamma_n = \gamma$ is constant, the rate in (19) corresponds to the known exact rate in (2) in the classic uniform Bahadur–Kiefer representation of sample quantiles. Refer to Deheuvels and Mason [4] for more results in this direction.

Remark 2. Note that in (19) smaller δ implies better rate, so it is enough to prove the statement for $0 < \delta$ small enough. Furthermore, if $\gamma_n \to 0$ and δ large it can happen that the rate in (19) tends to infinity. Since (18) holds with $\eta < 1/(2H)$, γ_n cannot tend to zero too fast. In the borderline case (which is not allowed in the statement of the theorem) $\gamma_n = n^{-1/(2H)}$, the rate would go to infinity.

Remark 3. Let $\ell_\infty([\gamma,T]\times[\rho,1-\rho])$ denote the class of bounded functions on $[\gamma,T]\times[\rho,1-\rho]$. Notice when $0 < \gamma \leq 1$ is fixed, we immediately get from (12) and (19) that

$$\left\{\frac{f(t,\tau_\alpha(t))\,u_n(t,\alpha)}{\sqrt{2\log\log n}} : (t,\alpha)\in[\gamma,T]\times[\rho,1-\rho]\right\}$$

is, w.p. 1, relatively compact in $\ell_\infty([\gamma,T]\times[\rho,1-\rho])$ and its limit set is the unit ball of the reproducing kernel Hilbert space determined by the covariance function defined for (t_1,α_1), $(t_2,\alpha_2)\in[\gamma,T]\times[\rho,1-\rho]$ by

$$\begin{aligned} K((t_1,\alpha_1),(t_2,\alpha_2)) &= E\left(G(t_1,\tau_{\alpha_1}(t_1))\,G(t_2,\tau_{\alpha_2}(t_2))\right) \\ &= P\left\{B^{(H)}(t_1) \leq t_1^H z_{\alpha_1}, B^{(H)}(t_2) \leq t_2^H z_{\alpha_2}\right\} - \alpha_1\alpha_2. \end{aligned}$$

Also we get when $0 < \gamma \leq 1$ is fixed the following strong approximation, namely on the probability space of Theorem 1,

$$\sup_{(t,\alpha)\in[\gamma,T]\times[\rho,1-\rho]}\left|\sqrt{n}\,f(t,\tau_\alpha(t))\,u_n(t,\alpha) + \sum_{i=1}^n G_i(t,\tau_\alpha(t))\right| = O\left(n^{1/2-\tau(\alpha)}(\log n)^{\tau_2}\right), \text{ a.s.,}$$

where $G_i(t,\tau_\alpha(t)) = \mathbb{G}^{(i)}_{(\gamma,T)}(h_{t,\tau_\alpha(t)})$. This follows from Theorems 1 and 3 by noting $\tau(\alpha) < 1/4$.

Corollary 1. *For any $0 < \rho < 1/2$, $T > 1$ and $\delta > 0$ we have*

$$\sup_{(t,\alpha)\in[0,T]\times[\rho,1-\rho]}\left|t^H v_n(t,\tau_\alpha(t)) + \frac{\exp\left(-\frac{z_\alpha^2}{2}\right)}{\sqrt{2\pi}}u_n(t,\alpha)\right| = O\left(n^{-1/6+\delta}\right), \ \ a.s. \tag{20}$$

Remark 4. Let $\ell_\infty([0,T]\times[\rho,1-\rho])$ denote the class of bounded functions on $[0,T]\times[\rho,1-\rho]$. Observe that (20) combined with the compact LIL pointed out in (13), immediately imply that

$$\left\{\frac{\exp\left(-\frac{z_\alpha^2}{2}\right)u_n(t,\alpha)}{\sqrt{2\pi}\sqrt{2\log\log n}} : (t,\alpha)\in[0,T]\times[\rho,1-\rho]\right\}$$

is, w.p. 1, relatively compact in $\ell_\infty([0,T]\times[\rho,1-\rho])$ and its limit set is the unit ball the reproducing kernel Hilbert space determined by the covariance function defined for $(t_1,\alpha_1),(t_2,\alpha_2)\in[0,T]\times[\rho,1-\rho]$ by

$$\begin{aligned}K\left((t_1,\alpha_1),(t_2,\alpha_2)\right)&=t_1^H t_2^H E\left(G(t_1,\tau_{\alpha_1}(t_1))G(t_2,\tau_{\alpha_2}(t_2))\right)\\&=t_1^H t_2^H\left(P\left\{B^{(H)}(t_1)\le t_1^H z_{\alpha_1},B^{(H)}(t_2)\le t_2^H z_{\alpha_2}\right\}-\alpha_1\alpha_2\right).\end{aligned}$$

We also get the following strong approximation, namely on the probability space of Theorem 2 with $\kappa=H$, for some $1/2>\xi>0$

$$\sup_{(t,\alpha)\in[0,T]\times[\rho,1-\rho]}\left|\frac{\exp\left(-\frac{z_\alpha^2}{2}\right)u_n(t,\alpha)}{\sqrt{2\pi}}+\frac{1}{\sqrt{n}}\sum_{i=1}^n t^H G_i(t,\tau_\alpha(t))\right|=O\left(n^{-\xi}\right),\ \text{a.s.},\tag{21}$$

where $G_i(t,\tau_\alpha(t))=\mathbb{G}^{(i)}_{(0,T)}\left(h_{t,\tau_\alpha(t)}\right)$. This follows from (11), noting that $\tau'(\alpha)>0$, combined with (20).

