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FACTORIZATION OF PROBABILITY MEASURES ON SYMMETRIC HYPERGROUPS

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Abstract

Generalizing known results for special examples, we derive a Khintchine type decomposition of probability measures on symmetric hypergroups. This result is based on a triangular central limit theorem and a discussion of conditions ensuring that the set of all factors of a probability measure is weakly compact. By our main result, a probability measure satisfying certain restrictions can be written as a product of indecomposable factors and a factor in $I_0(K)$, the set of all measures having decomposable factors only. Some contributions to the classification of $I_0(K)$ are given for general symmetric hypergroups and applied to several families of examples like finite symmetric hypergroups and hypergroup joins. Furthermore, all results are discussed in detail for a class of discrete symmetric hypergroups which are generated by infinitely many joins, for a class of countable compact hypergroups, for Sturm-Liouville hypergroups on $[0, \infty[$ and, finally, for polynomial hypergroups.

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Introduction

Several authors have proved factorization theorems for the commutative topological semigroup $(M^1(K), *)$ of all probability measures on some special symmetric hypergroups, where the notation of hypergroups was usually

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suppressed (see, for example, Bingham [4, 5], Finkh [18], Kingman [27], Lamperti [29], Ostrovskii [36], Ostrovskii and Truhina [37], Schwartz [41], Truhina [45] and, for a more general background, Rusza and Szekely [40]). Having discussed infinitely divisible probability measures on commutative hypergroups in [47], we wish to extend these investigations and establish factorization results for arbitrary commutative hypergroups. But, since we are still far from being able to transfer many crucial properties of locally compact abelian groups to arbitrary commutative hypergroups, we shall prove here the main results on decompositions of probability measures, infinitely divisible measures and probability measures having no indecomposable factors only for symmetric hypergroups satisfying an additional condition. These restrictions ensure that the factor sets of probability measures on K are compact in the usual sense. For locally compact groups this compactness is equivalent to the compactness of the group, and therefore the main results contained in this paper cannot be applied to most locally compact abelian groups. Nevertheless, our restrictions seem to be admissible and convenient, since Khintchine type decomposition theorems are well known for locally compact abelian groups (see Parthasarathy [38]), and since the conditions mentioned above are satisfied for many other symmetric hypergroups. For instance, the main results of this paper can be applied to the important families of Sturm-Liouville hypergroups on $\mathbb{R}_+$, polynomial hypergroups and compact symmetric hypergroups. In particular, the factorization results contained in [4, 5, 18, 29, 41] appear as immediate consequences of the unified treatment in this paper.

We briefly outline the structure of this article: In the first Section we introduce some preliminary notations and facts about commutative hypergroups. For further details and notations used throughout this article we refer to the surveys of Heyer [22, 23], Jewett [25] and, in particular, Voit [47]. In Section 2 we investigate conditions which imply the compactness of the set of all factors of probability measures on K. These results are used in Section 3 to derive the announced Khintchine type decomposition of probability measures into an infinitely divisible part and a product of indecomposable factors. Section 4 contains some supplementary general results about the set of all probability measures having no indecomposable factors. Since this set cannot be characterized for general symmetric hypergroups, we shall discuss only some particular and preparatory general results in Section 4. This section deals also with some special results for hypergroup joins and finite symmetric hypergroups. A complete determination of $I_0(K)$ is contained in Sections 5–8 separately for different further families of examples. In this way, we shall treat symmetric hypergroup structures on $\mathbb{N}_0$ which are constructed by an infinite chain of hypergroup joins (Section 5), their dual compact hyper-

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group structures (Section 6), Sturm-Liouville hypergroups on $\mathbb{R}_+$ (Section 7) and, lastly, polynomial hypergroups (Section 8). For these classes of examples we shall also reformulate the general decomposition theorems of Section 3 and the Levy-Khintchine representation for infinitely divisible probability measures.

It should be finally noted that Section 8 also contains some results concerning the dual space and the support of the Plancherel measure of a polynomial hypergroup. These results may be of interest independently of our decomposition results, but some of these facts are needed to apply the theorems of Sections 2 and 3 to polynomial hypergroups.

1. Preliminaries

A hypergroup $(K, *)$ (or K, for short) consists of a locally compact space K and a convolution $*$ on the Banach space $M_b(K)$ of all bounded measures on K such that $(M_b(K), *)$ becomes a Banach algebra and such that there exist a neutral element $e \in K$ (that is, the Dirac measure δ_e is the neutral element of the Banach algebra $M_b(K)$) and a continuous involution $\bar{\ }$: $K \to K$ that corresponds to the inverse mapping on a topological group. If this involution is the identity mapping on K, then K is said to be a symmetric (or hermitian) hypergroup. From the hypergroup axioms (see, for example, Jewett [25]) it follows immediately that every symmetric hypergroup is commutative, that is, the convolution $*$ is commutative.

If L is a locally compact space, let $M_b(L)$ be the space of all bounded Borel measures on L, $M_b^+(L)$ the space of all nonnegative bounded measures on L and $M^1(L)$ the space of all probability measures on L. The spaces $C(L)$, $C_b(L)$, $C_0(L)$ and $C_c(L)$ will be as usually defined.

For a given commutative hypergroup K let $\widehat{K}$ be the dual space of K, m (up to a multiplicative constant) the Haar measure on K and π the Plancherel measure on $\widehat{K}$ associated with m.

Finally we introduce two important characters: $\mathbf{1}$ stands for the identity character and α_0 for the uniquely determined positive character that is contained in $\operatorname{supp}\pi$ (for details we refer to Voit [46]). In many cases (for example, for groups) we have $\alpha_0 = \mathbf{1}$. In Sections 2 and 3 the existence of α_0 will be used to derive results for general commutative hypergroups from results for commutative hypergroups that satisfy $\mathbf{1} \in \operatorname{supp}\pi$. The existence of α_0 was applied earlier by Schwartz [41] to obtain factorization results for polynomial hypergroups.

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1.1. Notation. Let K be a commutative hypergroup. Then $M^1(K)$ equipped with the weak topology is a commutative topological semigroup. We shall introduce some natural notations for elements of $M^1(K)$.

(1) $\nu \in M^1(K)$ is said to be a factor of $\mu \in M^1(K)$ if $\nu * \rho = \mu$ for some measure $\rho \in M^1(K)$. For a measure $\mu \in M^1(K)$, $D(\mu) \subset M^1(K)$ stands for the set of all factors of μ. For $L \subset M^1(K)$ let $D(L) := \bigcup_{\mu \in L} D(\mu)$.

(2) The Dirac measure δ_e is the neutral element of $M^1(K)$.

(3) $\rho \in M^1(K)$ is called a unit if there exists $\nu \in M^1(K)$ such that $\rho * \nu = \delta_e$. The set U of all units is a subgroup of the semigroup $M^1(K)$ and consists exactly of the Dirac measures δ_x with x ranging over the maximal subgroup of K (see Jewett [25, 10.4]).

(4) $\mu \in M^1(K)$ is called an idempotent if $\mu = \mu * \mu$ and $\mu \neq \delta_e$. By Jewett [25, 10.2E], $\mu \in M^1(K)$ is an idempotent if and only if μ is the normalized Haar measure of a compact proper subhypergroup L of K ('proper' is taken to mean $L \neq \{e\}$).

(5) $\mu \in M^1(K)$ is called infinitely divisible if for each $n \in \mathbb{N}$ there exists $\mu_n \in M^1(K)$ such that $\mu = \mu_n^n$. Let $I(K)$ be the set of all infinitely divisible probability measures on K.

(6) $\mu \in M^1(K)$ is called indecomposable if $D(\mu) = U \cup \{\mu * \rho : \rho \in U\}$ and if μ is not a unit. Let $I_0(K)$ be the set of all probability measures on K having no indecomposable factors.

1.2. Proposition. *Let K be a second countable commutative hypergroup. Then $\widehat{K}$ is a second countable locally compact space.*

Proof. Considering Jewett [25, 7.3], we only have to verify that $\widehat{K}$ is second countable. But, if K is second countable, then $L^1(K)$ is obviously a separable Banach algebra. Thus the unit ball $B := \{\varphi \in L^1(K)^* : \|\varphi\| \leq 1\}$ of all linear functionals on $L^1(K)$ that are bounded by 1 is a compact and metrizable space with respect to the $\sigma(L^1(K)^*, L^1(K))$-topology. Since any compact metric space is second countable and since $\widehat{K}$ is homeomorphic to the structure space $\Delta(L^1(K)) \subset B$ of $L^1(K)$ (Jewett [25, 6.3]), the assertion follows.

The following proposition is useful to discuss some examples of the Sections 5 and 8.

1.3. Proposition. *Let K be a commutative hypergroup, and let $\alpha \in \widehat{K} \cap L^1(K)$. Then α is a discrete point in $\widehat{K}$, α is contained in* $\operatorname{supp} \pi$ *and $\alpha/\|\alpha\|_2^2$ is a nontrivial idempotent in $L^1(K)$.*

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PROOF. Since

$$\begin{aligned}\alpha(x)\cdot\int_K \alpha(y)\beta(\bar{y})\,dm(y) &= \int_K \alpha(x*y)\beta(\bar{y})\,dm(y)\\ &= \int_K \alpha(y)\beta(x*\bar{y})\,dm(y)\\ &= \beta(x)\cdot\int_K \alpha(y)\beta(\bar{y})\,dm(y)\end{aligned}$$

for all $x \in K$ and $\beta \in \widehat{K}$, it follows that $\hat{\alpha}(\beta) = 0$ for every $\beta \in \widehat{K}\backslash\{\alpha\}$. Therefore, since $\hat{\alpha}(\alpha) = \|\alpha\|_2^2 > 0$ and since $\hat{\alpha}$ is continuous, we see that α is isolated in $\widehat{K}$ and that $\hat{\alpha} \in L^1(\widehat{K})$. Now the inversion theorem (Jewett [25, 12.2C]) implies $\alpha = \hat{\alpha}^{\vee}$ and hence $\alpha \in \operatorname{supp}\pi$. Finally, $(\hat{\alpha}/\|\alpha\|_2^2)^2 = \hat{\alpha}/\|\alpha\|_2^2$ and the injectivity of the Fourier transform (Jewett [25, 7.3E]) show that $\alpha/\|\alpha\|_2^2$ is a nontrivial idempotent in $L^1(K)$.

2. Compactness of the set of all factors of a probability measure

In this section we discuss some conditions for commutative hypergroups which ensure that $D(L)$ is weakly compact for every compact set $L \subset M^1(K)$. These conditions will be applied in the Sections 5–8 to several families of hypergroups in order to verify the compactness conditions which are needed for the factorization results in the Sections 3 and 4. We first present a property of a measure $\mu \in M^1(K)$ which is, under certain additional assumptions, equivalent to the weak compactness of $D(\mu)$:

2.1. DEFINITION. Let K be a symmetric hypergroup. Then $\mu \in M^1(K)$ has Property (D) if there exists a neighbourhood W of α_0 in $\operatorname{supp}\pi$ such that $\hat{\nu}$ is positive on W for each $\nu \in D(\mu)$. We say K has Property (D) if every $\mu \in M^1(K)$ has Property (D).

2.2. PROPOSITION. *Let K be a second countable commutative hypergroup. If $\mathbf{1} \in \operatorname{supp}\pi$, then for every set $L \subset M^1(K)$ the following statements are equivalent.*

(1) *L is relatively compact with respect to the weak topology.*

(2) *For every $\varepsilon > 0$ there exists a compact set $H \subset K$ such that $\mu(K\backslash H) < \varepsilon$ for every $\mu \in L$.*

(3) *For every $\varepsilon > 0$ there exists a neighbourhood W of $\mathbf{1}$ in $\operatorname{supp}\pi$ such that $\operatorname{Re}\hat{\mu}(\alpha) \geq 1-\varepsilon$ for all $\mu \in L$ and $\alpha \in W$.*

Moreover, if $\mathbf{1} \notin \operatorname{supp}\pi$, then (1) and (2) are equivalent, and (1) implies that for every $\varepsilon > 0$ there is a neighbourhood W of $\mathbf{1}$ in $\widehat{K}$ such that $\operatorname{Re}\hat{\mu}(\alpha) \geq 1-\varepsilon$ for all $\mu \in L$ and $\alpha \in W$.

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PROOF. For (1) $\Leftrightarrow$ (2) see Parthasarathy [38, Theorem II.6.7].

Now assume (2). Fix $\varepsilon > 0$ and choose H as described in (2). Then the set $W := \{\alpha \in \operatorname{supp}\pi : |\alpha(x) - 1| \leq \varepsilon \ \ \forall x \in H\}$ is a neighbourhood of $\mathbf{1}$ in $\operatorname{supp}\pi$ (or in $\widehat{K}$ respectively) and satisfies

$$\begin{aligned}\operatorname{Re}\hat{\mu}(\alpha) = \int_K \operatorname{Re}(\alpha(x))\,d\mu(x) &\geq \int_H \operatorname{Re}(\alpha(x))\,d\mu(x) - \mu(K\backslash H)\\ &\geq (1-\varepsilon)\cdot\mu(H) - \varepsilon \geq 1 - 3\varepsilon + \varepsilon^2\end{aligned}$$

for any $\alpha \in W$ and any $\mu \in L$. Thus (3) and the additional assertion follow.

Now let us assume that $\mathbf{1} \in \operatorname{supp}\pi$ and that for a given $\varepsilon > 0$ there exists a neighbourhood W as described in (3). Take a function $h \in C_c(\widehat{K})$ such that $h \geq 0$, $\operatorname{supp} h \cap \operatorname{supp}\pi \subset W$, $\int_{\widehat{K}} h\,d\pi = 1$ and $h(\alpha) = h(\bar{\alpha})$ for all $\alpha \in \widehat{K}$. Then $\check{h} \in C_0(K)$ is a real valued function satisfying $\check{h} \leq \check{h}(e) = 1$ (see [7, Theorem 2.4.1]). In particular, $H := \{\check{h} \geq 1/2\} \subset K$ is compact and satisfies

$$\tfrac{1}{2}\mu(K\backslash H) \leq \int_K (1-\check{h})\,d\mu = \int_{\widehat{K}} (1-\hat{\mu})h\,d\pi = \int_{\widehat{K}} (1 - \operatorname{Re}\hat{\mu})h\,d\pi \leq \varepsilon.$$

This finishes the proof of Proposition 2.2.

2.3. LEMMA. *Let K be a second countable commutative hypergroup and L a weakly closed set in $M^1(K)$. If $D(L)$ is relatively compact, then $D(L)$ is compact.*

PROOF. We have to show that $D(L)$ is closed. To do this, take a sequence $(\nu_n)_{n\in\mathbb{N}} \subset D(L)$ that converges weakly to $\nu \in M^1(K)$. Then there exists a sequence $(\rho_n)_{n\in\mathbb{N}} \subset D(L)$ such that $\rho_n * \nu_n = \mu_n \in L$ for all $n \in \mathbb{N}$. Since $M^1(K)$ is a complete separable metric space with respect to the weak topology [38, Theorems II.6.2 and II.6.5], we may assume that $(\rho_n)_{n\in\mathbb{N}}$ converges to a measure $\rho \in M^1(K)$. Thus,

$$\nu * \rho = \left(\tau_w - \lim_{n\to\infty} \nu_n\right) * \left(\tau_w - \lim_{n\to\infty} \rho_n\right) = \tau_w - \lim_{n\to\infty} (\nu_n * \rho_n) \in L$$

which proves $\nu \in D(L)$ as claimed.

2.4. COROLLARY. *Let K be a second countable symmetric hypergroup with $\mathbf{1} \in \operatorname{supp}\pi$. Then $\mu \in M^1(K)$ has Property* (D) *if and only if $D(\mu)$ is compact.*

PROOF. We first note that the symmetry guarantees that all characters on K are real valued. Assume that $\mu \in M^1(K)$ has Property (D). Fix $\varepsilon > 0$

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and define

$$W := \{\alpha \in \widehat{K} : \hat{\mu}(\alpha) \geq 1 - \varepsilon \text{ and } \hat{\nu}(\alpha) > 0 \qquad \forall \nu \in D(\mu)\}.$$

Since $\hat{\mu} \in C_b(\widehat{K})$ and $\hat{\mu}(\mathbf{1}) = 1$, W is a neighbourhood of $\mathbf{1}$ in $\widehat{K}$. Take $\nu_1, \nu_2 \in M^1(K)$ with $\nu_1 * \nu_2 = \mu$. Thus, since $\|\hat{\nu}_1\|_\infty = \|\hat{\nu}_2\|_\infty = 1$, we get $\hat{\nu}_1(\alpha) \geq \hat{\mu}(\alpha) \geq 1 - \varepsilon$ for every $\alpha \in W$. Therefore, by Proposition 2.2, $D(\mu)$ is relatively compact with respect to the weak topology. Now Lemma 2.3 completes the first part of the proof.

The converse statement is an immediate consequence of Proposition 2.2.

In order to establish analogous results in Section 3 for commutative hypergroups satisfying $\mathbf{1} \notin \operatorname{supp} \pi$ we introduce a modified hypergroup convolution $\bullet$ on K that is connected with $*$ in an intimate way. Let α_0 be the positive character that is contained in $\operatorname{supp} \pi$. Defining

$$\delta_x \bullet \delta_y := (\alpha_0(x) \cdot \alpha_0(y))^{-1} \cdot \alpha_0 \cdot (\delta_x * \delta_y) \qquad (x, y \in K),$$

we get a new commutative convolution on $M_b(K)$ by bilinear and continuous extension [46, Theorem 2.2]. Then $(K, \bullet)$ is a commutative hypergroup for which $\mathbf{1} \in \operatorname{supp} \pi_\bullet$ holds ($\pi_\bullet$ stands for the Plancherel measure of $(K, \bullet)$; see Voit [46]). The following lemma describes some connections between the semigroups $(M^1(K), *)$ and $(M^1(K), \bullet)$:

2.5. Lemma. *$\varphi : (M^1(K), *) \to (M^1(K), \bullet)$, $\mu \mapsto \alpha_0\mu/\hat{\mu}(\alpha_0)$, is a weakly continuous injective semigroup homomorphism; $\varphi(M^1(K))$ is given by*

$$\left\{\nu \in M^1(K) : \frac{1}{\alpha_0}\nu \in M_b(K)\right\}.$$

Furthermore, $\varphi(D(\mu)) = D_\bullet(\varphi(\mu))$ for every $\mu \in M^1(K)$, where $D_\bullet(\nu)$ denotes the set of factors of a measure $\nu \in M^1(K)$ with respect to $\bullet$.

Proof. For $\mu, \nu \in M^1(K)$ and $f \in C_c(K)$ we have

$$\begin{aligned}
\varphi(\mu * \nu)(f) &= \int_K \frac{f(z)\alpha_0(z)}{(\mu * \nu)\hat{}(\alpha_0)}\, d(\mu * \nu)(z) \\
&= \int_K \int_K \int_K \frac{f(z)\alpha_0(z)}{\hat{\mu}(\alpha_0)\hat{\nu}(\alpha_0)} d(\delta_x * \delta_y)(z)\, d\mu(x)\, d\nu(y) \\
&= \int_K \int_K \int_K f(z) d(\delta_x \bullet \delta_y)(z) d\left(\frac{\alpha_0 \cdot \mu}{\hat{\mu}(\alpha_0)}\right)(x) d\left(\frac{\alpha_0 \cdot \nu}{\hat{\nu}(\alpha_0)}\right)(y) \\
&= (\varphi(\mu) \bullet \varphi(\nu))(f).
\end{aligned}$$

Thus φ is a homomorphism. The injectivity of φ and the description of $\varphi(M^1(K))$ are obvious.

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In order to prove that φ is weakly continuous, we first note that the mapping $M^1(K) \to \mathbb{R}$, $\mu \mapsto \hat{\mu}(\alpha_0) = \int_K \alpha_0\, d\mu$, is weakly continuous. Furthermore, the mapping $M^1(K) \to M_b^+(K)$, $\mu \mapsto \alpha_0\mu$, is continuous with respect to the vague topology. Thus, since on $M^1(K)$ the weak topology coincides with the vague topology, the weak continuity of φ follows.

Now let $\mu, \nu_1, \nu_2 \in M^1(K)$ satisfy $\nu_1 \bullet \nu_2 = \varphi(\mu)$. Then $(1/\alpha_0)\nu_1$ and $(1/\alpha_0)\nu_2$ are positive Radon measures on K and satisfy

$$\int_K \frac{1}{\alpha_0}\, d\nu_1 \cdot \int_K \frac{1}{\alpha_0}\, d\nu_2 = \int_K \int_K \int_K \frac{1}{\alpha_0(z)} d(\delta_x \bullet \delta_y)(z)\, d\nu_1(x)\, d\nu_2(y)$$
$$= \int_K \frac{1}{\alpha_0(z)} d(\varphi(\mu))(z) = \frac{1}{\hat{\mu}(\alpha_0)} < \infty.$$

Thus from $(1/\alpha_0)\nu_1$, $(1/\alpha_0)\nu_2 \in M_b^+(K)$ it follows that $\nu_1, \nu_2 \in \varphi(D(\mu))$. Therefore $D_\bullet(\varphi(\mu)) \subset \varphi(D(\mu))$. The converse inclusion holds trivially.

