Math. Proc. Camb. Phil. Soc. (1999), **127**, 133 133
Printed in the United Kingdom

Simple motion systems and Banach spaces associated to uniformly bounded representations

By MATTHIAS MAYER and CHRISTIAN SALLER
Technische Universität München, Zentrum Mathematik, D-80333 München, Germany
e-mail: mayerm@mathematik.tu-muenchen.de
e-mail: sallerc@mathematik.tu-muenchen.de

(*Received* 29 *October* 1997; *revised* 26 *May* 1998)

Abstract

Given a uniformly bounded representation of a locally compact group, we consider the closed circled convex hull K of the orbit of a vector. We call K a simple motion system (SMS) and endow its linear hull with the Minkowski functional of K. The representation theory on these 'SMS-spaces' is discussed, in particular for C_0-representations, for irreducible representations of connected groups and for integrable representations. As an application we give a criterion for the decomposibility of representations.

1. *Introduction*

We describe the behaviour of a vector x under the action of a uniformly bounded representation π of a locally compact group in terms of an associated Banach-space, the 'SMS-space' to π and x. This is done in a geometric way.

We consider the closed circled convex hull K of the orbit of x which we call the simple motion system (SMS) and endow its linear hull with the Minkowski functional of K. Obviously this space is invariant under G however the restriction of π is not usually strongly continuous with respect to this Minkowski-norm.

In Section 2 we present this construction and discuss the representation theory on general SMS-spaces. Section 3 is devoted to the study of C_0-representations. Here the SMS-space and the space of strongly continuous vectors are revealed as the orbits of x under the measure algebra $\mathrm{M}(G)$ and the group algebra $\mathrm{L}^1(G)$, respectively.

Sections 4 and 5 are concerned with certain canonical SMS-spaces associated to irreducible GCR-representations of connected groups and to integrable representations of unimodular groups, respectively.

Finally Section 6 generalizes the concept of SMS-spaces. We show that the SMS-spaces associated with unitary representations are dual spaces. Furthermore the theory applies to certain subspaces of the Fourier–Stieltjes algebra. Using a theorem of Taylor, we give a criterion for the decomposibility of a representation into irreducible ones.

2. *Simple motion systems associated to group representations*

Throughout this paper let G be a locally compact second countable topological group and $(\pi, \mathscr{B}_\pi)$ be a uniformly bounded, strongly continuous representation of G on a Banach space $\mathscr{B}_\pi$. In addition we assume $(\pi, \mathscr{B}_\pi)$ to be cyclic.

Definition 2·1. Let $x \in \mathscr{B}_\pi$ be a cyclic vector with respect to $(\pi, \mathscr{B}_\pi)$. We call the closed circled convex hull of the orbit of x

$$K \coloneqq \mathrm{cl}\,(\mathrm{cco}\,\pi(G)x)$$

the *simple motion system* associated to π and x.

The dense subspace

$$E \coloneqq \mathrm{SMS}\,(\pi, x) \coloneqq \mathrm{span}\,K = \mathbb{R}^+ . K$$

of $\mathscr{B}_\pi$ endowed with the Minkowski functional of K

$$\|y\|_E \coloneqq \mu_K(y) \coloneqq \inf\{\lambda \in \mathbb{R}^+ \mid y \in \lambda . K\}$$

is called the *SMS-space* associated to π and x, the vector x its *starting vector*.

Example 2·2.

(i) Let $\lambda_{\mathbb{Z}}$ be the regular representation of the integers on $\ell^2(\mathbb{Z})$ and consider the starting vector $x \coloneqq \delta_0$. We find

$$\mathrm{cco}\,\lambda_{\mathbb{Z}}(\mathbb{Z})x = \left\{ \sum_{i=-n}^{n} \lambda_i \delta_i \mid n \in \mathbb{N}, \sum_{i=-n}^{n} |\lambda_i| \leqslant 1 \right\},$$

hence the unit ball $\ell^1(\mathbb{Z})_{\leqslant 1}$ of $\ell^1(\mathbb{Z})$ is densely contained in the simple motion system $K = \mathrm{cl}\,(\mathrm{cco}\,(\lambda_{\mathbb{Z}}(\mathbb{Z})\delta_0))$. Since the inclusion $\iota\colon\ \ell^1(\mathbb{Z}) \hookrightarrow \ell^2(\mathbb{Z})$ is weak-$*$ weak continuous, $\ell^1(\mathbb{Z})_{\leqslant 1}$ is weakly-compact, whence it is norm-closed in $\ell^2(\mathbb{Z})$. This yields $K = \ell^1(\mathbb{Z})_{\leqslant 1}$ and $\mathrm{SMS}\,(\pi, \delta_0) = \ell^1(\mathbb{Z})$.

(ii) Now consider the regular representation $(\lambda_{\mathbb{R}}, \mathrm{L}^2(\mathbb{R}))$ of the reals. A function $x \in \mathrm{L}^2(\mathbb{R})$ is cyclic if and only if its Plancherel-transform vanishes at most on a null set. We will see in the next section that

$$K = \{\mu * x \mid \mu \in \mathrm{M}(\mathbb{R})_{\leqslant 1}\}$$
$$E \coloneqq \mathrm{SMS}\,(\lambda_{\mathbb{R}}, x) \simeq \mathrm{M}(\mathbb{R}) \quad \text{(isometrically isomorphic).}$$

PROPOSITION 2·3. *With the notations of Definition* 2·1 *and* $M \coloneqq \sup_{g\in G} \|\pi(g)\|$ *we have*

(i) *The Minkowski functional* μ_K *defines a norm on* $E = \mathrm{SMS}\,(\pi, x)$.

(ii) *The unit ball* $E_{\leqslant 1}$ *of* E *coincides with the simple motion system* K.

(iii) *The embedding* $\iota\colon\ E \hookrightarrow \mathscr{B}_\pi$ *has norm less or equal* $M\|x\|_{\mathscr{B}_\pi}$. *In particular the transposed embedding* $\iota^{\mathrm{t}}\colon\ \mathscr{B}_\pi' \to E'$ *has norm less or equal* $M\|x\|_{\mathscr{B}_\pi}$.

(iv) *The normed space* $(E, \|\,.\,\|_E)$ *is complete.*

Proof. (i)–(iii) are immediate, since K is closed, circled convex and bounded. To prove (iv) consider a Cauchy sequence $\{y_n\}_{n\in\mathbb{N}}$ with respect to $\|\,.\,\|_E$. Then $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to $\|\,.\,\|_{\mathscr{B}_\pi}$ by (iii), hence has a limit y in $\mathscr{B}_\pi$ with respect to $\|\,.\,\|_{\mathscr{B}_\pi}$. But the Cauchy-property with respect to $\|\,.\,\|_E$ assures that for $\varepsilon > 0$ and a large enough number $N \in \mathbb{N}$

$$y_m \in y_n + \varepsilon . K \quad \forall m, n \geqslant N.$$

Since K is closed in $\mathscr{B}_\pi$ this yields $y \in y_n + \varepsilon . K$, hence $y \in E$ and $y_n \to y$ with respect to $\|\,.\,\|_E$. □

Thus the representation $(\pi, \mathscr{B}_\pi)$ associates to the starting vector x a dense subspace E in $\mathscr{B}_\pi$ carrying a Banach-norm which is finer than the original one. Obviously K is G-invariant and G acts on E via $\tilde{\pi}(g) := \pi(g)|_E$.