Remark 5. Let $\ell_\infty([0,T])$ denote the class of bounded functions on $[0,T]$. Applying the compact LIL pointed out in the previous remark with $H=1/2$, to the median process considered by Swanson [13], i.e.

$$\sqrt{n}M_n(t)=u_n(t,1/2)=\sqrt{n}\tau^n_{1/2}(t),\ t\ge 0,$$

we get for any $T>0$, that

$$\left\{\frac{\sqrt{n}M_n(t)}{\sqrt{2\log\log n}}:t\in[0,T]\right\}$$

is, w.p. 1, relatively compact in $\ell_\infty([0,T])$, and its limit set is the unit ball of the reproducing kernel Hilbert space determined by the covariance function defined for $t_1,t_2\in[0,T]$

$$\begin{aligned}2\pi K(t_1,t_2)&=2\pi\sqrt{t_1t_2}E\left(G(t_1,0)G(t_2,0)\right)\\&=2\pi\sqrt{t_1t_2}\left(P\left\{B^{(1/2)}(t_1)\le 0,B^{(1/2)}(t_2)\le 0\right\}-1/4\right),\end{aligned}$$

which equals

$$\sqrt{t_1t_2}\sin^{-1}\left(\frac{t_1\wedge t_2}{\sqrt{t_1t_2}}\right).\tag{22}$$

In particular we get

$$\limsup_{n\to\infty}\frac{\|\sqrt{n}M_n\|_{[0,T]}}{\sqrt{2\log\log n}}=\sqrt{T\sin^{-1}(1)}=\sqrt{T\pi/2},\ \text{a.s.}$$

Moreover, since a mean zero Gaussian process $X(t)$, $t\ge 0$, with covariance function (22) is equal in distribution to $-\sqrt{2\pi t}G(t,0)$, $t\ge 0$, we see from (21) that there exist a sequence $B_1^{(1/2)},B_2^{(1/2)},\dots$, i.i.d. $B^{(1/2)}$ and a sequence of processes $X^{(1)},X^{(2)},\dots$, i.i.d. X sitting on the same probability space such that, a.s.

$$\left\|\sqrt{n}M_n-\frac{1}{\sqrt{n}}\sum_{i=1}^n X^{(i)}\right\|_{[0,T]}=o(1).$$

Of course, this implies the Swanson result that $\sqrt{n}M_n$ converges weakly on $[0,T]$ to the process X.

4 Proofs of Theorem 3 and Corollary 1

To ease the notation we suppress the upper index from the fractional Brownian motions, that is, in the following $B, B_1, B_2, \ldots$ are i.i.d. fractional Brownian motions with Hurst index H.

4.1 Proof of Theorem 3

Before we can prove Theorem 3 we must first gather together some facts about $\tau_\alpha^n(t)$, defined in (16).

Proposition 1. *With probability* 1 *for any choice of* $0 < \rho < 1/2$ *uniformly in* $t > 0$, $n \geq 1$ *and* $0 < \rho \leq \alpha \leq 1 - \rho$

$$0 \leq F_n\left(t, \tau_\alpha^n(t)\right) - \alpha \leq m/n,$$

where $m = 2(\lceil 2/H \rceil + 1)$.

Proof We require a lemma.

Lemma 1. *Let* B_j, $j = 1, \ldots, n$, *be i.i.d. fractional Brownian motions on* $[0, \infty)$ *with Hurst index* $0 < H < 1$, *where* $n \geq 2\lceil 2/H \rceil + 2$, *then w.p. zero does there exist a subset* $\{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}$, *where* $m = 2\lceil 2/H \rceil + 2$, *such that for some* $t > 0$

$$B_{i_1}(t) = \cdots = B_{i_m}(t).$$

Proof If such a subset exists then the paths of the independent fractional Brownian motions in $\mathbb{R}^k$ with $2k = m$,

$$X^1 = \left(B_{i_1}, \ldots, B_{i_k}\right) \text{ and } X^2 = \left(B_{i_{k+1}}, \ldots, B_{i_{2k}}\right) \tag{23}$$

would have non-empty intersection except at 0, which, since $k > 2/H$, contradicts the following special case of Theorem 3.2 in Xiao [18]:

Theorem. ***(Xiao)*** *Let* $X^1(t)$, $t \geq 0$, *and* $X^2(t)$, $t \geq 0$, *be two independent fractional Brownian motions in* $\mathbb{R}^d$ *with index* $0 < H < 1$. *If* $2/H \leq d$, *then w.p.* 1,

$$X^1\left([0, \infty)\right) \cap X^2\left((0, \infty)\right) = \emptyset.$$

We apply this result with X^1 and X^2 as in (23). □

Returning to the proof of Proposition 1, choose $n \geq 2\lceil 2/H \rceil + 2$ and for any choice of $t > 0$ let $B_{(1)}(t) \leq \cdots \leq B_{(n)}(t)$ denote the order statistics of $B_1(t), \ldots, B_n(t)$. We see that for any $\alpha \in (0, 1)$,

$$F_n\left(t, B_{(\lceil \alpha n \rceil)}(t)\right) \geq \lceil \alpha n \rceil / n \geq \alpha$$

and

$$F_n\left(t, B_{(\lceil \alpha n \rceil)}(t)-\right) \leq \left(\lceil \alpha n \rceil - 1\right)/n < \alpha.$$