2.6. LEMMA. *Let K be a second countable commutative hypergroup. Assume that there exists a sequence $(\alpha_k)_{k\in\mathbb{N}} \subset C_0(K) \cap \widehat{K}$ consisting of positive characters that converges to $\mathbf{1}$. Then, for every compact set $L \subset M^1(K)$, $D(L)$ is compact and the mapping $\varphi : D(L) \to \varphi(D(L))$ as defined in Lemma 2.5 is a homeomorphism.*

PROOF. Since $\{\nu \in M_b^+(K),\, \|\nu\| \leq 1\}$ is compact and metrizable with respect to the vague topology, every sequence $(\mu_n)_{n\in\mathbb{N}} \subset D(L)$ has a subsequence $(\mu_{n_l})_{l\in\mathbb{N}}$ that converges vaguely to a limit $\nu \in M_b^+(K)$ with $\|\nu\| \leq 1$. Since

$$\lim_{l\to\infty} \hat{\mu}_{n_l}(\alpha_k) = \lim_{l\to\infty} \int_K \alpha_k\, d\mu_{n_l} = \int_K \alpha_k\, d\nu = \hat{\nu}(\alpha_k)$$

for every $k \in \mathbb{N}$, we see that $(\hat{\mu}_{n_l})_{n\in\mathbb{N}}$ converges pointwise to $\hat{\nu}$ on

$$\{\alpha_k : k \in \mathbb{N}\} \subset \widehat{K}.$$

Proposition 2.2 shows that for every $\varepsilon > 0$ there is a neighbourhood W of $\mathbf{1}$ such that $\operatorname{Re}\hat{\mu}(\alpha) \geq 1 - \varepsilon$ for all $\mu \in L$ and $\alpha \in W$. Therefore, if we use the properties of the α_k, for every $\varepsilon > 0$ there exists an index k_0 such that $\hat{\rho}(\alpha_k) \geq \hat{\mu}(\alpha_k) \geq 1 - \varepsilon$ for all $k \geq k_0$, $\mu \in L$ and $\rho \in D(\mu)$. It follows that $\hat{\nu}(\alpha_k) \geq 1 - \varepsilon$ for $k \geq k_0$, and thus $\|\nu\| = \hat{\nu}(\mathbf{1}) = 1$. Hence $(\mu_{n_l})_{l\in\mathbb{N}} \subset D(L)$ tends weakly to $\nu \in M^1(K)$ which proves that $D(L)$ is relatively compact. Now Lemma 2.3 yields that $D(L)$ is weakly compact. Since φ is continuous by Lemma 2.5, the proof is complete.

At the end of this section we present a condition which implies Property (D):

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2.7. Lemma. *Let K be a second countable symmetric hypergroup such that there exists a net $(W_a)_{a\in A}$ of connected subsets of $\widehat{K}$ such that $(W_a \cap \operatorname{supp}\pi)_{a\in A}$ is a neighbourhood base of α_0 in $\operatorname{supp}\pi$. Then K has Property* (D).

Proof. We first note that $\hat{\mu}(\alpha_0) > 0$ and $\hat{\mu} \in C_b(\widehat{K})$ for any $\mu \in M^1(K)$. Thus there exists W_a on which $\hat{\mu}$ is positive. Hence $\hat{\nu}|_{W_a} \neq 0$ for every $\nu \in D(\mu)$, and, since W_a is connected, $\hat{\nu}|_{W_a} > 0$.

2.8. Remark. Using Lemma 2.3 and Lemma 2.7, we see that for symmetric hypergroups with $1 \in \operatorname{supp}\pi$ and with the connectedness property of Lemma 2.7, $D(L)$ is compact for every compact set $L \subset M^1(K)$. Furthermore, since Property (D) and the connectedness condition are preserved by the modification introduced above, Lemma 2.5 tells us that we can reduce pure algebraic factorization questions to the case $1 \in \operatorname{supp}\pi$. This preservation property is the reason for having introduced Property (D) in the form above.

3. A Khintchine type decomposition theorem for symmetric hypergroups

This section is devoted to a Khintchine type decomposition theorem for probability measures on symmetric hypergroups. To derive this main result which is formulated in Theorem 3.8 we shall need some preparatory results, one of them being the following central limit theorem which is a consequence of the fact that the set of all infinitely divisible probability measures is weakly closed.

3.1. Proposition. *Let K be a symmetric hypergroup. Let $(n_j)_{j\in\mathbb{N}} \subset \mathbb{N}$ be a sequence satisfying $\lim_{j\to\infty} n_j = \infty$, and let the measures μ, $\mu_{i,j} \in M^1(K)$ $(i, j \in \mathbb{N},\ 1 \le i \le n_j)$ satisfy*

$$\lim_{j\to\infty} \prod_{i=1}^{n_j} \hat{\mu}_{i,j}(\alpha) = \hat{\mu}(\alpha) \quad \textit{and} \quad \lim_{j\to\infty} \max_{1\le i\le n_j} |\hat{\mu}_{i,j}(\alpha) - 1| = 0$$

for every $\alpha \in \widehat{K}$. Then μ is infinitely divisible.

Proof. For $\alpha \in \widehat{K}$ we have

$$\hat{\mu}(\alpha) = \lim_{j\to\infty} \exp\left(\sum_{i=1}^{n_j} \ln(\hat{\mu}_{i,j}(\alpha))\right) = \lim_{j\to\infty} \exp\left(\sum_{i=1}^{n_j} (\hat{\mu}_{i,j}(\alpha) - 1)\right).$$

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Thus, by [47, Theorem 2.5], the sequence $(\exp(\sum_{i=1}^{n_j}(\mu_{i,j}-1)))_{j\in\mathbf{N}} \subset M^1(K)$ converges vaguely (and hence weakly) to $\mu \in M^1(K)$, and this sequence consists of infinitely divisible measures. Theorem 4.7 of Voit [47] finishes the proof.

We next introduce some notation needed for the subsequent results.

3.2. NOTATION. Let K be a second countable commutative hypergroup and $\mu \in M^1(K)$ a fixed probability measure. Since then $\widehat{K}$ is also second countable (see Proposition 1.2), there exists a function $l \in L^1(\widehat{K}) \cap C_b(\widehat{K})$ such that $l(\alpha) > 0$ and $l(\bar{\alpha}) = \overline{l(\alpha)}$ for all $\alpha \in \widehat{K}$.

Now, using the convention $1/\infty = 0$, we define the function

$$h := l \cdot \min(1, -1/\ln|\hat{\mu}|).$$

Then h and $h \cdot \ln|\hat{\mu}|$ are contained in the space $L^1(\widehat{K}) \cap C_b(\widehat{K})$ and, since $0 \geq \ln|\hat{\nu}| \geq \ln|\hat{\mu}|$ for any $\nu \in D(\mu)$, the function

$$\theta(\nu) := -\int_{\widehat{K}} \ln|\hat{\nu}| \cdot h\, d\pi$$

is well defined and weakly continuous on $D(\mu)$. Let us note two obvious properties of θ:

(1) $\theta(\nu) \geq 0$, for all $\nu \in D(\mu)$;

(2) $\theta(\nu_1 * \nu_2) = \theta(\nu_1) + \theta(\nu_2)$, for all $\nu_1, \nu_2 \in D(\mu)$.

3.3. LEMMA. *In the situation above let $\mu \in M^1(K)$ be a measure without idempotent factors. Then, for every compactum $L \subset \widehat{K}$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\nu \in D(\mu),\ \ \theta(\nu) \leq \delta \Rightarrow |\hat{\nu}(\alpha)| \geq 1-\varepsilon \qquad \forall \alpha \in L.$$

In particular, $\nu \in D(\mu)$ is a unit if and only if $\theta(\nu) = 0$.

PROOF. Fix $\varepsilon > 0$. If h is given as in 3.2, then $\check{h} \in C_0(K)$ is real valued and satisfies $|\check{h}| \leq \check{h}(e)$ [7, Theorem 2.4.1]. Also $W := \{x \in K : |\alpha(x) - 1| < \varepsilon/2$ for all $\alpha \in L\}$ is an open neighbourhood of $e \in K$. Moreover we have $c := \inf\{\check{h}(e) - \check{h}(x) : x \in K\backslash W\} > 0$.

In order to prove this fact, we first note that

$$\{x \in K : \check{h}(x) = \check{h}(e)\} = \{x \in K : \beta(x) = 1 \qquad \forall \beta \in \operatorname{supp} h\pi\}$$

is a compact subhypergroup of K (see Bloom and Heyer [8, Proposition 3.1]) where the associated normalized Haar measure ω satisfies

$$\operatorname{supp} \hat{\mu}\pi = \operatorname{supp} h\pi \subset \{\beta \in \widehat{K} : \hat{\omega}(\beta) = 1\}.$$

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Hence, using [47, Theorem 2.3], we see that ω is an idempotent factor of μ. Thus $\omega = \delta_e$, and therefore $\check{h}(x) \neq \check{h}(e)$ for every $x \in K\backslash\{e\}$. Lastly, since $S := \{x \in K : \check{h}(x) \geq 1/2\}$ and $S\backslash W$ are compact, we conclude $c > 0$ as claimed.

Now if we define $\delta := \varepsilon \cdot c/8$, then $\nu \in D(\mu)$ and $\theta(\nu) \leq \delta$ imply that

$$\begin{aligned}\int_{K\backslash W} 1 d(\nu * \nu^*) &\leq \int_K \frac{\check{h}(e) - \check{h}}{c} d(\nu * \nu^*) = \frac{1}{c}\int_{\widehat{K}} (1 - |\hat{\nu}^2|) h\, d\pi \\ &\leq \frac{-2}{c}\int_{\widehat{K}} \ln(|\hat{\nu}|) \cdot h\, d\pi \leq \frac{2}{c}\delta \leq \varepsilon/4\end{aligned}$$

and

$$\begin{aligned}1 - |\hat{\nu}(\alpha)| \leq 1 - |\hat{\nu}(\alpha)|^2 &\leq \int_W (1 - \bar{\alpha}) d(\nu * \nu^*) + \int_{K\backslash W} (1 - \bar{\alpha}) d(\nu * \nu^*) \\ &\leq \varepsilon/2 + \varepsilon/2 = \varepsilon\end{aligned}$$

for every $\alpha \in L$.

The last assertion also follows since $\nu \in M^1(K)$ is a unit if and only if $|\hat{\nu}| = 1$ on $\widehat{K}$.

The elementary assertion of the following remark is needed for the proof of Theorem 3.5.

3.4. Remark. For $n \in \mathbb{N}$ consider the group $G = (\mathbb{Z}/2\mathbb{Z})^n$. Since the product of all 2^n elements of G is equal to the neutral element e of G, any product of 2^n elements of G that is not equal to e contains at least two equal factors. We shall use this fact below via the obvious isomorphism between G and the n-tuple of all signs of n nonzero real numbers.

3.5. Theorem. *Let K be a second countable symmetric hypergroup. Let $\mu \in M^1(K)$ be a measure with Property* (D) *such that no factor of μ is idempotent or indecomposable. Then there exists a unit $\rho \in M^1(K)$ such that $\mu * \rho$ is infinitely divisible.*

Proof. (1) Using Lemma 2.5, we may assume without loss of generality that $\mathbf{1} \in \operatorname{supp} \pi$ (for the interrelation between $(K, *)$ and $(K, \bullet)$ see Voit [46]). Then $D(\mu)$ is weakly compact by Corollary 2.4. Using Notation 3.2, we first observe that

$$\inf_{\nu \in D(\mu),\, \theta(\nu) \neq 0} \theta(\nu) = 0,$$

since otherwise the infimum would be attained at an indecomposable factor of μ. Thus, since every factor of μ satisfies the conditions of Theorem 3.5,

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we get

$$\inf_{\nu\in D(\mu),\,\theta(\nu)\geq\theta(\mu)/2}\theta(\nu)=\theta(\mu)/2.$$

Hence there exist $\mu_1, \mu_2 \in D(\mu)$, such that $\mu_1 * \mu_2 = \mu$ and $\theta(\mu_1) = \theta(\mu_2)$. A repetition of these arguments shows that for every $n \in \mathbb{N}$ there exist factors $\nu_{n,1}, \nu_{n,2}, \ldots, \nu_{n,2^n} \in D(\mu)$ such that

$$\nu_{n,1} * \cdots * \nu_{n,2^n} = \mu \quad \text{and} \quad \theta(\nu_{n,k}) = 2^{-n}\cdot\theta(\mu) \qquad (k = 1, 2, \ldots, 2^n).$$

(2) Fix $m \in \mathbb{N}$ and let $\alpha_1, \ldots, \alpha_m$ be different points of $\widehat{K}$ such that $\hat{\mu}(\alpha_i) \neq 0$ for $i = 1, \ldots, m$. We shall prove that there exists a unit $\rho \in M^1(K)$ such that $\hat{\rho}\hat{\mu}$ is positive on $\{\alpha_1, \ldots, \alpha_m\}$. We may assume that $\hat{\mu}$ takes negative values on this set (otherwise take $\rho = \delta_e$). Then, if the measures $\nu_{m,1}, \nu_{m,2}, \ldots, \nu_{m,2^m} \in D(\mu)$ are given as described at the end of part (1), Remark 3.4 shows that we may assume that $\hat{\nu}_{m,1}\hat{\nu}_{m,2}$ is positive on $\{\alpha_1, \ldots, \alpha_m\}$. Therefore $\lambda_1 := \nu_{m,3} * \cdots * \nu_{m,2^m} \in D(\mu)$ has the property that $\hat{\lambda}$ has the same sign as $\hat{\mu}$ on $\{\alpha_1, \ldots, \alpha_m\}$, and that $\theta(\lambda_1) = (1 - 2^{1-m})\cdot\theta(\mu)$. Since every factor of μ satisfies the conditions of Theorem 3.5, a repetition of our arguments implies that for any $k \in \mathbb{N}$ there exist measures $\lambda_k \in D(\mu)$ such that $\theta(\lambda_k) = (1 - 2^{1-m})^k\cdot\theta(\mu)$ and such that $\hat{\lambda}_k$ has the same sign as $\hat{\mu}$ on $\{\alpha_1, \ldots, \alpha_m\}$. Since $D(\mu)$ is compact and metrizable, there exists a subsequence $(\lambda_{k_l})_{l\in\mathbb{N}}$ that converges weakly to a measure $\rho \in D(\mu)$. Since $\theta(\rho) = 0$, Lemma 3.3 shows that ρ is a unit. Moreover, since $(\hat{\lambda}_{k_l})_{l\in\mathbb{N}}$ converges pointwise on $\widehat{K}$ to $\hat{\rho}$ [47, Theorem 2.6], $\hat{\rho}$ has the same sign as $\hat{\mu}$ on $\{\alpha_1, \ldots, \alpha_n\}$.

(3) By Proposition 1.2 there exists a countable dense subset $\{\alpha_1, \alpha_2, \ldots\}$ of $\widehat{K}$. For any $m \in \mathbb{N}$ there exists a unit ρ_m such that $\hat{\rho}_m\cdot\hat{\mu}$ is nonnegative on $\{\alpha_1, \ldots, \alpha_m\}$. Since $D(\mu)$ is compact, the sequence $(\rho_m)_{m\in\mathbb{N}}$ has a convergent subsequence where the limit ρ is again a unit and where $\hat{\rho}$ has the same sign as $\hat{\mu}$ on $\{\alpha_1, \alpha_2, \ldots\}$. Since $\hat{\rho}$ and $\hat{\mu}$ are continuous on $\widehat{K}$, this remains valid on $\widehat{K}$.

(4) Since the conditions of Theorem 3.5 hold for every factor of μ, parts (1) and (3) show that for every $n \in \mathbb{N}$ there exist measures $\nu_{n,1}, \nu_{n,2}, \ldots, \nu_{n,2^n} \in D(\mu)$ such that $\nu_{n,1} * \cdots * \nu_{n,2^n} = \mu * \rho$ (ρ defined as at the end of part (3)) and such that

$$\theta(\nu_{n,k}) = 2^{-n}\cdot\theta(\mu) \quad \text{and} \quad \hat{\nu}_{n,k}(\alpha) \geq 0 \qquad (\alpha \in \widehat{K}, k = 1, 2, \ldots, 2^n).$$

Now Lemma 3.3 and Proposition 3.1 complete the proof.

3.6. Lemma. *Let K be a commutative hypergroup. If $\mu, \nu \in M^1(K)$ have no idempotent factors and satisfy $\mu \in D(\nu)$ and $\nu \in D(\mu)$, then there exists a unit $\rho \in M^1(K)$ such that $\mu = \rho * \nu$.*

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Proof. Using the function θ on $D(\mu)$ according to Section 3.2, from $\mu = \nu*\lambda_1$ and $\nu = \mu*\lambda_2$ we obtain $\theta(\mu) = \theta(\nu)+\theta(\lambda_1) = \theta(\mu)+\theta(\lambda_1)+\theta(\lambda_2)$. Since $\theta(\lambda_1), \theta(\lambda_2) \geq 0$, $\theta(\lambda_1) = \theta(\lambda_2) = 0$ follows. Therefore, by Lemma 3.3, λ_1 and λ_2 are units.

3.7. **Corollary.** *Let K be a second countable commutative hypergroup. Let $\mu \in M^1(K)$ be without idempotent factors such that $D(\mu)$ is compact. If the sequence $(\nu_n)_{n\in\mathbb{N}} \subset D(\mu)$ satisfies $\nu_n \in D(\nu_{n+1})$ for every $n \in \mathbb{N}$, then there exists a sequence $(\rho_n)_{n\in\mathbb{N}}$ of units such that $(\rho_n * \nu_n)_{n\in\mathbb{N}}$ is weakly convergent.*

Proof. (cf. Parthasarathy [38, Theorem III.5.3]) $D(\mu)$ is a compact, metrizable space with a suitable metric d. Let $\nu \in D(\mu)$ be the weak limit of a convergent subsequence of $(\nu_n)_{n\in\mathbb{N}}$. Denoting the set of units by U, we show that

$$\lim_{n\to\infty} \inf_{\rho\in U} d(\nu_n * \rho, \nu) = 0$$

which yields the assertion of the corollary. To show this, take a further subsequence $(\nu_{n_k})_{k\in\mathbb{N}}$ which converges, without loss of generality, to a measure $\tilde{\nu} \in D(\mu)$. Since $D(\nu)$ and $D(\tilde{\nu})$ are closed, our assumption implies that $\nu \in D(\tilde{\nu})$ and $\tilde{\nu} \in D(\nu)$. Therefore, by Lemma 3.6, $\nu = \rho*\tilde{\nu}$ where $\rho \in U$. Since this is true for every subsequence, the proof is complete.

3.8. **Theorem.** *Let K be a second countable symmetric hypergroup. Then every measure $\mu \in M^1(K)$ with Property* (D) *and without idempotent factor can be written as*

$$\mu = \nu_0 * \nu_1 * \nu_2$$

where ν_0 is a unit, $\nu_1 \in M^1(K)$ is an infinitely divisible measure without indecomposable or idempotent factors and, lastly, $\nu_2 \in M^1(K)$ is the convolution product of at most a countable number of indecomposable factors of μ.

If ν_2 consists of infinitely many indecomposable factors $(\lambda_i)_{i\in\mathbb{N}}$, then the partial products $(\lambda_1 * \cdots * \lambda_n)_{n\in\mathbb{N}}$ converge to ν_2 in the following sense: $(\varphi(\lambda_1 * \cdots * \lambda_n))_{n\in\mathbb{N}}$ converges weakly to $\varphi(\nu_2)$ where the mapping $\varphi : (M^1(K), *) \to (M^1(K), \bullet)$ is as described in Lemma 2.5.

If $\mathbf{1} \in \operatorname{supp}\pi$ or if there exists a sequence $(\alpha_k)_{k\in\mathbb{N}}$ of positive characters vanishing at infinity such that $(\alpha_k)_{k\in\mathbb{N}}$ converges to $\mathbf{1} \notin \operatorname{supp}\pi$, then $(\lambda_1 * \cdots * \lambda_n)_{n\in\mathbb{N}}$ itself converges weakly to ν_2.

Proof. Using Lemma 2.5 and Lemma 2.6, we may without loss of generality assume that $\mathbf{1} \in \operatorname{supp}\pi$. Moreover, Corollary 2.4 shows that the

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assumptions of Corollary 3.7 are true. Now, utilizing Theorem 3.5 and Corollary 3.7, we can apply the arguments of Parthasarathy [38, Theorem IV.11.3] in order to obtain the desired decomposition.