THEOREM 2·4. *We keep the notation of Definition* 2·1. *Then G acts on E by isometries. In fact, the unit ball K is invariant under the simple motions, that is*

$$\pi(\mu) . K \subseteq K \quad \textit{for } \mu \in \mathrm{M}(G)_{\leqslant 1}.$$

In particular the representation

$$\tilde{\pi}\colon \mathrm{M}(G) \to \mathrm{BL}(E), \quad \mu \mapsto \pi(\mu)|_E$$

is norm-decreasing.

Proof. Clearly, G acts by isometries. As to the remainder of the theorem, assume the statement not to be true. Then there exist $y \in K$ and $\mu \in \mathrm{M}(G)_{\leqslant 1}$ with $\pi(\mu)y \notin K$. But K is circled convex and closed in $\mathscr{B}_\pi$, whence the Hahn–Banach theorem forces a $\lambda \in \mathscr{B}'_\pi$ with

$$|\lambda(k)| \leqslant 1 \;\forall k \in K \quad \text{but} \quad |\lambda(\pi(\mu)y)| > 1.$$

This implies

$$1 < |\lambda(\pi(\mu))y| = \left|\int_G \lambda(\pi(g)y)\, d\mu(g)\right| \leqslant \int_G |\lambda(\pi(g)y)|\, d|\mu|(g) \leqslant \int_G d|\mu|(g) \leqslant 1,$$

a contradiction. □

Observe that it is not a priori clear that $\pi(\mu)$ is an element of E, which would simplify the proof above.

Our next aim is to describe the space

$$\tilde{E} := \{y \in E \mid g \mapsto \tilde{\pi}(g)y \text{ is continuous}\}$$

of the strongly continuous vectors for $\tilde{\pi}$ (which is closed by boundedness of $\tilde{\pi}$).

PROPOSITION 2·5. *For all* $\mu \in \mathrm{M}(G)$ *the integrated representation* $\tilde{\pi}(\mu)$ *of* $\tilde{\pi}|_{\tilde{E}}$ *coincides with* $\pi(\mu)|_{\tilde{E}}$.

Proof. For $\lambda \in \mathscr{B}'_\pi \subseteq E'$ and $y \in E$ we have

$$\lambda(\tilde{\pi}(\mu)y) = \int \lambda(\tilde{\pi}(g)y)d\mu(g) = \int \lambda(\pi(g)y)d\mu(g) = \lambda(\pi(\mu)y).$$

This shows the claim, since $\mathscr{B}'_\pi$ separates the points in $E \subseteq \mathscr{B}_\pi$. □

If E, F is a dual pair of vector spaces, we denote by $\sigma(E, F)$ and $\tau(E, F)$ the weak and the Mackey topology on E, respectively.

THEOREM 2·6. *We keep the notation of Definition* 2·1. *Then the space*

$$\tilde{E} := \{y \in E \mid g \mapsto \pi(g)y \textit{ is continuous}\}$$

of the strongly continuous vectors coincides with

$$\pi(\mathrm{L}^1(G))\,.\,E.$$

Thus $\tilde{\pi}$ *is strongly continuous on a* $\tau(E, \mathscr{B}'_\pi)$*-dense, norm closed, subspace of* E.

Proof. By Theorem 2·4 we have a continuous representation of $\mathrm{M}(G)$ on E, and furthermore the regular representation of G on $\mathrm{L}^1(G)$ is strongly continuous. Thus $\tilde{\pi}$ is strongly continuous on $\pi(\mathrm{L}^1(G))\,.\,E$.

On the other hand the factorization theorem (e.g. [**7**, **11**·**10**],) together with Proposition 2·5 shows

$$\tilde{E} = \tilde{\pi}(\mathrm{L}^1(G))\,.\,\tilde{E} = \pi(\mathrm{L}^1(G))\,.\,\tilde{E} \subseteq \pi(\mathrm{L}^1(G))\,.\,E,$$

which settles the first part of the theorem.

As to the Mackey-density, observe that $\pi(\mathrm{L}^1(G))E$ is dense in $\mathscr{B}_\pi$ with respect to the norm topology, hence $\sigma(\mathscr{B}_\pi, \mathscr{B}'_\pi)$-dense in $\mathscr{B}_\pi$. Therefore it is $\sigma(E, \mathscr{B}'_\pi)$-dense in E, thus $\tau(E, \mathscr{B}'_\pi)$-dense in E, by convexity. □

Note that the continuity properties on the SMS-space correspond to the continuity properties of the regular representation λ_G of G on $\mathrm{M}(G)$: the space of strongly continuous vectors of λ_G is exactly $\mathrm{L}^1(G) = \lambda_G(\mathrm{L}^1(G))\,.\,\mathrm{M}(G)$.

Thus in Example 2·2(ii) the representation $\tilde{\pi}$ is not strongly continuous on the whole SMS-space.

3. *SMS-spaces associated to* C_0*-representations*

In this section, we present a rather satisfactory description of SMS-spaces associated to an important class of representations, the C_0-representations.

Definition 3·1. A cyclic uniformly bounded representation $(\pi, \mathscr{B}_\pi)$ on a Banach space $\mathscr{B}_\pi$ is called a C_0*-representation*, if there exists a cyclic vector $x \in \mathscr{B}_\pi$, such that the matrix coefficients

$$v_{\lambda,x}(g) := \lambda(\pi(g)x)$$

belong to the space $C_0(G)$ of functions vanishing at infinity for all continuous linear functionals $\lambda \in \mathscr{B}'_\pi$. This implies immediately that for all $\lambda \in \mathscr{B}'_\pi, z \in \mathscr{B}_\pi$ the matrix coefficient $v_{\lambda,z}$ is in $C_0(G)$. If $(\pi, \mathscr{H}_\pi)$ is a unitary representation on a Hilbert space $\mathscr{H}_\pi$ this is equivalent to the fact that, for some cyclic vector $x \in \mathscr{H}_\pi$, the positive definite function

$$v_{x,x}(g) := \langle x, \pi(g)x\rangle$$

vanishes at infinity.

Example 3·2.

(i) The regular representation is a C_0-representation (cf. [**9**, 3·7]). In particular, the Duflo-Moore theorem ([**8**]) implies that square-integrable and integrable representations are C_0-representations.

(ii) If G is a semisimple connected Lie group with finite centre, every strongly continuous unitary representation on a Hilbert space $(\pi, \mathscr{H}_\pi)$ is a C_0-representation, provided that the restriction $\pi|_S$ of π to every simple noncompact component $S \lhd G$ does not contain the trivial representation of S. This is due to Howe and Moore ([**12**]).

(iii) A faithful irreducible representation of a minimal analytic group, i.e. a connected Lie group with compact centre and a closed adjoint group, has C_0-coefficients (cf. [**15**]). Similar statements are valid for connected so called totally minimal groups ([**16**])

We now calculate the SMS-space associated to a C_0-representation.