Thus

$$\tau_\alpha^n(t) = \inf\left\{x : F_n(t, x) \geq \alpha\right\} = B_{(\lceil \alpha n \rceil)}(t).$$

Since by the above lemma, w.p. 1, for all $t > 0$

$$\sum_{j=1}^{n} 1\left\{B_j(t) = B_{(\lceil \alpha n \rceil)}(t)\right\} < m = 2\lceil 2/H \rceil + 2,$$

we see that

$$\alpha \leq \lceil \alpha n \rceil / n \leq F_n\left(t, \tau_\alpha^n(t)\right) \leq \left(\lceil \alpha n \rceil + m - 1\right) / n \leq \alpha + m/n.$$

Thus w.p. 1 for any choice of $0 < \rho < 1/2$ uniformly in $t > 0$, $n \geq 2\lceil 2/H \rceil + 2$ and $0 < \rho \leq \alpha \leq 1 - \rho$

$$0 \leq F_n\left(t, \tau_\alpha^n(t)\right) - \alpha \leq m/n.$$

Note that this bound is trivially true for $1 \leq n < 2\lceil 2/H \rceil + 2$. □

Proposition 2. *For any $H \geq \delta > 0$ and $\rho \in (0, 1/2)$ there is a $D_0 = D_0(\rho, T) > 0$ (depending only on ρ and T) such that, w.p. 1 there is an $n_0 = n_0(\delta)$, such that for all $n > n_0$, uniformly in $(\alpha, t) \in [\rho, 1-\rho] \times (a_n(\delta), T]$,*

$$\left|\tau_\alpha(t) - \tau_\alpha^n(t)\right| \leq \frac{t^{H-\delta} D_0 \sqrt{\log\log n}}{\sqrt{n}},$$

with

$$a_n = a_n(\delta) = C\left(\frac{\log\log n}{n}\right)^{1/(2\delta)}, \tag{24}$$

where $C = C(\delta, \rho, T)$ depends only on δ, ρ and T.

Proof By Proposition 1, w.p. 1,

$$\sup_{(\alpha,t)\in[\rho,1-\rho]\times(0,T]} \left|F_n\left(t, \tau_\alpha^n(t)\right) - \alpha\right| \leq m/n. \tag{25}$$

We see by (14) that for any $H \geq \delta > 0$ w.p. 1 there is an n_0, such that for all $n > n_0$

$$\sup_{(\alpha,t)\in[\rho,1-\rho]\times(0,T]} t^\delta \left|F_n\left(t, \tau_\alpha^n(t)\right) - F\left(t, \tau_\alpha^n(t)\right)\right| \leq \frac{2\sigma_\delta(T)\sqrt{\log\log n}}{\sqrt{n}},$$

where, as in (15), $\sigma_\delta^2(T) = \frac{T^{2\delta}}{4} \leq \frac{T^2}{4}$. Thus by (25) and noting that $F\left(t, \tau_\alpha(t)\right) = \alpha$ we have w.p. 1 for all large enough n

$$\sup_{(\alpha,t)\in[\rho,1-\rho]\times(0,T]} t^\delta \left|F\left(t, \tau_\alpha(t)\right) - F\left(t, \tau_\alpha^n(t)\right)\right| \leq \frac{2T\sqrt{\log\log n}}{\sqrt{n}}. \tag{26}$$

Recall the notation in (17). Notice that whenever $t^H x - \tau_\alpha(t) > t^H/8$, for some $t > 0$ and $\alpha \in [\rho, 1-\rho]$,

$$\begin{aligned}
\left|F\left(t, \tau_\alpha(t)\right) - F\left(t, t^H x\right)\right| &= \int_{\tau_\alpha(t)}^{t^H x} \frac{1}{t^H\sqrt{2\pi}} \exp\left(-\frac{y^2}{2t^{2H}}\right) dy \\
&= \int_{z_\alpha}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du \\
&> \int_{z_\alpha}^{z_\alpha+1/8} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du \geq d_1 > 0,
\end{aligned}$$

where

$$d_1 = \inf\left\{\int_{z_\alpha}^{z_\alpha+1/8} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du : \alpha \in [\rho, 1-\rho]\right\}.$$

Similarly, whenever $\tau_\alpha(t) - t^H x > t^H/8$ for some $t > 0$ and $\alpha \in [\rho, 1-\rho]$,

$$\left|F(t, \tau_\alpha(t)) - F\left(t, t^H x\right)\right| \geq d_1.$$

We have shown that whenever $|t^H x - \tau_\alpha(t)| > t^H/8$, for some $t > 0$, and $\alpha \in [\rho, 1-\rho]$, then

$$|F(t, \tau_\alpha(t)) - F(t, t^H x)| > d_1 > 0.$$

Choose $C(\delta, \rho, T) = (2T/d_1)^{1/\delta}$ in (24). Then

$$\frac{2T\sqrt{\log\log n}}{\sqrt{n}} a_n^{-\delta} = \frac{2T}{C^\delta} = d_1. \tag{27}$$