3.9. REMARKS. (1) There are no known examples of symmetric hypergroups for which, in the preceding theorem, the partial products

$$(\lambda_1 * \cdots * \lambda_n)_{n\in\mathbf{N}}$$

of indecomposable factors are not convergent to ν_2 with respect to the weak topology.

(2) Let the maximal subgroup G of a commutative hypergroup K be noncompact. Then $\{\delta_x : x \in G\}$ is a closed and noncompact subset of $D(\mu)$ for every $\mu \in M^1(K)$, and thus no measure $\mu \in M^1(K)$ has Property (D). Therefore the theorems contained in Section 3 cannot be applied to hypergroups with a noncompact closed subgroup.

(3) Since $M^1(K)$ is weakly compact for every compact space K [21, Corollary 1.1.4], Lemma 2.3 shows that $D(\mu)$ is compact for every probability measure μ on a compact commutative hypergroup K. Since for such a hypergroup $\operatorname{supp}\pi = \widehat{K}$ is also true, all the conditions of Theorems 3.8 are satisfied for compact symmetric hypergroups. A family of examples is contained in Section 6, and a further family is discussed briefly now:

(4) On the compact interval $[-1, 1]$ there exist many hypergroup structures, and every hypergroup structure on this compact interval must be symmetric (see Schwartz [42] and Zeuner [53]). The best known examples are induced by linearization formulas for Jacobi polynomials $(P_n^{(\alpha,\beta)})_{n\in\mathbf{N}_0}$ (for details see, for instance, Lasser [30, Section 4]).

In particular, for $\alpha = \beta \geq -1/2$ we have hypergroups associated with ultraspherical polynomials. For this special case, factorization theorems and results concerning the set $I_0(K)$ were proved by Lamperti [29] and Truhina [45]. Also the Levy-Khintchine representation for infinitely divisible measures follows for these examples immediately from the general results of Lasser [32]. Further examples of hypergroup structures on $[-1, 1]$ that are associated with eigenfunctions of Sturm-Liouville differential equations can be found in some recent papers of Connett, Markett and Schwartz [14]. For these examples, the Levy-Khintchine representation and the classification of $I_0(K)$ seem to be still unknown.

The purpose of the next proposition is to generalize the preceding decomposition results to measures having idempotent factors. But, unfortunately, it is in general not possible to reduce the factorization problem completely, and thus we are only able to prove the following partial result:

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3.10. **Theorem.** *Let K be a second countable commutative hypergroup and $\mu \in M^1(K)$. Then there exists a compact subhypergroup H of K such that the associated Haar measure ω_H is the maximal idempotent factor of μ, that is, $\omega_L \in D(\omega_H)$ holds for all idempotent factors ω_L of μ. Moreover, if $\{x\} \neq \{x\} * L$ for every compact proper subhypergroup $L \subset H$ and every $x \in K$, then μ can be written as $\mu = \omega_H * \nu$ where $\nu \in M^1(K)$ has no idempotent factor.*

Conversely, if there are a compact proper subhypergroup L and $x \in K$ such that $\{x\} = \{x\} * L$, then δ_x has ω_L as maximal idempotent factor and the convolution equation $\omega_L * \nu = \delta_x$ has in $M^1(K)$ the unique solution $\nu = \delta_x$.

Proof (cf. Parthasarathy [38, Theorem IV.11.1]). The set $W := \{\alpha \in \widehat{K} : \hat{\mu}(\alpha) \neq 0\}$ satisfies $\pi(W) > 0$. Hence, by Bloom and Heyer [8, Proposition 3.1], the set $H := \{x \in K : \alpha(x) = 1 \quad \forall \alpha \in W\}$ is a compact subhypergroup of K. For ω_H, the normalized Haar measure of H, we have $\mu = \omega_H * \mu$. In order to show that ω_H is the maximal idempotent factor of μ, we take a further compact subhypergroup L satisfying $\omega_L \in D(\mu)$. Since then $\delta_x * \mu = \mu$ for every $x \in L$ (Bloom and Heyer [7, Theorem 3.3.5]), we have $\alpha(x) = 1$ for every $\alpha \in W$ and every $x \in L$. Thus $L \subset H$, which completes the proof of the first part.

To prove the second part, we denote the canonical projection from K onto the second countable commutative hypergroup K/H (see Jewett [25, Section 14]) by π. By a Lemma of Mackey (see, for example, [38, Lemma I.5.1]) there exists a Borel set $B \subset K$ such that $\pi|_B$ is bijective. Then, by [38, Corollary I.3.3], the inverse mapping $\pi^{-1} : K/H \mapsto B$ is Borel measurable. Now we define the measure $\nu \in M^1(K)$ by $\nu(A) := \mu(\pi^{-1}(\pi(A \cap B)))$, A being a Borel set in K. Since obviously $\omega_H * \nu = \mu$, it suffices to show that ν has no idempotent factor in order to complete the proof. To do this, we take a proper compact subhypergroup L of K where we can assume $L \subset H$. Since then for every $x \in K$ the set $B \cap (\{x\} * L)$ consists of at most one point, our assumption on L implies that

$$0 < 1 - \omega_L * \delta_x(B \cap (L * \{x\})) = \omega_L * \delta_x(K \backslash B) = \omega_L * \chi_{K\backslash B}(x)$$

for all $x \in K$, $\chi_{K\backslash B}$ denoting the characteristic function of the set $K \backslash B$. Hence

$$0 < \int_K \omega_L * \chi_{K\backslash B}\, d\nu = \omega_L * \nu(K\backslash B).$$

Therefore, since $\nu(K\backslash B) = 0$, we obtain $\omega_L \notin D(\mu)$ as desired.

The last part of the proposition is obvious.

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3.11. REMARKS. (1) Remark 4.6(2) contains an example of a finite symmetric hypergroup K with a proper subhypergroup L and an $x \in K$ satisfying $\{x\} * L = \{x\}$. Therefore, in contrast to locally compact abelian groups, it is not always possible to decompose every probability measure into an idempotent factor and a part without idempotent factors.

(2) It is not difficult to improve the assertions of Proposition 3.10 and to find weaker assumptions concerning the measure μ which ensure the desired decomposition of μ. But, since necessary and sufficient conditions for this decomposition admit a complicated description only, and since the idempotent factors can be treated ad hoc for many examples, we omit further details here.

4. Some supplementary general results about $I_0(K)$

After having proved the Decomposition Theorem 3.8, we would now like also to find Levy-Khintchine representations for infinitely divisible measures and, after that, an explicit description of $I_0(K)$, the set of all probability measures having no indecomposable factors. The first problem is still unsolved for arbitrary commutative or symmetric hypergroups. The Levy-Khintchine representation is known only for locally compact abelian groups (see, for instance, Berg and Forst [3]), for discrete symmetric hypergroups (see Proposition 4.1), for hypergroups which are intimately connected with special differential equations (see Chebli [11] and Section 7) and for hypergroups whose dual spaces admit natural dual hypergroup structures (see Lasser [32]). Since the Levy-Khintchine representation seems to be one essential basis for finding $I_0(K)$ explicitly, and since explicit descriptions of $I_0(K)$ vary strongly for different families of hypergroups (see Sections 5–8), it seems to be impossible to determine $I_0(K)$ for general symmetric hypergroups. On the other hand, since the Levy-Khintchine representation is available for many examples, we can compute $I_0(K)$ for many families of symmetric hypergroups (see Sections 5–8). In order to make the discussion of the special cases as easy as possible, we shall derive some general results about infinitely divisible measures and measures without indecomposable factors in this section.

The first result we here present is a Levy-Khintchine representation for infinitely divisible measures on discrete hypergroups. The proposition slightly generalizes well known results of this type (see Gallardo and Gebuhrer [19] and Voit [47]). It is remarkable that it is possible here to separate the idempotent part of an infinitely divisible measure completely (in contrast to the situation of Proposition 3.10).

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4.1. Proposition. *Let K be a discrete symmetric hypergroup and let $\mu \in M^1(K)$ be infinitely divisible. Then there exists an idempotent $\omega_H \in M^1(K)$, $\nu \in M^1(K)$ and $t \geq 0$ such that*

$$\mu = \omega_H * (e^{-t} \cdot \exp t\nu),$$

exp *denoting the usual exponential function in the Banach algebra $M_b(K)$.*

Proof. (cf. Parthasarathy [38, Theorem IV.7.2]) Since the subhypergroup of K generated by $\operatorname{supp}\mu$ is at most countable, we may without loss of generality assume that K itself is at most countable. Let ω_H be the maximal idempotent factor of μ where $H = \{e\}$ is possible. Taking the notation of Jewett [25, Section 14], we note that the canonical mapping

$$\pi^* : M_b(K/H) \mapsto M(K \mid H) := \{\nu \in M_b(K) : \nu * \omega_H = \nu\}$$

is an isomorphism (Jewett [25, 14.2E]). Since $(\pi^*)^{-1}(\mu)$ is an infinitely divisible measure on the discrete symmetric hypergroup K/H and has no idempotent factors, Theorem 4.3 and Theorem 3.9 of Voit [47] imply that $(\pi^*)^{-1}(\mu) = e^{-t} \cdot \exp t\nu$ with $t \geq 0$ and $\nu \in M^1(K/H)$. It follows that

$$\mu = \omega_H * e^{-t} \exp(t\pi^*(\nu)).$$

4.2. Proposition. *Let K be a second countable symmetric hypergroup, and let $\nu \in M_b(K)$ such that $\nu + c\delta_e \notin M^+(K)$ for all $c \in \mathbb{R}$. If $\mu \in M^1(K)$ can be written as $\mu = e^{-\nu(K)} \cdot \exp \nu$, then μ is not infinitely divisible. Moreover, if μ has Property* (D) *then μ has an indecomposable factor.*

Proof. Let μ be given as described above, and assume μ to be infinitely divisible. Since $\hat{\mu} = e^{\hat{\nu} - \hat{\nu}(1)}$ takes only positive values on $\widehat{K}$, μ has no idempotent factor. Thus Theorem 4.3 and Theorem 3.7 of Voit [47] imply that μ admits a representation $\hat{\mu} = e^{-f}$ where $f \in C(\widehat{K})$ is a real valued, strongly negative definite function (in the sense of [47]). Comparing this with the preceding representation of $\hat{\mu}$, we obtain $f = \hat{\nu}(\mathbf{1}) - \hat{\nu}$. In particular, f must be bounded and can therefore be written as $f = \hat{\rho}(\mathbf{1}) - \hat{\rho}$, ρ being a positive bounded measure on K (see Voit [47, Theorem 3.9]). It follows that $\nu = \rho + (\hat{\rho}(\mathbf{1}) - \hat{\nu}(\mathbf{1}))\delta_e$ contradicting our assumptions. Thus μ cannot be infinitely divisible.

Since every unit $\delta_x \in M^1(K)$ has the property that $\hat{\delta}_x(\alpha) \in \{\pm 1\}$ for all $\alpha \in \widehat{K}$, the injectivity of the Fourier transform implies that $\hat{\delta}_x$ takes negative values on $\widehat{K}$ for every unit $\delta_x \neq \delta_e$. Therefore, since every infinitely divisible measure takes only nonnegative values, the measure $\mu * \delta_x$ cannot be infinitely divisible for all units $\delta_x \neq \delta_e$. Since this fact is also true for δ_e

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by the considerations above, Theorem 3.5 entails that μ has indecomposable factors.

4.3. THEOREM. *Let K be a second countable symmetric hypergroup and ω_H the normalized Haar measure of a compact subhypergroup H which consists of more than two elements. If $\mu \in M^1(K)$ satisfies* $\operatorname{supp}\mu \subset H$, *is minorized by ω_H (that is, there is a constant $0 < a < 1$ such that $\mu(E) \geq a\omega_H(E)$ for all Borel sets $E \subset K$) and has Property* (D), *then μ is decomposable and has indecomposable factors.*

PROOF. Putting $\lambda := (\mu - a\omega_H)/(1-a)$, we have $\lambda \in M^1(K)$, $\lambda * \omega_H = \omega_H$ and therefore

$$\mu = (1-a)\lambda + a\omega_H = ((1-a)\delta_e + a\omega_H) * \lambda.$$

Hence, μ is decomposable and has a factor of the form

$$e^{-t}\delta_e + (1-e^{-t})\omega_H = e^{-t}\exp(t\omega_H) =: \nu \in M^1(K)$$

where $1 - e^{-t} = a$. Since H contains at least 3 elements, we can choose a closed set $D \subset K$ satisfying $e \notin D$ and $0 < \omega_H(D) =: d < 1/2$. Now, defining ρ as the restriction of ω_H to D, we obtain that the measures

$$(\omega_H - (1+\varepsilon)\rho)^2 = \omega_H(1 - 2d(1+\varepsilon)) + (1+\varepsilon)^2\rho^2,$$

$$\begin{aligned}(\omega_H - (1+\varepsilon)\rho) &+ \frac{t}{2}(\omega_H - (1+\varepsilon)\rho)^2 \\ &= \omega_H\left(1 + \frac{t}{2}(1 - 2d(1+\varepsilon))\right) - (1+\varepsilon)\rho + (1+\varepsilon)^2\rho^2\end{aligned}$$

and

$$(\omega_H - (1+\varepsilon)\rho)^3 = \omega_H(1 - 3d(1+\varepsilon) + 3d^2(1+\varepsilon)^2) - (1+\varepsilon)^3\rho^3$$

are positive if $\varepsilon > 0$ is sufficiently small. This statement is obvious for the first two measures. For the third measure it is an immediate consequence of the fact that $\rho^3 = (\chi_D * \chi_D * \chi_D)\omega_H$ (χ_D being the characteristic function of D) and that

$$\|\chi_D * \chi_D * \chi_D\|_\infty \leq \|\chi_D\|_\infty \|\chi_D\|_1^2 = d^2.$$

The positivity of the three measures above implies that

$$\begin{aligned}&e^{t((1+\varepsilon)d-1)}\left(\delta_e + \sum_{n=1}^{\infty}\frac{t^n}{n!}(\omega_H - (1+\varepsilon)\rho)^n\right) \\ &\qquad = e^{t((1+\varepsilon)d-1)}\exp t(\omega_H - (1+\varepsilon)\rho)\end{aligned}$$

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is a probability measure. Since this measure is a factor of μ and since it has indecomposable factors by Proposition 4.2 (note that this measure has Property (D)), the proof is finished.

4.4. REMARKS. (1) Fel′dman has presented a similar result for locally compact abelian groups in [17]. Since Fel′dman's paper contains only vague references to the proof, we have included the complete proof of Theorem 4.3. Also the following Corollary 4.5 was given by Fel′dman [17, Theorem 4] for abelian groups.

(2) The assertion of Theorem 4.3 may fail for certain subhypergroups H which consist of two elements. To present an example, we note that on the set $H = \{0, 1\}$ a hypergroup structure with neutral element 0 is determined uniquely by

$$\delta_1 * \delta_1 = b\delta_0 + (1-b)\delta_1,$$

$b \in]0, 1]$ being a fixed parameter. Now $\widehat{H}$ is given by $\widehat{H} = \{\mathbf{1}, \alpha\}$ where the character α satisfies $\alpha(0) = 1$ and $\alpha(1) = -b$. Moreover, via the mapping $\mu \mapsto \hat{\mu}(\alpha)$ we obtain a semigroup isomorphism between $M^1(H)$ and $[-b, 1]$ where the second space is a semigroup with respect to the usual multiplication. If $b \in]0, 1[$, then $-b$ is the only indecomposable factor of $[-b, 1]$ and $x \in [-b, 1]$ has $-b$ as indecomposable factor if and only if $x \in [-b, b^2]$. It follows that $I_0(H) = \{r\delta_0 + (1-r)\delta_1 ; r > b\}$ for $b \in]0, 1[$. Furthermore, for $b = 1$, we observe that $M^1(H) \simeq [-1, 1]$ has no indecomposable factor and thus that $I_0(H) = M^1(H)$ holds. In summary, we see that the assertion of Theorem 4.3 fails in both cases.

Let us apply the preceding theorem to two classes of examples for which $I_0(K)$ can be determined more precisely. The first class consists of the symmetric discrete hypergroups in which every element is contained in a finite subhypergroup.

4.5. COROLLARY. *Let K be a discrete symmetric hypergroup such that every $x \in K$ is contained in a finite subhypergroup of K. If $\mu \in M^1(K)$ has Property* (D) *and has no idempotent or indecomposable factor, then there exists a unit $\delta_x \in M^1(K)$ such that* $\operatorname{supp}(\delta_x * \mu)$ *is a subhypergroup of K consisting of at most two elements.*

In particular, if K is finite and symmetric, then this conclusion is true for every $\mu \in M^1(K)$ without idempotent and indecomposable factors.

PROOF. Since the subhypergroup of K generated by $\operatorname{supp}\mu$ is at most countable, we may assume that K itself is at most countable. Now, using

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Theorem 3.5 and Proposition 4.1, we can write μ as $\mu = \delta_x * e^{-t}\exp(t\nu)$ where δ_x is a unit, $t \geq 0$ and $\nu \in M^1(K)$. Take $x_1, x_2 \in \operatorname{supp}\nu\backslash\{e\}$. Then there exists a factor $\rho = e^{-\tau_1-\tau_2}\cdot\exp(\tau_1\delta_{x_1} + \tau_2\delta_{x_2})$ $(\tau_1, \tau_2 > 0)$ of μ which has Property (D) and which has no idempotent or indecomposable factors. Since

$$\operatorname{supp}\rho = \left(\bigcup_{n\in\mathbb{N}_0}\operatorname{supp}\delta_{x_1}^n\right) * \left(\bigcup_{n\in\mathbb{N}_0}\operatorname{supp}\delta_{x_2}^n\right)$$

is a finite subhypergroup by our assumption, Theorem 4.3 implies that $\operatorname{supp}\rho$ consists of at most two points. Thus $\operatorname{supp}\nu\backslash\{e\}$ contains at most one element, and $\operatorname{supp}(e^{-t}\exp(t\nu))$ is a subhypergroup consisting of at most two elements.

4.6. Remarks. (1) Since the hypergroups consisting of two elements are very easy to handle (see Remark 4.4(2)), it is not difficult to determine the set of all measures without idempotent or indecomposable factors explicitly for concrete finite symmetric hypergroups.

(2) The assertion of Corollary 4.5 may fail for measures $\mu \in I_0(K)$ which have idempotent factors. To illustrate this, we define a symmetric hypergroup structure on $K = \{0, 1, 2\}$ with neutral element 0 by

$$\delta_1 * \delta_1 = \delta_0, \qquad \delta_1 * \delta_2 = \delta_2 * \delta_1 = \delta_2, \qquad \delta_2 * \delta_2 = (\delta_0 + \delta_1)/2.$$

It is easy to see that $\{0, 1\}$ is a subgroup isomorphic to $\mathbb{Z}_2$, that every $\mu \in M := \{\nu \in M^1(K) : \operatorname{supp}\nu \subset \{0, 1\}\} \cup \{\delta_2\}$ is decomposable and that $D(\mu) \subset M$ for all $\mu \in M$. We thus have $I_0(K) = M$ and, in particular, $\delta_2 \in I_0(K)$.

The behaviour of the example considered in Remark 4.6(2) is intimately connected with the fact that this hypergroup is just the hypergroup join $\mathbb{Z}_2 \vee \mathbb{Z}_2$.

At the end of this section we shall deal with $I_0(K)$ for hypergroup joins $K = H \vee L$, H being a compact and L being a discrete symmetric hypergroup. For details on the definition of the hypergroup join we refer to Jewett [25, Section 10.5], and to Vrem [52]. Before we present the main result, we prove a technical lemma.

4.7. Lemma. *Let $K := H \vee L$ where $H = \{0, 1\}$ is a hypergroup with two elements and 0 as neutral element and where L is an at most countable discrete symmetric hypergroup. If $\mu \in M^1(K)$ has Property* (D) *and if μ can be written as $\mu = e^{-t}\exp t\nu$ with $t > 0$, $\nu \in M^1(K)$ and $\operatorname{supp}\nu \not\subset H$, then μ has an indecomposable factor.*

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Proof. Denote the neutral element of K by 0. Since K is discrete, we may without loss of generality assume that $\nu = \delta_x$ with $x \notin H$. Moreover,

$$\mu = e^{-t}\exp t\nu = e^{-t}\cdot\left(\delta_e + t\nu + \frac{t^2}{2}\nu * \nu + \cdots\right)$$

implies that H is contained in, but not equal to $\operatorname{supp}\mu$.