THEOREM 3·3. *Let $(\pi, \mathscr{B}_\pi)$ be a C_0-representation with cyclic vector x. Then*

$$K = \mathrm{cl}\,(\mathrm{cco}\,\pi(G)x) = \pi(\mathrm{M}(G)_{\leqslant 1})x.$$

In particular, $\mathrm{SMS}\,(\pi, x)$ *is the orbit of x under the action of* $\mathrm{M}(G)$.

Proof. We know from Theorem 2·4 that $\pi(\mathrm{M}(G)_{\leqslant 1})x \subseteq K$. To show the other inclusion, take $y \in \mathrm{cl}\,(\mathrm{cco}\,\pi(G)x)$ and a sequence $\{x_k\}_{k\in\mathbb{N}} \subseteq \mathrm{cco}\,\pi(G)x$ converging to y. Thus

$$x_k = \sum_{i=1}^{n_k} c_{ik}\pi(g_{ik})x \quad \text{with } n_k \in \mathbb{N},\ g_{ik} \in G,\ \sum_{i=1}^{n_k} |c_{ik}| \leqslant 1.$$

Hence with $\mu_k := \sum_{i=1}^{n_k} c_{ik}\delta_{g_{ik}} \in \mathrm{M}(G)_{\leqslant 1}$ we have

$$x_k = \pi(\mu_k)x.$$

Now, since G is separable, the unit ball $\mathrm{M}(G)_{\leqslant 1}$ is weak-$*$ sequentially compact, hence we may assume that $\{\mu_k\}_{k\in\mathbb{N}}$ converges to a $\mu \in \mathrm{M}(G)_{\leqslant 1}$ in the weak-$*$ topology. Together with the C_0-property of π, this implies for $\lambda \in \mathscr{B}'_\pi$

$$\begin{aligned}\lambda(y) &= \lim_{k\to\infty} \lambda(x_k) = \lim_{k\to\infty} \lambda(\pi(\mu_k)x)\\ &= \lim_{k\to\infty} \int_G \lambda(\pi(g)x)d\mu_k(g) = \lambda(\pi(\mu)x).\end{aligned}$$

An application of the Hahn-Banach theorem finishes the proof. □

To identify the Banach space structure of $(E, \|\,.\,\|_E)$ observe the following:

LEMMA 3·4. *Let E, F be Banach-spaces and $T\colon E \to F$ a continuous linear onto mapping. Then the factor space $E/\ker T$ endowed with the quotient norm is isometrically isomorphic to F if and only if $T(E_{\leqslant 1}) = F_{\leqslant 1}$.*

Proof. Standard.

THEOREM 3·5. *Let $(\pi, \mathscr{B}_\pi)$ be a C_0-representation and x a cyclic vector for π and $E := \mathrm{SMS}\,(\pi, x)$. Define $\mathscr{F}_x(\pi) := \{v_{\lambda,x} \mid \lambda \in \mathscr{B}'_\pi\} \subset C_0(G)$ and its polar $I := \mathscr{F}_x(\pi)^{\circ}$ in $\mathrm{M}(G)$. Then*

$$I = \{\mu \in \mathrm{M}(G) : \pi(\mu)x = 0\}$$

and I is a weak-$*$-closed left ideal in $\mathrm{M}(G)$. We have the following isometric isomorphisms:

$$E \underset{}{\overset{\text{isometrically}}{\cong}} M(G)/I \overset{\text{isometrically}}{\cong} \mathscr{F}_x(\pi)'$$

via

$$\mathrm{M}(G)/I \ni \mu + I \mapsto \pi(\mu)x \in E$$

and

$$\mathrm{M}(G)/I \ni \mu + I \mapsto (v_{\lambda,x} \mapsto \mu(v_{\lambda,x}) = \lambda(\pi(\mu)x)) \in \mathscr{F}_x(\pi)',$$

respectively. Furthermore, for $\lambda \in \mathscr{B}'_\pi \subset E'$ (*cf. Proposition* 2·3) *one has*

$$\|\lambda\|_{E'} = \|v_{\lambda,x}\|_\infty.$$

Proof. We have

$$\mu \in \mathscr{F}_x(\pi)^{\mathrm{o}} \iff 0 = \int v_{\lambda,x} d\mu(x) = \lambda(\pi(\mu)x) \quad \forall \lambda \in \mathscr{B}'_\pi$$
$$\iff \pi(\mu)x = 0.$$

Therefore, as a polar, I is weak-$*$-closed while by the second description it is a left ideal.

Now by Theorem 3·3 and the previous lemma the first isomorphism follows. The second one is due to general functional analysis.

Finally, for $\lambda \in \mathscr{B}'_\pi$, Theorem 3·3 implies:

$$\|\lambda\|_{E'} = \sup_{y \in E_{\leqslant 1}} |\lambda(y)| = \sup_{\mu \in M(G)_{\leqslant 1}} |\lambda(\pi(\mu)x)| = \sup_{\mu \in M(G)_{\leqslant 1}} |\mu(v_{\lambda,x})| = \|v_{\lambda,x}\|_\infty. \quad \square$$

The next theorem identifies the space of the strongly continuous vectors in E as the $\mathrm{L}^1(G)$-orbit of the starting vector x.

THEOREM 3·6. *In the situation of Theorem* 3·3 *the space* $\tilde{E}$ *of* $\tilde{\pi}$*-strongly continuous vectors in* E *coincides with the* $\mathrm{L}^1(G)$*-orbit of* x.

Proof. By Theorems 2·6 and 3·3 one has

$$\tilde{E} = \pi(\mathrm{L}^1(G))E = \pi(\mathrm{L}^1(G))\pi(\mathrm{M}(G))x = \pi(\mathrm{L}^1(G) * \mathrm{M}(G))x = \pi(\mathrm{L}^1(G))x.$$

COROLLARY 3·7. *The starting vector* x *is a strongly continuous vector for* $\tilde{\pi}$ *in its own* SMS*-space* $E = \mathrm{SMS}\,(\pi, x)$ *if and only if there is* $f \in \mathrm{L}^1(G)$ *with* $\pi(f)x = x$. *In this case,* $\tilde{\pi}$ *is strongly continuous on* E *and*

$$E = \pi(\mathrm{L}^1(G))x.$$

Example 3·8.

(i) Consider the regular representation $(\lambda_{\mathbb{R}}, \mathrm{L}^2(\mathbb{R}))$ of the reals and a cyclic vector $x \in \mathrm{L}^2(\mathbb{R})$, so that the Plancherel-transform $\hat{x}$ of x satisfies $\hat{x}(t) \neq 0$ a.e. Then for $\mu \in \mathrm{M}(\mathbb{R})$: $\lambda_{\mathbb{R}}(\mu)x = \mu * x = 0$ if and only if the Fourier–Stieltjes transform $\hat{\mu} = 0$, thus $I = \{0\}$. Therefore we find

$$E = \mathrm{SMS}\;(\lambda, x) \overset{\text{isometrically}}{\cong} \mathrm{M}(\mathbb{R}),$$
$$\tilde{E} \overset{\text{isometrically}}{\cong} \mathrm{L}^1(\mathbb{R}).$$

In particular the starting vector x is not strongly continuous in its SMS-space.