Now, (26) implies that w.p. 1 for all large n we have $|\tau_\alpha(t) - \tau_\alpha^n(t)| \leq t^H/8$, whenever $t > a_n$, which together with $\alpha \in [\rho, 1-\rho]$ implies that

$$\tau_\alpha(t), \tau_\alpha^n(t) \in t^H[z_\rho - 1/8, z_{1-\rho} + 1/8] =: t^H [a, b]. \tag{28}$$

We get for $t > a_n$

$$|F(t, \tau_\alpha(t)) - F(t, \tau_\alpha^n(t))| = \left|\Phi(\tau_\alpha(t)t^{-H}) - \Phi(\tau_\alpha^n(t)t^{-H})\right| = t^{-H}|\tau_\alpha(t) - \tau_\alpha^n(t)|\varphi(\xi),$$

where $\xi \in [z_\rho - 1/8, z_{1-\rho} + 1/8]$, φ is the standard normal density and

$$\varphi(\xi) \geq \min_{y \in [a,b]} \varphi(y) =: d_2 > 0.$$

Therefore by (26), w.p. 1, for all large n, for $t > a_n$ and $\alpha \in [\rho, 1-\rho]$

$$|\tau_\alpha(t) - \tau_\alpha^n(t)| \leq \frac{2T}{d_2} \frac{t^{H-\delta}\sqrt{\log\log n}}{\sqrt{n}},$$

so the statement is proved, with $D_0 = 2T/d_2$. □

For future reference we point out here that for any $a_n(\delta)$ as in (24) and $1 \geq \gamma_n > 0$ satisfying (18) for some $\eta < \frac{1}{2H}$

$$\lim_{n\to\infty} \frac{-\log a_n(\delta)}{\log n} = \frac{1}{2\delta} \geq \frac{1}{2H} > \lim_{n\to\infty} \frac{-\log \gamma_n}{\log n} = \eta.$$

Thus for all n sufficiently large

$$a_n(\delta) < \gamma_n. \tag{29}$$

Note that

$$v_n(t, \tau_\alpha^n(t)) - \sqrt{n}\{\alpha - F(t, \tau_\alpha^n(t))\} = \sqrt{n}(F_n(t, \tau_\alpha^n(t)) - \alpha) =: \Delta_n(t, \alpha), \tag{30}$$

for which by Proposition 1 we have

$$|\Delta_n(t, \alpha)| \leq \frac{m}{\sqrt{n}}, \text{ uniformly in } t > 0,\ 0 < \rho \leq \alpha \leq 1-\rho \text{ and } n \geq 1. \tag{31}$$

Rewriting (30) as

$$v_n(t, \tau_\alpha^n(t)) = -\sqrt{n}\{F(t, \tau_\alpha^n(t)) - \alpha\} + \Delta_n(t, \alpha),$$

we get using a Taylor expansion applied to $F(t,\tau_\alpha^n(t))-\alpha$,

$$v_n(t,\tau_\alpha^n(t)) = -\sqrt{n}f(t,\tau_\alpha(t))\left(\tau_\alpha^n(t)-\tau_\alpha(t)\right) - \frac{1}{2}\sqrt{n}f'\left(t,\theta_\alpha^n(t)\right)\left(\tau_\alpha^n(t)-\tau_\alpha(t)\right)^2 + \Delta_n(t,\alpha), \tag{32}$$

where $\theta_\alpha^n(t)$ is between $\tau_\alpha(t)$ and $\tau_\alpha^n(t)$ and $f'(t,x) = \partial f(t,x)/\partial x$. Write

$$\sqrt{n}f'\left(t,\theta_\alpha^n(t)\right)\left(\tau_\alpha^n(t)-\tau_\alpha(t)\right)^2 = \sqrt{n}t^{2H}f'\left(t,\theta_\alpha^n(t)\right)t^{-2H}\left(\tau_\alpha^n(t)-\tau_\alpha(t)\right)^2.$$

Observe that by (29) with $[a,b]$ as given in (28), w.p. 1, for all large n

$$\begin{aligned}\sup\left\{\left|t^{2H}f'\left(t,\theta_\alpha^n(t)\right)\right| : (\alpha,t)\in[\rho,1-\rho]\times[\gamma_n,T]\right\} &\le \sup_{t\in(0,T]}\sup\left\{t^{2H}\left|f'(t,x)\right| : x\in t^H[a,b]\right\}\\ &= \sup\left\{\left|f'(1,x)\right| : x\in[a,b]\right\}<\infty.\end{aligned}$$

Further by (29), we can apply Proposition 2 with $\delta = H/4$ to get, w.p. 1,

$$\sup_{(\alpha,t)\in[\rho,1-\rho]\times[\gamma_n,T]} t^{-2H}\left(\tau_\alpha^n(t)-\tau_\alpha(t)\right)^2 = O\left(\frac{\gamma_n^{-H/2}\log\log n}{n}\right).$$

Therefore, substituting back into (32) from the definition of u_n and from (31) we see that w.p. 1,

$$\sup_{(\alpha,t)\in[\rho,1-\rho]\times[\gamma_n,T]}\left|v_n\left(t,\tau_\alpha^n(t)\right)+f\left(t,\tau_\alpha(t)\right)u_n(t,\alpha)\right| = O\left(\frac{\gamma_n^{-H/2}\log\log n}{\sqrt{n}}\right). \tag{33}$$

Our next goal is to control the size of $v_n(t,\tau_\alpha(t)) - v_n(t,\tau_\alpha^n(t))$ uniformly in $(\alpha,t)\in[\rho,1-\rho]\times[\gamma_n,T]$ for appropriate $0<\gamma_n\le 1$. For this purpose we need to introduce some more notation.