We next show that μ has factors of the form $(1-c)\delta_0 + c\delta_1$ where $c > 0$ must be sufficiently small. To do this, we write μ as $\mu = c_0\delta_0 + c_1\delta_1 + c_2\lambda$ where $c_0, c_1, c_2 > 0$ and $\lambda \in M^1(K)$ with $\operatorname{supp}\lambda \cap H = \varnothing$. For fixed $c > 0$ we consider the convolution equation $((1-c)\delta_0 + c\delta_1) * \rho = \mu$. After writing ρ as $\rho = d_0\delta_0 + d_1\delta_1 + d_2\tilde{\lambda}$ ($\tilde{\lambda} \in M_b(K)$ with $\operatorname{supp}\tilde{\lambda} \cap H = \varnothing$), we see that this convolution equation is equivalent to

$$(1-c)\rho + cd_0\delta_1 + cd_1(\delta_1 * \delta_1) + cd_2\tilde{\lambda} = c_0\delta_0 + c_1\delta_1 + c_2\lambda.$$

Using Remark 4.4(2), we know that there is a constant $b \in]0, 1]$ such that $\delta_1 * \delta_1 = b\delta_0 + (1-b)\delta_1$. Therefore the convolution equation above is equivalent to

$$\lambda = \tilde{\lambda}, \quad c_0 = (1-c)d_0 + bcd_1, \quad c_1 = ((1-c) + c(1-b))d_1 + cd_0 \quad \text{and} \quad c_2 = d_2.$$

Since for $c = 0$ we have $d_0 = c_0 > 0$ and $d_1 = c_1 > 0$, an easy continuity argument yields an $\varepsilon > 0$ such that $\rho = \rho(c)$ is a positive measure for all $c \in [0, \varepsilon]$.

By a further continuity argument we see that there exists $r > 0$ such that $\kappa := e^{-r}\exp r\delta_1 = c(r)\delta_0 + (1 - c(r))\delta_1$ where $0 < c(r) < \varepsilon$. Now the results above imply that $\kappa \in D(\mu)$ and, in particular, $e^{r-t}\exp(t\nu - r\delta_1) \in D(\mu)$. Since this measure has indecomposable factors by Proposition 4.2, the proof is complete.

4.8. Theorem. *Let $K := H \vee L$ where $H \neq \{e\}$ is a second countable compact symmetric hypergroup and L is a symmetric discrete hypergroup. If $\mu \in M^1(K)$ has Property* (D) *and has no indecomposable or idempotent factor, then* $\operatorname{supp}\mu \subset H$.

Proof. Since there is an at most countable subhypergroup $\widetilde{L}$ of L such that $\operatorname{supp}\mu$ is contained in $H \vee \widetilde{L} \subset H \vee L$, we can assume that without loss of generality L is at most countable and K second countable.

We have to consider two cases separately.

We first assume that H contains at least three elements. Denote, as usual, the neutral element by e. Using Theorem 3.5, we can write μ as $\mu = \delta_x * \lambda * \lambda$ where δ_x is a unit and $\lambda \in M^1(K)$. It is clear that $x \in H$ for every unit δ_x. Therefore, in order to complete the proof, it suffices to show that $\operatorname{supp}\lambda \not\subset H$

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implies that $\lambda * \lambda$ has indecomposable factors. Assuming $\operatorname{supp}\lambda \not\subset H$, we can write λ as

$$\lambda = a_1\nu + \sum_{x\in L\setminus\{e\}} a_x\delta_x$$

where $\nu \in M^1(K)$, $\operatorname{supp}\nu \subset H$, $a_1 < 1$ and $a_x \geq 0$. Therefore

$$\begin{aligned}\lambda * \lambda &= a_1^2\nu * \nu + 2a_1 \sum_{x\in L\setminus\{e\}} a_x\delta_x + \sum_{x,y\in L\setminus\{e\}} a_xa_y\delta_x * \delta_y \\ &= c_1\nu * \nu + c_2\omega_H + c_3\varphi\end{aligned}$$

where $\varphi \in M^1(K)$, $\operatorname{supp}\varphi\cap H = \varnothing$, $c_1 \geq 0$, $c_2, c_2 > 0$ and $c_1+c_2+c_3 = 1$. We next consider the convolution equation

$$(c\omega_H + (1-c)\delta_e) * \rho = \lambda * \lambda$$

for $c \in [0, 1[$. Writing ρ as $d_1\nu*\nu+d_2\omega_H+d_3\varphi$, we see that the convolution equation above is a consequence of

$$d_1(1-c)\nu * \nu + (cd_1 + d_2)\omega_H + d_3\varphi = c_1\nu * \nu + c_2\omega_H + c_3\varphi.$$

Since $c_2 > 0$, we obtain that

$$d_1 = c_1/(1-c), \qquad d_2 = c_2 - cd_1 \quad \text{and} \quad d_3 = c_3$$

are nonnegative for sufficiently small $c > 0$. Therefore, the measures $c\omega_H + (1-c)\delta_e$ are factors of $\lambda * \lambda$ for sufficiently small $c > 0$. Since H consists of at least 3 elements, Theorem 4.3 implies that $\lambda*\lambda$ has an indecomposable factor.

Now assume that $|H| = 2$. Since then K is discrete, we can write μ as $\mu = \delta_x * \delta_x * e^{-r}\exp r\nu$ (δ_x a unit, $\nu \in M^1(K)$ and $r \geq 0$) by using Theorem 3.5 and Proposition 4.1. Now Lemma 4.7 and the fact $x \in H$ finish the proof.

4.9. Remark. The example presented in Remark 4.6(2) shows that the assertions of Theorem 4.8 may fail for measures $\mu \in I_0(K)$ having idempotent factors.

5. Examples: a class of discrete symmetric hypergroups

In this section we shall discuss a class of symmetric hypergroup structures on $\mathbb{N}_0$ which are constructed by an infinite number of hypergroup joins. Since for these examples the subsets $\{0, 1, \ldots, n\}$, $n \in \mathbb{N}$, form an increasing sequence of finite hypergroups, this class of hypergroups is very interesting to illustrate the preceding general decomposition results and also the

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problems which are connected with idempotent factors. For these examples which were introduced by Jewett [25, Section 15.1D] we shall reformulate the general results of the preceding sections and compute the sets $I(K)$ and $I_0(K)$.

We start with the following

5.1. DEFINITION. For each $n \in \mathbb{N}$ let b_n be a number such that $0 < b_n \leq 1$. Let $c_0 = 1$ and define numbers c_n inductively by

$$c_n = (1/b_n)(c_0 + c_1 + \cdots + c_{n-1}).$$

Then a symmetric hypergroup structure on $K := \mathbb{N}_0$ is given by

$$\delta_m * \delta_n = \delta_n * \delta_m = \delta_n \quad \text{for } 0 \leq m < n \text{ and}$$
$$\delta_n * \delta_n = \frac{c_0}{c_n}\delta_0 + \frac{c_1}{c_n}\delta_1 + \cdots + \frac{c_{n-1}}{c_n}\delta_{n-1} + (1 - b_n)\delta_n$$

where 0 is the neutral element.

Next we prove some preparatory results.

5.2. LEMMA. (1) $\operatorname{supp}\pi = \widehat{K} = \{\mathbf{1}\} \cup \{\alpha_n ; n \in \mathbb{N}\}$ *where*

$$\alpha_n(k) = \begin{cases} 1 & \text{for } k < n, \\ -b_n & \text{for } k = n, \\ 0 & \text{for } k > n. \end{cases}$$

(2) $D(\mu)$ *is compact for all* $\mu \in M^1(K)$.

PROOF. (1) We first take an arbitrary character $\alpha \in \widehat{K}\backslash\{\mathbf{1}\}$. Then there exists $n \in \mathbb{N}$ such that $\alpha(n) \neq 1$ and $\alpha(m) = 1$ for every $m < n$. Since $\alpha(m) \cdot \alpha(n) = \alpha(m)$ for $n < m$, we have $\alpha(m) = 0$ for $m > n$. Lastly, $\alpha(n)^2 = \delta_n * \delta_n(\alpha) = (1 - b_n)\alpha(n) + b_n \cdot 1$ and $\alpha(n) \neq 1$ imply $\alpha(n) = -b_n$. In particular, we have shown that every character $\alpha \neq \mathbf{1}$ is contained in $C_c(K)$. Now Proposition 1.3 shows that α is a discrete point contained in $\operatorname{supp}\pi$. On the other hand, by Voit [46, Theorem 2.11], $\operatorname{supp}\pi$ itself is a nondiscrete space so we must have $\mathbf{1} \in \operatorname{supp}\pi = \widehat{K}$ which completes the proof of the first part.

(2) Fix $\mu \in M^1(K)$. Since $c_{n-1} \geq c_0 + \cdots + c_{n-2}$ implies that $(c_{n-1}/c_n) + (1 - b_n) \geq 1/2$ for every $n \geq 2$, we obtain $\delta_k * \delta_l([n, \infty[) \geq \frac{1}{2}\delta_k([n+1, \infty[))$ for all $k, l, n \in \mathbb{N}_0$. In particular, we have $\nu_1 * \nu_2([n, \infty[) \geq \frac{1}{2}\nu_1([n+1, \infty[)$ for every $n \in \mathbb{N}_0$ and $\nu_1, \nu_2 \in M^1(K)$. Now, if $\varepsilon > 0$ is given, choose $n \in \mathbb{N}$ such that $\mu([n-1, \infty[) < \varepsilon/2$. Then the considerations above show that $\nu([n, \infty[) < \varepsilon$ for every $\nu \in D(\mu)$. Therefore $D(\mu)$ is relatively compact (Proposition 2.2) and closed (Lemma 2.3) with respect to the weak topology.

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We note that it is easy to see that $\widehat{K}$ admits a canonical dual hypergroup structure. This dual convolution structure will be studied in Section 6.

The following theorem summarizes the main factorization results for the discrete hypergroups considered in this section:

5.3. THEOREM. (1) *For every infinitely divisible measure $\mu \in M^1(K)$ there exist $t \geq 0$, $\nu \in M^1(K)$ and an idempotent $\omega_H \in M^1(K)$ such that $\mu = \omega_H * (e^{-t} \exp t\nu)$.*

(2) *If $b_1 \in]0, 1[$, then $I_0(K)$ consists exactly of the measures $\mu \in M^1(K)$ of the form $\mu = r\delta_0 + (1-r)\delta_1$ with $r \in]b_1, 1]$. If $b_1 = 1$, then*

$$I_0(K) = \{\nu \in M^1(K) : \operatorname{supp} \nu \subset \{0, 1\}\} \cup \{\delta_2\}.$$

(3) *Every $\mu \in M^1(K)$ can be written as*

$$\mu = \nu_0 * \nu_1 * \nu_2 * \rho$$

where ν_0 is a unit, $\nu_1 \in I_0(K)$, ν_2 is the weakly convergent convolution product of an at most countable number of indecomposable factors of μ and, lastly, ρ is either the normalized Haar measure of a compact subhypergroup of K or ρ is equal to δ_n where $n \geq 2$.

PROOF. (1) This is a direct consequence of Proposition 4.1.

(2) Since $K = \{0, 1\} \vee \{0, 2, 3, 4, \dots\}$ is a hypergroup join, and since every factor of $\mu \in \{\nu \in M^1(K) : \operatorname{supp} \nu \subset \{0, 1\}, \nu \neq \omega_{\{0,1\}}\}$ is supported by $\{0, 1\}$, we see that the classification of all measures having no indecomposable or idempotent factors is an immediate consequence of Theorem 4.8 and Remark 4.4(2). Now let us classify the elements of $I_0(K)$ which have idempotent factors. To do this, we first note that then $\omega_{\{0,1\}}$ is a factor, too. Since for $b_1 < 1$ the idempotent $\omega_{\{0,1\}}$ has an indecomposable factor (see Remark 4.4(2)), the classification is finished for $b_1 < 1$. Now, during the rest of the proof of part (2), we assume $b_1 = 1$. Then, defining $M := \{\mu \in M^1(K): \operatorname{supp} \mu \subset \{0, 1\}\}$ and observing

$$M \subset D(\omega_{\{0,1\}}) \subset D(\delta_2) \subset M \cup \{\delta_2\},$$

we get $M \subset I_0(K)$ and $\delta_2 \in I_0(K)$. In order to complete the proof, we shall show that every $\mu = \sum_{k=0}^{\infty} \mu_k \delta_k \in M^1(K)$ satisfying $\omega_{\{0,1\}} \in D(\mu)$, $\mu_2 < 1$ and $\mu_0 + \mu_1 < 1$ cannot be contained in $I_0(K)$. To do this, we consider three cases. First, if $\mu_0 = \mu_1 \neq 0$, then the measure $\nu := (\mu_0 + \mu_1)\delta_0 + \sum_{k=2}^{\infty} \mu_k \delta_k$ satisfies $\nu * \omega_{\{0,1\}} = \mu$ and $\operatorname{supp} \nu \not\subset \{0, 1\}$ and has no idempotent factors. Therefore, by the results proved above, $\nu \notin I_0(K)$ and $\mu \notin I_0(K)$. Moreover, if $\mu_0 = \mu_1 = 0$ and $\mu_2 > 0$, then the measure $\nu := \mu_2\delta_1 + \sum_{k=3}^{\infty} \mu_k \delta_k$ satisfies

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$\nu * \delta_2 = \mu$ and $\operatorname{supp}\nu \subset \{0, 1\}$ and has no idempotent factors. Again it follows that $\mu \notin I_0(K)$. Lastly, if $\mu_0 = \mu_1 = \mu_2 = 0$, then $\omega_{\{0,1,2\}}$ is a factor of μ which must have an indecomposable factor by Theorem 4.3. This completes the proof of part (2).

(3) If μ has no idempotent factors, then the assertion follows from Lemma 5.2 and Theorem 3.8. If μ has ω_H as maximal idempotent factor with $H = \{0, 1, \dots, n\}$, then μ can be represented as $\mu = c\omega_H + (1-c)\rho$ where $c \in [0, 1]$, $\rho \in M^1(K)$ and $\operatorname{supp}\rho \cap H \neq \varnothing$. If $c > 0$, then μ can be written as $\mu = \omega_H * \nu$ where $\nu := c\delta_1 + (1-c)\rho$ has no idempotent factors. Now the considerations above yield the desired decomposition for $c > 0$. If $c = 0$, then the maximality of ω_H implies $\mu(\{n+1\}) > 0$. Hence we can write μ as $\mu = \delta_{n+1} * \nu$ where $\nu := \mu(\{n+1\})\delta_0 + \sum_{k=n+2}^{\infty} \mu(\{k\})\delta_k$ has no idempotent factors which completes the proof.

6. Examples: a class of countable compact symmetric hypergroups

6.1. For a fixed sequence $(b_n)_{n\in\mathbb{N}}$ let the discrete hypergroup K be defined as in Section 5. We show in this section that $\widehat{K}$ is in a natural way a compact symmetric hypergroup, and we discuss some factorization results for these examples. To determine the dual convolution structure, we first remember Lemma 5.2 and identify $\widehat{K}$ (as a topological space) with $\mathbb{N} \cup \{\mathbf{1}\}$, the one-point-compactification of $\mathbb{N}$, in the obvious way. Next, using the convention that an empty sum is equal to 0 and an empty product equal to 1, we note the formula

$$(1-b_n) + \sum_{k=n+1}^{r} \frac{b_n}{b_k} \prod_{l=n+1}^{k} \frac{b_l}{1+b_l} = 1 - b_n \prod_{l=n+1}^{r} \frac{b_l}{1+b_l}$$

for $r, n \in \mathbb{N}$, $r \geq n$, which can be proved by straightforward induction. Therefore

$$\alpha_n \cdot \alpha_n = (1-b_n)\alpha_n + \sum_{k=n+1}^{\infty} \frac{b_n}{b_k} \prod_{l=n+1}^{k} \frac{b_l}{1+b_l} \cdot \alpha_k$$

for every $n \in \mathbb{N}$. Thus, using $\alpha_n \cdot \alpha_m = \alpha_n$ for $n < m$, and taking the character $\mathbf{1}$ as identity, we get an associative convolution structure on $\mathbb{N} \cup \{\mathbf{1}\}$ with

$$\delta_n * \delta_m = \delta_n \qquad (n < m)$$

and

$$\delta_n * \delta_n = (1-b_n)\delta_n + \sum_{k=n+1}^{\infty} \frac{b_n}{b_k} \prod_{l=n+1}^{k} \frac{b_l}{1+b_l} \delta_k \qquad (n \in \mathbb{N}).$$

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Moreover, some straightforward arguments show that the remaining hypergroup axioms are satisfied for $(\mathbb{N} \cup \{\mathbf{1}\}, *)$.

In summary, we have found a family of compact symmetric hypergroup structures on $\mathbb{N} \cup \{\mathbf{1}\}$.

At the end of this introduction, we note that Proposition 2 of Lasser [30] implies that K and $\widehat{K} = \mathbb{N} \cup \{\mathbf{1}\}$ are strong hypergroups.

6.2. THEOREM. (1) *Every infinitely divisible measure $\mu \in M^1(\mathbb{N} \cup \{\mathbf{1}\})$ without idempotent factor admits a Levy-Khintchine representation of the form*

$$\hat{\mu}(\alpha) = \exp\left(-\sum_{k=0}^{\infty}(1-\alpha(k))\nu(\{k\})\right) \qquad (\alpha \in (\mathbb{N} \cup \{\mathbf{1}\})^{\widehat{}} \equiv K)$$

where the Levy measure ν can be an arbitrary positive, not necessarily bounded measure on $K = \mathbb{N}_0$.

(2) *Every infinitely divisible measure $\mu \in M^1(\mathbb{N} \cup \{\mathbf{1}\})$ with maximal idempotent factor $\omega_H \neq \delta_1$ can be written as $\mu = \omega_H * (e^{-t} \exp t\nu)$ where $\nu \in M^1(\mathbb{N} \cup \{\mathbf{1}\})$ satisfies $\operatorname{supp} \nu \subset \{\mathbf{1}\} \cup (\mathbb{N} \backslash H)$.*

(3) $I_0(\mathbb{N} \cup \{\mathbf{1}\}) = \{\delta_1\}$.

(4) *Every $\mu \in M^1(\mathbb{N} \cup \{\mathbf{1}\})$ can be written as $\mu = \nu_1 * \nu_2$ where ν_1 is the weakly convergent convolution product of an at most countable number of indecomposable factors of μ and ν_2 is an idempotent or $\nu_2 = \delta_n$ $(n \in \mathbb{N})$.*

PROOF. (1) The Levy-Khintchine representation is a consequence of Voit [47, Theorem 4.3], of Lasser [32, Theorem 3.9] and of the fact that the dual hypergroup $(\mathbb{N} \cup \{\mathbf{1}\})^{\widehat{}} \equiv K$ has no quadratic forms or homomorphisms (in the sense of Lasser [32]) except for the trivial one. This fact follows immediately from the convolution defined in Section 5. Lastly, since for every character $\alpha \in (\mathbb{N} \cup \{\mathbf{1}\})^{\widehat{}}$ there is a neighbourhood of $\mathbf{1}$ on which $\alpha = 1$ holds, we obtain that every positive measure on $\mathbb{N}$ can appear as a Levy measure (see Lasser [32, Lemma 3.8]).

(2) Using the notation and methods introduced in the proof of Proposition 4.1, we can conclude from the finiteness of $(\mathbb{N} \cup \{\mathbf{1}\})/H$ that $\pi_*(\mu) = e^{-t} \exp t\tilde{\nu}$ holds where $\tilde{\nu} \in M^1((\mathbb{N} \cup \{\mathbf{1}\})/H)$. Now, defining $\nu \in M^1(\mathbb{N} \cup \{\mathbf{1}\})$ such that $\pi_*(\nu) = \tilde{\nu}$ and $\nu(H \backslash \{\mathbf{1}\}) = 0$, we obtain $\mu = \omega_H * (e^{-t} \exp t\nu)$ as desired.

(3) Since $\mathbb{N} \cup \{\mathbf{1}\}$ is compact, this hypergroup has Property (D). Since every subhypergroup $(\neq \{\mathbf{1}\})$ consists of infinitely many elements, from Theorem 4.3 it follows that every nontrivial idempotent contains indecomposable factors. Thus, since for every $n \in \mathbb{N}$ the hypergroup $\mathbb{N} \cup \{\mathbf{1}\}$ can be written as the join

$$\mathbb{N} \cup \{\mathbf{1}\} = \{n, n+1, \ldots, e\} \vee \{0, 1, \ldots, n-1, e\}$$

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(e being the neutral element), we can apply Theorem 4.8 and get

$$I_0(\mathbb{N} \cup \{\mathbf{1}\}) = \{\delta_1\}.$$

(4) Write $\mu \in M^1(\mathbb{N} \cup \{\mathbf{1}\})$ as $\mu = c\omega_H + (1-c)\nu$ where $c \in [0, 1]$, ω_H is the maximal idempotent factor of μ and $\nu \in M^1(\mathbb{N} \cup \{\mathbf{1}\})$ with $H \cap \operatorname{supp} \nu = \varnothing$. If $c > 0$, then we have $\mu = \omega_H * \rho$ where $\rho := c\delta_1 + (1-c)\nu$ has no idempotent factor. If $c = 0$, then the maximality of ω_H ensures that $\nu(\{n\}) > 0$ where $H = \{n+1, n+2, \dots, \mathbf{1}\}$. Thus $\rho := \nu(\{n\})\delta_1 + \nu|_{\mathbb{N}\setminus\{n\}}$ has no idempotent factor and satisfies $\rho * \delta_n = \mu$. Now, if we apply Theorem 3.8 and part (3) to ρ, then the proof is completed.