(ii) Now consider the representation of (i) as a representation of the discretisized group $\mathbb{R}_d$. The SMS-space remains of course unchanged, hence

$$\mathrm{SMS}\,(\lambda_{\mathbb{R}}, x) = \pi(\mathrm{M}(\mathbb{R}))x \supsetneq \pi(\ell^1(\mathbb{R}))x = \pi(\mathrm{M}(\mathbb{R}_d))x.$$

Thus Theorem 3·3 fails if the C_0-property is not satisfied.

(iii) Let $(\pi, \mathscr{B}_\pi)$ be an infinite dimensional cyclic representation of a compact group G with cyclic vector x. Then x is not strongly continuous in $E = \mathrm{SMS}\,(\pi, x)$. As to the proof, assume x to be strongly continuous. Since the circled convex hull of a compact set is totally bounded by Mazur's theorem ([**3**, VI·4·8]),

$$\tilde{K} \coloneqq \mathrm{cls}\,(\mathrm{cco}\,\tilde{\pi}(G)x, \|\,.\,\|_E)$$

is compact. As the topology on E is finer than that on $\mathscr{B}_\pi$, we have

$$\tilde{K} \subset K = E_{\leqslant 1}$$

On the other hand Proposition 2·5 and Theorem 3·3 imply:

$$K = \pi(\mathrm{M}(G)_{\leqslant 1})x = \tilde{\pi}(\mathrm{M}(G)_{\leqslant 1})x \subset \tilde{K}.$$

Hence the closed unit ball of E would be compact, forcing E to be finite dimensional. But this is a contradiction to x being cyclic for π.

(iv) If $(\pi, \mathscr{B}_\pi)$ is an irreducible C_0-representation and if $x \in \mathscr{B}_\pi$ such that the SMS-space $E = \mathrm{SMS}\,(\pi, x)$ is minimal with respect to inclusion among SMS-subspaces, then $\tilde{\pi}$ is strongly continuous on E. Take $0 \neq y \in \tilde{E}$; then, by Theorem 3·6, $y = \pi(f)x$ for a suitable $f \in \mathrm{L}^1(G)$. Since $y \in E$, the minimality of E implies that $E = \mathrm{SMS}\,(\pi, y) = \pi(\mathrm{M}(G))y$. Thus there is a $\mu \in \mathrm{M}(G)$ with $\pi(\mu)y = x$, whence one has $x = \pi(\mu)\pi(f)x = \pi(\mu * f)x$. But $\mu * f \in \mathrm{L}^1(G)$ and Corollary 3·7 shows the claim. (Note that irreducibility of π is needed to assure that the vector y used in the proof is cyclic, so that $\mathrm{SMS}(\pi, y)$ is well-defined.)

(v) On the other hand, if $(\pi, \mathscr{B}_\pi)$ is a C_0-representation and $x \in \mathscr{B}_\pi$ is a cyclic vector such that $E = \mathrm{SMS}\,(\pi, x)$ is maximal with respect to inclusion among SMS-subspaces, then again $\tilde{\pi}$ is strongly continuous on E. In fact, by the factorization theorem, there are $y \in \mathscr{B}_\pi$ and $f \in \mathrm{L}^1(G)$ with $x = \pi(f)y$, therefore y is cyclic for π and $x \in \mathrm{SMS}\,(\pi, y)$, so the maximality of E implies $\mathrm{SMS}\,(\pi, y) = E = \pi(\mathrm{M}(G))x$ and again we see $x \in \pi(\mathrm{L}^1(G))x$. Combined with (i) this shows that for every cyclic $f \in \mathrm{L}^2(\mathbb{R})$ there is a sequence $f_n \in \mathrm{L}^2(\mathbb{R})$ such that

$$\pi(\mathrm{M}(\mathbb{R}))f \subsetneq \pi(\mathrm{M}(\mathbb{R}))f_1 \subsetneq \pi(\mathrm{M}(\mathbb{R}))f_2 \ldots$$

(In other words, there is a strictly increasing sequence of subspaces of $\mathrm{L}^2(\mathbb{R})$ that are in a natural way algebraically isomorphic to $\mathrm{M}(\mathbb{R})$.)

4. *SMS-spaces of irreducible GCR-representations*

In this Section we are concerned with SMS-spaces associated to irreducible unitary GCR-representations. The significant fact here is the existence of a unique minimal SMS-space. The most important tool is the following result which is an easy generalization of a very deep one, due to Poguntke [**17**].

THEOREM 4·1. *Let $(\pi, \mathscr{H}_\pi)$ be an irreducible unitary representation of a connected locally compact group G. Furthermore let $(\pi, \mathscr{H}_\pi)$ be* GCR, *i.e. the operation of the C^*-algebra contains a compact operator. Then there exists $f \in \mathrm{L}^1(G)$ such that $\pi(f)$ is a nontrivial finite dimensional operator.*

Proof. The result in [**17**] shows the statement in the Lie case. Now let G be an arbitrary connected locally compact group. There exists a net $\{K_\alpha\}_\alpha$ of compact normal subgroups $K_\alpha \triangleleft G$ such that G is the projective limit of the Lie groups G/K_α. By [**2**, lemma 2] there is an α_0 with $K_\alpha \subseteq \ker \pi$ for all $\alpha \geqslant \alpha_0$. In particular for such an α the canonical representation π' of G/K_α with $\pi := \pi' \circ p_\alpha$ (where p_α denotes the canonical projection) is irreducible and Poguntke's result guarantees an $f \in \mathrm{L}^1(G/K_\alpha)$ such that $\pi'(f)$ is a nontrivial finite dimensional operator. Now the canonical onto homomorphism $\sigma : \mathrm{L}^1(G) \to \mathrm{L}^1(G/K_\alpha)$ fulfills $\pi = \pi' \circ \sigma$. Hence there exists $g \in \mathrm{L}^1(G)$ satisfying $\pi(g) = \pi'(\sigma(g)) = \pi'(f)$ and shows the statement. □

Our interest on L^1-functions that operate as finite rank operators is based on the following fact:

PROPOSITION 4·2. *Let $(\pi, \mathscr{H}_\pi)$ be an irreducible representation of a locally compact group and assume that*

$$J_\pi := \{f \in \mathrm{L}^1(G) \mid \pi(f) \textit{ has finite rank}\}$$

is not the kernel of π. Then

$$\mathscr{H}_{\mathrm{fin}} := \mathrm{span}\{\pi(f)x \mid x \in \mathscr{H}_\pi,\ f \in J_\pi\}$$

is a dense subspace and $\mathscr{H}_{\mathrm{fin}} = \pi(J_\pi)x$ for all $x \in \mathscr{H}_\pi$. In particular, $\mathrm{L}^1(G)$ acts algebraically irreducibly on $\mathscr{H}_{\mathrm{fin}}$. Furthermore $\mathscr{H}_{\mathrm{fin}}$ is the only subspace of $\mathscr{H}_\pi$ with the latter property.

Proof. The proof is based on that of theorem 2 in [**6**]. It is worked out in [**14**].