Recall notation (5). For any $K\ge 1$ denote the class of real-valued functions on $[0,T]$,

$$\mathcal{C}(K) = \left\{g:\ |g(s)-g(t)|\le Kf_H(|s-t|), 0\le s,t\le T\right\}.$$

One readily checks that $\mathcal{C}(K)$ is closed in $\mathcal{C}[0,T]$. The following class of functions $\mathcal{C}[0,T]\to\mathbb{R}$ will play an essential role in our proof:

$$\mathcal{F}(K,\gamma) := \left\{h_{t,x}^{(K)}(g) = 1\left\{g(t)\le x, g\in\mathcal{C}(K)\right\} : (t,x)\in\mathcal{T}(\gamma)\right\}.$$

For any $c>0$, $n>e$ and $1<T$ denote the class of real-valued functions on $[0,T]$,

$$\mathcal{C}_n := \mathcal{C}(\sqrt{c\log n}) = \left\{g:\ |g(s)-g(t)|\le\sqrt{c\log n}f_H(|s-t|),\ 0\le s,t\le T\right\}. \tag{34}$$

Define the class of functions $\mathcal{C}[0,T]\to\mathbb{R}$ indexed by $[\gamma_n,T]\times\mathbb{R} = \mathcal{T}(\gamma_n)$

$$\mathcal{F}_n = \left\{h_{t,x}^{(\sqrt{c\log n})}(g) = 1\left\{g(t)\le x, g\in\mathcal{C}_n\right\} : (t,x)\in\mathcal{T}(\gamma_n)\right\}.$$

To simplify our previous notation we shall write here

$$h_{t,x}^{(n)}(g) = h_{t,x}^{(\sqrt{c\log n})}(g)$$

For $h_{t,x}^{(n)} \in \mathcal{F}_n$ write

$$\alpha_n(h_{t,x}^{(n)}) = \sum_{i=1}^{n} \frac{1\left\{B_i(t) \le x, B_i \in \mathcal{C}_n\right\} - P\left\{B(t) \le x, B \in \mathcal{C}_n\right\}}{\sqrt{n}}.$$

Using (8), note that for each $(t,x) \in \mathcal{T}(\gamma_n)$, when $B_i \in \mathcal{C}_n$, for $i = 1, \ldots, n$,

$$\begin{aligned} \alpha_n\left(h_{t,x}^{(n)}\right) &= v_n(t,x) + \sqrt{n} P\left\{B(t) \le x, B \notin \mathcal{C}_n\right\} \\ &= \alpha_n(h_{t,x}) + \sqrt{n} P\left\{B(t) \le x, B \notin \mathcal{C}_n\right\}. \end{aligned}$$

Set

$$\mathcal{F}_n(\varepsilon) = \left\{\left(f, f'\right) \in \mathcal{F}_n^2 : d_P\left(f, f'\right) < \varepsilon\right\}$$

and

$$\mathcal{G}_n(\varepsilon) = \left\{f - f' : \left(f, f'\right) \in \mathcal{F}_n(\varepsilon)\right\},$$

where

$$d_P\left(f, f'\right) = \sqrt{E\left(f(B) - f'(B)\right)^2}.$$

By the arguments given in the Appendix of Kevei and Mason [5] the classes $\mathcal{F}_n(\varepsilon)$ and $\mathcal{G}_n(\varepsilon)$ are *pointwise measurable*. This means that the use of Talagrand's inequality below is justified.

Fix $n \ge 1$. Let $B_1, \ldots, B_n$ be i.i.d. B, and $\epsilon_1, \ldots, \epsilon_n$ be independent Rademacher random variables mutually independent of $B_1, \ldots, B_n$. Write for $\varepsilon > 0$,

$$\mu_n^S(\varepsilon) = E\left\{\sup_{f - f' \in \mathcal{G}_n(\varepsilon)} \left|\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \epsilon_i \left(f - f'\right)(B_i)\right|\right\},$$

Observe that as long as $\varepsilon = \varepsilon_n$ and $\gamma = \gamma_n$ satisfy

$$\sqrt{n}\varepsilon_n / \sqrt{\log n} \to \infty \tag{35}$$

and

$$\log\left(\frac{\log n}{\varepsilon_n \gamma_n}\right) / \log n \to \varsigma > 0, \text{ as } n \to \infty, \tag{36}$$

we have

$$\sqrt{n}\varepsilon_n / \sqrt{\log\left(\frac{\log n}{\varepsilon_n \gamma_n}\right)} \to \infty, \text{ as } n \to \infty,$$

which by (57) in [5] implies that for all large enough n for a suitable $A_1 > 0$

$$\mu_n^S(\varepsilon_n) \le A_1 \varepsilon_n \sqrt{\log\left(\frac{\log n}{\varepsilon_n \gamma_n}\right)}.$$