6.3. Remark. The compact hypergroups considered in this section were first introduced by Dunkl and Ramirez [16] for the special case $b_n = a/(1-a)$ ($0 < a \leq 1/2$, $n \in \mathbb{N}$). These examples are studied also in Vrem [52]. In particular, for $b_n = 1/(p-1)$ ($p \in \mathbb{N}$ being prime and $n \in \mathbb{N}$) these hypergroups are isomorphic to the hypergroups which arise as orbit spaces, when the compact groups Δ_p consisting of the units of the p-adic integers act on the additive groups of the p-adic integers (for details see Dunkl and Ramirez [16]).

7. Examples: Sturm-Liouville hypergroups on $\mathbb{R}_+ = [0, \infty[$

The class of symmetric hypergroups studied in this section generalizes the family of double coset hypergroups which are associated with noncompact Riemannian symmetric spaces of rank 1. The convolution structures on $\mathbb{R}_+ := [0, \infty[$ which are derived from special Sturm-Liouville differential equations on $[0, \infty[$ were first introduced and investigated by Chebli [11,12]. Zeuner [53] recently proved that these convolution structures yield in fact symmetric hypergroups structures on $\mathbb{R}_+$. Following the notation of Zeuner, we name these hypergroups Sturm-Liouville hypergroups (on $\mathbb{R}_+$). Before we prove some factorization results for these hypergroups which generalize some known results for some special examples (see Bingham [5], Ostrovskii [36] and Finkh [18]), we recall some basic facts that are contained in the papers of Chebli and Zeuner:

7.1. Preliminaries. Let $A : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing C^2-function that satisfies the following conditions: $A(0) = 0$; $\lim_{x\to\infty} A(x) = \infty$; A'/A is decreasing on $]0, \infty[$ and converges to $2\rho \geq 0$ for $x \to \infty$; $A'(x)/A(x) = \alpha/x + B(x)$ in a neighbourhood of 0, where $\alpha > 0$ and B is an odd C^∞-function on $\mathbb{R}$.

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Let Φ_λ, $\lambda \in \mathbb{C}$, be the unique solution of the Sturm-Liouville equation on $\mathbb{R}_+$

$$\Phi_\lambda'' + \frac{A'}{A}\Phi_\lambda' + (\lambda^2 + \rho^2)\Phi_\lambda = 0, \qquad \Phi_\lambda(0) = 1, \qquad \Phi_\lambda'(0) = 0.$$

By Zeuner [53] (see also Chebli [11]) there exists a unique symmetric hypergroup structure $(\mathbb{R}_+, *)$ with neutral element $e = 0$ such that

$$\operatorname{supp}(\delta_x * \delta_y) = [|x-y|, x+y]$$

and

$$\Phi_\lambda(x)\Phi_\lambda(y) = \int \Phi_\lambda(z)d(\delta_x * \delta_y)(z) \qquad (x, y \in \mathbb{R}_+, \lambda \in \mathbb{C}).$$

In particular using Theorem 4 of Chebli [12] and Theorem 4.3 of Voit [46], we have

$$\widehat{K} = \{\Phi_\lambda : \lambda \geq 0 \text{ or } \lambda = i\tau, \tau \in [0, \rho]\} \quad \text{and} \quad \operatorname{supp}\pi = \{\Phi_\lambda : \lambda \geq 0\}.$$

In particular, $\Phi_{i\rho} = \mathbf{1}$ is the trivial character, and Φ_0 is the uniquely determined positive character which is contained in $\operatorname{supp}\pi$. Moreover, Proposition 2 of Chebli [12] implies that a character Φ_λ is positive if and only if $\lambda = i\tau$ where $\tau \in [0, \rho]$. From [46, Equation (17)], it follows that $\lim_{x\to\infty} \Phi_\lambda'(x)/\Phi_\lambda(x) = |\lambda| - \rho \leq 0$ for $\lambda = i\tau$, $\tau \in [0, \rho[$. Thus $\Phi_\lambda \in C_0(\mathbb{R}_+)$ for $\lambda = i\tau$, $\tau \in [0, \rho[$.

Lastly we note that the mapping $\mathbb{R}_+ \cup \{i\tau : \tau \in [0, \rho]\} \to \widehat{K}$, $\lambda \mapsto \Phi_\lambda$, is a homeomorphism. Via this homeomorphism we shall now identify $\widehat{K}$ with the set $\mathbb{R}_+ \cup \{i\tau : \tau \in [0, \rho]\}$.

7.2. THEOREM. (1) *Every $\mu \in M^1(K)$ can be written as $\mu = \nu_1 * \nu_2$ where ν_1 is infinitely divisible and has no indecomposable factors and ν_2 is the weakly convergent product of an at most countable number of indecomposable factors of μ.*

(2) *Every infinitely divisible measure $\mu \in M^1(K)$ admits a Levy-Khintchine representation of the form*

$$\hat{\mu}(\lambda) = e^{-c(\lambda^2+\rho^2)} \cdot \exp\left(\int_{]0,\infty[} (1 - \Phi_\lambda(x))\, d\tau(x)\right) \qquad (\lambda \in \widehat{K})$$

where $c \geq 0$ and τ is a positive measure on $]0, \infty[$ satisfying

$$\int_0^\infty \frac{x^2}{1+x^2}\, d\tau(x) < \infty.$$

(3) *If $\mu \in M^1(K)$ is Gaussian with respect to this convolution structure, which means the Fourier transform of μ can be written as*

$$\hat{\mu}(\lambda) = e^{-c(\lambda^2+\rho^2)} \qquad (c \geq 0 \textit{ fixed}, \lambda \in \widehat{K}),$$

then μ has no indecomposable factor.

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(4) *Assume that the hypergroup convolution* $*$ *satisfies the following condition: there exists a continuous function* $c :]0, \infty[\times]0, \infty[\mapsto]0, \infty[$ *and a constant* $0 < R < 1/4$ *such that*

$$\delta_x * \delta_y - c(x, y)\lambda_{[|x-y|+R(x+y), (1-R)(x+y)]}$$

is a nonnegative measure for every $x, y > 0$ *(where* λ_A *means the restriction of the Lebesgue measure to the measurable set* $A \subset \mathbb{R}_+$*). Then* $I_0(K)$ *coincides with the set of all Gaussian measures on* $\mathbb{R}_+$.

PROOF. (1) Since $\operatorname{supp} \pi$ is homeomorphic to $\mathbb{R}_+$, 2.7 shows that K has Property (D). Furthermore the facts contained in Section 7.1 yield that either $\mathbf{1} \in \operatorname{supp} \pi$ (for $\rho = 0$) or that there is a sequence of positive characters vanishing at infinity which converges to $\mathbf{1}$ (for $\rho > 1$). Thus we can apply Theorem 3.8 in its strongest version. The proof is completed by observing that there exist no idempotents or units except for the trivial one.

(2) This was proved by Chebli [11, Theorem 7]. Note just that Chebli uses another parametrization of $\widehat{K}$.

(3) It suffices to show that every factor of a Gaussian measure is Gaussian. To do this, we assume μ is given as in the theorem and $\mu = \nu_1 * \nu_2$ where $\nu_1, \nu_2 \in M^1(K)$. Then $\hat{\nu}_1 \cdot \hat{\nu}_2 = \hat{\mu}$ on $\widehat{K}$. Defining $\tilde{\nu}_i(x) := \hat{\nu}_i(|x|)$ $(x \in \mathbb{R}, i = 1, 2)$ and $\tilde{\mu}(x) := \hat{\mu}(|x|)$ $(x \in \mathbb{R})$, we get real valued, continuous functions on $\mathbb{R}$. It is an immediate consequence of Chebli [12, Proposition 2] that for every $x \in \mathbb{R}_+$ there exists a positive symmetric measure $\nu_x \in M_b(\mathbb{R})$ such that

$$\Phi_\lambda(x) = \int_{-x}^{x} e^{-i\lambda t}\, d\nu_x(t) \quad \text{for all } \lambda \in \mathbb{R}.$$

Thus we have

$$\tilde{\nu}_i(z) = \hat{\nu}_i(|z|) = \int_{-\infty}^{\infty} \int_{-x}^{x} e^{izt}\, d\nu_x(t)\, d\nu_i(x) \qquad (i = 1, 2, \ z \in \mathbb{R})$$

which implies that for suitable constants $a_i > 0$ the functions $a_i\tilde{\nu}_i$ are characteristic functions on the group $\mathbb{R}$ (in the usual sense) and satisfy $a_1\tilde{\nu}_1(z) \cdot a_2\tilde{\nu}_2(z) = e^{-cz^2}$ $(z \in \mathbb{R})$. Now the Theorem of Cramer shows that $a_i\tilde{\nu}_i(z) = e^{-c_i z^2}$ $(z \in \mathbb{R}, \ i = 1, 2)$ where $c_1 \geq 0$ and $c_2 \geq 0$ are suitable constants. Finally, since a nonnegative bounded measure on $\mathbb{R}_+$ is determined uniquely by the restriction of its Fourier transform to $\operatorname{supp} \pi \simeq \mathbb{R}_+$ (see, for instance, Voit [47, Theorem 2.3]) and since $\hat{\nu}_i(\mathbf{1}) = \hat{\nu}_i(i\rho) = 1$, we obtain $\hat{\nu}_i(\lambda) = e^{-c_i(\lambda^2+\rho^2)}$ $(\lambda \in \widehat{K}, \ i = 1, 2)$ as desired.

(4) We shall prove that the condition stated in part (4) of the theorem implies the following statement: for every $x > 0$ there exists $y > x$ such

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that for every $\mu \in M_b^+(K)$ with $\operatorname{supp}\mu \subset [x, y]$ the measure $e^{-\mu(\mathbb{R}_+)} \exp \mu \in M^1(K)$ has indecomposable factors.

In fact, the assertion of part (4) then follows from the preceding parts of this theorem and from Theorem 3.5.

To prove the statement above, fix $x > 0$. Since $R < 1/4$, we may choose constants $y > x$ and $a, b > 0$ such that the inequalities

$$b > y/2, \qquad b > a \geq y - x + 2Ry, \qquad b \leq 2x(1-R) - y$$

and

$$b \leq 2x - y - R(x+y)$$

hold. This follows immediately from the fact that for $R = 1/4$ and $y = x$ the right hand parts of the four inequalities above are equal to $x/2$.

Now, after having fixed y, we take an arbitrary $\mu \in M_b^+(K)$ with $\operatorname{supp}\mu \subset [x, y]$. Let

$$c_\mu := \int_x^y \int_x^y c(u, v)\, d\mu(u)\, d\mu(v) > 0.$$

Now, using the assumption of part (4) and the inequalities

$$2x(1-R) \leq (1-R)(u+v) \quad \text{and} \quad y - x + 2Ry \geq |u-v| + R(u+v)$$

for $u, v \in [x, y]$, we observe that

$$\begin{aligned}
&(\mu * \mu - c_\mu \cdot \lambda)(f) \\
&= \int_x^y \int_x^y \left[\delta_u * \delta_v(f) - c(u, v) \int_{y-x+2Ry}^{2x(1-R)} f(z)\, d\lambda(z)\right] d\mu(u)\, d\mu(v) \\
&\geq \int_x^y \int_x^y \left[\delta_u * \delta_v(f) - c(u, v) \int_{|u-v|+R(u+v)}^{(1-R)(u+v)} f(z)\, d\lambda(z)\right] d\mu(u)\, d\mu(v) \\
&\geq 0
\end{aligned}$$

for every $f \in C_c(\mathbb{R}_+)$ satisfying $f \geq 0$ and $\operatorname{supp} f \subset [y - x + 2Ry, 2x(1-R)]$. Thus $\rho := \mu * \mu - c_\mu \cdot \lambda_{[y-x+2Ry, 2x(1-R)]}$ is a positive measure.

We next take a function $g \in C_c(\mathbb{R}_+)$ with $\operatorname{supp} g = [a, b]$ and $g > 0$ on $]a, b[$. Since $[a, b] \subset [y - x + 2Ry, 2x(1-R)]$ by our assumption, the positivity of the measure ρ shows that $\mu * \mu - \varepsilon g \cdot \lambda$ is positive for every sufficiently small $\varepsilon > 0$. The above inequalities also yield $[x - b, y + b] \subset [y - x + 2Ry, 2x(1-R)]$. Hence, again from the positivity of ρ it follows that $\mu * \mu - \varepsilon (g\lambda) * \mu$ is positive for every sufficiently small $\varepsilon > 0$. In this conclusion we have used the fact that $(g\lambda) * \mu \in C_c(\mathbb{R}_+)$ is supported by $[x - b, y + b]$. Lastly, the inequalities above imply $x < y < 2b$. This inequality and some straightforward considerations of the convolution $*$ show that $g * g * \mu \in C_c(\mathbb{R}_+)$ is positive on $[0, 2b + y[$. Since

$$\operatorname{supp}(g * g * g) \subset [0, 3b] \subset [0, 2b + y[,$$

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it follows that $(g\lambda) * (g\lambda) * \mu - \varepsilon(g\lambda)^3$ is positive for every sufficiently small $\varepsilon > 0$. Now, summarizing the positivity results above, we obtain that the measures

$$-\varepsilon g\lambda + (\mu - \varepsilon g\lambda)^2 \quad \text{and} \quad (\mu - \varepsilon g\lambda)^3$$

are nonnegative for a sufficiently small $\varepsilon > 0$. In particular, we see that for this ε the measures

$$(\mu - \varepsilon g\lambda)^n \qquad (n \geq 2) \quad \text{and thus} \quad \exp(\mu - \varepsilon g\lambda) = \delta_0 + \sum_{n=1}^{\infty} (\mu - \varepsilon g\lambda)^n$$

are positive. Now Proposition 4.2 shows that the probability measure

$$e^{-\mu(\mathbb{R}_+) + \varepsilon(g\lambda)(\mathbb{R}_+)} \cdot \exp(\mu - \varepsilon g\lambda)$$

has indecomposable factors. Since this measure is a factor of μ, the proof is finished.

7.3. Remark. It is unknown whether the condition used in Theorem 7.2(4) is satisfied for all Sturm-Liouville hypergroups on $\mathbb{R}_+$ or whether the assertion of part (4) can be proved without assuming this restriction. On the other hand, the condition of Theorem 7.2(4) is true for all examples for which $*$ is explicitly known. Examples which are associated with Bessel and Jacobi functions are presented below.

In order to make the check of the restriction used in Theorem 7.2(4) easier, we note a further condition which implies the restriction above and which is easier to verify:

7.4. Remark. If there exists a function $K :]0, \infty[^3 \mapsto [0, \infty[$ such that $\delta_x * \delta_y$ has the form $K(x, y, \cdot)\lambda$ for all $x, y > 0$ and such that the kernel K is continuous and positive on $\{(x, y, z) : x, y > 0, |x-y| < z < x+y\}$, then the condition introduced in Theorem 7.2(4) is true. In fact, for the proof we have only to verify that

$$c(x, y) := \min_{z \in [|x-y| + R(x+y), (x+y)(1-R)]} K(x, y, z)$$

has the properties required in Theorem 7.2(4).

7.5. Examples: hypergroups associated with Bessel functions. Fix $\alpha > 1/2$ and let $A(x) := x^{2\alpha+1}$. Then we obtain a Sturm-Liouville hypergroup on $\mathbb{R}_+$ with $\rho = 0$ such that the associated eigenfunctions Φ_λ are modified Bessel functions. The convolution $*$ is given by

$$d(\delta_x * \delta_y)(z) = K(x, y, z) z^{2\alpha+1} \, d\lambda|_{[|x-y|, x+y]}(z)$$

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where

$$K(x, y, z) = \frac{\Gamma(\alpha+1)[(z^2-(x-y)^2)((x+y^2)-z^2)]^{\alpha-1/2}}{\Gamma(1/2)\Gamma(\alpha+1/2)2^{2\alpha-1}(xyz)^{2\alpha}}.$$

Using Remark 7.4, we see that the assumptions of Theorem 7.2(4) are satisfied. In fact, for this family of examples the assertions of Theorem 7.2 are well known and can be found in the papers of Bingham [5], Kingman [27] and Ostrovskii [36] and in the thesis of Finkh [18].

7.6. EXAMPLES: HYPERGROUPS ASSOCIATED WITH JACOBI FUNCTIONS. Fix $\alpha \geq \beta \geq -1/2$ with $\alpha \neq -1/2$ and let $A(x) := (\sinh x)^{2\alpha+1} \cdot (\cosh x)^{2\beta+1}$. Then we get a Sturm-Liouville hypergroup on $\mathbb{R}_+$ with $\rho = \alpha+\beta+1$ such that the associated characters Φ_λ are Jacobi functions. The convolution $*$ is given by

$$d(\delta_x * \delta_y)(z) = K(x, y, z)A(z)\, d\lambda|_{[|x-y|, x+y]}(z)$$

where

$$K(x, y, z) = \frac{2^{1-2\rho}\Gamma(\alpha+1)}{\pi^{1/2}\Gamma(\alpha-\beta)\Gamma(\beta+1/2)} \cdot (\sinh x \cdot \sinh y \cdot \sinh z)^{-2\alpha}$$
$$\cdot \int_0^\pi (1-\cosh^2 x - \cosh^2 y - \cosh^2 z + 2\cosh x \cosh y \cosh z \cos t)_+^{\alpha-\beta-1} \cdot (\sin t)^{2\beta}\, dt$$

(for details see, for instance, Koornwinder [28, Equation (7.12)]). An easy calculation and Remark 7.4 imply again that the assumptions of Theorem 7.2(4) are satisfied.

7.7. EXAMPLES: MODIFICATION OF STURM-LIOUVILLE HYPERGROUPS. Let $(\mathbb{R}_+, *)$ be a hypergroup structure as introduced in 7.5 or 7.6. Fix a constant $\rho_0 \in [0, \infty[$ and define a new Sturm-Liouville hypergroup structure $(\mathbb{R}_+, \bullet)$ by

$$d(\delta_x \bullet \delta_y)(z) := \frac{\Phi_{i\rho_0}(z)}{\Phi_{i\rho_0}(x)\Phi_{i\rho_0}(y)} d(\delta_x * \delta_y)(z) \qquad (x, y, z \geq 0).$$

For details on this modified convolution structure we refer to Voit [46, Section 4]. Using Remark 7.4, the properties of the kernels introduced in 7.5 and 7.6 and, lastly, the continuity and positivity of the semicharacter $\Phi_{i\rho_0}$, we obtain that all the assertions of Theorem 7.2 are true also for this modified examples.

7.8. REMARK. It is known (see, for instance, Zeuner [53]) that all hypergroup structures on $\mathbb{R}_+$ which are isomorphic to double coset hypergroups

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are associated with noncompact Riemannian symmetric spaces of rank 1 and that these hypergroups are contained in the both families studied in 7.5 and 7.6. For example, for $\beta = 0$ and $\alpha = n-1$, $n \in \mathbb{N}\backslash\{1\}$, the hypergroups investigated in 7.6 are isomorphic to the double coset hypergroups $S = SO(n, 1)//SO(n)$ where $SO(n, 1)$ is the Lorentz group of dimension $n+1$. Furthermore, for $\alpha = n/2-1$ $(n \in \mathbb{N}\backslash\{1\})$ the hypergroups presented in 7.5 coincide with the double coset hypergroups $M(n)//SO(n)$, $M(n)$ denoting the Euclidean motion group in $\mathbb{R}^n$.

8. Examples: polynomial hypergroups

In this section we first introduce polynomial hypergroups and some technical conditions satisfied by almost every polynomial hypergroup. Then, after having discussed some fundamental properties of the dual space of a polynomial hypergroup (Theorem 8.2) and after having established some preparatory results, we summarize the main factorization results for polynomial hypergroups in Theorem 8.8. The examples contained in the Sections 8.11–8.18 show that the restrictions assumed in Theorem 8.8 are true for almost every known polynomial hypergroup. Lastly, Theorem 8.19 is devoted to prove that the set of all indecomposable measures is dense in $M^1(\mathbb{N}_0)$.