Remark 4·3. If G is unimodular, the vectors in $\mathscr{H}_{\mathrm{fin}}$ are the best integrable vectors in the following sense. If there exists a nontrivial coefficient function $v_{x,y} \in \mathrm{L}^p(G)$, $p \geqslant 1$, then for all u, w in $\mathscr{H}_{\mathrm{fin}}$ we have

$$v_{u,w} \in \mathrm{L}^p(G).$$

Indeed, take $x, y \in \mathscr{H}_\pi \setminus \{0\}$ with $v_{x,y} \in \mathrm{L}^p(G)$ and $u, w \in \mathscr{H}_{\mathrm{fin}}$. By the above, there are integrable functions f_u and f_w with

$$\pi(f_u)x = u \quad \text{and} \quad \pi(f_w)y = w.$$

Thus we find

$$v_{u,w} = v_{\pi(f_u)x, \pi(f_w)y} = f_u * v_{x,y} * \tilde{f}_w,$$

where $\tilde{f}(g) := \bar{f}(g^{-1})$. Now the usual convolution formulas (e.g. [**11**, 2·39]) show $v_{u,w} \in \mathrm{L}^p(G)$.

THEOREM 4·4. *Keep the assumptions and notations of Proposition 4·2. Then*

(i) *For all $x \in \mathscr{H}_\pi \setminus \{0\}$ the space $\mathscr{H}_{\mathrm{fin}}$ is contained in the strongly continuous part of* SMS (π, x) *and the restriction of $\tilde{\pi}$ to the closure of $\mathscr{H}_{\mathrm{fin}}$ is irreducible.*

(ii) *For every $y \in \mathscr{H}_{\mathrm{fin}}$ the* SMS*-space* SMS (π, y) *coincides with $\mathscr{H}_{\mathrm{fin}}$ and all the* SMS*-norms are equivalent.*

In particular $\mathscr{H}_{\mathrm{fin}}$ is the only minimal SMS*-space.*

Proof.

(i) By Theorem 2·6, $\tilde{E} \supseteq \pi(\mathrm{L}^1(G))E \supseteq \pi(J_\pi)x = \mathscr{H}_{\mathrm{fin}}$. The second statement is a consequence of the fact that $\pi(\mathrm{L}^1(G)).y$ contains $\mathscr{H}_{\mathrm{fin}}$ for all $y \in \mathscr{H}_\pi$.

(ii) This is a consequence of the algebraic irreducibility of $\mathscr{H}_{\mathrm{fin}}$ under $\mathrm{L}^1(G)$. The second assertion is trivial.

Remark 4·5.

(i) If the ideal J_π is dense in $\mathrm{L}^1(G)$, the representation $\tilde{\pi}$ is irreducible on the continuous part of every SMS-space. Indeed, for $x \in \mathscr{H}_\pi \setminus \{0\}$, the strongly continuous part of $E := \mathrm{SMS}(\pi, x)$ is $\tilde{E} = \pi(\mathrm{L}^1(G))E$, by Theorem 2·6. Thus the density of J_π and the strong continuity show that $\mathscr{H}_{\mathrm{fin}}$ is dense in $\tilde{E}$ and Theorem 4·4(i) is applicable. It is known that J_π is dense in $\mathrm{L}^1(G)$ if G is a connected Lie group which is nilpotent (cf. [**6**, theorem 2]) or semisimple (cf. Remark 4·5(ii)). But for the affine group J_π is not dense ([**13**, theorem 6]). More generally J_π is not dense in $\mathrm{L}^1(G)$ if the representation is not CCR.

(ii) Let $K < G$ be a compact subgroup and let $(\pi, \mathscr{H}_\pi)$ be a K-*finite representation*, i.e. the isotypic components of the K-irreducible subrepresentations are all finite dimensional. This is the case for irreducible representations of connected semisimple groups and for the euclidean motion groups (cf. [**19**, chapter 4·5·2]) if K is a maximal compact subgroup. Now, for all irreducible subrepresentations $\sigma < \pi|_K$, the orthogonal projection onto its (by assumption finite dimensional) isotypic component is given by $(1/d_\sigma)\pi(\overline{\chi}_\sigma)$, where χ_σ is the character of σ. Thus $\pi(f * \overline{\chi}_\sigma)$ has finite rank for all $f \in \mathrm{L}^1(G)$. In particular, the space of K-finite vectors with respect to π is contained in $\mathscr{H}_{\mathrm{fin}}$. By the Peter–Weyl theorem, the unique minimal SMS-space is the closure of the K-finite vectors with respect to the SMS-topology associated to any nonzero K-finite vector.

(iii) Observe that in the Lie case $\mathscr{H}_{\mathrm{fin}}$ contains a dense set of C^∞-vectors (cf. Theorem 4·1). But in general not all vectors in $\mathscr{H}_{\mathrm{fin}}$ are differentiable. Thus a G-orbit of a vector does not span $\mathscr{H}_{\mathrm{fin}}$, in general.

5. *SMS-spaces associated to integrable representations*

As in the last Section we reveal a canonical SMS-space associated to an irreducible representation $(\pi, \mathscr{H}_\pi)$, now described by a growth condition on the matrix coefficients.

We assume G to be *unimodular* but not necessarily connected and fix a Haar-measure μ_G on G.

Definition 5·1. A unitary irreducible representation $(\pi, \mathscr{H}_\pi)$ is *integrable* if there exists a vector $x \neq 0$ in $\mathscr{H}_\pi$ with $v_{x,x} \in \mathrm{L}^1(G)$. Any such x is called an *integrable* vector.

By Example 3·2(i), integrable representations are C_0-representations. We summarize some facts about integrable representations. Proofs and further results are contained in [**5**, sections 14·3, 14·4], for instance. Let d_π be the formal dimension of π. We have the following orthogonality relations (with respect to the inner products of $\mathrm{L}^2(G)$ and $\mathscr{H}_\pi$, respectively):

(i) $\langle v_{x,y}, v_{x',y'} \rangle = (1/d_\pi)\langle x, x' \rangle \langle y', y \rangle$;

(ii) $v_{x,y} * v_{x',y'} = (1/d_\pi)\langle x', y\rangle v_{x,y'}$;
(iii) $v_{y,x} * v_{x,x} = v_{y,x}$ if $\|x\| = \sqrt{d_\pi}$.