This, in turn, by (36) gives for all large enough n, for some $A_1' > 0$

$$\mu_n^S(\varepsilon_n) \le A_1' \varepsilon_n \sqrt{\log n}.$$

Therefore by Talagrand's inequality (45) applied with $M = 1$, we have for suitable finite positive constants D_1, D_1', $D_2 0$ and for all $z > 0$,

$$P\left\{||\sqrt{n}\alpha_n||_{\mathcal{G}_n(\varepsilon_n)} \geq D_1'(\varepsilon_n\sqrt{n\log n} + z)\right\} \leq P\left\{||\sqrt{n}\alpha_n||_{\mathcal{G}_n(\varepsilon_n)} \geq D_1(\sqrt{n}\mu_n^S(\varepsilon_n) + z)\right\}$$
$$\leq 2\left\{\exp\left(-\frac{D_2 z^2}{n\sigma^2_{\mathcal{G}_n(\varepsilon_n)}}\right) + \exp(-D_2 z)\right\}. \tag{37}$$

Let

$$\varepsilon_n = c_1\gamma_n^{-H/2}\left(\log\log n/n\right)^{1/4}, \text{ for some } c_1 > 0. \tag{38}$$

Recall that γ_n satisfies (18) with $\eta < 1/(2H)$, which implies $\varepsilon_n \to 0$. Further, ε_n fulfills (35) and

$$\frac{\log\left(\frac{\log n}{\varepsilon_n\gamma_n}\right)}{\log n} \to \frac{1}{4} + \eta\left(1 - \frac{H}{2}\right) =: \varsigma > 0,$$

which says that (36) holds. Also

$$n\sigma^2_{\mathcal{G}_n(\varepsilon_n)} = n\sup_{g\in\mathcal{G}_n(\varepsilon_n)}\mathrm{Var}(g(B)) \leq n\varepsilon_n^2.$$

Hence,

$$2\left\{\exp\left(-\frac{D_2 z^2}{n\sigma^2_{\mathcal{G}_n(\varepsilon_n)}}\right) + \exp\left(-D_2 z\right)\right\} \leq 2\left\{\exp\left(-\frac{D_2 z^2}{n\varepsilon_n^2}\right) + \exp\left(-D_2 z\right)\right\},$$

which, with $z = \varepsilon_n\sqrt{dn\log n/D_2}$ for some $d > 0$, is

$$\leq 2\left\{\exp\left(-d\log n\right) + \exp\left(-\sqrt{dD_2}\varepsilon_n\sqrt{n\log n}\right)\right\}.$$

By choosing $d > 0$ large enough, (37) combined with the Borel–Cantelli lemma gives that, w.p. 1,

$$||\alpha_n||_{\mathcal{G}_n(\varepsilon_n)} = \sup\left\{\left|\alpha_n\left(h_{s,x}^{(n)} - h_{t,y}^{(n)}\right)\right| : (s,x), (t,y) \in \mathcal{T}(\gamma_n),\ d_P^2\left(h_{s,x}^{(n)}, h_{t,y}^{(n)}\right) < \varepsilon_n^2\right\}$$
$$= O\left(n^{-1/4}\gamma_n^{-H/2}\left(\log\log n\right)^{1/4}\left(\log n\right)^{1/2}\right).$$

Recall that $\mathcal{T}(\gamma_n) = [\gamma_n, T] \times \mathbb{R}$. Since for $\gamma_n \leq t \leq T$

$$d_P^2\left(h_{t,x}^{(n)}, h_{t,y}^{(n)}\right) = E\left[\left(1\{B(t) \leq x\} - 1\{B(t) \leq y\}\right)1\{B \in \mathcal{C}_n\}\right]^2$$

$$\leq E\left|1\{B(t) \leq x\} - 1\{B(t) \leq y\}\right| = |F(t,x) - F(t,y)| \leq \gamma_n^{-H}|x - y|,$$

i.e. $|x - y| \leq c_1^2\sqrt{(\log\log n)/n}$ implies that $h_{t,x}^{(n)} - h_{t,y}^{(n)} \in \mathcal{G}_n(\varepsilon_n)$. This says that, w.p. 1, with c_1 as in (38),

$$\sup\left\{\left|\alpha_n\left(h_{t,x}^{(n)} - h_{t,y}^{(n)}\right)\right| : t \in [\gamma_n, T], |x - y| < \frac{c_1^2\sqrt{\log\log n}}{\sqrt{n}}\right\} \leq ||\alpha_n||_{\mathcal{G}_n(\varepsilon_n)},$$

where w.p. 1,

$$||\alpha_n||_{\mathcal{G}_n(\varepsilon_n)} = O\left(n^{-1/4}\gamma_n^{-H/2}\left(\log\log n\right)^{1/4}\left(\log n\right)^{1/2}\right).$$

Next note that

$$\Lambda_n := \sup\left\{\left|\alpha_n\left(h_{t,x}\right)-\alpha_n\left(h_{t,x}^{(n)}\right)\right| : (t,x)\in\mathcal{T}\left(\gamma_n\right)\right\}$$
$$\leq \sqrt{n}\sum_{i=1}^{n} 1\left\{B_i\notin\mathcal{C}_n\right\}+\sqrt{n}P\left\{B\notin\mathcal{C}_n\right\}.$$