8.1. Let $(a_n)_{n\in\mathbf{N}}$, $(b_n)_{n\in\mathbf{N}}$ and $(c_n)_{n\in\mathbf{N}}$ be sequences of real numbers satisfying $a_n, c_n > 0$, $b_n \geq 0$ and $a_n + b_n + c_n = 1$ $(n \in \mathbb{N})$. Moreover assume that $\alpha := \lim_{n\to\infty} a_n$, $\beta := \lim_{n\to\infty} b_n$ and $\gamma := \lim_{n\to\infty} c_n$ exist and satisfy $\alpha, \gamma > 0$. Now, defining

$$P_0 = 1, \qquad P_1(x) = 2\sqrt{\alpha\gamma}\cdot x + \beta \quad \text{and} \quad P_{n+1} = \frac{1}{a_n}((P_1 - b_n)P_n - c_nP_{n-1}),$$

we get a sequence of orthogonal polynomials by Favard's theorem (see, for instance, Chihara [13]). If in the linearization $P_m \cdot P_n = \sum_{k=|m-n|}^{m+n} g_{m,n,k}P_k$ all the linearization coefficients $g_{m,n,k}$ are nonnegative, then

$$\delta_m * \delta_n = \sum_{k=|m-n|}^{m+n} g_{m,n,k}\delta_k \qquad (m, n \in \mathbb{N}_0)$$

and bilinear continuous extension to $M_b(\mathbb{N}_0)$ define a symmetric hypergroup structure on $\mathbb{N}_0$, called a polynomial hypergroup. Furthermore, the mapping

$$\chi : D_s := \{x \in \mathbb{R} : (P_n(x))_{n\in\mathbf{N}_0} \text{ is bounded}\} \to \widehat{\mathbb{N}}_0,\ x \mapsto \alpha_x,$$

with $\alpha_x(n) := P_n(x)$ establishes a homeomorphism between $\widehat{\mathbb{N}}_0$ and D_s. In addition, up to this homeomorphism and up to normalization, the Plancherel measure π on $\widehat{\mathbb{N}}_0$ agrees with the orthogonality measure associated with

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the sequence $(P_n)_{n\in\mathbb{N}_0}$ (for the proofs of these facts see Lasser [30]). In particular, since the dual space of a discrete hypergroup is compact, D_s must be compact. In order to simplify the notation, we shall identify D_s with $\widehat{\mathbb{N}}_0$ and π with the orthogonality measure.

8.2. THEOREM. *In the above situation we have the following.* (1) $\alpha \geq \gamma$.

(2) $\operatorname{supp}\pi = [-1, 1] \cup L$ *and* $\widehat{\mathbb{N}}_0 = [-x_0, x_0] \cup L$ *where* $x_0 := (1-\beta)/2\sqrt{\alpha\gamma} \geq 1$ *and where* $L \subset [-1-\beta/\sqrt{\alpha\gamma}, -1[$ *is at most countable and* -1 *is the only possible limit point of* L. *In particular, if* $\beta = 0$ *then* $\operatorname{supp}\pi = [-1, 1]$ *and* $\widehat{\mathbb{N}}_0 = [-x_0, x_0]$. *Furthermore,* x_0 *corresponds to the identity character and* 1 *corresponds to* α_0.

(3) $\alpha = \gamma$ *is equivalent to* $\mathbf{1} \in \operatorname{supp}\pi$ *and this is equivalent to* $\widehat{K} = \operatorname{supp}\pi$.

(4) $[1, x_0]$ *coincides with the set of all positive characters on* $(\mathbb{N}_0, *)$, *and the characters contained in* $[1, x_0[$ *are zero at infinity.*

(5) $(\mathbb{N}_0, *)$ *has Property* (D).

PROOF. (1) By Lasser [30, page 191], the Haar measure on $(\mathbb{N}_0, *)$ is given by

$$h(0) = 1, \qquad h(n) = \prod_{k=1}^{n-1} a_k \Big/ \prod_{k=1}^{n} c_k \qquad (n \in \mathbb{N})$$

and satisfies $h(n) = g_{n,n,0}^{-1} \geq 1$ $(n \in \mathbb{N})$. This fact yields $\alpha \geq \gamma$.

(2) From the definition of the $(P_n)_{n\in\mathbb{N}_0}$ and from the Plancherel formula we see that the normalized polynomials $(\widetilde{P}_n)_{n\in\mathbb{N}_0} := (P_n \cdot \sqrt{h_n})_{n\in\mathbb{N}_0}$ are orthonormal with respect to π. They satisfy the recursion relation

$$x\cdot\widetilde{P}_n(x) = \frac{\sqrt{a_n c_{n+1}}}{2\sqrt{\alpha\gamma}}\widetilde{P}_{n+1}(x) + \frac{1}{2\sqrt{\alpha\gamma}}(b_n - \beta)\widetilde{P}_n(x) + \frac{\sqrt{a_{n-1}c_n}}{2\sqrt{\alpha\gamma}}\widetilde{P}_{n-1}(x) \quad (n\in\mathbb{N}).$$

Therefore the sequence $(\widetilde{P}_n)_{n\in\mathbb{N}_0}$ is contained in the class $M(0, 1)$ of Nevai [35, Section 3.1, Definition 6]. Now Nevai [35, Section 3.3, Theorem 7], ensures that $\operatorname{supp}\pi = [-1, 1] \cup L$ where L is at most countable and ± 1 are the only possible limit points of L. Furthermore, if $L \cap]1, \infty[\neq \emptyset$, then $x_1 := \sup_{x\in L} x \in L \subset \operatorname{supp}\pi$ corresponds to a positive character on $(\mathbb{N}_0, *)$ (see Chihara [13, Chapter 1.5]); but this is inconsistent with the fact that x_1 cannot be an isolated point in $\operatorname{supp}\pi$ (see Voit [46, Theorem 2.11]). Hence it follows that $L \subset]-\infty, -1[$.

In order to characterize D_s we note first that $L \cup [-1, 1] = \operatorname{supp}\pi \subset D_s$.

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Now fix $x \in \mathbb{R} \setminus \operatorname{supp} \pi$. Then Nevai [35, Section 4.1, Theorem 13], yields

$$\lim_{n\to\infty} \left| \frac{P_{n+1}(x)}{P_n(x)} \right| = \sqrt{\frac{\gamma}{\alpha}} \lim_{n\to\infty} \left| \frac{\tilde{P}_{n+1}(x)}{\tilde{P}_n(x)} \right|$$
$$= \sqrt{\frac{\gamma}{\alpha}} \cdot \left(|x| + \sqrt{x^2 - 1} \right) \begin{cases} > 1 & \text{for } |x| > x_0, \\ < 1 & \text{for } |x| < x_0. \end{cases}$$

It follows that $D_s = L \cup [-x_0, x_0]$. Since $\operatorname{supp} \pi \subset]-\infty, 1]$, we have $P_n(1) > 0$ for every $n \in \mathbb{N}_0$ (see Chihara [13, Chapter 1.5]). Hence α_0 corresponds to the real number 1. Moreover it is easy to check that $P_n(x_0) = 1$ for every $n \in \mathbb{N}_0$ and thus the identity character corresponds to x_0. Finally, since $|\alpha(z)| \le \alpha_0(z)$ for all $\alpha \in \operatorname{supp} \pi$ and $z \in K = \mathbb{N}_0$ (see Voit [46, Corollary 2.8]), the definition of P_1 entails

$$L \subset \operatorname{supp} \pi \subset \{x \in \mathbb{R} : |P_1(x)| \le P_1(1)\} = [-1 - \beta/\sqrt{\alpha\gamma}, 1],$$

which completes the proof of part (2).

(3) This is an immediate consequence of (2).

(4) The equation at the end of the proof of part (2) shows also that for $x_0 > 1$ all the characters that are contained in $[1, x_0[$ are zero at infinity. Moreover, since $\operatorname{supp} \pi \subset]-\infty, 1]$, it is clear that $P_n(x) > 0$ for any $n \in \mathbb{N}_0$ and $x \ge 1$. Thus, in order to complete the proof, we only have to show that $P_n(x) > 0$, for all $n \in \mathbb{N}_0$, implies $x \ge 1$. Since there exists only one positive character that is contained in $\operatorname{supp} \pi$ (cf. Voit [46, Theorem 2.11]), and since it is given by 1, we may assume $x \notin \operatorname{supp} \pi$. But $x < -1$ and $x \notin \operatorname{supp} \pi$ together imply

$$\lim_{n\to\infty} \frac{P_{n+1}(x)}{P_n(x)} = \sqrt{\frac{\gamma}{\alpha}} \lim_{n\to\infty} \frac{\tilde{P}_{n+1}(x)}{\tilde{P}_n(x)} = \sqrt{\frac{\gamma}{\alpha}} \cdot \left(-x - \sqrt{x^2 - 1} \right) < 0$$

(see Nevai [35, Section 4.1, Theorem 13]) which is impossible. This finishes the proof of part (4).

(5) This follows from Lemma 2.7, part (2) and the fact that $1 \in \mathbb{R}$ corresponds with the positive character contained in $\operatorname{supp} \pi$.

8.3. Modification of polynomial hypergroups. Let $(\mathbb{N}_0, *)$ be a polynomial hypergroup as introduced in Section 8.1. If $r \ge 1$ is fixed, then we have $P_n(r) > 0$ for all $n \in \mathbb{N}_0$, and the sequences

$$\tilde{a}_n := a_n \cdot \frac{P_{n+1}(r)}{P_n(r) P_1(r)}, \qquad \tilde{b}_n := b_n / P_1(r), \qquad \tilde{c}_n := c_n \cdot \frac{P_{n-1}(r)}{P_n(r) P_1(r)} \qquad (n \in \mathbb{N})$$

define a new modified polynomial hypergroup with

$$\tilde{\alpha} = \frac{\sqrt{\alpha\gamma} \cdot (r + \sqrt{r^2 + 1})}{2r\sqrt{\alpha\gamma} + \beta}, \quad \tilde{\beta} = \frac{\beta}{2r\sqrt{\alpha\gamma} + \beta}, \quad \tilde{\gamma} = \frac{\sqrt{\alpha\gamma} \cdot (r - \sqrt{r^2 + 1})}{2r\sqrt{\alpha\gamma} + \beta};$$

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for details see Voit [51, Section 2.12] and [46, Section 2.18]. In particular, $r = 1$ is equivalent to $\tilde{\alpha} = \tilde{\gamma}$ and this is equivalent to the fact that $\mathbf{1} \in \operatorname{supp} \pi$ for the modified convolution structure. Thus, for $r = 1$, the modification procedure above agrees with the general modification of commutative hypergroups which was introduced in Section 2.

8.4. Property (T). We say a polynomial hypergroup (or the associated sequence of orthogonal polynomials) has Property (T) if the following holds.

(T) Let $(T_n)_{n\in\mathbb{N}_0}$ be the Tchebichef polynomials of the first kind which are defined by $T_n(x) = \cos(n \arccos x)$, $x \in [-1, 1]$. Then the connection coefficients $h_{n,k}$, $n, k \in \mathbb{N}_0$, $n \geq k$, which are uniquely determined by $P_n = \sum_{k=0}^{n} h_{n,k} T_k$, are nonnegative.

This condition was successfully applied in some recent papers of the author [48, 49, 51] to derive some limit theorems for random walks on polynomial hypergroups. This condition is true for every polynomial hypergroup known to the author; see Sections 8.11–8.18 and the papers mentioned above.

Property (T) is needed here in order to ensure the following inequality. If $E_r \subset \mathbb{C}$ is the closed ellipse with foci ± 1 and radius $r > 1$, then $|P_n(z)| \leq P_n(r)$ for all $r > 1$, $n \in \mathbb{N}_0$ and $z \in E_r$. In fact, writing P_n as $P_n = \sum_{k=0}^{n} h_{n,k} T_k$ with $h_{n,k} \geq 0$, we obtain

$$\begin{aligned} |P_n(\cos\theta)| &\leq \sum_{k=0}^{n} h_{n,k} |T_k(\cos\theta)| \\ &\leq \sum_{k=0}^{n} h_{n,k} \cosh(k \operatorname{Im}\theta) = P_n(\cos(i \operatorname{Im}\theta)) \end{aligned}$$

for $n \in \mathbb{N}_0$ and $\theta \in \mathbb{C}$. Since $\cos\theta \in E_r$ is equivalent to $\cos(i \operatorname{Im}\theta) \leq r$ $(r > 1)$, the inequality above follows.

We conclude this section by noting that Property (T) is preserved by the modification procedure introduced in Section 8.3.

8.5. Proposition. *Let* $(\mathbb{N}_0, *)$ *be a polynomial hypergroup with Property* (T). *Let* $n \in \mathbb{N}$ *and let* $\mu \in M^1(\mathbb{N}_0)$ *be a probability measure of the form*

$$\mu = \exp\left(\sum_{i=1}^{n} c_i(\delta_i - \delta_0)\right)$$

where $c_1, c_2, \ldots, c_n \geq 0$. *Then every factor of* μ *is of the form*

$$\exp\left(\sum_{i=1}^{n} d_i(\delta_i - \delta_0)\right) \quad \text{where } d_1, \ldots, d_{n-1} \in \mathbb{R},\ d_n \geq 0.$$

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Proof. Using Remark 8.3 and Lemma 2.5, we can assume without loss of generality that $1 \in \operatorname{supp} \pi$.

For $r > 1$ let $\bullet_r$ be the modified convolution structure associated with the positive semicharacter α_r. By Lemma 2.5 we know that

$$\varphi_r : (M^1(\mathbb{N}_0), \bullet_r) \to (M^1(\mathbb{N}_0), *)$$

$$\rho = \sum_{k=0}^{\infty} \rho(k)\delta_k \mapsto \varphi(\rho) = \sum_{k=0}^{\infty} \frac{\rho(k)}{P_k(r)}\delta_k \Big/ \sum_{k=0}^{\infty} \frac{\rho(k)}{P_k(r)}$$

is a weakly continuous semigroup homomorphism satisfying

$$\varphi_r(M^1(\mathbb{N}_0)) = \left\{ \rho \in M^1(\mathbb{N}_0) : \sum_{k=0}^{\infty} P_k(r)\rho(k) < \infty \right\}.$$

If we define $t = \sum_{i=0}^{n} c_i$, from $\nu = \sum_{i=0}^{n} (c_i/t)\delta_i \in \varphi_r(M^1(\mathbb{N}_0))$ and the continuity of φ it follows that $\mu = e^{-t}\exp(t\nu) \in \varphi_r(M^1(\mathbb{N}_0))$. Since thus $D(\mu) \subset \varphi_r(M^1(\mathbb{N}_0))$ by Lemma 2.5, we have shown that

$$\sum_{k=0}^{\infty} P_k(r)\rho(k) < \infty \quad \text{for all } \rho \in D(\mu) \quad \text{and} \quad r > 1.$$

Since $|P_k(x)| \le P_k(r)$ for all $x \in E_r$ (see Section 8.4), $\tilde\rho(z) := \sum_{k=0}^{\infty} P_k(z)\rho(k)$ $(z \in \mathbb{C})$ defines an entire function on $\mathbb{C}$ for every $\rho \in D(\mu)$. Moreover, for $\rho_1, \rho_2 \in D(\mu)$ with $\rho_1 * \rho_2 = \mu$ we have

$$\tilde\rho_1(x) \cdot \tilde\rho_2(x) = \exp\left(\sum_{k=0}^{n} c_k (P_k(x) - 1) \right) \qquad (x \in [-1, 1]).$$

Analytical continuation shows that this equation is true for all $x \in \mathbb{C}$. In particular, since $\tilde\rho_2(r) \ge 1$ for $r > 1$ and $\rho_2 \in D(\mu)$, it follows that

$$\tilde\rho(r) \le \exp\left(\sum_{k=0}^{n} c_k (P_k(r) - 1) \right)$$

for all $\rho \in D(\mu)$ and $r > 1$. Therefore, using $|P_k(x)| \le P_k(r)$ for $x \in E_r$ and $\{z \in \mathbb{C} : |z| < r\} \subset E_{\sqrt{r^2+1}}$, we have

$$|\tilde\rho(x)| \le \tilde\rho\left(\sqrt{r^2+1}\right) \le \exp\left(\sum_{k=0}^{n} c_k \left(P_k\left(\sqrt{r^2+1}\right) - 1 \right) \right)$$

for $\rho \in D(\mu)$ and $x \in \mathbb{C}$ satisfying $|x| \le r$. Since $\tilde\rho$ has no zeros, Hadamard's factorization theorem (see, for instance, Holland [24, Theorem 4.9]) implies that $\tilde\rho(x) = \exp(P(x))$, P being a polynomial of degree not greater than n. Since $\tilde\rho = \hat\rho$ on $\widehat{\mathbb{N}}_0$, the injectivity of the Fourier transform yields the desired representation of ρ where $d_n < 0$ is impossible, since otherwise $\lim_{r\to\infty} \tilde\rho(r) = 0$ would be true.

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8.6. LEMMA. *Let* $(\mathbb{N}_0, *)$ *be a polynomial hypergroup. If for every* $n \in \mathbb{N}$, $n \geq 2$, *there exists* $m \in \mathbb{N}$ *such that* $\operatorname{supp} \delta_n^m = \{0, 1, 2, \dots, nm\}$, *then the probability measure* $e^{-t}\exp(t\delta_n)$ *has an indecomposable factor for every* $t > 0$ *and every* $n \geq 2$.

PROOF. We shall prove that there exists $\varepsilon > 0$ such that the measure $\mu(\varepsilon) := \exp(t(\delta_n - \varepsilon\delta_1))$ is nonnegative. Using Proposition 4.2, we can then conclude that $e^{-t(1-\varepsilon)} \cdot \exp(t(\delta_n - \varepsilon\delta_1)) \in D(e^{-t} \cdot \exp(t\delta_n))$ has an indecomposable factor which finishes the proof.

In order to prove the nonnegativity of $\mu(\varepsilon)$ for small $\varepsilon > 0$, we write $\mu(\varepsilon)$ as

$$\mu(\varepsilon) = \delta_0 + \sum_{k=1}^{\infty} t^k \sum_{l=0}^{k} \frac{1}{l!(k-l)!} \delta_n^l * (-\varepsilon\delta_1)^{k-l}.$$

Since the linearization coefficients $g_{u,v,|u-v|}$ and $g_{u,v,u+v}$ are positive for all $u, v \in \mathbb{N}_0$, we have

$$\{0, 1, \dots, n \cdot k\} * \{n\} = \{0, 1, \dots, n \cdot (k+1)\}.$$

Therefore the assumption of the lemma yields that

$$\operatorname{supp} \delta_n^k = \{0, 1, \dots, n \cdot k\} \quad \text{for all } k \geq m.$$

Since thus

$$\operatorname{supp} \delta_n^k \supset \bigcup_{l=1}^{k-1} \operatorname{supp}(\delta_k^l * (-\varepsilon\delta_1)^{k-l}) \qquad (k \geq m)$$

and

$$\operatorname{supp} \delta_n^m \supset \bigcup_{l=1}^{m} \operatorname{supp}(\delta_n - \varepsilon\delta_1)^l = \bigcup_{l=1}^{m} \bigcup_{i=0}^{l} \operatorname{supp}(\delta_n^i * (-\varepsilon\delta_1)^{l-i}),$$

and since the negative measures appearing on the right sides of these two formulas are always multiplied by powers of ε, we can find an $\varepsilon > 0$ sufficiently small such that the measures

$$(\delta_n - \varepsilon\delta_1)^k = \sum_{l=0}^{k} \binom{k}{l} \delta_n^l * (-\varepsilon\delta_1)^{k-l} \quad \text{for } k = m+1,\ m+2, \dots, 2m$$

and the measure

$$\delta_0 + \sum_{l=1}^{m} (\delta_n - \varepsilon\delta_1)^l \cdot \frac{t^l}{l!}$$

are non-negative. Since every $k \in \mathbb{N}$ with $k \geq m+1$ can be written as $k = \sum_{j=m+1}^{2m} j \cdot a_{k,j}$ with $a_{k,j} \in \mathbb{N}_0$, the measures $(\delta_n - \varepsilon\delta_1)^k$ are nonnegative for all $k \geq m+1$. In summary, we obtain that $\mu(\varepsilon) = \sum_{k=0}^{\infty} \frac{t^k}{k!}(\delta_n - \varepsilon\delta_1)^k$ is nonnegative as claimed.

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8.7. Lemma. *Let* $(\mathbb{N}_0, *)$ *be a polynomial hypergroup such that* $b_k = 0$ *for all* $k \in \mathbb{N}$. *For* $k \in \mathbb{N}$ *let*

$$H_k := \begin{cases} \{0, 2, 4, \dots, k\} & \text{if } k \text{ is even} \\ \{1, 3, 5, \dots, k\} & \text{if } k \text{ is odd.} \end{cases}$$

If for every $n \in \mathbb{N}$, $n \geq 3$, *there exists* $m \in \mathbb{N}$ *such that* $\operatorname{supp} \delta_n^m = H_{nm}$, *then the probability measure* $e^{-t} \exp(t\delta_n)$ *has an indecomposable factor for every* $t > 0$ *and every* $n \geq 3$.