Furthermore in case $\|x\| = \sqrt{d_\pi}$ the right-convolution in $L^2(G)$ with the positive definite function $v_{x,x}$ is the orthogonal projection onto the space

$$\mathscr{K} := \{v_{h,x} \mid h \in \mathscr{H}_\pi\}$$

and the wavelet-transform $\mathscr{H}_\pi \ni h \mapsto v_{h,x}$ defines a unitary intertwining operator between π and the restriction of the regular representation to $\mathscr{K}$. We need the following easy observation:

PROPOSITION 5·2. *Let $x_1,\ x_2 \in \mathscr{H}_\pi \setminus \{0\}$ with $v_{x_i,x_i} \in L^1(G),\ i = 1, 2$. Then*

$$\pi(L^1(G))x_1 = \{y \mid v_{y,x_1} \in L^1(G)\} = \{z \mid v_{z,z} \in L^1(G)\} = \pi(L^1(G))x_2.$$

Proof. This is easily seen by [**10**, lemma 4·2]. □

This yields immediately

COROLLARY 5·3. *Let $\mathscr{H}_\pi$ be an integrable representation of the unimodular group G. Then*

(i) *For all integrable $x \in \mathscr{H}_\pi \setminus \{0\}$ the* SMS*-spaces coincide and have equivalent* SMS*-norms.*
(ii) *The* SMS*-space of the integrable vectors $E_{\rm int}$ is a minimal* SMS*-space. The canonical mapping*

$$L^1(G)/\tilde{I} \ni f + \tilde{I} \mapsto \pi(f)x, \quad (x \neq 0)$$

is norm-decreasing, where $\tilde{I} := \{f \in L^1(G) \mid \pi(f)x = 0\}$. The representation on $E_{\rm int}$ is strongly continuous.
(iii) *If, in addition, G is connected, $E_{\rm int}$ is the unique minimal* SMS*-space and is contained in all* SMS*-spaces. In particular, for every $y \in \mathscr{H}_\pi$ exists an $f \in L^1(G)$ such that $\pi(f)y$ is integrable.*

Proof.

(i) Is obvious by the above.
(ii) We assume without loss of generality $\|x\| = \sqrt{d_\pi}$. Thus we have

$$E := \mathrm{SMS}\,(\pi, x) = \pi(M(G))x = \pi(M(G) * v_{x,x})x \subseteq \pi(L^1(G))x.$$

The minimality follows from Proposition 5·2. Let $I := \{\mu \in M(G), \pi(\mu)x = 0\}$. Then

$$\|\pi(f)x\|_{\mathrm{SMS}\,(\pi,x)} = \|f + I\|_{M(G)} \leqslant \|f + \tilde{I}\|_1.$$

(iii) Now assume G to be connected. Since $(\pi, \mathscr{H}_\pi)$ is integrable, $\pi(f)$ is a Hilbert–Schmidt operator for every $f \in L^1(G) \cap L^2(G)$ (cf. [**5**, 14·4·3]). Therefore π is GCR and Theorem 4·4 shows that $E_{\rm int}$ coincides with $\mathscr{H}_{\rm fin}$. □

For noncompact groups it is not possible that all matrix coefficients are integrable (see for instance [**4**, p. 233]). In particular the canonical subspace $E_{\rm int}$ is always a proper subspace.

6. *SMS-type spaces*

In Theorem 3·5 we saw that the SMS-space E associated with a C_0-representation $(\pi, \mathscr{B}_\pi)$ is the dual of a Banach space, namely of the closure of $\mathscr{B}'_\pi$ with respect to the norm of E'. This result remains true in a more general functional analytic setting that is introduced below. These generalized SMS-spaces provide more information for rather general representations.

Moreover this theory applies to the concept of Fourier–Stieltjes algebras and Arsac-spaces. This yields, for instance, a necessary and sufficient criterium for a unitary representation on a separable Hilbert space to split into irreducible ones.

Definition 6·1. Let E and B be Banach spaces and $\iota\colon\ E \hookrightarrow B$ be a one-to-one continuous mapping with dense image such that

$$\iota(E_{\leqslant 1}) \text{ is closed in } B. \tag{$*$}$$

Then E is called an SMS-space in B. Very often we identify E with its dense image in B.

Definition 6·1 gives a natural generalization of the objects of our first five sections. To find new examples we need the following

PROPOSITION 6·2. *Let E, B be Banach spaces and $\iota\colon\ E \hookrightarrow B$ be continuous, one-to-one with dense image.*

(i) *If E is in addition the dual space of a Banach space $'E$ and $\iota^{\mathrm{t}}\colon\ B' \hookrightarrow E'$ is the transposed mapping, then E is an* SMS*-space in B, provided that*

$$\iota^{\mathrm{t}}(B') \subseteq {}'E \subseteq E',$$

where we identify $'E$ with the corresponding subspace of the bidual E'.

(ii) *If E and B are dual spaces of Banach spaces $'E$ and $'B$, respectively, then E is an* SMS*-space in B if the mapping ι is continuous with respect to the weak-$*$ topologies.*

Proof.

(i) The unit ball $E_{\leqslant 1}$ is compact with respect to $\sigma(E, \iota^{\mathrm{t}}(B'))$ since it is $\sigma(E, {}'E)$-compact by the Banach-Alaoglu theorem. Now ι is $\sigma(E, \iota^{\mathrm{t}}(B')) - \sigma(B, B')$-continuous, forcing $\iota(E_{\leqslant 1})$ to be $\sigma(B, B')$-closed. In particular $\iota(E_{\leqslant 1})$ is norm closed, proving the claim.

(ii) Is an immediate consequence of the Banach-Alaoglu theorem.

Example 6·3.

(i) In case $1 \leqslant r \leqslant p < \infty$ the sequence-space ℓ^r is an SMS-space in ℓ^p with respect to the canonical injection.

(ii) Let (X, μ) be a probability space. Then for $1 \leqslant r \leqslant p \leqslant \infty$ the Lebesgue-space $\mathrm{L}^p(X)$ is an SMS-space in $\mathrm{L}^r(X)$.

(iii) Let H be a Hilbert space and $\mathrm{L}^1(H)$, $\mathrm{L}^2(H)$, $\mathscr{K}(H)$ be the spaces of trace-class operators, Hilbert–Schmidt operators and compact operators, respectively. We have $\mathrm{L}^1(H) \subseteq \mathrm{L}^2(H) \subseteq \mathscr{K}(H) \subseteq \mathrm{BL}(H)$ and

$$\mathrm{L}^2(H)' = \mathrm{L}^2(H), \quad \mathscr{K}(H)' = \mathrm{L}^1(H).$$

This reveals $\mathrm{L}^1(H)$ as an SMS-space in $\mathrm{L}^2(H)$ and as an SMS-space in $\mathscr{K}(H)$ and $\mathrm{L}^2(H)$ as an SMS-space in $\mathscr{K}(H)$.

If we strengthen the condition $(*)$ to

$$\iota(E_{\leqslant 1}) \text{ is weakly-compact in } B, \qquad (**)$$

we find the following striking theorem:

THEOREM 6·4. *Let E be a* SMS*-space in the Banach space B satisfying the weak-compactness hypothesis* $(**)$. *Then $(E, \|\,.\,\|_E)$ is isometrically isomorphic to the dual*

$$(B', \|\,.\,\|_{E'})'$$

via the canonical mapping

$$E \ni e \mapsto (\lambda \mapsto \lambda(e)).$$

More precisely: *E is* (*canonically isometrically isomorphic to*) *the dual space of B' endowed with the norm inherited from E'* (*cf. Proposition* 2·3).

Proof.

(1) Let M equal $\{f \in E'' : f(\lambda) = 0 \text{ for all } \lambda \in B' \subset E'\}$, let $\Phi : E \to E''$ be the canonical embedding and let $\pi_M : E'' \to E''/M$ be the canonical projection. Consider the mapping $\psi = \pi_M \circ \Phi : E \to E''/M$.