We readily get using inequality (46) that for any $\omega > 2$ there exists a $c > 0$ in (34) such that $P\{B \notin \mathcal{C}_n\} \leq n^{-\omega}$, which implies

$$P\left\{\Lambda_n > \sqrt{n}n^{-\omega}\right\} \leq n^{1-\omega}.$$

Thus we readily see by using the Borel–Cantelli lemma that, w.p. 1,

$$\begin{aligned}&\sup\left\{\left|\alpha_n\left(h_{t,x}-h_{t,y}\right)\right| : t\in\left[\gamma_n,T\right],|x-y|<\frac{c_1^2\sqrt{\log\log n}}{\sqrt{n}}\right\}\\&=O\left(n^{-1/4}\gamma_n^{-H/2}\left(\log\log n\right)^{1/4}\left(\log n\right)^{1/2}\right).\end{aligned} \tag{39}$$

Applying Proposition 2 with $\delta = H/4$, keeping (29) in mind, and by choosing $c_1 > 0$ large enough in the definition of ε_n, we see that, w.p. 1, for all large n

$$\sup_{(\alpha,t)\in[\rho,1-\rho]\times[\gamma_n,T]}\left|\tau_\alpha^n(t)-\tau_\alpha(t)\right| \leq \frac{T^{3H/4}D_0\sqrt{\log\log n}}{\sqrt{n}} \leq \frac{c_1^2\sqrt{\log\log n}}{\sqrt{n}},$$

which says that, w.p. 1, for all large enough n uniformly in $(\alpha,t)\in[\rho,1-\rho]\times[\gamma_n,T]$,

$$\sup\left\{\left|v_n\left(t,\tau_\alpha(t)\right)-v_n\left(t,\tau_\alpha^n(t)\right)\right| : (\alpha,t)\in[\rho,1-\rho]\times[\gamma_n,T]\right\}$$
$$\leq \sup\left\{\left|\alpha_n\left(h_{t,x}-h_{t,y}\right)\right| : t\in\left[\gamma_n,T\right],|x-y|<\frac{c_1^2\sqrt{\log\log n}}{\sqrt{n}}\right\}.$$

Thus by (39), w.p. 1, for large enough $c > 0$ and $c_1 > 0$,

$$\sup\left\{\left|v_n\left(t,\tau_\alpha(t)\right)-v_n\left(t,\tau_\alpha^n(t)\right)\right| : (\alpha,t)\in[\rho,1-\rho]\times[\gamma_n,T]\right\}$$
$$=O\left(n^{-1/4}\gamma_n^{-H/2}\left(\log\log n\right)^{1/4}\left(\log n\right)^{1/2}\right).$$

On account of (33) this finishes the proof of Theorem 3. □

4.2 Proof of Corollary 1

Let $\gamma_n = n^{-\eta}$, where $0 < \eta < 1/(2H)$ to be determined later. By Theorem 3

$$\begin{aligned}&\sup_{(t,\alpha)\in[\gamma_n,T]\times[\rho,1-\rho]}\left|t^H v_n\left(t,\tau_\alpha(t)\right)+\frac{\exp\left(-\frac{z_\alpha^2}{2}\right)}{\sqrt{2\pi}}u_n(t,\alpha)\right|\\&=O\left(\left(\log\log n\right)^{1/4}\left(\log n\right)^{1/2}n^{-\frac{1}{4}+\eta\frac{H}{2}}\right),\text{ a.s.}\end{aligned} \tag{40}$$

Next

$$\sup_{(t,\alpha)\in[0,\gamma_n]\times[\rho,1-\rho]}\left|t^H v_n\left(t,\tau_\alpha(t)\right)\right| \leq \sup\left\{\left|t^H\alpha_n\left(h_{t,x}\right)\right| : (t,x)\in[0,\gamma_n]\times\mathbb{R}\right\}. \tag{41}$$

Now by a simple Borel–Cantelli argument based on inequality (47) the right side of (41) is equal to

$$= O\left((\log n)^{1/2} n^{-\eta H}\right), \text{ a.s.} \tag{42}$$

Next, by Proposition 2, for any $0 < \delta < H$

$$\sup\left\{|u_n(t,\alpha)| : (\alpha,t) \in [\rho, 1-\rho] \times (a_n, \gamma_n]\right\} = O\left((\log\log n)^{1/2} n^{-\eta(H-\delta)}\right), \text{ a.s.}$$

so the same holds without the logarithmic factor

$$\sup\left\{|u_n(t,\alpha)| : (\alpha,t) \in [\rho, 1-\rho] \times (a_n, \gamma_n]\right\} = O\left(n^{-\eta(H-\delta)}\right), \text{ a.s.} \tag{43}$$

The latter rate is larger than the one in (42). Furthermore, comparing (40) and (43) one sees that the optimal choice for η is $\eta = 1/(6H)$, and the best rate is $n^{-1/6+\delta}$ (due to the arbitrariness of δ in the last step we changed $\eta\delta$ to δ). That is the statement is proved for $t \geq a_n$.