Proof. Assume first that n is even and $n \neq 2$. Similarly to the proof of Lemma 8.6 it suffices to show that

$$\mu(\varepsilon) := \exp(t(\delta_n - \varepsilon\delta_2)) = \sum_{k=0}^{\infty} \frac{t^k}{k!} (\delta_n - \varepsilon\delta_2)^k$$

is nonnegative for sufficiently small $\varepsilon > 0$. Since $g_{u,v,|u-v|}$ and $g_{u,v,v+u}$ are positive for all $u, v \in \mathbb{N}_0$, we have $H_{nk} * \{n\} = H_{n(k+1)}$. This fact and the assumptions of the lemma imply that

$$\operatorname{supp} \delta_n^k \supset \bigcup_{l=0}^{k-1} \operatorname{supp}(\delta_n^l * (-\varepsilon\delta_2)^{k-l}) \qquad (k \geq m)$$

and

$$\operatorname{supp} \delta_n^m \supset \bigcup_{l=1}^{k-1} \operatorname{supp}(\delta_n - \varepsilon\delta_2)^l .$$

Now from the arguments used in the proof of Lemma 8.6 it follows that $\mu(\varepsilon)$ is nonnegative for sufficiently small $\varepsilon > 0$.

If n is odd and $n \neq 1$, then the methods used above also imply that $\mu(\varepsilon) := \exp(t(\delta_n - \varepsilon\delta_1))$ is nonnegative for sufficiently small $\varepsilon > 0$. Thus the proof is complete.

8.8 Theorem. *Let* $(\mathbb{N}_0, *)$ *be a polynomial hypergroup.*

(1) *Every* $\mu \in M^1(K)$ *can be written as* $\mu = \nu_1 * \nu_2$ *where* ν_1 *is infinitely divisible and has no indecomposable factors, and* ν_2 *is the weakly convergent product of at most countably many indecomposable factors of* μ.

(2) *Every infinitely divisible measure* $\mu \in M^1(K)$ *admits a Levy-Khintchine representation of the form*

$$\mu = e^{-t} \cdot \exp(t\nu) \quad \text{where } t \geq 0 \quad \text{and} \quad \nu \in M^1(\mathbb{N}_0).$$

(3) *If* $(\mathbb{N}_0, *)$ *has Property* (T) *and if for every* $n \in \mathbb{N}\backslash\{1\}$ *there exists* $m \in \mathbb{N}$ *such that* $\operatorname{supp} \delta_n^m = \{0, 1, 2, \dots, nm\}$, *then*

$$I_0(\mathbb{N}_0) = \{\mu \in M^1(\mathbb{N}_0) : \mu = \exp(t(\delta_1 - \delta_0)), t \geq 0\}.$$

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(4) *If* $(\mathbb{N}_0, *)$ *has Property* (T), *if* $b_n = 0$ *for all* $n \in \mathbb{N}$, *and if for every* $n \in \mathbb{N}\backslash\{1, 2\}$ *there exists* $m \in \mathbb{N}$ *such that* $\operatorname{supp}\delta_n^m = H_{nm}$ *(H_{nm} given as in Lemma* 8.7), *then*

$$I_0(\mathbb{N}_0) = \{\mu \in M^1(\mathbb{N}_0) : \mu = \exp(t_1\delta_1 + t_2\delta_2 - (t_1 + t_2)\delta_0), t_1, t_2 \geq 0\}.$$

PROOF. (1) By Theorem 8.2(5), $(\mathbb{N}_0, *)$ has Property (D). Since for $\alpha = \gamma$, $\mathbf{1} \in \operatorname{supp}\pi$ is true and since for $\alpha > \gamma$ the assumptions of Lemma 2.6 hold, part (1) follows immediately from Theorem 3.8.

(2) This follows from Proposition 4.1.

(3) Let $t \geq 0$. Using Proposition 8.5, we know that every factor of $\mu = \exp(t(\delta_1 - \delta_0))$ has the form $\exp(\tau(\delta_1 - \delta_0))$ where $\tau \geq 0$. Hence $\mu \in I_0(\mathbb{N}_0)$.

In order to prove the converse inclusion, assume $\mu \in M^1(\mathbb{N}_0)$ has no indecomposable factor. Then Theorem 3.5 and part (2) imply that $\mu = \exp(\sum_{k=1}^{\infty} t_k(\delta_1 - \delta_0))$ where $t_k \geq 0$ $(k \in \mathbb{N})$ and $\sum_{k=1}^{\infty} t_k < \infty$. Since for $k \geq 2$ and $t_k > 0$ the measure $\exp(t_k(\delta_k - \delta_0)) \in D(\mu)$ has indecomposable factors by Lemma 8.6, we must have $t_k = 0$ for $k \geq 2$. Thus the proof of part (3) is finished.

(4) Let $t_1 \geq 0$ and $t_2 \geq 0$. Using Proposition 8.5, we know that every factor ν of $\mu = \exp(t_1(\delta_1 - \delta_0) + t_2(\delta_2 - \delta_0))$ has the form $\nu = \exp(\tau_1(\delta_1 - \delta_0) + \tau_2(\delta_2 - \delta_0))$ where $\tau_1 \in \mathbb{R}$ and $\tau_2 \geq 0$. Since $b_n = 0$ for all $n \in \mathbb{N}$, $P_1(-1) = -1$ and $P_2(-1) = 1$ are true (see Section 8.1). Thus $\hat{\nu}(-1) = \exp(-2d_1)$. Thus, using $|\hat{\nu}(-1)| \leq 1$ for the probability measure ν, we get $\tau_1 \geq 0$. Therefore, ν has the same form as μ which implies $\mu \in I_0(\mathbb{N}_0)$.

In order to establish the converse inclusion, assume $\mu \in M^1(\mathbb{N}_0)$ has no indecomposable factor. Then Theorem 3.5 and part (2) imply that $\mu = \exp(\sum_{k=1}^{\infty} t_k(\delta_1 - \delta_0))$ where $t_k \geq 0$ $(k \in \mathbb{N})$ and $\sum_{k=1}^{\infty} t_k < \infty$. Since for $k \geq 3$ and $t_k > 0$ the measure $\exp(t_k(\delta_k - \delta_0)) \in D(\mu)$ has indecomposable factors by Lemma 8.7, we must have $t_k = 0$ for $k \geq 3$. Thus the proof of Theorem 8.8 is finished.

8.9. REMARKS. (1) The factorization result contained in Theorem 8.8(1) was stated earlier by Schwartz [41, Theorem 9]. However the proof there contains a little gap, which is filled here by using part (2) of Theorem 8.2.

(2) For the special case of polynomial hypergroups associated with ultraspherical polynomials, the results of Theorem 8.8 were established earlier by Bingham [4] and Ostrovskii and Truhina [37]. In particular, the ideas of the proof of Proposition 8.5 are taken from the paper of Ostrovskii and Truhina.

(3) Either part (3) or part (4) of Theorem 8.8 can be applied to almost every known polynomial hypergroup. In this way, Property (T) holds for

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every polynomial hypergroup known to the author. The second condition of part (3) or part (4) which claims that sufficiently many $g_{m,n,k}$ are positive is fulfilled for every known polynomial hypergroup except for the, unfortunately very important, hypergroups which are associated with Tchebichef polynomials of the first kind.

(4) It follows easily from Proposition 8.5 and the methods used in the proof of Theorem 8.8(4) that for polynomial hypergroups associated with Tchebichef polynomials of the first kind $\exp(t\delta_{2^k}) \cdot e^{-t} \in I_0(K)$ for all $t \geq 0$ and $k \in \mathbb{N}$. Therefore, one part of the assertion of Theorem 8.8(4) does not hold in general.

To make the check of the assumptions of part (3) and (4) in Theorem 8.8 easier, we establish a lemma which treats these conditions. In 8.11–8.18 we then give some examples.

8.10. Lemma. *Let $(\mathbb{N}_0, *)$ be a polynomial hypergroup.* (1) *If $g_{n,n,1} > 0$ and $g_{n,n,2n-1} > 0$ for all $n \geq 2$, then the assumptions of Lemma 8.6 hold. Moreover, $g_{n,n,1} > 0$ and $g_{n,n,2n-1} > 0$ are true for all $n \geq 2$ if and only if*

$$b_n > 0 \quad \text{and} \quad b_n + \sum_{k=1}^{n-1} (b_{n+k} - b_k) > 0 \quad \text{for all } n \geq 2.$$

(2) *Let $b_n = 0$ for all $n \in \mathbb{N}$. If $g_{n,n,2} > 0$ and $g_{n,n,2n-2} > 0$ for all $n \geq 3$, then the assumptions of Lemma 8.7 hold. Moreover, $g_{n,n,2} > 0$ and $g_{n,n,2n-2} > 0$ hold for all $n \geq 3$ if and only if*

$$a_n c_{n+1} + a_{n-1} c_n - c_1 > 0 \quad \text{and}$$
$$a_n c_{n+1} + a_{n-1} c_n - c_1 + \sum_{k=1}^{n-2} (a_{n+k} c_{n+k+1} - a_k c_{k+1}) > 0$$

for all $n \geq 3$.

Proof. (1) By the assumption we have $\operatorname{supp} \delta_n^2 \supset \{0, 1, 2n-1, 2n\}$ for $n \geq 2$. We next show that

$$\operatorname{supp} \delta_n^{2k} \supset \{0, 1, \ldots, k\} \cup \bigcup_{l=1}^{k} \{2ln - k, 2ln - k + 1, \ldots, 2ln\}$$

for all $k \in \mathbb{N}$ by induction. In fact, this statement is true for $k = 1$ and the

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induction step is an immediate consequence of

$$\left(\{0, 1, \ldots, k\} \cup \bigcup_{l=1}^{k}\{2ln-k, 2ln-k+1, \ldots, 2ln\}\right) * \{0, 1, 2n-1, 2n\}$$
$$\supset \{0, 1, \ldots, k, k+1\} \cup \bigcup_{l=1}^{k+1}\{2ln-k-1, 2ln-k, \ldots, 2ln\}.$$

Now, taking $k = 2n$, we obtain $\{0, 1, \ldots, 4n^2\} \subset \operatorname{supp}\delta_n^{4n}$ as desired. In order to prove the equivalence of the conditions mentioned in Lemma 8.10(1), we first note that

$$g_{n,n,1} = \int_{\mathbf{R}} P_1 P_n P_n \, d\pi \Big/ \int_{\mathbf{R}} P_1^2 \, d\pi = b_n \cdot \int_{\mathbf{R}} P_n^2 \, d\pi \Big/ \int_{\mathbf{R}} P_1^2 \, d\pi$$

which establishes that $g_{n,n,1} > 0$ and $b_n > 0$ are equivalent for all $n \in \mathbb{N}$. Moreover, using $g_{1,n,n+1} = a_n$, $g_{1,n,n} = b_n$, the recursion formulas

$$g_{m,n,m+n} = g_{m-1,n,m+n-1} \cdot \frac{a_{n+m-1}}{a_{m-1}}$$

and

$$g_{m,n,m+n-1} = g_{m-1,n,m+n-2} \cdot \frac{a_{n+m-2}}{a_{m-1}} + g_{m-1,n,m+n-1} \cdot \frac{b_{m+n-1} - b_{n-1}}{a_{m-1}}$$

(for $m, n \in \mathbb{N}$, $m \geq 2$; see Lasser [30, page 188]) and induction, we obtain

$$g_{m,n,m+n} = \frac{a_n a_{n+1} \cdots a_{m+n-1}}{a_1 a_2 \cdots a_{m-1}}$$

and

$$g_{m,n,m+n-1} = \frac{a_n a_{n+1} \cdots a_{m+n-2}}{a_1 a_2 \cdots a_{m-1}} \cdot \left(b_n + \sum_{k=1}^{n-1}(b_{n+k} - b_k)\right).$$

Since $a_k > 0$ for all $k \in \mathbb{N}$, the latter equation yields the equivalence of the second pair of conditions in part (1).

(2) Since by the assumption $\operatorname{supp}\delta_n^2 \supset \{0, 2, 2n-2, 2n\}$ for $n \geq 3$, the same methods as used in the proof of part (1) imply that

$$\operatorname{supp}\delta_n^{2k} \supset \{0, 2, 4, \ldots, 2k\} \cup \bigcup_{l=1}^{k}\{2ln-2k, 2ln-k+2, \ldots, 2ln-2, 2ln\}$$

for all $k \in \mathbb{N}$. Now, taking $k = n$, we obtain $\{0, 2, \ldots, 2n^2\} \subset \operatorname{supp}\delta_n^{2n}$ as claimed. In order to establish that the conditions mentioned in Lemma 8.10(2) are equivalent, we first note that

$$g_{n,n,2} = \int_{\mathbf{R}} P_2 P_n P_n \, d\pi \Big/ \int_{\mathbf{R}} P_2^2 \, d\pi = g_{2,n,n} \cdot \int_{\mathbf{R}} P_n^2 \, d\pi \Big/ \int_{\mathbf{R}} P_2^2 \, d\pi$$

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and

$$g_{2,n,n} = (a_n c_{n+1} - a_{n-1}c_n - c_1)/a_1$$

is true for $n \geq 2$. These facts ensure the equivalence of $g_{n,n,2} > 0$ and $a_n c_{n+1} - a_{n-1}c_n - c_1 > 0$. Moreover, using $g_{2,n,n} = (a_n c_{n+1} - a_{n-1}c_n - c_1)/a_1$,

$$g_{m,n,m+n} = \frac{a_n a_{n+1} \cdots a_{m+n-1}}{a_1 a_2 \cdots a_{m-1}},$$

the recursion formula

$$\begin{aligned} g_{m,n,m+n-2} = {} & g_{m-1,n,m+n-3} \cdot \frac{a_{n+m-3}}{a_{m-1}} \\ & + g_{m-1,n,m+n-1} \cdot \frac{c_{m+n-1}}{a_{m-1}} - g_{m-2,n,m+n-2} \cdot \frac{c_{m-1}}{a_{m-1}} \end{aligned}$$

(for $m, n \in \mathbb{N}$, $m \geq 2$; see Lasser [30, page 188]) and induction, we obtain

$$\begin{aligned} g_{m,n,m+n-2} = {} & \frac{a_n a_{n+1} \cdots a_{m+n-3}}{a_1 a_2 \cdots a_{m-1}} \\ & \cdot \left(a_n c_{n+1} + a_{n-1}c_n - c_1 + \sum_{k=1}^{n-2} (a_{n+k}c_{n+k+1} - a_k c_{k+1}) \right) \end{aligned}$$

for all $m, n \in \mathbb{N}$ with $m \geq 2$. This shows that the second couples of conditions in part (2) are equivalent.

8.11. Examples: Jacobi polynomials. Let $\alpha, \beta \in \mathbb{R}$ with $\alpha \geq \beta > -1$ and

$$\begin{aligned} & (\alpha+\beta+1)(\alpha+\beta+4)^2(\alpha+\beta+6) \\ & \qquad \geq (\alpha-\beta)^2[(\alpha+\beta+1)^2 - 7(\alpha+\beta+1) - 24]. \end{aligned}$$

We define

$$a_n = \frac{2(n+\alpha+\beta+1)(n+\alpha+1)(\alpha+\beta+2)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+1)2(\alpha+1)},$$

$$b_n = \frac{\alpha-\beta}{2(\alpha+1)} \left(1 - \frac{(\alpha+\beta+2)(\alpha+\beta)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta)} \right)$$

and

$$c_n = \frac{2n(n+\beta)(\alpha+\beta+2)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta)2(\alpha+1)} \qquad (n \in \mathbb{N}).$$

These sequences induce a polynomial hypergroup on $\mathbb{N}_0$ which is associated with Jacobi polynomials $(P_n^{(\alpha,\beta)})_{n \in \mathbb{N}_0}$. Moreover, the sequences defined above define a polynomial hypergroup if and only if the indices α and β satisfy the restrictions above (for details see the survey of Gasper [20, Theorem 3], and Lasser [30, Section 3(a)]). From Gasper [20, Theorem 4], it

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follows that every such polynomial hypergroup has Property (T). This fact was earlier used in Voit [49, Section 5.1].

If $\alpha = \beta$, then we have ultraspherical polynomials and the inequality above can be reduced to $\alpha \geq -1/2$. Since for $\alpha > -1/2$ the linearization coefficients $g_{m,n,m+n-2k}$ are known and positive ($m, n \in \mathbb{N}_0$, $k = 0, 1, \ldots, \min(m, n)$; see, for instance, Lasser [30, Section 3(a)], we can apply Lemma 8.10(2) and obtain that the conclusions of Theorem 8.8(4) are true. In fact, this result is just the main result of the paper of Ostrovskii and Truhina [37]. For $\alpha = \beta = -1/2$ we have Tchebichef polynomials of the first kind.

If $\alpha > \beta$, then the sequence $(b_n)_{n\in\mathbb{N}}$ is positive and increasing. Therefore, the assumptions of Lemma 8.10(1) are fulfilled. Thus we can apply Theorem 8.8(3) here.

8.12. Examples: generalized Tchebichef polynomials. Let $\beta > -1$ and $\alpha \geq \beta + 1$. Define

$$a_n = \begin{cases} \dfrac{l+\alpha+\beta-1}{2l+\alpha+\beta+1} & \text{if } n = 2l,\ l \in \mathbb{N}, \\ \dfrac{l+\alpha+1}{2l+\alpha+\beta+2} & \text{if } n = 2l+1,\ l \in \mathbb{N}_0, \end{cases}$$

$b_n = 0$ and $c_n = 1 - a_n$ $(n \in \mathbb{N}_0)$. Then we have a polynomial hypergroup with Property (T) whose associated orthogonal polynomials are generalized Tchebichef polynomials (for details see Lasser [30, Section 3(f)] and Voit [48, Section 5.2]).

Let $g_{m,n,k}$ and $g^{(\alpha,\beta)}_{m,n,k}$ respectively be the linearization coefficients of the generalized Tchebichef polynomials and of the Jacobi polynomials (with indices α and β) respectively. Then, using Equation (18.1) of Lasser [30] and the facts $g^{(\alpha,\beta)}_{j,j,2j-1} > 0$ and $g^{(\alpha,\beta)}_{j,j,1} > 0$ for $\alpha > 0$ (see Section 8.11), we get

$$g_{2j,2j,4j-2} = g^{(\alpha,\beta)}_{j,j,2j-1} > 0 \quad \text{and} \quad g_{2j,2j,2} = g^{(\alpha,\beta)}_{j,j,1} > 0.$$

Similarly, from [30, Equation (18.4)] it follows that

$$g_{2j+1,2j+1,4j} \geq \frac{2j+\beta+1}{4j+\alpha+\beta+2} \cdot g^{(\alpha,\beta+1)}_{j,j,2j} > 0$$

and

$$g_{2j+1,2j+1,2} \geq \frac{\alpha+1}{\alpha+\beta+1} \cdot g^{(\alpha,\beta+1)}_{j,j,0} > 0.$$

Therefore, we can apply Lemma 8.10(2) and obtain that the conclusion of Theorem 8.8(4) is true for generalized Tchebichef polynomials.

8.13. Examples: Grinspun polynomials. Fix $a \in \mathbb{R}$, $a \geq 2$, and define $a_1 = (a-1)/a$, $c_1 = 1/a$ and $a_n = c_n = 1/2$ $(n = 2, 3, \ldots)$. This

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sequence defines a polynomial hypergroup which is associated with Grinspun polynomials (see Lasser [30, 3(g)(ii)]). From Askey [1, Theorem 5], it follows that this hypergroup has Property (T). Moreover, the explicit representation of the linearization coefficients $g_{m,n,k}$ given by Lasser [30, 3(g)(ii)] and Lemma 8.10(2) ensure that part (4) of Theorem 8.8 can be applied for $a > 2$. For $a = 2$ we have again Tchebichef polynomials of the first kind.

8.14. EXAMPLES: q-ULTRASPHERICAL POLYNOMIALS. Fix β, q with $-1 < \beta < 1$ and $0 < q < 1$, and define

$$\widetilde{A}_n = \frac{1 - q^{n+1}}{2(1 - \beta q^n)}, \qquad n \in \mathbb{N}_0, \quad \text{and} \quad \widetilde{C}_n = \frac{1 - \beta^2 q^{n-1}}{2(1 - \beta q^n)}, \qquad n \in \mathbb{N}.$$

Then the q-ultraspherical polynomials $C_n(x; \beta \mid q)$ are given by

$$C_{-1}(x; \beta|q) = 0, \;\; C_0(x; \beta|q) = 1,$$
$$x \cdot C_n(x; \beta|q) = \widetilde{A}_n C_{n+1}(x; \beta|q) + \widetilde{C}_n C_{n-1}(x; \beta|q)$$

(see, for example, Bressoud [10]). The normalized orthogonal polynomials $\widetilde{C}_n(x; \beta|q) = C_n(x; \beta|q)/C_n(1; \beta|q)$ $(n \in \mathbb{N}_0)$ then define a polynomial hypergroup with Property (T) where the associated linearization coefficients $g_{m,n,m+n-2k}$ $(m, n \in \mathbb{N}_0,\; k = 0, 1, 2, \dots, \min(m, n))$ are positive (see Voit [49, Section 5.3]; the positivity of the linearization coefficients follows from Bressoud [10, Theorem 1]). Therefore the conclusion of Theorem 8.8(4) is true for q-ultraspherical polynomials.