(*a*) ψ is one-to-one. If $\psi(x) = 0$ then $\Phi(x) \in M$, thus $0 = \Phi(x)(\lambda) = \lambda(x)$ for all $\lambda \in B'$, hence $x = 0$.

(*b*) ψ is onto. Consider the following topological spaces.

- E endowed with $\sigma(E, B')$ (which is possible, since $i' : B' \to E'$ is one-to-one);
- E'' with $\sigma(E'', B')$ (this is possible, since $i' : B' \to E' \subset E'''$ is one-to-one);
- E''/M with $\sigma(E''/M, B')$ (which is possible for the following reasons: as $M \subset \ker(\lambda)$ for all $\lambda \in B' \subset E' \subset E'''$, the mapping $\tilde{\lambda} : x + M \mapsto \lambda(x)$ is well-defined for all $\lambda \in B'$. Also, the mapping $B' \ni \lambda \mapsto \tilde{\lambda} \in (E''/M)^*$ is one-to-one. For if $\tilde{\lambda} = 0$ one has $\tilde{\lambda}(x + M) = 0$ for all $x \in E''$, hence $\lambda(x) = 0$ for all $x \in E''$, especially for all $x \in E$; as E is dense in B, it follows that $\lambda = 0$.)

Now, observe that:

- $(E, \sigma(E, B'))$ is Hausdorff as B' separates points of $E \subseteq B$;
- $(E''/M, \sigma(E''/M, B'))$ is Hausdorff. Assume that for $x + M \in E''/M$, $\lambda(x + M) = 0$ for all $\lambda \in B'$. Then $x(\lambda) = 0$ for all $\lambda \in B'$, hence $x \in M$ and $x + M = 0_{E''/M}$; therefore B' separates points of E''/M;
- $\Phi : (E, \sigma(E, B')) \to (E'', \sigma(E'', B'))$ is clearly continuous;
- $\pi_M : (E'', \sigma(E'', B')) \to (E''/M, \sigma(E''/M, B'))$ is continuous, as $\sigma(E''/M, B')$ is just the final topology on E''/M with respect to π_M if E'' is endowed with $\sigma(E'', B')$.

Since $E_{\leqslant 1}$ is $\sigma(B, B')$-compact, it follows that it is $\sigma(E, B')$-compact and $\psi(E_{\leqslant 1}) = \pi_M \circ \Phi(E_{\leqslant 1})$ is $\sigma(E''/M, B')$-compact, in particular closed. On the other hand, $\Phi(E_{\leqslant 1})$ is $\sigma(E'', E')$-dense in $E''_{\leqslant 1}$, so by $B' \subset E'$ it is also $\sigma(E'', B')$-dense in $E''_{\leqslant 1}$. As π_M is continuous and onto, $\psi(E_{\leqslant 1}) = \pi_M \circ \Phi(E_{\leqslant 1})$ is $\sigma(E''/M, B')$-dense in $\pi_M(E''_{\leqslant 1})$. By closedness of $\psi(E_{\leqslant 1})$ with respect to $\sigma(E''/M, B')$, it follows that $\pi_M(E''_{\leqslant 1}) \subset \psi(E_{\leqslant 1})$, and hence $E''/M = \pi_M(E'') \subset \psi(E)$. Thus ψ is onto.

(2) By (1) we have a mapping $E \to E''/M$ that is one-to-one and onto. By the definiton of M and a well-known theorem in functional analysis, it follows that

$$E''/M \cong (B, \|\,.\,\|_{E'})'.$$

Hence the mapping

$$\kappa\colon (E, \|\,.\,\|_E) \to (B', \|\,.\,\|_{E'}), \quad \kappa(e)(\lambda) = \lambda(e) = i'(\lambda)(e)$$

is one-to-one and onto.

(3) κ is isometric: as $E''/M \cong (B, \|\,.\,\|_{E'})'$ is isometric, we have $\|\kappa\| = \|\psi\| = \|\pi_M \circ \Phi\| \leqslant \|\pi_M\|\,\|\Phi\| \leqslant 1$; whence for all $e \in E$ $\|\kappa(e)\| \leqslant \|e\|_E$. On the other hand, by definition, one has $\|\kappa(e)\| = \sup\{|\lambda(e)|\,|\,\|\lambda\|_{E'} \leqslant 1,\ \lambda \in B'\}$. Now take $e \in E$ with $\|e\|_E = 1$, then $te \notin E_{\leqslant 1}$ for every $t > 1$, hence by the Hahn-Banach theorem, for all $t > 1$ exists $\lambda_t \in B'$ with $|\lambda_t(k)| \leqslant 1$ for all $k \in E_{\leqslant 1}$, but $|\lambda_t(te)| \geqslant 1$. Thus $\|\lambda_t\|_{E'} \leqslant 1$, but $|\kappa(e)(\lambda_t)| \geqslant 1/t$. It follows that

$$\|\kappa(e)\| = \sup_{\|\lambda\|_{E'} \leqslant 1} |\lambda(e)| \geqslant \sup_{t>1} |\lambda_t(e)| \geqslant \sup_{t>1} |1/t| = 1 = \|e\|_E, \quad \text{as required.}$$

Example 6·5.

(i) If $(\pi, \mathscr{B}_\pi)$ is a uniformly bounded representation on a reflexive Banach space with cyclic vector x, SMS$\,(\pi, x)$ is an SMS-space in $\mathscr{B}_\pi$ satisfying $(**)$, whence it is a dual space. If $(\pi, \mathscr{B}_\pi)$ is a C_0-representation the simple motion system K is weakly compact, due to the fact that the mapping $\mathrm{M}(G) \ni \mu \mapsto \pi(\mu)x$ is weak-$*$ weak continuous. Furthermore, Theorem 3·5 shows that the space whose dual is SMS$\,(\pi, x)$ by Theorem 6·4, coincides with that one we have computed in Section 3.

(ii) More generally, in case that E is an SMS-space in a reflexive Banach space B, Theorem 6·4 shows

$$E = (B', \|\,.\,\|_{E'})'.$$

In particular, if B is a Hilbert space, the density of E in B yields

$$E = (B, \|\,.\,\|_{E'})' = (E, \|\,.\,\|_{E'})'.$$

We conclude this paper with an application of our results to Fourier–Stieltjes algebras. Let us recall at first some results concerning the Arsac-spaces associated to a unitary representation $(\pi, \mathscr{H}_\pi)$.

Denote by W_π the von Neumann algebra generated by the operators $\pi(g)$, $g \in G$ and consider the predual $\mathrm{W}_{\pi *}$, that is the dual of W_π with respect to the ultraweak topology $\sigma(\mathrm{BL}(\mathscr{H}_\pi), \mathrm{L}^1(\mathscr{H}_\pi))$. Then $\mathrm{W}_{\pi *}$ is canonically isomorphic to the quotient of $\mathrm{L}^1(\mathscr{H}_\pi)$ by the ultraweak polar W_π° of W_π.