Now we handle the $t < a_n$ case. Put

$$\Delta_n(a_n) := \sup\left\{|u_n(t,\alpha)| : 0 \leq t \leq a_n, 0 < \rho \leq \alpha \leq 1-\rho\right\}.$$

Observe that for all $0 \leq t \leq a_n$ and $0 < \rho \leq \alpha \leq 1-\rho$,

$$|\tau_\alpha^n(t)| \leq \max_{1\leq i\leq n} M_i(a_n),$$

where for $1 \leq i \leq n$, $M_i(a_n) = \sup\{|B_i(a_n s)| : 0 \leq s \leq 1\}$. Notice that $B(a_n s), 0 \leq s \leq 1$, is equal in distribution to $a_n^H B(s)$, $0 \leq s \leq 1$. Further, as an application of the Landau–Shepp theorem we have for some $c > 0$ and $d > 0$

$$P\left\{\sup_{0\leq s\leq 1} |B(s)| > y\right\} \leq d\exp\left(-cy^2\right), \text{ for all } y > 0. \tag{44}$$

We get now using a simple Borel–Cantelli lemma argument based on inequality (44) and

$$M_1(a_n) \stackrel{\mathrm{D}}{=} a_n^H \sup\left\{|B_1(s)| : 0 \leq s \leq 1\right\},$$

that for some $D > 0$, w.p. 1, for all n sufficiently large,

$$\max_{1\leq i\leq n} M_i(a_n) \leq D a_n^H \sqrt{\log n}.$$

Hence, w.p. 1, uniformly $0 \leq t \leq a_n$ and $0 < \rho \leq \alpha \leq 1-\rho$, for all large enough n,

$$|\tau_\alpha^n(t)| \leq D a_n^H \sqrt{\log n}.$$

Also trivially we have uniformly $0 \leq t \leq a_n$ and $0 < \rho \leq \alpha \leq 1-\rho$

$$|\tau_\alpha(t)| = t^H |z_\alpha| \leq a_n^H z_{1-\rho}.$$

Thus, w.p. 1, uniformly $0 \leq t \leq a_n$ and $0 < \rho \leq \alpha \leq 1-\rho$, for all large enough n,

$$\Delta_n(a_n) \leq 2D a_n^H \sqrt{n\log n}.$$

Note that

$$\rho_n := 2Da_n^H \sqrt{n \log n} = 2DC^H \left(\frac{\log \log n}{n} \right)^{H/(2\delta)} \sqrt{n \log n},$$

satisfies

$$\frac{-\log \rho_n}{\log n} \to \frac{H}{2\delta} - \frac{1}{2} = \frac{H - \delta}{2\delta} > 0.$$

For some $\delta > 0$ small enough $(H - \delta)/(2\delta) > 1/6$, therefore

$$\Delta_n(a_n) = O(n^{-1/6}), \text{ a.s.}$$

which together with (40), and (42) finish the proof of the corollary. □

5 Appendix: Useful inequalities

5.1 Talagrand's inequality

We shall be using the following exponential inequality due to Talagrand [14].

Talagrand Inequality. *Let $\mathcal{G}$ be a pointwise measurable class of measurable real-valued functions defined on a measure space $(S, \mathcal{S})$ satisfying $||g||_\infty \le M,\ g \in \mathcal{G}$, for some $0 < M < \infty$. Let $X, X_n,\ n \ge 1$, be a sequence of i.i.d. random variables defined on a probability space $(\Omega, \mathcal{A}, P)$ and taking values in S, then for all $z > 0$ we have for suitable finite constants $D_1, D_2 > 0$,*

$$P\left\{ ||\sqrt{n}\alpha_n||_{\mathcal{G}} \ge D_1 \left(E \left\| \sum_{i=1}^{n} \epsilon_i g(X_i) \right\|_{\mathcal{G}} + z \right) \right\} \le 2\exp\left(-\frac{D_2 z^2}{n\sigma_{\mathcal{G}}^2} \right) + 2\exp\left(-\frac{D_2 z}{M} \right), \quad (45)$$

where $\sigma_{\mathcal{G}}^2 = \sup_{g \in \mathcal{G}} \mathrm{Var}(g(X))$ and $\epsilon_n,\ n \ge 1$, are independent Rademacher random variables mutually independent of $X_n,\ n \ge 1$.

5.2 Application of Landau–Shepp Theorem

By the Lévy modulus of continuity theorem for fractional Brownian motion $B^{(H)}$ with Hurst index $0 < H < 1$ (see Corollary 1.1 of Wang [17]), we have for any $0 < T < \infty$, w.p. 1,

$$\sup_{0 \le s \le t \le T} \frac{\left| B^{(H)}(t) - B^{(H)}(s) \right|}{f_H(t - s)} =: L < \infty.$$

Therefore we can apply the Landau and Shepp [11] theorem (also see Sato [12] and Proposition A.2.3 in [15]) to infer that for appropriate constants $C > 0$ and $D > 0$, for all $z > 0$,

$$P\{L > z\} \le C \exp\left(-Dz^2\right). \quad (46)$$

5.3 A maximal inequality

The following inequality is proved in Kevei and Mason [5], where it is Inequality 2.

Inequality *For all* $0 < \gamma \le 1$ *and* $\tau > 0$ *we have for some* $E(\tau)$ *and for suitable finite positive constants* $D_3, D_4 > 0$*, for all* $z > 0$

$$\begin{aligned} &P\left\{\max_{1\le m\le n} \sup_{(t,x)\in[0,\gamma]\times\mathbb{R}} |\sqrt{m}t^{\tau}\alpha_m\left(h_{t,x}\right)| \ge D_3\sqrt{n}\left(E(\tau)(2\gamma)^{\tau}+z\right)\right\} \\ &\le 2\left\{\exp\left(-D_4 z^2\left(2\gamma\right)^{-2\tau}\right)+\exp\left(-D_4 z\sqrt{n}\left(2\gamma\right)^{-\tau}\right)\right\}. \end{aligned} \tag{47}$$

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