8.15. EXAMPLES: ASSOCIATED LEGENDRE POLYNOMIALS. For a fixed $\nu \geq 0$ the associated Legendre polynomials $P_n^{\nu}(x)$ as defined in Section 3(b) of Lasser [30] bear a polynomial hypergroup structure with Property (T) (Property (T) is proved in Voit [49, Section 5.5]). Moreover, for all $m, n \in \mathbb{N}_0$ and $k = 0, 1, 2, \dots, \min(m, n)$, the linearization coefficients $g_{m,n,m+n-2k}$ are positive and thus the conclusions of part (4) of Theorem 8.8 are true for associated Legendre polynomials.

To prove the positivity of the linearization coefficients, we first note that up to positive normalization factors the $g_{m,n,m+n-2k}$ are equal to the linearization coefficients $b_{m,n,m+n-2k}$ introduced in Lasser [32]. By the theorem in Lasser [32], these coefficients can be written as

$$b_{m,n,m+n-2k} = R_{m,0} \cdot c(m, n, m+n-2k)$$
$$+ \sum_{j=1}^{k-1} R_{m,j} \cdot c(m-2j, n, m+n-2j) + R_{m,k}$$

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where $R_{m,0} = 1$, $R_{m,j} \geq 0$ $(j = 1, \dots, k-1$; see [32, page 407]), $c(m, n, m+n-2k) > 0$ and $c(m-2j, n, m+n-2k) \geq 0$ $(j = 1, \dots, k-1$; see [32, Proposition 1]). Hence $b_{m,n,m+n-2k} > 0$ as claimed.

8.16. EXAMPLES: GERONIMUS POLYNOMIALS. Fix $a, b \in \mathbb{R}$ with $a+b \leq 0$ and $ab+1 \geq 0$. If

$$a_n = \frac{n(a-1)(b-1)+2-a-b}{(2-a-b)(n(a-1)(b-1)+1-ab)} > 0,$$

$$b_n = -\frac{a+b}{2-a-b} \geq 0, \quad \text{and} \quad c_n = 1-a_n-b_n > 0 \qquad (n \in \mathbb{N}),$$

then the associated orthogonal polynomials are Geronimus polynomials which define a polynomial hypergroup structure with Property (T); for details we refer to Voit [49, Section 5.4].

For $a = -b$ the coefficients b_n are equal to 0, and we have $g_{n,n,2} > 0$ and $g_{n,n,2n-2} > 0$ by using [49, Equation (5.15)]. Therefore, part (4) of Theorem 8.8 can be applied.

For $a \neq -b$ we have $g_{n,n,1} > 0$ and $g_{n,n,2n-1} > 0$ again by using Equation (5.15) of [49]. Thus Lemma 8.10(1) shows that part (3) of Theorem 8.8 is applicable.

For $b = 0$ and $a = -k$ $(k \in \mathbb{N}_0)$, the semigroup $(M^1(\mathbb{N}_0), *)$ is isomorphic to the semigroup of all isotropic probability measures on a special infinite, finitely generated semigroup. For details we refer to Voit [48, Section 5.1.2]. There are further infinite, finitely generated semigroups for which the set of all isotropic probability measures are semigroups with respect to the usual convolution and for which this semigroups are isomorphic to polynomial hypergroups; see Soardi [43]. Since the polynomial hypergroups of Soardi [43] are always connected with Tchebichef polynomials of the second kind, Part (4) of Theorem 8.8 can be applied here (see Section 8.11, too).

8.17. EXAMPLES: POLYNOMIAL HYPERGROUPS ASSOCIATED WITH INFINITE DISTANCE-TRANSITIVE GRAPHS. Fix $a, b \in \mathbb{R}$ with $a, b \geq 2$. Taking

$$a_n = \frac{a-1}{a}, \qquad b_n = \frac{b-2}{a(b-1)}, \qquad c_n = \frac{1}{a(b-1)} \qquad (n \in \mathbb{N}),$$

we obtain a polynomial hypergroup with Property (T) which is associated with generalized Cartier polynomials (see Voit [51, Sections 5.3 and 5.6]).

If $b = 2$ and $a > 2$, then $b_n = 0$ for all $n \in \mathbb{N}$, and $g_{m,n,m+n-2k} > 0$ for $m, n \in \mathbb{N}_0$ and $k = 0, 1, 2, \dots, \min(m, n)$ by Voit [51, Equation (5.6)]. Therefore part (4) of Theorem 8.8 can be applied.

If $b > 2$, then $g_{m,n,m+n-1} > 0$ and $g_{m,n,|m-n|+1} > 0$ by [51, Equation (5.6)]. Therefore, Lemma 8.10(1) ensures that Theorem 8.8(3) is applicable.

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For $a = b = 2$ we have Tchebichef polynomials of the first kind.

We conclude this section by noting that for $a, b \in \mathbb{N}$, $a, b \geq 2$, the polynomial hypergroups above are isomorphic to the double coset hypergroups which arise when we consider the automorphism groups of infinite distance-transitive graphs and the stabilizer subgroups of fixed vertices. Moreover, all polynomial hypergroups which are isomorphic to double coset hypergroups are contained in the class introduced in this section; for details and further references we refer to Voit [51, Section 5].

8.18. EXAMPLES: POLYNOMIAL HYPERGROUPS SATISFYING A CONDITION OF ASKEY. Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ satisfy $a_n, c_n > 0$, $b_n \geq 0$, $a_n + b_n + c_n = 1$ and

$$a_1c_2 \geq c_1, \qquad a_nc_{n+1} \geq a_{n-1}c_n, \qquad b_n \geq b_{n-1}, \quad \text{for } n = 2, 3, \ldots .$$

Then these sequences define a polynomial hypergroup with Property (T) (see Voit [49, Section 5.7]).

If $b_n = 0$ for all $n \in \mathbb{N}$, then the condition above also implies that the assumptions of Lemma 8.10(2) are true. Therefore, the conclusion of part (4) of Theorem 8.8 is true for the symmetric case. In particular, Theorem 8.8(4) can be applied to Pollaczek polynomials, since Askey's condition holds for these orthogonal polynomials (see Lasser [30, Section 3(e)]).

Furthermore, if $b_2 > 0$ (and thus $b_n > 0$ for $n \geq 2$), then Askey's condition entails the assumptions of Lemma 8.10(1). Hence, the conclusion of part (3) of Theorem 8.8 is true for this nonsymmetric case.

We finish the treatment of the arithmetic of polynomial hypergroups by noting the following theorem about indecomposable measures. As the preceding discussion of examples shows, the assumptions of the theorem are true for all examples except for the polynomial hypergroups associated with Tchebichef polynomials of the first kind.

8.19. THEOREM. *Let* $(\mathbb{N}_0, *)$ *be a polynomial hypergroup. If either* $g_{m,n,m+n-1} > 0$ *for all* $m, n \in \mathbb{N}_0$ *or* $b_n = 0$ *and* $g_{m,n,m+n-2} > 0$ *for all* $m, n \in \mathbb{N}$, *then the set of all indecomposable factors is dense in* $M^1(\mathbb{N}_0)$ *with respect to the norm* $\|\cdot\|$.

PROOF. (cf. Ostrovskii and Truhina [37, Theorem 2]) We first assume that $g_{m,n,m+n-1} > 0$ for all $m, n \in \mathbb{N}$. If $\mu \in M^1(\mathbb{N}_0)$ satisfies $\operatorname{supp}\mu \subset \{0, 1, \ldots, n\}$, $n \in \operatorname{supp}\mu$ and $n-1 \notin \operatorname{supp}\mu$ for some $n \geq 1$, then μ is indecomposable. In fact, $\nu_1 * \nu_2 = \mu$ with $\nu_1 \neq \delta_0 \neq \nu_2$ would imply that ν_1 and ν_2 are finitely supported and that the numbers $n_i := \max\operatorname{supp}\nu_i$

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$(i = 1, 2)$ satisfy $n_1 + n_2 = n$ and $n_1 + n_2 - 1 \in \operatorname{supp}(\delta_{n_1} * \delta_{n_2}) \subset \operatorname{supp}\mu$ contradicting our assumption. Therefore, for arbitrary $\nu = \sum_{k=0}^{\infty} c_k\delta_k \in M^1(\mathbb{N}_0)$ and $N \in \mathbb{N}$ sufficiently large, the measure

$$\nu_N := (1 - 1/N) \cdot \left(\sum_{k=0}^{N} c_k\right)^{-1} \cdot \sum_{k=0}^{N} c_k\delta_k + \frac{1}{N}\delta_{N+2} \in M^1(\mathbb{N}_0)$$

is indecomposable. Since $\|\nu - \nu_N\| \to 0$ for $N \to \infty$, the proof of the first part is finished.

Since the proof of the second part is very similar to the first one (see also [37, Theorem 2]), we omit it.

To conclude this section we would like to point out a remarkable property of characters that are contained in the discrete set L:

8.20. Proposition. *For every* $x \in \operatorname{supp}\pi\backslash[-1, 1]$ *the following statements are equivalent:*

(1) $x < -x_0$;

(2) x *is isolated in* $D_s \cong \widehat{\mathbb{N}}_0$;

(3) *the character* $\alpha_x \in \widehat{\mathbb{N}}_0$ *corresponding to* x *(that is,* $\alpha_x(n) = P_n(x)$ *for* $n \in \mathbb{N}_0$*) is contained in* $L^1(\mathbb{N}_0, m)$, *which means*

$$\sum_{n=0}^{\infty} |P_n(x)| \cdot h_n = \|\alpha_x\|_1 < \infty.$$

Proof. The equivalence of (1) and (2) follows from Theorem 8.2. Furthermore, it follows from Nevai [35, Section 4.1, Theorem 18], that

$$\begin{aligned}\lim_{n\to\infty} \frac{h_{n+1}}{h_n}\left|\frac{P_{n+1}(x)}{P_n(x)}\right| &= \sqrt{\frac{\alpha}{\gamma}} \lim_{n\to\infty}\left|\frac{\widetilde{P}_{n+1}(x)}{\widetilde{P}_n(x)}\right| \\ &= \sqrt{\frac{\alpha}{\gamma}} \cdot \left(|x| + \sqrt{x^2 - 1}\right)^{-1} \begin{cases} > 1 & \text{for } x \in]-x_0, -1[\\ < 1 & \text{for } x < -x_0 \end{cases}\end{aligned}$$

for $x \in \operatorname{supp}\pi\backslash[-1, 1]$. This implies that (1) and (3) are equivalent for $x \neq -x_0$. Finally, since $-x_0$ is not isolated in D_s, Proposition 1.3 yields $\alpha_{-x_0} \notin L^1(K)$.

8.21. Remark. For some parameters, the polynomial hypergroups introduced in Section 8.16 and Section 8.17 have an isolated character α which is integrable with respect to the Haar measure by Proposition 8.20 (for details on $\operatorname{supp}\pi$ and $\widehat{\mathbb{N}}_0$ see Askey and Wilson [2, Equations (4.28)–(4.30)], Voit

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[49, Section 5.4], and Voit [51, Section 5.6]). In particular, from Proposition 1.3 it follows that $\alpha/\|\alpha\|_2^2 \cdot m$ is a nontrivial idempotent measure. This yields examples of polynomial hypergroups, and therefore of commutative hypergroups without proper compact subhypergroups, that have nontrivial idempotent measures. This fact shows that the classification of the idempotent measures on commutative hypergroups is much more complicated than in the case of locally compact abelian groups (for groups see Rudin [39]). For the classification of idempotent measures $\mu \in M_b(K)$ with $\|\mu\| \leq 1$ on a commutative hypergroup see Bloom [6].

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References

[1] R. Askey, 'Orthogonal polynomials and positivity', in *Studies in Applied Mathematics* **6**, Wave Propagation and Special Functions, Ludwig, D. and Olver, F. W. J., eds., 64–85, Philadelphia, SIAM 1970.

[2] R. Askey and J. A. Wilson, 'Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials', *Memoirs Amer. Math. Soc.* **319** (1985).

[3] C. Berg and G. Forst, *Potential Theory on Locally Compact Abelian Groups*, Springer, Berlin, Heidelberg, New York, 1975.

[4] N. H. Bingham, 'Positive definite functions on spheres', *Proc. Cambridge Philos. Soc.* **73** (1973), 145–156.

[5] N. H. Bingham, 'Factorization theory and domains of attraction for generalized convolution algebras', *Proc. London Math. Soc.* **23** (1971), 16–30.

[6] W. Bloom, 'Idempotent measures on commutative hypergroups', *Probability Measures on Groups* VIII, Proc. Conf., Oberwolfach, 1985, 13–23. Lecture Notes in Math. 1210, Springer, Berlin, Heidelberg, New York, 1986.

[7] W. Bloom and H. Heyer, 'The Fourier transform of probability measures on hypergroups', *Rend. Mat.* **2** (1982), 315–334.

[8] W. Bloom and H. Heyer, 'Convolution semigroups and resolvent families of measures on hypergroups', *Math. Z.* **188** (1985), 449–474.

[9] W. Bloom and H. Heyer, 'Continuity of convolution semigroups on hypergroups', *J. Theoret. Prob.* **1** (1988), 271–286.

[10] D. M. Bressoud, 'Linearization and related formulas for q-ultraspherical polynomials', *SIAM J. Math. Anal.* **12** (1981), 161–168.

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[11] H. Chebli, 'Opérateurs de translation généralisee et semigroupes de convolution'. *Théorie du Potentiel et Analyse Harmonique*, Proc. Conf. Strasbourg, 1973, pp. 33–59. Springer, Lecture Notes in Math. 404, Berlin, Heidelberg, New York; 1974.

[12] H. Chebli, 'Sur un theoreme de Paley-Wiener associe a la decomposition spectrale d'un operateur de Sturm-Liouville sur $]0, \infty[$', *J. Funct. Anal.* **17** (1974), 447–461.

[13] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.

[14] W. C. Connett and A. L. Schwartz, 'Analysis of a class of probability preserving measure algebras on compact intervals', *Trans. Amer. Math. Soc.*, **317** (1990), 371–393.

[15] C. F. Dunkl, 'The measure algebra of a locally compact hypergroup', *Trans. Amer. Math. Soc.* **179** (1973), 331–348.

[16] C. F. Dunkl and D. E. Ramirez, 'A family of countable compact P^*-hypergroups', *Trans. Amer. Math. Soc.* **202** (1975), 339–356.

[17] G. M. Fel′dman, 'On generalized Poisson distributions on groups', *Theory Probab. and its Appl.* **20** (1975), 641–644.

[18] U. Finkh, *Beiträge zur Wahrscheinlichkeitstheorie auf einer Kingman Struktur*, Dissertation. Tübingen, 1986.

[19] L. Gallardo and O. Gebuhrer, 'Lois de probabilite infiniment divisibles sur les hypergroupes commutatifs, discrets, denomerables', *Probability Measures on Groups VII*, Proc. Conf., Oberwolfach, 1983, 116–130. Springer, Lecture Notes in Math. 1064, Berlin, Heidelberg, New York, 1984.

[20] G. Gasper, 'Positivity and special functions', in *Theory and Applications of Special Functions*, ed. R. Askey, pp. 375–434, Academic Press, New York, 1975.

[21] H. Heyer, *Probability Measures on Locally Compact Groups*, Springer, Berlin, Heidelberg, New York, 1977.

[22] H. Heyer, 'Probability theory on hypergroups: a survey', *Probability Measures on Groups VII*, Proc. Conf., Oberwolfach, 1983, 481–550, Springer, Lecture Notes in Math. 1064, Berlin, Heidelberg, New York, 1984.

[23] H. Heyer, 'Convolution semigroups and potential kernels on a commutative hypergroup', in K. H. Hofmann et al. (eds.), *The Analytical and Topological Theory of Semigroups*, pp. 273–312, de Gruyter Berlin, New York, 1990.

[24] A. S. B. Holland, *Introduction to the Theory of Entire Functions*, Academic Press, New York, London, 1973.

[25] R. I. Jewett, 'Spaces with an abstract convolution of measures', *Adv. Math.* **18** (1975), 1–101.

[26] D. Kendall, 'Delphic semigroups, infinitely divisible regenerative phenomena, and the arithmetic of p-functions', *Z. Wahrscheinlichkeitsth. Verw. Geb.* **9** (1968), 163–195.

[27] J. F. C. Kingman, 'Random walks with spherical symmetry', *Acta Math.* **109** (1963), 11–53.

[28] T. Koornwinder, 'Jacobi functions and analysis of noncompact semisimple Lie groups', in R. A. Askey et al. (eds.), *Special Functions: Group Theoretical Aspects and Applications*, pp. 1–85, Reidel Dordrecht, Boston, Lancaster, 1984.

[29] J. Lamperti, 'The arithmetic of certain semi-groups of positive operators', *Proc. Cambridge Philos. Soc.* **64** (1968), 161–166.

[30] R. Lasser, 'Orthogonal polynomials and hypergroups', *Rend. Math. Appl.* **2** (1983), 185–209.

[31] R. Lasser, 'Bochner theorems for hypergroups and their application to orthogonal polynomial expansions', *J. Approx. Theory* **37** (1983), 311–327.

[32] R. Lasser, 'Convolution semigroups on hypergroups', *Pacific J. Math.* **127** (1987), 353–371.

[33] R. Lasser, 'Linearization of the product of associated Legendre polynomials', *SIAM J. Math. Anal.* **14** (1983), 403–408.

[34] Yu. V. Linnik, *Decomposition of Probability Distributions*, Oliver and Boyd, Edinburgh, London, 1964.

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[35] P. Nevai, 'Orthogonal polynomials', *Mem. Amer. Math. Soc.* **213** (1979).
[36] I. V. Ostrovskii, 'Description of the I_0 class in a special semigroup of probability measures', *Soviet Math. Dokl.* **14** (1973), 525–529.
[37] I. V. Ostrovskii and I. R. Truhina, 'The arithmetic of Schoenberg-Kennedy semigroups'. in *Questions of mathematical physics and functional analysis* (*Proc. Res. Sem. Inst. Low Temp. Phys. Engrg.*, Kharkov), pp. 11–19, 171, "Naukova Dunka", Kiev, 1976.
[38] K. R. Parthasarathy, *Probability Measures on Metric Spaces*, Academic Press, New York, London, 1967.
[39] W. Rudin, *Fourier analysis on Groups*, Interscience, New York, 1962.
[40] I. Z. Rusza and G. J. Szekely, *Algebraic Probability Theory*, Wiley, Chichester, New York, 1988.
[41] A. L. Schwartz, 'l^1-convolution algebras: representation and factorization', *Z. Wahrscheinlichkeitsth. Verw. Geb.* **41** (1977), 161–176.
[42] A. L. Schwartz, 'Classification of one-dimensional hypergroups', *Proc. Amer. Math. Soc.* **239** (1988), 1073–1081.
[43] P. M. Soardi, 'Limit theorems for random walks on discrete semigroups related to nonhomogeneous trees and Chebyshef polynomials', *Math. Z.* **200** (1989), 313–327.
[44] G. Szegö, *Orthogonal Polynomials, Amer. Math. Soc. Coll. Publ.* **23**, Providence, R.I., 1959.
[45] I. R. Truhina, 'A problem related to the arithmetic of probability measures on the sphere', *J. Soviet. Math.* **17** (1981), 2321–2333.
[46] M. Voit, 'Positive characters on commutative hypergroups and some applications', *Math. Z.* **198** (1988), 405–421.
[47] M. Voit, 'Positive and negative definite functions on the dual space of a commutative hypergroup', Analysis **9** (1989), 371–387.
[48] M. Voit, 'Laws of large numbers for polynomial hypergroups and some applications', *J. Theoret. Prob.* **3** (1990), 245–266.
[49] M. Voit, 'Central limit theorems for a class of polynomial hypergroups', *Adv. Appl. Prob.* **22** (1990), 66–87.
[50] M. Voit, 'Negative definite functions on commutative hypergroups', *Probability Measures on Groups IX*, Proc. Conf., Oberwolfach, 1988, 376–388, Springer, Lecture Notes in Math. **1379**, Berlin, Heidelberg, New York, 1989.
[51] M. Voit, 'Central limit theorems for random ralks on $\mathbb{N}_0$ that are associated with a sequence of orthogonal polynomials', *J. Multivariate Anal.* **34** (1990), 290–322.
[52] C. Vrem, 'Hypergroup joins and their dual objects', *Pacific J. Math.* **111** (1984), 483–495.
[53] Hm. Zeuner, 'One-dimensional hypergroups', *Adv. Math.* **76** (1989), 1–18.
[54] Hm. Zeuner, 'Properties of the cosh hypergroup', *Probability Measures on Groups IX*, Proc. Conf., Oberwolfach, 1988, 425–434, Springer Lecture Notes in Math. 1379, Berlin, Heidelberg, New York, 1989.

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