Taking account of the fact that $\mathrm{L}^1(\mathscr{H}_\pi)$ is the projective tensor product of $\mathscr{H}_\pi$ with its dual Hilbert space $\overline{\mathscr{H}_\pi}$ we see that the canonical mapping

$$v_{x,y} \mapsto (T \mapsto \langle x, \pi(T)y\rangle), \quad x, y \in \mathscr{H}_\pi$$

extends to an isometric isomorphism between A_π, the closure of the space of matrix coefficients with respect to the norm of the Fourier–Stieltjes algebra $\mathrm{B}(G)$, and the predual $\mathrm{W}_{\pi *}$ (cf. [**1**, chapter 1]).

THEOREM 6·6. *Consider a unitary representation* $(\pi, \mathscr{H}_\pi)$ *on a separable Hilbert space* $\mathscr{H}_\pi$. *Then* π *splits into irreducible subrepresentations if and only if* $\mathrm{W}_\pi^{\mathrm{o}} \subseteq \mathrm{L}^1(\mathscr{H}_\pi)$ *is closed in* $\mathrm{L}^1(\mathscr{H}_\pi)$ *with respect to the Hilbert–Schmidt norm.*

Proof.
'$\Leftarrow$' Assume that the condition holds. Let $\mathrm{cl}\,(\mathrm{W}_\pi^{\mathrm{o}})$ be the closure of $\mathrm{W}_\pi^{\mathrm{o}}$ in $\mathrm{L}^2(\mathscr{H}_\pi)$. By assumption, $\mathrm{W}_\pi^{\mathrm{o}} = \mathrm{L}^1(\mathscr{H}_\pi) \bigcap \mathrm{cl}\,(\mathrm{W}_\pi^{\mathrm{o}})$, thus we have a norm-decreasing one-to-one mapping

$$\iota\colon \mathrm{A}_\pi \simeq \mathrm{L}^1(\mathscr{H}_\pi)/\mathrm{W}_\pi^{\mathrm{o}} \to \mathrm{L}^2(\mathscr{H}_\pi)/(\mathrm{cl}\,(\mathrm{W}_\pi^{\mathrm{o}})), \quad T + \mathrm{W}_\pi^{\mathrm{o}} \mapsto T + \mathrm{cl}\,(\mathrm{W}_\pi^{\mathrm{o}}).$$

Since $\mathrm{L}^1(\mathscr{H}_\pi)_{\leqslant 1}$ is weakly compact in $\mathrm{L}^2(\mathscr{H}_\pi)$, we have

$$\iota((\mathrm{L}^1(\mathscr{H}_\pi)/(\mathrm{W}_\pi^{\mathrm{o}}))_{\leqslant 1}) = \iota(\mathrm{L}^1(\mathscr{H}_\pi)_{\leqslant 1}) + \mathrm{cl}\,(\mathrm{W}_\pi^{\mathrm{o}})$$

is weakly compact and A_π is an SMS-space in $\mathrm{L}^2/(\mathrm{cl}\,(\mathrm{W}_\pi^{\mathrm{o}}))$. Hence Theorem 6·4 shows that A_π is a dual space and by a theorem of Taylor π splits (cf. [**18**, theorem 3·5]).

'$\Rightarrow$' We may assume π to be multiplicity free. Then

$$\pi \simeq \begin{pmatrix} \boxed{\rho_1} & & \mathbf{0} \\ & \boxed{\rho_2} & \\ \mathbf{0} & & \ddots \end{pmatrix}$$

with inequivalent, irreducible (and therefore disjoint) representations ρ_k. This implies

$$\mathrm{W}_\pi = \begin{pmatrix} \boxed{\mathrm{BL}(\mathscr{H}_{\rho_1})} & & \mathbf{0} \\ & \boxed{\mathrm{BL}(\mathscr{H}_{\rho_2})} & \\ \mathbf{0} & & \ddots \end{pmatrix}$$

and the operators in $\mathrm{W}_\pi^{\mathrm{o}}$ are those trace class operators on $\mathscr{H}_\pi$ with zeros on the diagonal, that is the operators of the form

$$T = \begin{pmatrix} \boxed{0} & & * \\ & \boxed{0} & \\ * & & \ddots \end{pmatrix}.$$

This space is closed in $\mathrm{L}^1(\mathscr{H}_\pi)$ with respect to the Hilbert–Schmidt norm.

Acknowledgements. We want to thank our teacher Professor G. Schlichting who motivated us to investigate simple motion systems.

REFERENCES

[**1**] G. ARSAC. Sur l'espace de Banach engendré par les coefficients d'une représentation unitaire. Thesis, Université de Lyon (1973).
[**2**] L. BAGETT and K. TAYLOR. Riemann–Lebesgue sets of $\mathbb{R}^n$ and representations which vanish at infinity. *J. Funct. Anal.* **28** (1978), 168–181.
[**3**] J. CONWAY. *A course in functional analysis* (Springer Verlag, 1985).
[**4**] M. COWLING. The Kunze Stein phenomenon. *Ann. Math.* **107** (1978), 209–234.
[**5**] J. DIXMIER. *C*-Algebras* (North-Holland, 1982).
[**6**] J. DIXMIER. Opérateurs de rang fini dans les répresentations unitaires. *Publ. Math. I.H.E.S.* **6** (1960), 305–317.

[**7**] F. F. Bonsall and J. Duncan. *Complete normed algebras* (Springer Verlag, 1973).

[**8**] M. Duflo and C. C. Moore. On the regular representation of a non unimodular locally compact group. *J. Funct. Anal.* **21** (1976), 209–243.

[**9**] P. Eymard. L'algébre de Fourier d'un groupe localement compact. *Bull. Soc. Math. France* **92** (1964), 181–263.

[**10**] H. G. Feichtinger and K. Gröchenig. A unified approach to atomic decompositions via integrable group representations, in *Function spaces and applications,* Lecture Notes in Math. vol. 1302 (Springer Verlag, 1988), pp. 52–73.

[**11**] G. Folland. *A course in abstract harmonic analysis* (Studies in advanced mathematics, CRC Press, 1995).

[**12**] R. Howe and C. C. Moore. Asymptotic properties of matrix coefficients. *J. Funct. Anal.* **32** (1979), 72–96.

[**13**] I. Khalil. Sur l'analyse harmonique du groupe affine de la droite. *Stud. Math.* **51** (1974), 139–167.

[**14**] A. Lüdeking and D. Poguntke. Cocycles on abelian groups and primitive ideals in group C^*-algebras of two-step nilpotent groups and connected Lie groups. *J. Lie Theory* **4** (1994), 39–103.

[**15**] M. Mayer. Asymptotics of matrix coefficients and closures of Fourier–Stieltjes algebras. *J. Funct. Anal.* **143** (1997), 42–54.

[**16**] M. Mayer. Strongly mixing groups. *Semigroup Forum* **54** (1997), 303–316.

[**17**] D. Poguntke. Unitary representations of Lie groups and operators of finite rank. *Ann. Math.* **140** (1994), 503–556.

[**18**] K. Taylor. Geometry of the Fourier algebras and locally compact groups with atomic unitary representations. *Math. Ann.* **262** (1983), 183–190.

[**19**] G. Warner. *Harmonic analysis on semisimple Lie groups* I (Springer Verlag, 1972).